Risk-Neutral Densities and their Application in the Piterbarg Framework

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Abstract

In this paper we consider two well-known interpolation schemes for the construction of the JSE Shareholder Weighted Top 40 implied volatility surface. We extend the Breeden and Litzenberger formula to the derivative pricing framework developed by Piterbarg post the 2007 financial crisis. Our results show that the statistical moments of the constructed risk-neutral densities are highly dependent on the choice of interpolation scheme. We show how the risk-neutral denity surface can be used to price options and briefly describe how the statistical moments can be used to inform trading strategies.

1 Introduction

It is well-known in Quantitative Finance literature that the volatility of an underlying asset is not constant as initially proposed by Black and Scholes (1973) [1]. Instead, for different strikes on the same underlying contract, the volatility displays a smile shape which is so significant that it has been termed the "volatility smile". Practitioners soon realised that inversion of the Black-Scholes (1973) [1] equation would lead to volatilities implied by market prices now known as "implied volatility". Nowadays it is convention to quote option contracts by implied volatility.

Implied volatility is a forward-looking measure which contains rich information if a continuum of these volatilities are available at different strikes and maturity dates. Unfortunately, implied volatility data is usually sparse and it is necessary to infer values by making use of interpolation/extrapolation techniques. Dumas et al. (1998) [7] and Gatheral (2004) [6] proposed parameterisation techniques for implied volatility with respect to strike which will be explained in Section 2.

Breeden and Litzenberger (1978) [5] show that the risk-neutral density function of an underlying asset based on the Black-Scholes (1973) [1] valuation formula can be obtained from a call option price surface by differentiating the call option prices twice with respect to strike . This is only possible if the call option price surface exists at a variety of strikes and maturity dates. We show the importance of the chosen interpolation/extrapolation scheme to create the surface, since different techniques lead to different risk-neutral densities. The risk-neutral density is a powerful tool which can be used to inform trading strategies.

More sophisticated pricing models have been developed after lessons learned from the 2007 global financial crisis. One such model is proposed by Piterbarg (2010) [2] where the author takes collateral into account when pricing derivatives. The literature regarding risk-neutral densities in the Black-Scholes (1973) [1] framework is well developed. However, not much research has been done on risk-neutral densities in the Piterbarg (2010) [2] framework. This paper attempts to bridge the gap. The rest of this paper is structured as follows: Section 2 introduces the interpolation/extrapolation schemes proposed by Dumas et al. (1998) [7] and Gatheral (2004) [6] respectively, Section 3 introduces the Breeden and Litzenberger (1978) [5] formula for risk-neutral densities in the Black-Scholes (1973) [1] framework which we extend to the Piterbarg (2010) [2] framework by making use of the work done by von Boetticher (2017) [3], Section 4 shows our numerical results applied to the JSE Shareholder Weighted Top 40 index as at 4 February 2019 and in Section 5 we conclude our findings.

2 Interpolation/Extrapolation Schemes

2.1 Dumas Quadratic Function

We define the implied volatility in the Black-Scholes (1973) [1] framework by $\sigma_{BS}(K,\tau)$ where K is the strike and $\tau = T - t$ is the time to expiry. Dumas et al. (1998) [7] proposed four equations for the shape of implied volatility:

$$\sigma_{BS}(K,\tau) = a_0, \tag{2.1}$$

$$\sigma_{BS}(K,\tau) = a_0 + a_1 K + a_2 K^2, \tag{2.2}$$

$$\sigma_{BS}(K,\tau) = a_0 + a_1 K + a_2 K^2 + a_3\tau + a_5 K\tau, \tag{2.3}$$

$$\sigma_{BS}(K,\tau) = a_0 + a_1 K + a_2 K^2 + a_3\tau + a_4\tau^2 + a_5 K\tau. \tag{2.4}$$

Equation (2.1) is the same constant volatility as in the Black-Scholes (1973) [1] model. Equation (2.2) is a function of strike only, which means that the model is fit to each time to expiration separately. Equations (2.3) and (2.4) are functions of both the strike and time to expiry.

Dumas et. al (1998) [7] conclude that model specifications with a time to maturity parameter perform the worst of all. Therefore, we only consider Equation (2.2) at this stage and will explain how to model the term structure of implied volatility in due course.

Equation (2.2) can be calibrated to market implied volatilities by minimising the sum of square error terms as follows:

$$\min_{\{a_0,a_1,a_2\}} \sum_{i=1}^{N} (\sigma_{Dumas_i} - \sigma_{BS_i})^2,$$

where σ_{Dumas_i} is the estimated implied volatility for strike i using Equation (2.2) proposed by Dumas et al. (1998) [7] and N is the number of observed market implied volatilities.

2.2 Gatheral SVI Parameterisation

The Stochastic Volatility Inspired (SVI) model introduced by Gatheral (2004) [6] is a popular model among Quantitative Finance practitioners. The model attemps to explain the volatility smile by making use of only 5 parameters. The "raw" SVI model initially proposed by Gatheral (2004) [6] takes the form:

$$w(x) = a + b\{\rho(x-m) + \sqrt{(x-m)^2 + \sigma^2}\}, \tag{2.5}$$

with left and right asymptotes:

$$w_{Left}(x) = a - b(1-\rho)(x-m),$$

$$w_{Right}(x) = a + b(1-\rho)(x-m),$$

where

- $x = \ln(\frac{K}{F_t})$ is the log forward moneyness with forward price $F_t = S_t e^{r\tau}$, S_t being the spot price of the underlying at time t and r the unique risk-free interest rate in the Black-Scholes framework.
- w is the total implied variance defined as $w(K,\tau) = \sigma_{BS}^2(K,\tau)\tau$.
- a gives the overall level of variance.
- b gives the angle between the left and right asymptotes.
- σ determines how smooth the vertex is.
- ρ determines the orientation of the graph.
- m shifts the graph left or right.

The model is calibrated in a similar fashion to the model proposed by Dumas et al. (1998) [7] by minimising the sum of square errors:

$$\min_{\{a,b,\rho,m,\sigma\}} \sum_{i=1}^{N} (\sigma_{SVI_i} - \sigma_{BS_i})^2,$$

where σ_{SVI_i} is the estimated implied volatility for strike i using the Gatheral (2004) [6] parameterisation and N is the number of observed market implied volatilities.

Both of the interpolation schemes described above only deal with interpolation/extrapolation in the strike (K) direction. Next we show how to interpolate implied volatilities in the time to maturity (τ) direction.

2.3 Interpolating Time to Maturity

For each fixed time to maturity, the techniques proposed by Dumas et al. (1998) [7] and Gatheral (2004) [6] allow us to infer implied volatilities for a wide range of strikes. Option contracts typically only trade at a limited number of maturities especially on the Johannesburg Stock Exchange (JSE) where options expire quarterly in March, June, September and December. It is therefore necessary to interpolate implied volatilities between sparse expiry dates in order for us to price contracts and obtain information regarding unknown expiries. We are interested in the work of Kahalé (2004) [4] where the author shows that the no-arbitrage condition with respect to time to maturity (τ) holds if and only if the total implied variance $\sigma^2_{BS}(K,\tau)\tau$ is an increasing function of τ. We apply the method as shown in his paper:

- For each time to maturity $\tau \in [\tau_i, \tau_{i+1}]$ and each strike K, we calculate the implied volatility $\sigma_{BS}(K,\tau)$ so that $\sigma^2_{BS}(K,\tau)\tau$ is a linear interpolation of $\sigma^2_{BS}(K,\tau_i)\tau_i$ and $\sigma^2_{BS}(K,\tau_{i+1})\tau_{i+1}$.

Once an implied volatility surface has been constructed covering a wide range of strikes and times to expiration, the next step is to build a risk-neutral density surface which contains information about the underlying asset through statistical moments i.e. mean, standard deviation or volatility, skewness and kurtosis. The next section reviews the work of Breeden and Litzenberger (1978) [5] and extends their equation to the Piterbarg (2010) [2] framework by making use of formulas derived by von Boetticher (2017) [3].

3 Risk-Neutral Densities

The risk-neutral probability meausure is crucial in Quantitative Finance as it allows us to price financial derivatives. The risk-neutral measure is not to be confused with the real-world probability measure. In fact, they can be vastly different since different investors require different risk premiums in the real-world, whereas all investors are assumed to be insensitive to risk in the risk-neutral world (Hull (2008) [11]). In the Black-Scholes (1973) [1] framework, the assumption is that there exists a unique constant risk-free rate r which can be used to price a European option V_t at time t as:

$$V_t = e^{-r\tau} E[f(S_T)], \tag{3.1}$$

where $f(S_T)$ is the terminal payoff function of a contract with underlying stock price S_T at maturity date T.

In the case of a European call option where:

$$\begin{aligned} f(S_T) &= max(S_T - K, 0), \\ &= (S_T - K)^+, \end{aligned}$$

Equation (3.1) can be written as:

$$\begin{aligned} V_t &= e^{-r\tau} \int_{-\infty}^{\infty} (S_T - K)^+ p(S_T) dS_T, \\ &= e^{-r\tau} \int_{K}^{\infty} (S_T - K) p(S_T) dS_T, \end{aligned}$$

where $p(S_T)$ is the risk-neutral probability density function of the underlying S_T at maturity T. Note the use of the words "risk-neutral". The reason being that the stock drifts at the risk-free rate r in the Black-Scholes (1973) [1] framework according to the stochastic differential equation:

$$dS_t = rS_t dt + \sigma_{BS} S_t dW_t, \tag{3.2}$$

where W_t is standard Brownian motion.

Breeden and Litzenberger (1978) [5] show that the risk-neutral probability density function $p(S_T)$ can be obtained by differentiating call prices twice with respect to strike (K). The proof is shown below:

First differentiate the market call price with respect to the strike (K) using the Leibniz integral rule:

$$\begin{aligned} \frac{\partial V}{\partial K} &= e^{-r\tau} \left\{ \int_{K}^{\infty} -p(S_T) dS_T \right\}, \\ \frac{\partial V}{\partial K} &= e^{-r\tau} \left\{ -(1 - \int_{0}^{K} p(S_T) dS_T) \right\}, \\ \frac{\partial V}{\partial K} &= e^{-r\tau} \left\{ \int_{0}^{K} p(S_T) dS_T - 1 \right\}, \\ e^{r\tau} \frac{\partial V}{\partial K} &= \left\{ \int_{0}^{K} p(S_T) dS_T - 1 \right\}. \end{aligned}$$

Differentiate again with respect to the strike (K) and use the Fundamental Theorem of Calculus to obtain the formula for the risk-neutral density function:

$$e^{r\tau} \frac{\partial^2 V}{\partial K^2} = p(K). \tag{3.3}$$

Given a call price surface at different strikes (K) and times to maturity (τ), Equation (3.3) allows us to construct a risk-neutral density surface on the domain (K, τ). This surface contains information about the level of the underlying at different forward-looking times. The interpolation/extrapolation algorithm that we choose to adopt plays a significant role in the construction of the risk-neutral density. We will explain this further in Section 4.

The assumption of a unique constant risk-free rate proposed in the Black-Scholes (1973) [1] model proved to be flawed after the 2007 global financial crisis. Borrowing and lending at the risk-free rate, ignoring the dynamics of the repo and collateral markets was a limiting assumption because "risk-free" no longer held true when Lehman Brothers defaulted. Piterbarg (2010) [2] extended the Black-Scholes (1973) [1] model by relaxing the assumption of a unique constant risk-free rate r and considered three different interest rates:

- r_F - the unsecured funding rate used to finance the derivative,
- r_C - the collateral rate paid on funds received,
- r_R^S - the rate gained in a repurchase agreement into which an underlying asset S is entered into.

The relationship between the three interest rates is given by $r_C \leq r_R^S \leq r_F$. Pitebarg (2010) [2] shows that in the absence of collateral, discounting occurs off the funding curve and in the presence of collateral, we discount off the collateral curve. We will denote zero collateral by ZC and full collateral by FC. The formulas for a European call option in the Piterbarg (2010) [2] framework and in the case of constant r_F and r_C are:

$$V_{ZC_t} = e^{-r_F \tau} E[(S_T - K)^+], \tag{3.4}$$

$$V_{FC_t} = e^{-r_C \tau} E[(S_T - K)^+]. \tag{3.5}$$

By using the same steps as in the proof of the Breeden and Litzenberger (1978) [5] formula in the Black-Scholes (1973) [1] framework, the formulas for the risk-neutral densities in the case of zero and full collateral in the Piterbarg (2010) [2] framework are given by:

$$e^{r_F \tau} \frac{\partial^2 V_{ZC}}{\partial K^2} = p(K), \tag{3.6}$$

$$e^{r_C \tau} \frac{\partial^2 V_{FC}}{\partial K^2} = p(K). \tag{3.7}$$

Note that the term "risk-neutral" in the Piterbarg (2010) [2] framework differs slightly compared to the Black-Scholes (1973) [1] framework since the stock drifts at the repurchase rate r_R^S. In this case we sell the underlying and agree to buy it back at a premium r_R^S, which leads to the following stochastic differential equation for the underlying:

$$dS_t = r_R^S S_t dt + \sigma_P S_t dW_t. \tag{3.8}$$

Hence, pricing occurs under the measure associated with the repurchase rate r_R^S. In Equation (3.8), σ_P refers to the implied volatility in the Piterbarg (2010) [2] framework which is a function of the strike (K) and the time to maturity (τ).

Levendis and Venter (2019) [8] show that the implied volatility in the Piterbarg (2010) [2] framework differs to the implied volatility in the Black-Scholes (1973) [1] framework, which leads to different prices for European call options. For the remainder of this paper we will consider the Piterbarg (2010) [2] framework and follow the steps below in our numerical testing:

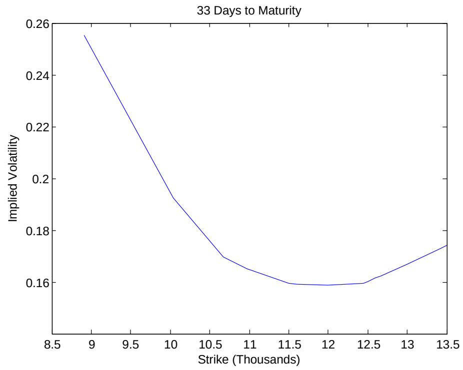
- Market implied volatilities will be calculated using the closed-form solution for European call options with zero collateral derived by von Boetticher (2017) [3]. Levendis and Venter (2019) [8] show how this is done in their paper.
- The interpolation/extrapolation schemes proposed by Dumas et. al (1998) [7] and Gatheral (2004) [6] will be used to interpolate the market implied volatilities in the strike (K) direction and the method proposed by Kahalé (2004) [4] will be used to interpolate in the time to maturity (τ) direction. We will create two implied volatility surfaces which we will then input into the Breeden and Litzenberger (1978) [5] formula which we extended to the Piterbarg (2010) [2] framework.
- From the constructed risk-neutral density surfaces, we will show how to price a binary option and we will tabulate the statistical moments including mean, volatility, skewness and kurtosis.

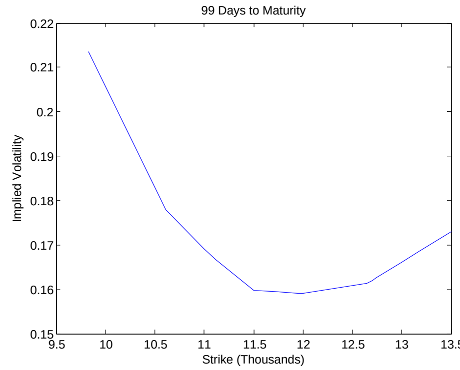
Our aim is to show that different interpolation/extrapolation algorithms lead to different statistical moments.

4 Numerical Results

4.1 Data

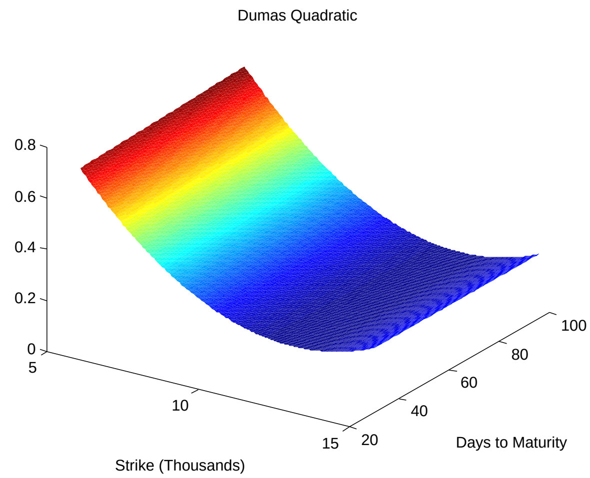
Data for the JSE Shareholder Weighted Top 40 Index as at 4 February 2019 was obtained from https://www.jse.co.za/downloadablefiles/equityderivatives/edmstats with spot $S_t = 10279$. Further, we assume that $r_R^S = 5\%$ and $r_F = 9\%$. The market implied volatility data is shown graphically for options expiring in 33 days and 99 days. Note that the JSE Shareholder Weighted Top 40 Index is quoted in Rands.

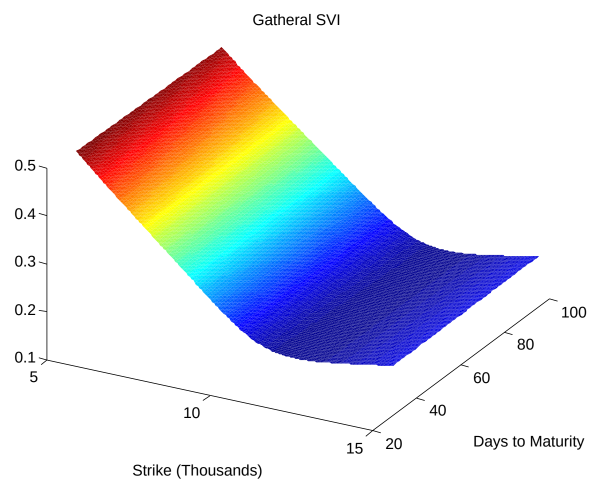




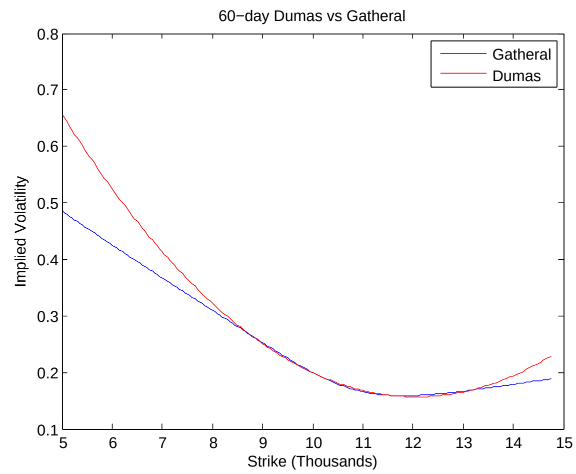
4.2 Constructing the Implied Volatility Surface

In this section we show the interpolated/extrapolated implied volatility surfaces using the data in Section 4.1. First, the algorithm proposed by Dumas et. al (1998) [7] making use of Kahalé (2004) [4] for time to maturity (τ) and then the algorithm by Gatheral (2004) [6] also utilising Kahalé (2004) [4].





The two implied volatility surfaces differ in the wings (deep in-the-money and deep out of the money options). The 60-day volatility smile is shown below as an illustration:



Gatheral (2004) [6] states that the total implied variance should increase linearly in the wings which is in line with our results shown above. The model proposed by Dumas et. al (1998) [7] leads to much higher implied volatilities at the wings. Next, we construct the call price surface which we input into the Breeden-Litzenberger (1978) [5] formula in order to obtain the risk-neutral densities.

4.3 Constructing the Call Price Surface

This section shows the required formulas for construction of the call option surface. The implied volatilities from the constructed implied volatility surface in Section 4.2 can be substituted into the closed-form solutions for European call options derived by von Boetticher (2017) [3]. The solutions for zero collateral and full collateral European call options are shown below:

$$V_{ZC_t} = e^{-\int_t^T r_F(u)du}(S_t e^{\int_t^T r_R^S(u)du} N(d_1) - KN(d_2)), \tag{4.1}$$

$$V_{FC_t} = e^{-\int_t^T r_C(u)du}(S_t e^{\int_t^T r_R^S(u)du} N(d_1) - KN(d_2)), \tag{4.2}$$

where

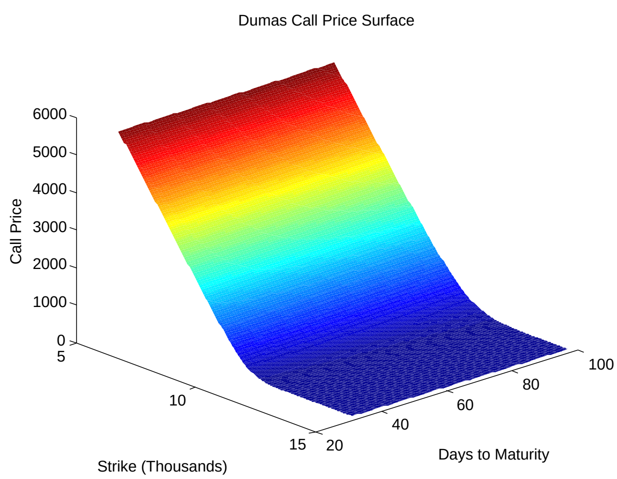
$$d_1 = \frac{\ln(\frac{S_t}{K}) + \int_t^T r_R^S(u)du + \frac{1}{2}\sigma_P^2\tau}{\sigma_P\sqrt{\tau}},$$

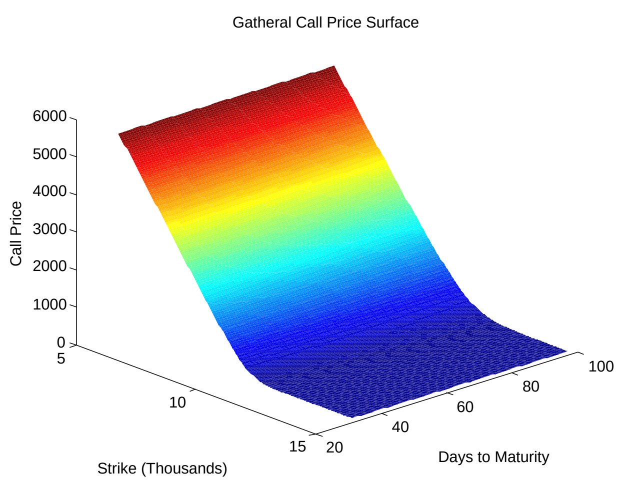
$$d_2 = d_1 - \sigma_P\sqrt{\tau},$$

and $N(\cdot)$ denotes the standard normal distribution function.

The work of von Boetticher (2017) [3] shows that a European call option that is fully collateralised is more expensive compared to a zero collateral call option. The interested reader is referred to [8] where a recent study confirms the result.

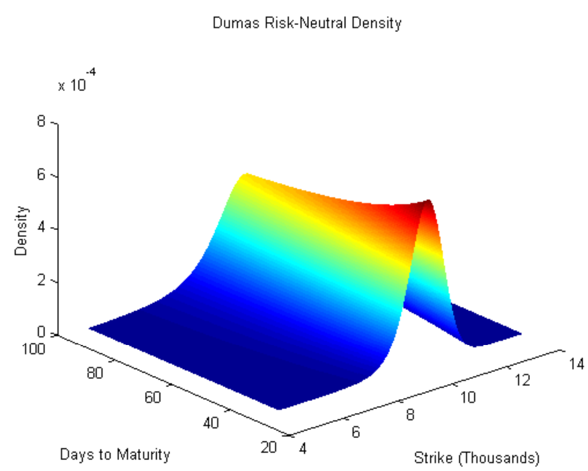
Using the constructed implied volatility surface by Dumas et. al (1998) [7] or Gatheral (2004) [6], the European call option surface for a zero collateral trade can be constructed by making use of Equation (4.1). Two call option surfaces are shown below by making use of the respective interpolation/extrapolation schemes. In the next section we introduce the risk-neutral density surface by using the call option surface as input into the Breeden-Litzenberger (1978) [5] formula.

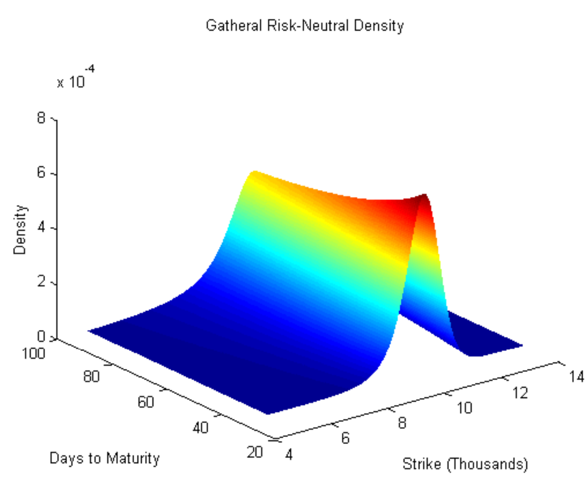




4.4 Constructing the Risk-Neutral Density Surface

In this section we will show how to construct the risk-neutral density surface and discuss its use. The call option surface from Section 4.3 can be used as input into Equation (3.6) in order to obtain the risk-neutral density surface in the Piterbarg (2010) [2] framework. The two risk-neutral density surfaces using the "Dumas Call Price Surface" and "Gatheral Call Price Surface" from Section 4.3 respectively are shown below:





The risk-neutral density surface can be used to price options. Consider a binary option as an example that pays 1 if the underlying is above a certain strike K and 0 otherwise. Mathematically, this can be written as:

$$f(S_T) = \begin{cases} 1 & S_T > K, \\ 0 & S_T \leq K. \end{cases}$$

Then, if the trade has no collateral, the price of a binary option in the Piterbarg (2010) [2] framework is given by:

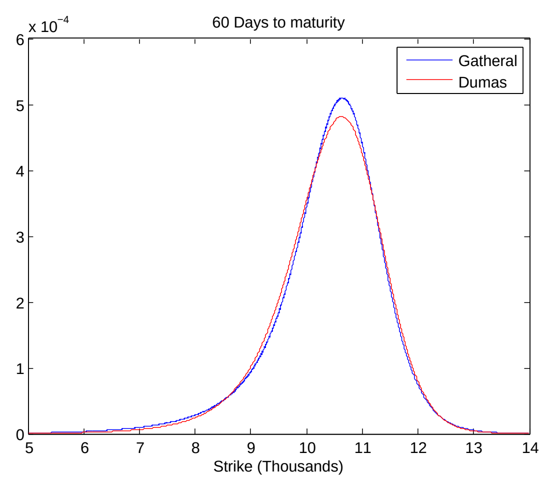
$$\begin{aligned} V_t &= e^{-r_F \tau} E[f(S_T)], \\ &= e^{-r_F \tau} \int_{-\infty}^{\infty} f(S_T) p(S_T) dS_T, \\ &= e^{-r_F \tau} \int_{K}^{\infty} (1) p(S_T) dS_T. \end{aligned}$$

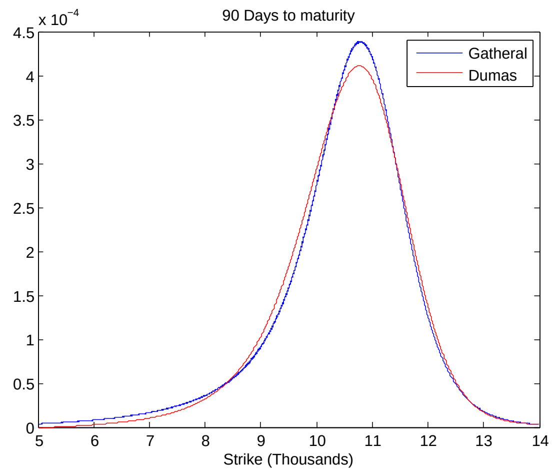
Here $p(S_T)$ is known from the risk-neutral density surface. The table below shows the prices for zero collateral binary options with strike $K = 10000$ expiring in 60 days and 90 days respectively:

	60 day price	90 day price
Dumas surface	0.69982	0.69948
Gatheral surface	0.70892	0.70522

The table indicates that the price of a binary option is dependent on the choice of interpolation/extrapolation scheme.

The graphs below compare the distribution of the underlying 60 days and 90 days in advance by using the respective interpolation/extrapolation algorithms:





The graphs indicate that different interpolation/extrapolation choices lead to different risk-neutral densities. Below we tabulate the statistical moments obtained from the respective algorithms:

Moment	60 days Dumas	60 days Gatheral	90 days Dumas	90 days Gatheral
Mean	10 402	10 398	10 498	10 439
Volatility	1 064	1 032	1 282	1 397
Skewness	-2.41	-1.61	-2.20	-2.38
Kurtosis	22.16	12.60	18.27	16.62

The table shows that the mean and volatility of the underlying do not differ much when comparing the two algorithms, but the skewness and kurtosis show some deviation. In a recent study, Flint and Maré (2017) [9] investigated a statistical moment trading strategy where their results were very promising. Their goal was to create a risk-neutral density surface weekly and based on the skewness or kurtosis, either hold the underlying or move to cash. If the skewness or kurtosis for the current week was higher compared to the previous week, then the investor would hold the underlying, else they would hold cash. Our results show that the choice of interpolation/extrapolation scheme could impact this trading strategy, especially if the trading strategy relies on the relative week-on-week movement in the skewness or kurtosis. We do not attempt to favour any interpolation/extrapolation algorithm because the choice of scheme will be unique to each problem.

5 Conclusion

In this paper we investigated two interpolation/extrapolation schemes for implied volatility based on JSE Top 40 data as at 4 February 2019. We showed how to create a risk-neutral density surface in the Piterbarg (2010) [2] framework from the implied volatility surface. The risk-neutral densities constructed from the Breeden-Litzenberger (1978) [5] formula lead to an elegant and simple way to price options with non-complex payoff features. Another powerful feauture is the forward-looking information contained in the densities as shown by Flint and Maré (2017) [9].

Our results show that the choice of interpolation/extrapolation scheme has an impact on the statistical moments and also leads to different option prices. Each scheme has its own strenghts and weaknesses regarding the complexity of implementation and accuracy. We do not attempt to favour a particular method, but merely point out that different techniques lead to different answers.

Areas for future research might include the implementation of the Ross (2015) [10] recovery theorem as shown in a recent paper by Flint and Maré (2017) [9], where the theorem provides a way in which real-world probabilities can be extracted.

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