Workshop on Logics of Dependence and Independence (LoDE 2020V)

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Preface

Logics of dependence and independence are novel non-classical logics aiming at characterizing dependence and independence notions in philosophy and in social and physical sciences. This field of research has grown rapidly in recent years. This family of logics has found applications in fields like database theory, linguistics, social choice, quantum physics along with other fields. This workshop brings together researchers from all these relevant areas and provide a snapshot of the state of the art of logics of dependence and independence.

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Knowability as continuous dependence

Alexandru Baltag

An ***empirical variable*** is one whose exact value may never be known, and instead only inexact approximations can be observed. Examples are in natural sciences, economics etc (where the inexact observations are some form of measurements), but also in the semantics of questions in natural language (where the inexact observations are partial answers). This leads to a topological conception of empirical variables, as ***maps from the state space into a topological space***. Here, the exact value of the variable is represented by the output of the map, while the open neighborhoods of this value represent the knowable approximations of the exact answer.

A central tenet in empirical sciences is establishing ***functional correlations*** between variables, with a view towards (1) establishing causality, but also (2) predicting the (approximate) value of a hard-to-measure variable Y when given (approximate) value(s) of easier-to-measure variable X. In interrogative terms, this is related to inquisitive implication: every partial answer to question Y is entailed by some partial answer to X. In this talk, I argue that ***knowability*** of a dependency amounts to the ***continuity*** of the given functional correlation. I give a ***learning-theoretic*** justification of this claim, connecting with Kevin Kelly's notion of gradual learnability, then I give some concrete examples. Next, I present a ***complete and decidable axiomatization*** of the minimal logic of continuous dependence, and briefly sketch the ideas behind the proofs.

Further, I discuss the distinction between ***knowing the dependence*** between X and Y, and ***knowing-how*** to determine Y (with any desired accuracy) from X: the later is a stronger notion of knowability, that requires the ability to find the accuracy that is needed for X-measurements (to determine Y with the given accuracy). I formalize this distinction in terms of ***continuity versus uniform continuity***, and go on to propose an axiomatization of strongly known dependence, in the framework of ***uniformity spaces*** (-Andrè Weil's qualitative generalization of metric spaces).

Time-permitting, I may also briefly describe an alternative setting that seems better fit for computing applications, where inexact observations correspond to approximate computations. Knowability is then given by ***Scott continuity*** (of the dependence between variables taking values in Scott domains). I show how this setting fits within the topological framework, and how the corresponding complete axiomatization requires an additional axiom.

This talk is based on ongoing joint work with Johan van Benthem.

Dependency: the question-based view

Ivano Ciardelli

Most literature on the logic dependency construes dependency as a relation between variables. I discuss an alternative perspective, stemming from inquisitive logic, which construes dependency instead as a relation between questions.

I will point out some interesting consequences of the alternative perspective. For instance, even in a context with just two variables x and y, there are many questions one can ask about those variables, and thus many dependencies that can be recognized beyond the standard "value of x determines value of y". Another example: the fact that questions can be syntactically composed and decomposed allows them to be manipulated in inferences in a meaningful way; thus, we can, e.g., formally prove a dependency of Q' on Q in natural deduction by assuming Q and deriving Q'.

I will make the general ideas concrete in the context of an extension of first-order predicate logic with questions and dependencies. If time permits, I will survey some recent results by Gianluca Grilletti and myself on the expressive limitations of this logic. Finally, I will highlight some important open problems in the area.

Strongly first order dependencies in team semantics

Pietro Galliani

In this talk I will introduce and discuss the problem of finding which dependency conditions between variables may be added to the language of First Order Logic without increasing its expressive power. This may be seen as an attempt to explore the boundary between First Order Logic and Higher Order Logic “from below” as well as to study expressively weak logics based on Team Semantics.

Embedding causal team languages into dependence logic

Fausto Barbero and Pietro Galliani

Causal team semantics has been recently introduced ([1, 2]) as a semantic framework in which contingent dependencies (such as those that are studied in team semantics, see e.g. [7]) can be studied together with causal dependencies. The notions of causal dependence that are considered in this context are those that were defined in the field of *causal inference* ([6, 5, 4]) in terms of *interventionist counterfactuals.*

The syntax of the causal languages from [1, 2] borrows a few elements from the literature on causal inference. Hence, it is a natural question how these languages relate to earlier formalisms based on team semantics, and how the models on which they are interpreted (causal teams) relate to the usual teams. We will see that (under an assumption called *recursivity*, that will be specified below) there is a systematical way of translating the causal languages into fragments of logics based on team semantics (over first-order structures of an appropriate signature). In particular, the basic counterfactual languages $\mathcal{CO}(\sigma)$ translate into the existential fragment of first-order logic, while the languages $\mathcal{COD}(\sigma)$ and $\mathcal{CO}_{\sqcup}(\sigma)$ (enriched, respectively, with dependence atoms and inquisitive disjunction) both translate into the existential fragment of dependence logic. As an application of the embeddings, we show the decidability of the satisfiability problem these languages, over signatures with a finite number of variables.

Notation. We use capital letters $X, Y, Z...$ for variables, and small letters $x, y, z...$ for constants (called *values*). Boldface letters $\mathbf{X}$, resp. $\mathbf{x}$, denote finite sets or sequences of variables, resp. of values.

Formal definitions and results in the field of causal inference often need to be formulated in relation to a *signature*, which describes which variables are taken into consideration and over what sets their values are allowed to vary. More precisely, a **signature** σ is a pair (Dom, Ran), where Dom is a nonempty set of variables and Ran is a function that associates to each variable $X \in Dom$ a nonempty set $Ran(X)$ of values (the *range* of X). Note that we write $Ran(\mathbf{X}) := Ran(X_1) \times Ran(X_n)$ when $\mathbf{X} = \langle X_1, \dots, X_n \rangle$.

An **assignment** of signature σ will be a mapping $s : Dom \to \bigcup_{X \in Dom} Ran(X)$ such that $s(X) \in Ran(X)$ for each $X \in Dom$. A **team** T of signature σ will be any set of such assignments. A graph, in this note, will be a pair $(\mathbf{V}, E)$, where $\mathbf{V}$ is a set of variables and $E \subseteq \mathbf{V} \times \mathbf{V}$. Given such a graph G, we denote as PA_V^G the set of parents of V in G (i.e. the set of variables X such that $(X, V) \in E$). We may omit the superscript.

Causal teams. A causal team enriches a team by isolating a set of functions which describe the causal mechanisms that link the variables. A graph is used to keep track of the domains of these functions.

Definition 0.1. *A* ***causal team*** T *of signature* $\sigma = (Dom, Ran)$ *with endogenous variables* $End(T) \subseteq Dom$ *is a triple* $T = (T^-, G_T, \mathcal{F}_T)$*, with*

1. T^- *is a team of signature* σ *(*team component *of* T*).*
2. $G_T = (Dom, E)$ *is an irreflexive graph over* Dom *(*graph component *of* T*) such that* $(X, Y) \in E \implies Y \notin End(T)$*.*
3. $\mathcal{F}_T$ *is a function* $\{(V_i, f_{V_i}) \mid V_i \in End(T)\}$ *(*function component *of* T*) that assigns to each endogenous variable a function* $f_V : Ran(PA_V) \to Ran(V)$*.*

which satisfies: (*) for all $s \in T^-$*, and all* $Y \in \mathbf{V}$*,* $s(Y) = f_Y(s(PA_Y))$*.*

A causal team is said to be **recursive** if its graph is acyclic. For simplicity of exposition, we will always restrict attention to recursive causal teams.

Causal languages. We will consider the following languages (parametrized by a signature $\sigma = (Dom, Ran)$), as introduced in [1]:

- $\mathcal{CO}(\sigma) : Y = y \mid Y \neq y \mid \alpha \wedge \alpha \mid \alpha \vee \alpha \mid \alpha \supset \alpha \mid \mathbf{X} = \mathbf{x} \;\Box\!\!\to \alpha$
- $\mathcal{COD}(\sigma) : Y = y \mid Y \neq y \mid \;=\!(\mathbf{X}; Y) \mid \psi \wedge \psi \mid \psi \vee \psi \mid \alpha \supset \psi \mid \mathbf{X} = \mathbf{x} \;\Box\!\!\to \psi$
- $\mathcal{CO}_{\sqcup}(\sigma) : Y = y \mid Y \neq y \mid \psi \wedge \psi \mid \psi \vee \psi \mid \psi \sqcup \psi \mid \alpha \supset \psi \mid \mathbf{X} = \mathbf{x} \;\Box\!\!\to \psi$

where $\{\mathbf{X}, Y\} \subseteq Dom$, $y \in Ran(Y)$, $\mathbf{x} \in Ran(\mathbf{X})$, $\alpha \in \mathcal{CO}(\sigma)$. $\mathbf{X} = \mathbf{x}$ is an abbreviation for a (finite) conjunction $X_1 = x_1 \wedge \ldots X_n = x_n$.

We point out that $\mathcal{COD}(\sigma)$ can be seen as a generalization of *propositional dependence logic* over propositional letters $p_1, \ldots, p_n, \ldots$ provided Dom contains corresponding variables $P_1, \ldots, P_n, \ldots$ with $Ran(P_i) = \{0, 1\}$. Then, the formula p_i can be identified with $P_i = 1$, $\neg p_i$ with $P_i = 0$, and so on.

Interventions on recursive causal teams. The *interventionist counterfactual* operator $\Box\!\!\to$ is given a meaning in terms of *interventions on a causal team*. The causal team $T_{\mathbf{X}=\mathbf{x}}$ produced after the intervention $do(\mathbf{X} = \mathbf{x})$ on $T = (T^-, G_T, \mathcal{F}_T)$ describes what would happen if we subtracted the variables $\mathbf{X}$ to their current causal mechanisms (i.e. the corresponding functions from $\mathcal{F}_T$) and forced them to take the constant values $\mathbf{x}$. In such event, the values of variables that are descendants of $\mathbf{X}$ in G_T need to be recomputed using the functions from $\mathcal{F}_T$. We illustrate the idea with an example (see [1, 2] for the formal definition).

We consider a causal team T with $Dom = \{U, X, Y, W, Z\}$, function component $\mathcal{F}(X) = U, \mathcal{F}(Y) = X + 1, \mathcal{F}(W) = X + 2, \mathcal{F}(Z) = U + 2 * W$ and team and graph component as in the picture below; we apply to T the intervention $do(X = 1)$.

T:

U	X	Y	W	Z
1	1	2	3	7
2	2	3	4	10

$\rightsquigarrow$

U	X	Y	W	Z
1	**1**	...	...	...
2	**1**	...	...	...

$\rightsquigarrow$

$\rightsquigarrow$

U	X	Y	W	Z
1	1	**2**	**3**	...
2	1	**2**	**3**	...

$\rightsquigarrow$ $T_{X=1}$:

U	X	Y	W	Z
1	1	2	3	**7**
2	1	2	3	**8**

$\mathcal{F}_{T_{X=1}}$ is $\mathcal{F}_T$ restricted to $End(T_{X=1}) := End(T) \setminus \{X\}$; and $G_{T_{X=1}}$ is obtained by removing from G_T all arrows that point to X (in this case, the arrow (U, X)).

More generally, the intervention $do(\mathbf{X} = \mathbf{x})$ can be defined whenever $\mathbf{X} = \mathbf{x}$ is **consistent**, i.e. it does not contain two conjuncts $X = x$ and $X = x'$ with $x \neq x'$.

Semantic clauses. We say that $S = (S^-, G_S, \mathcal{F}_S)$ is a **causal subteam** of $T = (T^-, G_T, \mathcal{F}_T)$, $S \leq T$, if $S^- \subseteq T^-$, $G_S = G_T$ and $\mathcal{F}_S = \mathcal{F}_T$. We use the improper notation $\{s\}$ for the causal subteam of T of team component $\{s\}$ whenever T is clear from the context.

Satisfaction of a formula by a team, $T \models \varphi$, is defined inductively as:

- $T \models Y = y$ (resp. $Y \neq y$) iff, for all $s \in T^-$, $s(Y) = y$ (resp. $s(Y) \neq y$).
- $T \models =\!(\mathbf{X}; Y)$ iff for all $s, s' \in T^-$, $s(\mathbf{X}) = s'(\mathbf{X})$ implies $s(Y) = s'(Y)$.
- $T \models \psi \wedge \chi$ iff $T \models \psi$ and $T \models \chi$.
- $T \models \psi \vee \chi$ iff there are $T_1, T_2 \leq T$ s.t. $T_1^- \cup T_2^- = T^-$, $T_1 \models \psi$ and $T_2 \models \chi$.
- $T \models \psi \sqcup \chi$ iff $T \models \psi$ or $T \models \chi$.
- $T \models \mathbf{X} = \mathbf{x} \;\Box\!\!\rightarrow \chi$ iff $\mathbf{X} = \mathbf{x}$ is inconsistent or $T_{\mathbf{X}=\mathbf{x}} \models \chi$.
- $T \models \alpha \supset \chi$ iff $T^\alpha \models \chi$, where T^α is the (unique) causal subteam of T with team component $\{s \in T^- \mid \{s\} \models \alpha\}$.

The languages $\mathcal{CO}(\sigma)$ are *flat* (see [7]), while $\mathcal{COD}(\sigma)$ and $\mathcal{CO}_{\sqcup}(\sigma)$ are just *causally downward closed* (i.e., if S is a *causal* subteam of T and $\varphi \in \mathcal{COD}(\sigma) \cup \mathcal{CO}_{\sqcup}(\sigma)$, $T \models \varphi$ entails $S \models \varphi$). We remark that the laguages $\mathcal{CO}(\sigma)$ are closed under dual negation, defined inductively: $(X = x)^d := X \neq x$, $(X \neq x)^d := X = x$, $(\psi \wedge \chi)^d := \psi^d \vee \chi^d$, $(\psi \vee \chi)^d := (\psi^d \wedge \chi^d)$, $(\psi \supset \chi)^d := \psi \wedge \chi^d$, $(\psi \;\Box\!\!\rightarrow \chi)^d := \psi \;\Box\!\!\rightarrow \chi^d$.

Lemma 0.2 ([2])**.** *Let T be a causal team and $\varphi \in \mathcal{CO}$. Then $T \models \varphi^d$ iff, for all $s \in T^-$, $\{s\} \not\models \alpha$.*

The translation. We want to compare the above languages with first-order languages, possibly extended with dependence atoms (*dependence logic*, [7]) or with $\sqcup$. We assume the reader is familiar with these languages, and we only review the clause for $\exists$:

- $M, T \models \exists x\psi$ if and only if for some $F : T \to M$ we have $T[F/x] \models \psi$, where $T[F/x] := \{s(F(s)/x) \mid s \in T\}$.

Definition 0.3. *To each causal team $T = (T^-, G_T, \mathcal{F}_T)$ of signature $\sigma = (Dom, Ran)$ and endogenous variables $End(T)$, we associate a first-order structure $M_T = (|M_T|, (c^{M_T})_{c\in|M_T|}, (f_V^{M_T})_{V\in End(T)})$, defined as $M_T = (|M_T|, (c^{M_T})_{c\in|M_T|}, (f_V^{M_T})_{V\in End(T)})$ where $|M_T| = \bigcup_{V\in Dom} Ran(V)$ and $f_V^{M_T}(\mathbf{c}) = \begin{cases} \mathcal{F}_T(V)(\mathbf{c}) & \text{if } \mathbf{c} \in Ran(PA_V) \\ \text{some } d \in Ran(V) & \text{otherwise.} \end{cases}$.*

Notice that the team component of T is not involved in the definition of M_T; therefore, a causal subteam S of T will have the same associated model $M_S = M_T$. Notice also that, for any consistent $\mathbf{X} = \mathbf{x}$, $M_{T_{\mathbf{X}=\mathbf{x}}}$ is a reduct of M_T.

We will show that, for an appropriate translation tr of $\mathcal{COD} \cup \mathcal{CO}_{\sqcup}$ formulas into formulas of predicate logics, we have: $T \models \varphi \iff M_T, T^- \models tr(\varphi)$.
More generally, for every graph $G \subseteq G_T$ we define, recursively on the syntax of φ, a translation $tr(\varphi, G)$. The idea is that, for each intervention $do(\mathbf{X} = \mathbf{x})$, the formula $tr(\varphi, G_{T_{\mathbf{X}=\mathbf{x}}})$ will encode the fact that φ holds in the modified causal team $T_{\mathbf{X}=\mathbf{x}}$. For each $G \subseteq G_T$, we define a (first-order) formula

$$Eq(G) := \bigwedge_{V\in End(G)} V = f_V(PA_V).$$

where $End(G)$ is the set of vertices V of G of indegree > 0. By this definition, $Eq(G_T)$ asserts that the system of equations $V = \mathcal{F}(V)(PA_V)$ $(V \in End(T))$ associated to T holds; and $Eq(G_{T_{\mathbf{X}=\mathbf{x}}})$ will similarly describe the reduced system of equations that is obtained after applying the intervention $do(\mathbf{X} = \mathbf{x})$ to T.

The (relativized) translation is defined by induction on the syntax of φ:

- $tr(\eta, G) = \eta$ if η is $X = x$, $X \neq x$ or $=(\mathbf{X}; Y)$.
- $tr(\psi \circ \chi, G) := (tr(\psi, G) \circ tr(\chi, G))$ for $\circ = \wedge, \vee$ or $\sqcup$
- $tr(\mathbf{X} = \mathbf{x} \,\Box\!\!\rightarrow \psi, G) := \exists \mathbf{X} \exists \mathbf{D_X}(\mathbf{X} = \mathbf{x} \wedge Eq(G_{\mathbf{X}=\mathbf{x}}) \wedge tr(\psi, G_{\mathbf{X}=\mathbf{x}}))$
  (where $\mathbf{D_X}$ is the set of descendants of $\mathbf{X}$, listed in an appropriate order)
- $tr(\theta \supset \chi, G) := tr(\theta^d, G) \vee tr(\chi, G)$.

The embedding result.

Theorem 0.4. *Let $T = (T^-, G, \mathcal{F})$ be a recursive causal team. Then:*

$$T \models \varphi \iff M_T, T^- \models tr(\varphi, G).$$

Corollary 0.5. *a) $\mathcal{CO}$ embeds into the existential fragment of FO.*
b, c) $\mathcal{COD}$ and $\mathcal{CO}_{\sqcup}$ embed into the existential fragment of dependence logic.

Statements a) and b) immediately follow from theorem 0.4. Statement c) is obtained by first applying the translation and then eliminating $\sqcup$ using a well-known equivalence: $\psi \sqcup \chi \equiv \exists P \exists Q(=(P) \wedge =(Q) \wedge (P = Q \to \psi) \wedge (P \neq Q \to \chi))$ (where $P = Q \to \psi$ abbreviates $P \neq Q \vee \psi$ and $P \neq Q \to \chi$ abbreviates $P = Q \vee \chi$).

Corollary 0.6. *Let Dom be a finite set of causal variables, and let φ be any $\mathcal{COD}$ or $\mathcal{CO}_{\sqcup}$ formula over the variables of Dom. Then it is decidable whether φ is satisfiable in some causal team over Dom.*

Proof sketch. If φ is satisfied by some nonempty causal team T then, by Theorem 0.4 and downward closure, for every $s \in T^-$ it holds that $M_T, \{s\} \models Eq(G_T) \wedge tr(\varphi, G_T)$, and conversely if $M, \{s\} \models Eq(G_T) \wedge tr(\varphi, G_T)$ then the team obtained from T by replacing T^- with $\{s\}$ satisfies φ (and, by the definition of $Eq(G_T)$, $\{s\}$ is also compatible with the causal graph G_T). Therefore, $\exists \mathbf{Z}(Eq(G_T) \wedge tr(\varphi, G_T))$ is a sentence of existential dependence logic that is satisfiable if and only if there is a causal team with graph G_T that satisfies φ. Therefore, $\bigvee_{G_T} \exists \mathbf{Z}(Eq(G_T) \wedge tr(\varphi, G_T))$, where G_T ranges over all acyclic graphs with the variables in *Dom* as nodes, is an existential dependence logic sentence that is satisfiable if and only if φ is satisfiable by some causal team of domain *Dom*. Each such sentence is equivalent to the first-order sentence ("flattening") that is obtained by removing all dependence atoms. This procedure is clearly computable, therefore our problem is reduced to satisfiability in existential first-order logic, which is decidable (e.g., by the results in [3]). For $\mathcal{CO}_{\sqcup}(\sigma)$, remove all $\sqcup$ from $tr(\varphi, G_T)$ as was done in 0.5.

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Separation Logic and Logics with Team Semantics

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Separation logic is a successful logical system for formal reasoning about programs that mutate their data structures. It goes back to work by Reynolds, O'Hearn, Pym and others [18, 10, 14] and builds on Hoare logic [8], a system for proving specifications of form $\{\textit{precondition}\}\text{code}\{\textit{postcondition}\}$ about how a piece of code changes the properties of the states of a computation. Traditional Hoare logic works very well for programs with simple fixed data types, but reasoning about programs with mutable data structures becomes complicated and problematic, and this is the main issue that is addressed by separation logic. Actually, separation logic is part of a larger family of logics with *bunched implications* [13, 16, 15], but to get the point of this paper across, which is the connection with team semantics, we consider a stripped down presentation of separation logic, as used for instance in [6, 17], viewing separation logic as an extension of first-order logic for reasoning about *heaps* (modelled as partial functions $h\colon A \to A^k$) whose expressive power arises from two non-standard logical connectives: the *separating conjunction*, and the *magic wand*. With these new connectives one can write concise specifications of recursive data structures such as doubly linked lists, trees with linked leaves and parent pointers, and so on, and reason in the style of Hoare logic about the semantics of programs with such data structures [18].

More precisely, we define for any $k \geq 1$ the separation logic SL^k as the extension of first-order logic by two new atomic formulae **emp** and $x \mapsto \overline{y}$, and the new connectives $\star$ and $-\!\!*$. A formula $\psi(\overline{z}) \in \mathrm{SL}^k$ of vocabulary τ is interpreted over a triple $(\mathfrak{A}, \mathfrak{h}, s)$, consisting of a τ-structure $\mathfrak{A}$, a finite partial function $\mathfrak{h}\colon A \to A^k$ on the universe of $\mathfrak{A}$, called a heap, and an assignment $s\colon \mathrm{free}(\psi) \to A$ mapping the free variables of ψ to elements of $\mathfrak{A}$. The atom **emp** expresses that the heap is empty, and $x \mapsto \overline{y}$ is true in $(\mathfrak{A}, \mathfrak{h}, s)$ if the heap $\mathfrak{h}$ consist of the single item $s(x) \mapsto s(y_1), \ldots, s(y_k)$. Using the traditional first-order connectives and quantifiers, together with the separating conjunction and the magic wand, one then builds powerful statements describing dynamic transformations of data structures. The separating conjunction $\psi \star \varphi$ asserts that there is a disjoint split of the heap $\mathfrak{h}$ into two disjoint heaps satisfying ψ and φ, respectively, and the *separating implication* or *magic wand* $\psi -\!\!* \varphi$ states that φ is true for any extension $\mathfrak{h} \,\cup\, \mathfrak{h}'$ of the given heap $\mathfrak{h}$ by a heap $\mathfrak{h}'$ that satisfies ψ.

Since separation logic is an extension of first-order logic, the fundamental algorithmic problems such as (finite) satisfiability, validity, or entailment are, of course, undecidable. For both theoretical and practical purposes it is interesting to classify fragments of separation logic for which such problems become decidable, and to determine their complexity, and on the other side to identify those fragments that are expressively complete, and thus as difficult to handle as full separation logic. Such work has been done for instance in [3, 4, 5, 6].

Team semantics, on the other side, is the mathematical basis of modern logics for reasoning about dependence, independence, and imperfect information. It originates in the work of Wilfrid Hodges [9], and relies on the idea to evaluate logical formulae $\varphi(x_1, \ldots, x_n)$ not for single assignments $s\colon \{x_1, \ldots, x_n\} \to A$ from the free variables to elements of a structure $\mathfrak{A}$, but for *sets of such assignments*. These sets, which may have arbitrary size, are now called *teams*. Together with the fundamental idea of Väänänen [19] to treat dependencies not as annotations of quantifiers (as in IF-logic), but as atomic properties of teams, this has lead to a lively interdisciplinary research area, involving not just first-order logics, but also logics on the propositional and modal level, see e.g. [1]. Team semantics admits reasoning about large sets of data, modelled by second-order objects such as sets of assignments, with a first-order syntax that does not explicitly refer to higher-order variables. In the presence of appropriate atomic team properties, such as dependence, inclusion and exclusion, or independence, team semantics can boost the expressiveness of first-order formalisms to the full power of existential second-order logic or, in the presence of further propositional operators such as different variants of implication or negation, even to full second-order logic (SO). There are several reasons for this high expressive power of logics of dependence and independence. One of them is the second-order nature of atomic dependencies in teams. For instance, saying that z depends on y in the team X means that there exists a function which, for all assignments $s \in X$, maps $s(y)$ to $s(z)$. A further reason is that in the context of teams, disjunctions and existential quantification are really second-order operations. Note, however, that only the combination of dependence atoms and disjunction/existential quantification leads to the expressive power of (existential) SO. We write $\mathfrak{A} \models_X \varphi$ to denote that φ is true in the structure $\mathfrak{A}$ for the team X. In this extended abstract we assume that the reader is familiar with the basic definitions of team semantics, and results are given without proofs. Detailed definitions and complete proofs will be given in the full version of this paper.

Connections and differences. Separation logic and team semantics have been introduced with quite different motivations, and are investigated by research communities with rather different backgrounds and objectives. Nevertheless, there are obvious similarities between these formalisms. First of all, both separation logic and logics with team semantics involve the manipulation of second-order objects, such as heaps and teams, by first-order syntax without reference to second-order variables. Moreover, these semantical objects are closely related; it is for instance obvious that a heap, i.e. a partial function $\mathfrak{h}\colon A \to A^k$, can be seen as a team with variables $x, y_1, \dots, y_k$ satisfying the atom that $\overline{y}$ depends on x. Even more strikingly, the separating conjunction of separation logic is (essentially) the same as the team-semantical disjunction; moreover several notions of implications have been studied for team semantics, so it seems natural to interpret also the magic wand in this context. Based on such similarities, the possible connections between separation logic and team semantics have been raised as a question at several occasions, and lead to informal discussions between these research communities. The objective of this paper is to make this connection precise, and to study its potential but also its obstacles and limitations. We remark that the point of connecting separation logic with team semantics is not just expressive power. Actually, both separation logic and the logics with team semantics that we need here can readily be embedded into second-order logic (SO), and it is not difficult to see that they indeed essentially have the full power of SO. But going through second-order logic does not provide informative and compositional translations between these frameworks, and would thus produce only very limited insights. Rather we aim for a natural set of team-semantical operators that admit us to construct a faithful, complete and compositional representation of separation logic into a suitable logic with team semantics.

At least when we consider logics of dependence and independence in their standard format, there are also important differences to the framework of separation logic. This standard format is based on a collection of atomic dependencies on teams, typically dependence, inclusion, exclusion and/or independence, together with the usual first-order literals, and extends these by conjunction, disjunction, existential and universal quantifiers. In particular, these logics are not closed under negation, which is the first essential difference to separation logic. In fact, in logics of dependence and independence, negation is applied only to first-order atoms, not to dependencies or to compound formulae. Although one can define, for any formula φ, a kind of negation $\varphi^{\neg}$, its meaning is not the same as the classical negation and, in particular, the law of the excluded middle (*tertium non datur*) does not hold, not even for atomic formulae. A second relevant issue is the *empty team property* of these logics:

every formula whatsoever is satisfied by the empty team. This is a source of some technical difficulties in translations from separation logic to logics with team semantics, and excludes in particular the representation of the empty heap by the empty team.

Team logic for separation logic. To make translations from separation logic into team semantics possible, we consider the syntactic extension of separation logic by the dual connectives to the separating conjunction and the magic wand, the *separating disjunction* $\psi \circ \varphi$ and the *septraction* $\psi \multimap \varphi$, so that we can write all formulae of separation logic in negation normal form. This is a conservative extension that does not change the expressive power of the logic. We then discuss which of the ingredients of logics with team semantics are needed for achieving the expressive power of separation logic, and in particular, how the standard framework should be extended so that all of separation logic can be translated in a natural way. Of specific importance for the the translation that we propose are the *non-emptiness atom* NE, the uniform quantifiers $\exists^1$ and $\forall^1$, classical and dependent disjunctions and the intuitionistic implication. Although these operators are not part of what we call the standard framework of logics of dependence and independence, they have been studied quite thoroughly in team semantics, for instance in [2, 7, 12, 20].

Nonemptiness: $\mathfrak{A} \models_X \mathsf{NE}$ if $X \neq \varnothing$.

Finiteness: $\mathfrak{A} \models_X \mathrm{Fin}(\overline{x})$ if $X(\overline{x})$ is finite.

Equiextension: $\mathfrak{A} \models_X \overline{x} \bowtie \overline{y}$ if $X(\overline{x}) = X(\overline{y})$.

Classical disjunction: $\mathfrak{A} \models_X \varphi \sqcup \psi$ if $\mathfrak{A} \models_X \varphi$ or $\mathfrak{A} \models_X \psi$.

Uniform quantification: These are the usual quantifiers of FO, lifted to the team level:

$\mathfrak{A} \models_X \exists^1 x\varphi$ if $\mathfrak{A} \models_{X[x\mapsto\{a\}]} \varphi$ for some $a \in A$, and

$\mathfrak{A} \models_X \forall^1 x\varphi$ if $\mathfrak{A} \models_{X[x\mapsto\{a\}]} \varphi$ for all $a \in A$.

Dependent disjunction: $\mathfrak{A} \models_X \varphi \curlyvee_{\overline{x}} \psi$ if there is a disjoint decomposition $X = X_1 \,\dot\cup\, X_2$ satisfying $\mathfrak{A} \models_{X_1} \varphi$ and $\mathfrak{A} \models_{X_2} \psi$ such that for all $s, s' \in X$, if $s \in X_i$ and $s(\overline{x}) = s'(\overline{x})$ then also $s' \in X_i$.

Intuitionistic implication: $\mathfrak{A} \models_X \varphi \rightarrow \psi$ if, for all teams $Y \subseteq X$ with $\mathfrak{A} \models_Y \varphi$, also $\mathfrak{A} \models_Y \psi$.

We remark that in dependence logic, i.e. first-order logic with dependence atoms $\mathrm{dep}(\overline{x}; \overline{y})$ some of these connectives are expressible. The addition of NE, classical and dependent disjunctions, and the uniform quantifiers produces an expressively modest extension of dependence logic that remains inside the existential fragment of second-order logic. The further addition of the intuitionistic implication $\varphi \rightarrow \psi$ changes this, but it is needed for expressing the magic wand and the separating disjunction, which are universal second-order connectives. We note that the finiteness atom (which is necessary when we consider finite heaps that take values in an infinite structure) is easily expressible through dependence, equiextension, and intuitionistic implication, by means of Dedekind-finiteness. To summarize, the specific logic with team semantics that we are going to use for a compositional translation of separation logic is defined as follows.

▶ **Definition 1.** *Team logic for separation logic, abbreviated* TLfSL*, is the extension of dependence logic by* $\mathsf{NE}, \sqcup, \forall^1$*, and the intuitionistic implication* $\rightarrow$*. Note that* $\exists^1$*, dependent disjunction, equiextension, and the finiteness atoms are definable in it, and will also be used.*

From heaps to teams. We next discuss how the semantic objects for separation logic, i.e. triples $(\mathfrak{A}, \mathfrak{h}, s)$ consisting of a structure $\mathfrak{A}$, a heap $\mathfrak{h}$, and an assignment s, should be represented by the semantic objects in team semantics, i.e. pairs $(\mathfrak{B}, X)$ consisting of a structure $\mathfrak{B}$ and a team X. We will then want to provide translations, mapping any formula

$\varphi \in \mathrm{SL}$ (in negation normal form) to a formula $\varphi^* \in$ TLfSL such that, whenever $(\mathfrak{B}, X)$ represents $(\mathfrak{A}, \mathfrak{h}, s)$, we have that $\mathfrak{A}, \mathfrak{h} \models_s \varphi$ if, and only if, $\mathfrak{B} \models_X \varphi^*$.

We start with the natural idea to view a heap $\mathfrak{h}\colon A \rightharpoonup_{\mathrm{fin}} A^k$ as a team over the variables $x, y_1, \dots, y_k$, and to represent a triple $(\mathfrak{A}, \mathfrak{h}, s)$ by a pair $(\mathfrak{A}, Y_{\mathfrak{h},s})$, leaving the structure $\mathfrak{A}$ unchanged and expanding the team representing the heap by the values representing the assignment s, to obtain an expanded team $Y_{\mathfrak{h},s}$.

▶ **Definition 2.** *For a heap $\mathfrak{h}\colon A \rightharpoonup_{fin} A^k$ and an assignment $s\colon \{z_1, \dots, z_m\} \to A$, the team $Y_{\mathfrak{h},s}$ consists of all assignments $t\colon \{x, y_1, \dots y_k\} \cup \{z_1, \dots, z_m\} \to A$ such that $t(x) \in \mathrm{dom}(\mathfrak{h})$, $t(\overline{y}) = \mathfrak{h}(t(x))$ and $t(z_i) = s(z_i)$. Notice that the team $Y_{\mathfrak{h},s}$ fulfils the dependence atom $\mathrm{dep}(x; y_1, \dots, y_k)$ and the constancy atoms $\mathrm{dep}(z_1), \dots, \mathrm{dep}(z_m)$.*

Although $Y_{\mathfrak{h},s}$ is a natural representation of the pair $(\mathfrak{h}, s)$, this idea is too simple to work well. Any pair $(\mathfrak{h}, s)$ where the heap is empty is represented by the empty team, so all information about the assignment s is lost. Moreover, the standard logics of dependence and independence have the empty team property, and even logics as strong as team logic, which have classical negation (and hence do not have the empty team property) cannot express anything useful about the given structure in the presence of the empty team [11]. To take care of the problems arising with the empty team we add a dummy element δ to $\mathfrak{A}$ to obtain the structure $\mathfrak{A}^\delta$ with universe $A \ \cup \ \{\delta\}$ such that, for every relation symbol $R \in \tau$, we set $R^{\mathfrak{A}^\delta} := R^{\mathfrak{A}}$ and for any function symbol $f \in \tau$, we let $f^{\mathfrak{A}^\delta}$ coincide with $f^{\mathfrak{A}}$ on all tuples from $\mathfrak{A}$, and map all other tuples to δ. We extend the vocabulary by a new constant symbol δ interpreting the dummy element and add a dummy assignment to the team.

▶ **Definition 3.** *Given a structure $\mathfrak{A}$ and some element $\delta \notin A$, a triple $(\mathfrak{A}, \mathfrak{h}, s)$ is now represented by the pair $(\mathfrak{A}^\delta, X_{\mathfrak{h},s})$ where $X_{\mathfrak{h},s} := Y_{\mathfrak{h},s} \cup \{s^\delta\}$ with $s^\delta(x) = s^\delta(y_1) = \dots = s^\delta(y_k) = \delta$ and $s^\delta(z_i) = s(z_i)$ for $i = 1, \dots, m$. Note that for any assignment $s\colon \{z_1, \dots, z_m\} \to A$ we have that $X_{\varnothing,s} = \{s^\delta\}$.*

Based on the presentation of triples $(\mathfrak{A}, \mathfrak{h}, s)$ by $(\mathfrak{A}^\delta, X_{\mathfrak{h},s})$ it is not difficult to translate the first-order part of separation logic to the extension of dependence logic with the uniform quantifiers $\exists^1$ and $\forall^1$, the classical disjunction $\sqcup$, the non-empty split disjunction and the non-emptiness predicate NE.

Splitting and extending heaps and teams. The translation of the separating conjunction $\psi \star \varphi$ into team semantics requires that we are able to talk about splits of a heap $\mathfrak{h}$ on the level of teams $X^*_{\mathfrak{h},s}$. We do this by defining the formula

$$\mathrm{split}(x, c) := [(c = \delta \wedge \mathsf{NE}) \vee (c \neq \delta \wedge \mathrm{dep}(c) \wedge \mathsf{NE})] \wedge [(x = \delta \wedge (\mathsf{NE} \curlyvee_c \mathsf{NE})) \vee (x \neq \delta \wedge \mathrm{dep}(x; c))].$$

For any triple $(\mathfrak{A}, \mathfrak{h}, s)$ and any function $F\colon X_{\mathfrak{h},s} \to \mathcal{P}^+(A \cup \{\delta\})$, we have that $\mathfrak{A}^\delta \models_{X_{\mathfrak{h},s}[c \mapsto F]}$ $\mathrm{split}(x, c)$ if, and only if, there is an element $a \in A$ and a split $(\mathfrak{h}_1, \mathfrak{h}_2)$ of $\mathfrak{h}$ such that $X_{\mathfrak{h},s}[c \mapsto F] = X_{\mathfrak{h}_1,s}[c \mapsto \delta] \cup X_{\mathfrak{h}_2,s}[c \mapsto a]$. The separating conjunction is translated as

$$(\varphi \star \psi)^* := \exists c \Big(\mathrm{split}(x, c) \wedge \big((c = \delta \wedge \varphi^*) \vee (c \neq \delta \wedge \psi^*) \big) \Big).$$

We next discuss the translation of the septraction $\psi \multimap \varphi$. Recall that $\mathfrak{A}, \mathfrak{h} \models_s \psi \multimap \varphi$ if there exists a disjoint extension $\mathfrak{h} \ \cup \ \mathfrak{h}'$ with $\mathfrak{A}, \mathfrak{h}' \models_s \psi$ and $\mathfrak{A}, \mathfrak{h} \ \cup \ \mathfrak{h}' \models_s \varphi$. We have to represent extensions of the given heap $\mathfrak{h}$ by appropriate extensions of the encoding team

$X_{\mathfrak{h},s}$. We need a formula that says that a team Y, restricted to variables $(x, \overline{y})$, correctly encodes a heap. This is achieved by

$$\mathrm{heap}(x, \overline{y}) := \mathrm{dep}(x; \overline{y}) \wedge \mathrm{Fin}(x) \wedge \Big((x = \delta \wedge \overline{y} = \overline{\delta} \wedge \mathsf{NE}) \vee (x \neq \delta \wedge \bigwedge_{i=1}^{k} y_i \neq \delta)\Big).$$

For a given formula φ of separation logic, and its translation to φ^* to a formula with team semantics, let $\varphi^*[u, \overline{v}]$ be obtained from φ^* by renaming $x, \overline{y}$ to new variables $u, \overline{v}$ (whereas the variables $\overline{z}$ representing the assignment s are left unchanged). We now construct a formula to talk about disjoint extensions of the given heap by a heap that satisfies φ.

$$\varphi_{\mathrm{ext}}(x, \overline{y}, u, \overline{v}) := \Big((x = \delta \wedge \mathrm{heap}(u, \overline{v}) \wedge \varphi^*[u, \overline{v}]) \vee (x \neq \delta \wedge u = x \wedge \overline{v} = \overline{y})\Big) \wedge \mathrm{dep}(u; x)$$

The translation of $\varphi \multimap \psi$ now asserts that ψ is true in some disjoint extension of the given heap by a heap that satisfies φ:

$$(\varphi \multimap \psi)^* := \exists u \exists \overline{v} (\varphi_{\mathrm{ext}}(x, \overline{y}, u, \overline{v}) \wedge \psi^*[u, \overline{v}]).$$

Translating the separating disjunction and the magic wand. We finally discuss the translation of the two connectives that involve a universal quantification about heaps. It is clear that these are not definable in the existential fragment of second-order logic, and the natural way to go is to extend dependence logic by the intuitionistic implication $\psi \rightarrow \varphi$. But notice that this is a quite different kind of implication than the magic wand, and the translation is far from obvious. Recall that

$$\begin{aligned} \mathfrak{A}, \mathfrak{h} \models_s \varphi \mathrel{-\!\!*} \psi \iff & \text{ for every heap } \mathfrak{h}' \text{ with } \mathfrak{h} \# \mathfrak{h}' \text{ and } \mathfrak{A}, \mathfrak{h}' \models_s \psi, \text{also } \mathfrak{A}, (\mathfrak{h} \ \cup \ \mathfrak{h}') \models_s \psi \\ \mathfrak{A} \models_X \varphi \rightarrow \psi \iff & \text{ for every subteam } Y \subseteq X \text{ with } \mathfrak{A} \models_Y \varphi, \text{ also } \mathfrak{A} \models_Y \psi \end{aligned}$$

where $\mathfrak{h} \# \mathfrak{h}'$ means that $\mathfrak{h}$ and $\mathfrak{h}'$ are disjoint, i.e. $\mathrm{dom}(\mathfrak{h}) \cap \mathrm{dom}(\mathfrak{h}') = \varnothing$.

We start with the idea to use a universal variant of the translation of septraction, i.e. $\forall u \forall \overline{v} (\varphi_{\mathrm{ext}}(x, \overline{y}, u, \overline{v}) \rightarrow \psi^*[u, \overline{v}])$. Intuitively, in the evaluation of this formula over $X_{\mathfrak{h},s}$, the universal quantification generates a team Y that represents a maximal extension of the given heap $\mathfrak{h}$. The implication then says that *all subteams* of Y that represent an extension by a heap $\mathfrak{h}'$ that satisfies φ must also satisfy ψ. But this is not really correct, because left side of the implication can also be true for subteams that do not contain all the data present in $\mathfrak{h}$ i.e. represent an extension not of $\mathfrak{h}$, but of some subheap. We thus have to restrict the left side of the implication so that it talks only about those subteams that contain the full information of the team $X_{\mathfrak{h},s}$. To achieve this, we construct a formula in two variables x, x' that enforces a cyclic permutation of the values of x in the team. We set

$$\mathrm{cycle}(x, x') := x \bowtie x' \wedge \mathrm{dep}(x; x') \wedge \mathrm{dep}(x'; x) \wedge (x \bowtie x' \rightarrow ((x = \delta \wedge \mathsf{NE}) \vee x \neq \delta)).$$

A finite team Z with $\mathrm{dom}(Z) = \{x, x'\}$ is a model of $\mathrm{cycle}(x, x')$ if, and only if, there is a cyclic permutation $(a_0, \ldots, a_{m-1})$ of $Z(x) = Z(x')$ such that $Z = \{s_0, \ldots, s_{m-1}\}$ with $s_i(x) = a_i$ and $s_i(x') = a_{i+1 (\mathrm{mod}\ m)}$. As a consequence, if $Y \subseteq Z$ is a non-empty subteam of such a model with $\models_Y x \bowtie x'$ then $Y = Z$. We then translate the magic wand $\varphi \mathrel{-\!\!*} \psi$ by

$$(\varphi \mathrel{-\!\!*} \psi)^* := \exists x' \Big(\mathrm{cycle}(x, x') \wedge \forall u \forall \overline{v} \big((\mathsf{NE} \wedge x \bowtie x' \wedge \varphi_{\mathrm{ext}}(x, \overline{y}, u, \overline{v})) \rightarrow \psi^*[u, \overline{v}]\big)\Big).$$

For the translation of the separating disjunction $\varphi \circ \psi$, a similar idea is used, based on the formula $\mathrm{split}(x, c)$. We put

$$(\varphi \circ \psi)^* := \exists x' \Big(\mathrm{cycle}(x, x') \wedge \forall c \big((\mathsf{NE} \wedge x \bowtie x' \wedge \mathrm{split}(x, c)) \rightarrow \big(((c = \delta \wedge \varphi^*) \vee c \neq \delta) \sqcup ((c \neq \delta \wedge \psi^*) \vee c = \delta)\big)\Big).$$

We summarize our findings:

► **Theorem 4.** *There is a compositional translation that maps any formula* $\varphi \in \mathrm{SL}$ *into a formula* $\varphi^* \in \mathrm{TLfSL}$ *such that* $(\mathfrak{A}, \mathfrak{h}) \models_s \varphi \iff \mathfrak{A}^\delta \models_{X_{\mathfrak{h},s}} \varphi^*$*, for every triple* $\mathfrak{A}, \mathfrak{h}, s$*.*

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First-order inquisitive logics of finite width

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Inquisitive pair semantics (previously known as inquisitive semantics) is a formalism introduced by Groenendijk to study logical relations between statements and questions ([7]). Mascarenhas gave a complete axiomatization for the corresponding propositional logic `InqL` and showed that this can be regarded as an pseudo-intermediate logic ([8]). Later, Sano presented a cut-free Gentzen-style calculus for `InqL` ([9]).

Ciardelli and Roelofsen introduced (what is now called) inquisitive semantics by generalizing the inquisitive pair semantics of Groenendijk ([5]; see also [4] for the state-of-the-art in the topic). This semantics allows to capture the inquisitive content of some complex questions, not correctly represented by the previous system (see for example [1, Ch. 5]). In [5] the authors proved that `InqB`—the propositional logic corresponding to this generalized semantics—is also an extension of intuitionistic logic and gave a complete axiomatization for it. `InqBQ`, a first-order version of `InqB`, was later introduced in [2] and it is still an open problem whether this logic is axiomatizable.

The precise connection between `InqL` and `InqB` was first explored Ciardelli in [1, Ch. 6]. There the author introduced a $\supseteq$-chain of logics $\langle \mathtt{InqB}_n \rangle_{n\in\mathbb{N}}$, converging to `InqB`—that is, $\mathtt{InqB} = \bigcap_{n\in\mathbb{N}} \mathtt{InqB}_n$—among which also inquisitive pair logic appears: $\mathtt{InqL} = \mathtt{InqB}_2$. Moreover, building on the results of Mascarenhas, it was shown that these logics are all finitely axiomatizable.

Following the same approach, in [10] Sano defined a $\supseteq$-chain $\langle \mathtt{InqBQ}_n \rangle_{n\in\mathbb{N}}$ of first-order logics bounded by `InqBQ`, among which $\mathtt{InqBQ}_2$ can be regarded as a first-order version of `InqL`. In the same paper, Sano axiomatized $\mathtt{InqBQ}_2$ by adapting the canonical model completeness technique for first-order intuitionistic logic with constant domain CD [6, Sec. 7.2]. Two questions were left open in Sano's paper: whether the other elements of the chain are axiomatizable, and whether first-order inquisitive logic is the limit of this chain: we tackle exactly these two questions.

Firstly, we present the family of formulas $\{\mathrm{C}_n\}_{n\in\mathbb{N}}$, characterizing the classes of models of bounded finite size—that is, the classes of models defining the logics $\mathtt{InqBQ}_n$. Then we show that `InqBQ` *is not* the limit of this chain, by exhibiting a formula in $(\bigcap_{n\in\mathbb{N}} \mathtt{InqBQ}_n) \setminus \mathtt{InqBQ}$. Finally, we give an explicit strongly complete axiomatization for every logic $\mathtt{InqBQ}_n$, using the formula C_n. Interestingly, this completeness proof also relies on the canonical model proof for the logic CD, but it is completely based on a semantical analysis of this model, in contrast with the proof-theoretic approach of Sano and Mascarenhas.

Background

For brevity and to simplify the presentation, we will only work with the first-order signature $\Sigma = \{P\}$ (consisting of only a unary predicate symbol P) in a language without the equality symbol. All the results here presented can be easily adapted to finite signatures in the language with equality. The syntax of `InqBQ` is given by the following grammar:

$$\varphi \quad ::= \quad P(x) \mid \bot \mid \varphi \wedge \varphi \mid \varphi \mathbin{\vee\!\!\!\vee} \varphi \mid \varphi \to \varphi \mid \forall x.\varphi \mid \exists\!\!\!\exists x.\varphi$$

We also introduce a shorthand for negation, defined as $\neg\varphi :\equiv \varphi \to \bot$.
An *information model*—used to interpret formulas of `InqBQ`—is a tuple $\mathcal{M} = \langle\, M_w \mid w \in W \,\rangle$, where W is a set (the *worlds of* $\mathcal{M}$) and each M_w is a first-order structures over the same

	w	w'	w''
a	■	■	□
b	■	□	■

(a) An example of information model. The domain is $D = \{a, b\}$ and the set of worlds is $W = \{w, w', w''\}$. The extension of P is represented by the black boxes: for example $a \in P_{w'}$ and $a \notin P_{w''}$.

$$
\begin{array}{lll}
\mathcal{M}, s \vDash_g P(x) & \iff & \forall w \in s.\ g(x) \in P_w \\
\mathcal{M}, s \vDash_g \bot & \iff & s = \emptyset \\
\mathcal{M}, s \vDash_g \varphi \land \psi & \iff & \mathcal{M}, s \vDash_g \varphi \text{ and } \mathcal{M}, s \vDash_g \psi \\
\mathcal{M}, s \vDash_g \varphi \mathbin{\rlap{$\vee$}\,\vee} \psi & \iff & \mathcal{M}, s \vDash_g \varphi \text{ or } \mathcal{M}, s \vDash_g \psi \\
\mathcal{M}, s \vDash_g \varphi \to \psi & \iff & \forall t \subseteq s.\ [\mathcal{M}, t \vDash_g \varphi \Rightarrow \mathcal{M}, t \vDash_g \psi] \\
\mathcal{M}, s \vDash_g \forall x.\varphi & \iff & \forall d \in D^{\mathcal{M}}.\ \mathcal{M}, s \vDash_{g[x \mapsto d]} \varphi \\
\mathcal{M}, s \vDash_g \overline{\exists} x.\varphi & \iff & \exists d \in D^{\mathcal{M}}.\ \mathcal{M}, s \vDash_{g[x \mapsto d]} \varphi
\end{array}
$$

(b) The semantics of `InqBQ`. In the clauses, g is an assignment with values in D.

domain D (the *domain of* $\mathcal{M}$). Additionally we require the structures M_w to be *pairwise distinct*, that is, to have pairwise distinct interpretations of the symbol P; we will indicate with P_w the extension of P in M_w. An example of information model is depicted in Figure (a).

Formulas of `InqBQ` are evaluated over *pointed models*, that is, pairs $\langle \mathcal{M}, s \rangle$ consisting of an information model $\mathcal{M}$ and a set $s \subseteq W$—we call s an *information state of* $\mathcal{M}$. The semantics of `InqBQ` is presented in Figure (b). We define as usual the corresponding logical consequence relation: we indicate with $\Phi \vDash \psi$ that for every pointed model $\langle \mathcal{M}, s \rangle$, if $\mathcal{M}, s \vDash \varphi$ for every $\varphi \in \Phi$ then $\mathcal{M}, s \vDash \psi$.

A chain of first-order inquisitive logics

In [10], Sano defined a chain of inquisitive-like logics by restricting the semantics to the classes of pointed models $\{ \langle \mathcal{M}, s \rangle \mid \#s \leq n \}$. We introduce a generalization of this hierarchy.

Definition 1. Let λ be an arbitrary cardinal. We define $\mathtt{InqBQ}_{<\lambda}$ as the logic of the class of pointed models $\{ \langle \mathcal{M}, s \rangle \mid \#s < \lambda \}$, that is, the set of formulas valid in all the pointed models in $\{ \langle \mathcal{M}, s \rangle \mid \#s < \lambda \}$. Moreover, we define $\mathtt{InqBQ}_{\lambda}$ as a shorthand for $\mathtt{InqBQ}_{<\lambda^+}$ (where λ^+ indicates the cardinal successor of λ).

As previously mentioned, $\mathtt{InqBQ}_2$—that is, the logic of information states with at most two worlds—coincides with first-order inquisitive pair logic introduced and axiomatized by Sano. We will call the logics $\mathtt{InqBQ}_n$ for $n \in \mathbb{N}$ *inquisitive logics of finite width.* Clearly $\mathtt{InqBQ}_{<\lambda} \supseteq \mathtt{InqBQ}_{<\kappa}$ for every $\lambda < \kappa$; for λ and κ finite, we can prove that the containment is strict.[1]

Lemma 2. Consider the formulas recursively defined as follows:

$$
\mathrm{C}_1 := \forall x.(P(x) \mathbin{\rlap{$\vee$}\,\vee} \neg P(x)) \qquad\qquad \mathrm{C}_{n+1} := \overline{\exists} x. \bigvee_{i=1}^{n} \Big[(P(x) \to \mathrm{C}_i) \land (\neg P(x) \to \mathrm{C}_{n+1-i}) \Big]
$$

Then a pointed model $\langle \mathcal{M}, s \rangle$ satisfies C_n iff $\#s \leq n$.

Proof. We prove this result by strong induction on n. The case of C_1 is simple, since a pointed model $\langle \mathcal{M}, s \rangle$ satisfies the formula iff every world of s agrees on the extension of P; since the structures in $\mathcal{M}$ are required to be pairwise distinct, the latter condition is equivalent to $\#s \leq 1$.

As for the inductive step, fix a pointed model $\langle \mathcal{M}, s \rangle$. Firstly, suppose the model satisfies C_n. Then there exists an element d and a value $k \in [1, n-1]$ such that $\mathcal{M}, s \vDash P(d) \to C_k$ and $\mathcal{M}, s \vDash \neg P(d) \to C_{n-k}$. By the inductive hypothesis, the statements are equivalent to the two conditions $s^+ := \#\{ w \in s \mid d \in P_w \} \leq k$ and $s^- := \#\{ w \in s \mid d \notin P_w \} \leq n - k$; and since $s = s^+ \cup s^-$, it follows that $\#s \leq n$.

Secondly, suppose $\#s \leq n$. If $\#s = 1$, then the statement is easy to verify; so suppose that $\#s > 1$, that is, that there exists an element d and two worlds $w, w' \in s$ such that

[1] This result also follows from the corresponding result for the propositional case ([1, Proposition 4.1.8]).

$d \in P_w \setminus P_{w'}$. Defining s^+ and s^- as above, it follows that $w \in s^+ \setminus s^-$ and $w' \in s^- \setminus s^+$; and since $n = \#s = \#s^+ + \#s^-$, we have that $\#s^+, \#s^- \in [1, n-1]$. By defining $k := \#s^+$ and by inductive hypothesis, we obtain $\mathcal{M}, s \vDash P(d) \to \mathrm{C}_k$ and $\mathcal{M}, s \vDash \neg P(d) \to \mathrm{C}_{n+1-k}$, from which $\mathcal{M}, s \vDash \mathrm{C}_{n+1}$ easily follows. □

Corollary 3. $\mathrm{C}_n \in \mathtt{InqBQ}_n \setminus \mathtt{InqBQ}_{n+1}$.

An interesting property of the propositional inquisitive hierarchy is that $\mathtt{InqB} = \bigcap_n \mathtt{InqB}_n$, that is, $\mathtt{InqB}$ is the *limit* of the chain $\langle\, \mathtt{InqB}_n \mid n \in \mathbb{N} \,\rangle$ ([1, Corollary 4.1.6]). This is not the case for the first-order version of the hierarchy.

Proposition 4. $\bigcap_{n\in\mathbb{N}} \mathtt{InqBQ}_n = \mathtt{InqBQ}_{<\aleph_0} \supsetneq \mathtt{InqBQ}$.

Proof. The equality on the left and the containment on the right follow by definition of $\mathtt{InqBQ}_n$ and $\mathtt{InqBQ}_{<\aleph_0}$. To prove that the containment is strict, we introduce the following formula:

$$\mathrm{Pc} := \forall x, y. [\, (P(x) \to P(y)) \mathbin{\vee\!\!\!\vee} (P(y) \to P(x))\,]\,.$$

To analyze Pc, we need to introduce a special kind of information models: We call a pointed model $\langle \mathcal{M}, s \rangle$ a *P-chain* if the relation $\preceq$ defined as $w \preceq w'$ iff $P_w \subseteq P_{w'}$ is a total order on s. We want to show that the formula Pc is satisfied by a pointed model $\langle \mathcal{M}, s \rangle$ iff $\langle \mathcal{M}, s \rangle$ is a P-chain. Firstly, suppose that $\mathcal{M}, s \nvDash \mathrm{Pc}$. Let a, b be elements such that $\mathcal{M}, s \nvDash P(a) \to P(b)$ and $\mathcal{M}, s \nvDash P(b) \to P(a)$. Both formulas are $\mathbin{\vee\!\!\!\vee}$-free and $\overline{\exists}$-free and so they are *flat*.[2] Thus there exist two worlds w, w' such that

$$\mathcal{M}, \{w\} \vDash P(a) \quad \mathcal{M}, \{w\} \nvDash P(b) \qquad\qquad \mathcal{M}, \{w'\} \nvDash P(a) \quad \mathcal{M}, \{w'\} \vDash P(b) \tag{1}$$

In particular, w and w' are incomparable under $\preceq$, and so $\preceq$ is not a total order.

Secondly, suppose that $\langle \mathcal{M}, s \rangle$ is not a P-chain. So there exist two incomparable worlds w, w' under $\preceq$; which in turns means there exist two elements a, b for which the relations in 1 hold. But from this it follows immediately that $\mathcal{M}, s \nvDash \mathrm{Pc}$, as wanted.

Given this property, the reader can easily verify that the formula $\psi := \mathrm{Pc} \to \overline{\exists} x.[P(x) \to C_1]$ is valid over models with finitely many worlds; but that there exist infinite P-chains that do not validate ψ. □

So this shows that $\mathtt{InqBQ}$ is not the limit of the inquisitive logics of finite width, unlike in the propositional case. However, it is still true that $\mathtt{InqBQ} = \mathtt{InqBQ}_{<\lambda}$ for some cardinal $\lambda > \aleph_0$— we just need to take $\lambda := \bigcup_{\varphi \notin \mathtt{InqBQ}} \min\{\, \#s \mid \mathcal{M}, s \nvDash \varphi \,\}$. This leads to the following question, which is currently open.

Open question 5. Is $\mathtt{InqBQ}$ equal to $\mathtt{InqBQ}_{\aleph_0}$, the logic of countable information states?

To recap: indicating with $\mathtt{CQC}$ classical first-order logic, for a certain uncountable λ we have

$$\mathtt{CQC} = \mathtt{InqBQ}_1 \supsetneq \mathtt{InqBQ}_2 \supsetneq \mathtt{InqBQ}_3 \ldots \supsetneq \mathtt{InqBQ}_{<\aleph_0} \supsetneq \mathtt{InqBQ}_{<\lambda} = \mathtt{InqBQ}$$

Axiomatizing inquisitive logics of finite-width

We focus now on the inquisitive logics of finite-width. As noticed in Lemma 2, C_n characterizes exactly the models with at most n worlds. This suggests the axiomatization in Figure (c) for the logic $\mathtt{InqBQ}_n$ (compare with the axiomatization for $\mathtt{InqBQ}$ proposed in [3, Ch. 4]). Our aim is to prove that this axiomatization is strongly complete, and to do so we will treat $\mathtt{InqBQ}_n$ as a first-order intuitionistic theory. The result that allows us to do so is the following:

Proposition 6 (Immediate Corollary of Proposition 6.6.11 of [1])**.** Consider a first-order intuitionistic Kripke model $\mathcal{N}$ (1) with constant domain, (2) based on a frame of the form

[2] Recall that a formula φ is called flat if $\mathcal{M}, s \vDash \varphi$ iff $\forall w \in s.\ \mathcal{M}, \{w\} \vDash \varphi$. See [3, Proposition 4.1.9] for a proof that $\mathbin{\vee\!\!\!\vee}$-free and $\overline{\exists}$-free formulas are flat.

Axioms of `IQC` (intuitionistic first-order logic)	CD **schema:** $\forall x.(\varphi \mathbin{\vee\!\!\!\vee} \psi) \to (\forall x.\varphi \mathbin{\vee\!\!\!\vee} \psi)$ for x not free in ψ
Modus Ponens Rule: $\varphi, \varphi \to \psi / \psi$	KP **schema:** $(\neg\varphi \to \psi \mathbin{\vee\!\!\!\vee} \chi) \to (\neg\varphi \to \psi) \mathbin{\vee\!\!\!\vee} (\neg\varphi \to \chi)$
UP **schema:** $(\neg\varphi \to \overline{\exists} x.\psi) \to \overline{\exists} x.(\neg\varphi \to \psi)$ for x not free in φ	DNE **axioms:** $\neg\neg\alpha \to \alpha$ for α $\mathbin{\vee\!\!\!\vee}$-free and $\overline{\exists}$-free
	C_n **formula**

(c) Hilbert-style axiomatization for $\texttt{InqBQ}_n$.

$\langle \mathcal{P}(W) \setminus \{\emptyset\}, \supseteq\rangle$ (for W an arbitrary set) and (3) satisfying the DNE axioms. There exists an information model $\mathcal{M}$ with set of worlds W such that $\mathcal{M} \vDash \varphi$ iff $\mathcal{N} \Vdash \varphi$.[3]

The strategy of the proof consists in studying $\texttt{InqBQ}_n$ as a theory in $\mathcal{N}^c_A$, the canonical model of the superintuitionistic logic $\mathrm{CD}+\mathrm{KP}+\mathrm{UP}$ over a countable domain A ([6, Sec. 7.2]). Recall that $\mathcal{N}^c_A$ is the constant-domain intuitionistic model, whose domain is A; whose points are the saturated theories in the signature $\Sigma[A]$ (that is, Σ extended with a constant symbol $\overline{a}$ for every element $a \in A$) ordered by inclusion; and in which the extension of P is defined by the clause $[\![P(a)]\!] = \{\Gamma | P(\overline{a}) \in \Gamma\}$. The defining property of the canonical model is that, for every sentence in the signature $\Sigma[A]$, $\mathcal{N}^c_A, \Gamma \vDash \varphi$ iff $\varphi \in \Gamma$.

Given a theory Γ in $\mathcal{N}^c_A$, we will indicate with E_Γ the set of endpoints of $\mathcal{N}^c_A$ that are successors of Γ. We can show that the rooted submodel of $\mathcal{N}^c_A$ having as root a theory $\Gamma \supseteq \texttt{InqBQ}_n$ has the properties (1-3) listed in Proposition 6, and so Γ is the theory of an information model with at most n worlds. Property (1) follows from the definition of canonical model and Property (3) follows from $\texttt{InqBQ}_n \subseteq \Gamma$, so the non-trivial part is showing that Property (2) holds. The following are the main technical results needed for the proof.

Proposition 7. Let Γ be a theory in $\mathcal{N}^c_A$ extending

$$\mathrm{DNE}_A := \{\, \neg\neg\alpha[\overline{a}/\overline{x}] \to \alpha[\overline{a}/\overline{x}] \mid \overline{a} \subseteq A \text{ and } \mathbin{\vee\!\!\!\vee} \text{ and } \overline{\exists} \text{ not appearing in } \alpha \,\}$$

Then Γ has at least one successor which is an endpoint of $\mathcal{N}^c_A$.

Lemma 8. Let Γ be a theory in $\mathcal{N}^c_A$ extending DNE_A such that $\mathcal{N}^c_A, \Gamma \Vdash \mathrm{C}_n$. Then $\#E_\Gamma \leq n$.

Lemma 9. Let Γ as in Lemma 8 and let E be a non-empty set of endpoints above Γ. Then there exists a point Θ successor of Γ such that $E_\Theta = E$.

As a proof of concept, we detail the proof of Lemma 8.

Proof of Lemma 8. The proof proceeds by strong induction over n. For $n = 1$, we have $\mathcal{N}^c_A, \Gamma \Vdash \forall x.(P(x) \mathbin{\vee\!\!\!\vee} \neg P(x))$, that is, for every element $a \in A$ in the domain of the model we have $\mathcal{N}^c_A, \Gamma \Vdash P(a)$ or $\mathcal{N}^c_A, \Gamma \Vdash \neg P(a)$. This is possible iff Γ is itself an endpoint of the canonical model, from which the thesis follows. As for the inductive step, suppose $\mathcal{N}^c_A, \Gamma \Vdash \mathrm{C}_n$. We are going to find formulas $\alpha_1, \ldots, \alpha_n$ such that (1) every endpoint of $\mathcal{N}^c_A$ satisfies exactly one among them and (2) $\mathcal{N}^c_A, \Gamma \Vdash \alpha_i \to \mathrm{C}_1$ for every $i \in [1, n]$. From these two properties, the inductive step follows easily. Let us start by unpacking the condition $\mathcal{N}^c_A, \Gamma \Vdash \mathrm{C}_n$:

$$\begin{array}{rl} & \mathcal{N}^c_A, \Gamma \Vdash \overline{\exists} x. \mathbin{\vee\!\!\!\vee}_{i=1}^{n-1} [(P(x) \to C_i) \wedge (\neg P(x) \to C_{n-i})] \\ \text{(for some } a_0 \in A,\ k_0 \leq n-1) \implies & \mathcal{N}^c_A, \Gamma \Vdash P(a_0) \to C_{k_0} \text{ and } \mathcal{M}^c_A, \Gamma \Vdash \neg P(a_0) \to C_{n-k_0} \end{array}$$

Notice that every endpoint of $\mathcal{N}^c_A$ satisfies exactly one among the formulas $P(a_0)$ and $\neg P(a_0)$. If $k_0 = 1$ we define $\alpha_1 := P(a_0)$; otherwise we proceed unpacking the formula C_{k_0}:

$$\begin{array}{rl} & \mathcal{N}^c_A, \Gamma \Vdash P(a_0) \to \overline{\exists} x. \mathbin{\vee\!\!\!\vee}_{i=1}^{k_0-1} [(P(x) \to C_i) \wedge (\neg P(x) \to C_{k_0-i})] \\ \text{(by DNE, UP and KP)} \implies & \mathcal{N}^c_A, \Gamma \Vdash \overline{\exists} x. \mathbin{\vee\!\!\!\vee}_{i=1}^{k_0-1} [(P(a_0) \wedge P(x) \to C_i) \wedge (P(a_0) \wedge \neg P(x) \to C_{k_0-i})] \\ \text{(for some } a_{00} \in A,\ k_{00} \leq k_0) \implies & \begin{cases} \mathcal{N}^c_A, \Gamma \Vdash P(a_0) \wedge P(a_{00}) \to C_{k_{00}} \\ \mathcal{N}^c_A, \Gamma \Vdash P(a_0) \wedge \neg P(a_{00}) \to C_{k_0-k_{00}} \end{cases} \end{array}$$

[3] $\Vdash$ indicates the usual intuitionistic forcing relation. $\mathbin{\vee\!\!\!\vee}$ and $\overline{\exists}$ are treated as intuitionistic disjunction and existential quantifier under $\Vdash$.

Similarly, if $n - k_0 = 1$ we define $\alpha_n := \neg P(a_0)$; otherwise we can unpack C_{n-k_0} and find $a_{01} \in A$ and $k_{01} \in [1, n-k_0]$ such that

$$\mathcal{N}_A^c, \Gamma \Vdash \neg P(a_0) \wedge P(a_{01}) \to C_{k_{01}} \quad \text{and} \quad \mathcal{N}_A^c, \Gamma \Vdash \neg P(a_0) \wedge \neg P(a_{01}) \to C_{(n-k_0)-k_{01}}$$

Notice that every endpoint of $\mathcal{N}_A^c$ satisfies exactly one among the formulas $P(a_0) \wedge P(a_{00})$, $P(a_0) \wedge \neg P(a_{00})$, $\neg P(a_0) \wedge P(a_{01})$ and $\neg P(a_0) \wedge \neg P(a_{01})$. By proceeding recursively with this "unpacking" procedure we can find a sequence of formulas satisfying properties (1) and (2).[4] □

The previous results entail that the subframe generated by Γ is isomorphic to $\langle \mathcal{P}(E_\Gamma) \setminus \{\emptyset\}, \supseteq \rangle$ and so, by Proposition 6, Γ is the theory of an information model with at most n worlds. This can be refined in the following proposition, which in turn allows to prove the completeness result.

Proposition 10. Given a finite set E of endpoints of $\mathcal{N}_A^c$, there exists a unique theory Γ extending DNE_A and such that $E_\Gamma = E$. Moreover, this is the theory of an inquisitive model with $\#E$ worlds.

Theorem 11. The axiomatization proposed is strongly complete for $\mathtt{InqBQ}_n$.

Proof. We will indicate with $\vdash_n$ the consequence relation for the system proposed in Figure (c), and with $\vdash_L$ the consequence relation for the superintuitionistic logic $L := \mathrm{CD} + \mathrm{KP} + \mathrm{UP}$.

Let $\Phi \cup \{\psi\}$ be a set of formulas and suppose that $\Phi \nvdash_n \psi$. By definition of $\vdash_n$, this is equivalent to $\Phi \cup \mathrm{DNE} \cup \mathrm{C}_n \nvdash_L \psi$. By [6, Lemma 7.2.3], this means that there exists a countable set of parameters A and a saturated theory Γ in $\mathcal{N}_A^c$ such that $\Phi \cup \mathrm{DNE} \cup \mathrm{C}_n \subseteq \Gamma$ and $\psi \notin \Gamma$. In particular, by Lemma 8 $\#E_\Gamma \leq n$; and by Proposition 10 this means that Γ is the theory of an information model with at most n worlds, meaning this model satisfies all the formulas in Φ and does not satisfies ψ. This amounts to the strong completeness of the system for $\mathtt{InqBQ}_n$. □

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[4] For example α_1 has the form $P(a_0) \wedge P(a_{00}) \wedge \cdots \wedge P(a_{0\ldots0})$ for $a_0, a_{00} \ldots$ the elements found at different stages of the proof.

Enumerating Teams in First-Order Team Logics

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1 Introduction

Decision problems in general ask for the existence of a solution to some problem instance. In contrast, for *enumeration problems* we aim at generating all solutions. For many—or maybe most—real-world tasks, enumeration is therefore more natural or practical to study; we only have to think of the domain of databases where the user is interested in all answer tuples to a database query. Other application areas include web search engines, data mining, web mining, bioinformatics and computational linguistics. From a theoretical point of view, maybe the most important problem is that of enumerating all satisfying assignments of a given propositional formula.

Clearly, even simple enumeration problems may produce a big output. The number of satisfying assignments of a formula can be exponential in the length of the formula. In [9], different notions of efficiency for enumeration problems were first proposed, the most important probably being DelP ("polynomial delay"), consisting of those enumeration problems where, for a given instance x, the time between outputting any two consecutive solutions as well as pre- and postcomputation times (see [13]) are polynomially bounded in $|x|$. Another notion of tractability is captured by the class IncP where the delay and post-computation time can also depend on the number of solutions that were already output. The separation DelP $\subsetneq$ IncP was mentioned in [14], although one should note that slighlty different definitions were used there. Several examples of membership results for tractable classes can be found in [11, 10, 4, 2, 1, 5]. As a notion of higher complexity, recently an analogue of the polynomial hierarchy for enumeration problems has been introduced [3]. For this Del$C$ (for a decision complexity class C) was defined as the class of enumeration problems that can be enumerated by a machine M with access to an oracle $L \in C$ with polynomial delay and polynomially bounded oracles queries. For proving hardness a new reducibility notion was introduced. The enumeration problem E_1 reduces to enumeration problem E_2 via D-reductions ($E_1 \leq_{\mathrm{D}} E_2$), if there is a machine that enumerates E_1 that has access to an E_2-oracle with polynomial delay. Lower bounds for enumeration problems are obtained by proving hardness under D-reductions in a level Σ_k^p of that hierarchy for some $k \geq 1$ and are regarded as evidence for intractability.

Here, we consider enumeration tasks for first-order-team-based logics with the inclusion ($\subseteq$), the dependence ($=\!(\dots)$) and the independence ($\perp$) atom with lax semantics. We assume the reader to be familiar with team-based logics.

For a fixed first-order formula and a given input structure, the complexity of the problem of counting all satisfying teams has been studied by Haak et al. [8], where completeness for classes such as $\# \cdot \mathrm{P}$ and $\# \cdot \mathrm{NP}$ was obtained. In the enumeration context, and in analogy to the case of classical propositional logic as above, it is now natural to ask for algorithms to enumerate all satisfying teams of a fixed formula in a given input structure. Enumerating teams for formulas with the above mentioned dependency atom thus means enumerating all sets of tuples in a relational database that fulfil the given Boolean combination of FO-statements and functional dependencies. In this paper, we consider this problem and initiate the study of enumeration complexity for team based logics. Notice that, the task of enumerating teams has been considered before in the propositional setting by Meier and Reinbold [12].

We study the problems of enumerating all satisfying teams ($\text{E-}\textsc{Sat}_\varphi^{\text{team}}$) or certain optimal satisfying teams, where optimal can mean maximal ($\text{E-}\textsc{MaxSat}_\varphi^{\text{team}}$) or minimal ($\text{E-}\textsc{MinSat}_\varphi^{\text{team}}$) with respect to inclusion or maximum ($\text{E-C}\textsc{MaxSat}_\varphi^{\text{team}}$) or minimum ($\text{E-C}\textsc{MinSat}_\varphi^{\text{team}}$) with respect to cardinality.

Problem:	$\text{E-}\textsc{Sat}_\varphi^{\text{team}}$, for $\varphi \in \mathrm{FO}(A)$, for $A \subseteq \{=(\dots), \subseteq, \perp\}$
Input:	Structure $\mathcal{A}$
Output:	$\{X \mid \mathcal{A} \models_X \varphi \wedge X \neq \emptyset\}$

Our results are summarised in Table 1 on p. 5. It is known that in terms of expressive power dependence logic corresponds to the class NP. Hence, one cannot expect efficient algorithms for enumerating teams, and in fact, we prove that the problem is DelNP-complete (i.e., $\mathrm{Del}\Sigma_1^p$-complete) in all variants (enumerating all or optimal satisfying teams). Analogous results hold for independence logic. Inclusion logic, however, in a model-theoretic sense is equal to the class P (at least in lax semantics) [7]). Consequently, inclusion logic is less expressive than dependence logic over sentences (under the assumption $\mathrm{P} \neq \mathrm{NP}$). While this is not the case over open formulas, the picture in the enumeration context mostly reflects the situation over sentences: We prove that for each inclusion logic formula, there is a polynomial-delay algorithm for enumerating all satisfying teams in a given structure. This is also true when we want to enumerate all maximal, minimal, or maximum satisfying teams. Interestingly, enumerating minimum satisfying teams is DelNP-complete, as for the other logics we consider.

2 Results

We start with our results for the class DelP. All those results are for inclusion logic and rely on the fact that $\textsc{MaxSubTeam}$—the problem to compute the maximal subteam of a given team satisfying a given inclusion logic formula in a given structure—is computable in polynomial time [7]. Note that in inclusion logic there is a unique maximum satisfying team due to union closure.

Problem:	$\textsc{MaxSubTeam}$
Input:	Structure $\mathcal{A}$, formula $\varphi \in \mathrm{FO}(\subseteq)$, team X
Output:	X' with $\mathcal{A} \models_{X'} \varphi \wedge X' \subseteq X \wedge \forall X'' \subseteq X : \lvert X'' \rvert > \lvert X' \rvert \Rightarrow \mathcal{A} \not\models_{X''} \varphi$

► **Theorem 1.** *Let* $\varphi \in \mathrm{FO}(\subseteq)$. *Then* $\text{E-}\textsc{Sat}_\varphi^{\text{team}}, \text{E-}\textsc{MinSat}_\varphi^{\text{team}}, \text{E-}\textsc{MaxSat}_\varphi^{\text{team}}$, $\text{E-C}\textsc{MaxSat}_\varphi^{\text{team}} \in \mathrm{DelP}$.

Proof. We will show this result for $\text{E-}\textsc{Sat}_\varphi^{\text{team}}$. The proofs for all other cases can be found in the full version of the paper. We construct a recursive algorithm with access to a

MaxSubTeam oracle that on input $(\mathcal{A}, X, Y)$ enumerates all satisfying subteams $X' \neq \emptyset$ of X with $Y \subseteq X'$. To compute for a given $\mathcal{A}$ all satisfying subteams, we then need to run this algorithm on input $(\mathcal{A}, \mathrm{dom}(\mathcal{A})^{|\mathrm{free}(\varphi)|}, \emptyset)$.

Algorithm 1: Algorithm used to show E-Sat$^{\mathrm{team}}_{\varphi}$ $\in$ DelP for $\varphi \in \mathrm{FO}(\subseteq)$

```
Function EnumerateSubteams(structure A, teams X, Y)
    X = MaxSubTeam(A, X)
    if X ≠ ∅ ∧ Y ⊆ X then
        output X
        for s ∈ X do
            Y = {s' | s' < s ∧ s' ∈ X}
            EnumerateSubteams(A, X \ {s}, Y)
```

The algorithm does not output any solution more than once. In the recursive calls, it only outputs solutions where at least one assignment is omitted from the maximal solution, which is the only solution output before. Also, when the assignment s is chosen in the for-loop, the next recursive call only outputs solutions that omit s, but contain all assignments $s' < s$ that were present in X. In contrast, in every solution found in previous recursive calls, at least one of the assignments $s' < s$ from X was omitted. On the other hand, the algorithm outputs every solution at least once. Every solution is a subset of the maximal satisfying subteam of X and the algorithm starts with that maximal solution and then recursively looks for all strict subsets of it. This can be seen by noticing that when choosing the assignment s in the for-loop, the next recursive call outputs all satisfying subteams of X that exclude s, except for those that also exclude some $s' < s$ from X and were hence output before. ◀

Next we show that for certain formulas the problem E-Sat$^{\mathrm{team}}_{\varphi}$ captures the class DelNP. Moreover, we will extend this result to all remaining cases, that is, all combinations of logics and problems we did not classify already.

▶ **Theorem 2.** *Let $A \subseteq \{=(\dots), \perp\}$. There exists a formula $\varphi \in \mathrm{FO}(A)$ such that the problem* Sat$^{\mathrm{team}}_{\varphi}$ *is* NP*-hard. Hence, the problems* E-Sat$^{\mathrm{team}}_{\varphi}$, E-MaxSat$^{\mathrm{team}}_{\varphi}$, E-CMaxSat$^{\mathrm{team}}_{\varphi}$, E-MinSat$^{\mathrm{team}}_{\varphi}$, E-CMinSat$^{\mathrm{team}}_{\varphi}$ *are* DelNP*-hard.*

For the next result, we need the model checking problem for first-order team logic formulas in the setting of data complexity (fixed formula) defined as follows: The problem VerifyTeam$_{\varphi}$ is the problem to decide, given a structure $\mathcal{A}$ and team X, whether X is non-empty and $\mathcal{A} \models_X \varphi$. This problem is contained in NP for the dependence, inclusion and independence logic.

▶ **Lemma 3.** *Let $A \subseteq \{\perp, \subseteq, =()\}$, $\varphi \in \mathrm{FO}(A)$. Then* VerifyTeam$_{\varphi}$ $\in$ NP.

▶ **Theorem 4.** *Let $A = \{\perp, =(\dots), \subseteq\}$, $\varphi \in \mathrm{FO}(A)$. Then* E-Sat$^{\mathrm{team}}_{\varphi}$, E-MaxSat$^{\mathrm{team}}_{\varphi}$, E-CMaxSat$^{\mathrm{team}}_{\varphi}$, E-MinSat$^{\mathrm{team}}_{\varphi}$, E-CMinSat$^{\mathrm{team}}_{\varphi}$ $\in$ DelNP.

Proof. We again give a proof for E-Sat$^{\mathrm{team}}_{\varphi}$ (the proofs for all other cases can be found in the full version of the paper). We give a recursive algorithm enumerating E-Sat$^{\mathrm{team}}_{\varphi}$ with polynomial delay, when given oracle access to ExtendTeam$_{\varphi}$ (for definition see below) and VerifyTeam$_{\varphi}$.

Problem:	ExtendTeam$_{\varphi}$
Input:	Structure $\mathcal{A}$, team X, set of assignments Y
Question:	$\{X' \mid \mathcal{A} \models_{X'} \varphi \wedge X \subsetneq X' \wedge X' \cap Y = \emptyset\} \neq \emptyset$?

$\textsc{ExtendTeam}_\varphi \in \mathrm{NP}$ for all φ: A team X' is guessed and $X \subsetneq X' \wedge X' \cap Y = \emptyset$ can be checked in polynomial time. Finally, $\mathcal{A} \models_{X'} \varphi$ can be decided in NP by Lemma 3.

We now construct an algorithm that gets a structure $\mathcal{A}$ and a team X as inputs and outputs all satisfying teams X' with $X \subseteq X'$ and $X' \setminus X \subseteq \{s \in \mathrm{dom}(\mathcal{A})^{|\mathrm{free}(\varphi)|} \mid s > \max(X)\}$, that is, X' only contains new assignments that are larger than the largest assignment in X. For this to work for $X = \emptyset$, we define $\max(\emptyset)$ to be the empty assignment. The algorithm searches these teams X' by using recursive calls where exactly one assignment $s > \max(X)$ is added to X. By design, the recursive call where s' is added only outputs teams that contain s' and no assignment between $\max(X)$ and s', ensuring that no team is output twice. To get all satisfying teams we run the algorithm on input $(\mathcal{A}, \emptyset)$.

Algorithm 2: Algorithm used to show $\text{E-}\textsc{Sat}^{\mathrm{team}}_\varphi \in \mathrm{DelNP}$ for $\varphi \in \mathrm{FO}(A)$

Function EnumerateSuperteams(*structure* $\mathcal{A}$, *team* X)
  $Y = \bigcup_{s < \max(X) \wedge s \notin X} s$
  if $X \neq \emptyset \wedge \textsc{VerifyTeam}_\varphi(\mathcal{A}, X)$ **then** output X
  if $\textsc{ExtendTeam}_\varphi(\mathcal{A}, X, Y)$ **then**
    forall $s > \max(X)$ **do**
      EnumerateSuperteams$(\mathcal{A}, X \cup \{s\})$

◀

Problem:	$\textsc{CMinSat}^{\mathrm{team}}_\varphi$
Input:	Structure $\mathcal{A}, k \in \mathbb{N}$
Question:	$\{X \mid \mathcal{A} \models_X \varphi \wedge |X| \leq k\} \neq \emptyset$?

▶ **Theorem 5.** *There is a formula* $\varphi \in \mathrm{FO}(\subseteq)$ *such that* $\textsc{CMinSat}^{\mathrm{team}}_\varphi$ *is* NP*-hard. Hence,* $\text{E-}\textsc{CMinSat}^{team}_\varphi$ *is* DelNP*-hard.*

▶ **Corollary 6.** *Let* $\mathcal{E} = \{\text{E-}\textsc{Sat}, \text{E-}\textsc{MaxSat}, \text{E-}\textsc{CMaxSat}, \text{E-}\textsc{MinSat}, \text{E-}\textsc{CMinSat}\}$.

1. *For all* $\mathrm{E} \in \mathcal{E}$ *and* $\varphi \in \mathrm{FO}(A)$ *with* $A \subseteq \{\perp, =\!(\dots), \subseteq\}$ $\mathrm{E}^{\mathrm{team}}_\varphi$ *is in* DelNP.
2. *There are formulas* $\varphi_1 \in \mathrm{FO}(=\!(\dots)), \varphi_2 \in \mathrm{FO}(\perp), \varphi_3 \in \mathrm{FO}(\subseteq)$ *such that for all* $\mathrm{E} \in \mathcal{E}$ *the problems* $\mathrm{E}^{\mathrm{team}}_{\varphi_1}$, $\mathrm{E}^{\mathrm{team}}_{\varphi_2}$ *and* $\text{E-}\textsc{CMinSat}^{\mathrm{team}}_{\varphi_3}$ *are* DelNP*-complete.*

By Corollary 6 we get a characterization of the class DelNP as the closure of the mentioned problems under the enumeration reducibility notion.

3 Conclusion

In Table 1, we summarise the complexity results we obtained in this paper. We completely classified all here considered enumeration problems and obtained either polynomial-delay algorithms or completeness for DelNP.

There are some open issues that immediately lead to questions for further research. First, all our results are obtained for a certain fixed set of generalised dependency relations. Our selection was motivated by those logics found in the literature, but essentially arbitrary. It will be interesting to see whether other atoms or combinations of atoms lead to different (higher?) complexity.

Also, there is a notion of *strict semantics* (see, e.g., the work of Galliani [6]). Our results do not immediately transfer to strict semantics, therefore it would be interesting to study the enumeration complexity of team logics in strict semantics.

	$\subseteq$	$=(\dots)$	$\perp$
E-Sat	$\in$ DelP	DelNP-complete	DelNP-complete
E-MaxSat	$\in$ DelP	DelNP-complete	DelNP-complete
E-MinSat	$\in$ DelP	DelNP-complete	DelNP-complete
E-CMaxSat	$\in$ DelP	DelNP-complete	DelNP-complete
E-CMinSat	DelNP-complete	DelNP-complete	DelNP-complete

Table 1 Summary of obtained complexity results

Maybe even more interesting is the extension of the logical language by the so called strong (or classical) negation (our logics only allow atomic negation). It is known that with full classical negation, many generalised dependency atoms can be simulated (in modal logic, negation is even complete in the sense that it can simulate any FO-expressible dependency). We consider it likely that enumeration problems for logics with classical negation will lead us out of the class DelNP and potentially even to arbitrary levels of the hierarchy.

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Complexity of probabilistic inclusion logic and additive real arithmetics

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Abstract. We study an adaptation of inclusion logic to probabilistic team semantics which is a novel framework for studying logical and probabilistic dependencies simultaneously. In terms of its computational properties we show that the data complexity of probabilistic inclusion logic is in polynomial time. We also consider probabilistic inclusion logic extended with dependence atoms, and show that this logic is strictly less expressive than probabilistic independence logic but captures a natural additive fragment of existential second-order logic, which in turn collapses to non-deterministic polynomial time over sentences. We also investigate the axiomatic properties of marginal identity atoms, and compare our findings to the axiomatization of inclusion dependencies well known in database literature.

1 Introduction

Team semantics is the semantical framework of modern logics of dependence and independence. Introduced by Hodges [12] and adapted to dependence logic by Väänänen [16], team semantics defines truth in reference to collections of assignments, called *teams*. Thus team semantics is particularly suitable for a formal analysis of properties, such as the functional dependence between two variables, which only arise in the presence of multiple assignments. In the past decade numerous research articles have, via re-adaptations of team semantics, shed more light into the interplay between logic and dependence. A common feature, and limitation, in all these endeavors has been their preoccupation with notions of dependence that are *qualitative* in nature. That is, notions of dependence and independence that make use of quantities, such as conditional independence in statistics, have usually fallen outside the scope of these studies.

In contrast to earlier literature there has recently been a gradual shift toward quantitative dependence in team semantics studies. Two parallel approaches have been identified. In *multiteam semantics* formulae are evaluated against multisets of variable assignments, called *multiteams* [4]. This approach, which is analogous to the *bag semantics* in databases, centers attention to application domains in which the actual multiplicities of values, and not just their ratios, are meaningful. Another approach comes from *probabilistic team semantics* in which the basic semantic units are probability distributions, called *probabilistic teams*. To be sure, the idea of adding a probability measure on a team is not new, as first ideas of probabilistic teams trace back to the works of Galliani [6] and Hyttinen et al. [13]. But a systematic study on the topic is quite recent. In [5] probabilistic team semantics was studied in relation to the dependence concept that is most central in statistics: conditional independence. Mirroring [7,9,15] the expressiveness of probabilistic independence logic ($\mathrm{FO}(\perp\!\!\!\perp_{\mathrm{c}})$), obtained by extending first-order logic with conditional independence, was in [5,11] characterised in terms of arithmetic variants of existential second-order logic. In [11] the data complexity of $\mathrm{FO}(\perp\!\!\!\perp_{\mathrm{c}})$ was also identified in the context of *Blum-Shub-Smale machines* [1] and the existential theory of the reals. In [10] the focus was shifted to the expressivity hierarchies between probabilistic logics defined in terms of different quantitative dependencies.

Of all the dependence concepts thus far investigated in team semantics, that of *inclusion* has arguably turned out to be the most intriguing and fruitful. One reason is that *inclusion logic*, which arises from this concept, can only define properties of teams that are decidable in polynomial time [8]. In contrast, other natural team-based logics, such as dependence and independence logic, capture non-deterministic polynomial time [7,15,16], and many variants, such as team logic, have an even higher complexity [14]. Thus it should come as no surprise if quantitative variants of many team-based logics turn out be intractable; in principle, adding arithmetical operations and/or counting cannot be a mitigating factor when it comes to complexity. Indeed, it has been recently shown that the data complexity of probabilistic independence logic over sentences is possibly even higher than NP; it can be characterised in terms of a fragment of the existential theory of the reals which is NP-hard but not necessarily in NP [11]. The least known upper bound is PSPACE, as this is the least known upper bound for the full existential theory of the reals [2].

In this paper we ask the following general question: what are the definability and complexity properties of inclusion logic, if defined in quantitative terms. In team semantics the *inclusion atom* $x \subseteq y$, for two variables x and y, expresses that each value a of x also appears as a value of y. A quantitative variant of this atom is obtained by considering a so-called *marginal identity atom* $x \approx y$, which states the probability (or multiplicity) of x being a is the same as the probability (or multiplicity) of y being a, for all possible values a [5]. Of the aforementioned two parallel approaches, our focus is in probabilistic team semantics.

We make the following contributions. First, we show that the data complexity of probabilistic inclusion logic ($\mathrm{FO}(\approx)$) over sentences is in P. Thus no complexity increase, at the sentence level, is here effected by the introduction of quantities. In contrast, as stated above, whether independence logic is defined in terms of probabilistic teams or plain teams bears a (possible) impact upon complexity. Second, we show that probabilistic inclusion logic extended with dependence atoms ($\mathrm{FO}(\approx, =\!(\cdot))$) captures an additive variant of existential second-order logic. Using this we also show that $\mathrm{FO}(\approx, =\!(\cdot))$ over sentences corresponds to NP. Third, we show that $\mathrm{FO}(\perp\!\!\!\perp_{\mathrm{c}})$ over open formulae is strictly more expressive than $\mathrm{FO}(\approx, =\!(\cdot))$.[4] From [10] we already know that $\mathrm{FO}(\approx, =\!(\cdot))$ is strictly more expressive than $\mathrm{FO}(\approx)$; the reason is that marginal identity atoms, but not dependence atoms, are closed under so-called *scaled unions* of probabilistic teams. Thus we obtain the following strict expressivity hierarchy:

[4] Results in the vein of the second and third item have been independently developed, but not yet published, in the context of multiteam semantics. While our results are here similar, the proof techniques are different.

$\mathrm{FO}(\approx) < \mathrm{FO}(\approx, =(\cdot)) < \mathrm{FO}(\perp\!\!\!\perp_{\mathrm{c}})$. Fourth, we consider the axiomatic properties of the marginal independence atom. That inclusion atoms enjoy simple sound and complete axioms is well known from database theory [3]; we will investigate whether the same axioms yield a complete characterisation of marginal identity atoms.

2 Preliminaries

2.1 Probabilistic team semantics

Let D be a finite set of first-order variables, A a finite set, and X a finite set of assignments (i.e., a *team*) from D to A. A *probabilistic team* $\mathbb{X}$ is then defined as a function

$$\mathbb{X}\colon X \to [0,1]$$

such that $\sum_{s\in X} \mathbb{X}(s) = 1$. Also the empty function is considered a probabilistic team. We call D and A the variable domain and value domain of $\mathbb{X}$, respectively.

Let $\mathbb{X}\colon X \to [0,1]$ be a probabilistic team, A a finite non-empty set, p_A the set of all probability distributions $d\colon A \to [0,1]$, and $F\colon X \to p_A$ a function. We now define the duplication and supplementation operations for probabilistic teams. We first introduce the following useful notation:

$$X[A/x] := \{s(a/x) \mid s \in X, a \in A\}.$$

Duplicate Team. We denote by $\mathbb{X}[A/x]$ *the duplicate team* $X[A/x] \to [0,1]$ defined such that

$$\mathbb{X}[A/x](s(a/x)) := \sum_{\substack{t\in X\\ t\upharpoonright(\mathrm{Var}_1\setminus\{x\})=s\upharpoonright(\mathrm{Var}_1\setminus\{x\})}} \mathbb{X}(t)\cdot\frac{1}{|A|},$$

for each $a \in A$ and $s \in X$. Note that if x is a fresh variable then the righthand side of the above definition is simply $\mathbb{X}(s)\cdot\frac{1}{|A|}$.

Supplement Team. We denote by $\mathbb{X}[F/x]$ *the supplement team* $X[A/x] \to [0,1]$ defined such that

$$\mathbb{X}[F/x](s(a/x)) := \sum_{\substack{t\in X\\ t\upharpoonright(\mathrm{Var}_1\setminus\{x\})=s\upharpoonright(\mathrm{Var}_1\setminus\{x\})}} \mathbb{X}(t)\cdot F(t)(a),$$

for each $a \in A$ and $s \in X$. Again, if x is a fresh variable, the righthand side of the above definition can be simplified to $\mathbb{X}(s)\cdot F(s)(a)$.

Function Arithmetics. Let α be a real number, and f and g be functions from a shared domain into real numbers. The scalar multiplication αf is a function defined by $(\alpha f)(x) := \alpha f(x)$. The addition $f+g$ is defined as $(f+g)(x) = f(x)+g(x)$, and the multiplication fg is defined as $(fg)(x) := f(x)g(x)$. In particular, if f and g are distributions and $\alpha+\beta = 1$, then $\alpha f + \beta g$ is a distribution.

We may now define probabilistic team semantics for first-order formulae. As is customary in the team semantics context, we restrict attention to first-order formulae in negation normal form.

Definition 1. *Let $\mathfrak{A}$ be a τ-structure over a finite domain A, and $\mathbb{X}\colon X \to [0,1]$ a probabilistic team of $\mathfrak{A}$. The satisfaction relation $\models_{\mathbb{X}}$ for first-order logic is defined as follows:*

$$
\begin{array}{lll}
\mathfrak{A} \models_{\mathbb{X}} x = y & \Leftrightarrow \forall s \in X : \textit{if } \mathbb{X}(s) > 0, \textit{ then } s(x) = s(y) \\
\mathfrak{A} \models_{\mathbb{X}} x \neq y & \Leftrightarrow \forall s \in X : \textit{if } \mathbb{X}(s) > 0, \textit{ then } s(x) \neq s(y) \\
\mathfrak{A} \models_{\mathbb{X}} R(\boldsymbol{x}) & \Leftrightarrow \forall s \in X : \textit{if } \mathbb{X}(s) > 0, \textit{ then } s(\boldsymbol{x}) \in R^{\mathfrak{A}} \\
\mathfrak{A} \models_{\mathbb{X}} \neg R(\boldsymbol{x}) & \Leftrightarrow \forall s \in X : \textit{if } \mathbb{X}(s) > 0, \textit{ then } s(\boldsymbol{x}) \notin R^{\mathfrak{A}} \\
\mathfrak{A} \models_{\mathbb{X}} (\psi \wedge \theta) & \Leftrightarrow \mathfrak{A} \models_{\mathbb{X}} \psi \textit{ and } \mathfrak{A} \models_{\mathbb{X}} \theta \\
\mathfrak{A} \models_{\mathbb{X}} (\psi \vee \theta) & \Leftrightarrow \mathfrak{A} \models_{\mathbb{Y}} \psi \textit{ and } \mathfrak{A} \models_{\mathbb{Z}} \theta \textit{ for some } \mathbb{Y}, \mathbb{Z}, \\
 & \quad \alpha \in [0,1] \textit{ such that } \alpha\mathbb{Y} + (1-\alpha)\mathbb{Z} = \mathbb{X} \\
\mathfrak{A} \models_{\mathbb{X}} \forall x\psi & \Leftrightarrow \mathfrak{A} \models_{\mathbb{X}[A/x]} \psi \\
\mathfrak{A} \models_{\mathbb{X}} \exists x\psi & \Leftrightarrow \mathfrak{A} \models_{\mathbb{X}[F/x]} \psi \textit{ holds for some } F\colon X \to p_A.
\end{array}
$$

Note that, in the case for disjunction for non-empty $\mathbb{X}$, $\alpha = 0$ when $\mathbb{Y}$ is the empty team and $\alpha = 1$ when $\mathbb{Z}$ is the empty team, for $\mathbb{X}$, $\mathbb{Y}$, and $\mathbb{Z}$ need to be distributions or empty.

Probabilistic independence logic ($\mathrm{FO}(\perp\!\!\!\perp_{\mathrm{c}})$) is now defined as the extension of first-order logic with probabilistic independence atoms $\boldsymbol{y} \perp\!\!\!\perp_{\boldsymbol{x}} \boldsymbol{z}$ whose semantics is the standard semantics of conditional independence in probability distributions. *Probabilistic inclusion logic* ($\mathrm{FO}(\approx)$), is obtained by extending first-order logic with *marginal identity atoms* $\boldsymbol{x} \approx \boldsymbol{y}$ which state that the marginal distributions on $\boldsymbol{x}$ and $\boldsymbol{y}$ are identically distributed.

2.2 $\mathbb{R}$-structures

In this paper we relate team semantics to structures that enrich finite relational structures by adding real numbers ($\mathbb{R}$) as a second domain sort and functions that map tuples over the finite domain to $\mathbb{R}$.

Definition 2. *Let τ and σ be a finite relational and a finite functional vocabulary, respectively. An $\mathbb{R}$-structure of vocabulary $\tau \cup \sigma$ is a tuple*

$$\mathfrak{A} = (A, \mathbb{R}, (R^{\mathfrak{A}})_{R \in \tau}, (g^{\mathfrak{A}})_{g \in \sigma}),$$

where the reduct of $\mathfrak{A}$ to τ is a finite relational structure, and each $g^{\mathfrak{A}}$ is a function from $A^{\mathrm{ar}(g)}$ to $\mathbb{R}$.

In particular, our focus is on a variant of functional existential second-order logic with numerical terms ($\mathrm{ESO}_{\mathbb{R}}$) that is designed to describe properties of $\mathbb{R}$ structures. As first-order terms we have only first-order variables. For a set σ of function symbols, the set of numerical σ-terms i is generated by the following grammar:

$$i ::= c \mid f(\boldsymbol{x}) \mid i \times i \mid i + i \mid \mathrm{SUM}_{\boldsymbol{y}}\, i,$$

where the interpretations of $+, \cdot, \sum$ are the standard addition, multiplication, and summation of real numbers, respectively.

Definition 3 (Syntax of $\mathrm{ESO}_{\mathbb{R}}$). *Let τ be a finite relational vocabulary and σ a finite functional vocabulary. Let $O \subseteq \{+, \times, \mathrm{SUM}\}$, $E \subseteq \{=, <, \leq\}$, and $C \subseteq \mathbb{R}$. The set of $\tau \cup \sigma$-formulae of $\mathrm{ESO}_{\mathbb{R}}[O, E, C]$ is defined via the grammar:*

$$\begin{aligned}\phi ::= x = y \mid \neg x = y \mid i\ e\ j \mid \neg i\ e\ j \mid R(\boldsymbol{x}) \mid \neg R(\boldsymbol{x}) \mid \\ \phi \wedge \phi \mid \phi \vee \phi \mid \exists x \phi \mid \forall x \phi \mid \exists f \psi,\end{aligned}$$

where i and j are numerical σ-terms constructed using operations from O and constants from C, and $e \in E$, $R \in \tau$ is a relation symbol, f is a function variable, $\boldsymbol{x}$ is a tuple of first-order variables, and ψ is a $\tau \cup (\sigma \cup \{f\})$-formula of $\mathrm{ESO}_{\mathbb{R}}[O, E, C]$.

Note that the syntax of $\mathrm{ESO}_{\mathbb{R}}[O, E, C]$ allows first-order subformulae to appear only in negation normal form, and the semantics of $\mathrm{ESO}_{\mathbb{R}}[O, E, C]$ is defined via $\mathbb{R}$-structures and assignments analogous to first-order logic in a standard way, but the existential quantification of a function variable f ranges over all functions $f\colon A^{\mathrm{ar}(f)} \to \mathbb{R}$. Furthermore, given $S \subseteq \mathbb{R}$, we define $\mathrm{ESO}_S[O, E, C]$ as the variant of $\mathrm{ESO}_{\mathbb{R}}[O, E, C]$ in which existential quantification ranges over $h\colon A^{\mathrm{ar}(f)} \to S$.

Loose fragment. For $S \subseteq \mathbb{R}$, define L-$\mathrm{ESO}_S[O, E, C]$ as the *loose fragment* of $\mathrm{ESO}_S[O, E, C]$ in which negated numerical atoms $\neg i\ e\ j$ are disallowed.

3 Results

We show that the data complexity problem of probabilistic inclusion logic is in P via an approach from linear systems.

Theorem 1 *Let $\phi \in \mathrm{FO}(\approx)$ be a sentence. Given a structure $\mathfrak{A}$, the problem of determining whether $\mathfrak{A} \models \phi$ is in* P.

We show that $\mathrm{FO}(\approx, =\!(\cdot))$ captures exactly those properties of probabilistic teams that can be defined in loose additive ESO.

Theorem 2 *Over open formulae,* $\mathrm{FO}(\approx, =\!(\cdot))$ *corresponds to* L-$\mathrm{ESO}_{[0,1]}[+, =, 0, 1]$.

Corollary 3 *Over sentences,* $\mathrm{FO}(\approx, =\!(\cdot))$ *corresponds to* NP.

In contrast, a recent result shows that $\mathrm{FO}(\perp\!\!\!\perp_{\mathrm{c}})$ defines exactly those properties of probabilistic teams that can be defined in L-$\mathrm{ESO}_{[0,1]}[+, \times, =, 0, 1]$ [11]. Thus we can show that $\mathrm{FO}(\approx, =\!(\cdot)) < \mathrm{FO}(\perp\!\!\!\perp_{\mathrm{c}})$ over open formulae by considering properties definable in corresponding variants of real arithmetics.

Theorem 4 *Over open formulae,* $\mathrm{FO}(\approx, =\!(\cdot)) < \mathrm{FO}(\perp\!\!\!\perp_{\mathrm{c}})$.

Finally, we consider the axiomatic properties of marginal identity atoms.

Conjecture 5 *Marginal identity atoms have a finite sound and complete axiomatization over probabilistic teams.*

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A Hierarchy of Dependencies[1]

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1 Introduction

Semantic frameworks are commonly based on the notion of truth that is captured as a relation between possible worlds and formulas. The team semantics for propositional dependence logic (Yang and Väänänen 2016, 2017) is based on the observation that propositional dependence cannot be defined in terms of truth relative to single possible worlds. Above the layer of possible worlds, one needs to add the extra layer of teams (sets of possible worlds) and define dependency relations among statements relative to these teams. We have here an example of a peculiar semantic relativity: While atomic statements are primarily evaluated with respect to possible worlds, dependence statements are primarily evaluated with respect to teams. This paper is motivated by the view that this kind of relativity is a more integral part of language than it might seem and in order to capture it in full generality one should go beyond the two-layered framework (involving just possible worlds and sets of possible worlds) and employ a whole hierarchy of other types of semantic objects. These new semantic objects allow us to capture higher-order dependencies as well as some tricky interaction between the dependence operator and other logical operators.

In the next section, we will further elaborate on this idea and formulate two principles that will govern our approach. The principles are formulated rather vaguely but their meaning is illustrated with examples and they are embodied in a precisely defined formal semantics that is introduced in the subsequent section.

2 Semantic relativity and syntactic sensitivity

The common logical approach to information can be called eliminative. Growth of information is represented as elimination of possibilities. It is usually assumed that there is only one type of "possibilities" that can be eliminated by a piece of information. In the most standard approach these possibilities are called possible worlds. Thus, a body of information is usually modelled as a set of possible worlds, those worlds that are compatible with the information (Bar Hillel and Carnap 1964). This approach is sometimes (for example, by van Benthem and Martinez 2008) called *Information as Range*. We will call it *Information as a Set of Possible Worlds*, or *ISPW*, for short.

More abstract frameworks that go beyond the *ISPW* approach to overcome its weaknesses are related to the development of various relational semantics for non-classical logics such as intuitionistic logic, relevant logic and other substructural logics (Kripke 1965, Wansing 1993, Bimbó 2016). These logics are usually equipped with some generalization of the possible world semantics. These generalized relational semantics typically replace possible worlds with other kinds of entities, for example situations or information states. Let us use for these generalized possible worlds a neutral generic term *states*. In contrast to possible worlds, states may be partial or, in some cases, even inconsistent (e.g. in relevant logic).

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Even though these general frameworks overcome some of the weaknesses of the *ISPW* approach, they typically share with it one important feature: There is only one type of information that can be conveyed by all kinds of declarative sentences. In analogy to *ISPW*, where every piece of information is associated with a set of possible worlds, in the relational semantics of the mainstream non-classical logics every piece of information is associated with a set of states of one particular kind (typically an upward closed set).

The general strategy to identify the informational content of a sentence with a set of states of one particular kind is without doubt very powerful. It allows to capture some complicated relations among sentences (e.g. entailment) via somewhat more perspicuous set-theoretic relations (e.g. inclusion), which is analogous to Venn diagrams for Aristotelian syllogistic that transform various relations among concepts into simple relations among sets. This approach represents the informational contents of all declarative sentences as semantic objects of the same kind. Such a uniform account of information is appealing and for some purposes it is sufficient.

However, propositional dependence logic (Yang and Väänänen 2016, 2017), clearly shows that there are statements that do not fit into such a simple picture. The propositions expressing functional dependence of statements correspond semantically to sets of sets of possible worlds rather than to simple sets of possible worlds. In other words, while an elementary statement p primarily classifies possible worlds into those in which p is true and those in which it is false, a dependence statement *q is functionally dependent on p* (formalized as $=(p,q)$) primarily classifies sets of possible worlds (usually called teams) into those that support $=(p,q)$ and those that do not support $=(p,q)$.

In this paper, we would like to argue that such statements requiring a more complex semantic representation are quite common in language and are not restricted to dependence claims. For example, a modal statement of the form *might p* (formalized as $\Diamond p$) can also be captured as a classifier of sets of possible worlds. It classifies sets of possible worlds into those in which p may be true (is true at least in one of its elements) and those in which it is false (it is false in all its elements). One can observe that in both mentioned cases (dependence statements and might-statements) the special informational character of the statement is determined by an application of a particular logical operator.

So, the first claim, on which our paper is based, can be expressed in the form of the following principle that is partly reflected in the standard semantics of propositional dependence logic.

Principle of semantic relativity: *Different kinds of sentences may classify different kinds of semantic objects. Moreover, this semantic diversity is generated by particular logical operators.*

The second claim on which our paper is based can be, rather vaguely, expressed in the form of the following principle:

Principle of syntactic sensitivity: *The behaviour of logical operators is sensitive to the syntactic features of the statements to which the operators are applied.*

For example, consider negations of two statements that are of substantially different forms:

(A) *It is not the case that p* (formalized as $\neg p$),

(B) *It is not the case that q is dependent on p* (formalized as $\neg{=}(p,q)$).

Negation in the first case operates primarily on the level of possible worlds. A natural semantic clause is this:

(A)* $\neg p$ holds (is true) in a world w iff p does not hold (is not true) in the world w.

It is then natural to expand this clause to the level of teams in the following way: $\neg p$ holds in (is supported by) a team X iff p does not hold (is not true) in any world of X. The situation seems quite different in the second case. The statement (B) does not have natural truth conditions relative to single possible worlds. It must be evaluated on the level of teams. Using the setting of standard propositional dependence logic we obtain that the statements of this form are contradictions, which seems rather unintuitive. Instead, we claim that a natural semantic clause for (B) would be:

(B)* $\neg{=}(p,q)$ holds in (is supported by) a team X iff $=(p,q)$ does not hold in the team X.

In this case what negation negates is a global property of the whole team and not the local property of its possible worlds. We can see that the form of the statement to which a negation is applied affects the behaviour of the negation, in particular, the level on which it operates. Similar effects can be observed for example in the case of disjunction.

Moreover, it seems that the operators that are responsible for the semantic relativity are also syntactically sensitive. If the dependence operator is applied, for example, to two elementary statements p and q, it can be naturally evaluated on the level of teams. But what if it is applied for example to two might-statements that are themselves team-relative. For example, the claim

(C) *might-q is functionally dependent on might-p* (formalized as $=(\Diamond p,\Diamond q)$),

or the claim

(D) *q might be functionally dependent on p* (formalized as $\Diamond{=}(p,q)$)

are naturally evaluated on an even higher level. They seem to be relative to sets of teams. For these cases, the following sematnic analysis would be adequate:

(C)* $=(\Diamond p,\Diamond q)$ holds in a set of teams Z iff for any two teams $X,Y\in Z$, if X and Y agree on the value of $\Diamond p$, they also agree on the value of $\Diamond q$.

(D)* $\Diamond{=}(p,q)$ holds in a set of teams Z iff there is a team $Y\in Z$ such that $=(p,q)$ holds in Y, i.e., for any worlds $v,w\in Y$, if v and w agree on the value of p, they also agree on the value of q.

These examples indicate that to obtain a fully general framework that reflects the principle of semantic relativity as well as the principle of syntactic sensitivity, we will need an infinite hierarchy of teams. The goal of this paper is to propose and explore such a framework.

3 Formal semantics

In this section we will define a formal semantics motivated by the observations presented in the previous section. Let us start with definitions. A possible world is a function that assigns to every atomic formula a unique truth value (either T, or F). Every possible world will be called a context of degree 0. A context of degree $n+1$ is defined as a nonempty set of contexts of degree n. The empty set is called a context of infinite degree. The degree of a context C will be denoted as $d(C)$.

We will work with a propositional language L containing formulas that are built out of atomic formulas by $\neg, \wedge, \vee, \to$ and the dependence connective. There are no restrictions concerning the application of this connective. So, if $\varphi_1, \dots, \varphi_n, \psi$ are formulas of L then we can apply this connective to obtain $=(\varphi_1, \dots, \varphi_n, \psi)$ as another formula of L. We also define a contradiction $\bot$ as $p \wedge \neg p$ (for an arbitrarily selected atom p) and the modality operators $\Diamond\varphi$ as $\neg(\varphi \to \bot)$, and $\Box\varphi$ as $\neg\varphi \to \bot$.

For every L-formula φ we define the degree of φ, denoted as $d(\varphi)$, in the following way:

$d(p) = 0$, for every atomic formula p,

$d(\neg\varphi) = d(\varphi)$,

$d(\varphi \wedge \psi) = d(\varphi \vee \psi) = max\{d(\varphi), d(\psi)\}$,

$d(\varphi \to \psi) = max\{d(\varphi) + 1, d(\psi)\}$,

$d(=(\varphi_1, \dots, \varphi_n, \psi)) = max\{d(\varphi_1), \dots d(\varphi_n), d(\psi)\} + 1$.

Now we define a relation $\Vdash$ of truth between contexts and formulas. However, we impose the following restriction:

$C \Vdash \varphi$ is defined if and only if $d(\varphi) \leq d(C)$.

In particular, dependence claims, conditionals and, consequently, also modal assertions are not evaluated in singular possible worlds. We state that

if $d(\varphi) < d(C)$, then $C \Vdash \varphi$ iff for all $D \in C$, $D \Vdash \varphi$.

In particular, if C is the empty set, then automatically, $C \Vdash \varphi$, for any formula φ. It remains to be defined $C \Vdash \varphi$ for the cases where $d(C) = d(\varphi)$. For this purpose, we will use the following notation. Let C be a context and φ an L-formula such that $d(\varphi) < d(C)$. Then C^φ denotes a context that is either empty or of the same degree as C, and that is defined as follows:

$C^\varphi = \{D \in C \mid D \Vdash \varphi\}$.

Moreover, let C, D be contexts and φ an L-formula such that $d(\varphi) \leq d(C) = d(D)$. Then we write $C(\varphi) = D(\varphi)$ iff

either $C \Vdash \varphi$ and $D \Vdash \varphi$, or $C \Vdash \neg\varphi$ and $D \Vdash \neg\varphi$.

Now we are prepared to formulate the semantic conditions for the cases, where the degree of the formula on the right is equal to the degree of the context on the left:

$C \Vdash p$ iff $C(p) = T$, for every atomic formula p,

$C \Vdash \neg\varphi$ iff $C \nVdash \varphi$,

$C \Vdash \varphi \wedge \psi$ iff $C \Vdash \varphi$ and $C \Vdash \psi$,

$C \Vdash \varphi \vee \psi$ iff $C \Vdash \varphi$ or $C \Vdash \psi$,

$C \Vdash \varphi \to \psi$ iff $C^\varphi \Vdash \psi$,

$C \Vdash =(\varphi_1, \dots, \varphi_n, \psi)$ iff for any $D, E \in C$, if $D(\varphi_i) = E(\varphi_i)$, for all i, then $D(\psi) = E(\psi)$.

The degree of a set of formulas is defined as the maximal degree of the formulas in the set. The degree of an argument Δ/φ is defined as the degree of the set $\Delta \cup \{\varphi\}$.

Let Δ/φ be an argument of the degree n. We say that the argument is valid if $C \Vdash \varphi$, for every context C such that $d(C) = n$ and $C \Vdash \psi$, for all $\psi \in \Delta$. Two L-formulas, φ and ψ are equivalent iff φ/ψ and ψ/φ are valid arguments.

Let us observe some consequences of these definitions. First, note that $C \Vdash \bot$ iff C is empty. Now, we can calculate the semantic clauses for the modal claims of the forms $\Diamond\varphi$ and $\Box\varphi$. If $d(\varphi) = n$ then $d(\Diamond\varphi) = d(\Box\varphi) = n+1$. Let C be a context of degree $n+1$ (note that C must be non-empty). Then $C \Vdash \Diamond\varphi$ iff $C \Vdash \neg(\varphi \to \bot)$ iff $C \nVdash \varphi \to \bot$ iff $C^{\varphi} \nVdash \bot$ iff C^{φ} is non-empty. Moreover, $C \Vdash \Box\psi$ iff $C \Vdash \neg\varphi \to \bot$ iff $C^{\neg\varphi} \Vdash \bot$ iff $C^{\neg\varphi}$ is empty. Thus we obtain:

$C \Vdash \Diamond\varphi$ iff there is $D \in C$ such that $D \Vdash \varphi$,

$C \Vdash \Box\varphi$ iff for all $D \in C$, $D \Vdash \varphi$.

One can observe that the particular cases $\neg p$, $\neg{=}(p,q)$, $=(\Diamond p, \Diamond q)$, $\Diamond{=}(p,q)$ discussed in the previous section are evaluated as expected, i.e. in accordance with (A)*, (B)*, (C)*, and (D)*.

The resulting framework differs in many aspects from the standard framework of propositional dependence logic. The background propositional logic for the language L^-, defined as L without the dependence connective, is non-classical. For example, though transitivity of implication holds for atomic sentences (the argument $p \to q, q \to r/p \to r$ is valid), it does not hold for some more complex cases like $p \to q, q \to \Diamond r/p \to \Diamond r$ (note that $\Diamond$ is defined using the connectives of L^- and thus this argument is formulated in L^-). One can observe that this result is desirable. Consider for example the following invalid natural language instance of the argument form: *If Ann is in Berlin, she is in Germany. If Ann is in Germany, she might be in Munich. Therefore, if Ann is in Berlin, she might be in Munich.*

The proposed framework allows us to capture in a natural way the meaning of claims that seem to be quite tricky from the perspective of the standard framework of propositional dependence logic. For example, we can represent naturally iterated dependencies or express claims such as *if r is not dependent on p then it is dependent on q*.

In our paper, we will explore this framework and especially how the dependence operator behaves in it, how it interacts with the modalities and the non-classical conditional. Moreover, we will present further natural language examples to which this framework can be reasonably applied.

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Algebraic Semantics for Propositional Dependence Logic*

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1. Introduction

In this work we introduce an algebraic semantics for propositional dependence logic and we show some possible applications of this novel semantic framework.

Dependence logic was introduced in 2007 by Jouko Väänänen [11] as an extension of first-order logic with dependence atoms. In its now standard formulation, dependence logic is defined *via* team semantics, originally introduced by Hodges in [5], which generalizes standard Tarski's semantics by teams, namely sets of assignments which map first-order variables to elements of the domain. In its propositional version, a team is simply a set of valuations mapping propositional atoms to either 1 or 0. Propositional dependence logic has been studied in [9], while [10] also considers other extensions of classical logic using team semantics.

There is no algebraic semantics for propositional dependence logic in the literature, though some study in the algebraic interpretations of dependence logic was initiated for instance in [1] and [6]. In this work we follow a partially different route and we introduce the algebraic semantics of dependence logic by introducing downward team algebras and relying on the relation between dependence and inquisitive logic. This relation has already been pointed out e.g. in [3, 9], while the algebraic semantics for inquisitive logic has been recently introduced in [2] and [8].

In the present work we define *downward team algebras* analogously to inquisitive algebras and we show they provide a sound and complete semantics for dependence logic. Moreover, we describe applications of this semantics to the study of extensions of dependence logic and we discuss ongoing work to extend the present setting to other logics based on team semantics, in particular to so-called team logic – see [6].

2. Downward Team Algebras and Algebraic Semantics

To give an algebraic semantics to dependence logic we shall first introduce downward team algebras. We first recall some basic definition. We say that a bounded distributive lattice H is a Heyting algebra if it can be endowed with an operation $\rightarrow$ such that $a \wedge b \leq c$ if and only if $a \leq b \rightarrow c$. The operation of negation $\neg$ is defined by $\neg x := x \rightarrow 0$. Moreover, recall that if H is a Heyting algebra, $H_\neg$ is its subset of *regular elements*, namely $H_\neg = \{x \in H : x = \neg\neg x\}$. It is easy to show that $H_\neg$ is a subalgebra of H with respect to the reduct $\{1, 0, \wedge, \rightarrow\}$ and that it forms a Boolean algebra if supplemented by a join $\vee$ defined, for all $a, b \in H_\neg$, as $a \vee b := \neg(\neg a \wedge_H \neg b)$. The logic $\mathtt{ND}$ is defined axiomatically as follows:

$$\mathtt{ND} = \mathtt{IPC} + \{(\neg p \rightarrow \bigvee_{i \leq k} \neg q_i) \rightarrow \bigvee_{i \leq k} (\neg p \rightarrow \neg q_i) : k \geq 2\}.$$

and was first introduced by Maximova in [7].

We can now define downward team algebras as follows.

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Definition 1 (Downward Team Algebra)**.** A *downward team algebra* H is a tuple $(H, \mathbin{\vee\!\!\!\vee}, \otimes, \wedge, \to, 0, H_\neg, \vee)$, where $(H, \mathbin{\vee\!\!\!\vee}, \otimes, \wedge, \to, 0)$ is a ND-algebra, $(H_\neg, \vee, \wedge, \to, 0)$ a Boolean algebra and, in addition, it satisfies the following equation:

$$(*) \quad x \otimes (y \mathbin{\vee\!\!\!\vee} z) \approx (x \otimes y) \mathbin{\vee\!\!\!\vee} (x \otimes z)$$
$$(**) \quad \neg x \otimes \neg y \approx \neg x \vee \neg y.$$

We can define negation over such algebras by $\neg x := x \to 0$, and also a partial operation $=\!\!(x)$ on elements of $H_\neg$ as $=\!\!(x) := x \mathbin{\vee\!\!\!\vee} \neg x$. Now, since for any Heyting algebra H we have already seen that $(H_\neg, \vee, \wedge, \to, 0)$ is always a Boolean algebra, the definition before amounts to say that $(H, \mathbin{\vee\!\!\!\vee}, \otimes, \wedge, \to, 0)$ is a ND-algebra provided with an extra operator $\otimes$ satisfying $(*)$ and $(**)$. A downward team algebra is thus a ND-algebra supplemented with a tensor operator $\otimes$ and a signed subset of elements, i.e. its regular elements $H_\neg$. We denote by **DTA** the class (and the category) of downward team algebras. We remark that a similar class of structures has been previously considered in [2, 8] to give an algebraic semantics to inquisitive logic.

We shall now supplement downward team algebras by suitable valuations of atomic formulas in order to obtain algebraic models for dependence logic. In particular, we will give a semantics for the version of dependence logic formulated in the signature $\{\top, \bot, \wedge, \otimes, \to, =\!\!(p)\}$, which is called *propositional intuitionistic dependence logic* in [9]. In particular, we will treat the dependency atom $=\!\!(\vec{p}, q)$ as a non-primitive symbol, which can be defined as follows:

$$=\!\!(\vec{p}, q) := (\bigwedge_{i \leq n} =\!\!(p_i)) \to =\!\!(q).$$

A *valuation* over a downward team algebra H is a function $V^\neg : \mathtt{AT} \to H_\neg$ which assigns to every atomic formula some regular element in $H_\neg$. Algebraic models for intuitionistic dependence logic are then defined as follows.

Definition 2 (Algebraic Downward Team Model)**.** An *algebraic downward team model* of intuitionistic dependence logic is a pair $M = (H, V^\neg)$ such that H is a downward team algebra and $V^\neg$ a valuation.

The interpretation of an arbitrary formula $\phi \in \mathcal{L}_{\mathbf{PD}}$ in an algebraic downward team model $M = (H, V^\neg)$ is then defined as follows.

Definition 3 (Interpretation of Arbitrary Formulas)**.** Given an algebraic downward team model M and a formula $\phi \in \mathcal{L}_{\mathbf{PD}}$, its *interpretation* $[\![\phi]\!]^M$ is defined as follows:

$$[\![p]\!]^M = V^\neg(p) \qquad [\![\bot]\!]^M = 0 \qquad [\![=\!\!(p)]\!]^M = [\![p]\!]^M \vee [\![\neg p]\!]^M$$
$$[\![\phi \wedge \psi]\!]^M = [\![\phi]\!]^M \wedge [\![\psi]\!]^M \qquad [\![\phi \otimes \psi]\!]^M = [\![\psi]\!]^M \otimes [\![\chi]\!]^M \qquad [\![\phi \to \psi]\!]^M = [\![\phi]\!]^M \to [\![\psi]\!]^M$$

This is a standard definition, *modulo* the fact that atomic formulas can be assigned only to elements of the signed subset $H_\neg$, and cannot be given arbitrary values of H. From the former definitions it is straightforward to adapt the usual definitions of truth at a model and validity. We say that a formula ϕ is *true under* $V^\neg$ *in* H or *true in the model* $M = (H, V^\neg)$ and write $M \vDash^\neg \phi$ if $[\![\phi]\!]^M = 1_H$. We say that ϕ is *valid in* H and write $H \vDash^\neg \phi$ if ϕ is true in every model $M = (H, V^\neg)$ over H. Finally, we way that ϕ is an *algebraic validity* of intuitionistic dependence logic if it is true in all algebraic downward team models, namely if $\mathbf{DTA} \vDash^\neg \phi$.

3. Characterisation of Downward Team Algebras

DTA is by our definition an equational class of algebras, hence by Birkhoff Theorem it is also a variety of algebraic structures, i.e. a class of algebras which is closed under subalgebras, products and homomorphic images. As a variety, we have that **DTA** is generated by its collection of subdirectly irreducible elements. However, in the case of downward team algebras it is actually possible to identify a smaller class of generators. Recall that a Heyting algebra H is said to be *regular* if $H = \langle H_\neg \rangle$, and let regular downward team algebras be defined analogously. We let $\mathbf{DTA}_{FRSI}$ be the collection of finite, regular, subdirectly irreducible downward team algebras and $Var_{FRSI}(\mathtt{ND})$ the collection of finite, regular, subdirectly irreducible ND-algebras. We can then prove the two following results:

Theorem 4. *Suppose* $\mathbf{DTA} \not\vDash^\neg \phi$*, then* $\mathbf{DTA}_{FRSI} \not\vDash^\neg \phi$*.*

Theorem 5. *The categories* $\mathbf{DTA}_{FRSI}$ *and* $Var_{FRSI}(\mathtt{ND})$ *are equivalent:* $\mathbf{DTA}_{FRSI} \cong Var_{FRSI}(\mathtt{ND})$*.*

By Theorem 4 we know that $\mathbf{DTA}_{FRSI}$ witnesses the falsity of formulas in the variety, while Theorem 5 tells us that any finite, regular, subdirectly irreducible ND-algebra can be extended in a unique way to a downward team algebra. In particular, the tensor disjunction $\otimes$ over finite, regular, subdirectly irreducible ND-algebra H is always defined as follows:

$$x \otimes y := \bigvee \{a \vee b : a \leq x, b \leq y \text{ and } a, b \in H_\neg\}.$$

Moreover, Theorem 5 allows us to relate downward team algebras to teams *via* the following observation. As done in [2], given a finite set s, we can construct a corresponding finite, regular. subdirectly irreducible ND-algebra H_s by first building the Boolean algebra $B = (\wp(s), \subseteq)$ and then considering the Heyting algebra of nonempty downsets $Dw^+(B)$. Conversely, it was proved in [2, 8] that any algebra in $Var_{FRSI}(\mathtt{ND})$ is isomorphic to one of the form H_s. We thus obtain the following equivalence between categories, extending 5. We denote by **FinSet** the category of finite sets with functions as arrows.

Theorem 6. $\mathbf{DTA}_{FRSI} \cong Var_{FRSI}(\mathtt{ND}) \cong \mathbf{FinSet}$.

Hence every finite set determines a finite, regular, downward team algebras up to isomorphism, and *vice versa*.

Finally, we remark that the previous theorem also allows us to relate the present setting to previous work on inquisitive logic. In fact, it was shown in [8] that $Var_{FRSI}(\mathtt{ND})$ is the class of generators of $Var^\neg(\mathtt{InqB})$, the so-called DNA-variety of Heyting algebras which validate the axioms of inquisitive logic under valuations restricted to regular elements. It is possible to show that DNA-varieties are standard varieties which are also closed under core superalgebras, where we say that a Heyting algebra K is a *core superalgebra* of H if $H_\neg = K_\neg$ and $H \preceq K$.[1] From these facts, one can prove that **DTA** is the least variety containing $\mathbf{DTA}_{FRSI}$ which is also closed under core superalgebra, thus showing in which sense the collection $\mathbf{DTA}_{FRSI}$ generates **DTA**.

4. Equivalence of Team and Algebraic Semantics

In the previous section we have shown how, given a finite set s, one can construct a downward team algebra H_s and we have proved that the category **FinSet** of finite sets is equivalent to

[1] We write $H \preceq K$ to denote that H is a subalgebra of K.

the category **DTA** of downward team algebras. Now we want to extend this result to take into account the situation where the starting set s is not an arbitrary finite set, but a finite team, i.e. a set of valuations. To this end, let **FinTeam** be the category of finite teams and **ADTM** the category of algebraic downward team models. We shall prove in this section the equivalence of these two categories.

Given a finite team s we define its corresponding *algebraic downward team model* M_s as $M_s = (H_s, V_s^{\neg})$, where $V_s^{\neg}$ is the canonical valuation defined, for all $p \in \mathtt{AT}$, as $V^{\neg}(p) = \wp^{+}\{\alpha \in s : \alpha(p) = 1\}$, where $\wp^{+}(x) = \wp(x) \setminus \{\emptyset\}$. By using the equivalence of categories of the previous section we can then prove the following proposition.

Proposition 7.

1. *Let $s \in \wp(2^{\mathtt{AT}})$ be a finite team and $M_s = (H_s, V_s)$ its corresponding finite, regular, subdirectly irreducible algebraic downward team model. We have that $s \vDash \varphi$ if and only if $M_s \vDash^{\neg} \varphi$.*

2. *Let $M = (H, V^{\neg})$ be a finite, regular, subdirectly irreducible algebraic downward team model and s_M its corresponding team. We have that $M \vDash^{\neg} \phi$ if and only if $s_M \vDash \phi$.*

Hence from the previous proposition and Theorem 4 above, we then obtain our main result, i.e. the equivalence of team and algebraic semantics.

Theorem 8 (Equivalence of Team and Algebraic Semantics)**.** *The team semantics and the algebraic semantics of* **PD** *are equivalent, i.e. $\wp(2^{\mathtt{AT}}) \vDash \phi$ if and only if* **DTA** $\vDash^{\neg} \phi$.

Which shows that downward team algebras give a complete semantics to (intuitionistic) dependence logic. Finally, the previous results also give us the following equivalence.

Theorem 9. $\mathbf{ADTM}_{FRSI} \cong \mathbf{FinTeam}$.

5. Applications, Extensions and Open Problems

We conclude our presentation of the algebraic semantics for propositional dependence logic by mentioning some further problems and applications.

Firstly, notice that the introduction of algebraic semantics for dependence logic turns especially useful to study the arity fragments of dependence logic. In fact, it was shown in [8] that the sublattice of DNA-logics extending propositional inquisitive logic InqB is dually isomorphic to $\omega + 1$. Moreover, every such extension corresponds to a fragment obtained by putting some constraints on the cardinality of the underlying algebras. The case of dependence logic is similar, in particular we can show that the lattice of extensions of dependence logic is isomorphic to the lattice of extensions of inquisitive logic.

Secondly, an important direction of investigation is whether the framework that we described here can be extended to other logics defined *via* team semantics. In particular, we propose the introduction of the following algebraic structures to provide an algebraic semantics to *team logic*, i.e. the system $\mathbf{PL}(\sim)$ described e.g. in [6].

Definition 10 (Full Team Algebra)**.** A *full team algebra* B^A is a tuple $(B, \mathbin{\vee\!\!\!\vee}, \oplus, \otimes, \sim, 0_B, A, \vee, \wedge, \neg, 0_A)$, such that:

- $(B, \mathbin{\vee\!\!\!\vee}, \oplus, \sim, 0_B)$ and $(A, \vee, \wedge, \neg, 0_A)$ are Boolean algebras;
- $A \subseteq B$;

- $\otimes \upharpoonright A^2 = \vee$, $\oplus \upharpoonright A^2 = \wedge$ and $\sim\upharpoonright A = \neg$.

And, in addition, B^A satisfies the following equation:

$$(*) \quad x \otimes (y \veebar z) \approx (x \otimes y) \veebar (x \otimes z).$$

Here a special role is then played by *regular* full team algebras, namely full team algebras B^A such that every element of B can be expressed as a join of elements of A. Interestingly, this is equivalent to say that B is an atomic Boolean algebra and that A is its set of atoms. Finally, we remark that such definition seems to capture in algebraic terms some previous work on the notion of *teamification*, developed especially in [6].

Finally, we mention two important open issues related to the algebraic semantics of dependence logic. On the one hand, we have seen that the category of finite, regular, subdirectly irreducible downward team algebras is equivalent to the categories of finite sets and of finite, regular, subdirectly irreducible ND-algebra. Is it possible to obtain a similar result for the full category of downward team algebras? On the other hand, there has been no attempt so far to give a topological semantics to dependence logic. The work done in this paper could shed some light in this direction, especially considering that in [2] a topological semantics for inquisitive logic was introduced. It is therefore interesting to investigate whether it is possible to extend such framework to dependence logic and possibly to other logics based on team semantics.

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On definability of team relations with k-invariant atoms

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Abstract

Given a team X, a k-tuple $\vec{y}$ of variables in $\mathrm{dom}(X)$ defines a corresponding k-ary (team) relation $X(\vec{y})$. The expressive power of a logic L with team semantics amounts to the set of properties of team relations which L-formulae can define. We introduce a concept of k-invariance which is a natural semantic restriction on logical atoms. Then we develop a proof method to show that, if L is an extension of FO with any k-invariant atoms, then there are such properties of $(k+1)$-ary team relations which cannot be defined in L. This method can be applied for the arity fragments of various logics with team semantics to prove undefinability results.

1 Background and related work

The origin of team semantics goes back to the work of Hodges [8] who presented it to give a compositional semantics as an alternative to game-theoretic semantics of IF-logic by Hintikka and Sandu ([6, 7]). In the compositional approach it was not sufficient to consider single assignments; instead there was a need to use *sets of assignments* which are nowadays called *teams*. Väänänen [12] developed this approach further by introducing *dependence atoms* and adding them to first order logic with team semantics. Later various other natural atoms from database theory have been added to this framework, such as *independence atoms* ([4]), *inclusion atoms* and *exclusion atoms* ([2]).

In this extended abstract we present a notion of *k-invariance* of atoms, which is closely related to the study of *arity fragments* of logics with team semanics. We list here some of the most relevant works related to the expressive power of such fragments (on the level of sentences). In [1], [3] it is shown that the arity fragments of dependence and independence logic correspond to the *functional* arity fragments of *existential second order logic*, ESO. In [11] it is shown that similarly the arity fragments of exclusion logic correspond to the *relational* arity fragments of ESO. Moreover, in [5] it is shown that inclusion logic has a strict arity hierarchy over graphs.

When the expressive power of logics with team semantics is studied on the level of all formulas (not just sentences), we need to examine which properties of *team relations* are definable. Galliani [2] has shown that with *inclusion-exclusion logic* one can define exactly those team relations which are ESO-definable. In [10] we show that the relationship between these two logics becomes more delicate when we consider k-ary inclusion-exclusion logic (INEX[k]) and k-ary relational fragment of ESO, (ESO[k]). Then all INEX[k]-definable properties are ESO[k]-definable and conversely all ESO[k]-definable properties *of k-ary team relations* are INEX[k]-definable. However, this leaves open what happens to INEX[k]-definability of team relations of higher arity. A partial answer to this question is given in this extended abstract as we show, in particular, that the $(k+1)$-*totality* of a team relation cannot be defined in INEX[k]. All the results here are based on PhD Thesis [9] by the author.

2 Preliminaries

Let $\mathcal{M}$ be a relational structure. A *team* X for $\mathcal{M}$ is any set of assignments s for $\mathcal{M}$ with a common domain – denoted by $\mathrm{dom}(X)$. Often $\mathrm{dom}(X)$ is assumed to be finite, but in this paper we may also allow teams with infinite domains. For any $\{y_1, \dots, y_k\} \subseteq \mathrm{dom}(X)$ we write

$$X(y_1 \dots y_k) = \{s(y_1 \dots y_k) \mid s \in X\}.$$

Hence every k-tuple $\vec{y}$ of variables in $\mathrm{dom}(X)$ naturally defines a corresponding k-ary *team relation* $X(\vec{y})$ in the model $\mathcal{M}$.

The semantics of first order logic (FO) can naturally be generalized from single assignments to sets of assignments. This leads to *team semantics* ([12]) where a team X satisfies an FO-formula φ (denoted by $\mathcal{M} \vDash_X \varphi$) if and only if every assignment $s \in X$ satisfies φ with the standard semantics of FO.

When FO with team semantics is extended with new logical atoms (or operators) we obtain more expressive logics. Some of the most common atoms studied are dependence atoms ([12]), independence atoms ([4]), inclusion atoms and exclusion atoms ([2]). In most natural extensions of FO the so-called *locality* property is preserved. That is, the truth of a formula φ is determined by only the values of those variables which occur in φ as free variables. Some atoms, such as dependence atoms and exclusion atoms, are also *closed downwards*. This means that if X satisfies the atom, then also all the subteams $Y \subseteq X$ satisfy the atom. If L is an extension of FO with only downwards closed atoms, then this property holds for all L-formulae, whence we say that L is closed downwards.

3 k-invariant atoms

A natural way to restrict the expressive power of logics with team semantics is to put restrictions on the complexity of logical atoms that can be used. By restricting the *arity* of atoms, shorter tuples of variables are allowed to be used and thus it suffices to check team relations of lower arity when evaluating the atom. For example, the truth condition of k-ary inclusion atoms is defined with respect to k-ary team relations – as follows:

$$\mathcal{M} \vDash_X x_1 \dots x_k \subseteq y_1 \dots y_k \quad \text{iff} \quad X(x_1 \dots x_k) \subseteq X(y_1 \dots y_k).$$

Definition 3.1. Let X, Y be teams for a model $\mathcal{M}$ such that X, Y have a shared domain D. We say that X and Y are *k-equivalent* if the following holds for all $\{y_1, \dots, y_k\} \subseteq D$:

$$X(y_1 \dots y_k) = Y(y_1 \dots y_k).$$

Moreover, we say that an atom A is *k-invariant* if we have

$$\mathcal{M} \vDash_X A \quad \text{iff} \quad \mathcal{M} \vDash_Y A,$$

for all models $\mathcal{M}$ and k-equivalent teams X and Y for $\mathcal{M}$.

The notion of k-invariance intuitively states that an atoms can only "see" the k-ary relations in the given team. Hence this property could also be called "k-dimensionality". Also note that this definition is very liberal since it allows e.g. atoms which are not invariant under isomorphisms. However, the results in Section 5 can be proven without any further restrictions on this definition.

Next we define quite a general class of logics with team semantics by setting the k-invariance restriction on certain atoms.

Definition 3.2. A logic L belongs to class $\mathcal{L}[k]$ if (i) L is an extension of FO with new atomic formulas so that L is local; and (ii) all atomic formulas in L belong to either (or both) of the following two classes:

(a) downwards closed atoms;

(b) k-invariant atoms.

Note that, in particular, all downward closed logics and all k-ary fragments of logics with team semantics (studied so far) belong to the class $\mathcal{L}[k]$.

4 Definability of team relations

By saying that a class $\mathcal{P}$ (i.e. a property) of k-ary team relations is definable in a logic L with team semantics, we mean that by fixing a tuple $y_1 \dots y_k$ of distinct variables, in the given order, there is a formula $\varphi(y_1 \dots y_k) \in$ L such that

$$\mathcal{M} \vDash_X \varphi \;\text{ iff }\; X(y_1 \dots y_k) \in \mathcal{P}.$$

It is important to note the difference between defining relations in a *model* and relations in a *team*. Consider e.g. the property of *symmetry* of binary relations which is FO-definable as a property of a relation in a model. However, since this property is not closed downwards, the corresponding team property is not definable in any downwards closed logic L – such as dependence logic ([12]).

Next focus our attention to k-ary inclusion-exclusion logic INEX[k] which extends FO with both k-ary inclusion and exclusion atoms. In [10] we showed that all ESO[k]-definable properties of k-ary team relations can be defined in INEX[k]. Hence the expressive power of INEX[2] is already very strong since we can define all ESO[2]-definable properties of binary team relations. However, this result tells us only very little about the expressive power of INEX[1].

By the results in [10] we also know that all INEX[k]-definable properties of team relations must be ESO[k]-definable. However, these results leave open whether INEX[1] can define some natural ESO[1]-definable properties of *binary* team relations, such as the following two:

(a) $X(y_1 y_2)$ is symmetric.

(b) $X(y_1 y_2)$ is c-colorable for a given $c \in \mathbb{Z}_+$.

(In (b) we consider the *undirected graph* that corresponds to $X(y_1y_2)$.)

It can be shown that the rather complex property (b) is definable in INEX[1], for any $c \in \mathbb{Z}_+$, with the following formula:

$$\gamma_{\leq c} \vee \exists x_1 \ldots \exists x_c \Big(\bigwedge_{i \neq j} x_i \,|\, x_j \,\wedge\, \forall y \,\big(\bigvee_{i \leq c} y \subseteq x_i\big) \\ \wedge \bigvee_{x_1,\ldots,x_c} \big\{ y_1 \subseteq x_i \,\wedge\, y_2 \,|\, x_i \;\mid\; i \leq c \big\} \Big),$$

where $\gamma_{\leq c}$ is an FO-sentence defining that the model has at most c elements.

However, interestingly it turns out that the much more simple property (a) cannot be defined in INEX[1]. This follows from a more general result which we present in the next section.

5 Theorem for proving undefinability results

Theorem 5.1 below can be used for proving undefinability results for any logic L in the class $\mathcal{L}[k]$ (recall Def. 3.2). A complete proof of this theorem is too long for this extended abstract, but the intuitive idea behind it can be described as follows. When removing assignments from a team, a k-invariant atom can "see" this change only if some of the k-ary team relations change. When a team X is large enough with respect a given L-formula φ, there must be some assignments whose removal does dot change any k-ary relations anywhere within the "evaluation process of φ" and thus the removal does not change the truth value of any atoms in L. (For a complete proof, see Section 5.2 in [9].)

Theorem 5.1. *Let $\mathcal{P}$ be a property of $(k+1)$-ary relations. Assume that there is a constant c such that for any finite model $\mathcal{M}$, with at least c elements, there are teams X and X^* for $\mathcal{M}$ such that the following conditions hold:*

1. $X^* \subseteq X$
2. $\mathrm{dom}(X) = \{y_1, \ldots, y_{k+1}\}$.
3. $X(y_1 \ldots y_{k+1})$ *has the property $\mathcal{P}$.*
4. $(X \setminus \{s\})(y_1 \ldots y_{k+1})$ *does not have the property $\mathcal{P}$ for any $s \in X^*$.*
5. $|X^*| \geq \frac{|\mathrm{dom}(\mathcal{M})|^{k+1}}{c}$.

Then the property $\mathcal{P}$ cannot be defined in any logic $\mathrm{L} \in \mathcal{L}[k]$.

The assumptions above may look quite technical, but the core idea is rather simple: *those properties of teams that are very sensitive to removal of assignments cannot be defined with logics in $\mathcal{L}[k]$*. By "sensitive" we mean that for any team X with the given property $\mathcal{P}$, there are several assignments (namely those in $X^* \subseteq X$) such that the removal of any single one of them makes X to lose the property $\mathcal{P}$. By "several" we mean that the number of such assignments ($|X^*|$) is at least $|\,\mathrm{dom}(\mathcal{M})^{k+1}|/c$, for some constant c. Theorem 5.1 can be used for proving various undefinability results. We present here two examples, where some very simple properties of team relations are shown to be undefinable.

Corollary 5.2. *For any $k \geq 1$, the $(k+1)$-totality of $X(y_1 \dots y_{k+1})$ (i.e. the property of it being the full relation $\mathrm{dom}(\mathcal{M})^{k+1}$) is undefinable in $\mathcal{L}[k]$.*

Proof. Let $c = 1$ and let $\mathcal{M}$ be any finite model. Let X be the team for $\mathcal{M}$ for which $\mathrm{dom}(X) = \{y_1, \dots, y_{k+1}\}$ and $X(y_1 \dots y_{k+1}) = \mathrm{dom}(\mathcal{M})^{k+1}$. Let $X^* = X$; now the claim follows immediately from Theorem 5.1. $\square$

Corollary 5.3. *Symmetry of $X(y_1 y_2)$ cannot be defined in $\mathcal{L}[1]$.*

Proof. Let $c = 2$ and let $\mathcal{M}$ be any finite model with at least 2 elements. Let X be the team for $\mathcal{M}$ s.t. $\mathrm{dom}(X) = \{y_1, y_2\}$ and $X(y_1 y_2) = \mathrm{dom}(\mathcal{M})^2$. We define

$$X^* := \{s \in X \mid s(x_1) \neq s(x_2)\}.$$

Now $X(y_1 y_2)$ is clearly symmetric by being the full binary relation. However, $(X \setminus \{s\})(y_1 y_2)$ is not symmetric for any $s \in X^*$ (as the removal of a *single* edge from any 2-cycle immediately violates the symmetry). We also have

$$|X^*| \,=\, |\mathrm{dom}(\mathcal{M})|^2 - |\mathrm{dom}(\mathcal{M})| \,\geq\, \frac{|\mathrm{dom}(\mathcal{M})|^2}{2} \,=\, \frac{|\mathrm{dom}(\mathcal{M})|^2}{c}.$$

Thus the claim follows from Theorem 5.1. $\square$

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Expressivity of Linear Temporal Logic under Team Semantics

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Introduction

Linear temporal logic (LTL) is a simple logic for formalising concepts of time. It has become important in theoretical computer science, where Amir Pnueli connected it to system verification in 1977, and within that context the logic has been studied extensively [6]. With regards to expressive power, a classic result by Hans Kamp from 1968 shows that LTL is expressively equivalent to $\mathsf{FO}^2(<)$ [4, 7].

LTL has found applications in the field of formal verification, where it is used to check whether a system fulfils its specifications. However, the logic cannot capture all of the interesting specifications a system may have, since it cannot express dependencies between its executions, known as traces. These properties, coined hyperproperties by Clarkson and Schneider in 2010, include properties important for cybersecurity such as noninterference and secure information flow [2]. Due to this background, extensions of LTL have recently been the focus of research.

HyperLTL is one of the most extensively studied of these extensions [1]. Its formulas are interpreted over sets of traces and the syntax extends LTL with quantification on traces. Among the many results for the logic, there are many expressivity results, that relate it to fragments of first, and even second order logic. In particular there is a translation from HyperLTL to $\mathsf{FO}(<, \mathsf{E})$, where E is an equal level predicate [3]. Here the sets of traces T are coded as $T \times \mathbb{N}$ for the domains of the first-order models.

On the other hand, there are alternative approaches to extending LTL to catch hyperproperties. Team semantics is a framework in which one moves on from considering truth through single assignments to regarding teams of assignments as the linchpin for the satisfaction of a formula. Clearly, this framework, when applied to LTL, provides an approach on the hyperproperties. Krebs et al in 2018 introduced two semantics for LTL under team semantics: the synchronous semantics and the asynchronous variant that differ on the interpretation of the temporal operators [5]. The same paper showed a variety of complexity and expressivity results for the two semantics, as well as that the asynchronous semantic has the flatness property, while the synchronous one does not. This article will follow the semantic definitions of that previous work.

In this article several translations between fragments of TeamLTL and FO under team semantics are introduced. Firstly, we define a translation from the asynchronous semantics to FO^3 under team semantics, which relies on the flatness of both logics. Next we develop this translation further, in order to accommodate for extensions of asynchronous TeamLTL which lack flatness, and we translate them to $\mathsf{FO}^3(=(\dots), \sim)$. We further evolve the previous translation to apply to the synchronous semantics, which in turn we translate to $\mathsf{FO}^4(=(\dots), \sim)$.

Preliminaries

Definition 1 (Traces)**.** Let Φ be a set of atomic propositions. A *trace* π over Φ is an infinite sequence $\pi \in (2^\Phi)^\omega$. We denote a trace as $\pi = (\pi(i))_{i=0}^\infty$, and given $j \geq 0$ we denote the suffix of π starting at the jth element $\pi[j,\infty) := (\pi(i))_{i=j}^\infty$.

Definition 2 (Linear Temporal Logic)**.** Formulas of LTL are defined by the grammar

$$\varphi := p \mid \neg p \mid \varphi \wedge \varphi \mid \varphi \vee \varphi \mid X\varphi \mid F\varphi \mid G\varphi \mid \varphi U \varphi \mid \varphi R \psi,$$

where $p \in \Phi$.

Definition 3 (Classical Semantics for LTL)**.** Given a trace π, proposition $p \in \Phi$, and LTL formulas φ and ψ, the semantics of linear temporal logic are as follows.

$$\pi \models p \Leftrightarrow p \in \pi(0)$$
$$\pi \models \neg p \Leftrightarrow p \notin \pi(0)$$
$$\pi \models \varphi \wedge \psi \Leftrightarrow \pi \models \varphi \text{ and } \pi \models \psi$$
$$\pi \models \varphi \vee \psi \Leftrightarrow \pi \models \varphi \text{ or } \pi \models \psi$$
$$\pi \models X\varphi \Leftrightarrow \pi[1,\infty) \models \varphi$$
$$\pi \models F\varphi \Leftrightarrow \exists k \geq 0 : \pi[k,\infty) \models \varphi$$
$$\pi \models G\varphi \Leftrightarrow \forall k \geq 0 : \pi[k,\infty) \models \varphi$$
$$\pi \models \varphi U \psi \Leftrightarrow \exists k \geq 0 : \pi[k,\infty) \models \psi \text{ and } \forall k' < k : \pi[k',\infty) \models \varphi$$
$$\pi \models \varphi R \psi \Leftrightarrow \forall k \geq 0 : \pi[k,\infty) \models \psi \text{ or } \exists k' < k : \pi[k',\infty) \models \varphi$$

The properties captured by LTL are sets of traces, known as trace properties. Hyperproperties are sets of trace properties, which intuitively means that hyperproperties can be captured by logics that are interpreted over sets of traces. HyperLTL is one such logic, however team semantics provides another approach.

A team is a set of traces. We denote $T[i,\infty) := \{t[i,\infty) \mid t \in T\}$. The upcoming definitions are from [5].

Definition 4 (Team Semantics for LTL)**.** Suppose T is a team, $p \in \Phi$ is a proposition, and φ and ψ are $\mathsf{TeamLTL}$ formulae. Then the semantics of $\mathsf{TeamLTL}$ are defined by the following.

$T \models p \Leftrightarrow p \in \pi(0)$ for all $\pi \in T$
$T \models \neg p \Leftrightarrow p \notin \pi(0)$ for all $\pi \in T$
$T \models \varphi \wedge \psi \Leftrightarrow T \models \varphi$ and $T \models \psi$
$T \models \varphi \vee \psi \Leftrightarrow$ there exists $T_1, T_2 \subseteq T$ such that $T_1 \cup T_2 = T$ and $T_1 \models \varphi$ and $T_2 \models \psi$
$T \models X\varphi \Leftrightarrow T[1,\infty) \models \varphi$
$T \models F^s\varphi \Leftrightarrow \exists k \geq 0$ such that $T[k,\infty) \models \varphi$
$T \models F^a\varphi \Leftrightarrow$ for all $\pi \in T$ $\exists k_\pi \geq 0$, such that $\{\pi[k_\pi,\infty) \mid \pi \in T\} \models \varphi$
$T \models G^s\varphi \Leftrightarrow \forall k \geq 0 : T[k,\infty) \models \varphi$
$T \models G^a\varphi \Leftrightarrow$ for all $\pi \in T$ and $\forall k_\pi \geq 0$ $\{\pi[k_\pi,\infty) \mid \pi \in T\} \models \varphi$
$T \models \varphi U^s \psi \Leftrightarrow \exists k \geq 0$ such that $T[k,\infty) \models \psi$ and $\forall k' < k$ such that $T[k',\infty) \models \varphi$
$T \models \varphi U^a \psi \Leftrightarrow \forall \pi \in T\ \exists k_\pi \geq 0 :$ $\{\pi[k_\pi,\infty) \mid \pi \in T\} \models \psi$ and $\forall k'_\pi < k_\pi$ such that $\{\pi[k'_\pi,\infty) \mid \pi \in T\} \models \varphi$
$T \models \varphi R^s \psi \Leftrightarrow \forall k \geq 0$ such that $T[k,\infty) \models \psi$ or $\exists k' < k$ such that $T[k',\infty) \models \varphi$
$T \models \varphi R^a \psi \Leftrightarrow \forall \pi \in T\ \forall k_\pi \geq 0 :$ $\{\pi[k_\pi,\infty) \mid \pi \in T\} \models \psi$ or $\exists k'_\pi < k_\pi$ such that $\{\pi[k'_\pi,\infty) \mid \pi \in T\} \models \varphi$

We denote the asynchronous and the synchronous fragments by $\mathsf{TeamLTL}^{\mathsf{a}}$ and $\mathsf{TeamLTL}^{\mathsf{s}}$, respectively. The two fragments of team $\mathsf{TeamLTL}$ have vastly different expressive power. $\mathsf{TeamLTL}^{\mathsf{a}}$ collapses to a universal fragment of HyperLTL, while $\mathsf{TeamLTL}^{\mathsf{s}}$ is incomparable to HyperLTL. The latter claim is witnessed by the fact that there is no HyperLTL sentence that is equivalent to F^sp [5].

Definition 5 (Flatness Property)**.** A logic L has the *flatness property*, if for all formulae φ of L and teams T it holds that $T \models \varphi$ if and only if $\{t\} \models \varphi$ for all $t \in T$.

A Translation of Asynchronous LTL to FO

Suppose $T = \{\pi_j \mid j \in J\}$ is a team of traces. Define $\mathcal{M}_T$ to be the following structure of vocabulary $\{\leq\} \cup \{P_i \mid p_i \in \Phi\}$ where

$$
\begin{aligned}
&Dom(\mathcal{M}_T) = T \times \mathbb{N} \\
&\leq^{\mathcal{M}_T} = \{((\pi_i, n), (\pi_j, m)) \mid i = j \text{ and } n \leq m\} \\
&P_i^{\mathcal{M}_T} = \{(\pi_k, j) \mid p_i \in \pi_k(j)\}.
\end{aligned}
$$

In addition we define a team $S_T = \{s_i \mid s_i(x) = (\pi_i, 0),\ \text{for all } \pi_i \in T\}$. We notate $\varphi \hookrightarrow \psi := \neg\varphi \vee (\varphi \wedge \psi)$.

Next we define inductively the translations ST_w, where $w \in \{x, y, z\}$, from $\mathsf{TeamLTL^a}$ to FO^3 under team semantics as follows:

$$
\begin{array}{ll}
ST_x(p_i) = P_i(x) & ST_x(G^a\varphi) = \forall y(x \leq y \hookrightarrow ST_y(\varphi)) \\
ST_x(\neg p_i) = \neg P_i(x) & ST_x(F^a\varphi) = \exists y(x \leq y \wedge ST_y(\varphi)) \\
ST_x(\varphi \wedge \psi) = ST_x(\varphi) \wedge ST_x(\psi) & ST_x(\varphi U^a \psi) = \exists y(x \leq y \wedge ST_y(\psi) \wedge \\
ST_x(\varphi \vee \psi) = ST_x(\varphi) \vee ST_x(\psi) & \quad \forall z((x \leq z \wedge z < y) \hookrightarrow ST_z(\varphi))) \\
ST_x(X\varphi) = \exists y(x < y \wedge ST_y(\varphi) \wedge & ST_x(\varphi R^a \psi) = \forall y(x \leq y \hookrightarrow (ST_y(\psi) \vee \\
\quad \forall z \neg(x < z \wedge z < y)) & \quad \exists z(x \leq z \wedge z < y \wedge ST_z(\varphi)))).
\end{array}
$$

Proposition 6. *For all* $\mathsf{TeamLTL^a}$ *formulas* φ, $T \models \varphi \Leftrightarrow \mathcal{M}_T \models_{S_T} ST_x(\varphi)$.

This proposition follows from the fact that both logics are downward closed and flat, and in fact, by the same argument, any translation from LTL to FO is also a translation for the asynchronous semantic.

Translations in the Absence of Flatness

The previous translation makes use of the fact that both $\mathsf{TeamLTL^a}$ and FO have the flatness property. However, flatness does not hold for $\mathsf{TeamLTL^s}$ or extensions of $\mathsf{TeamLTL^a}$. Thus the translation needs to be modified to accommodate for these cases.

Let $\mathcal{M}_T$ and S_T be as previously. We define a translation of $\mathsf{TeamLTL^a}$ formulas to $\mathsf{FO}^3(=(\dots), \sim)$ as follows:

The translation is analogous to the previous translation for the atomic propositions, $\wedge$, $\vee$, and X.

$$
\begin{aligned}
ST^*_x(F^a\varphi) &= \exists y(x \leq y \wedge {=}(x,y) \wedge ST^*_y(\varphi)) \\
ST^*_x(G^a\varphi) &= \sim \exists y(x \leq y \wedge {=}(x,y) \wedge \sim ST^*_y(\varphi)) \\
ST^*_x(\varphi U^a \psi) &= \exists y(x \leq y \wedge {=}(x,y) \wedge ST^*_y(\psi) \wedge \\
&\qquad \sim \exists z(x \leq z \wedge z \leq y \wedge {=}(x,z) \wedge \sim ST^*_z(\varphi))) \\
ST^*_x(\varphi R^a \psi) &= \sim \exists y(x \leq y \wedge {=}(x,y) \wedge \sim ST^*_y(\psi) \wedge \\
&\qquad \exists z(x \leq z \wedge z < y \wedge {=}(x,z) \wedge \sim ST^*_z(\varphi))).
\end{aligned}
$$

Theorem 7. *For all* $\mathsf{TeamLTL^a}$ *formulas* φ, $T \models \varphi \Leftrightarrow \mathcal{M}_T \models_{S_T} ST^*_x(\varphi)$.

This result can now easily be expanded to extensions of $\mathsf{TeamLTL^a}$ which are not flat, by providing a translation for the extending atoms or operators. For instance, the dependence atom satisfies the equivalence ${=}(p, q) \equiv$

$(p \wedge (q \varovee \neg q) \vee (\neg p \wedge (q \varovee \neg q)$, which uses the Boolean disjunction $\varovee$ that can be expressed in $\mathsf{FO}(=(\dots), \sim)$. Thus by using this equivalence we can translate any formula of $\mathsf{TeamLTL}^{\mathsf{a}}(=(\dots))$ to $\mathsf{FO}^3(=(\dots), \sim)$ using the previous translation.

Corollary 8. *For all* $\mathsf{TeamLTL}^{\mathsf{a}}(=(\dots))$ *formulas* φ, $T \models \varphi \Leftrightarrow \mathcal{M}_T \models_{S_T} ST^*_x(\varphi)$.

Translation for Synchronous Team LTL

The synchronous team semantics for LTL does not have the flatness property [5]. Armed with the previous translation, we need to capture the equal level teams on the first-order side. This can be done as for HyperLTL, by introducing an equal level predicate E [3].

Let $\mathcal{M}_T$ and S_T be as above, with the addition of the equal level predicate E together with its negation, both defined in the usual way by $E^{\mathcal{M}_T} = \{((\pi_i, k), (\pi_j, k)) \mid i, j \in J \text{ and } k \in \mathbb{N}\}$. Next we define a translation from $\mathsf{TeamLTL}^{\mathsf{s}}$ to $\mathsf{FO}^4(=(\dots), \sim)$ as follows: The translation is analogous to the previous translations for the atomic propositions, $\wedge$, $\vee$, and X.

$$
\begin{aligned}
ST^*_x(F^s\varphi) &= \exists y(=(y) \wedge \exists z(E(y,z) \wedge x \le z \wedge ST^*_z(\varphi))) \\
ST^*_x(G^s\varphi) &= \sim \exists y(=(y) \wedge \exists z(E(y,z) \wedge x \le z \wedge \sim ST^*_z(\varphi))) \\
ST^*_x(\varphi U^s \psi) &= \exists y(=(y) \wedge \exists z(E(y,z) \wedge x \le z \wedge ST^*_z(\psi) \wedge \\
&\quad \sim \exists y(=(y) \wedge \exists w(E(y,w) \wedge x \le w \wedge w \le z \wedge ST^*_w(\varphi))))) \\
ST^*_x(\varphi R^s \psi) &= \sim \exists y(=(y) \wedge \exists z(E(y,z) \wedge x \le z \wedge \sim ST^*_z(\psi) \wedge \\
&\quad \exists y(=(y) \wedge \exists w(E(y,w) \wedge x \le w \wedge w < z \wedge \sim ST^*_w(\varphi))))).
\end{aligned}
$$

Theorem 9. *For all* $\mathsf{TeamLTL}^{\mathsf{s}}$ *formulas* φ, $T \models \varphi \Leftrightarrow \mathcal{M}_T \models_{S_T} ST^*_x(\varphi)$.

In future research the translations presented in this article can be used to further study the expressivity and complexity of $\mathsf{TeamLTL}$.

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On the Presburger Fragment of Logics with Multiteam Semantics

Richard Wilke

Modern logics for arguing about dependence and independence are based on team semantics [Vää07]. From a purely logical point of view these logics have clean theoretical properties, as for example inclusion and exclusion logic corresponds to independence logic which again is equivalent to existential second-order logic. However, in these logics data is represented as teams which are *sets of assignments*, hence one can only argue about the *presence or absence* of data. As in many real-world applications the *multiplicities* are a key factor (e.g. in databases) different logics that incorporate such information have been proposed [HPV15, HPV17, DHK+18]. In this article we consider *multisets of assignments*, called *multiteams*, which extend teams by the number of occurrences of each assignment. Notions such as independence in this setting only make sense if the multiplicities are natural numbers, hence we consider only finite multiteams and structures.

Logics with team semantics without negation are embeddable in existential second-order logic Σ^1_1, similarly logics with multiteam semantics can be embedded into the second-order logic $\mathrm{ESO}^{\mathsf{mts}}[+,\cdot]$ with built-in features for dealing with arithmetic. Formally, all structures are extended by a numerical sort and second-order quantifiers over functions $f : A^k \longrightarrow \mathbb{N}$ mapping tuples of elements of the universe A of a structure $\mathfrak{A}$ to natural numbers are added to first-order logic. Additionally, basic arithmetic $+$ and $\cdot$ is available allowing terms of the kind $f\bar{x} + g\bar{y}$. This note intends to discuss the Presburger fragment $\mathrm{ESO}^{\mathsf{mts}}[+]$ of this logic, i.e. the restriction where only addition is allowed, but no multiplication. As it has turned out on the level of multiteam semantics this logic is equivalent to $\mathrm{FO}^{\mathsf{M}}[\Subset, \mid\,]$, that is multiteam inclusion / exclusion logic. The focus of the present work is on multiteam logics, and due to the space limitations we will not investigate the second-order logic $\mathrm{ESO}^{\mathsf{mts}}[+]$; for the same reason most proofs are omitted, sometimes when we translate a logic into another we present the formula that expresses an atom of one logics in the other, but do not argue for its correctness. We aim at discussing the logic $\mathrm{FO}^{\mathsf{M}}[\Subset, \mid\,]$ in more detail which includes finding an atom α such that $\mathrm{FO}^{\mathsf{M}}[\Subset, \mid\,] \equiv \mathrm{FO}^{\mathsf{M}}[\alpha]$. In team semantics independence logic is equivalent to inclusion / exclusion logic, which – as we will discover – is not the case under multiteam semantics, but multiteam independence can still express both multiteam inclusion and exclusion. It turns out that in multiteam logics with a *forking* atom $\prec_{=1/2}$ both inclusion and exclusion are expressible. This leads to a further analysis of the different variants of forking $\prec_{\leq p}$ and $\prec_{\geq p}$ for some $p \in [0, 1]$.

§1 Multiteam Semantics

A multiset $M = (S, n)$ is a tuple of a (finite) set S together with a function $n : S \longrightarrow \mathbb{N}_{>0}$ assigning every element its *multiplicity*. We write $|M|$ for the size of M, that is $\sum_{x\in S} n(x)$. The additive union of two multisets $(S, n) \uplus (S', n')$ is $(S \cup S', n + n')$, where $n + n'(x) := n(x) + n'(x)$ with the convention that $n(x) = 0$ in case $x \notin S$ and analogously for n'. Inclusion $(S, n) \Subset (S', n')$ means that $n(s) \leq n'(s)$ holds for all $s \in S$. For a number k the multiple kM is $\biguplus_{i<k} M$. A *multiteam* M is a multiset (X, n) such that X is a team. We fix some notation. The *support*, or *underlying team*, of M is $M^{\mathsf{T}} := X$; the evaluation of M on a tuple $\bar{x}$, written $M(\bar{x})$, is the multiset $\{\!\{ s(\bar{x}) : s \in M \}\!\}$, where $\{\!\{ \lambda(s) : s \in M \}\!\}$ is a notation for $\biguplus_{s\in M^{\mathsf{T}}} n(s)\{\!\{ \lambda(s) \}\!\}$; the *restriction* $M{\restriction}_\rho$ is $\{\!\{ s \in M : s \models \rho \}\!\}$; the *probability* $\Pr_M(\bar{x} = \bar{a})$ that the variable

$\bar{x}$ takes value $\bar{a}$ in M is defined as $|M\upharpoonright_{\bar{x}=\bar{a}}|/|M|$, and the conditional probability $\Pr_M(\bar{x} = \bar{a} \mid \bar{y} = \bar{b})$ is defined similarly. Moreover, if ζ is a first-order formula we write $\zeta \longrightarrow \xi$ as a shorthand for $\neg\zeta \vee (\zeta \wedge \xi)$.

The dependency concepts known from team semantics or database theory can be understood in a natural way under multiteam semantics. Further, the access to multiplicities gives rise to additional notions. The following lists the most important ones that are considered throughout this abstract.

Definition 1 (Multidependence Atoms)**.** Let $\mathfrak{A}$ be a finite structure and M a multiteam.

Dependence: $\mathfrak{A} \vDash_M \operatorname{dep}(\bar{x}, y) :\Longleftrightarrow \mathfrak{A} \vDash_{M^{\mathsf{T}}} \operatorname{dep}(\bar{x}, y)$
Exclusion: $\mathfrak{A} \vDash_M \bar{x} \mid \bar{y} :\Longleftrightarrow \mathfrak{A} \vDash_{M^{\mathsf{T}}} \bar{x} \mid \bar{y}$
Inclusion: $\mathfrak{A} \vDash_M \bar{x} \Subset \bar{y} :\Longleftrightarrow M(\bar{x}) \Subset M(\bar{y})$
Statistical independence: $\mathfrak{A} \vDash_M \bar{x} \perp\!\!\!\perp \bar{y}$ holds if, and only if, $\Pr_M(\bar{x} = \bar{a}) = \Pr_M(\bar{x} = \bar{a} \mid \bar{y} = \bar{b})$ for all $\bar{a} \in M(\bar{x})$ and $\bar{b} \in M(\bar{y})$. An equivalent condition is that $M(\bar{x}) \times M(\bar{y}) = |M| \cdot M(\bar{x}\bar{y})$.
Conditional independence: $\mathfrak{A} \vDash_M \bar{x} \perp\!\!\!\perp_{\bar{z}} \bar{y}$ if $\Pr_M(\bar{x} = \bar{a} \mid \bar{z} = \bar{c}) = \Pr_M(\bar{x} = \bar{a} \mid \bar{y}\bar{z} = \bar{b}\bar{c})$ for all $\bar{a} \in A^{|\bar{x}|}$, $\bar{b} \in A^{|\bar{y}|}$ and $\bar{c} \in A^{|\bar{z}|}$.

$\lhd$

First-order operators can be defined as either being *strict*, i.e. using each assignment exactly once, or *lax*. In team semantics lax operators turned out to be the correct choice, which intuitively is based on the fact that only the information whether or not an assignment is present is available in a team. The situation is different under multiteam semantics since the multiplicities are accessible and an analysis has shown that indeed strict semantics should be assumed [GW]. For a set of multiteam dependency notions Ω, its closure under first-order operators is denoted by $\mathrm{FO}^{\mathsf{M}}[\Omega]$.

Definition 2 (Multiteam Semantics)**.** Multiteam semantics is defined by the following rules. Let Ω be a set of multidependency atoms, $\mathfrak{A}$ a structure, M a multiteam over A and $\psi, \psi_1, \psi_2 \in \mathrm{FO}^{\mathsf{M}}[\Omega]$.

- $\mathfrak{A} \vDash_M \psi_1 \wedge \psi_2$ if $\mathfrak{A} \vDash_M \psi_1$ and $\mathfrak{A} \vDash_M \psi_2$;
- $\mathfrak{A} \vDash_M \psi_1 \vee \psi_2$ if there are $M_1 \uplus M_2 = M$ with $\mathfrak{A} \vDash_{M_i} \psi_i$;
- $\mathfrak{A} \vDash_M \forall x \psi$ if $\mathfrak{A} \vDash_{M[x \mapsto A]} \psi$;
- $\mathfrak{A} \vDash_M \exists x \psi$ if $\mathfrak{A} \vDash_{M[x \mapsto F]} \psi$ for some function $F : M \longrightarrow A$.

Where $M[x \mapsto A] = \{\!\{ s[x \mapsto a] : s \in M, a \in A \}\!\}$, i.e. every assignment in M is updated with every value of A, thus $|M[x \mapsto A]| = |A| \cdot |M|$. The function F maps every assignment $s \in M$ to a value $F(s) \in A$. If an assignment s is present more than once in M each copy may or may not receive a different value from F. Accordingly $M[x \mapsto F]$ denotes $\{\!\{ s[x \mapsto F(s)] : s \in M \}\!\}$, especially $|M[x \mapsto F]| = |M|$. $\lhd$

Downwards- and union closure are defined analogously to team semantics, i.e. φ is downwards closed if $\mathfrak{A} \vDash_M \varphi$ implies $\mathfrak{A} \vDash_R \varphi$ for all $R \Subset M$ and ψ is union closed in case $\mathfrak{A} \vDash_M \psi$ and $\mathfrak{A} \vDash_R \psi$ implies $\mathfrak{A} \vDash_{M \uplus R} \psi$. To avoid confusion between team and multiteam semantics we write FO^{T} for first-order team logic and accordingly FO^{M} for first-order multiteam logic.

§2 Between Inclusion, Exclusion and Independence

Let us start by repeating the picture in team semantics. Independence logic $\mathrm{FO}^{\mathsf{T}}[\perp]$ and conditional independence logic $\mathrm{FO}^{\mathsf{T}}[\perp_{\mathsf{c}}]$ coincide, as was shown by Galliani [Gal12]. The proof provides translations of exclusion and inclusion atoms into independence logic and a formula that expresses conditional independence by means of inclusion / exclusion, i.e. $\mathrm{FO}^{\mathsf{T}}[\perp_{\mathsf{c}}] \preceq \mathrm{FO}^{\mathsf{T}}[\subseteq, \mid] \preceq \mathrm{FO}^{\mathsf{T}}[\perp]$ and hence $\mathrm{FO}^{\mathsf{T}}[\perp] \equiv \mathrm{FO}^{\mathsf{T}}[\perp_{\mathsf{c}}] \equiv \mathrm{FO}^{\mathsf{T}}[\subseteq, \mid]$. We observe that instead of going through this chain of translations, conditional independence can be defined by using just a single independence atom in team semantics.

Example 3. The formula $\varphi_{\perp_c}(\bar{x}, \bar{y}, \bar{z}) \in \mathrm{FO}^{\mathsf{T}}[\perp]$ is equivalent to $\bar{x} \perp_{\bar{z}} \bar{y}$, where

$$\varphi_{\perp_c}(\bar{x}, \bar{y}, \bar{z}) := \forall \bar{p} \exists \bar{u} \exists \bar{w} \big((\bar{z} = \bar{p} \longrightarrow \bar{u}\bar{w} = \bar{x}\bar{y}) \wedge (\bar{z} \neq \bar{p} \vee \bar{z}\bar{u} \perp \bar{p}\bar{w}) \big).$$

Intuitively this formula builds from a given team X an extension Y such that $Y\restriction_{\bar{p}=\bar{z}=\bar{a}}(\bar{u}, \bar{w}) = X\restriction_{\bar{z}=\bar{a}}(\bar{x}, \bar{y})$. Further, no restriction on Y is imposed whenever $\bar{p}$ and $\bar{z}$ differ, hence all possible combinations may be present which implies that $Y \models \bar{z}\bar{u} \perp \bar{p}\bar{w}$ holds if and only if $X \models \bar{x} \perp_{\bar{z}} \bar{y}$. $\diamond$

A similar technique however fails under multiteam semantics. Nevertheless for the special case of dependence $\mathrm{dep}(\bar{x}, y)$, which is equivalent to $y \perp_{\bar{x}} y$ and $y \perp\!\!\!\perp_{\bar{x}} y$, this idea is applicable for multiteams.

Proposition 4. $\mathrm{FO}^{\mathsf{M}}[\mathrm{dep}] \leq \mathrm{FO}^{\mathsf{M}}[\perp\!\!\!\perp]$.

Proof. As already stated $\mathrm{dep}(\bar{x}, y) \equiv y \perp\!\!\!\perp_{\bar{x}} y$ which we claim to be equivalent to

$$\psi := \forall \bar{p} \exists u \big((\bar{x} = \bar{p} \longrightarrow u = y) \wedge \bar{x} y \perp\!\!\!\perp \bar{p} u \big).$$

Assume $\mathfrak{A} \not\models \mathrm{dep}(\bar{x}, y)$. Thus there are $s, s' \in M^{\mathsf{T}}$ with $s(\bar{x}) = s'(\bar{x}) = \bar{a}$ but $b = s(y) \neq s'(y) = c$. Towards a contradiction assert that $\mathfrak{A} \models_M \psi$. Let R be the multiteam $M[\bar{p} \mapsto A^k][u \mapsto F]$ for an appropriate F such that $R \models (\bar{x} = \bar{p} \longrightarrow u = y)$. Observe that $0 = \Pr_R(\bar{x}y = \bar{a}b \mid \bar{p}u = \bar{a}c) < \Pr_R(\bar{x}y = \bar{a}b)$.

On the other hand assume $\mathfrak{A} \models_M \mathrm{dep}(\bar{x}, y)$, i.e. $f : M^{\mathsf{T}}(\bar{x}) \longrightarrow A$ exists such that for all assignments $s \in M$ holds $s(y) = f(s(\bar{x}))$. We describe how the values for u can be chosen such that we witness $\mathfrak{A} \models_M \psi$. The choice is clear for all assignments in which $\bar{x}$ and $\bar{p}$ agree. If $s(\bar{x}) \neq s(\bar{p})$ we put $s(u) = f(s(\bar{p}))$, in case $s(\bar{p})$ is a value occurring in $M(\bar{x})$ and $s(u) = c$ for an arbitrary fixed value $c \in A$ otherwise. Let S be the resulting multiteam. Certainly $\Pr_S(\bar{x}y = \bar{a}b) = \Pr_S(\bar{x}y = \bar{a}b \mid \bar{p}u = \bar{c}d)$ for all $\bar{a}, \bar{c}, b, d \in A^*$. $\square$

In our favour we can prove that multiteam inclusion is expressible in statistical independence logic by a slight modification of the formula that defines inclusion via independence in team semantics [Gal12].

Proposition 5. $\mathrm{FO}^{\mathsf{M}}[\Subset] \leq \mathrm{FO}^{\mathsf{M}}[\perp\!\!\!\perp]$.

Proof. Without going into detail we claim that $\bar{x} \Subset \bar{y}$ is equivalent to the formula $\varphi_{\Subset}(\bar{x}, \bar{y})$, where $\varphi_{\Subset}(\bar{x}, \bar{y}) := \forall a, b, \bar{z} \big((\bar{z} \neq \bar{x} \wedge \bar{z} \neq \bar{y}) \vee (\bar{z} \neq \bar{y} \wedge a \neq b) \vee (\bar{z} = \bar{y} \wedge a = b) \vee ((\bar{z} = \bar{y} \vee a = b)) \wedge \bar{z} \perp\!\!\!\perp ab \big)$. $\square$

Hence, similarly to team semantics multiteam independence can express dependence (and hence exclusion) and inclusion. However, multiteam inclusion and exclusion logic is less expressive than statistical independence logic as is demonstrated by the upcoming theorem that we state without proof (cf. [GW]). In fact we find that no combination of downwards closed and union closed atomic dependency notions is able to express independence under multiteam semantics.

Theorem 6. Let $\bar{\alpha}$ be any collection of downwards closed atoms and $\bar{\beta}$ be any collection of union closed atoms. There is no formula $\psi \in \mathrm{FO}^{\mathsf{M}}[\bar{\alpha}, \bar{\beta}]$ with $x \perp\!\!\!\perp y \equiv \psi(x, y)$.

This leaves open two questions, first: Can statistical independence logic $\mathrm{FO}^{\mathsf{M}}[\perp\!\!\!\perp]$ express conditional independence $\bar{x} \perp\!\!\!\perp_{\bar{z}} \bar{y}$? As the previous statements demonstrate the methods applicable in team semantics that show $\mathrm{FO}^{\mathsf{T}}[\perp] \equiv \mathrm{FO}^{\mathsf{T}}[\perp_c]$ do not translate to multiteam semantics. The second issue is whether there is a (natural) atomic formula α such that $\mathrm{FO}^{\mathsf{M}}[\Subset, \mid] \equiv \mathrm{FO}^{\mathsf{M}}[\alpha]$? We leave open the first question and give a positive answer to the second question in the following paragraph.

§3 Forking / Anonymity

We now turn our attention to atoms that can count assignments by their *forking degree*, i.e. such an atom may state that depending on a variable the values of another one all occur with probability at least (at most) a given threshold. Grädel and Hegselmann [GH16] investigated the notion of forking in context of team semantics. To do so they augmented each structure by a number sort which enabled writing formulae such as $x\not<^{\leq\lambda}y$ which states that for each value for x at most λ different values for y occur, where λ is a variable over the number sort. Since handling natural numbers is a built-in feature of multiteam semantics we do not need to consider two sorted structures in order to define a meaningful concept. For technical reasons we assume all structures to contain at least two elements in the following.

Definition 7 (Forking)**.** Let M be a multiteam over some finite set A and $p \in [0,1]$. For $\lhd \in \{\leq, =, \geq\}$ the *forking atom* $\not<_{\lhd p}$ is defined via $M \models \bar{x}\not<_{\lhd p}\bar{y}$, if $\Pr_M(\bar{y} = \bar{b} \mid \bar{x} = \bar{a}) \lhd p$ for all $\bar{a}, \bar{b} \in A^*$ whenever $\Pr_M(\bar{y} = \bar{b} \mid \bar{x} = \bar{a}) > 0$. ⊿

The forking atom $\not<_{\leq 1/2}$ resembles a multiteam version of the anonymity atom that was introduced by Väänänen [GKKV19]. It states that the values a certain variable takes do not suffice to determine the value of another. More formally, $x\Upsilon y$ is satisfied in a team X whenever for every value a that x takes in X there are (at least) two assignments s and s' such that $s(x) = s'(x) = a$ but $s(y) \neq s'(y)$. This atom is in fact equivalent to non-dependence [Gal15]. In multiteam semantics we may further impose the degree p of anonymity in $\not<_{\leq p}$ giving us a natural atom defining the concept of anonymity.

Let us start the analysis of the forking atoms by examining the closure properties of the different forking variants.

Proposition 8. Let $p \in (0,1)$, $q \in (0,1/2]$ and $r \in \{\frac{1}{n} \ : \ n > 1\}$. $\not<_{\leq p}$ is union- but not downwards closed, while $\not<_{\geq q}$ and $\not<_{=r}$ are neither union-, nor downwards closed.

There are two conspicuousnesses of this proposition. First, the thresholds for which the statements hold exclude certain cases. For some of these values the forking atoms trivialise; indeed we observe that $\bar{x}\not<_{\leq 0}y \equiv \bar{x}\not<_{=0}y \equiv$ false, $\bar{x}\not<_{\leq 1}y \equiv \bar{x}\not<_{\geq 0}y \equiv$ true and furthermore $\bar{x}\not<_{=p}\bar{y} \equiv$ false for all $p \neq 1/n$. The remaining atoms, i.e. $\bar{x}\not<_{=1}\bar{y}$ and $\bar{x}\not<_{\geq p}\bar{y}$ for $p > 1/2$, all coincide with the dependence atom $\mathrm{dep}(\bar{x};\bar{y})$.

This explains the choice of the thresholds. Secondly, one might expect a symmetry between $\not<_{\leq p}$ and $\not<_{\geq p}$ like, for example, one being union closed and the other downwards closed. While this is not true we find that $\not<_{\geq p}$ is in fact *weakly downwards closed* for all $p \in [0,1]$.

Definition 9. A formula φ is *weakly downwards closed*, or *downwards closed in the team semantical sense*, if $\mathfrak{A} \models_{(X,n)} \varphi$ implies $\mathfrak{A} \models_{(Y,m)} \varphi$ for all $(Y,m) \Subset (X,n)$ such that $n(s) = m(s)$ for all $s \in Y$. ⊿

Since the other forking atoms are not weakly downwards closed we obtain the following relationship.

Corollary 10. The logics $\mathrm{FO}^{\mathsf{M}}[\not<_{\leq p}]$ and $\mathrm{FO}^{\mathsf{M}}[\not<_{\geq q}]$ are incomparable for all $p \in (0,1)$ and $q \in (0,1]$.

Let us continue our analysis by comparing forking logics to more well known logics with multiteam semantics. Because of the severe space limitations and since the formulae arising in the upcoming proofs are too long and difficult to parse we state the relationships without presenting even the formulae used in the translations. However, we hope that the closure properties provide enough intuition for the reader to believe the statements.

Theorem 11. (1) $\mathrm{FO}^{\mathsf{M}}[\mathrm{dep}] \preceq \mathrm{FO}^{\mathsf{M}}[\not<_{\geq 1/2}], \mathrm{FO}^{\mathsf{M}}[\not<_{=1/2}]$.
(2) $\mathrm{FO}^{\mathsf{M}}[\Subset] \preceq \mathrm{FO}^{\mathsf{M}}[\not<_{\leq 1/2}], \mathrm{FO}^{\mathsf{M}}[\not<_{=1/2}]$.
(3) $\mathrm{FO}^{\mathsf{M}}[\not<_{\leq 1/2}] \preceq \mathrm{FO}^{\mathsf{M}}[\Subset]$.

Corollary 12. $\mathrm{FO}^{\mathsf{M}}[\not<_{\leq 1/2}] \equiv \mathrm{FO}^{\mathsf{M}}[\Subset]$.

This enables us to identify $\not\leq_{=1/2}$ as the atom equivalent to inclusion / exclusion in multiteam semantics.

Theorem 13. $\mathrm{FO}^{\mathsf{M}}[\not\leq_{\geq 1/2}] \lneq \mathrm{FO}^{\mathsf{M}}[\mathrm{dep}, \Subset] \equiv \mathrm{FO}^{\mathsf{M}}[\not\leq_{=1/2}]$.

Proof. By Theorem 11 we may use $\not\leq_{\leq 1/2}$ as it is available in $\mathrm{FO}^{\mathsf{M}}[\Subset]$. Then $\bar{x} \not\leq_{=1/2} \bar{y} \equiv (\mathrm{dep}(\bar{x}, \bar{y}) \vee \mathrm{dep}(\bar{x}, \bar{y})) \wedge \bar{x} \not\leq_{\leq 1/2} \bar{y}$.

$\bar{x} \not\leq_{\geq 1/2} \bar{y} \equiv \mathrm{dep}(\bar{x}, \bar{y}) \vee_{\bar{x}} \bar{x} \not\leq_{=1/2} \bar{y}$, where $\mathfrak{A} \models_M \varphi \vee_{\bar{x}} \psi \iff$ there are $R \uplus S = M$ such that $\mathfrak{A} \models_R \varphi$, $\mathfrak{A} \models_S \psi$ and for all $s, s' \in M$ if $s(\bar{x}) = s'(\bar{x})$ then both s and s' belong either to R or to S. It is easy to define this kind of disjunction using dependence atoms. $\square$

Notice that $\mathrm{FO}^{\mathsf{M}}[\not\leq_{=1/2}] \equiv \mathrm{FO}^{\mathsf{M}}[\not\leq_{\leq 1/2}, \not\leq_{\geq 1/2}]$ follows as a corollary. Let us end this section by demonstrating that using $\not\leq_{=1/2}$ one can express $\not\leq_{=1/n}$ for all $n \in \mathbb{N}_{>0}$.

Proposition 14. $\mathrm{FO}^{\mathsf{M}}[\not\leq_{=1/n}] \leq \mathrm{FO}^{\mathsf{M}}[\not\leq_{=1/2}]$ for all $n \in \mathbb{N}_{>0}$.

Proof. The special case $n = 1$ was handled in Theorem 11 which also allows us to make use of dependence atoms. Let $n > 1$ (of course $n = 2$ is trivial but also covered by the upcoming construction). We claim that $\bar{x} \not\leq_{=1/n} \bar{y}$ is equivalent to the formula η:

$$\exists \bar{y}_1 \cdots \exists \bar{y}_n (\bigwedge_{i<n} \mathrm{dep}(\bar{x}, \bar{y}_i) \wedge \bigwedge_{i \neq j} \bar{y}_i \neq \bar{y}_j \wedge (\bigvee_{i<n} \bar{y} = \bar{y}_i) \wedge \bigwedge_{i \neq j} (\bar{y} = \bar{y}_i \vee \bar{y} = \bar{y}_j \longrightarrow \bar{x} \not\leq_{=1/2} \bar{y})).$$

Before we start the analysis, notice that the formula η is $\bar{x}$-guarded, that is $\mathfrak{A} \models_M \eta$ holds if, and only if, $\mathfrak{A} \models_{M\restriction_{\bar{x}=\bar{a}}} \eta$ for all $\bar{a} \in M^T(\bar{x})$. In fact, instead of η one may consider its unguarded version, that is the formula η, where $\mathrm{dep}(\bar{x}, \bar{y}_i)$ is replaced by $\mathrm{dep}(\bar{y}_i)$ and $\bar{x} \not\leq_{=1/2} \bar{y}$ by $\not\leq_{=1/2} \bar{y}$. Since $\bar{x} \not\leq_{=1/n} \bar{y}$ is also $\bar{x}$-guarded we will for the sake of simplicity in the following consider $\not\leq_{=1/n} \bar{y}$ and multiteams M with domain $\bar{y}$. Assume $\mathfrak{A} \models_M \not\leq_{=1/n} \bar{y}$. Thus, $\Pr_M(\bar{y} = \bar{b}) = 1/n$ for each $\bar{b} \in M^T(\bar{y})$, implying that $|M^T(\bar{y})| = n$. Let us write this set as $\{\bar{b}_1, \dots, \bar{b}_n\}$. To show the claim $\mathfrak{A} \models_M \eta$, let M' be the extension of M by values for $\bar{y}_1, \dots, \bar{y}_n$ such that for all $s \in M^T$ holds $s(\bar{y}_i) = \bar{b}_i$. By construction $\mathfrak{A} \models_{M'} \bigwedge_{i<n} \mathrm{dep}(\bar{y}_i) \wedge \bigwedge_{i \neq j} \bar{y}_i \neq \bar{y}_j \wedge (\bigvee_{i<n} \bar{y} = \bar{y}_i)$. Hence it remains to verify $\mathfrak{A} \models_{M'} \bigwedge_{i \neq j} (\bar{y} = \bar{y}_i \vee \bar{y} = \bar{y}_j \longrightarrow \not\leq_{=1/2} \bar{y})$, which is the case if for all $i \neq j$ holds $\Pr_R(\bar{y} = \bar{b}_i) = \Pr_R(\bar{y} = \bar{b}_j) = 1/2$ where $R = M'\restriction_{\bar{y} \in \{\bar{b}_i, \bar{b}_j\}}$. This is equivalent to $|M'\restriction_{\bar{y}=\bar{b}_i}| = |M'\restriction_{\bar{y}=\bar{b}_j}|$. By assumption the probability that $\bar{y}$ takes any value equals $1/n$. Thus all values for $\bar{y}$ must be equally distributed whence $|M'\restriction_{\bar{y}=\bar{b}_i}| = |M'\restriction_{\bar{y}=\bar{b}_j}|$ and hence $\mathfrak{A} \models_M \eta$ follows.

Conversely let $\mathfrak{A} \models_M \eta$. Thus there is an extension M' of M by (constant) values $\bar{b}_1, \dots, \bar{b}_n$ for $\bar{y}_1$ through $\bar{y}_n$ such that $M' \models \bigwedge_{i<n} \mathrm{dep}(\bar{y}_i) \wedge \bigwedge_{i \neq j} \bar{y}_i \neq \bar{y}_j \wedge (\bigvee_{i<n} \bar{y} = \bar{y}_i)$. Hence $|M^T(\bar{y})| \leq n$. Moreover $M' \models \bigwedge_{i \neq j} (\bar{y} = \bar{y}_i \vee \bar{y} = \bar{y}_j \longrightarrow \not\leq_{=1/2} \bar{y})$, implying that for all $i \neq j$ we have $|M\restriction_{\bar{y}=\bar{b}_i}| = |M\restriction_{\bar{y}=\bar{b}_j}|$. Therefore $|M^T(\bar{y})| = n$ (if there are less than n values one of these multisets is empty and hence not equivalent to another non empty one, which must exist). Since M contains the same amount of assignments that map $\bar{y}$ to $\bar{b}_i$ as those that map $\bar{y}$ to $\bar{b}_j$ we conclude that $\Pr_M(\bar{y} = \bar{b}_i) = 1/n$ for $i \in \{1, \dots, n\}$. $\square$

§4 Summary

Figure 1 displays the relationships of the various logics considered in this note and shows the corresponding relations for logics with team semantics.

$$
\begin{tikzcd}[column sep=small]
 & & & & \mathrm{ESO}^{\mathsf{mts}}[+,\cdot] & & \\
 & & & & \mathrm{FO}^{\mathsf{M}}[\perp\!\!\!\perp_{\mathsf{c}}] \arrow[u] & & \\
 & & & & \mathrm{FO}^{\mathsf{M}}[\perp\!\!\!\perp] \arrow[u, "?"'] & & \\
 & \Sigma^1_1 \arrow[d, leftrightarrow] & & & \mathrm{FO}^{\mathsf{M}}[\nless_{\le 1/2}, \nless_{\ge 1/2}] \equiv \mathrm{FO}^{\mathsf{M}}[\nless_{=1/2}] \equiv \mathrm{FO}^{\mathsf{M}}[\Subset, \mid\,] \arrow[u] \arrow[rr, leftrightarrow] & & \mathrm{ESO}^{\mathsf{mts}}[+] \\
 & \mathrm{FO}^{\mathsf{T}}[\perp] \equiv \mathrm{FO}^{\mathsf{T}}[\subseteq, \mid\,] & & \mathrm{FO}^{\mathsf{M}}[\nless_{\ge 1/2}] \arrow[ur] & & & \\
 & & & & \mathrm{FO}^{\mathsf{M}}[\nless_{=1/n}] \arrow[uu] & & \\
 \mathrm{FO}^{\mathsf{T}}[\subseteq] \arrow[uur] & & \mathrm{FO}^{\mathsf{T}}[\mid\,] \equiv \mathrm{FO}^{\mathsf{M}}[\mathrm{dep}] \arrow[uul] \arrow[uur] & & & \mathrm{FO}^{\mathsf{M}}[\Subset] \equiv \mathrm{FO}^{\mathsf{M}}[\nless_{\le 1/2}] \arrow[uuul] & \\
 & & \mathrm{FO}^{\mathsf{T}} \equiv \mathrm{FO}^{\mathsf{M}} \arrow[u] \arrow[ull, bend left] \arrow[uurr, bend right] \arrow[urrr, bend right] & & & &
\end{tikzcd}
$$

Figure 1: An arrow $L \longrightarrow R$ means $L \lneq R$ and $L \leftrightarrow R$ stands for $L \equiv R$. The precise relationship between statistical independence and conditional independence logic remains open.

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