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University of Helsinki
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Master's Thesis

Existence and uniqueness of a solution for the semilinear heat equation

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Tiivistelmä – Referat – Abstract

This thesis is about the existence and uniqueness of a solution for the semilinear heat equation of polynomial type. The extensive study of properties for these equations started off in the 1960s, when Hiroshi Fujita published his results that the existence and uniqueness of solutions depends critically on the exponent of the nonlinear term. In this thesis we expose some of the basic methods used in the theory of linear, constant coefficient partial differential equations. These considerations lay out the groundwork for the main result of the thesis, which is the existence and uniqueness of a solution to the generalized heat equation.

In Chapter 2 we expose the basics of functional analysis. We start off by defining Banach spaces and provide some examples of them. Then, we state the very useful Banach fixed point theorem, which guarantees the existence and uniqueness of a solution to certain types of integral equations. Next, we consider linear maps between normed spaces, with a focus on linear isomorphisms, which are linear maps preserving completeness. The isomorphisms prove to be very useful, when we consider weighted spaces. This is because for certain types of weights, we can identify the multiplication by weight with a linear isomorphism.

In Chapter 3 we introduce the Fourier transform, which is a highly useful tool for studying linear partial differential equations. We go through its basic mapping properties, such as, interaction with derivatives and convolution. Then, we consider useful spaces in Fourier analysis.

Chapter 4 is on the regular, inhomogeneous heat equation. A common method for deriving the solution to heat equation is formally applying the Fourier transform to it. This way we obtain a first order, linear ordinary differential equation, for which there is a known solution. The derived solution will serve as a motivator for how to approach the semilinear case. Also, in the end we will solve explicitly a slight generalization of the heat equation.

In Chapter 5 we prove the main result of this thesis: existence and uniqueness of a generalized solution for the semilinear heat equation. The methods we use in the proof are quite elementary in the sense that we do not need heavy mathematical machinery. We reformulate the generalized semilinear heat equation using an operator and show that it satisfies the conditions of the Banach fixed point theorem in a small, closed ball of a suitable Banach space.

We also include an appendix, in which we discuss differentiability properties of the generalized solution. It is possible to apply methods used in the proof of the generalized case to prove continuous differentiability. We provide some ideas on how one should approach the time differentiability of the solution by estimating the difference quotient of the integral operator.

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Chapter 1

Introduction

In the theory of partial differential equations, one of the most important equations is the heat equation, which describes the evolution of the heat distribution in a medium. The study of the heat equation can be traced back to Joseph Fourier, who published his seminal work on heat flow in [3]. In this work, he derived solutions for the equation in simple domains by applying the eponymous Fourier series, which still prove to be useful in different areas of mathematics. Since then, methods for studying the equation have advanced, and the interest has shifted from finding particular solutions to existence and uniqueness of solutions. From the mathematical point of view, an interesting generalization is to add a nonlinear term of polynomial type to the heat equation, making it into a semilinear partial differential equation. In the 1960s Hiroshi Fujita studied this generalization in his papers [4] and [5], and showed that the existence and non-existence of solutions depend critically on the exponent of the polynomial perturbation. The understanding of the semilinear heat equation has naturally grown ever since, and for the purposes of this thesis, we mention the paper [10] by Jari Taskinen, since we will apply similar techniques. In his paper he proves among more general cases, that for exponents $p \geq 4$ with a sufficiently small initial condition, there exists a unique solution to the one-dimensional semilinear heat equation.

In Chapter 2 we expose the basics of functional analysis. We start off by defining Banach spaces and provide some examples of them. Then, we state the very useful Banach fixed point theorem, which guarantees the existence and uniqueness of a solution to certain types of integral equations. Next, we consider linear maps between normed spaces, with a focus on linear isomorphisms, which are linear maps preserving completeness. The isomorphisms prove to be very useful, when we consider weighted spaces. This is due to the fact that for certain types of weights, we can identify the multiplication by weight with a linear isomorphism.

In Chapter 3 we consider the Fourier transform

$$\mathcal{F}(f)(\xi) = \int_{\mathbb{R}^d} f(x)e^{-i\xi\cdot x}dx,$$

which is a highly useful tool for studying linear partial differential equations. We expose its basic mapping properties, such as, interaction with derivatives and convolution. Then, we consider useful spaces in Fourier analysis, like the Schwartz space $S(\mathbb{R}^d)$ of rapidly decreasing functions, which is an example of a space that is preserved under the Fourier transform.

Chapter 4 is on the regular, inhomogeneous heat equation

$$\partial_t u(x,t) - \Delta u(x,t) = f(x,t).$$

A common method for deriving the solution to heat equation is formally applying the Fourier transform to it. This way we obtain a first order, linear ordinary differential equation, for which there is a known solution. The derived solution will actually serve as a motivator for how to approach the semilinear case. Also, in the end we will solve explicitly a slight generalization of the heat equation.

In Chapter 5 we prove the main result of this thesis: existence and uniqueness of a generalized solution for the semilinear heat equation. The methods we use in the proof are quite elementary in the sense that we do not need heavy mathematical machinery. We reformulate the generalized semilinear heat equation using the operator:

$$G(u)(x,t) = \frac{1}{\sqrt{4\pi t}}\int_{\mathbb{R}} e^{-\frac{(x-y)^2}{4t}} f(y)dy + \int_0^t \frac{1}{\sqrt{t-s}}\int_{\mathbb{R}} e^{-\frac{(x-y)^2}{4(t-s)}}|u(y,s)|^p dyds,$$

and show that it satisfies the conditions of the Banach fixed point theorem in a suitable Banach space of continuous functions, endowed with the sup-norm:

$$\|f\|_w = \sup_{x\in\mathbb{R}} \sup_{t\in\mathbb{R}_+^0} \sqrt{t+1}\left(1 + \frac{|x|}{\sqrt{t+1}}\right)^m |f(x,t)|.$$

We also include an appendix, in which we discuss differentiability properties of the generalized solution. It is possible to apply methods used in the proof of the generalized case to prove continuous differentiability. We provide some ideas on how one should approach the time differentiability of the solution by estimating the difference quotient of the integral operator.

Chapter 2

Normed spaces

Since this thesis is about the existence and uniqueness of a solution for a partial differential equation, it is natural to consider properties of normed spaces, namely complete normed spaces.

Basic normed space theory

We begin by recalling some basic theory of normed spaces, and then give some concrete and useful examples of them. We start from the definition of a normed space, and then move onto defining the necessary topological concepts.

Definition 2.1. If E is a vector space and $\|\cdot\|$ is a norm on E, then we call the pair $(E, \|\cdot\|)$ a normed space.

For future reference, we usually use a short-hand notation for $(E, \|\cdot\|) = E$. This is due to the fact that sometimes the norm is arbitrary (especially in theorems about normed spaces) and also because in concrete examples the norm is known.

Definition 2.2. A sequence (x_n) in a normed space E is called a Cauchy sequence, if for all $\epsilon > 0$ there exists a natural number n_0 such that $\|x_n - x_m\| < \epsilon$ for all $n, m \geq n_0$.

Definition 2.3. A normed space E is called complete, if every Cauchy sequence in E converges to an element in E. We usually call complete normed spaces Banach spaces.

Let us now present some examples of Banach spaces, the most basic one being the space of real numbers endowed with the usual Euclidean norm. Then, we have the usual spaces of continuous functions endowed with the supremum norm, which can be divided into two cases: one with compact domain (where compactness of X implies boundedness), and the general non-compact domain case, where the boundedness of the functions has to be assumed. We will not be giving any proofs, since they are not the main interest, but they can be found in any basic functional analysis book. For Finnish references, the proofs are found in the lecture notes of Real analysis [7] and Functional analysis [1].

Example 2.4. If K is a compact metric space, we denote

$$(C(K, \mathbb{R}), \|\cdot\|_\infty) = \{f : K \to \mathbb{R} : \|f\|_\infty < \infty\}.$$

Example 2.5. If X is a metric space, we denote

$$C_b := (C_b(X, \mathbb{R}), \|\cdot\|_\infty) = \{f : X \to \mathbb{R} \text{ bounded and continuous} : \|f\|_\infty < \infty\}.$$

If we take $X = \mathbb{R}^d$, there are many important subspaces of continuous functions. In the study of partial differential equations, we are particularly interested in functions with differentiability properties. Let us begin by defining the multi-index $\alpha = (\alpha_1, \alpha_2, ..., \alpha_d)$, $\alpha \in \mathbb{N}^d$, where $\mathbb{N} = \{0, 1, 2, 3...\}$. For our purposes, it encodes how many times and in which direction a transformation is applied. With this in mind, let us define the αth derivative of a function $f : \mathbb{R}^d \to \mathbb{R}$:

$$\partial_x^\alpha f(x) = \partial_{x_1}^{\alpha_1} \partial_{x_2}^{\alpha_2} ... \partial_{x_d}^{\alpha_d} f(x), \tag{2.6}$$

where $\partial_{x_i}^{\alpha_i}$, $i = 1, 2, 3, .., d$ means differentiating α_i times with respect to variable x_i. If $\alpha_i = 0$, it means we do not differentiate the function with respect to x_i, and if $\alpha_i = 1$ we denote $\partial_{x_i}^{\alpha_i} = \partial_{x_i}$. We may now define the space of k times continuously differentiable functions:

$$C^k(\mathbb{R}^d) := C^k(\mathbb{R}^d, \mathbb{R}) = \{f \in C(\mathbb{R}^d, \mathbb{R}) : \partial_x^\alpha f \in C(\mathbb{R}^d, \mathbb{R}) \ \forall \alpha \in \mathbb{N}^d \text{ such that } |\alpha| \le k\},$$

where $|\alpha| = \sum_{i=1}^d \alpha_i$.

Now we may define the space of infinitely many times differentiable functions, which is also called the space of smooth functions:

$$C^\infty(\mathbb{R}^d) := C^\infty(\mathbb{R}^d, \mathbb{R}) = \{f \in C(\mathbb{R}^d, \mathbb{R}) : \partial_x^\alpha f \in C(\mathbb{R}^d, \mathbb{R}) \ \forall \alpha \in \mathbb{N}^d\}.$$

Note that neither of the vector spaces $C^k(\mathbb{R}^d)$ and $C^\infty(\mathbb{R}^d)$ can be endowed with the sup-norm, since $\mathbb{R}^d$ is not compact and we do not assume boundedness of functions in these spaces. In order for us to introduce an important subspace of $C_b(\mathbb{R}^d)$ with nice differentiability properties, we need to define the closed support of a function $f : X \to \mathbb{R}$:

$$spt(f) = \overline{\{x \in X : f(x) \neq 0\}}.$$

We say that f is compactly supported, if $spt(f)$ is compact. Now we may define the space of smooth functions with compact support:

$$C_0^\infty(\mathbb{R}^d) = \{f \in C^\infty(\mathbb{R}^d) : spt(f) \text{ is compact}\}.$$

Using simple arguments one can show that $C_0^\infty(\mathbb{R}^d)$ is a linear subspace of $C_b(\mathbb{R}^d)$.

We turn to L^p-spaces, which are harder to describe precisely. For simplicity, we consider the case where the functions are defined on the d-dimensional Euclidean space or any good enough subspace X of it. The space L^p consists of equivalence classes of measurable functions and it can be endowed with the following norm to obtain a complete normed space,

$$\|f\|_p = \left(\int_X |f(x)|^p dx\right)^{\frac{1}{p}},$$

where $1 \leq p < \infty$ and $X \subseteq \mathbb{R}^d$. One can also define the space L^p for $p = \infty$: the space consists of essentially bounded functions, and it is endowed with the essential sup-norm. Another special case is $p = 2$, where the norm is induced by the following inner product:

$$(f, g)_{L^2(X)} = \int_X f(x)g(x)dx,$$

which makes $L^2(X)$ into a complete inner product space, also called a Hilbert space. These exhibit a lot of useful properties, which Banach spaces do not necessarily have, but these considerations are not important for our purposes.

Next, we introduce a very powerful theorem called Banach fixed point theorem. First, we define the Lipschitz-maps

Definition 2.7. Let E and F be normed spaces. We say that the mapping (not necessarily linear) $f : E \to F$ is M-Lipschitz, if there exists a constant $M \geq 0$ such that $\|f(x) - f(y)\|_F \leq M \|x - y\|_E$ for all $x, y \in E$. And if $M < 1$, then f is called a contraction.

Theorem 2.8 (Banach fixed point theorem)**.** *Let X be a closed, non-empty subset of complete normed space E, and $T : X \to X$ be a contraction. Then, T has a unique fixed point $a \in X$. In addition, if $x \in X$, then the sequence $(T^k(x))_{k=1}^\infty$ converges to a. Here, the fixed point a means an element of X such that $T(a) = a$. Moreover, T^k is the k^{th} iterate of T: $T^2 = T \circ T$, $T^3 = T \circ T \circ T$, $T^k = \underbrace{T \circ T ... T \circ T}_{k \text{ times}}$.*

Proof. Let us first prove that the fixed point is unique. Let $a, b \in X$ be distinct fixed points of $T : X \to X$. Since we assumed T to be a contraction, we obtain:

$$\|a - b\| = \|T(a) - T(b)\| \leq q \|a - b\|,$$

where $q < 1$. This gives us $a = b$, which is a contradiction. Thus, if T has a fixed point, it has to be unique.

Next, we prove that there exists a fixed point and that the sequence $(T(x), T^2(x), .., T^k(x), ..)$ converges to a. Let $x_0 \in X$. We form a sequence $(x_n)_{n=0}^{\infty}$, where $x_{n+1} = Tx_n$ $\forall n \in \mathbb{N}$. We first show, that $(x_n)_{n=0}^{\infty}$ is a Cauchy sequence. For all $n \in \mathbb{N}$ we have

$$\|x_n - x_{n+1}\| = \|T(x_{n-1}) - T(x_n)\| \leq q \|x_{n-1} - x_n\| .$$

By using a simple induction we obtain:

$$\|x_n - x_{n+1}\| \leq q^n \|x_0 - x_1\| .$$

Let $1 \leq n < k$. Using the formula for the geometric sum, we obtain:

$$\begin{aligned}\|x_n - x_k\| &\leq \|x_n - x_{n+1}\| + \|x_{n+1} - x_{n+2}\| + ... + \|x_{k-1} - x_k\| \\ &\leq (q^n + q^{n+1} + ... + q^{k-1}) \|x_0 - x_1\| \\ &\leq \frac{q^n \|x_0 - x_1\|}{1-q}.\end{aligned}$$

Using $q < 1$ we know that $q^n \to 0$ as $n \to \infty$. This gives us that the sequence $(x_n)_{n=0}^{\infty}$ is Cauchy. We recall from course Topology I [11], that a closed subset $X \subset E$ is a complete metric space. Thus, the sequence $(x_n)_{n=0}^{\infty}$ converges towards some $a \in X$. Since T is continuous, we have that $T(x_n) \to T(a)$. On the other hand, $T(x_n) = x_{n+1}$, so $T(x_n) \to a$. This gives us that $T(a) = a$. $\square$

It is known that the differential operator $\frac{d}{dx}$ is not continuous in any reasonable normed space. Even though this seems bad in view of the applications of the fixed point theorem, there is a way of reformulating particular differential equations as integral operator equations. This helps us immensely, since integral operators tend to be Lipschitz-continuous, so what is left to prove is the contractivity and that it maps the set E into itself.

Linear isomorphisms and weighted spaces

In Functional analysis it is important to study mappings which preserve the completeness property of a normed space. For this we need to recall a couple of definitions and results of basic functional analysis. A good place to begin is to consider linear maps. We recall that if X and Y are vector spaces, then $T : X \to Y$ is a linear mapping if it satisfies the equation $T(ax + by) = aT(x) + bT(y)$ for all $x, y \in X$. Since we are working in Banach spaces, we can also speak of continuity, and it turns out that continuity and boundedness are equivalent for linear maps. For future reference, we use the terms linear map, linear operator and linear transformation to describe the same thing.

Theorem 2.9. *Let E and F be Banach spaces and $T : E \to F$ linear. Then, the following are equivalent:*

(1) T is continuous

(2) There exists a constant $C > 0$ such that $\|Tx\|_F \leq C\|x\|_E$.

There are many canonical examples of bounded linear maps, but for our purposes a particularly interesting example is the usual integral operator with a nice enough kernel function $K(x, y)$.

Example 2.10. Let us define the linear integral operator $T : C([0, 1], \mathbb{R}) \to C([0, 1], \mathbb{R})$,

$$T(f)(x) = \int_0^1 K(x, y)f(y)dy,\ K(x, y) \in C([0, 1]^2, \mathbb{R})$$

The linearity is obvious and the boundedness proof is quite elementary, it is based on the fact that the kernel is a continuous function in a compact space (and thus has a uniform upper bound $M \geq 0$). So we get that

$$\|T(f)\|_\infty \leq \int_0^1 \|K(x, y)\|_\infty \|f\|_\infty dx \leq M(1 - 0)\|f\|_\infty = M\|f\|_\infty.$$

A canonical example of a discontinuous linear map is the differential operator mentioned earlier. The discontinuity is due to the following fact: $e^{inx} \in C([0, 1], \mathbb{C})$ with $\|e^{inx}\|_\infty = 1$, but the norm of the derivative clearly is not bounded:

$$\left\|\frac{de^{inx}}{dx}\right\|_\infty = \|ne^{inx}\|_\infty = n.$$

Let us denote the space of continuous linear maps from E to F by $\mathcal{L}(E, F)$. This is a normed space, if we endow it with the so called operator norm, and even a Banach space if F is Banach. The operator norm is defined as follows

$$\|T\| = \sup_{\|x\|_E \leq 1} \|Tx\|_F.$$

We note that, the infimum of constants $C > 0$ in Theorem 2.9 equals the operator norm.

An important property of the space $\mathcal{L}(E, F)$ is the so called submultiplicity of composition of continuous linear maps. The following theorem will give further explanation on this:

Theorem 2.11. *Let E, F and G be normed spaces and $T \in \mathcal{L}(E, F)$ and $S \in \mathcal{L}(F, G)$ be bounded linear operators. Then, the composition of functions $ST = S \circ T \in \mathcal{L}(E, G)$ and*

$$\|ST\| \leq \|S\|\|T\|. \tag{2.12}$$

Proof. By basic linear algebra, a composition of two linear maps is linear. If $x \in E$ and $\|x\|_E \leq 1$, then

$$\|STx\|_G = \|S(Tx)\|_G \leq \|S\| \, \|Tx\|_F \leq \|S\| \, \|T\| \, .$$

Thus, $ST \in \mathcal{L}(E, G)$ and $\|ST\| \leq \|S\| \, \|T\|$. $\square$

Even though completeness is not a topological property per sé, it turns out to be useful for us to consider the necessary condition for a linear map to be a homeomorphism between two normed spaces. We already know that boundedness is equivalent to continuity, but now we state the following theorem, which says that boundedness from below implies the continuity of the inverse map.

Theorem 2.13. *Let E, F be normed spaces and $T : E \to F$ be a linear bijection. Then, T^{-1} is linear and*

(2.14) $\quad T^{-1}$ *is continuous* $\Leftrightarrow$ *There exists* $\alpha > 0$ *such that* $\|Tx\|_F \geq \alpha \|x\|_E \, , \forall x \in E.$

Proof. Let $x, y \in F$ and $\lambda, \mu \in \mathbb{R}$. Then, we have the following:

$$T(\lambda T^{-1}x + \mu T^{-1}y) = \lambda T T^{-1}x + \mu T T^{-1}y = \lambda x + \mu y = T(T^{-1}(\lambda x + \mu y))$$

Since T is a bijection, it follows that T^{-1} is linear:

$$\lambda T^{-1}x + \mu T^{-1}y = T^{-1}(\lambda x + \mu y).$$

Then, we prove that (2.14) holds.

"$\Rightarrow$" Since $T^{-1}x_0 = 0 \Leftrightarrow x_0 = 0$, we may assume that E and F are non-trivial normed spaces. This means that $E, F \neq \{0\}$. Thus, there exists a non-zero $x_0 \in F$, so that

$$0 < \frac{\|T^{-1}x_0\|_E}{\|x_0\|_F} \leq \left\|T^{-1}\right\| .$$

Thus, we obtain that $0 < \|T^{-1}\| < \infty$, which gives us that $\frac{1}{\|T^{-1}\|} < \infty$.

If $x \in E$ is arbitrary, then

$$\|x\|_E = \left\|T^{-1}Tx\right\|_E \leq \left\|T^{-1}\right\| \|Tx\|_F \, .$$

Now, we obtain

$$\left\|T^{-1}\right\|^{-1} \|x\|_E \leq \|Tx\|_F \, , \; x \in E.$$

By choosing $\alpha = \|T^{-1}\|^{-1}$ we conclude the proof of this direction.

"$\Leftarrow$" Assume that $\|Tx\|_F \geq \alpha \|x\|_E$ for all $x \in E$. If $y \in F$ is arbitrary, we choose $x = T^{-1}y$. Then, $Tx = y$ and by assumption:

$$\left\|T^{-1}y\right\|_E = \|x\|_E \leq \frac{1}{\alpha} \|Tx\|_F = \frac{1}{\alpha} \|y\|_F ,$$

which holds for all $y \in F$. Thus, we have $\|T^{-1}\| \leq \frac{1}{\alpha} < \infty$. By Theorem 2.9, T^{-1} is a continuous linear map. $\square$

We obtain the following theorem as a corollary:

Theorem 2.15. *Let E, F be normed spaces and $T : E \to F$ be a linear bijection. Then, T is a homeomorphism if and only if there exist constants $\alpha, \beta > 0$ such that*

$$\alpha \|x\|_E \leq \|Tx\|_F \leq \beta \|x\|_E , \forall x \in E. \tag{2.16}$$

We take the conditions (2.16) of the Theorem 2.15 to define linear isomorphisms.

Definition 2.17. If $T \in \mathcal{L}(E, F)$ satisfies the inequalities (2.16) of Theorem 2.15, then we say that T is a linear isomorphism and that E and F are linearly isomorphic.

Theorem 2.18. *Let E and F be normed spaces and $T \in \mathcal{L}(E, F)$ be a linear isomorphism. Then, we have that E is complete if and only if F is complete.*

Proof. Due to symmetry it suffices to prove the claim in only one direction. Let (x_n) be a Cauchy sequence in F, then we have

$$\left\|T^{-1}x_n - T^{-1}x_m\right\| = \left\|T^{-1}(x_n - x_m)\right\| \leq \left\|T^{-1}\right\| \|x_n - x_m\|$$

so $(T^{-1}x_k)$ is also Cauchy in E. Since E is complete, there exists $y \in E$ such that $T^{-1}x_k \to y$, and

$$\|x_k - Ty\| = \left\|T(T^{-1}x_k - y)\right\| \leq \|T\| \left\|T^{-1}x_k - y\right\| \to 0.$$

Thus, F is complete. $\square$

Now that we have essentially characterized mappings preserving the completeness of normed spaces, we may move onto one of the central subjects of this thesis, weighted normed spaces. These can be defined quite generally, but we are only interested in subspaces of $C(X, \mathbb{R})$. This space is an algebra, because for elements $f, g \in C(X, \mathbb{R})$ we have a well-defined multiplication $(fg)(x) = f(x)g(x) \in C(X, \mathbb{R})$. The same holds if we replace $C(X, \mathbb{R})$ with $C_b(X, \mathbb{R})$, which is a Banach space. These considerations give us a natural candidate for what we call a weight, and what we mean by weighted space.

Definition 2.19. Let w be an element of the vector space $C(X, \mathbb{R})$. Then, we define the weighted space associated with the weight w as follows

$$C_w = (C_w(X, \mathbb{R}), \|\cdot\|_w) =: \{u \in C(X, \mathbb{R}) : \|wu\|_\infty < \infty\}.$$

The definition gives us a large collection of functions in $C(X, \mathbb{R})$, and we proceed to specify the most useful weights for us. We first need to demand that for $w \in C(X, \mathbb{R})$ there exists a fixed $\delta > 0$ such that $|w(x)| \geq \delta$ for all $x \in X$. This gives us that $w^{-1} \in C_b(X, \mathbb{R})$, where $w^{-1}(x) = \frac{1}{w(x)}$ for all $x \in X$. Thus, we get the following result, that if the weight is "invertible", then the corresponding weighted space is complete.

Theorem 2.20. *Let w be an element belonging to the space $C(X, \mathbb{R})$. If there exists a fixed $\delta > 0$ such that $|w(x)| \geq \delta$ for all $x \in X$, then the corresponding weighted space $C_w = (C_w(X, \mathbb{R}), \|\cdot\|_w)$ is complete.*

Proof. Let $u \in C_w$ and define a linear mapping $T_{w^{-1}} : C_b \to C_w$ as follows: $T_{w^{-1}}u = w^{-1}u$. It is clearly linear, and we have that

$$\|T_{w^{-1}}u\|_w = \left\|ww^{-1}u\right\|_\infty = \|u\|_\infty . \tag{2.21}$$

This means that $T_{w^{-1}}$ is an isometry, but more importantly, it satisfies the conditions of being a linear isomorphism with constants $\alpha = \beta = 1$. Thus, by Theorem 2.18, we have that C_w is also complete. □

Chapter 3

Fourier transform

In this chapter we will review the basic properties concerning the Fourier transform, which is of huge importance in the theory of linear, constant coefficient partial differential equations. It provides a lot of insight about how the solutions of these equations should look like. The proofs will be mostly omitted, since Fourier analysis is not the main focus of this thesis and they are easily found in any basic literature, for example [9].

We begin by defining the Fourier transform $\mathcal{F}$ for integrable functions.

Definition 3.1. Suppose that $f : \mathbb{R}^d \to \mathbb{C}$ is in $L^1(\mathbb{R}^d)$. We define

$$\mathcal{F}(f)(\xi) = \int_{\mathbb{R}^d} f(x)e^{-i\xi\cdot x}dx,\ \ \xi, x \in \mathbb{R}^d. \tag{3.2}$$

The above expression is clearly well-defined, and $\|\mathcal{F}(f)\|_\infty \le \|f\|_1$. The Fourier transform can have useful properties in other spaces too, more on this later.

There are other ways of defining the Fourier transform, for example we could replace the exponent $-i\xi\cdot x$ with $-i2\pi\xi\cdot x$. The definitions give equivalent results, but depending on the context, the normalizing constants may be better. We will see that the inversion formula for our exponent has an unpleasant normalizing constant, which the other definition does not have. But the reason we choose (3.2) as our definition is, that the Fourier transform of derivatives have a nicer form than for the exponent $-i2\pi\xi \cdot x$.

Elementary properties of the Fourier transform

Let us begin by considering elementary properties of the Fourier transform and associated concepts. We first consider whether one can recover the function which has been Fourier transformed. Turns out there is an explicit inversion formula, which is akin to the regular Fourier transform. Let us state the inversion theorem:

Theorem 3.3 (Fourier inversion theorem)**.** *If $f \in L^1(\mathbb{R}^d)$ and $\mathcal{F}f \in L^1(\mathbb{R}^d)$, then we have a well-defined inverse Fourier transform and it can be calculated as follows*

$$\mathcal{F}^{-1}(f)(x) = \frac{1}{(2\pi)^d}\int_{\mathbb{R}^d} \mathcal{F}(f)(\xi)e^{i\xi\cdot x}d\xi,\ \ \xi, x \in \mathbb{R}^d. \tag{3.4}$$

Now that we have the inversion formula, we next consider operations, which have nice properties under the Fourier transform. First one of these is the convolution, which is very useful in partial differential equations and analysis in general. For example it is used to prove that the subspace $C_0^\infty(\mathbb{R}^d)$ is dense in all $L^p(\mathbb{R}^d)$ space. This means that if $f \in L^p(\mathbb{R}^d)$, it can be approximated by a function $g \in C_0^\infty(\mathbb{R}^d)$ (in the L^p-norm). A proof for this result is found in [7].

Definition 3.5. Suppose that f and g belong to $L^1(\mathbb{R}^d)$. We define the convolution as follows:

$$(f * g)(x) = \int_{\mathbb{R}^d} f(x-y)g(y)dy = \int_{\mathbb{R}^d} f(y)g(x-y)dy,\ \ x \in \mathbb{R}^d. \tag{3.6}$$

The reason for introducing the convolution is the following theorem, which relates the Fourier transform of the convolution to the product of Fourier transforms:

Theorem 3.7 (Convolution theorem)**.** *Suppose $f, g \in L^1(\mathbb{R}^d)$. Then,*

(1) $(f * g) \in L^1(\mathbb{R}^d)$ *with* $\|f * g\|_1 \leq \|f\|_1 \|g\|_1$

(2) The Fourier transform of $f * g$ satisfies the following equation:

$$\mathcal{F}(f * g)(\xi) = \mathcal{F}(f)(\xi)\mathcal{F}(g)(\xi),\ \ \xi \in \mathbb{R}^d. \tag{3.8}$$

Proof. Let us begin by proving statement (1). If $f, g \in L^1(\mathbb{R}^d)$, then

$$\begin{aligned}
\|f * g\|_1 &= \int_{\mathbb{R}^d}\left|\int_{\mathbb{R}^d} f(x-y)g(y)dy\right| dx \\
&\leq \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} |f(x-y)g(y)|dydx \\
&= \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} |f(x-y)g(y)|dxdy \qquad (3.9) \\
&= \int_{\mathbb{R}^d} |g(y)| \int_{\mathbb{R}^d} |f(x-y)|dxdy \\
&= \int_{\mathbb{R}^d} |g(y)|\, \|f\|_1\, dy \qquad (3.10) \\
&= \|f\|_1 \|g\|_1 ,
\end{aligned}$$

where in the equality (3.9) we applied Fubini's theorem and in the inequality (3.10) we used that integral over the whole space $\mathbb{R}^d$ is translation invariant.

Now, we prove the equation (3.8). To this end, let us calculate the Fourier Transform $\mathcal{F}$ of the function $(f * g) \in L^1(\mathbb{R}^d)$:

$$
\begin{aligned}
\mathcal{F}(f * g)(\xi) &= \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} f(x-y)g(y)dy\, e^{-i\xi\cdot x}dx \\
&= \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} f(x-y)g(y)e^{-i\xi\cdot y}e^{i\xi\cdot y}e^{-i\xi\cdot x}dydx \\
&= \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} f(x-y)g(y)e^{-i\xi\cdot y}e^{-i\xi\cdot(x-y)}dxdy \qquad (3.11) \\
&= \int_{\mathbb{R}^d} g(y)e^{-i\xi\cdot y} \int_{\mathbb{R}^d} f(x-y)e^{-i\xi\cdot(x-y)}dxdy \\
&= \int_{\mathbb{R}^d} g(y)e^{-i\xi\cdot y}\mathcal{F}(f)(\xi)dy \qquad (3.12) \\
&= \mathcal{F}(f)(\xi)\mathcal{F}(g)(\xi),
\end{aligned}
$$

where we applied Fubini once again in (3.11), and in (3.12) we used that integral over the whole space $\mathbb{R}^d$ is translation invariant. □

As a corollary of Convolution theorem 3.7 and Fourier inversion theorem 3.3 we obtain the following:

Corollary 3.13. *Suppose $f, g \in L^1(\mathbb{R}^d)$ and in addition that $\mathcal{F}(f)\mathcal{F}(g) \in L^1(\mathbb{R}^d)$. Then, we have the following identity*

$$\mathcal{F}^{-1}(\mathcal{F}(f)\mathcal{F}(g))(x) = (f * g)(x), \ x \in \mathbb{R}^d.$$

Another useful property of the Fourier transform is how it maps partial derivatives of a function. Let us begin by recalling that a monomial is defined analogously to the αth derivative (2.6):

$$x^\alpha = x_1^{\alpha_1}x_2^{\alpha_2}...x_d^{\alpha_d}, \ x \in \mathbb{R}^d, \ \alpha \in \mathbb{N}^d.$$

This will be useful in the next theorem which connects the Fourier transform of a derivative of a function to multiplying transformed function by a monomial.

Theorem 3.14. *Let $f \in C_0^\infty(\mathbb{R}^d)$ and α be a multi-index. Then, we have the following relationship:*

$$\mathcal{F}(\partial_x^\alpha f(x))(\xi) = (i\xi)^\alpha \mathcal{F}(f)(\xi), \ \xi, x \in \mathbb{R}^d. \qquad (3.15)$$

It is possible to relax the conditions of the function f and its differentiability, to obtain similar results, for example, in Sobolev spaces. The book [2] is excellent for learning about these spaces.

In view of the above theorems we can deduce that with suitable integrability and differentiability conditions, the first theorem can be inverted. This means that multiplying by a monomial in x-space corresponds to differentiation in ξ-space.

Theorem 3.16. *Let $|x^\alpha||f(x)|$ be an element in $L^1(\mathbb{R}^d)$. Then, we have for the αth derivative of the Fourier transform $\partial_\xi^\alpha \mathcal{F}(f) \in C(\mathbb{R}^d)$ and*

$$\partial_\xi^\alpha \mathcal{F}(f)(\xi) = \mathcal{F}((-ix)^\alpha f(x)), \ \xi, x \in \mathbb{R}^d. \tag{3.17}$$

Let us illustrate the usefulness of the material covered so far. We consider the following second order, linear differential equation in $\mathbb{R}$ for the unknown function $u : \mathbb{R} \to \mathbb{R}$:

$$-\partial_x^2 u(x) + u(x) = f(x), \ x \in \mathbb{R}, \tag{3.18}$$

where $f \in C_0^\infty(\mathbb{R})$ is a known function. If we assume that (3.18) has a solution, then we may apply the Fourier transform on both sides to obtain the following:

$$\begin{aligned}
\mathcal{F}(f)(\xi) &= \mathcal{F}(-\partial_x^2 u)(\xi) + \mathcal{F}(u)(\xi) \\
&\overset{*}{=} \xi^2 \mathcal{F}(u)(\xi) + \mathcal{F}(u)(\xi) \\
&= (1 + \xi^2)\mathcal{F}(u)(\xi),
\end{aligned}$$

where $\xi, x \in \mathbb{R}$. In the equality (*) we applied Theorem 3.14. Since $1 + \xi^2 > 0$, we may divide both sides with it and obtain

$$\mathcal{F}(u)(\xi) = \frac{\mathcal{F}(f)(\xi)}{1 + \xi^2}. \tag{3.19}$$

Using Corollary 3.13 and the black box knowledge that $\mathcal{F}(Ce^{-|\cdot|})(\xi) = \frac{1}{1+\xi^2}$, we get a solution for (3.18):

$$u(x) = \int_{\mathbb{R}} Ce^{-|y|} f(x - y) dy. \tag{3.20}$$

Here C is some constant depending on the definition of Fourier transform. The solution above might not be in an explicit form, but it still demonstrates how one can construct a solution to a linear, inhomogeneous, constant coefficient differential equation. An interesting thing to note is that both e^{-t} and e^t solve the homogeneous version of (3.18), and in the above integral representation there is the function $e^{-|x|}$. This is not a coincidence, since $e^{-|x|}$ is the so called fundamental solution of the differential operator $-\partial_x^2 + I$.

It is also wise to note that the method above does not work as such in all cases: it may for example happen that after applying the Fourier transform we obtain

$$p(\xi)\mathcal{F}(u)(\xi) = \mathcal{F}(f)(\xi),$$

where p is a polynomial with zeroes in $\mathbb{R}$. Questions regarding problems like this, what it means precisely to be a fundamental solution, are answered by distribution theory.

Useful spaces in Fourier analysis

Next, we consider spaces which are useful in Fourier analysis. We are especially interested in the mapping properties of Fourier transform: if the function f has some properties, what properties does the transformed function $\mathcal{F}f$ have? Especially, are some essential properties preserved? Or even better, do we have spaces which are completely preserved under the Fourier transform? It is natural to start the considerations with L^p spaces. We already established that if $f \in L^1(\mathbb{R}^d)$, then $\mathcal{F}f \in L^\infty(\mathbb{R}^d)$ with the estimate $\|\mathcal{F}(f)\|_\infty \leq \|f\|_1$. Considering purely the basic theory of partial differential equations, cases $p \neq 2$ are not that essential. Thus, we focus on L^2, which is particularly useful, because it is a Hilbert space. What exactly are the Fourier mapping properties of $L^2(\mathbb{R}^d)$? The essential feature is that Fourier transform is a so called unitary operator $L^2 \to L^2$. This is the same as being a continuous linear bijection, but with the added caveat that the norm is preserved. What we mean by this is:

Theorem 3.21. *Let f be an element in $L^2(\mathbb{R}^d)$. Then, the Fourier transform $\mathcal{F}f$ is also in $L^2(\mathbb{R}^d)$. Furthermore, the Fourier transform is a continuous linear bijection $L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d)$, which is even a unitary operator:*

$$\|f\|_{L^2(\mathbb{R}^d)} = \|\mathcal{F}f\|_{L^2(\mathbb{R}^d)}\,. \tag{3.22}$$

What about spaces with differentiability and compact support properties? One can ask whether or not it is true that $\mathcal{F}f \in C_0^\infty(\mathbb{R})$ for $f \in C_0^\infty(\mathbb{R})$. It turns out that the only function satisfying the aforementioned property is $f \equiv 0$.

Even though the compactly supported case is a dead end, there is a nice class of functions called the Schwartz functions, whose derivatives are rapidly decreasing. We denote and define the space as follows:

Definition 3.23. $S(\mathbb{R}^d) =: \{f \in C^\infty(\mathbb{R}^d) : \sup_{N\in\mathbb{N}} \sup_{\alpha\in\mathbb{N}^d} \sup_{x\in\mathbb{R}^d} |(1+x)^{2N}\partial_x^\alpha f(x)| < \infty\}.$

We first observe that if $f \in S(\mathbb{R}^d)$, then $f \in L^p(\mathbb{R}^d)$ for all $1 \leq p < \infty$, and also, if $f \in C_0^\infty(\mathbb{R}^d)$, then it is in $S(\mathbb{R}^d)$. The canonical example of a function in $S(\mathbb{R}^d)$ is the Gaussian function $\phi(x) = e^{-x^2}$, which turns out to be useful in the study of the solution for heat equation. This, in part, is due to the fact that the Gaussian is an eigenfunction of the Fourier transform:

Lemma 3.24. *Let $a > 0$ and $\phi(x) = e^{-a|x|^2}$. Then,*

$$\mathcal{F}(f)(\xi) = \left(\frac{\pi}{a}\right)^{\frac{d}{2}} e^{\frac{-|\xi|^2}{4a}}, \ \xi \in \mathbb{R}^d. \tag{3.25}$$

It turns out that the Fourier transform is a linear isomorphism $\mathcal{F} : S(\mathbb{R}^d) \to S(\mathbb{R}^d)$. Here the topology of the space is more delicate than just a normed space: it is given by a countable family of seminorms $\|\cdot\|_n$, and more specifically by a metric:

$$d(f, g) = \sum_{n \in \mathbb{N}} 2^{-n} \frac{\|f - g\|_n}{1 + \|f - g\|_n}, \tag{3.26}$$

where $f, g \in S(\mathbb{R}^d)$. We skip the detailed description of the seminorms (as they appear in the Definition 3.23.) The continuity of the Fourier transform in $S(\mathbb{R}^d)$ is analogous to the case in $L^2(\mathbb{R}^d)$:

Theorem 3.27. *If $f \in S(\mathbb{R}^d)$ then the Fourier transform $\mathcal{F}f$ also belongs to $S(\mathbb{R}^d)$. Furthermore the Fourier transform is a continuous linear bijection $S(\mathbb{R}^d) \to S(\mathbb{R}^d)$.*

Chapter 4

The solution of the classical heat equation

In this chapter we will discuss the solution to the heat equation. This will serve both as an introduction to the equation and as a motivation to the reformulation of the semilinear heat equation. Let $\Delta = \sum_{i=1}^{n} \partial_{x_i}^2$ be the usual Laplace operator in $\mathbb{R}^d$ and let ∂_t be the partial derivative with respect to variable t, which is often called the time variable. From now on, we denote $\mathbb{R}_+ = (0, \infty)$ and $\mathbb{R}_+^0 = [0, \infty)$.

The (inhomogeneous) heat equation reads as:

$$\partial_t u(x,t) - \Delta u(x,t) = f(x,t), \ x \in \mathbb{R}^d, t \in \mathbb{R}_+.$$

We say that the heat equation is homogeneous, if $f \equiv 0$ in $\mathbb{R}^d \times \mathbb{R}_+$. The following is the associated initial value problem (IVP), also known as the Cauchy problem for the heat equation:

$$\begin{cases} \partial_t u(x,t) - \Delta u(x,t) = f(x,t), & (x,t) \in \mathbb{R}^d \times \mathbb{R}_+ \\ u(x,0) = g(x), & x \in \mathbb{R}^d, \end{cases} \tag{4.1}$$

where the function $u : (\mathbb{R}^d \times \mathbb{R}_+^0) \to \mathbb{R}$ is an unknown function, $g \in C_0^\infty(\mathbb{R}^d)$ and $f(\cdot, t) \in C_0^\infty(\mathbb{R}^d)$, for every $t \in \mathbb{R}_+$, are known functions. The function spaces where f and g belong can be relaxed. Let us start by some heuristics, to get a sense of what the solution to the initial value problem might look like. To keep things simple, let us assume that both f and g are as above, and thus in their respective Schwartz spaces. Now we may apply the Fourier transform with respect to the space variable x on both sides, and obtain the following:

$$\begin{cases} \partial_t \mathcal{F}(u)(\xi,t) + |\xi|^2 \mathcal{F}(u)(\xi,t) = \mathcal{F}(f)(\xi,t), & (\xi,t) \in \mathbb{R}^d \times \mathbb{R}_+ \\ \mathcal{F}(u)(\xi,0) = \mathcal{F}(g)(\xi), & \xi \in \mathbb{R}^d, \end{cases}$$

where, similarly to methods for obtaining a solution for the equation (3.18), we used Theorem 3.14.

We have essentially reduced the seemingly difficult partial differential equation to the Cauchy problem of a first order linear differential equation, for which there is a known solution:

$$\begin{aligned}\mathcal{F}u(\xi,t) =&(\mathcal{F}g)(\xi)e^{-\int_0^t|\xi|^2dr}+\int_0^t e^{-\int_s^t|\xi|^2dr}(\mathcal{F}f)(\xi,s)ds\\ =&(\mathcal{F}g)(\xi)e^{-t|\xi|^2}+\int_0^t e^{-(t-s)|\xi|^2}(\mathcal{F}f)(\xi,s)ds\\ =&\left(\frac{1}{4\pi t}\right)^{\frac{d}{2}}\mathcal{F}\left(g*e^{-\frac{|x|^2}{4t}}\right)(\xi)+\int_0^t\left(\frac{1}{4\pi(t-s)}\right)^{\frac{d}{2}}\mathcal{F}\left(f*e^{-\frac{|x|^2}{4(t-s)}}\right)(\xi,s)ds.\end{aligned}$$

Now using Corollary 3.13 and Lemma 3.24 we get:

$$u(x,t)=\left(\frac{1}{4\pi t}\right)^{\frac{d}{2}}\int_{\mathbb{R}^d}e^{-\frac{|x-y|^2}{4t}}g(y)dy+\int_0^t\left(\frac{1}{4\pi(t-s)}\right)^{\frac{d}{2}}\int_{\mathbb{R}^d}e^{-\frac{|x-y|^2}{4(t-s)}}f(y,s)dyds. \tag{4.2}$$

The term $\left(\frac{1}{4\pi t}\right)^{\frac{d}{2}}e^{-\frac{|x|^2}{4t}}$ is called the (d-dimensional) heat kernel, and is usually denoted as $\Phi(x,t)$. It has multiple useful properties, and we shall consider them briefly. Obviously $\Phi(\cdot,t)\in S(\mathbb{R}^d)$ for all $t\in\mathbb{R}_+$. It is a solution to the homogeneous heat equation, and moreover, if the reader is comfortable with the notion of Dirac delta δ_x, then Φ solves the following initial value problem:

$$\begin{cases}\partial_t\Phi(x,t)-\Delta\Phi(x,t)=0, & (x,t)\in\mathbb{R}^d\times\mathbb{R}_+\\ \Phi(x,0)=\delta_x, & x\in\mathbb{R}^d\end{cases} \tag{4.3}$$

Let us show that for times $t\in\mathbb{R}_+$, the heat kernel Φ solves the homogeneous heat equation. We begin by calculating the derivative with respect to t:

$$\begin{aligned}\partial_t\Phi(x,t)&=\partial_t\left[\left(\frac{1}{4\pi t}\right)^{\frac{d}{2}}\right]e^{-\frac{|x|^2}{4t}}+\left(\frac{1}{4\pi t}\right)^{\frac{d}{2}}\partial_t\left(e^{-\frac{|x|^2}{4t}}\right)\\ &=\left(-\frac{d}{2t}\right)\left(\frac{1}{4\pi t}\right)^{\frac{d}{2}}e^{-\frac{|x|^2}{4t}}+\left(\frac{1}{4\pi t}\right)^{\frac{d}{2}}\frac{|x|^2}{4t^2}e^{-\frac{|x|^2}{4t}}\\ &=\left[\left(\frac{1}{4\pi t}\right)^{\frac{d}{2}}\frac{|x|^2}{4t^2}-\frac{d}{2t}\left(\frac{1}{4\pi t}\right)^{\frac{d}{2}}\right]e^{-\frac{|x|^2}{4t}},\end{aligned}$$

Let us then calculate the second partial derivative with respect to space variable x_i. First,

$$\partial_{x_i}\Phi(x,t) = \left(\frac{1}{4\pi t}\right)^{\frac{d}{2}}\left(-\frac{x_i}{2t}\right)e^{-\frac{|x|^2}{4t}},$$

and then the second derivative,

$$\begin{aligned}
\partial_{x_i}^2\Phi(x,t) &= \partial_{x_i}\left[\left(\frac{1}{4\pi t}\right)^{\frac{d}{2}}\left(-\frac{x_i}{2t}\right)e^{-\frac{|x|^2}{4t}}\right] \\
&= \left(\frac{1}{4\pi t}\right)^{\frac{d}{2}}\left(-\frac{1}{2t}\right)e^{-\frac{|x|^2}{4t}} + \left(\frac{1}{4\pi t}\right)^{\frac{d}{2}}\left(\frac{x_i^2}{4t^2}\right)e^{-\frac{|x|^2}{4t}} \\
&= \left[\left(\frac{1}{4\pi t}\right)^{\frac{d}{2}}\frac{x_i^2}{4t^2} - \frac{1}{2t}\left(\frac{1}{4\pi t}\right)^{\frac{d}{2}}\right]e^{-\frac{|x|^2}{4t}}.
\end{aligned}$$

Now, we obtain the Laplacian of the heat kernel by summing the second derivatives from 1 to d:

$$\Delta\Phi(x,t) = \sum_{i=1}^{d}\partial_{x_i}^2\Phi(x,t) = \left[\left(\frac{1}{4\pi t}\right)^{\frac{d}{2}}\frac{|x|^2}{4t^2} - \frac{d}{2t}\left(\frac{1}{4\pi t}\right)^{\frac{d}{2}}\right]e^{-\frac{|x|^2}{4t}}.$$

By noticing that this is the same as $\partial_t\Phi(x,t)$ we get that $\Delta\Phi(x,t) - \partial_t\Phi(x,t) = 0$.

To assign some meaning to (4.3), the heat kernel is a so called fundamental solution to the Cauchy problem (4.1). As the equation (4.2) suggests, the solution is comprised of convolution integrals, and it is very similar to what we had for the solution (3.20) of the equation (3.18). The last thing, and very much not the least, is the following property:

$$\int_{\mathbb{R}^d}\Phi(z,t)dz = \int_{\mathbb{R}^d}\Phi(x-y,t)dy = 1,$$

where $x \in \mathbb{R}^d$ and $t > 0$. This property is crucial for proving that (4.2) is a solution to the original Cauchy problem. We will not prove this, but we will discuss it a bit, do a formal calculation to justify it being a solution and provide some references. There are some technical difficulties concerning this problem, namely that using basic analysis and measure theory, we can only prove that (4.2) is a solution if we assume that the inhomogeneous term $f(x,t)$ is in addition to being sufficiently differentiable (atleast twice) we need $f(\cdot,t)$ to be compactly supported in $\mathbb{R}^d$ for all $t > 0$. A proof for this can be found in [2].

If we want to relax the assumptions of the inhomogeneous term by assuming $f(\cdot, t) \in C_b(\mathbb{R}^d)$, $\forall t > 0$, we need a considerably more powerful semigroup theory, which in turn requires an in-depth understanding of theory of unbounded operators. For further reading of these types of considerations, [8] is a good place to start.

Let us now provide the formal calculation mentioned before. Functions f and g are assumed to be the same as in the beginning of this chapter, but let us assume for further simplicity that $d = 1$. First, we calculate derivative of (4.2) with respect to t:

$$\begin{aligned}
\partial_t u(x,t) = &\int_{\mathbb{R}} \partial_t \Phi(x-y,t) g(y) dy \\
&+ \int_0^t \int_{\mathbb{R}} \partial_t \Phi(x-y,t-s) f(y,s) dyds + \int_{\mathbb{R}^d} \Phi(x-y,t-t) f(y,t) dy \\
= &\int_{\mathbb{R}} \partial_t \Phi(x-y,t) g(y) dy \\
&+ \int_0^t \int_{\mathbb{R}} \partial_t \Phi(x-y,t-s) f(y,s) dyds + \int_{\mathbb{R}^d} \delta(x-y) f(y,t) dy \\
= &\int_{\mathbb{R}} \partial_t \Phi(x-y,t) g(y) dy \\
&+ \int_0^t \int_{\mathbb{R}} \partial_t \Phi(x-y,t-s) f(y,s) dyds + f(x,t).
\end{aligned}$$

Now, the second derivative with respect to x:

$$\partial_x^2 u(x,t) = \int_{\mathbb{R}} \partial_x^2 \Phi(x-y,t) g(y) dy + \int_0^t \int_{\mathbb{R}^d} \partial_x^2 \Phi(x-y,t-s) f(y,s) dyds.$$

Now using the fact that Φ solves the initial value problem (4.3) we get that:

$$\partial_t u(x,t) - \partial_x^2 u(x,t) = f(x,t), \;\; x \in \mathbb{R}, t \in \mathbb{R}_+.$$

Let us end this chapter by considering a (linear) generalization of the heat equation, the solution of which can be obtained from the previous equation (4.2) by adding an exponential factor. The problem reads as:

$$\begin{cases} \partial_t u(x,t) - \Delta u(x,t) + cu(x,t) = f(x,t), & (x,t) \in \mathbb{R}^d \times \mathbb{R}_+ \\ u(x,0) = g(x), & x \in \mathbb{R}^d, \end{cases}$$

where $c \in \mathbb{R}$. There is a concrete way of obtaining a solution to the Cauchy problem above. Consider the following equation:

$$\begin{cases} \partial_t v(x,t) - \Delta v(x,t) = e^{ct} f(x,t), & (x,t) \in \mathbb{R}^d \times \mathbb{R}_+ \\ v(x,0) = g(x), & x \in \mathbb{R}^d, \end{cases} \tag{4.4}$$

We remark that this is a special case of (4.1). Thus, a modified version of the formula (4.2) is a solution to (4.4). Now if v solves this, then $u(x,t) = e^{-ct}v(x,t)$ will solve the generalized version. Let us calculate:

$$\begin{aligned}
\partial_t u(x,t) - \Delta u(x,t) + cu(x,t) &= \partial_t(v(x,t)e^{-ct}) - \Delta(v(x,t)e^{-ct}) + cu(x,t) \\
&= e^{-ct}\partial_t v(x,t) - cv(x,t)e^{-ct} - e^{-ct}\Delta v(x,t) + cu(x,t) \\
&= f(x,t) - cu(x,t) + cu(x,t) \\
&= f(x,t).
\end{aligned}$$

Also, setting $t = 0$ we see that the initial values match.

Chapter 5

Existence and uniqueness of the generalized solution to the semilinear heat equation

In this section we prove the main result of this thesis: existence and uniqueness of the generalized solution for the semilinear heat equation. Let us begin by stating the classical problem:

$$\begin{cases} \partial_t u(x,t) - \partial_x^2 u(x,t) = |u(x,t)|^p, & (x,t) \in \mathbb{R} \times \mathbb{R}_+ \\ u(x,0) = f(x), & x \in \mathbb{R} \end{cases} \tag{5.1}$$

where $u : (\mathbb{R} \times \mathbb{R}_+^0) \to \mathbb{R}$ is unknown and the known $f : \mathbb{R} \to \mathbb{R}$ is at least continuous and integrable. A classical solution u of (5.1) must be two times continuously differentiable with respect to $x \in \mathbb{R}$, continuously differentiable with respect to $t \in \mathbb{R}_+$ and continuous for all $t \in \mathbb{R}_+^0$. By recalling the formula (4.2) we define the generalized formulation of (5.1), by replacing the inhomogeneous term by $|u|^p$, as the following integral equation:

$$u(x,t) = \frac{1}{\sqrt{4\pi t}} \int_{\mathbb{R}} e^{-\frac{(x-y)^2}{4t}} f(y) dy + \int_0^t \frac{1}{\sqrt{t-s}} \int_{\mathbb{R}} e^{-\frac{(x-y)^2}{4(t-s)}} |u(y,s)|^p dy ds. \tag{5.2}$$

One can show that a classical solution u of (5.1) also solves (5.2) by using calculations similar to the end of Chapter 4.

The proof for the existence and uniqueness of the generalized solution relies heavily on the completeness of a certain weighted space of continuous functions. The corresponding weight is defined as follows:

$$w(x,t) = \sqrt{t+1} \left(1 + \frac{|x|}{\sqrt{t+1}}\right)^m,$$

where $x \in \mathbb{R}$, $t \in \mathbb{R}_+^0$ and we choose $m = 3$. It is quite easy to see, that $w(x,t) \geq 1$ $\forall (x,t) \in (\mathbb{R} \times \mathbb{R}_+^0)$. Thus, we have by Theorem 2.20 that the corresponding weighted space

$$C_w = C_w(\mathbb{R} \times \mathbb{R}_+^0) = \{f \in C(\mathbb{R} \times \mathbb{R}_+^0) : \|wg\|_\infty = \sup_{x\in\mathbb{R}} \sup_{t\in\mathbb{R}_+^0} |w(x,t)g(x,t)| < \infty\} \tag{5.3}$$

is complete. Moreover, by basic metric topology, every closed ball in C_w with radius $\delta > 0$

$$B_\delta = \{u \in C_w : \|u\|_w \leq \delta\}$$

is also complete.

Now we may formulate the main theorem:

Theorem 5.4. *Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function such that*

$$|f(x)| \leq \frac{\delta}{K(1+|x|)^{m+2}},$$

where $m = 3$ and K is sufficiently large. If $p \geq 4$, then (5.2) has a unique solution in B_δ. The $\delta > 0$ will be chosen to be small enough in the course of the proof.

Since the proof of Theorem 5.4 is quite technical, it is useful to prepare the forthcoming calculations. Some of them are quite trivial, but they are used repeatedly throughout the proof.

Let us begin by considering the heat kernel or Gaussian: there exists a constant $C > 0$ such that

$$e^{-z^2} \leq C(1+|z|)^{-m},\ z \in \mathbb{R} \tag{5.5}$$

This is due to the fact that the Gaussian belongs to $S(\mathbb{R})$. Also recall the following formula for the integral of Gaussian:

$$\int_{\mathbb{R}} e^{-ax^2} dx = \sqrt{\frac{\pi}{a}},\ a > 0.$$

If $|x-y| \geq \frac{|x|}{2}$ and $r > 0$, then

$$e^{-\frac{r(x-y)^2}{4t}} \leq e^{-\frac{rx^2}{16(t+1)}},\ \ x, y \in \mathbb{R},\ t \in \mathbb{R}_+. \tag{5.6}$$

If $|x-y| \leq \frac{|x|}{2}$, then

$$(1+|y|)^{-m-2} \leq \left(1 + \frac{|y|}{\sqrt{t+1}}\right)^{-m} \leq 2^m \left(1 + \frac{|x|}{\sqrt{t+1}}\right)^{-m}. \tag{5.7}$$

The first inequality of course holds on the whole real line. The proof for the second inequality is based on the fact that if $|x-y| \leq \frac{|x|}{2}$, then by triangle inequality $|y| \geq \frac{|x|}{2}$.

We also need some estimates for time integrals. If $p \geq 4$, then

$$\int_0^{\frac{t}{2}} \frac{1}{(s+1)^{\frac{p-1}{2}}} ds \leq 2\sqrt{\frac{t}{t+1}} \tag{5.8}$$

and

$$\int_{\frac{t}{2}}^{t} \frac{1}{(s+1)^{\frac{p}{2}}} ds \leq \frac{t}{2} \frac{1}{(\frac{t}{2}+1)^{\frac{p}{2}}} \leq \frac{C}{\sqrt{t+1}}. \tag{5.9}$$

Now we start the existence and uniqueness proof. The basic idea is that instead of "solving" the integral equation, we define the following mapping G:

$$G(u)(x,t) = \frac{1}{\sqrt{4\pi t}} \int_{\mathbb{R}} e^{-\frac{(x-y)^2}{4t}} f(y)dy + \int_0^t \frac{1}{\sqrt{t-s}} \int_{\mathbb{R}} e^{-\frac{(x-y)^2}{4(t-s)}} |u(y,s)|^p dyds, \tag{5.10}$$

and prove that it has a fixed point $G(u) = u$. The goal is to find a sufficiently small ball B_δ in our weighted space, which G maps to itself. After this, we have to prove that $G : B_\delta \to B_\delta$ is a contraction, and thus we may apply the Banach fixed point theorem 2.8. This is achieved by splitting the integrals into different domains. Namely, we split $\mathbb{R}$ into sets $\{|x-y| \geq \frac{|x|}{2}\}$ and $\{|x-y| \leq \frac{|x|}{2}\}$ and the time interval $[0,t]$ into $[0, \frac{t}{2}]$ and $[\frac{t}{2}, t]$. Also, since the heat kernel is not too well-behaved for small times t, we split the time into cases where $t \leq 1$ and $t > 1$. Keeping all of the above in mind, let us begin the calculations.

We begin the proof of Theorem 5.4 by estimating the first integral of (5.10) for times $t \leq 1$, and also we split $\mathbb{R}$ in the aforementioned way. Let us begin by looking at the integral over the set $\{|x-y| \geq \frac{|x|}{2}\}$.

$$
\begin{aligned}
&\frac{1}{\sqrt{4\pi t}}\int_{|x-y|\geq\frac{|x|}{2}} e^{-\frac{(x-y)^2}{4t}}|f(y)|dy\\
&=\frac{1}{\sqrt{4\pi t}}\int_{|x-y|\geq\frac{|x|}{2}} e^{-\frac{1}{2}\frac{(x-y)^2}{4t}}e^{-\frac{1}{2}\frac{(x-y)^2}{4t}}|f(y)|dy\\
&\overset{*}{\leq}\frac{1}{\sqrt{4\pi t}}e^{-\frac{x^2}{32(t+1)}}\int_{|x-y|\geq\frac{|x|}{2}} e^{-\frac{(x-y)^2}{8t}}|f(y)|dy\\
&\overset{**}{\leq} C_1\left(1+\frac{|x|}{\sqrt{t+1}}\right)^{-m}\frac{1}{\sqrt{4\pi t}}\int_{\mathbb{R}} e^{\frac{-(x-y)^2}{8t}}|f(y)|dy\\
&\leq C_1\left(1+\frac{|x|}{\sqrt{t+1}}\right)^{-m}\frac{1}{\sqrt{4\pi t}}\int_{\mathbb{R}} e^{-\frac{(x-y)^2}{8t}}\frac{\delta}{K(1+|y|)^{m+2}}dy\\
&\leq C_1\left(1+\frac{|x|}{\sqrt{t+1}}\right)^{-m}\frac{\delta}{K}\frac{1}{\sqrt{4\pi t}}\sqrt{8\pi t}\\
&\leq \frac{\sqrt{2}C_1}{K}\delta\left(1+\frac{|x|}{\sqrt{t+1}}\right)^{-m}.
\end{aligned}
$$

In the inequalities (*) and (**) we used (5.6) and (5.5), respectively. Next, we consider the integral over the domain $\{|x-y|\leq\frac{|x|}{2}\}$:

$$
\begin{aligned}
&\frac{1}{\sqrt{4\pi t}}\int_{|x-y|\leq\frac{|x|}{2}} e^{-\frac{(x-y)^2}{4t}}|f(y)|dy\\
&\leq\frac{1}{\sqrt{4\pi t}}\int_{|x-y|\leq\frac{|x|}{2}} e^{-\frac{(x-y)^2}{4t}}\frac{\delta}{K}(1+|y|)^{-m-2}dy\\
&\overset{1.}{\leq}\frac{\delta}{K}\frac{1}{\sqrt{4\pi t}}\int_{|x-y|\leq\frac{|x|}{2}} e^{-\frac{(x-y)^2}{4t}}(1+|y|)^{-2}2^m\left(1+\frac{|x|}{\sqrt{t+1}}\right)^{-m}dy\\
&\leq 2^m\frac{\delta}{K}\left(1+\frac{|x|}{\sqrt{t+1}}\right)^{-m}\frac{1}{\sqrt{4\pi t}}\int_{\mathbb{R}} e^{-\frac{(x-y)^2}{4t}}(1+|y|)^{-2}dy\\
&\leq 2^m\frac{\delta}{K}\left(1+\frac{|x|}{\sqrt{t+1}}\right)^{-m}\frac{1}{\sqrt{4\pi t}}\sqrt{4\pi t}\\
&= 2^m\frac{\delta}{K}\left(1+\frac{|x|}{\sqrt{t+1}}\right)^{-m}.
\end{aligned}
$$

In the inequality (1.) above, we used (5.7). Now we have for times $t \le 1$ the following estimate:

$$\begin{aligned}
&\frac{1}{\sqrt{4\pi t}}\int_{\mathbb{R}} e^{-\frac{(x-y)^2}{4t}}|f(y)|dy\\
&\le \left(1+\frac{|x|}{\sqrt{t+1}}\right)^{-m}\delta\left(\frac{2^m}{K}+\frac{\sqrt{2}C_1}{K}\right)\\
&\le \left(1+\frac{|x|}{\sqrt{t+1}}\right)^{-m}\frac{C'\delta}{K}.
\end{aligned}$$

The last inequality is just absorbing the constants.

Let us now estimate the same integral for times $t > 1$. First,

$$\begin{aligned}
&\frac{1}{\sqrt{4\pi t}}\int_{|x-y|\le\frac{|x|}{2}} e^{-\frac{(x-y)^2}{4t}}|f(y)|dy\\
&\le \frac{1}{\sqrt{4\pi t}}\int_{|x-y|\le\frac{|x|}{2}} e^{-\frac{(x-y)^2}{4t}}\frac{\delta}{K}(1+|y|)^{-m-2}dy\\
&\le \frac{1}{\sqrt{4\pi t}}\frac{\delta}{K}2^m\left(1+\frac{|x|}{\sqrt{t+1}}\right)^{-m}\int_{\mathbb{R}}(1+|y|)^{-2}dy\\
&\le 2^{m+1}\frac{\delta}{K}\left(1+\frac{|x|}{\sqrt{t+1}}\right)^{-m}\frac{1}{\sqrt{4\pi t}}
\end{aligned} \tag{5.11}$$

and second,

$$\begin{aligned}
&\frac{1}{\sqrt{4\pi t}}\int_{|x-y|\ge\frac{|x|}{2}} e^{-\frac{(x-y)^2}{4t}}|f(y)|dy\\
&\le \frac{1}{\sqrt{4\pi t}}\int_{|x-y|\ge\frac{|x|}{2}} e^{-\frac{x^2}{16(t+1)}}|f(y)|dy\\
&\le \frac{1}{\sqrt{4\pi t}}\frac{\delta}{K}C_2\left(1+\frac{|x|}{\sqrt{t+1}}\right)^{-m}\int_{\mathbb{R}}(1+|y|)^{-m-2}dy\\
&= \frac{C_2}{\sqrt{4\pi t}}\frac{\delta}{K}\left(1+\frac{|x|}{\sqrt{t+1}}\right)^{-m}\frac{2}{m+1}.
\end{aligned} \tag{5.12}$$

Thus, we obtain the following final estimate for the first integral for times $t > 1$:

$$
\begin{aligned}
&\frac{1}{\sqrt{4\pi t}}\int_{\mathbb{R}} e^{-\frac{(x-y)^2}{4t}}|f(y)|dy \\
&\leq \left(\frac{2}{m+1}+2^{m+1}\right)\frac{1}{\sqrt{4\pi t}}\frac{\delta}{K}\left(1+\frac{|x|}{\sqrt{t+1}}\right)^{-m} \\
&\leq \frac{C_m}{\sqrt{4\pi t}}\frac{\delta}{K}\left(1+\frac{|x|}{\sqrt{t+1}}\right)^{-m}. \qquad (5.13)
\end{aligned}
$$

Next, we estimate the second integral:

$$
\int_0^t \frac{1}{\sqrt{4\pi(t-s)}}\int_{\mathbb{R}} e^{-\frac{(x-y)^2}{4(t-s)}}|u(y,s)|^p dyds. \qquad (5.14)
$$

We begin from the inner integral, and split it up in the same way as in the first integral. Let us look at the set $\{|x-y| \geq \frac{|x|}{2}\}$. First, we note that since $u \in B_\delta$ we have that

$$
|u(y,s)|^p \leq \delta^p\left(1+\frac{|y|}{\sqrt{s+1}}\right)^{-pm}\frac{1}{(s+1)^{\frac{p}{2}}}, \qquad (5.15)
$$

which we will use in every upcoming estimate of the inner integral. After the following calculation, we will not show explicitly that we used (5.15).

$$
\begin{aligned}
&\int_{|x-y|\geq\frac{|x|}{2}} e^{-\frac{(x-y)^2}{4(t-s)}}|u(y,s)|^p dy \\
&\leq \int_{|x-y|\geq\frac{|x|}{2}} e^{-\frac{(x-y)^2}{4(t-s)}}\delta^p\left(1+\frac{|y|}{\sqrt{s+1}}\right)^{-pm}\frac{1}{(s+1)^{\frac{p}{2}}}dy \\
&\leq \delta^p C\left(1+\frac{|x|}{\sqrt{t+1}}\right)^{-m}\int_{\mathbb{R}}(1+|\tilde{y}|)^{-pm}\frac{1}{(s+1)^{\frac{(p-1)}{2}}}d\tilde{y} \\
&= \delta^p C\left(1+\frac{|x|}{\sqrt{t+1}}\right)^{-m}\frac{1}{(s+1)^{\frac{(p-1)}{2}}}\frac{2}{pm-1}. \qquad (5.16)
\end{aligned}
$$

In the inequality above, we used (5.5) and made a change of variables $\tilde{y} = \frac{y}{\sqrt{s+1}}$. Also, the last line is not singular since $pm \geq 12 > 1$. Let us also calculate another estimate for the same integral:

$$\begin{aligned}
&\int_{|x-y|\geq\frac{|x|}{2}} e^{-\frac{(x-y)^2}{4(t-s)}}\delta^p\left(1+\frac{|y|}{\sqrt{s+1}}\right)^{-pm}\frac{1}{(s+1)^{\frac{p}{2}}}dy\\
&\leq \delta^p e^{-\frac{x^2}{32(t+1)}}\int_{|x-y|\geq\frac{|x|}{2}} e^{-\frac{(x-y)^2}{8(t-s)}}\frac{1}{(s+1)^{\frac{p}{2}}}dy\\
&\leq \delta^p C_1\left(1+\frac{|x|}{\sqrt{t+1}}\right)^{-m}\int_{|x-y|\geq\frac{|x|}{2}} e^{-\frac{(x-y)^2}{8(t-s)}}\frac{1}{(s+1)^{\frac{p}{2}}}dy\\
&\leq C_1\delta^p\left(1+\frac{|x|}{\sqrt{t+1}}\right)^{-m}\frac{\sqrt{8\pi(t-s)}}{(s+1)^{\frac{p}{2}}}. \qquad (5.17)
\end{aligned}$$

We may move to the case where $\{|x-y|\leq\frac{|x|}{2}\}$. It is good to keep (5.7) in mind while working in this integration domain:

$$\begin{aligned}
&\int_{|x-y|\leq\frac{|x|}{2}} e^{-\frac{(x-y)^2}{4(t-s)}}\delta^p\left(1+\frac{|y|}{\sqrt{s+1}}\right)^{-pm}\frac{1}{(s+1)^{\frac{p}{2}}}dy\\
&\leq \int_{|x-y|\leq\frac{|x|}{2}} e^{-\frac{(x-y)^2}{4(t-s)}}\delta^p\left(1+\frac{|y|}{\sqrt{s+1}}\right)^{-(p-1)m}2^m\left(1+\frac{|x|}{\sqrt{t+1}}\right)^{-m}\frac{1}{(s+1)^{\frac{p}{2}}}dy\\
&\leq \delta^p\left(1+\frac{|x|}{\sqrt{t+1}}\right)^{-m}\int_{\mathbb{R}}(1+|\tilde{y}|)^{-(p-1)m}\frac{2^m}{(s+1)^{\frac{(p-1)}{2}}}d\tilde{y}\\
&= \delta^p\left(1+\frac{|x|}{\sqrt{t+1}}\right)^{-m}\frac{2^{m+1}}{(p-1)m-1}\frac{1}{(s+1)^{\frac{(p-1)}{2}}}. \qquad (5.18)
\end{aligned}$$

Similarly to (5.16), we have no singularity, since $(p-1)m\geq 9>1$.

The following estimate is quite straight-forward, and to avoid constant repetition of calculations, we just state it:

(5.19)

$$\int_{|x-y|\leq\frac{|x|}{2}} e^{-\frac{(x-y)^2}{4(t-s)}}\delta^p\left(1+\frac{|y|}{\sqrt{s+1}}\right)^{-pm}\frac{1}{(s+1)^{\frac{p}{2}}}dy\leq 2^m\delta^p\left(1+\frac{|x|}{\sqrt{t+1}}\right)^{-m}\frac{\sqrt{4\pi(t-s)}}{(s+1)^{\frac{p}{2}}}.$$

We have the necessary estimates for the inner integral, so we can proceed to the time integral. We first consider times $[0,\frac{t}{2}]$. In this case, we use the two integral estimates

(5.16) and (5.18) to establish the following

$$
\begin{aligned}
&\int_0^{\frac{t}{2}} \frac{1}{\sqrt{4\pi(t-s)}} \int_{\mathbb{R}} e^{-\frac{(x-y)^2}{4(t-s)}} |u(y,s)|^p dyds \\
&\leq \delta^p \left(1+\frac{|x|}{\sqrt{t+1}}\right)^{-m} \left(\frac{2^{m+1}}{(p-1)m-1}\right) \int_0^{\frac{t}{2}} \frac{1}{\sqrt{4\pi(t-s)}} \frac{1}{(s+1)^{\frac{p-1}{2}}} ds \\
&+ \delta^p \left(1+\frac{|x|}{\sqrt{t+1}}\right)^{-m} \left(\frac{2}{pm-1}\right) \int_0^{\frac{t}{2}} \frac{1}{\sqrt{4\pi(t-s)}} \frac{1}{(s+1)^{\frac{(p-1)}{2}}} ds \\
&\leq \delta^p \left(1+\frac{|x|}{\sqrt{t+1}}\right)^{-m} \left(\frac{2^{m+1}}{(p-1)m-1} + \frac{2}{pm-1}\right) \int_0^{\frac{t}{2}} \frac{1}{\sqrt{\frac{4\pi t}{2}}} \frac{1}{(s+1)^{\frac{(p-1)}{2}}} ds \\
&\leq 3\delta^p \left(1+\frac{|x|}{\sqrt{t+1}}\right)^{-m} \sqrt{\frac{2}{4\pi t}} 2\sqrt{\frac{t}{t+1}} \\
&\leq \frac{3\delta^p}{\sqrt{t+1}} \left(1+\frac{|x|}{\sqrt{t+1}}\right)^{-m}.
\end{aligned}
$$

Here, the third inequality follows from (5.8), and we also used the fact that

$$
\left(\frac{2^{m+1}}{(p-1)m-1} + \frac{2}{pm-1}\right) \leq 3.
$$

Now we calculate the estimates for $[\frac{t}{2}, t]$ using (5.17) and (5.19):

$$
\begin{aligned}
&\int_{\frac{t}{2}}^{t} \frac{1}{\sqrt{4\pi(t-s)}} \int_{\mathbb{R}} e^{-\frac{(x-y)^2}{4(t-s)}} |u(y,s)|^p dyds \\
&\leq \int_{\frac{t}{2}}^{t} \frac{1}{\sqrt{4\pi(t-s)}} 2^m \delta^p \left(1+\frac{|x|}{\sqrt{t+1}}\right)^{-m} \frac{\sqrt{4\pi(t-s)}}{(s+1)^{\frac{p}{2}}} ds \\
&+ \int_{\frac{t}{2}}^{t} \frac{C_1}{\sqrt{4\pi(t-s)}} \delta^p \left(1+\frac{|x|}{\sqrt{t+1}}\right)^{-m} \frac{\sqrt{8\pi(t-s)}}{(s+1)^{\frac{p}{2}}} ds \\
&\leq C_4 \delta^p \left(1+\frac{|x|}{\sqrt{t+1}}\right)^{-m} \frac{1}{\sqrt{t+1}}.
\end{aligned}
$$

The last inequality is the usual absorbing of constants combined with the estimate (5.9).

Now, combining the estimates for the second integral, we obtain:

$$
\int_0^t \frac{1}{\sqrt{t-s}} \int_{\mathbb{R}} e^{-\frac{(x-y)^2}{4(t-s)}} |u(y,s)|^p dyds \leq (3+C_4)\delta^p \left(1+\frac{|x|}{\sqrt{t+1}}\right)^{-m} \frac{1}{\sqrt{t+1}}. \tag{5.20}
$$

Thus, giving us a concrete upper bound for $G(u)(x,t)$ when $t \leq 1$:

$$|G(u)(x,t)| \leq \frac{C'\delta}{K}\left(1+\frac{|x|}{\sqrt{t+1}}\right)^{-m} + (3+C_4)\delta^p\left(1+\frac{|x|}{\sqrt{t+1}}\right)^{-m}\frac{1}{\sqrt{t+1}}.$$

Multiplying the equation by the weight $w(x,t) = \sqrt{t+1}\left(1+\frac{|x|}{\sqrt{t+1}}\right)^m$, we get

$$w(x,t)|G(u)(x,t)| \leq \frac{C'\delta}{K}\sqrt{t+1} + (3+C_4)\delta^p \leq \frac{C'''}{K}\delta + (3+C_4)\delta^p, \tag{5.21}$$

where the last inequality is just absorbing $\sqrt{t+1} \leq \sqrt{2}$ into the constant.

The form above is promising, and leads us to solve the following inequality, which will be needed to guarantee that $G(u)$ is a map $B_\delta \to B_\delta$, where $\delta > 0$:

$$\frac{C'''}{K}\delta + (3+C_4)\delta^p \leq \delta \Leftrightarrow \delta\left(\frac{C'''-K}{K} + (3+C_4)\delta^{p-1}\right) \leq 0.$$

So if $\delta > 0$, then we need that $\left(\frac{c'''-K}{K} + (3+C_4)\delta^{p-1}\right) \leq 0$, which requires:

$$\delta \leq \left(\frac{K-C'''}{K(3+C_4)}\right)^{\frac{1}{p-1}}. \tag{5.22}$$

The right-hand side above can be made positive by choosing a large enough $K > 0$.

Let us now do calculations for $t > 1$:

$$|G(u)(x,t)| \leq \frac{1}{\sqrt{4\pi t}}\frac{C_m}{K}\left(1+\frac{|x|}{\sqrt{t+1}}\right)^{-m}\delta + (3+C_4)\delta^p\left(1+\frac{|x|}{\sqrt{t+1}}\right)^{-m}\frac{1}{\sqrt{t+1}}$$

Multiplying again by the weight yields

$$\begin{aligned} w(x,t)|G(u)(x,t)| &\leq \frac{\sqrt{t+1}}{\sqrt{4\pi t}}\frac{C_m}{K}\delta + (3+C_4)\delta^p \\ &\leq \frac{C_m'}{K}\delta + (3+C_4)\delta^p. \end{aligned} \tag{5.23}$$

The second inequality is just absorbing the constant and using the fact that $\frac{\sqrt{t+1}}{\sqrt{4\pi t}}$ is uniformly bounded when $t > 1$. Now proceeding as in the case $t \leq 1$, we need to ensure that

$$\frac{C_m'}{K}\delta + (3+C_4)\delta^p \leq \delta.$$

And similarly to above, we see that this holds, if

$$\delta \leq \left(\frac{K - C'_m}{K(3 + C_4)} \right)^{\frac{1}{p-1}}. \tag{5.24}$$

We choose K such that in addition to (5.22), also the right-hand side of (5.24) is positive. Then, we choose

$$\delta \leq \min \left(\left(\frac{K - C'''}{K(3 + C_4)} \right)^{\frac{1}{p-1}}, \left(\frac{K - C'_m}{K(3 + C_4)} \right)^{\frac{1}{p-1}} \right), \tag{5.25}$$

and take supremum with respect to x and t in (5.21) and (5.23) to obtain $\|G(u)\|_w \leq \delta$. Thus, we have that there exists a $\delta > 0$ such that $G : B_\delta \to B_\delta$.

All that remains is to prove that G is a contraction. For this, we prove the following lemma:

Lemma 5.26. *If $u, v \in B_\delta$ and $p \geq 4$, then we have the following estimate:*

$$|u(y,s)|u(y,s)|^{p-1} - v(y,s)|v(y,s)|^{p-1}| \leq \|u - v\|_\infty p\delta^{p-1} \left(1 + \frac{|y|}{\sqrt{s+1}} \right)^{-(p-1)m}. \tag{5.27}$$

Proof. The claim is trivial if either u or v is zero. Also if $u = \pm v$ the claim is trivial.

Now working in the non-trivial cases, consider the fact that the function $f(x) = x|x|^{p-1}$ is increasing and continuously differentiable, with derivative $f'(x) = p|x|^{p-1}$. Without loss of generality, we may assume that $u > v$. We get for some $\xi(y,s) \in (v(y,s), u(y.s))$

$$\begin{aligned} &f(u(y,s)) - f(v(y,s)) = f'(\xi(y,s))(u(y,s) - v(y,s)) \\ &\Leftrightarrow |f(u(y,s)) - f(v(y,s))| = |u(y,s) - v(y,s)|p|\xi(y,s)|^{p-1}. \end{aligned}$$

This implies:

$$|f(u(y,s)) - f(v(y,s))| \leq \|u - v\|_\infty \delta^{p-1} \left(1 + \frac{|y|}{\sqrt{s+1}} \right)^{-(p-1)m}. \tag{5.28}$$

□

Note that the estimate (5.27) holds also when we replace $u|u|^{p-1}$ by $|u|^p$.

The proof for the contractivity uses the same methods as the proof that $G : B_\delta \to B_\delta$:

$$\begin{aligned} &|G(u)(x,t) - G(v)(x,t)| \\ &\leq \int_0^t \frac{1}{\sqrt{4\pi(t-s)}} \int_{\mathbb{R}} e^{-\frac{(x-y)^2}{4(t-s)}} ||u(y,s)|^p - |v(y,s)|^p| \, dyds \\ &\leq p\delta^{p-1} \|u - v\|_\infty \int_0^t \frac{1}{\sqrt{4\pi(t-s)}} \int_{\mathbb{R}} e^{-\frac{(x-y)^2}{4(t-s)}} \left(1 + \frac{|y|}{\sqrt{s+1}} \right)^{-(p-1)m} \frac{1}{(s+1)^{\frac{p-1}{2}}} dyds, \end{aligned}$$

in the above calculations we used the modified version of the estimate (5.27) (replacing $u|u|^{p-1}$ by $|u|^p$).

So, in similar fashion as for the previous proof, we split up $\mathbb{R}$ into sets $\{|x-y| \leq \frac{|x|}{2}\}$ and $\{|x-y| \geq \frac{|x|}{2}\}$. Let us begin with the set $\{|x-y| \leq \frac{|x|}{2}\}$:

$$\begin{aligned}
&\int_{|x-y|\leq\frac{|x|}{2}} e^{-\frac{(x-y)^2}{4(t-s)}} \left(1+\frac{|y|}{\sqrt{s+1}}\right)^{-(p-1)m} \frac{1}{(s+1)^{\frac{p-1}{2}}} dy \\
&\leq 2^m \left(1+\frac{|x|}{\sqrt{t+1}}\right)^{-(p-1)m} \int_{\mathbb{R}} e^{-\frac{(x-y)^2}{4(t-s)}} \frac{1}{(s+1)^{\frac{p-1}{2}}} dy \\
&\leq 2^m \left(1+\frac{|x|}{\sqrt{t+1}}\right)^{-m} \frac{\sqrt{4\pi(t-s)}}{(s+1)^{\frac{p-1}{2}}}.
\end{aligned}$$

Then, for $\{|x-y| \geq \frac{|x|}{2}\}$

$$\begin{aligned}
&\int_{|x-y|\geq\frac{|x|}{2}} e^{-\frac{(x-y)^2}{4(t-s)}} \left(1+\frac{|y|}{\sqrt{s+1}}\right)^{-(p-1)m} \frac{1}{(s+1)^{\frac{p-1}{2}}} dy \\
&\leq \int_{|x-y|\geq\frac{|x|}{2}} e^{-\frac{(x-y)^2}{8(t-s)}} e^{-\frac{(x-y)^2}{8(t-s)}} \frac{1}{(s+1)^{\frac{p-1}{2}}} dy \\
&\leq C_1 \left(1+\frac{|x|}{\sqrt{t+1}}\right)^{-m} \frac{\sqrt{8\pi(t-s)}}{(s+1)^{\frac{p-1}{2}}}.
\end{aligned}$$

Now combining the above calculations we get the following estimate:

$$\begin{aligned}
&\int_0^t \int_{\mathbb{R}} e^{-\frac{(x-y)^2}{4(t-s)}} \left(1+\frac{|y|}{\sqrt{s+1}}\right)^{-(p-1)m} \frac{1}{(s+1)^{\frac{p-1}{2}}} dyds \\
&\leq 2^m \left(1+\frac{|x|}{\sqrt{t+1}}\right)^{-m} \int_0^t \frac{\sqrt{(4\pi(t-s)}}{\sqrt{(4\pi(t-s)}} \frac{1}{(s+1)^{\frac{p-1}{2}}} ds \\
&+ C_1 \left(1+\frac{|x|}{\sqrt{t+1}}\right)^{-m} \int_0^t \frac{\sqrt{8\pi(t-s)}}{\sqrt{4\pi(t-s)}} \frac{1}{(s+1)^{\frac{p-1}{2}}} ds \\
&= C_1' \left(1+\frac{|x|}{\sqrt{t+1}}\right)^{-m} \frac{1}{\sqrt{t+1}}.
\end{aligned}$$

The last line is just an absorption of constants.

Combining everything we obtain the following explicit inequality:

$$\begin{aligned}|G(u)(x,t)-G(v)(x,t)| &\le C_1'\left(1+\frac{|x|}{\sqrt{t+1}}\right)^{-m}\frac{1}{\sqrt{t+1}}p\delta^{p-1}\left\|u-v\right\|_\infty \\ &\le C_1'\left(1+\frac{|x|}{\sqrt{t+1}}\right)^{-m}\frac{1}{\sqrt{t+1}}p\delta^{p-1}\left\|u-v\right\|_w,\end{aligned}$$

where the last inequality follows from the fact that $\|g\|_\infty \le \|g\|_w$, $\forall g \in C_w$. Now multiplying by the weight $w(x,t)$ and taking the supremum with respect to $x \in \mathbb{R}$ and $t \in \mathbb{R}_+^0$ on both sides yields

$$\|G(u)-G(v)\|_w \le p\delta^{p-1}C_1'\left\|u-v\right\|_w,$$

where $\delta > 0$ can be chosen in such a way that it satisfies (5.25) and $p\delta^{p-1}C_1' < 1$. Thus, $G: B_\delta \to B_\delta$ is a contraction. So, by Banach fixed point theorem 2.8, there exists a unique $u \in B_\delta$ such that $G(u) = u$. This proves Theorem 5.4.

It might not come as a surprise, but this proof can be easily generalized to $\mathbb{R}^d$. This is due to the fact that the d-dimensional heat kernel $\Phi(x,t) = (\frac{1}{4\pi t})^{\frac{d}{2}}e^{-\frac{|x-y|^2}{4t}}$ can be decomposed into a product of d one-dimensional heat kernels. Also the weight $w(x,t) = \sqrt{t+1}\left(1+\frac{|x|}{\sqrt{t+1}}\right)^{-m}$ is easily generalized to $\mathbb{R}^d \times \mathbb{R}_+$. The reason we did not do the proof in $\mathbb{R}^d$ is that it is essentially the same as in $\mathbb{R}$ but with added technical difficulty, which does not necessarily add any further insights. It is good to note that based on our estimates we may replace the nonlinear term $|u|^p$ by $u|u|^{p-1}$, and the proof for Theorem 5.4 would still work.

Appendix A

Discussion on differentiability of the generalized solution

The proof that a generalized solution is a classical solution is a bit trickier, but very similar to the proof of 5.4. It might be that the semigroup methods mentioned in Chapter 4 could work here, since u belongs to $C_b(\mathbb{R} \times \mathbb{R}_+^0)$.

In this appendix we will discuss the case of differentiability with respect to t. For this, we consider a slightly modified version of the original weighted space $C_w(\mathbb{R} \times \mathbb{R}_+^0)$. To this end, let $\epsilon > 0$ and $\tau > \epsilon$ be fixed, and we denote $I_\tau = (\tau - \epsilon, \tau + \epsilon)$. We define the following space:

$$\begin{aligned} C_w^\tau(\mathbb{R} \times I_\tau) = \{ & g \in C_w(\mathbb{R} \times \mathbb{R}_+^0) : \partial_t g \text{ exists, is continuous on the} \\ & \text{interval } t \in \bar{I}_\tau \text{ and } \|g\|_\tau = \sup_{x \in \mathbb{R}} \sup_{t \in \bar{I}_\tau} w(x,t) |\partial_t g(x,t)| < \infty \}, \end{aligned}$$

where $\bar{I}_\tau$ is the closure of I_τ. We endow the space C_w^τ with the norm

$$\|g\|_{w,\tau} = \|wg\|_\infty + \|g\|_\tau \,, \tag{A.1}$$

so that it becomes a Banach space, by well known arguments.

We denote by $B_{\delta'}$ a closed ball in $C_w^\tau(\mathbb{R} \times I_\tau)$ centered at 0 and radius $\delta' > 0$. As in the proof for Theorem 5.4, the goal is to prove that there exists such a $\delta' > 0$ that $G : B_{\delta'} \to B_{\delta'}$ is a contraction. If this holds, then it is not hard to see that by choosing $\delta' > 0$ to be small enough, the differentiable solution will coincide with the generalized solution in the Theorem 5.4. We will provide some ideas on how to approach this. It is sufficient to only consider the derivative term $\|g\|_\tau$ of the norm, since the proof of 5.4 covers the term $\|wg\|_\infty$. Let $t \in I_\tau$, $u \in B_{\delta'}$ and $h > 0$ be such that $t + h \in I_\tau$, and let us calculate the difference quotient of $G(u)(x,t)$ (see (5.10)) with respect to t:

$$
\begin{aligned}
&\frac{G(u)(x,t+h)-G(u)(x,t)}{h} \\
&= \int_{\mathbb{R}} \frac{(\Phi(x-y,t+h)-\Phi(x-y,t))}{h} f(y)dy \\
&+ \frac{1}{h}\int_0^{t+h}\int_{\mathbb{R}} \Phi(x-y,t+h-s)|u(y,s)|^p dyds \\
&- \frac{1}{h}\int_0^{t}\int_{\mathbb{R}} \Phi(x-y,t-s)|u(y,s)|^p dyds \\
&= I_1+I_2+I_3.
\end{aligned}
$$

The case I_1 is the simplest, if we consider purely the differentiability, since the heat kernel $\phi(\cdot,t)\in S(\mathbb{R})$ for every $t\in\mathbb{R}_+$:

$$
\begin{aligned}
&\int_{\mathbb{R}} \frac{(\Phi(x-y,t+h)-\Phi(x-y,t))}{h} f(y)dy \\
&= \int_{\mathbb{R}} \partial_t(\Phi(x-y,t+\nu_1 h)f(y)dy \\
&= \int_{\mathbb{R}} \Phi(x-y,t+\nu_1 h)\left(\frac{(x-y)^2}{4(t+\nu_1 h)^2}-\frac{1}{2(t+\nu_1 h)}\right) f(y)dy, \\
&\leq \int_{\mathbb{R}} \Phi(x-y,t+\nu_1 h)\frac{(x-y)^2}{4(t+\nu_1 h)^2}\frac{\delta}{K}(1+|y|)^{-m-2}dy \\
&+ \int_{\mathbb{R}} \Phi(x-y,t+\nu_1 h)\frac{1}{2(t+\nu_1 h)}\frac{\delta}{K}(1+|y|)^{-m-2}dy \\
&= I'+I''
\end{aligned}
$$

where $|\nu_1|\leq 1$, and for future reference, for all $j=1,...,n$ $|\nu_j|\leq 1$ is a constant related to applying the mean value theorem. We should also justify that we can bound I' and I'' sufficiently. Let us start from noticing that

$$
\frac{1}{2(t+\nu_1 h)} \leq C_{\epsilon,\tau}, \tag{A.2}
$$

where $C_{\epsilon,\tau}$ is a uniform constant depending on τ and ϵ. Thus, I'' can be estimated in a similar fashion as the calculations (5.11), (5.12) and (5.13) (the first integral with times $t>1$). For I' we use (A.2) too, and in addition to this, we should split the integral domain $\mathbb{R}$ into sets $B_1=\{R(x,y,t)=\frac{(x-y)^2}{4(t+\nu_1 h)}\leq 1\}$ and $B_2=\{R(x,y,t)>1\}$. In the integration domain B_1 we simply estimate $R(x,y,t) \leq 1$, and estimate similarly as in the case I''. In the domain B_2 we use the fact that $R(x,y,t)\Phi(x-y,t+\nu_1 h)$ is uniformly bounded, and proceed with the estimates as usual. We may now move onto working with I_2:

$$
\begin{aligned}
&\frac{1}{h}\int_0^{t+h}\int_{\mathbb{R}} \Phi(x-y,t+h-s)|u(y,s)|^p dyds \\
&= \frac{1}{h}\int_{-h}^{t}\int_{\mathbb{R}} \Phi(x-y,t-s)|u(y,s+h)|^p dyds \\
&= \frac{1}{h}\int_{-h}^{0}\int_{\mathbb{R}} \Phi(x-y,t-s)|u(y,s+h)|^p dyds \\
&+ \frac{1}{h}\int_{0}^{t}\int_{\mathbb{R}} \Phi(x-y,t-s)|u(y,s+h)|^p dyds.
\end{aligned}
$$

We used a simple change of variables to move the h inside the function u. Now, combining I_2 and I_3, we obtain:

$$
\begin{aligned}
&I_2 + I_3 \\
&= \frac{1}{h}\int_{-h}^{0}\int_{\mathbb{R}} \Phi(x-y,t-s)|u(y,s+h)|^p dyds \\
&+ \int_0^t \int_{\mathbb{R}} \Phi(x-y,t-s)\left(\frac{|u(y,s+h)|^p - |u(y,s)|^p}{h}\right) dyds \qquad \text{(A.3)} \\
&= \frac{1}{h}\int_{-h}^{0}\int_{\mathbb{R}} \Phi(x-y,t-s)|u(y,s+h)|^p dyds \\
&+ \int_0^{t-\epsilon_2} \int_{\mathbb{R}} \Phi(x-y,t-s)\left(\frac{|u(y,s+h)|^p - |u(y,s)|^p}{h}\right) dyds \\
&+ \int_{t-\epsilon_2}^{t} \int_{\mathbb{R}} \Phi(x-y,t-s)\left(\frac{|u(y,s+h)|^p - |u(y,s)|^p}{h}\right) dyds \qquad \text{(A.4)} \\
&= \frac{1}{h}\int_{-h}^{0}\int_{\mathbb{R}} \Phi(x-y,t-s)|u(y,s+h)|^p dyds \\
&+ \int_0^{t-\epsilon_2} \int_{\mathbb{R}} \Phi(x-y,t-s)\left(\frac{|u(y,s+h)|^p - |u(y,s)|^p}{h}\right) dyds \\
&+ \int_{t-\epsilon_2}^{t} \int_{\mathbb{R}} \Phi(x-y,t-s)\left(p|u(y,s+\nu_2 h)|^{p-1}\partial_s u(y,s+\nu_2 h)\right) dyds \\
&= J_1 + J_2 + J_3.
\end{aligned}
$$

To explain the calculations above, we moved the difference quotient into the function $u \in B_{\delta'}$. Then, we split up the integral domain of (A.3) into $[0, t-\epsilon_2]$ and $[t-\epsilon_2, t]$, where $\epsilon_2 > 0$ is fixed and chosen in such a way that $t + h - \epsilon_2 \in I_\tau$. Thus, we may apply mean value theorem to the difference quotient in (A.4).

Let us now consider, how we should go about estimating J_1, J_2 and J_3 separately. First we note, that J_1 is very similar to (5.14), but the integral with respect to s has length h. This means, that we may apply similar estimates and the length of the interval will cancel out the term h^{-1}. The case of J_3 is similar to J_1, since it is still close enough to (5.14) to have similar estimates. This is due to the fact that we assumed the time derivative of u to also belong to a similar weighted space. In addition to all this, we may choose ϵ_2 to be very small, which will be helpful too.

Lastly, we consider J_2, which is the trickiest of the three integrals. We begin by noticing that the heat kernel $\Phi(x-y, t-s)$ is not singular in the set $\mathbb{R} \times [0, t-\epsilon_2]$ and u is not assumed to be differentiable in the domain. Thus, we should apply change of variables once again so that we have a difference quotient of the heat kernel inside the integral. Then, we do some splitting up of the integrals to obtain the following expression for J_2:

$$\begin{aligned}
J_2 &= \int_h^{t-\epsilon_2} \int_{\mathbb{R}} \partial_t \Phi(x-y, t-s+\nu_3 h) |u(y,s)|^p dyds \\
&+ \int_{t-\epsilon_2}^{t-\epsilon_2+h} \int_{\mathbb{R}} \Phi(x-y, t+h-s) \left(\frac{|u(y,s)|^p}{h} \right) dyds \\
&- \int_0^h \int_{\mathbb{R}} \Phi(x-y, t-s) \left(\frac{|u(y,s)|^p}{h} \right) dyds \\
&= I_4 + I_5 + I_6.
\end{aligned}$$

First, we see that both I_5 and I_6 are similar to J_1 in that their integrals with respect to s has a length of h, thus canceling the term h^{-1}. Also, their integrands can be estimated in a similar manner. As for the integral I_4, we use similar methods as in the estimates for integral I_1 combined with the usual estimates of (5.14).

Bibliography

[1] Kari Astala, Petteri Piiroinen ja Hans-Olav Tylli: *Funktionaalianalyysin peruskurssi, lecture notes, University of Helsinki, 2010.*

[2] Lawrence Evans: *Partial Differential Equations, Graduate Studies in Mathematics, Vol. 19, AMS, Providence, 1991.*

[3] Joseph Fourier: *Théorie analytique de la chaleur, 1822*

[4] Hiroshi Fujita: *On the blowing up of solutions of the Cauchy problem for* $u_t = \Delta u + u^{1+\alpha}$ In: J. Fac. Sci. Univ. Tokyo, sect. 1 13 (1966), pp. 109-124.

[5] Hiroshi Fujita: *On some nonexistence and nonuniqueness theorems for nonlinear parabolic equations* In: Proc. Symp. Pure Math., Vol. 18, Part I, Amer. Math. Soc., 1968, pp. 138-161.

[6] Mats Gyllenberg, Lasse Lamberg, Petri Ola ja Petteri Piiroinen: *DY I-II, lecture notes, University of Helsinki, 2014.*

[7] Ilkka Holopainen: *Reaalianalyysi I, Lecture notes, 2012.*

[8] Alessandra Lunardi: *Linear and nonlinear diffusion problems, Lecture notes, 2004*

[9] Eero Saksman: *Fourier analyysi, lecture notes, University of Helsinki, 2016.*

[10] Jari Taskinen: *Asymptotical behaviour of a class of semilinear diffusion equations* In: Journal of Evolution Equations 7 (2007), pp. 429-447

[11] Jussi Väisälä: *Topologia I, Helsinki, Limes ry, 2011.*

[12] Fred B. Weissler *Existence and non-existence of global solutions for a semilinear heat equation* In: Israel journal of mathematics, Vol. 38, Nos. 12, 1981 pp. 29-39