Discrete Mathematics 340 (2017) 1008–1011

Contents lists available at ScienceDirect

Discrete Mathematics

journal homepage: www.elsevier.com/locate/disc

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Clique coloring B_1-EPG graphs

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ARTICLE INFO

Article history:
Received 28 February 2016
Received in revised form 16 January 2017
Accepted 17 January 2017

Keywords:
Clique coloring
Edge intersection graphs
Paths on grids
Polynomial time algorithm

ABSTRACT

We consider the problem of clique coloring, that is, coloring the vertices of a given graph such that no (maximal) clique of size at least two is monocolored. It is known that interval graphs are 2-clique colorable. In this paper we prove that B_1-EPG graphs (edge intersection graphs of paths on a grid, where each path has at most one bend) are 4-clique colorable. Moreover, given a B_1-EPG representation of a graph, we provide a linear time algorithm that constructs a 4-clique coloring of it.

1. Introduction

An *EPG representation* $\langle \mathcal{P}, \mathcal{G} \rangle$ of a graph G, is a collection of paths $\mathcal{P}$ of the two-dimensional grid $\mathcal{G}$, where the paths represent the vertices of G in such a way that two vertices are adjacent in G if and only if the corresponding paths share at least one edge of the grid. A graph which has an EPG representation is called an *EPG graph* (EPG stands for Edge-intersection of Paths on a Grid). In this paper, we consider the subclass B_1-EPG. A *B_1-EPG representation* of G is an EPG representation in which each path in the representation has at most one *bend* (turn on a grid point). Recognizing B_1-EPG graphs is an NP-complete problem [11]. Also, both the coloring and the maximum independent set problem are NP-complete for B_1-EPG graphs [9].

EPG graphs have a practical use, for example, in the context of circuit layout setting, which may be modeled as paths (wires) on a grid. In the knock-knee layout model, two wires may either cross or bend (turn) at a common point grid, but are not allowed to share a grid edge; that is, overlap of wires is not allowed. In this context, some of the classical optimization graph problems are relevant, for example, maximum independent set and coloring. More precisely, the layout of a circuit may have multiple layers, each of which contains no overlapping paths. Referring to a corresponding EPG graph, then each layer is an independent set and a valid partitioning into layers corresponds to a proper coloring.

In this paper, we consider the problem of *clique coloring*, that is, coloring the vertices of a given graph such that no (maximal) clique of size at least two is monocolored. Clique coloring can be seen also as coloring the clique hypergraph of a graph. The question of coloring clique hypergraphs was raised by Duffus et al. in [8].

We prove that B_1-EPG graphs are 4-clique colorable. Moreover, given a B_1-EPG representation of a graph, we provide a linear time algorithm that constructs a 4-clique coloring of it.

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http://dx.doi.org/10.1016/j.disc.2017.01.019

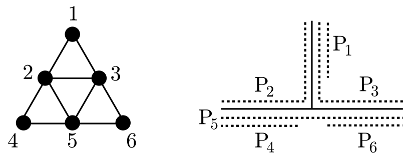

Fig. 1. A B_1-EPG representation of the 3-sun. The central triangle {2, 3, 5} is a claw clique; the other three triangles are edge cliques.
Source: Figure from [10].

2. Preliminaries

All graphs considered here are connected, finite and simple, we follow the notation of [2]. The vertex set of a graph G is denoted by $V(G)$. A *complete graph* is a graph that has all possible edges. A *clique* of a graph G is a maximal complete subgraph of G.

A *k-coloring* of a graph G is a function $f : V(G) \to \{1, 2, \ldots, k\}$ such that $f(v) \neq f(w)$ for adjacent vertices $v, w \in V(G)$. The *chromatic number* $\chi(G)$ of a graph G is the smallest positive integer k such that G has a k-coloring. A *k-clique coloring* of a graph G is a function $f : V(G) \to \{1, 2, \ldots, k\}$ such that no clique of G with size at least two is monocolored. A graph G is *k-clique colorable* if G has a k-clique coloring. The *clique chromatic number* of G, denoted by $\chi_c(G)$, is the smallest k such that G has a k-clique coloring.

Clique coloring has some similarities with usual coloring. For example, every k-coloring is also a k-clique coloring, and $\chi(G)$ and $\chi_c(G)$ coincide if G is triangle-free. But there are also essential differences, for example, a clique coloring of a graph need not be a clique coloring for its subgraphs. Indeed, subgraphs may have a greater clique chromatic number than the original graph. Another difference is that even a 2-clique colorable graph can contain an arbitrarily large clique. It is known that the 2-clique coloring problem is NP-complete, even under different constraints [1,12].

Many families of graphs are 3-clique colorable, for example, comparability graphs, co-comparability graphs, circular arc graphs and the k-powers of cycles [3,4,7,8]. In [1], Bacsó et al. proved that almost all perfect graphs are 3-clique colorable and conjectured that all perfect graphs are 3-clique colorable. This conjecture was recently disproved by Charbit et al. [6], who show that there exist perfect graphs with arbitrarily large clique chromatic number. Previously known families of graphs having unbounded clique chromatic number are, for example, triangle-free graphs, UE graphs (edge intersection graphs of paths in a tree), and line graphs [1,5,13].

It has been proved that chordal graphs, and in particular interval graphs, are 2-clique colorable [14]. Moreover, the following result holds for strongly perfect graphs, a superclass of chordal graphs.

Lemma 1 (*Bacsó et al. [1]*). *Every strongly perfect graph admits a 2-clique coloring in which one of the color classes is an independent set.*

For chordal graphs, such a coloring can be easily obtained in linear time, by a slight modification of the 2-clique coloring algorithm for chordal graphs proposed in [14]. Namely, let $v_1, \ldots, v_n$ be a perfect elimination ordering of the vertices of a chordal graph G, i.e., for each i, $N[v_i]$ is a clique of $G[\{v_i, \ldots, v_n\}]$; color the vertices from v_n to v_1 with colors a and b in such a way that v_n gets color a and v_i gets color b if and only if all of its neighbors that are already colored got color a. A perfect elimination ordering of the vertices of a chordal graph can be computed in linear time [15].

3. B_1-EPG graphs are 4-clique colorable

In this section, we prove that B_1-EPG graphs are 4-clique colorable. We need the following definitions and theorem.

Let $\langle \mathcal{P}, \mathcal{G} \rangle$ be a B_1-EPG representation of a graph G. A clique C of G is an *edge clique* of $\langle \mathcal{P}, \mathcal{G} \rangle$ if all the paths of $\mathcal{P}$ that correspond to the vertices of C share a common edge of the grid $\mathcal{G}$. A clique C of G is a *claw clique* of $\langle \mathcal{P}, \mathcal{G} \rangle$ if there is a point x of the grid and three edges of the grid sharing x (they may be shaped $\perp$, $\top$, $\vdash$, or $\dashv$), such that each path of $\mathcal{P}$ that corresponds to a vertex of C contains two of these three edges, and every pair of these three edges is contained in at least one path P of $\mathcal{P}$ (so, it is not an edge clique). We say that the claw clique is *centered at x*, or that x is the *center* of the claw clique. An example can be seen in Fig. 1.

Theorem 2 (*Golumbic et al. [10]*). *Let $\langle \mathcal{P}, \mathcal{G} \rangle$ be a B_1-EPG representation of a graph G. Every clique in G corresponds to either an edge clique or a claw clique in $\langle \mathcal{P}, \mathcal{G} \rangle$.*

Now we can prove the main result of this paper.

Theorem 3. *Let G be a B_1-EPG graph. Then, G is 4-clique colorable. Moreover, given a B_1-EPG representation of G, a 4-clique coloring of G in which one of the color classes is an independent set can be obtained in linear time on the number of vertices and edges.*

Proof. Let $\langle \mathcal{P}, \mathcal{G} \rangle$ be a B_1-EPG representation of the graph G. Each path of $\mathcal{P}$ is composed of either a single segment, formed by one or more edges on the same row or column of the grid $\mathcal{G}$, or of two segments sharing a point of the grid, one horizontal (i.e., in a row) and one vertical (i.e., in a column). We will first assign colors independently to the horizontal and vertical segments of each path, and then we will show how to combine those colors into a single color for each path, as required.

First, we use Lemma 1 to color the segments on each row and each column of $\mathcal{G}$ as if they were vertices of an interval graph, with two colors a and b, such that the segments colored b form an independent set, i.e., a pairwise non intersecting set, on each row (respectively column).

We thus obtain four types of paths according to the colors given to their corresponding segments: (a, a), (a, b), (b, a), (b, b), where the first component corresponds to the horizontal segment of the path and the second component corresponds to the vertical segment of the path; if one of these parts does not exist, we assign an a to the missing component.

Observe that

the color class (b, b) is an independent set. (1)

Let us now investigate which cliques could be monocolored. Edge cliques of $\langle \mathcal{P}, \mathcal{G} \rangle$ are also cliques of the interval graph corresponding to the row of the grid, respectively column of the grid, to which the edge (where all the paths of the clique intersect) belongs. Thus, the colors of the paths in such a clique have to be different in the horizontal component (respectively in the vertical component), that is, the clique is not monocolored.

Let us now turn to the claw cliques. Suppose that there is a claw clique which is monocolored. Then this clique contains at least two paths whose horizontal segments overlap, and have the same color, and the same is true for vertical segments. Since the horizontal segments colored b form an independent set, and the same is true for the vertical segments, the only possible coloring of the paths in our monocolored claw clique is (a, a).

Now, for each point x of the grid which is the center of one or more claw cliques monocolored (a, a), we will perform a recoloring of one or two paths having a bend at x. In this way, each path will be recolored at most once, as it has at most one bend. Paths without bends will not be recolored.

The order in which we process the points x of the grid does not matter: The recolorings are independent of recolorings at other grid points. In the recoloring we will assign color b to some segments that were originally colored a, obeying the following rules, for any fixed point x of the grid:

(I) the recolored paths either get color (a, b) or (b, a),
(II) every segment of a path with a bend at x that is recolored b is contained in a segment of a path with a bend at x that is colored a;
(III) if we recolor two paths with a bend at x, they only share x (i.e., they are shaped $\llcorner$ and $\urcorner$ or $\ulcorner$ and $\lrcorner$);
(IV) after recoloring, there is no claw clique colored (a, a) centered at x.

It will be explained below how to construct a recoloring obeying rules (I)–(IV). Once such a recoloring is found, the segments colored b may no longer be an independent set, but properties (I)–(III) prevent us from creating new monocolored cliques. Indeed, property (II) ensures that we create no monocolored edge clique. We claim that properties (I)–(III) guarantee we have no new monocolored claw clique. Assume instead that a claw clique centered at a grid point x gets monocolored after the process, and by symmetry assume it is shaped $\perp$. By (I), it is either monocolored (a, b) or (b, a). In the first case, the vertical segment of one of the paths having a bend at x, let us say P, has to be recolored, because the segments that were originally colored b formed an independent set. Property (II) implies that there is a path Q having a bend at x and whose vertical segment contains the vertical segment of P and is colored a; This leads to a contradiction, because by maximality, Q belongs to the clique. In the second case, since by (III) at most one of the paths of the clique that have a bend at x was recolored, and the segments that were originally colored b formed an independent set, the horizontal segments of all the paths that belong to the clique and do not have a bend at x were recolored b. Property (II) implies that there is a path belonging to the clique whose horizontal segment is colored a, a contradiction as well. These observations and property (IV) ensure that after going through all grid points, we have found a 4-clique coloring of G. By (1) and by Property (I), the coloring has an independent color class.

Let us now explain how we find the recoloring with properties (I), (II), (III) and (IV), for a fixed grid point x. We distinguish three cases. Let us say a shape is *missing* at x if either there is no path of this shape with a bend at x or there is at least one path of this shape with a bend at x that is not colored (a, a).

Case 1: Two or more of the shapes $\lrcorner$, $\llcorner$, $\urcorner$, $\ulcorner$ are missing at x.

If there is no (a, a)-colored claw clique centered at x, we do not recolor anything. Clearly, (I)–(IV) hold. Otherwise, there is a unique (a, a)-colored claw clique at x, and symmetry allows us to assume this clique is shaped $\perp$. Both shapes $\urcorner$ and $\ulcorner$ are missing at x. Of all $\lrcorner$- or $\llcorner$-shaped paths with bend at x, choose the one with the shortest vertical segment, and recolor it (a, b). Then (I)–(IV) hold.

Case 2: Exactly one of the shapes $\lrcorner$, $\llcorner$, $\urcorner$, $\ulcorner$ is missing at x.

By symmetry, we can assume $\ulcorner$ is missing at x. Let $\mathcal{P}$ be the set of all paths with bend at x that have the shape $\lrcorner$. If there is a path $P \in \mathcal{P}$ whose horizontal segment is contained in another path with bend at x, then recolor P with (b, a). Otherwise, if there is a path P in $\mathcal{P}$ whose vertical segment is contained in another path with bend at x, then recolor P with (a, b). In both cases, the choice of P ensures that (I)–(IV) hold.

It remains to consider the case that for each of the paths in $\mathcal{P}$, their horizontal (vertical) segment strictly contains all horizontal (vertical) segments of paths with bend at x (in particular, $|\mathcal{P}| = 1$). Then, choose any ⌞-shaped path P_1 and any ⌝-shaped path P_2 with bend at x, recolor P_1 with (a, b) and P_2 with (b, a) and observe that (I)–(IV) hold by the choice of P_1 and P_2.

Case 3: None of the shapes ⌟, ⌞, ⌝, ⌜ is missing at x.

Consider the shortest of all segments of paths with bend at x (or one of them if there is more than one), and let Q be the path it belongs to. By symmetry, we may assume Q is shaped ⌜, and the shortest segment is the horizontal segment. As in the previous case, let $\mathcal{P}$ be the set of all ⌟-shaped paths with bend at x. If there is a path $P \in \mathcal{P}$ whose horizontal (vertical) segment is contained in another path with bend at x, then recolor P with (b, a) (or with (a, b), respectively), and recolor Q with (b, a). The choice of P and Q guarantees (I)–(IV).

Otherwise, for each of the paths in $\mathcal{P}$, their horizontal (vertical) segment strictly contains all horizontal (vertical) segments of paths with bend at x. Choose any ⌞-shaped path P_1 and any ⌝-shaped path P_2 with bend at x, and recolor P_1 with (a, b) and P_2 with (b, a) (Q is not recolored in this case). Again, (I)–(IV) hold.

The presented algorithm gives a 4-clique coloring of G, where one color class is an independent set. The algorithm can be implemented to run in linear time in the number of vertices and edges of G. □

4. Conclusion

In this paper we have proved that B_1-EPG graphs are 4-clique colorable, and that such coloring can be obtained in linear time in the number of vertices and edges of the graph, given a B_1-EPG representation of it. This algorithm may use four colors in graphs that are in fact 2-clique colorable or 3-clique colorable.

We conjecture that indeed B_1-EPG graphs are 3-clique colorable. Examples of B_1-EPG graphs that require three colors for a clique coloring are the odd chordless cycles, and we could not find examples of B_1-EPG graphs having clique chromatic number 4 (The Mycielski graph with chromatic number 4, line graphs of big cliques and the graph in [6] having clique chromatic number 4 are not B_1-EPG).

Further open questions are the computational complexity of 2-clique coloring and 3-clique coloring on B_1-EPG graphs.

Acknowledgments

This work was partially supported by UBACyT Grant 20020130100808BA, CONICET PIP 122-01001-00310 and 112-201201-00450CO, and ANPCyT PICT 2012-1324 and 2015-2218 (Argentina), Fondecyt grant 1140766 and "Nucleo Milenio Información y Coordinación en Redes" ICM/FIC P10-024F (Chile), and MathAmSud Project 13MATH-07 (Argentina–Brazil–Chile–France).

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