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BRIEF PAPER

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Stabilization of switched neural networks with time-varying delay via bumpless transfer control

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Abstract

This paper investigates the stabilization of switched neural networks with time-varying delay. In order to overcome the drawback that the classical switching state feedback controller may generate the bumps at switching time, a new switching feedback controller which can smooth effectively the bumps is proposed. According to mode-dependent average dwell time, new exponential stabilization results are deduced for switched neural networks under the proposed feedback controller. Based on a simple corollary, the procedures which are used to calculate the feedback control gain matrices are also obtained. Two simple numerical examples are employed to demonstrate the effectiveness of the proposed results.

KEYWORDS
bumpless transfer, delay, stabilization, switched neural network

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1 | INTRODUCTION

1.1 | Background and research status

Neural networks, which are used to solve many practical problems and show perfectly intelligent features, have been applied to many fields such as pattern recognition, intelligent robots, automatic control, predictive estimation, biology, medicine, economics and so on. Their superiorities include self-learning function, associative storage function and the excellent ability to find the optimal solution rapidly. As we have seen, in the real applications, neural networks are usually required to have the desired dynamic behavior. This implies that the dynamic characteristic of neural networks is a crucial research focus. Stability, which represents the performance that the initial deviation state restores to the orginal equilibrium state after the vanishing of disturbance, is the basic structural characteristic of dynamical systems. Generally speaking, unstable systems does not have regulating ability and are not available for applications. For neural networks, stability is also the basic requirement to guarantee the normal work of neural network circuits. Therefore, stability is the most essential and significant issue in the analysis and design of neural networks.

Up to now, the stability of neural networks has been extensively studied and many novel stability results for neural networks with or without time delay have been proposed via many innovative approaches and effective tools. For example, based on the dynamic delay interval method, the asymptotical stability results of neural networks with two delay components are proposed in [1]. According to the relaxed Lyapunov-Krasovskii functional, less conservative stability criteria for neural networks with distributed delay are presented in [2]. By developing a new integral inequality, the delay-dependent stability criteria in terms of linear matrix inequalities (LMIs) for neural networks with time-varying delay are derived in [3]. For unstable neural networks, we should employ some effective control strategies to stabilize them, which is called as the stabilization problem. As a typical control method, feedback control, whose control input is persistent, is widely used for the stabilization of unstable neural networks. For instance, by using Wirtinger-type integral inequalities, the stabilization of neural networks time-varying delay are investigated in [4,5]. Due to quadratic linear combination and double-integral inequality, Z Wang et al. design state-dependent switching control law to realize the stabilization of delayed memristive neural networks [6]. On the basis of Lyapunov functional method, the stabilization results for memristive neural networks with time-varying delay are presented in [7].

Switched neural networks are a special category of switched systems which include several alternative neural networks and a switching signal produced by switching device. In contrast to ordinary neural networks, switched neural networks may switch from one mode to another one at switching time under switching signal. Some undesired dynamic behaviors may be generated by the switching, even if the dynamic behaviors of all alternative neural networks satisfy the practical engineering requirements. This indicates that the investigation of switched neural networks is more complicated because both switching signal and subsystems must be concerned. Many effective research methods of switched systems, such as integral-type multiple Lyapunov functions [8], average dwell time [9], variational method [10], discretized multiple Lyapunov-Krasovskii functional [11], are also valid for switched neural networks. Up to now, the stabilization problem for switched neural networks by feedback control technique has been well discussed. In [12], the delay-independent and delay-dependent mean square exponential stabilization results for stochastic neural networks with Markovian switching are proposed. The H_∞ controller is designed for uncertain switched neural networks [13]. In [14], the researchers present a memoryless state feedback controller to stabilize stochastic Cohen-Grossbery neural networks with mode-dependent mixed time delay and Markovian switching. The state feedback controller is designed in [15] to realize the finite-time stabilization of uncertain switched neural networks with time-varying delay. The robust finite-time H_∞ control problem is solved in [16] for uncertain neural-type switched neural networks with distributed delay.

The multi-controller is widely employed to stabilize unstable switched systems. As we know, the switching among different controllers may generate the transient which causes the resonance effect that is harmful and even dangerous in some circumstances [17]. For example, in aero-engine control systems, the oscillation of control input may directly destroy the attitude stability of aircraft. Therefore, the smooth transition of control signal, which is called as bumpless transfer, must be considered in many real applications. A bumpless transfer controller for discrete-time linear switched systems is presented in [18,19] by distributing the bumps over some samples. CS Cheong deduces a bumpless transfer technique for adaptive switching control [20]. The dynamic bumpless transfer compensator design is proposed in [21] for uncertain linear switched systems. The anti-windup bumpless transfer control structure is presented in [22] for smooth switching control. The asynchronous bumpless transfer, which is divided into robust performance bumpless transfer and robust control bumpless transfer, is dealt with in [23] for linear switched systems in which the switching signal is not consistent with the switching process of controllers. The bumpless transfer problem for switched systems with partial actuator failures is addressed in [24] to guarantee smooth output transition. However, up to now, there is no bumpless transfer result for nonlinear switched systems with time delay.

1.2 | Motivations

The bumps of control input may interfere the neural network circuit, which yields that the smooth transition of control input is a significant research topic. Unfortunately, until now this topic is still not addressed for switched neural networks. For example, in [15] the researchers introduce the switching state feedback controller as $u(t) = K_{\sigma(t)}x(t)$, which implies that the control input switches instantaneously from $K_{\sigma(t_{k-1})}x(t_k)$ to $K_{\sigma(t_k)}x(t_k)$ at switching time t_k. Due to the fact that $K_i \neq K_j$, $K_i x$ may not equal to $K_j x$ for some x. Therefore, the dump may be generated by this controller at switching time, which illustrates that the classical switching feedback controllers cannot work for the case that the smooth control input is preferred.

The existing results on bumpless transfer merely focus on linear switched systems [18–24]. As we know, the nonlinearity and the delay may severely affect the dynamic behavior, which implies that these results presented in [18–24] cannot be applied directly to switched neural networks with time delay. Therefore, it is necessary to propose new switching feedback feedback controller to handle the bumpless transfer control for delayed switched neural networks.

1.3 | Our work and contributions

Motivated by the above discussion, this paper copes with the stabilization of switched neural networks with time-varying delay and bumpless transfer control. A novel switching state feedback controller, which is smooth at switching time, is employed. Under mode-dependent average dwell time (MDADT), new stabilization results, which ensure the closed-loop system is exponentially stable, are obtained. Based on a simple corollary, the procedures which are used to solve the feedback control gain matrices are also presented. The effectiveness of the proposed results is demonstrated by numerical examples.

The main contributions of this paper are listed as follows. First, the bumpless transfer of switched neural networks with time-varying delay is first coped with and a novel switching state feedback controller is designed. Second, novel stabilization results, which can guarantee the smooth transitions of control input, are presented. Last, under a simple corollary, the procedures for calculating the control gain matrices are also proposed.

Notation. N and R are the set of nonnegative integers and real numbers, respectively. R^n is n-dimensional real vector space, $M = \{1, 2, \dots, m\}$ is the index set of subsystems and $V_p(t)$ is the Lyapunov function of p-th subsystem. σ is the switching signal taking value in index set M. In this paper, we always assume that $\sigma(t)$ is right-continuous. Namely, $\sigma(t) = \sigma\left(t^+\right)$. If $\sigma(t) \neq \sigma(t^-)$, we say time t is a switching time. The k-th switching time is denoted as t_k. We also assume that there exist positive constants T_{min} and T_{max} such that $T_{min} \leq t_{k+1} - t_k \leq T_{max}$ for $k \in N$. $\lambda_{max}(\cdot)$ and $\lambda_{min}(\cdot)$ denote the maximum and the minimum eigenvalue of corresponding matrix, respectively. For symmetric matrices X_1 and X_2, $X_1 \leq X_2$ is equivalent to that $X_1 - X_2$ is a symmetric non-positive definite matrix, $\|\cdot\|$ denotes the Euclidean norm of corresponding vector.

2 | PRELIMINARIES

In this paper, we consider the switched neural networks with time delay as follows:

$$\begin{cases} \dot{x}(t) = -A_{\sigma(t)}x(t) + B_{\sigma(t)}f_{\sigma(t)}(x(t-\tau(t))) + u(t), \\ x(t_0 + s) = \phi(s), s \in [-\bar{\tau}, 0], \end{cases} \tag{1}$$

where $x(t) \in R^n$ is the state vector, $f_p(y) = \left(f_{p1}(y_1), \dots, f_{pn}(y_n)\right)$ is a known activation function, $\tau(t)$ is the time-varying delay such that $0 < \tau(t) \leq \bar{\tau}$, $A_p = diag\left(a_1^p, \dots, a_n^p\right)$, $p \in M$, is a diagonal matrix with positive entries, which denotes the decay rates of the neurons, $B_p = \left(b_{jl}^p\right)_{n\times n}$ is the delayed connection weight matrix, $\phi(s)$ is a bounded continuous function, $u(t)$ is the control input.

In order to stabilize the system (1), we can employ the following classical switching state feedback controller [15]

$$u(t) = K_{\sigma(t)}x(t), \tag{2}$$

where K_p, $p \in M$, is the control gain matrix. Then, we can deduce the stabilization results for the system (1) with the feedback controller (2) via some typical stability or stabilization results for switched systems (see [15,25]). A distinctive feature of the controller (2) is the occurrence of bumps because the control input $K_{\sigma(t_{k-1})}x(t_k)$ switches instantaneously to the control input $K_{\sigma(t_k)}x(t_k)$ at switching time t_k. In many practical applications, these dumps are undesired because they cannot satisfy the rigorous requirements of specifications and may generate some negative consequences. Therefore, the switching among sub-controllers is expected to be smooth to eliminate the bumps.

To achieve this purpose, we hope the transition can switch from $K_{\sigma(t_{k-1})}x$ to $K_{\sigma(t_k)}x$ smoothly. Intuitively, a simple method is to smooth the "jumps" of control input over some sub-interval of the activated time interval $[t_k, t_{k+1})$. For simplicity, in this paper we introduce the following switching state feedback controller

$$u(t)=\begin{cases}\left(\frac{t-t_k}{\theta_{\sigma(t_k)}T_k}K_{\sigma(t_k)}+\left(1-\frac{t-t_k}{\theta_{\sigma(t_k)}T_k}\right)\right.\\ \qquad\left.\times K_{\sigma(t_{k-1})}\right)x(t),t\in\left[t_k,t_k+\theta_{\sigma(t_k)}T_k\right),\\ K_{\sigma(t_k)}x(t),t\in\left[t_k+\theta_{\sigma(t_k)}T_k,t_{k+1}\right),\end{cases}\tag{3}$$

where $\theta_p \in (0,1)$, $T_k = t_{k+1} - t_k$. The feedback control gain on time interval $\left[t_k, t_k + \theta_{\sigma(t_k)}T_k\right)$ is time-varying and is the linear combination of $K_{\sigma(t_{k-1})}$ and $K_{\sigma(t_k)}$. The smooth transition from $K_{\sigma(t_{k-1})}x$ to $K_{\sigma(t_k)}x$ is enabled on time interval $\left[t_k, t_k + \theta_{\sigma(t_k)}T_k\right)$. Obviously, $u(t)$ is continuous on time interval $[t_k, t_{k+1})$. Moreover, we have from (3) that

$$u(t_k)=K_{\sigma(t_{k-1})}x(t_k)=K_{\sigma(t_{k-1})}x\left(t_k^-\right)=u\left(t_k^-\right),$$

which is smooth and indicates that there is no bump at switching time t_k. For convenience, we say that $\left[t_k, t_k + \theta_{\sigma(t_k)}T_k\right)$ is the transitional time interval, $\theta_{\sigma(t_k)}T_k$ is the transitional time length and θ_p is the transitional time rate of the p-th subsystem, respectively. Under the controller (3), the closed-loop system of (1) can be written as

$$\begin{cases}\dot{x}(t)=\left(-A_{\sigma(t_k)}+\frac{\Gamma_k^1(t)K_{\sigma(t_k)}+\Gamma_k^2(t)K_{\sigma(t_{k-1})}}{\theta_{\sigma(t_k)}T_k}\right)\\ \qquad\times x(t)+B_{\sigma(t_k)}f_{\sigma(t_k)}(x(t-\tau(t))),t\in\left[t_k,\tilde{t}_k\right),\\ \dot{x}(t)=\left(-A_{\sigma(t_k)}+K_{\sigma(t_k)}\right)x(t)\\ \qquad+B_{\sigma(t_k)}f_{\sigma(t_k)}(x(t-\tau(t))),t\in\left[\tilde{t}_k,t_{k+1}\right),\\ x(t_0+s)=\phi(s),s\in[-\bar{\tau},0],\end{cases}\tag{4}$$

where $\tilde{t}_k = t_k + \theta_{\sigma(t_k)}T_k$, $\Gamma_k^1(t) = t - t_k$, $\Gamma_k^2(t) = \tilde{t}_k - t$.

As usual, we give the following assumptions.

(A1) There exists positive constant l_p^j such that $\frac{f_{pj}(y_1)-f_{pj}(y_2)}{y_1-y_2} \le l_p^j$ for any $p \in M$, $j = 1, 2, \dots, n$, $y_1, y_2 \in R$ and $y_1 \neq y_2$.

(A2) $f_{pj}(0) = 0$ for any $p \in M$ and $j = 1, 2, \dots, n$.

For convenience, we denote $L_p = diag(l_p^1, l_p^2, \dots, l_p^n)$.

Similar to [14,26], we give the definitions of stability and stabilization with bumpless transfer for the switched neural network (1).

Definition 1. The zero solution of switched neural network (1), where $u(t) = 0$, is said to be exponentially stable if there exist positive constants γ and χ such that

$$\|x(t)\| \le \chi\|\phi\|_{\bar{\tau}}e^{-\gamma(t-t_0)}, t \ge t_0,$$

where $\|\phi\|_{\bar{\tau}} = \sup_{-\bar{\tau}\le s\le 0}\|\phi(s)\|$.

Definition 2. The switched neural network (1) is said to be exponentially stabilizable with bumpless transfer under the switching state feedback controller (3), if the closed-loop system (4) is exponentially stable.

Definition 3 ([27]). For a switching signal $\sigma(t)$ and $T \ge t \ge t_0$, let $N_{\sigma p}(T, t)$ be the switching numbers that the p-th subsystem is activated over the time interval $[t, T)$ and $T_p(T, t)$ denotes the total running time of the p-th subsystem over time interval $[t, T)$. We say that $\sigma(t)$ has a mode-dependent dwell average time τ_{ap}, if there exist positive numbers N_{0p} and τ_{ap} such that

$$N_{\sigma p}(T,t) \le N_{0p} + \frac{T_p(T,t)}{\tau_{ap}}, \forall T \ge t \ge t_0.$$

Lemma 1. *Let nonnegative piecewise continuous function $y(t)$, $t \in [t_0 - \bar{\tau}, \infty)$, such that*

$$\begin{cases}D^+y(t)\le ay(t)+by(t-\tau(t)),t\in[t_k,t_{k+1}),\\ y(t_{k+1})\le c_{k+1}y\left(t_{k+1}^-\right),k\ge 0,\end{cases}\tag{5}$$

where $b > 0$, $a > -b$, $c_k \ge 1$. Then, we have

$$y(t)\le e^{\hat{\varepsilon}\bar{\tau}}\prod_{i=0}^{k}c_i\bar{y}(t_0)e^{\hat{\varepsilon}(t-t_0)},t\in[t_k,t_{k+1}),\tag{6}$$

where $c_0 = 1$, $\bar{y}(t_0) = \sup_{s\in[-\bar{\tau},0]}y(t_0+s)$, $\hat{\varepsilon} = a + b$.

Proof. Obviously, for $t \in [t_0 - \bar{\tau}, t_0)$, we have

$$y(t)\le y(\bar{t}_0)\le e^{\hat{\varepsilon}\bar{\tau}}c_0\bar{y}(t_0)e^{\hat{\varepsilon}(t-t_0)}.$$

If (6) is not true for $k = 0$, there must exist $\hat{t}_1 \in [t_0, t_1)$ such that

$$\begin{cases}y\left(\hat{t}_1\right)=e^{\hat{\varepsilon}\bar{\tau}}c_0\bar{y}(t_0)e^{\hat{\varepsilon}(\hat{t}_1-t_0)},\\ y(t)\le e^{\hat{\varepsilon}\bar{\tau}}c_0\bar{y}(t_0)e^{\hat{\varepsilon}(t-t_0)},t\in\left[t_0,\hat{t}_1\right],\\ D^+y\left(\hat{t}_1\right)>e^{\hat{\varepsilon}\bar{\tau}}\hat{\varepsilon}c_0\bar{y}(t_0)e^{\hat{\varepsilon}(\hat{t}_1-t_0)}.\end{cases}\tag{7}$$

Based on (5) and (7), we deduce

$$\begin{aligned}D^+y\left(\hat{t}_1\right)\le&ay\left(\hat{t}_1\right)+by\left(\hat{t}_1-\tau\left(\hat{t}_1\right)\right)\\ \le&ae^{\hat{\varepsilon}\bar{\tau}}c_0\bar{y}(t_0)e^{\hat{\varepsilon}(\hat{t}_1-t_0)}\\ &+be^{\hat{\varepsilon}\bar{\tau}}c_0\bar{y}(t_0)e^{\hat{\varepsilon}(\hat{t}_1-t_0-\tau(\hat{t}_1))}\\ =&\left(a+be^{-\hat{\varepsilon}\tau(\hat{t}_1)}\right)e^{\hat{\varepsilon}\bar{\tau}}c_0\bar{y}(t_0)e^{\hat{\varepsilon}(\hat{t}_1-t_0)}\\ \le&\hat{\varepsilon}e^{\hat{\varepsilon}\bar{\tau}}c_0\bar{y}(t_0)e^{\hat{\varepsilon}(\hat{t}_1-t_0)}.\end{aligned}$$

The above inequality contradicts (7), which indicates (6) holds for $k = 0$. Then, due to (5) we have

$$y(t_1) \le c_1 y\left(t_1^-\right) = e^{\hat{\varepsilon}\bar{\tau}} c_0 c_1 \bar{y}(t_0) e^{\hat{\varepsilon}(t_1 - t_0)}.$$

If (6) is not satisfied for $k = 1$, there must exist some $\hat{t}_2 \in [t_1, t_2)$ such that

$$\begin{cases} y\left(\hat{t}_2\right) = e^{\hat{\varepsilon}\bar{\tau}} c_0 c_1 \bar{y}(t_0) e^{\hat{\varepsilon}\left(\hat{t}_2 - t_0\right)}, \\ y(t) \le e^{\hat{\varepsilon}\bar{\tau}} c_0 c_1 \bar{y}(t_0) e^{\hat{\varepsilon}(t - t_0)}, t \in \left[t_1, \hat{t}_2\right], \\ D^+ y\left(\hat{t}_2\right) > \hat{\varepsilon} e^{\hat{\varepsilon}\bar{\tau}} c_0 c_1 \bar{y}(t_0) e^{\hat{\varepsilon}\left(\hat{t}_2 - t_0\right)}. \end{cases} \tag{8}$$

If $\hat{t}_2 - \tau\left(\hat{t}_2\right) \in [t_1, \hat{t}_2]$, we have

$$y\left(\hat{t}_2 - \tau\left(\hat{t}_2\right)\right) \le e^{\hat{\varepsilon}\bar{\tau}} c_0 c_1 \bar{y}(t_0) e^{\hat{\varepsilon}\left(\hat{t}_2 - t_0 - \tau\left(\hat{t}_2\right)\right)}. \tag{9}$$

If $\hat{t}_2 - \tau\left(\hat{t}_2\right) < t_1$, we have

$$\begin{aligned} y\left(\hat{t}_2 - \tau\left(\hat{t}_2\right)\right) &\le c_0 e^{\hat{\varepsilon}\bar{\tau}} \bar{y}(t_0) e^{\hat{\varepsilon}\left(\hat{t}_2 - t_0 - \tau\left(\hat{t}_2\right)\right)} \\ &\le e^{\hat{\varepsilon}\bar{\tau}} c_0 c_1 \bar{y}(t_0) e^{\hat{\varepsilon}\left(\hat{t}_2 - t_0 - \tau\left(\hat{t}_2\right)\right)}. \end{aligned} \tag{10}$$

According to (5), (8), (9) and (10), we obtain

$$\begin{aligned} D^+ y\left(\hat{t}_2\right) &\le \left(a + b e^{-\hat{\varepsilon}\tau\left(\hat{t}_2\right)}\right) e^{\hat{\varepsilon}\bar{\tau}} c_0 c_1 \bar{y}(t_0) e^{\hat{\varepsilon}\left(\hat{t}_2 - t_0\right)} \\ &\le \hat{\varepsilon} e^{\hat{\varepsilon}\bar{\tau}} c_0 c_1 \bar{y}(t_0) e^{\hat{\varepsilon}\left(\hat{t}_2 - t_0\right)}, \end{aligned}$$

which contradicts (8). Therefore, (6) holds for $k = 1$. Under mathematical induction, we know that (6) is true for all $k \ge 0$. □

3 | MAIN RESULTS

In this section, according to MDADT we present the stabilization results for the system (1) under the switching state feedback controller (3).

Theorem 1. *Assume that for any $p \in M$, there exist symmetric positive definite matrix P_p, positive definite matrix Q_p, positive constants $\mu_p > 1$, α_p, β_p, ξ_p, $\tilde{\varepsilon}_p$, $\bar{\varepsilon}_p$, constant $\tilde{\alpha}_p > -\beta_p$, such that:*

(i) $\begin{cases} -A_p^T P_p - P_p A_p - \mu_q^{-1}\left(Q_q^T + Q_q\right) \\ + \xi_p^{-1} P_p B_p B_p^T P_p \le \tilde{\alpha}_p P_p, q \ne p, q \in M, \\ - A_p^T P_p - P_p A_p - \left(Q_p^T + Q_p\right) \\ + \xi_p^{-1} P_p B_p B_p^T P_p \le -\alpha_p P_p; \end{cases}$

(ii) $\xi_p L_p^T L_p \le \beta_p P_p$;

(iii) $P_p \le \mu_p P_q, q \ne p, q \in M$;

(iv) $\bar{\varepsilon}_p\left(1 - 0.5\theta_p\right) - 0.5\tilde{\varepsilon}_p \theta_p - \frac{\ln \mu_p}{\tau_{ap}} > 0$;

(v) $\begin{cases} \tilde{\alpha}_p + \beta_p e^{\Delta} \le \tilde{\varepsilon}_p, \\ -\alpha_p + \beta_p e^{\Delta} \le -\bar{\varepsilon}_p; \end{cases}$

where $\Delta = \max_{t \ge t_0}\left(\sum_{p \in M} \bar{\varepsilon}_p T_p(t - \bar{\tau}, t)\right)$. Then, the system (1) is exponentially stabilizable under the controller (3) with $K_p = -P_p^{-1} Q_p$.

Proof. For convenience, we denote $\rho(k) = \sigma(t_k)$, $u_0 = 1$ and $u_k = \mu_{\sigma(t_k)}$ for $k \ge 1$. We choose the candidate Lyapunov fucntion as follows:

$$V_p(t) = x^T(t) P_p x(t), p \in M. \tag{11}$$

For $t \in \left[t_k, \tilde{t}_k\right)$, we have

$$\begin{aligned} &D^+ V_{\rho(k)}(t) \\ &= x^T(t)\left(-A_{\rho(k)}^T P_{\rho(k)} - P_{\rho(k)} A_{\rho(k)}\right. \\ &\quad - \frac{t - t_k}{\theta_{\rho(k)} T_k}\left(Q_{\rho(k)}^T + Q_{\rho(k)}\right) - \left(1 - \frac{t - t_k}{\theta_{\rho(k)} T_k}\right) \\ &\quad \times \left.\left(Q_{\rho(k-1)}^T P_{\rho(k-1)}^{-1} P_{\rho(k)} + P_{\rho(k)} P_{\rho(k-1)}^{-1} Q_{\rho(k-1)}\right)\right) x(t) \\ &\quad + f_{\rho(k)}^T (x(t - \tau(t))) B_{\rho(k)}^T P_{\rho(k)} x(t) \\ &\quad + x^T(t) P_{\rho(k)} B_{\rho(k)} f_{\rho(k)}(x(t - \tau(t))) \\ &\le x^T(t)\left(-A_{\rho(k)}^T P_{\rho(k)} - P_{\rho(k)} A_{\rho(k)} - \frac{t - t_k}{\theta_{\rho(k)} T_k}\right. \\ &\quad \times \left(Q_{\rho(k)}^T + Q_{\rho(k)}\right) + \xi_{\rho(k)}^{-1} P_{\rho(k)} B_{\rho(k)} B_{\rho(k)}^T P_{\rho(k)} \\ &\quad - \left(1 - \frac{t - t_k}{\theta_{\rho(k)} T_k}\right)\left(Q_{\rho(k-1)}^T P_{\rho(k-1)}^{-1} P_{\rho(k)}\right. \\ &\quad \left.\left. + P_{\rho(k)} P_{\rho(k-1)}^{-1} Q_{\rho(k-1)}\right)\right) x(t) \\ &\quad + \xi_{\rho(k)} x^T(t - \tau(t)) L_{\rho(k)}^T L_{\rho(k)} x(t - \tau(t)) \\ &\le x^T(t)\left(-A_{\rho(k)}^T P_{\rho(k)} - P_{\rho(k)} A_{\rho(k)} - \frac{t - t_k}{\theta_{\rho(k)} T_k}\right. \\ &\quad \times \left(Q_{\rho(k)}^T + Q_{\rho(k)}\right) + \xi_{\rho(k)}^{-1} P_{\rho(k)} B_{\rho(k)} B_{\rho(k)}^T P_{\rho(k)} \\ &\quad \left. - u_{k-1}^{-1}\left(1 - \frac{t - t_k}{\theta_{\rho(k)} T_k}\right)\left(Q_{\rho(k-1)}^T + Q_{\rho(k-1)}\right)\right) x(t) \\ &\quad + \xi_{\rho(k)} x^T(t - \tau(t)) L_{\rho(k)}^T L_{\rho(k)} x(t - \tau(t)). \end{aligned} \tag{12}$$

By Condition (i), (12) can be continued as

$$\begin{aligned} &D^+ V_{\rho(k)}(t) \\ &\le \left(\left(1 - \frac{t - t_k}{\theta_{\rho(k)} T_k}\right)\tilde{\alpha}_{\rho(k)} - \frac{t - t_k}{\theta_{\rho(k)} T_k}\alpha_{\rho(k)}\right) V_{\rho(k)}(t) \\ &\quad + \beta_{\rho(k)} V_{\rho(k)}(t - \tau(t)) \\ &= \hat{\alpha}_{\rho(k)}(t) V_{\rho(k)}(t) + \beta_{\rho(k)} V_{\rho(k)}(t - \tau(t)), \end{aligned} \tag{13}$$

where $\hat{\alpha}_k(t) = \left(1 - \frac{t - t_k}{\theta_{\rho(k)} T_k}\right)\tilde{\alpha}_{\rho(k)} - \frac{t - t_k}{\theta_{\rho(k)} T_k}\alpha_{\rho(k)}$. Similarly, for $t \in \left[\tilde{t}_k, t_{k+1}\right)$, we have

$$D^+ V_{\rho(k)}(t) \le -\alpha_{\rho(k)} V_{\rho(k)}(t) + \beta_{\rho(k)} V_{\rho(k)}(t - \tau(t)). \tag{14}$$

For $t = t_{k+1}$, we obtain

$$\begin{aligned} &V_{\rho(k+1)}(t_{k+1}) = x^T(t_{k+1}) P_{\rho(k+1)} x(t_{k+1}) \\ &\le u_{k+1} x^T(t_{k+1}^-) P_{\rho(k)} x(t_{k+1}^-) = u_{k+1} V_{\rho(k)}\left(t_{k+1}^-\right). \end{aligned} \tag{15}$$

We derive from (13), (14) and (15) that

$$\begin{cases} D^{+}V_{\rho(k)}(t) \\ \leq \tilde{\alpha}V_{\rho(k)}(t) + \beta V_{\rho(k)}(t-\tau(t)), t \in [t_k, t_{k+1}) \\ V_{\rho(k+1)}(t_{k+1}) \leq u_{k+1}V_{\rho(k)}\left(t_{k+1}^{-}\right), k \geq 0, \end{cases} \tag{16}$$

where $\tilde{\alpha} = \max_{p\in M}\left\{\tilde{\alpha}_p\right\}$ and $\beta = \max_{p\in M}\left\{\beta_p\right\}$. According to Condition (iii) and (11), we know that

$$V_{\rho(k)}(t-\tau(t)) \leq \mu V_{\rho(l)}(t-\tau(t)), \tag{17}$$

where $\mu = \max_{p\in M}\left\{\mu_p\right\}$, $l = \begin{cases} 0, \text{ if } t-\tau(t) < t_1, \\ h, \text{ if } t_h \leq t-\tau(t) < t_{h+1}, h \geq 1. \end{cases}$ Denote $V(t_0+s) = V_{\rho(0)}(t_0+s)$. For any $t \in [t_k, t_{k+1})$, $k \in N$, we let $V(t) = V_{\rho(k)}(t)$. Under (16) and (17), we obtain

$$\begin{cases} D^{+}V(t) \leq \tilde{\alpha}V(t) + \beta\mu V(t-\tau(t)), t \in [t_k, t_{k+1}), \\ V(t_{k+1}) \leq u_{k+1}V\left(t_{k+1}^{-}\right), k \geq 0. \end{cases} \tag{18}$$

Then, owing to Lemma 1, we derive that

$$V(t) \leq e^{\check{\varepsilon}\bar{\tau}}\prod_{i=0}^{k}u_iV_0e^{\check{\varepsilon}(t-t_0)}, t \in [t_k, t_{k+1}), \tag{19}$$

where $V_0 = \sup_{s\in[-\bar{\tau},0]}V_{\rho(0)}(t_0+s)$, $\check{\varepsilon} = \tilde{\alpha} + \beta\mu$.

Let k^* be the smallest positive integer such that $t_{k^*} - \bar{\tau} \geq t_1$. For any $t \in [t_k, t_{k+1})$, $0 \leq k \leq k^*-1$, we have from (19) that

$$\begin{cases} V_{\rho(k)}(t) \leq GV_0\prod_{i=0}^{k}u_ie^{\sum_{i=0}^{k-1}\eta_iT_i}e^{\int_{t_k}^{t}\check{\varepsilon}_k(s)ds}, \\ \qquad t \in \left[t_k, \tilde{t}_k\right), \\ V_{\rho(k)}(t) \leq GV_0\prod_{i=0}^{k}u_ie^{\sum_{i=0}^{k-1}\eta_iT_i-\upsilon_kT_k} \\ e^{-\bar{\varepsilon}_{\rho(k)}(t-\tilde{t}_k)}, t \in \left[\tilde{t}_k, t_{k+1}\right), \end{cases} \tag{20}$$

where $G = e^{\left(\check{\varepsilon}+\max_{i\in M}\left\{\bar{\varepsilon}_i\right\}\right)(t_{k^*}-t_0)+\check{\varepsilon}\bar{\tau}}$, $\eta_i = 0.5\theta_{\rho(i)}\tilde{\varepsilon}_{\rho(i)} - \bar{\varepsilon}_{\rho(i)}\left(1-0.5\theta_{\rho(i)}\right)$, $\upsilon_i = 0.5\theta_{\rho(i)}\bar{\varepsilon}_{\rho(i)}$, $\check{\varepsilon}_k(t) = \frac{\tilde{\varepsilon}_{\rho(k)}(\tilde{t}_k-t)-\bar{\varepsilon}_{\rho(k)}(t-t_k)}{\theta_{\rho(k)}T_k}$.

It follows from (15) and (20) that

$$V_{\rho(k^*)}(t_{k^*}) \leq GV_0\prod_{i=0}^{k^*}u_ie^{\sum_{i=0}^{k^*-1}\eta_iT_i}.$$

For $t \in \left[t_{k^*}, \tilde{t}_{k^*}\right)$, we claim that

$$V_{\rho(k^*)}(t) \leq GV_0\prod_{i=0}^{k^*}u_ie^{\sum_{i=0}^{k^*-1}\eta_iT_i}e^{\int_{t_k^*}^{t}\check{\varepsilon}_{k^*}(s)ds}. \tag{21}$$

If (21) is not satisfied, there must exist some $t^* \in \left[t_{k^*}, \tilde{t}_{k^*}\right)$ such that

$$\begin{cases} V_{\rho(k^*)}(t^*) = GV_0\prod_{i=0}^{k^*}u_ie^{\sum_{i=0}^{k^*-1}\eta_iT_i}e^{\int_{t_k^*}^{t^*}\check{\varepsilon}_{k^*}(s)ds}, \\ V_{\rho(k^*)}(t) \leq GV_0\prod_{i=0}^{k^*}u_ie^{\sum_{i=0}^{k^*-1}\eta_iT_i}e^{\int_{t_k^*}^{t}\check{\varepsilon}_{k^*}(s)ds}, \\ \qquad t \in [t_{k^*}, t^*], \\ D^{+}V_{\rho(k^*)}(t^*) > \check{\varepsilon}_{k^*}(t^*)GV_0\prod_{i=0}^{k^*}u_ie^{\sum_{i=0}^{k^*-1}\eta_iT_i}\times \\ \qquad e^{\int_{t_k^*}^{t^*}\check{\varepsilon}_{k^*}(s)ds}, \end{cases} \tag{22}$$

When $t^*-\tau(t^*) \geq t_{k^*}$, we have

$$\begin{aligned} &V_{\rho(k^*)}(t^*-\tau(t^*)) \\ &\leq GV_0\prod_{i=0}^{k^*}u_ie^{\sum_{i=0}^{k^*-1}\eta_iT_i}e^{\tilde{\varepsilon}_{\rho(k^*)}(t^*-t_{k^*})}e^{-\tilde{\varepsilon}_{\rho(k^*)}\tau(t^*)} \\ &\leq e^{\Delta}V_{\rho(k^*)}(t^*). \end{aligned} \tag{23}$$

When $t^*-\tau(t^*) \in \left[t_{k^*-1}+\theta_{\rho(k^*-1)}T_{k^*-1}, t_{k^*}\right)$, we have

$$\begin{aligned} &V_{\rho(k^*)}(t^*-\tau(t^*)) \\ &\leq u_{k^*}V_{\rho(k^*-1)}(t^*-\tau(t^*)) \\ &\leq GV_0\prod_{i=0}^{k^*}u_ie^{\sum_{i=0}^{k^*-2}\eta_iT_i-\upsilon_{k^*-1}T_{k^*-1}}\times \\ &e^{-\bar{\varepsilon}_{\rho(k^*-1)}\left(t^*-\tau(t^*)-\left(t_{k^*-1}+\theta_{\rho(k^*-1)}T_{k^*-1}\right)\right)}, \\ &\leq GV_0\prod_{i=0}^{k^*}u_ie^{\sum_{i=0}^{k^*-1}\eta_iT_i}e^{\tilde{\varepsilon}_{\rho(k^*)}(t^*-t_{k^*})}\times \\ &e^{\bar{\varepsilon}_{\rho(k^*-1)}(t_{k^*}-(t^*-\tau(t^*)))}e^{-\tilde{\varepsilon}_{\rho(k^*)}(t^*-t_{k^*})} \\ &\leq e^{\Delta}V_{\rho(k^*)}(t^*). \end{aligned} \tag{24}$$

When $t^*-\tau(t^*) \in \left[t_{k^*-1}, t_{k^*-1}+\theta_{\rho(k^*-1)}T_{k^*-1}\right)$, we have

$$\begin{aligned} &V_{\rho(k^*)}(t^*-\tau(t^*)) \\ &\leq u_{k^*}V_{\rho(k^*-1)}(t^*-\tau(t^*)) \\ &\leq GV_0\prod_{i=0}^{k^*}u_ie^{\sum_{i=0}^{k^*-2}\eta_iT_i}e^{\tilde{\varepsilon}_{\rho(k^*-1)}(t^*-\tau(t^*)-t_{k^*-1})} \\ &\leq GV_0\prod_{i=0}^{k^*}u_ie^{\sum_{i=0}^{k^*-1}\eta_iT_i}e^{\tilde{\varepsilon}_{\rho(k^*)}(t^*-t_{k^*})}\times \\ &e^{-\tilde{\varepsilon}_{\rho(k^*-1)}\left(t_{k^*-1}+\theta_{\rho(k^*-1)}T_{k^*-1}-(t^*-\tau(t^*))\right)}\times \\ &e^{-\tilde{\varepsilon}_{\rho(k^*)}(t^*-t_{k^*})}e^{v_{k^*-1}T_{k^*-1}} \\ &\leq e^{\Delta}V_{\rho(k^*)}(t^*), \end{aligned} \tag{25}$$

where $v_i = \bar{\varepsilon}_{\rho(i)}\left(1-\theta_{\rho(i)}\right)$.

When $t-\tau(t^*) \in \left[t_l+\theta_{\rho(t_l)}T_l, t_{l+1}\right)$, $l < k^*-1$, we have

$$
\begin{aligned}
&V_{\rho(k^*)}\left(t^*-\tau\left(t^*\right)\right)\\
&\leq u_{k^*}V_{\rho(l)}\left(t^*-\tau\left(t^*\right)\right)\\
&\leq u_{k^*}GV_0\prod_{i=0}^{l}u_i e^{\sum_{i=0}^{l-1}\eta_i T_i-\upsilon_l T_l}\\
&\quad\times e^{-\bar{\varepsilon}_{\rho(l)}\left(t^*-\tau(t^*)-\left(t_l+\left(1-\theta_{\rho(l)}\right)T_l\right)\right)}\\
&\leq GV_0\prod_{i=0}^{k^*}u_i e^{\sum_{i=0}^{k^*-1}\eta_i T_i}e^{\tilde{\varepsilon}_{\rho(k^*)}\left(t^*-t_{k^*}\right)}e^{-\tilde{\varepsilon}_{\rho(k^*)}\left(t^*-t_{k^*}\right)}\\
&\quad\times\prod_{i=l+1}^{k^*-1}u_i^{-1}e^{-\sum_{i=l+1}^{k^*-1}\eta_i T_i}e^{\bar{\varepsilon}_{\rho(l)}\left(t_{l+1}-\left(t^*-\tau(t^*)\right)\right)}\\
&\leq e^{\Delta}V_{\rho(k^*)}\left(t^*\right).
\end{aligned}
\tag{26}
$$

When $t-\tau\left(t^*\right)\in\left[t_l,t_l+\theta_{\rho(t_l)}T_l\right)$, $l<k^*-1$, we have

$$
\begin{aligned}
&V_{\rho(k^*)}\left(t^*-\tau\left(t^*\right)\right)\\
&\leq u_{k^*}V_{\rho(l)}\left(t^*-\tau\left(t^*\right)\right)\\
&\leq u_{k^*}GV_0\prod_{i=0}^{l-1}u_i e^{\sum_{i=0}^{l-1}\eta_i T_i}e^{\tilde{\varepsilon}_{\rho(l)}\left(t^*-\tau(t^*)-t_l\right)}\\
&\leq GV_0\prod_{i=0}^{k^*}u_i e^{\sum_{i=0}^{k^*-1}\eta_i T_i}e^{\tilde{\varepsilon}_{\rho(k^*)}\left(t^*-t_{k^*}\right)}e^{-\tilde{\varepsilon}_{\rho(k^*)}\left(t^*-t_{k^*}\right)}\\
&\quad\times\prod_{i=l+1}^{k^*-1}u_i^{-1}e^{-\sum_{i=l}^{k^*-1}\eta_i T_i}e^{\tilde{\varepsilon}_{\rho(l)}\left(t^*-\tau(t^*)-t_l\right)}\\
&\leq e^{\Delta}V_{\rho(k^*)}\left(t^*\right).
\end{aligned}
\tag{27}
$$

It follows from (13), (24)-(27) that

$$
\begin{aligned}
&D^{+}V_{\rho(k^*)}\left(t^*\right)\\
&\leq\hat{\alpha}_{\rho(k^*)}\left(t^*\right)V_{\rho(k^*)}\left(t^*\right)+\beta_{\rho(k^*)}e^{\Delta}V_{\rho(k^*)}\left(t^*\right)\\
&=\left(\frac{\tilde{t}_{k^*}-t^*}{\theta_{\rho(k)}T_{k^*}}\left(\tilde{\alpha}_{\rho(k^*)}+\beta_{\rho(k^*)}e^{\Delta}\right)\right.\\
&\quad\left.+\frac{t^*-t_{k^*}}{\theta_{\rho(k^*)}T_{k^*}}\left(-\alpha_{\rho(k^*)}+\beta_{\rho(k^*)}e^{\Delta}\right)\right)V_{\rho(k^*)}\left(t^*\right)\\
&\leq\breve{\varepsilon}_{k^*}\left(t^*\right)GV_0\prod_{i=0}^{k^*}u_i e^{\sum_{i=0}^{k^*-1}\eta_i T_i}e^{\int_{t_k^*}^{t^*}\breve{\varepsilon}_{k^*}(s)ds},
\end{aligned}
$$

which contradicts (22). Therefore, the first inequality in (20) is true for $k=k^*$.

Similarly, we can obtain from Condition (i) that the second inequality in (20) is also satisfied for $k=k^*$. Therefore, (20) holds for $k=k^*$. Then, under mathematical induction, we know that (20) is true for any $k\geq 0$.

In addition, we can obtain

$$
\begin{aligned}
&\prod_{i=0}^{k}u_i e^{\sum_{i=0}^{k}\eta_i T_i}\\
&\leq\prod_{p\in M}\mu_p^{N_{\sigma p}\left(t_{k+1},t_0\right)}e^{-\sum_{p\in M}\varepsilon_p T_p\left(t_{k+1},t_0\right)}\\
&\leq\prod_{p\in M}\mu_p^{N_{0p}}e^{-\sum_{p\in M}\left(\varepsilon_p-\frac{\ln\mu_p}{\tau_{ap}}\right)T_p\left(t_{k+1},t_0\right)}
\end{aligned}
\tag{28}
$$

where $\varepsilon_p=\left(1-0.5\theta_p\right)\bar{\varepsilon}_p-0.5\theta_p\tilde{\varepsilon}_p$. In the light of (20) and (28), we get for $t\in[t_k,t_{k+1})$ that

$$
\begin{aligned}
&V_{\rho(k)}(t)\\
&\leq GV_0\prod_{p\in M}\mu_p^{N_{0p}}e^{-\sum_{p\in M}\left(\varepsilon_p-\frac{\ln\mu_p}{\tau_{ap}}\right)T_p(t,t_0)}\\
&\quad\times u_k e^{\left(\max_{p\in M}\left\{\bar{\varepsilon}_p\right\}+\max_{p\in M}\left\{\tilde{\varepsilon}_p\right\}\right)(t-t_k)}\\
&\leq GG_1V_0\prod_{p\in M}\mu_p^{N_{0p}}e^{-\min_{p\in M}\left\{\varepsilon_p-\frac{\ln\mu_p}{\tau_{ap}}\right\}(t-t_0)},
\end{aligned}
$$

where $G_1=\max_{p\in M}\left\{\mu_p\right\}e^{\left(\max_{p\in M}\left\{\bar{\varepsilon}_p\right\}+\max_{p\in M}\left\{\tilde{\varepsilon}_p\right\}\right)T_{max}}$. By (11), the above inequality can be continued as

$$
\begin{aligned}
\|x(t)\|\leq&\sqrt{\frac{GG_1\max_{p\in M}\left\{\lambda_{max}\left(P_p\right)\right\}}{\min_{p\in M}\left\{\lambda_{min}P_p\right\}}}\prod_{p\in M}\mu_p^{0.5N_{0p}}\times\\
&\|\phi\|_{\bar{\tau}}e^{-0.5\min_{p\in M}\left\{\varepsilon_p-\frac{\ln\mu_p}{\tau_{ap}}\right\}(t-t_0)},
\end{aligned}
\tag{29}
$$

which indicates that the system (1) can be exponentially stabilized under the controller (3) with $K_p=P_p^{-1}Q_p$, $p\in M$. □

Remark 1. Because the feedback control input $u(t)$ is incompletely matched on $\left[t_k,\tilde{t}_k\right)$, the closed-loop system (4) may be divergent on $\left[t_k,\tilde{t}_k\right)$. In addition, the switching among different subsystems may also generate destabilizing effect. Obviously, these destabilizing effect caused by smooth transition and switching must be counteracted by the convergent effect existing on time interval $\left[\tilde{t}_k,t_{k+1}\right)$, which requires that the activated time of feedback control $u(t)=K_{\sigma(t_k)}x(t)$ must be generous. According to Condition (iv) in Theorem 1, one can obtain that the activated time of feedback control $u(t)=K_{\sigma(t_k)}x(t)$ must satisfy

$$
t_{k+1}-\tilde{t}_k\geq\frac{\left(\tilde{\varepsilon}_p-\bar{\varepsilon}_p+\frac{2\ln\mu_p}{\tau_{ap}}\right)\left(t_{k+1}-t_k\right)}{\bar{\varepsilon}_p+\tilde{\varepsilon}_p}.
$$

Remark 2. In some cases, we should restrict that the system (1) must be stabilized with a required convergent rate. If the convergent rate is specified as 0.5ε, we can replace Condition (iv) in Theorem 1 with

$$\bar{\varepsilon}_p\left(1-0.5\theta_p\right)-0.5\tilde{\varepsilon}_p\theta_p-\frac{\ln\mu_p}{\tau_{ap}}\geq\varepsilon. \tag{30}$$

It follows from Condition (i) in Theorem 1 that P_p is dependent on Q_q, $q \neq p$, which implies that K_p is relevant to K_q. Generally speaking, this relation may result in a huge amount of calculation if M is big enough. Therefore, we give the following corollary derived from Theorem 1.

Corollary 1. *Assume that for any $p \in M$, there exist symmetric positive definite matrix P_p, positive definite matrix Q_p, positive constants $\mu_p > 1$, α_p, β_p, $\tilde{\varepsilon}_p, \bar{\varepsilon}_p$, constant $\tilde{\alpha}_p > -\beta_p$, such that:*

(i) $$\begin{cases}-A_p^TP_p-P_pA_p+P_pB_pB_p^TP_p\leq\tilde{\alpha}_pP_p,\\ -A_p^TP_p-P_pA_p-\left(Q_p^T+Q_p\right)+P_pB_pB_p^TP_p\\ \qquad\qquad\qquad\leq-\alpha_pP_p;\end{cases}$$

(ii) $L_p^TL_p\leq\beta_pP_p$;

(iii) $\bar{\varepsilon}_p\left(1-0.5\theta_p\right)-0.5\tilde{\varepsilon}_p\theta_p\geq\varepsilon+\frac{\ln\mu_p}{\tau_{ap}}$;

(iv) $P_p \leq \mu_pP_q$, $q \in M, q \neq p$;

where $\tilde{\varepsilon}_p = \tilde{\alpha}_p + \beta_pe^{\Delta}$ and $\bar{\varepsilon}_p \leq \alpha_p - \beta_pe^{\Delta}$ with $\Delta = \max_{t\geq t_0}\left(\sum_{p\in M}\bar{\varepsilon}_pT_p\left(t-\bar{\tau},t\right)\right)$, respectively. Then, the system (1) can be exponentially stabilized under the controller (3) with $K_p = -P_p^{-1}Q_p$ and convergent rate 0.5ε.

Remark 3. Although there exist some stabilization results for switched systems via bumpless transfer control. However, these results are only valid for linear switched systems without time delay. Obviously, the switched neural network (1) is a nonlinear switched system with time-varying delay. Therefore, the stabilization results presented in [18,19,21–24] are invalid for the system (1).

Remark 4. Based on Schur complement [28], the matrix inequalities of Condition (i) in Theorem 1 and Corollary 1 can be transformed into LMIs easily. For example, the second matrix inequality of Condition (i) of Corollary 1 can be rewritten as

$$\begin{pmatrix}-A_p^TP_p-P_pA_p-\left(Q_p+Q_p^T\right)+\alpha_pP_p & P_pB_p\\ B_p^TP_p & -I\end{pmatrix}\leq 0.$$

Obviously, the other matrix inequalities of the proposed results are ordinary LMIs. Therefore, all the matrix inequalities can be solved conveniently by the LMI toolbox of Matlab.

It is obvious that P_p is only dependent on Q_p, which indicates the convenience for finding control gain matrices. Based on Corollary 1, for switching signal $\sigma(t)$ and required convergent rate 0.5ε, the control gain matrix K_p can be obtained by the following procedures.

1. Obtain the MDADT τ_{ap} in terms of the switching signal $\sigma(t)$.
2. Choose appropriate parameters $\tilde{\alpha}_p$, β_p and μ_p, and then find P_p by solving Conditions (ii), (iv) and the first matrix inequality of Condition (i) in Corollary 1.
3. Choose a constant $\varepsilon_M > \varepsilon + \frac{\ln\mu_p}{\tau_p}$ and let $\bar{\varepsilon}_p = \varepsilon_M$, $\tilde{\varepsilon}_p = \tilde{\alpha}_p + \beta_pe^{\varepsilon_M\bar{\tau}}$.
4. According to $\theta_p \leq \frac{2\left(\varepsilon_M-\varepsilon-\frac{\ln\mu_p}{\tau_{ap}}\right)}{\varepsilon_M+\tilde{\varepsilon}_p}$, get appropriate θ_p.
5. Calculate $\alpha_p = \beta_pe^{\varepsilon_M\bar{\tau}} + \varepsilon_M$.
6. By solving the second matrix inequality of Condition (i) in Corollary 1, we can obtain the matrix Q_p. Then, the control gain matrix can be derived by $K_p = -P_p^{-1}Q_p$.

4 | NUMERICAL SIMULATION

Example 1. Consider the switched neural network (1) with $m = 2$, $A_1 = diag(1,1)$, $A_2 = diag(0.5,0.5)$, $f_1(x) = f_2(x) = (\sin(x_1), \sin(x_2))^T$, $\tau(t) = 0.5 + 0.2\sin t$, $B_1 = \begin{pmatrix}0.5 & -1.3\\ 1 & 0.9\end{pmatrix}$, $B_2 = \begin{pmatrix}0.1 & 0.8\\ -0.8 & 0.5\end{pmatrix}$, $t_{k+1} = t_k + 0.7 + 0.1(-1)^k$,

$$\sigma(t)=\begin{cases}1, t\in\left[t_{2l},t_{2l+1}\right), l\in N,\\ 2, t\in\left[t_{2l+1},t_{2l+2}\right), l\in N.\end{cases} \tag{31}$$

These two subsystems are unstable (the oscillating time response curves of the unstable subsystems are shown in Figure 1). It is obvious that $L_1 = L_2 = diag(1,1)$, $\bar{\tau} = 0.7$.

For given convergent rate $0.5\varepsilon = 0.05$, we could obtain the feedback controller (3) by the procedures presented in Section 3.

1. According to the switching signal (31), we know that $\tau_{a1} = 0.6$ and $\tau_{a2} = 0.8$.
2. By choosing $\tilde{\alpha}_1 = 0.3$, $\tilde{\alpha}_2 = 0.3$, $\beta_1 = 1.11$, $\beta_2 = 0.88$, $\mu_1 = 1.2$, $\mu_2 = 1.3$ and solving Conditions (ii), (iv) and the first LMI of Condition (i) in Corollary 1, we have
$$P_1=\begin{pmatrix}1.0410 & 0.1540\\ 0.1540 & 1.0714\end{pmatrix},$$
$$P_2=\begin{pmatrix}1.1564 & -0.0001\\ -0.0001 & 1.1518\end{pmatrix}.$$
3. Choose $\varepsilon_M = 1$ such that $\varepsilon_M > \varepsilon + \frac{\ln\mu_1}{\tau_{a1}}$ and $\varepsilon_M > \varepsilon + \frac{\ln\mu_2}{\tau_{a2}}$. Let $\bar{\varepsilon}_1 = \bar{\varepsilon}_2 = \varepsilon_M$ and $\tilde{\varepsilon}_1 = \tilde{\alpha}_1 + \beta_1e^{\varepsilon_M\bar{\tau}} = 2.5353$ and $\tilde{\varepsilon}_2 = \tilde{\alpha}_2 + \beta_1e^{\varepsilon_M\bar{\tau}} = 2.0721$.

4. According to $\theta_1 \leq \frac{2\left(\varepsilon_M-\varepsilon-\frac{\ln \mu_1}{\tau_{a1}}\right)}{\varepsilon_M+\tilde{\varepsilon}_1} = 0.3372$ and $\theta_2 \leq \frac{\left(\varepsilon_M-\varepsilon-\frac{\ln \mu_2}{\tau_{a2}}\right)}{\varepsilon_M+\tilde{\varepsilon}_2} = 0.3724$, we can choose $\theta_1 = \theta_2 = 0.3$.
5. Compute $\alpha_1 = \beta_1 e^{\varepsilon_M \bar{\tau}} + \varepsilon_M = 3.2353$ and $\alpha_2 = \beta_2 e^{\varepsilon_M \bar{\tau}} + \varepsilon_M = 2.7721$.
6. By solving the second LMI of Condition (i) in Corollary 1, we can obtain the feasible solution

$$Q_1 = \begin{pmatrix} 2.1623 & 0.0147 \\ 0.0147 & 2.1652 \end{pmatrix},$$

$$Q_2 = \begin{pmatrix} 2.0421 & 0.1703 \\ 0.1703 & 2.1634 \end{pmatrix}.$$

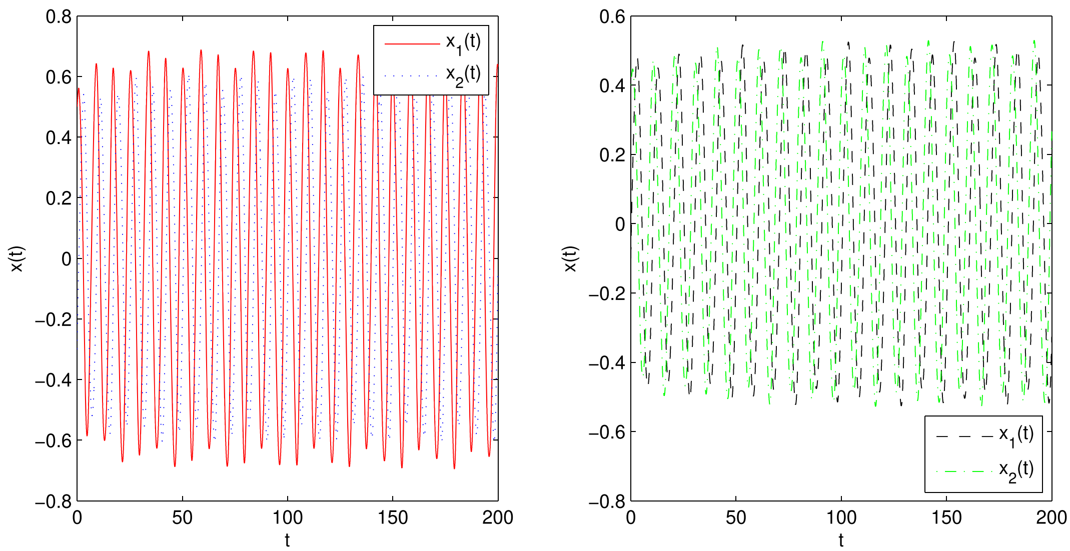

FIGURE 1 The oscillating time response curves of subsystems [Color figure can be viewed at wileyonlinelibrary.com]

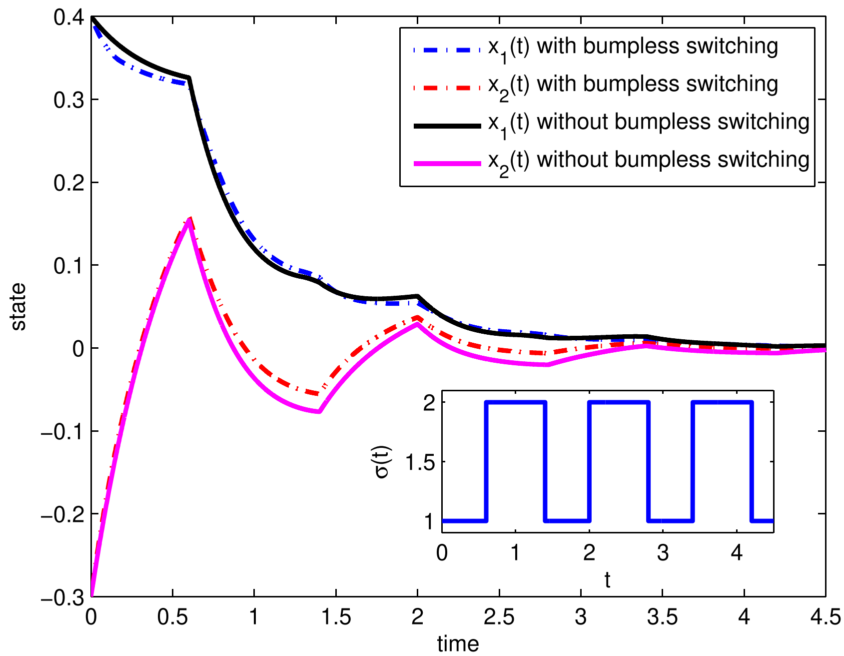

FIGURE 2 Stable time response curves and switching signal of switched neural network with the controller (2) and the controller (3) [Color figure can be viewed at wileyonlinelibrary.com]

Therefore, the feedback control gain matrices of the controller (3) are

$$K_1 = -P_1^{-1}Q_1 = \begin{pmatrix} -0.9815 & 0.1411 \\ 0.1411 & -0.9536 \end{pmatrix},$$

$$K_2 = -P_1^{-1}Q_2 = \begin{pmatrix} -1.7295 & -0.0002 \\ -0.0002 & -1.7346 \end{pmatrix},$$

respectively. According to the stability or stabilization results presented in [7,25,26,29], the above control gains can also guarantee that the switched neural networks is exponential stabilizable under the classical switching feedback controller (2).

Figures 2 and 3 show that the stable time response curves of this neural network with the feedback controller (3) and the curves of control input of the controller (3), respectively. In order to give the comparison results between bumpless transfer control and the non-bumpless transfer control [7,26,29], we have also plotted the time response curves of this switched neural network with the controller (2) and the curves of control input of the controller (2). As shown in Figure 2, we know that this switched neural network can be stabilized by both the controller (2) and the controller (3). However, because of the noncontinuity of control input, the controller (2) may generate bumps at switching time. As can be seen from the sub-figure of Figure 3, under the controller (2), the control components u_1 and u_2 jump from -0.2958 and -0.0722 to -0.5519 and -0.2142 at $t_1 = 0.6$, respectively, which indicates the occurrence of bumps. Clearly, under the controller (3), the control input is smooth at switching time, which demonstrates the bumpless transfer control law can effectively avoid the occurrence of bumps which exists in the classical switching feedback control strategy [7,26,29].

Example 2. Now we introduce a simple practical simulation example to shows the effectiveness of the proposed results. Consider the multi-loop model of aero-engine [30,31]

$$\begin{pmatrix} \dot{n}_h \\ \dot{n}_l \end{pmatrix} = A_{\sigma(t)} \begin{pmatrix} n_h \\ n_l \end{pmatrix} + B_{\sigma(t)} \begin{pmatrix} mf \\ A_e \end{pmatrix}, \tag{32}$$

where $M = 2$, n_h and n_l are the rotational speed of the high and low pressure rotor, respectively, mf and A_e, which are the control input, are the fuel flow and the area of tail nozzle, respectively. According to Theorem 1, we know that this system is exponentially stabiliz able under the controller (3) with $K_p = Q_p\bar{P}_p^{-1}$ if there exist symmetric positive definite matrix $\bar{P}_p$, matrix Q_p, positive constants $\mu_p > 1$, α_p, constant $\tilde{\alpha}_p$, such that

$$\begin{cases} \bar{P}_pA_p^T + A_p\bar{P}_p - \tilde{\alpha}_p \le 0, \\ \bar{P}_pA_p^T + A_p\bar{P}_p + Q_p^T + B_pQ_p + \alpha_p\bar{P}_p \le 0, \end{cases} \tag{33}$$

$$\bar{P}_q \le \mu_p\bar{P}_p, (p,q) \in M \times M, \tag{34}$$

$$\alpha_p\left(1 - 0.5\theta_p\right) - 0.5\tilde{\alpha}_p\theta_p - \frac{\ln \mu_p}{\tau_{ap}} > 0. \tag{35}$$

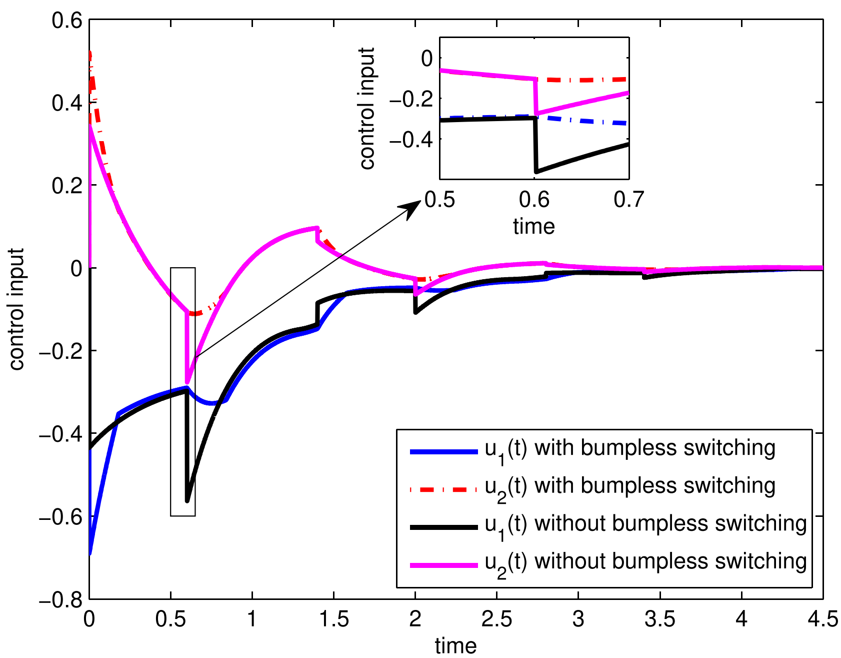

FIGURE 3 Control input curves of switched neural network with the controller (2) and the controller (3) [Color figure can be viewed at wileyonlinelibrary.com]

For numerical simulation, we assume that

$$A_1 = \begin{pmatrix} -2 & 2 \\ 0.5 & 3 \end{pmatrix}, A_2 = \begin{pmatrix} -1.1789470 & 2.119459 \\ 2.46103 & -3.679685 \end{pmatrix},$$

$$B_1 = \begin{pmatrix} 0.8 & 1 \\ 0.8 & 0.6 \end{pmatrix}, B_2 = \begin{pmatrix} 0.3129523 & 0.1576769 \\ 0.5631366 & 0.8378436 \end{pmatrix}$$

which are borrowed from [30,31]. Obviously, each subsystem without control input is unstable. By choosing $\tilde{\alpha}_1 = 6.4$, $\tilde{\alpha}_2 = 2.4$, $\alpha_1 = \alpha_2 = 3$, $\mu_1 = \mu_2 = 2$, and solving (33)-(34), we obtain

$$\bar{P}_1 = \begin{pmatrix} 0.1460 & 0.0784 \\ 0.0784 & 0.2248 \end{pmatrix}, \bar{P}_2 = \begin{pmatrix} 0.1821 & 0.0910 \\ 0.0910 & 0.1761 \end{pmatrix},$$

$$Q_1 = \begin{pmatrix} 2.3192 & -1.4207 \\ -2.1749 & -0.2499 \end{pmatrix}, Q_2 = \begin{pmatrix} -1.8018 & -0.1526 \\ 0.4873 & 0.0121 \end{pmatrix}.$$

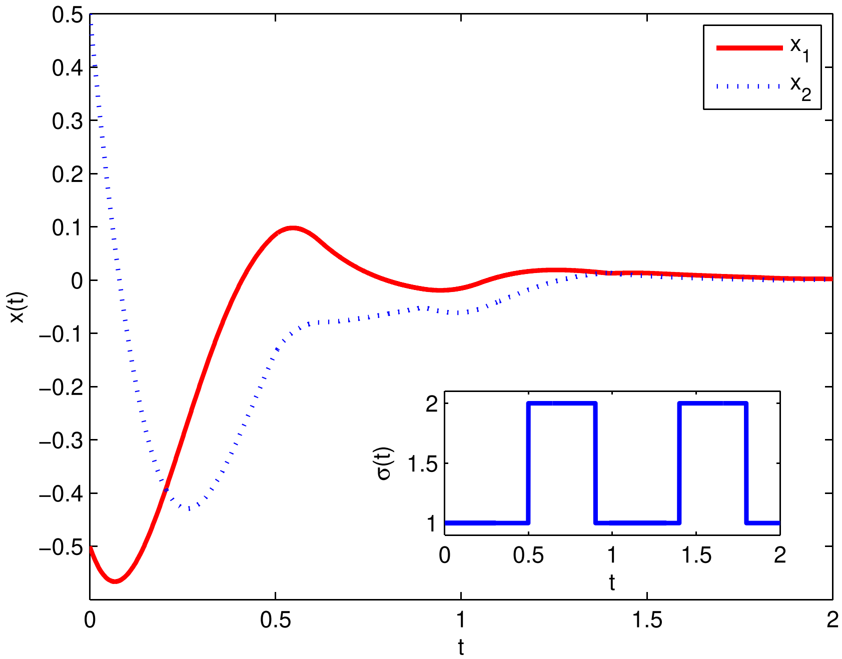

FIGURE 4 The stable time response curves of the system (32) under the switching signal (36) and the controller (3) [Color figure can be viewed at wileyonlinelibrary.com]

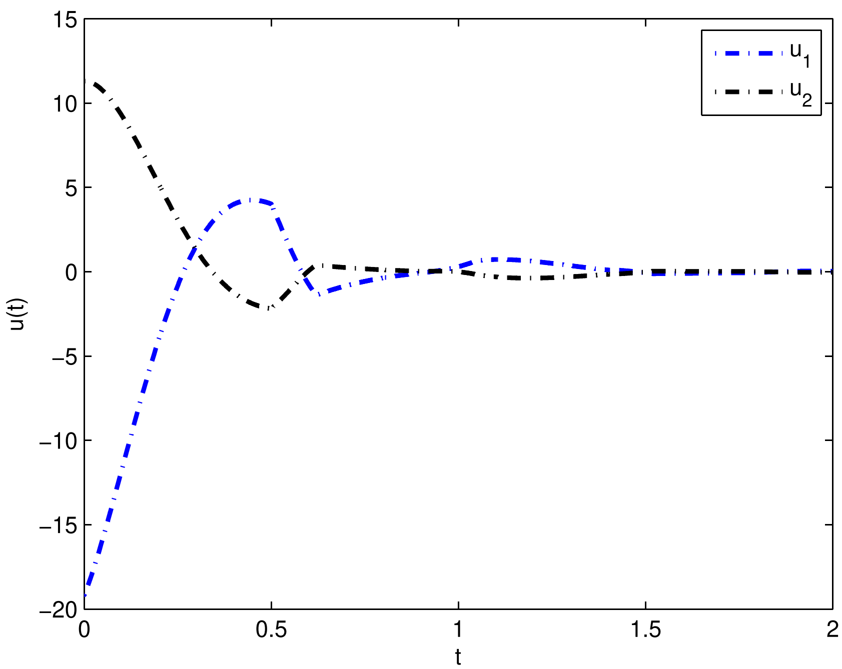

FIGURE 5 The time curves of control input in the controller (3) in Example 2 [Color figure can be viewed at wileyonlinelibrary.com]

Then, owing to (35), this system is is exponential stabilizable under the feedback controller (3) with $\theta_1 < 0.3433$, $\theta_2 < 0.59977$, and

$$K_1 = \begin{pmatrix} 23.7164 & -14.5903 \\ -17.5922 & 5.0226 \end{pmatrix},$$
$$K_2 = \begin{pmatrix} -12.7551 & 5.7211 \\ 3.5613 & -1.7707 \end{pmatrix}.$$

For $\theta_1 = \theta_2 = 0.3$, and

$$\sigma(t) = \begin{cases} 1, t \in [0.9l, 0.9l + 0.5), l \in N, \\ 2, t \in [0.9l + 0.5, 0.9l + 0.9), l \in N, \end{cases} \tag{36}$$

we have plotted the stable time response curves for the system (32) with the feedback controller (3) and the time curves of control input of the controller (3) in Figures 4 and 5, respectively. It is obvious that the control input is continuous at switching instants, which shows the effectiveness of the proposed bumpless transfer control.

5 | CONCLUSIONS

This paper has coped with the stabilization problem of switched neural networks with time-varying delay. A new switching state feedback controller whose control input is smooth at switching time is designed. According to MDADT, the theoretical results that ensure the closed-loop system is exponentially stable are established. The procedures that can be applied to calculating the control gain matrices are also proposed. Two simple numerical examples are employed to show effectiveness of the presented results. In the future work, we will concentrate on the output stabilization of nonlinear switched systems with time delay by bumpless transfer control.

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