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131

SUMAIRA REHMAN

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UNIVERSITY OF TARTU
Press

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Dissertation has been accepted for the commencement of the degree of Doctor of Philosophy (PhD) in mathematics on June 19, 2020 by the Council of the Institute of Mathematics and Statistics, Faculty of Science and Technology, University of Tartu.

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The public defense will take place on August 27, 2020 at 13:30 in Narva 18-1020.

Publication of this dissertation has been granted by the Institute of Mathematics and Statistics of the University of Tartu.

ISSN 1024-4212
ISBN 978-9949-03-402-4 (print)
ISBN 978-9949-03-403-1 (PDF)

University of Tartu Press
http://www.tyk.ee/

Contents

Chapter 1

Introduction

In the present thesis we discuss the bounds of fast solving Fredholm integral equations of the second kind with weakly singular kernels when the information about the smooth coefficient functions in the kernel and about the free term is restricted to a given number of their sample values. We shall consider linear integral equations, that is, equations involving an unknown function which appears under one or more integral signs and where the dependence on this function is linear.

The early history of integral equations goes back to the special integral equations studied by a number of scientists of eighteenth and nineteenth centuries [12]. At the end of the nineteenth century the interest in integral equations increased, mainly because of their intimate relationship with differential equations: initial and boundary value problems for differential equations can often be converted into integral equations and there are usually significant advantages to be gained from making use of this conversation.

Systematic study of integral equations started from the works of Italian mathematician Vito Volterra (1860–1940) and Swedish mathematician Ivar Igor Fredholm (1866–1927) at the late 19th and early 20th century, see the relevant references to their works in the monographs [3, 10–12]. In particular, in the early 1900s, Fredholm gave necessary and sufficient conditions for the solvability of a large class of integral equations. Especially, the Fredholm theory applies to the equations of the form

$$u(x) = \int_0^1 K(x,y)u(y)dy + f(x), \quad 0 \le x \le 1, \qquad (1.0.1)$$

which are nowadays usually referred to as linear Fredholm integral equations of the second kind. Here K and f are given functions and u is unknown function which we have to find. Function K is called the kernel of the equation (1.0.1) and the function f is referred to as free term, or as forcing function of (1.0.1). A condition such as $0 \leq x \leq 1$ following an equation in (1.0.1) indicates that this equation must hold for all values of the interval $[0, 1]$. The word "kind" in the terminology "second kind integral equation" refers to the location of the unknown function: first kind equations have the unknown function presented under the integral sign only, second kind equations also have the unknown function outside the integral. Although the first kind integral equation

$$\int_0^1 K(x, y)u(y)dy = f(x), \quad 0 \leq x \leq 1,$$

would seem to be of the same general type as the second kind integral equation (1.0.1), it occurs that these two equations are in fact fundamentally different in character. Integral equations of the first kind are usually classified as ill-conditioned, because their solutions u are sensitive to small changes in data functions K and f [3]. Loosely speaking, an ill-conditioned problem is one in which small changes in the data can lead to very large changes in the solution and therefore special methods for solving such problems are needed. For introduction to this topic, see [25, 26, 40, 68]. First kind integral equations are not studied in the present thesis.

Integral equations arise naturally in many mathematical models of various "real-world" phenomena. Other problems whose natural formulation are in terms of differential equations also provide a plentiful supply of integral equations. In particular, integral equations occur in areas such as study of epidemics [88], financial mathematics [42], viscoelasticity [16, 32], potential problems [33, 43], nuclear physics [7], atmosphere physics [5, 29, 73, 74] and in radiative heat exchange [90]. These equations also arise naturally in the theory of signal processing [65] and inverse problems [28, 30, 31].

If $K(x, y) = 0$ for $0 \leq x \leq y \leq 1$, then Fredholm integral equation (1.0.1) takes the form

$$u(x) = \int_0^x K(x, y)u(y)dy + f(x), \quad 0 \leq x \leq 1, \tag{1.0.2}$$

which is usually referred to as linear Volterra integral equation of the second kind. The classical Fredholm theory therefore also applies to Volterra equations, but, as pointed out in [41], it loses much of its power: the relationship between the two varieties of equations (1.0.1) and (1.0.2) is a useful one, but it is wrong to infer that the differences between them are minimal. Frequently a direct study of Volterra integral equations yields many results which cannot be obtained for Fredholm equations, see, for example, [80, 81]. A reader interested in studies on Volterra integral equations may consult the monographs [10–12, 41] which give a good picture of these developments and contain an extensive bibliography.

The main objects of study in the present thesis are fast methods for the numerical solution of Fredholm integral equations (1.0.1) with kernels of the form

$$K(x,y) = a(x,y)|\, x-y\,|^{-\nu} + b(x,y), \tag{1.0.3}$$

where $a(x,y)$ and $b(x,y)$ are some continuous functions for $(x,y)\in[0,1]\times[0,1]$ and $\nu\in(0,1)$ is a fixed parameter. It is easy to see that a kernel $K(x,y)$ of the from (1.0.3) is weakly singular [40, 63], that is, $K(x,y)$ is continuous for $y\neq x$ and may have at most a weak singularity as $y\to x$:

$$|\, K(x,y)\,| \leq c|\, x-y\,|^{-\nu}, \quad x,y\in[0,1], \quad y\neq x, \tag{1.0.4}$$

where $c>0$ and $\nu\in(0,1)$ are some constants. It must be remarked that the class of continuous kernels on $[0,1]\times[0,1]$ make up a subclass of weakly singular kernels (1.0.3): if in (1.0.3) the coefficient $a(x,y)=0$ for all $(x,y)\in[0,1]\times[0,1]$, then the kernel $K=b$ is a continuous function on the square $[0,1]\times[0,1]$. Note also that the condition $\nu<1$ in (1.0.3) and (1.0.4) is substantial since in this case the kernel $K(x,y)$ is integrable as a function of y and the following estimate holds:

$$\sup_{0\leq x\leq 1}\int_0^1 |\, K(x,y)\,|\, dy < \infty.$$

This is no longer true for kernels $K(x,y)$ which are strongly singular (Cauchy singular) for $y=x$. These are often typified by

$$K(x,y) = \frac{a(x,y)}{x-y},$$

where $a(x, y)$ is a bounded function for $(x, y) \in [0, 1] \times [0, 1]$. In this case the integral in (1.0.1) is divergent and therefore another concept of an integral is needed, see [23, 44, 46, 59, 60]. The integral equations with strongly singular kernels will not be considered in this thesis.

When working with integral equations that are modeling some problems from real world applications, it is only rarely possible to find the solution of a given integral equation on closed form. Therefore, in general, numerical methods are required for solving integral equations. As a consequence, various methods for the numerical solution of integral equations have been developed by many researchers in the past. In particular, among the monographs discussing the numerical solution of Fredholm integral equations of the second kind are the ones by Atkinson [3], Chen, Micchelli and Xu [14], Hackbusch [27], Krasnoselskii, Vainikko et al [39], Kress [40], Mikhlin [43], Saranen and Vainikko [63], Vainikko [74], Vainikko, Pedas and Uba [84], see also the survey papers [2, 9, 87].

When analyzing the convergence of a numerical method for a given integral equation one needs information about the smoothness of its exact solution on the domain on which the equation is to be solved. For Fredholm integral equations (1.0.1) with bounded kernels, the smoothness of the kernel K and the forcing function f determines the smoothness of the exact solution u (if it exist) on the closed interval of integration $[0, 1]$. A typical result here is formulated as follows. Let $m \in \mathbb{N}_0 := \{0, 1, \dots, \}$ be a fixed integer and let $K \in C^m([0, 1] \times [0, 1])$, $f \in C^m[0, 1]$, that is, K and f are some m times continuously differentiable functions on $[0, 1] \times [0, 1]$ and $[0, 1]$, respectively. Finally suppose that equation (1.0.1) has a solution u. Then $u \in C^m[0, 1]$.

However, if we admit weakly singular kernels (1.0.3), this result is no longer true: for $a, b \in C^m([0, 1] \times [0, 1])$, $f \in C^m[0, 1]$, the solution u of equation (1.0.1) in general belongs only to $C[0, 1] \backslash C^1[0, 1]$, regardless of the value $m \geq 1$. This will be made precise by Theorem 3.1.3 in Chapter 3, see also [24, 51, 53, 64, 74, 83, 84]. Thus the solutions $u(x)$ of Fredholm integral equations (1.0.1) with weakly singular kernels (1.0.3) are typically non-smooth at the endpoints $x = 0$ and $x = 1$ of the interval $[0, 1]$ (where, their derivatives become unbounded). This means that if a numerical method is possess a high order of convergence one has to take into account, in some way, the singular behavior of the exact solution near the boundary of $[0, 1]$. In piecewise polynomial projection methods this behavior can be reflected in the construction of special graded grids where the grid points are more densely clustered near the boundary of the interval $[0, 1]$, see [3, 27, 36, 50, 66, 67, 74, 84, 85]. However, using strongly graded non-uniform grids may create significant round-off errors in the calculations and lead to implementation difficulties. Thus, if one wants to retain uniform grids, then one will have to abandon polynomial spline spaces in favor of certain non-polynomial spline spaces whose dimension will be considerably larger [12]. Another way to avoid graded grids is to proceed as follows: first we regularize the exact solution of the underlying weakly singular integral equation by a suitable smoothing transformation and after that we apply a numerical method to the transformed integral equation on a uniform grid, cf., [19, 45, 48, 49, 72].

In the present thesis we use a smoothing transformation to regularize the exact solution of the underlying problem together with periodization techniques in order to construct fast solvers for Fredholm integral equations (1.0.1) with weakly singular kernels (1.0.3).

In a fast solver, the conditions of optimal accuracy and minimal arithmetical operations (complexity of the solver) are met. We mean the order optimality and order minimal work on a class of problems. In our case the class of problems is defined by the smoothness conditions which we will set on the free term $f(x)$ and the kernel $K(x, y) = a(x, y) \mid x - y \mid^{-\nu} + b(x, y)$ of equation (1.0.1). We introduce the notions of fast and quasifast solvers more exactly (see Definitions 3.2.1 and 3.2.2) in Chapter 3.

In particular, we consider a class of equations (1.0.1) where the free term f of (1.0.1) belongs to $C^{m,\nu}(0, 1)$, $m \in \mathbb{N} = \{1, 2, \dots, \}$, $\nu \in (0, 1)$, the kernel K is determined by the formula (1.0.3), with $a, b \in C^{2m}([0, 1] \times [0, 1])$, and equation (1.0.1) is assumed to possess a unique solution $u \in C = C[0, 1]$. By $C^{m,\nu} = C^{m,\nu}(0, 1)$ we denote the Banach space of functions f which are continuous on the closed interval $[0, 1]$ and m times continuously differentiable on the open interval $(0, 1)$ and such that

$$\| f \|_{C^{m,\nu}} := \max_{0 \le x \le 1} \mid f(x) \mid + \sum_{k=1}^{m} \sup_{0 < x < 1} [x(1 - x)]^{k-1+\nu} \mid f^{(k)}(x) \mid < \infty.$$

Thus, if $f \in C^{m,\nu}(0, 1)$, then $f \in C[0, 1]$, but the derivatives of f may be unbounded near the boundary of $[0, 1]$:

$$\mid f^{(k)}(x) \mid \le c[x(1 - x)]^{1-\nu-k}, \ k = 1, \dots, m,$$

where $c = c(f)$ is a positive constant not depending on $x \in (0, 1)$.

Our aim is to produce approximate solutions u_n $(n \in \mathbb{N})$ to equations

$$u(x) = \int_0^1 [a(x,y) \mid x - y \mid^{-\nu} + b(x,y)]u(y) + f(x), \quad 0 \le x \le 1, \quad (1.0.5)$$

such that

1) given the values of f, a, b at $O(n_\star)$ suitable chosen points (where $n_\star = n_\star(n) \to \infty$ as $n \to \infty$) the parameters of u_n are available at the cost of $O(n_\star)$ flops, and the accuracy

$$\sup_{0 \le x \le 1} \mid u(x) - u_n(x) \mid \le c_m {n_\star}^{-m} \| f \|_{C^{m,\nu}} \quad (1.0.6)$$

is achieved, where u is the exact solution of equation (1.0.5) and c_m is a positive constant that is independent of n and f;

2) having determined the parameters of u_n, the value of $u_n(x)$ at any particular point $x \in [0, 1]$ is available at the cost of $O(1)$ flops.

We call such methods fast $(C, C^{m,\nu})$ solvers of equation (1.0.5). Thus, a fast $(C, C^{m,\nu})$ solver is a method of optimal accuracy order to solve (1.0.5); the convergence order (1.0.6) is the best that one can achieve for $f \in C^{m,\nu}(0, 1)$ and a, $b \in C^{2m}([0, 1] \times [0, 1])$ by a method of $O(n_\star)$ flops. It is optimal also in the sense of information: to obtain the convergence order (1.0.6) for all $f \in C^{m,\nu}(0, 1)$, $\| f \|_{C^{m,\nu}} \le 1$, at least $O(n_\star)$ values of f, a, b must be involved.

Further, we speak about a $(L^p, C^{m,\nu})$-fast solver $(1 \le p < \infty)$ if we have instead of (1.0.6) that

$$\| u - u_n \|_p \le c_m {n_\star}^{-m} \| f \|_{C^{m,\nu}},$$

where $\| u \|_p = (\int_0^1 \mid u(x) \mid^p dx)^{1/p}$ is the norm of u in $L^p(0, 1)$; similarly, a solver is (C, C^m) fast if in (1.0.5) the norm $\| f \|_{C^{m,\nu}}$ is replaced by $\| f \|_{C^m}$, where $\| f \|_{C^m}$ is the standard norm of f in $C^m[0, 1]$, see Section 2.1.

We speak about a $(C, C^{m,\nu})$ quasi-fast solver of (1.0.5) if the accuracy requirement (1.0.6) is replaced by

$$\sup_{0 \le x \le 1} \mid u(x) - u_n(x) \mid \le c_m {n_\star}^{-m} (\log n) \| f \|_{C^{m,\nu}},$$

that is, if in the right hand side of (1.0.6) a supplementary factor $\log n$ is allowed.

An intensive investigation of optimal algorithms and complexity for various problems was started in [22, 69, 70]. In these works the fundamentals of general theory of optimal algorithms were introduced. In [54–56, 89] the complexity of the approximate solutions of Fredholm integral equations of the second kind with smooth kernels has been studied. Actually, also in [77, 78, 82, 86] fast or quasi-fast solvers are constructed, but only for integral equations without singularities; for (1.0.5) this corresponds to the case $a(x, y) \equiv 0$. The fast or quasi-fast solving of periodic weakly singular integral equations of the second kind has been undertaken in [52], see also [17, 21, 57, 63]

The construction of fast and quasi-fast $(C, C^{m,\nu})$ solvers for non-periodic integral equations of the second kind with weakly singular kernels was an open problem for a long time. Only in the papers [61] and [62] this question has been discussed and a fast (or, at least, quasi-fast) solver for (1.0.5) has been constructed. In Chapter 3 and 4 of this thesis we follow the ideas of [61] and [62], respectively.

The thesis consists of four chapters.
Chapter 1 and 2 have an introductory character. In Chapter 2 we introduce some basic notions and results which we need in this thesis.

In Chapter 3 we first give the definitions of fast and quasi-fast methods for solving (1.0.5). After that we reduce the problem (1.0.5) into a periodic problem and construct an approximate method for the periodized problem. In the course of Sections 3.3–3.7 we obtain a solver which under the conditions introduced above is $(C, C^{m,\nu})$ quasi-fast and $(L^p, C^{m,\nu})$ fast for $1 \leq p < \infty$. Moreover, we show that this solver is $(C, C^{m,\nu})$ fast under some strengthened conditions on the functions a and b in (1.0.5).

In Chapter 4 we modify the approach used in Chapter 3. In Chapter 3 a solver is presented and justified which is of optional order in the sense of accuracy and minimal amount of arithmetical work to realize it numerically. This solver remains to be of optimal order also for the solutions of problems (1.0.5) with a, $b \in C^{2m}([0, 1] \times [0, 1])$ and $f \in C^{m+1}[0, 1]$, but the restriction $f \in C^{m+1}[0, 1]$ allows to modify the solver so that its numerical realization is essentially easier than the realization of the previously constructed solver. This is achieved by the approximation of f in (1.0.5) by trigonometric interpolation projection $Q_n f$ instead of orthogonal projection $P_n f$ (their definitions are given in Subsection 2.6.3) and Chapter 4 is devoted to the justification of this idea. We emphasize that the complexity order of the problem (1.0.5) with $f \in C^{m+1}[0,1]$ remains the same as that one with f from the more general class $C^{m,\nu}(0,1)$ in Chapter 3. So our main result in Chapter 4 concerns not the improvement of the error estimates in the definitions of fast and quasi-fast solvers, but the simplification of the numerical algorithm of a fast or quasi-fast solver in the case $f \in C^{m+1}[0,1]$.

Thus, in Chapter 4 we construct a quasi-fast (C, C^{m+1}) solver for the problem (1.0.5) which is (L^p, C^{m+1}) fast, $1 \leq p < \infty$. This new solver has an advantage compared with the method considered in Chapter 3, since the grid values of the free term in the matrix form of this new method are easily available from the values of f, which is not the case in Chapter 3.

Chapter 2

Preliminary Results

In this chapter we collect some notions and basic known results which we need later.

2.1 Some basic notations and results

.

Throughout this work $c, c', c_0, c_1, \dots$ denote positive constants which may have different values in different occurrences. By $\mathbb{N} = \{1, 2, \dots\}$ we denote the set of all positive integers, by $\mathbb{N}_0 = \{0\} \cup \mathbb{N}$ the set of non-negative integers, by $\mathbb{Z} = \{\dots, -1, 0, 1, 2, \dots\}$ the set of integers, by $\mathbb{R} = (\infty, +\infty)$ the set of real numbers and by $\mathbb{C}$ set of complex numbers.

Let $m \in \mathbb{N}_0$ be a given integer. We will use the following notations for spaces and norms in them.

By $C[0,1]$ we denote the Banach space of continuous functions u on $[0,1]$ with the norm

$$\| u \|_{C[0,1]} = \| u \|_\infty = \max_{0 \le x \le 1} | u(x) |;$$

$C^m = C^m[0,1]$ is the Banach space of m times $(m \geq 1)$ continuously differentiable functions u on $[0,1]$,

$$\| u \|_{C^m[0,1]} = \sum_{k=0}^{m} \| u^{(k)} \|_\infty = \sum_{k=0}^{m} \max_{0 \le x \le 1} | u^{(k)}(x) |;$$

$\widetilde{C}^m = \widetilde{C}^m(\mathbb{R})$ is the space of 1-periodic functions $\widetilde{u}$ on $\mathbb{R}$ with the same norm $\| \widetilde{u} \|_{C^m}$; $C = C[0,1] = C^0[0,1]$, $\quad \widetilde{C} = \widetilde{C}(\mathbb{R}) = \widetilde{C}^0(\mathbb{R})$; recall that a function $\widetilde{u} = \widetilde{u}(x)$ is 1-periodic on $\mathbb{R}$ if $\widetilde{u}(x+1) = \widetilde{u}(x)$ for all $x \in \mathbb{R}$.

By $L^p(0,1)$ $(1 \leq p < \infty)$ we denote the Banach space of measurable functions u on $[0,1]$ such that

$$\| u \|_p = \| u \|_{L^p(0,1)} = \left(\int_0^1 | u(x) |^p dx \right)^{1/p} < \infty;$$

$\widetilde{L}^p = \widetilde{L}^p(\mathbb{R})$, $1 \leq p < \infty$, is the space of 1-periodic functions $\widetilde{u}$ on $\mathbb{R}$ with the same norm: $\| \widetilde{u} \|_p = \left(\int_0^1 | \widetilde{u}(x) |^p \, dx \right)^{1/p}$, $1 \leq p < \infty$.

By $L^\infty(0,1)$ we denote the Banach space of measurable functions u on $[0,1]$ such that

$$\inf_{\Omega \subset [0,1]; \mu(\Omega)=0} \sup_{x \in [0,1] \setminus \Omega} | u(x) | < \infty,$$

where $\mu(\Omega)$ is the Lebesgue measure of set Ω; the norm of this space is defined as

$$\| u \|_{L^\infty(0,1)} = \| u \|_\infty = \inf_{\Omega \subset [0,1]; \mu(\Omega)=0} \sup_{x \in [0,1] \setminus \Omega} | u(x) |;$$

$\widetilde{L}^\infty = \widetilde{L}^\infty(\mathbb{R})$ is the space of 1-periodic function $\widetilde{u}$ on $\mathbb{R}$ with the same norm:

$$\| \widetilde{u} \|_{L^\infty(\mathbb{R})} = \| \widetilde{u} \|_\infty = \inf_{\Omega \subset [0,1]; \mu(\Omega)=0} \sup_{x \in [0,1] \setminus \Omega} | \widetilde{u}(x) | .$$

By $W^{m,p}(0,1)$, $1 \leq p \leq \infty$, $m \in \mathbb{N}$ is denoted the Sobolev space of functions u on $(0,1)$ equipped with the norm

$$\| u \|_{W^{m,p}} = \left(\sum_{k=0}^m \| u^{(k)} \|_p^p \right)^{1/p}, \quad 1 \leq p < \infty; \quad \| u \|_{W^{m,\infty}} = \sum_{k=0}^m \| u^{(k)} \|_\infty;$$

$\widetilde{W}^{m,p} = \widetilde{W}^{m,p}(\mathbb{R})$, $1 \leq p \leq \infty$, is the Sobolev space of 1-periodic functions $\widetilde{u}$ with the same norm $\| \widetilde{u} \|_{W^{m,p}}$.

By $\mathcal{H}^{m,\lambda} = \mathcal{H}^{m,\lambda}[0,1]$, $0 < \lambda \leq 1$, is denoted the Hölder space of functions $u \in C^m[0,1]$ with $u^{(m)}$ satisfying the Hölder condition with the exponent λ,

$$\| u \|_{\mathcal{H}^{m,\lambda}} = \| u \|_{C^m} + \sup \left\{ \frac{| u^{(m)}(x) - u^{(m)}(y) |}{| x - y |^\lambda} \ : \ x, y \in [0,1], \, x \neq y \right\};$$

$\widetilde{\mathcal{H}}^{m,\lambda}(\mathbb{R})$, $0 < \lambda \leq 1$, is the Hölder space of 1-periodic functions $\widetilde{u}$ with the same norm $\| \widetilde{u} \|_{\mathcal{H}^{m,\lambda}}$.

Definition 2.1.1. Let X be a Banach space. A subset $X_1 \subset X$ is called relatively compact in X if any sequence $(x_n) \subset X_1$ contains a subsequence converging in X.

A helpful result in connection with such sets in $C[0,1]$ is the following theorem; its proof can be found in many textbooks e.g. in [40].

Theorem 2.1.1. (*Arzela–Ascoli*) *A set $S \subset C[0,1]$ is relatively compact in $C[0,1]$ if and only if the following two conditions are fulfilled:*
(i) the functions $u \in S$ are uniformly bounded, i.e., there is a constant c such that $|u(x)| \leq c$ for all $x \in [0,1], u \in S$;
(ii) the functions u are equicontinuous, i.e., for every $\epsilon > 0$ there exists a $\delta > 0$ such that $x_1, x_2 \in [0,1]$, $|x_1 - x_2| \leq \delta$ implies $|u(x_1) - u(x_2)| \leq \epsilon$ for all $u \in S$.

An important rule for differentiating composite functions is the chain rule: if u and φ are two differentiable functions, then

$$\frac{d}{dx} u(\varphi(x)) = u'(\varphi(x))\varphi'(x).$$

This rule can be generalized to the case of *mth* derivatives with arbitrary $m \in \mathbb{N}$. The result is known as Faà di Bruno's formula. It can be formulated as follows, see [35, 38].

Theorem 2.1.2. (*Faà di Bruno*) *Let $m \in \mathbb{N}$ and u be an m times continuously differentiable function on an interval which contains the values of $\varphi \in C^m[0,1]$. Then the composite function $u(\varphi(x))$ is m times continuously differentiable for $x \in [0,1]$ and the derivatives of the composition function at any point $x \in [0,1]$ can be expressed by Faà di Bruno's differentiation formula*

$$\left(\frac{d}{dx}\right)^j u(\varphi(x)) = \sum_{k_1+2k_2+\cdots+jk_j=j} \frac{j!}{k_1! \ldots k_j!} \quad u^{(k)}(\varphi(x)) \left(\frac{\varphi'(x)}{1!}\right)^{k_1} \cdots \left(\frac{\varphi^{(j)}(x)}{j!}\right)^{k_j}, \tag{2.1.1}$$

where $x \in [0,1]$, $k = k_1 + \cdots + k_j$, *and the sum is taken over all non-negative integers* $k_1, \ldots, k_j$, *such that* $k_1 + 2k_2 + \cdots + jk_j = j$, $j = 1, \ldots, m$.

Next we recall the definitions and some properties of Euler's gamma and beta functions, see, for example, [1, 58].

The gamma function $\Gamma(x)$ is defined by the formula

$$\Gamma(x) = \int_0^\infty e^{-s} s^{x-1} ds, \quad 0 < x < \infty. \tag{2.1.2}$$

Elementary considerations from the theory of improper integrals reveal that the integral in (2.1.2) exists for any $x > 0$. Moreover, setting $x = 1$, we easily see that

$$\Gamma(1) = \int_0^\infty e^{-s} ds = \lim_{z\to\infty} \int_0^z e^{-s} ds = \lim_{z\to\infty} (1 - e^{-z}) = 1. \tag{2.1.3}$$

Additionally we may, for arbitrary $x > 0$, manipulate the integral in the definition of $\Gamma(x)$ by mean of a partial integration. This yields

$$\begin{aligned} \Gamma(x+1) &= \int_0^\infty e^{-s} s^x ds = \lim_{z\to\infty,\, y\to 0} \int_y^z e^{-s} s^x ds \\ &= x \int_0^\infty e^{-s} s^{x-1} ds = x\Gamma(x), \quad x > 0. \end{aligned}$$

We have thus shown that

$$\Gamma(x+1) = x\Gamma(x), \quad x > 0, \tag{2.1.4}$$

which together with (2.1.3) yields

$$\Gamma(n+1) = n!\,, \quad n \in \mathbb{N}. \tag{2.1.5}$$

One important result that we shall need later is the integral identity

$$\int_0^1 s^{x-1}(1-s)^{y-1} ds = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}, \tag{2.1.6}$$

where $x > 0$ and $y > 0$ are given. The integral in the left side of (2.1.6) is known as Euler's integral of the first kind or Euler's beta function $B(x,y)$:

$$B(x,y) = \int_0^1 s^{x-1}(1-s)^{y-1} ds, \quad x, y \in (0,\infty). \tag{2.1.7}$$

It follows from (2.1.6) and (2.1.7) that gamma and beta functions are related by the equality

$$B(x,y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}, \quad x, y \in (0,\infty). \tag{2.1.8}$$

In particular, with the help of (2.1.3), (2.1.4) and (2.1.8) we get

$$B(n,n) = \frac{[(n-1)!\,]^2}{(2n-1)!}, \quad n \in \mathbb{N}. \tag{2.1.9}$$

2.2 Linear operators and Fredholm alternative

In this section we will introduce some results from the theory of linear operators, see [4, 40].

Let X and Y be Banach spaces. A linear operator $A : X \to Y$ is called bounded if there exists a positive constant c such that

$$\| Ax \|_Y \le c \| x \|_X \quad \forall x \in X.$$

An operator $A : X \to Y$ is said to be continuous if

$$\| x_n - x \|_X \to 0, \quad n \to \infty$$

implies

$$\| Ax_n - Ax \|_Y \to 0, \quad n \to \infty.$$

A linear operator $A : X \to Y$ is continuous if and only if it is bounded.

One says that a linear operator $A : X \to Y$ has the inverse $A^{-1} : Y \to X$ if $A^{-1}A = I_X$ and $AA^{-1} = I_Y$ where I_X and I_Y are the identity mappings in X and Y, respectively.

For a linear operator $A : X \to Y$ we denote by

$$\mathcal{N}(A) = \{x \in X : Ax = 0\}$$

the null space of A, and by

$$\mathcal{R}(A) = \{y \in Y : y = Ax\}$$

the range of A. By $\mathcal{L}(X,Y)$ we denote the Banach space of linear bounded operators $A : X \to Y$ with the norm

$$\| A \|_{\mathcal{L}(X,Y)} = \{\sup \| Ax \|_Y : x \in X, \| x \|_X \leq 1\}.$$

Later instead of $\| A \|_{\mathcal{L}(X,Y)}$ we will also use the notation $\| A \|_{X \to Y}$.

Clearly, $\mathcal{N}(A) \subset X$ and $\mathcal{R}(A) \subset Y$ are subspaces; if $A \in \mathcal{L}(X, Y)$, then $\mathcal{N}(A)$ is closed.

Theorem 2.2.1. (*Banach*) *Let X and Y be Banach spaces and $A \in \mathcal{L}(X, Y)$. If $\mathcal{N}(A) = 0$ and $\mathcal{R}(A) = Y$ then A has the inverse $A^{-1} \in \mathcal{L}(Y, X)$.*

Theorem 2.2.2. (*Banach–Steinhaus*) *Let $A : X \to Y$ be a linear bounded operator and let (A_n), $n \in \mathbb{N}$, be a sequence of linear bounded operators $A_n : X \to Y$ from a Banach space X into a Banach space Y. For pointwise convergence*

$$A_n x \to Ax,\ n \to \infty \quad \textit{for all} \quad x \in X,$$

it is necessary and sufficient that

$$\| A_n \|_{\mathcal{L}(X,Y)} \leq c \quad \textit{for all} \quad n \in \mathbb{N}$$

with some constant c and that

$$A_n x \to Ax, n \to \infty \quad \textit{for all} \quad x \in V,$$

where V is some dense subset of X.

Theorem 2.2.3. *Let X be a Banach space, and let $A \in \mathcal{L}(X, X)$ be a bounded linear operator from X into X with $\| A \|_{\mathcal{L}(X,X)} < 1$. Then there exists $(I - A)^{-1} \in \mathcal{L}(X, X)$, and*

$$\| (I - A)^{-1} \|_{\mathcal{L}(X,X)} \leq \frac{1}{1 - \| A \|_{\mathcal{L}(X,X)}},$$

where I is the identity mapping in X.

Theorem 2.2.4. *Let X and Y be Banach spaces. If the operators $A, B \in \mathcal{L}(X, Y)$ are such that A has a bounded inverse $A^{-1} \in \mathcal{L}(Y, X)$ and*

$$\| B \|_{\mathcal{L}(X,Y)} \| A^{-1} \|_{\mathcal{L}(Y,X)} < 1,$$

then $A+B$ has a bounded inverse $(A+B)^{-1} \in \mathcal{L}(Y,X)$ and

$$\| (A+B)^{-1} \|_{\mathcal{L}(Y,X)} \leq \frac{\| A^{-1} \|_{\mathcal{L}(Y,X)}}{1 - \| B \|_{\mathcal{L}(X,Y)} \| A^{-1} \|_{\mathcal{L}(Y,X)}}.$$

Definition 2.2.1. Let X and Y be Banach spaces. A linear operator $A : X \to Y$ is called compact if A transforms every bounded set of X into a relatively compact set of Y .

Equivalently, $A : X \to Y$ is compact if for every bounded sequence $(u_n) \subset X$, the sequence (Au_n) contains a sub-sequence that converges in Y.

A linear compact operator $A : X \to Y$ from a Banach space X into a Banach space Y is bounded and thus continuous.

Theorem 2.2.5. *Let $A_n : X \to Y$, $n = 1, 2, \ldots,$ be linear compact operators, $A : X \to Y$ a linear bounded operator, and let $\| A_n - A \|_{\mathcal{L}(X,Y)} \to 0$ as $n \to \infty$. Then $A : X \to Y$ is compact.*

Theorem 2.2.6. (*Fredholm alternative*) *Let X be a Banach space, and let $A \in \mathcal{L}(X,X)$ be a compact operator. Then the equation $x = Ax + f$ with $f \in X$ has a unique solution $x \in X$ if and only if the homogeneous equation $x = Ax$ has only the trivial solution $x = 0$. In such a case, the operator $I - A$ has a bounded inverse $(I - A)^{-1} \in \mathcal{L}(X,X)$.*

2.3 Weighted space $C^{m,\nu}(0,1)$

Let $m \in \mathbb{N}$ and $\nu \in (0,1)$ be given. In order to describe the smoothness properties of a solution of a weakly singular integral equation we introduce a weighted space $C^{m,\nu}(0,1)$ of smooth functions on $(0,1)$. As a matter of fact, $C^{m,\nu}(0,1)$ is an adaptation of a class of functions that were introduced by Vainikko in [74] for describing the possible singular behavior of solutions of multidimensional weakly singular integral equations, see also [13, 51, 53].

For given $m \in \mathbb{N}$ and $\nu \in (0,1)$ by $C^{m,\nu} = C^{m,\nu}(0,1)$ we denote the set of continuous functions f on $[0,1]$ which are m times continuously differentiable in $(0,1)$ and such that

$$| f^{(j)}(x) | \leq c[x(1-x)]^{1-\nu-j}, \quad 0 < x < 1, \quad j = 1, \ldots, m, \qquad (2.3.1)$$

where $c = c(f) > 0$ is a constant not depending on x. In other words,

$f \in C^{m,\nu}(0,1)$ if $f \in C[0,1] \cap C^m(0,1)$ and

$$\sum_{j=1}^{m} \sup_{0<x<1} [x(1-x)]^{j+\nu-1} \mid f^{(j)}(x) \mid < \infty.$$

Equipped with the norm

$$\begin{aligned} \| f \|_{C^{m,\nu}} &= \max_{0 \le x \le 1} \mid f(x) \mid \\ &+ \sum_{j=1}^{m} \sup_{0<x<1} [x(1-x)]^{j+\nu-1} \mid f^{(j)}(x) \mid, \; f \in C^{m,\nu}(0,1), \end{aligned} \tag{2.3.2}$$

the set $C^{m,\nu}(0,1)$ becomes a Banach space.

Note that $C^m[0,1]$ $(m \in \mathbb{N})$ belongs to $C^{m,\nu}(0,1)$ for arbitrary ν, $0 < \nu < 1$. Some other examples are given by

$$f_1(x) = [x(1-x)]^{1/2},$$

$$f_2(x) = x^{1/10}$$

and

$$f_3(x) = (1-x)^{3/4},$$

where $0 \le x \le 1$. Clearly,

$$f_1 \in C^{m,1/2}(0,1), \quad f_2 \in C^{m,9/10}(0,1), \quad f_3 \in C^{m,1/4}(0,1),$$

with arbitrary $m \in \mathbb{N}$. Further, it is easy to see that, for q, $m \in \mathbb{N}$

$$C^q[0,1] \subset C^{q,\nu}(0,1) \subset C^{m,\mu}(0,1) \subset C[0,1], \; q \ge m \ge 1, \; 0 < \nu \le \mu < 1. \tag{2.3.3}$$

Observe that as ν increases so does the singularity order of the derivatives of the functions in $C^{m,\nu}(0,1)$.

It follows from [51] the following result, see also [74].

Theorem 2.3.1. *Let $m \in \mathbb{N}$ and $\nu \in (0,1)$ be given. Define an integral* *operator T by the formula*

$$(Tu)(x) = \int_0^1 \Big[a(x,y) |\, x - y \,|^{-\nu} + b(x,y) \Big] u(y) dy, \quad x \in [0,1],$$

where $a, b \in C^m([0,1] \times [0,1])$. Then T maps $C^{m,\nu}(0,1)$ into itself and T is compact as an operator from $C^{m,\nu}(0,1)$ into $C^{m,\nu}(0,1)$.

2.4 Smoothing transformation

Let $\varphi : [0,1] \to [0,1]$ be defined by the formula

$$\varphi(t) = \frac{1}{B(r,r)} \int_0^t \sigma^{r-1}(1-\sigma)^{r-1} d\sigma, \quad t \in [0,1], \quad r \in \mathbb{N}, \qquad (2.4.1)$$

where

$$B(r,r) = \int_0^1 \sigma^{r-1}(1-\sigma)^{r-1} d\sigma$$

is the Euler's beta function (2.1.7), with $a = b = r$, see Section 2.1.

In the case $r = 1$ it follows from (2.4.1) that $\varphi(t) = t$. We are interested in transformations (2.4.1) with $r \geq 2$, since such transformations then will possess a smoothing property for a function $f \in C^{m,\nu}(0,1)$, see Theorem 2.4.1 below. The smoothing parameter $r \geq 1$ could also be a real number, but for simplicity of the following presentation we restrict ourselves to $r \in \mathbb{N}$. Note also that it follows from the formula (2.1.9) that the coefficient before the integral in (2.4.1) can be calculated as follows:

$$\frac{1}{B(r,r)} = \frac{(2r-1)!}{[(r-1)!\,]^2}.$$

From the definition (2.4.1) we get

$$\varphi(0) = 0, \quad \varphi(1) = 1, \quad \varphi'(t) = t^{r-1}(1-t)^{r-1} > 0 \quad \text{for} \quad 0 < t < 1,$$

that is, φ is a strictly increasing function on $[0,1]$. Thus, there exists a continuous inverse $\varphi^{-1} : [0,1] \to [0,1]$, with $\varphi^{-1}(0) = 0$ and $\varphi^{-1}(1) = 1$. Moreover, for a given $r \in \mathbb{N}$ it follows from (2.4.1) that $\varphi \in C^r[0,1]$ and, if

$r \geq 2$, then

$$\varphi^{(j)}(0) = \varphi^{(j)}(1) = 0, \quad j = 1, \ldots, r-1; \quad \varphi^{(r)}(0) \neq 0, \quad \varphi^{(r)}(1) \neq 0. \tag{2.4.2}$$

Theorem 2.4.1. *Suppose that $f \in C^{m,\nu}(0,1)$, with $m \in \mathbb{N}$ and $\nu \in (0,1)$. Denote $\mathsf{h}(t) = f(\varphi(t))$, where $0 \leq t \leq 1$ and φ is defined by the formula (2.4.1). Let $r \in \mathbb{N}$ be such that*

$$r > \frac{m}{1-\nu}. \tag{2.4.3}$$

Then $\mathsf{h} \in C^m[0,1]$ and

$$\mathsf{h}^{(j)}(0) = \mathsf{h}^{(j)}(1) = 0, \quad j = 1, \ldots, m. \tag{2.4.4}$$

Proof. Clearly, $\mathsf{h} \in C[0,1] \cap C^m(0,1)$. This yields that we have to study only the boundary behavior of the derivatives of the function h. Actually, we need to show that

$$\mathsf{h}^{(j)}(0) := \lim_{t \to 0} \mathsf{h}^{(j)}(t) = 0, \quad j = 1, \ldots, m, \tag{2.4.5}$$

and

$$\mathsf{h}^{(j)}(1) := \lim_{t \to 1} \mathsf{h}^{(j)}(t) = 0, \quad j = 1, \ldots, m. \tag{2.4.6}$$

We establish the relation (2.4.5); for (2.4.6) the argument is similar.

Let $0 < t \leq 1/2$. Then

$$\varphi^{(j)}(t) \leq c_j t^{r-j}, \quad j = 0, 1, \ldots, r, \tag{2.4.7}$$

where $c_0, \ldots, c_r$ are some positive constants which are independent of t. Since $f \in C^{m,\nu}(0,1)$, it follows from (2.3.1) and (2.4.7) by the formula of

Faà di Bruno (2.1.1) (see Theorem 2.1.2) that

$$
\begin{aligned}
\mid \mathsf{h}^{(j)}(t) \mid &= \left| \left(\frac{d}{dt}\right)^j f(\varphi(t)) \right| \\
\leq \sum_{\substack{k_1,\dots,k_j \in \mathbb{N}_0: \\ k_1+2k_2+\dots+jk_j=j}} & \frac{j!}{k_1!\dots k_j!} f^{(k_1+\dots+k_j)}(\varphi(t)) \left(\frac{\varphi'(t)}{1!}\right)^{k_1} \dots \left(\frac{\varphi^{(j)}(t)}{j!}\right)^{k_j} \\
&\leq c \sum_{\substack{k_1,\dots,k_j \in \mathbb{N}_0: \\ k_1+2k_2+\dots+jk_j=j}} t^{r(1-\nu-k_1-\dots-k_j)} t^{(r-1)k_1} \dots t^{(r-j)k_j} \\
&\leq c_1 t^{r(1-\nu)-j},
\end{aligned}
$$

where $j = 1, \dots, m$. This together with (2.4.3) yields (2.4.5). □

2.5 Interpolation by polynomials

For n, $m \in \mathbb{N}$ denote $h = 1/n$ and $\mathbb{Z}_m = \{k \in \mathbb{Z} : -\frac{m}{2} < k \leq \frac{m}{2}\}$. To $u \in \widetilde{C}(\mathbb{R})$, assign the piecewise polynomial interpolant $\Pi_{h,m} u \in \widetilde{C}(\mathbb{R})$ which is defined on every subinterval $[ih, (i+1)h]$ independently of other subintervals as an algebraic polynomial u_i of degree $m-1$ (of order m) such that $u_i(jh) = u(jh)$ for $j - i \in \mathbb{Z}_m$. Well known error estimates for a polynomial interpolant (cf. [4, 48]) yield the following results.

Theorem 2.5.1. *(i) For $u \in \widetilde{C}^m(\mathbb{R})$ $(m \in \mathbb{N})$, it holds*

$$\| u - \Pi_{h,m} u \|_\infty \leq c n^{-m} \| u^{(m)} \|_\infty,$$

$$\max_{x \in \mathbb{R}} |u^{(k)}(x) - (\Pi_{h,m} u)^{(k)}(x)| \leq c n^{-(m-k)} \| u^{(m)} \|_\infty, \quad k = 0, \dots, m-1;$$

at $x = ih$, $i \in \mathbb{Z}$, this holds for both one side limits of $(\Pi_{h,m} u)^{(k)}(x)$.

(ii) For $u \in \widetilde{\mathcal{H}}^{m,\lambda}(\mathbb{R})$, $m \in \{0\} \cup \mathbb{N}$, $0 < \lambda < 1$, it holds

$$\| u - \Pi_{h,m+1} u \|_\infty \leq c n^{-m-\lambda} \| u^{(m)} \|_{\mathcal{H}^{m,\lambda}} .$$

2.6 Trigonometric interpolation

In this section we present shortly some known results about trigonometric interpolation which will be used later. For a more detailed discussion see [40, 63, 75, 91]. We start from the standard results on Fourier series in one variable.

2.6.1 Fourier series

For a function $v \in L^2(0,1)$ the series

$$\sum_{n=-\infty}^{\infty} \hat{v}(n) e^{\mathrm{i}n2\pi t}, \tag{2.6.1.1}$$

where $\mathrm{i} = \sqrt{-1}$ and

$$\hat{v}(n) = \int_0^1 v(t) e^{-\mathrm{i}n2\pi t} dt \quad (n \in \mathbb{Z}),$$

is called the Fourier series of v, its coefficients $\hat{v}(n)$ are called the Fourier coefficients of v. On $L^2(0,1)$, as usual, the mean square norm $\| v \|_{L^2(0,1)} = (v,v)^{1/2}$ of an element $v \in L^2(0,1)$ is introduced by the scalar product

$$(v,w) = \int_0^1 v(t)\, \overline{w(t)}\, dt; \quad v, w \in L^2(0,1).$$

Let us denote by h_n trigonometric monomials

$$h_n(t) = e^{\mathrm{i}n2\pi t}$$

for $t \in \mathbb{R}$ and $n \in \mathbb{Z}$. Then the set $\{h_n : n \in \mathbb{Z}\}$ is an orthogonal system. By the Weirstrass approximation theorem (see [18]), the set of trigonometric polynomials

$$\left\{ \sum_{k \in \mathbb{Z}: -\frac{n}{2} < k \leq \frac{n}{2}} e^{\mathrm{i}k2\pi t}, \quad c_k \in \mathbb{C},\ t \in \mathbb{R} \right\}$$

is dense with respect to the maximum norm in the space of 1-periodic continuous functions $\widetilde{C}(\mathbb{R})$, and $C[0,1]$ is dense in $L^2(0,1)$ in the mean square norm. Therefore (see [40]), the orthogonal system $\{h_n\}$ is complete and the Fourier series (2.6.1.1) converges in the mean square norm to v.

Thus, every function $v \in L^2(0,1)$ can be represented by its Fourier series

$$v(t) = \sum_{n\in\mathbb{Z}} \hat{v}(n) e^{\mathrm{i}n2\pi t}$$

which converges in $L^2(0,1)$.

2.6.2 Representation forms of trigonometric polynomials

For $n \in \mathbb{N}$, we denote

$$\mathbb{Z}_n = \{k \in \mathbb{Z} : -\frac{n}{2} < k \leq \frac{n}{2}\}, \qquad \mathcal{T}_n = \operatorname{span}\{e^{\mathrm{i}k2\pi t} : k \in \mathbb{Z}_n\}.$$

Thus $\mathcal{T}_n$ consists of trigonometric polynomials, $\dim \mathcal{T}_n = n$.

There are two possible representations of $v_n \in \mathcal{T}_n$ – through its Fourier coefficients $\hat{v}_n(k)$,

$$\hat{v}_n(k) = \int_0^1 v_n(t) e^{-\mathrm{i}k2\pi t} dt,\ k \in \mathbb{Z}_n,$$

and through its nodal values $v_n(jn^{-1}),\quad j = 0, \ldots, n-1$:

$$v_n(t) = \sum_{k\in\mathbb{Z}_n} \hat{v}_n(k) e^{\mathrm{i}k2\pi t}; \tag{2.6.2.2}$$

$$v_n(t) = \sum_{j=0}^{n-1} v_n(jn^{-1}) \phi_{n,j}(t), \quad \phi_{n,j}(t) = \frac{1}{n} \sum_{k\in\mathbb{Z}_n} e^{\mathrm{i}k2\pi(t-jn^{-1})}.$$

The functions $\phi_{n,j} \in \mathcal{T}_n$, $j = 0, 1, \ldots, n-1$, called *fundamental trigonometric polynomials*, satisfy

$$\phi_{n,j}(ln^{-1}) = \delta_{j,l}, \quad j, l = 0, \ldots, n-1, \tag{2.6.2.3}$$

where $\delta_{j,l}$ is the Kronecker symbol:

$$\delta_{i,j} = \begin{cases} 1, & \text{for } i = j, \\ 0, & \text{for } i \neq j. \end{cases}$$

Having the nodal values of $v_n \in \mathcal{T}_n$ in hand, its Fourier coefficients are given by

$$\hat{v}_n(k) = \int_0^1 v_n(t)e^{-\mathrm{i}k2\pi t}dt = \frac{1}{n}\sum_{j=0}^{n-1} v_n(jn^{-1})e^{-\mathrm{i}k2\pi jn^{-1}},\ k \in \mathbb{Z}_n,$$

or

$$\hat{v}_n = \mathcal{F}_n \underline{v}_n,$$

where $\hat{v}_n$ is the vector with components $\hat{v}_n(k)$, $k \in \mathbb{Z}_n$, $\underline{v}_n$ is the vector with components $v_n(jn^{-1}), j = 0, \dots, n-1$, and $\mathcal{F}_n$ is the *discrete Fourier transform.*

Conversely, having the Fourier coefficients in hand, the nodal values of $v_n \in \mathcal{T}_n$ are given by the *inverse discrete Fourier transform* following from (2.6.2.2) :

$$v_n(jn^{-1}) = \sum_{k\in\mathbb{Z}_n} \hat{v}_n(k)e^{\mathrm{i}k2\pi jn^{-1}},\ j = 0, \dots, n-1,$$

or

$$\underline{v}_n = \mathcal{F}_n^{-1}\hat{v}_n.$$

So we can change the form of representation of a trigonometric polynomial where needed. In usual matrix calculus, an application of $\mathcal{F}_n$ or $\mathcal{F}_n^{-1}$ costs n^2 flops. Using *fast Fourier transform* (FFT) techniques, both transforms can be implemented in $\mathcal{O}(n \log n)$ flops. For FFT, see [15]. See also [6, 8].

2.6.3 Trigonometric orthogonal and interpolation projections

For $v \in \widetilde{L}^1(0, 1)$ and $n \in \mathbb{N}$ the Fourier projection $P_n v$ is defined by

$$(P_n v)(t) = \sum_{k\in\mathbb{Z}_n} \hat{v}(k)e^{\mathrm{i}k2\pi t}, \quad t \in \mathbb{R},$$

$$\hat{v}(k) = \int_0^1 v(t)e^{-\mathrm{i}k2\pi t}dt,\ k \in \mathbb{Z}.$$

For $v \in \widetilde{L}^2(0, 1)$, $P_n v$ is the orthogonal projection of v onto $\mathcal{T}_n$.

Observe that $(P_n v)^{(m)} = P_n v^{(m)}$ for $v \in \widetilde{C}^m(\mathbb{R})$, $m \in \mathbb{R}$.

Indeed, we have for any $t \in \mathbb{R}$ that

$$(P_n v)(t) = \sum_{k\in\mathbb{Z}_n} \Big(\int_0^1 v(s)e^{-\mathrm{i}k2\pi s}ds\Big)e^{\mathrm{i}k2\pi t}$$

and

$$(P_n v)^{(m)}(t) = \sum_{k\in\mathbb{Z}_n} \Big(\int_0^1 v(s)e^{-\mathrm{i}k2\pi s}ds\Big)(\mathrm{i}k2\pi)^m e^{\mathrm{i}k2\pi t}.$$

On the other hand,

$$(P_n v^{(m)})(t) = \sum_{k\in\mathbb{Z}_n} \Big(\int_0^1 v^{(m)}(s)e^{-\mathrm{i}k2\pi s}ds\Big)e^{\mathrm{i}k2\pi t}.$$

Integrating $\int_0^1 v^{(m)}(s)e^{-\mathrm{i}k2\pi s}ds$ by parts we obtain

$$\begin{aligned}\int_0^1 v^{(m)}(s)e^{-\mathrm{i}k2\pi s}ds &= e^{-\mathrm{i}k2\pi s}v^{(m-1)}(s)\Big|_{s=0}^{s=1} + \mathrm{i}k2\pi \int_0^1 v^{(m-1)}(s)e^{-\mathrm{i}k2\pi s}ds \\ &= \mathrm{i}k2\pi \int_0^1 v^{(m-1)}(s)e^{-\mathrm{i}k2\pi s}ds\end{aligned}$$

Proceeding in this way gives

$$(P_n v^{(m)})(t) = (\mathrm{i}k2\pi)^m \sum_{k\in\mathbb{Z}_n} \Big(\int_0^1 v(s)e^{-\mathrm{i}k2\pi s}ds\Big)e^{\mathrm{i}k2\pi t}, \quad t \in \mathbb{R}. \quad \square$$

For $v \in \widetilde{C}(\mathbb{R})$, the interpolation projection $Q_n v$ is defined by the requirements

$$Q_n v \in \mathcal{T}_n, \quad (Q_n v)(jn^{-1}) = v(jn^{-1}), \quad j = 0, \dots, n-1.$$

Due to (2.6.2.3),

$$(Q_n v)(t) = \sum_{j=0}^{n-1} v(jn^{-1})\phi_{n,j}(t),$$

hence the Fourier coefficients $(\widehat{Q_n v})(k)$, $k \in \mathbb{Z}_n$, of $Q_n v$ can be computed from the sample values $v(jn^{-1})$, $j = 0, \dots, n-1$, by FFT in $\mathcal{O}(n \log n)$ flops.

It is known (see, for example, [91]) that

$$||P_n||_{\widetilde{C}(\mathbb{R})\to\widetilde{C}(\mathbb{R})} \leq c \log n, \quad ||P_n||_{L^p(0,1)\to L^p(0,1)} \leq c_p, \quad 1 < p < \infty, \tag{2.6.3.4}$$

$$||Q_n||_{\widetilde{C}(\mathbb{R})\to\widetilde{C}(\mathbb{R})}\leq c\log n, \quad ||Q_n||_{\widetilde{C}(\mathbb{R})\to L^p(0,1)}\leq c_p, \quad 1\leq p<\infty \tag{2.6.3.5}$$

Here the constants c and c_p are independent of n; we always assume that $n \geq 2$ in order to simplify the citing of estimates containing the factor $\log n$.

The following estimates are direct consequences of (2.6.3.4) and (2.6.3.5):

$$\| v - P_n v \|_\infty \leq c_m n^{-m}(\log n) \ \| v^{(m)} \|_\infty, \quad v \in \widetilde{C}^m(\mathbb{R}),\ m \in \mathbb{N}, \tag{2.6.3.6}$$

$$\| v - P_n v \|_p \leq c_{m,p} n^{-m} \ \| v^{(m)} \|_p, \quad v \in \widetilde{W}^{m,p}(\mathbb{R}),\ m \in \mathbb{N},\ 1 < p < \infty, \tag{2.6.3.7}$$

$$\| v - Q_n v \|_\infty \leq c_m n^{-m}(\log n) \ \| v^{(m)} \|_\infty, \quad v \in \widetilde{C}^m(\mathbb{R}),\ m \in \mathbb{N}, \tag{2.6.3.8}$$

$$\| v - Q_n v \|_\infty \leq c_{m,\lambda} n^{-m-\lambda}(\log n) \ \| v^{(m)} \|_{\mathcal{H}^{m,\lambda}}, \quad v \in \widetilde{\mathcal{H}}^{m,\lambda},\ m \geq 0, \tag{2.6.3.9}$$

$$\| v - Q_n v \|_p \leq c_{m,p} n^{-m} \ \| v^{(m)} \|_\infty, \quad v \in \widetilde{C}^m(\mathbb{R}), \quad m \in \mathbb{N},\ 1 \leq p < \infty. \tag{2.6.3.10}$$

The constants in the estimates (2.6.3.6)-(2.6.3.10) are independent of n and v.

Chapter 3

Fast and Quasi-Fast Solvers

In this chapter we shall propose and justify a possibility to construct fast and quasi-fast solvers for Fredholm integral equations of the second kind with weakly singular kernels. This chapter is the core of the thesis and is based on the paper [61].

3.1 Basic class of problems

Consider the weakly singular Fredholm integral equation of the second kind

$$u(x) = \int_0^1 \big(a(x,y)|x-y|^{-\nu} + b(x,y)\big)\, u(y)dy + f(x), \quad 0 \le x \le 1, \quad (3.1.1)$$

where $0 < \nu < 1$.

In the final results of this chapter for given $m \in \mathbb{N}$ and $\nu \in (0,1)$, we assume that

$(A1)$ $f \in C^{m,\nu}(0,1)$;
$(A2)$ $a, b \in C^{2m}([0,1] \times [0,1])$;
$(A3)$ the homogeneous equation

$$u(x) = \int_0^1 \big(a(x,y)|x-y|^{-\nu} + b(x,y)\big)\, u(y)dy,$$

corresponding to (3.1.1) has in $C[0,1]$ only the trivial solution $u = 0$.

Recall that $C^{m,\nu} = C^{m,\nu}(0,1)$ consists of functions $f \in C[0,1] \cap C^{m,\nu}(0,1)$ satisfying the conditions (2.3.1); it is a Banach space with respect to the norm (2.3.2).

By $C^l([0,1]\times[0,1])$ $(l\in\mathbb{N}_0)$ we denote the Banach space of functions $v=v(x,y)$ that have continuous partial derivatives $(\frac{\partial}{\partial x})^j(\frac{\partial}{\partial y})^k v(x,y)$ $(j,k=0,1,\dots,l)$ on the square $[0,1]\times[0,1]$,

$$\|v\|_{C^l([0,1]\times[0,1])}=\sum_{j=0}^{l}\sum_{k=0}^{l}\max_{(x,y)\in[0,1]\times[0,1]}\left|\left(\frac{\partial}{\partial x}\right)^j\left(\frac{\partial}{\partial y}\right)^k v(x,y)\right|.$$

Introduce also the following strengthened smoothness condition for functions a and b:
$(A2')$ $a,b\in\mathcal{H}^{2m,\mu}([0,1]\times[0,1])$ with a $\mu\in(0,1]$.

The Hölder space $\mathcal{H}^{2m,\mu}([0,1]\times[0,1])$ consists of functions $v\in C^{2m}([0,1]\times[0,1])$ with $(\frac{\partial}{\partial x})^i(\frac{\partial}{\partial y})^j v(x,y)$, $i+j=2m$, satisfying the Hölder condition with the exponent $\mu\in(0,1]$:

$$|\, v\,|_{2m,\mu}:=\sup_{\substack{x_1,x_2,y_1,y_2\in[0,1],\\ i+j=2m\\ (x_1,y_1)\neq(x_2,y_2)}}\frac{\left|(\frac{\partial}{\partial x})^i(\frac{\partial}{\partial y})^j v(x_1,y_1)-(\frac{\partial}{\partial x})^i(\frac{\partial}{\partial y})^j v(x_2,y_2)\right|}{(|x_2-x_1|+|y_2-y_1|)^\mu}<\infty;$$

$\mathcal{H}^{2m,\mu}([0,1]\times[0,1])$ is a Banach space with the norm

$$\|\, v\,\|_{\mathcal{H}^{2m,\mu}([0,1]\times[0,1])}=\|\, v\,\|_{\mathcal{C}^{2m}([0,1]\times[0,1])}+|\, v\,|_{2m,\mu}.$$

Denote by

$$T=A+B \tag{3.1.2}$$

the integral operator of equation (3.1.1) with operators A and B defined by

$$(Au)(x)=\int_0^1 a(x,y)|x-y|^{-\nu}u(y)dy,\quad (Bu)(x)=\int_0^1 b(x,y)u(y)dy, \tag{3.1.3}$$

where $0<\nu<1$. We have the following result (see, for example, [40]).

Theorem 3.1.1. *Assume that $a,b\in C[0,1]$. Then T defined by $\{(3.1.2),(3.1.3)\}$ maps $L^\infty(0,1)$ into $C[0,1]$ and is compact as an operator from $L^\infty(0,1)$ into $C[0,1]$, hence also as an operator from $C[0,1]$ into $C[0,1]$.*

Due to $\{(3.1.2), (3.1.3)\}$ equation (3.1.1) can be rewritten in the form

$$u = Tu + f, \tag{3.1.4}$$

where $a, b \in C([0,1] \times [0,1])$ and $f \in C[0,1]$.

The existence, uniqueness and regularity of the solution to equation (3.1.4) (equation (3.1.1)) can be characterized by the following two theorems.

Theorem 3.1.2. *Assume that* $a, b \in C([0,1] \times [0,1])$ *and* $f \in C[0,1]$. *Moreover, assume that* $(A3)$ *holds, that is, the homogeneous equation* $u = Tu$ *possesses in* $C[0,1]$ *only the trivial solution* $u = 0$. *Then equation* (3.1.4) (*equation* (3.1.1)) *is uniquely solvable and its solution* u *belongs to* $C[0,1]$.

Proof. This result is a consequence of Theorems 2.2.6 and 3.1.1. □

Theorem 3.1.3. *Assume that* $a, b \in C^m([0,1] \times [0,1])$, $f \in C^{m,\nu}(0,1)$, $m \in \mathbb{N}$, $\nu \in (0,1)$. *Moreover, assume that* $(A3)$ *holds. Then equation* (3.1.4) (*equation* (3.1.1)) *has a solution* $u \in C^{m,\nu}(0,1)$ *which is unique in* $C[0,1]$. *If* $a \equiv 0$ *and* $f \in C^m[0,1]$, *then* $u \in C^m[0,1]$.

Proof. This result is a consequence of Theorems 2.2.6 , 2.3.1 and 3.1.2. □

Corollary 3.1.1. *Let* $\nu \in (0,1)$ *and* $m \in \mathbb{N}$ *be given. Let the assumptions* $(A1) - (A3)$ *be fulfilled. Then equation* (3.1.1) *has a unique solution* u *in* $C[0,1]$ *and it belongs to* $C^{m,\nu}(0,1)$.

The results of Theorem 3.1.3 and Corollary 3.1.1 are of fundamental importance in the subsequent analysis of Chapters 3 and 4. In particular, it follows from Theorem 3.1.3 that, in general, we cannot expect that u, the solution to equation (3.1.1), belongs to $C^1[0,1]$, even when we have a, $b \in C^m([0,1] \times [0,1])$ and $f \in C^m([0,1]$ for some $m \in \mathbb{N}$. We may only say that $u \in C^{m,\nu}(0,1)$, that is, solution u can, in general, exhibit singular behavior near the boundary of the interval $[0,1]$, where its derivatives may become unbounded:

$$| u^{(j)}(x) | \leq c[x(1-x)]^{1-\nu-j}, \quad 0 < x < 1, \quad j = 1, \ldots, m.$$

In order to make clear that boundary singularities of the derivatives of a solution to a second kind weakly singular integral equation are typical for such equations, let us consider an integral equation of the form

$$u(x) = \int_0^1 K(x,y)u(y)dy + f(x), \quad 0 \le x \le 1, \tag{3.1.5}$$

where $f \in C^m[0,1]$ $(m \in \mathbb{N})$ and

$$K(x,y) = \mid x-y \mid^{-\nu}, \quad x, y \in [0,1], \quad x \neq y, \quad 0 < \nu < 1.$$

Let us assume, that equation (3.1.5) has a solution $u \in C[0,1]$. Then, in general, $u \notin C^1[0,1]$.

Indeed, assuming that $u \in C^1[0,1]$, we can rewrite (3.1.5) in the form

$$u(x) = \int_0^x (x-y)^{-\nu}u(y)dy + \int_x^1 (y-x)^{-\nu}u(y)dy + f(x),$$

or

$$u(x) = \int_0^x \tau^{-\nu}u(x-\tau)d\tau + \int_0^{1-x} \tau^{-\nu}u(x+\tau)d\tau + f(x), \quad 0 < x < 1, \tag{3.1.6}$$

from which it follows that we can differentiate (3.1.6) with respect to x. After differentiating we obtain an equation of the form

$$\begin{aligned} u'(x) &= x^{-\nu}u(0) + \int_0^x \tau^{-\nu}u'(x-\tau)d\tau \\ &- (1-x)^{-\nu}u(1) + \int_0^{1-x} \tau^{-\nu}u'(x+\tau)d\tau + f'(x), \end{aligned}$$

or, after some reorganizing,

$$u'(x) = \int_0^1 \mid x-y \mid^{-\nu}u'(y)dy + f'(x) + u(0)x^{-\nu} - u(1)(1-x)^{-\nu}, \tag{3.1.7}$$

where $0 < x < 1$. We see that on the $l.h.s$ of this equation is $u'(x)$ which is a continuous function for all $x \in C[0,1]$, but its $r.h.s$ is typically unbounded near the boundary of $[0,1]$. Indeed, since $u' \in C[0,1]$, the integral term

$$\int_0^1 \mid x-y \mid^{-\nu}u'(y)dy$$

in (3.1.7) is a continuous function for $x \in [0,1]$, see, for example, [40].

Thus, the first two terms on the *r.h.s.* of (3.1.7) are bounded continuous functions on $[0, 1]$. However, this is not the case for the last two terms on the *r.h.s.* of (3.1.7): the term $u(0)x^{-\nu}$ has the singularity at $x = 0$ provided that $u(0) \neq 0$ and the term $u(1)(1 - x)^{-\nu}$ has the singularity at $x = 1$ if $u(1) \neq 0$. Thus, the assumption $u \in C^1[0, 1]$ leads to a contradiction if $u(0) \neq 0$ and/or $u(1) \neq 0$; these inequalities take place for most of $f \in C^m[0, 1]$.

Thus, when constructing high order numerical methods for weakly singular integral equations, one should take into account the possible nonsmooth behavior of the derivatives of an exact solution. This becomes even more significant since our aim is to work out algorithms with the optimal order of their convergence. For us information about the regularity of the solution of (3.1.1) given by Theorem 3.1.3 and Corollary 3.1.1 is extremely important since our approach below (see Section 3.3) is based on an idea of killing the singularities of the derivatives of the exact solution to the underlying problem by a suitable smoothing transformation.

The main purpose of the present thesis is to construct fast and quasi-fast solvers for equation (3.1.1). In a fast solver, the conditions of optimal accuracy and minimal arithmetical operation are met. We mean the order optimality and order minimal work on a class of problems. In our case the class of problems is defined by the smoothness conditions which we set above on the free term $f(x)$ and the kernel $K(x, y) = a(x, y)| x - y |^{-\nu} + b(x, y)$ of equation (3.1.1), see $(A1), (A2)$ or $(A1)$, $(A2')$.

3.2 Notions of fast and quasi-fast solvers

We will use the following notions of fast and quasi-fast solvers.

Definition 3.2.1. By a $(C, C^{m,\nu})$ fast solver of equation (3.1.1) we mean a solver which produces approximate solutions $u_n \in C[0, 1], n \in \mathbb{N}$, such that

- given the values of a, b and f, each at not more than $n_\star$ points (depending on the solver, with $n_\star \to \infty$ as $n \to \infty$), the parameters of u_n can be determined at the cost of $\gamma_m n_\star$ arithmetical operations and an accuracy

$$\|u - u_n\|_\infty = \max_{0 \leq x \leq 1} |u(x) - u_n(x)| \leq c_m n_\star^{-m} \|f\|_{C^{m,\nu}} \tag{3.2.1}$$

is achieved where u is the solution of (3.1.1);

- having the parameters of u_n in hand, the value of u_n at any point $x \in [0,1]$ is available at the cost of γ'_m arithmetical operations.

Here the constants c_m, γ_m, γ'_m are independent of f and n.

Note that estimate (3.2.1) is information optimal even in the case where $a(x,y) = 0$ for $(x,y) \in [0,1] \times [0,1]$ (in this case the solution of (3.1.1) can be presented in the form $u = (I-B)^{-1} f$, where I is the identity mapping and $(I-B)^{-1}$ is the inverse of the operator $I - B$). Namely (see original work [89] or lecture notes [79]), for any m, $m' \in \mathbb{N}$, $\beta > 0$, $\bar{\beta} > 1$ and any solver of (3.1.1) with $a(x,y) \equiv 0$ depending on $n_\star$ evaluation points for $f \in C^m = C^m[0,1]$ and $b \in C^{m'}([0,1] \times [0,1])$, there is a "bad" pair f, b satisfying

$$\|f\|_{C^m} = 1, \quad \|b\|_{C^{m'}([0,1]\times[0,1])} \leq \beta, \quad \|(I-B)^{-1}\|_{C\to C} \leq \bar{\beta},$$

and such that, disregarding the amount of arithmetical work, the lower error bound

$$\|u - u_n\|_\infty \geq c_o n_\star^{-\min\{m, m'/2\}}$$

holds where c_0 is a positive constant depending only on m, m', β and $\bar{\beta}$. Thus, under traditional assumptions $f \in C^m[0,1]$, $b \in C^m([0,1] \times [0,1])$, i.e. for $m' = m$, only the accuracy order

$$\|u - u_n\|_\infty \leq c n_\star^{-m/2} \|f\|_{C^m}$$

can be achieved by any solver (this partial result has been established already in [20]). A further consequence of the lower error bound is that accuracy (3.2.1) is possible only if $m' \geq 2m$, and this explains the constellation of our assumption $(A2)$ with $m' = 2m$.

We speak about a $(L^p, C^{m,\nu})$ fast solver $(1 \leq p < \infty)$ if the accuracy requirement (3.2.1) in Definition 3.2.1 is replaced by

$$\|u - u_n\|_p \leq c_m n_\star^{-m} \|f\|_{C^{m,\nu}},$$

where $\|u\|_p$ is the norm of u in $L^p(0,1)$. Similarly, we obtain a (C, C^m) fast solver if in the accuracy requirement (3.2.1) the norm $\|f\|_{C^{m,\nu}}$ is replaced by $\|f\|_{C^m}$.

Definition 3.2.2. In a $(C, C^{m,\nu})$ quasi-fast solver, the accuracy requirement (3.2.1) is replaced by

$$\|u - u_n\|_\infty \leq c_m n_\star^{-m} (\log n_\star) \|f\|_{C^{m,\nu}} \tag{3.2.2}$$

maintaining other requirements of Definition 3.2.1.

Remark 3.2.1. The estimate (3.2.1) can be rewritten with respect to the *complexity*

$$n_{\star\star} := \gamma_m n_\star$$

of the solver in the equivalent form

$$\|u - u_n\|_\infty \leq \bar{c}_m n_{\star\star}^{-m} \|f\|_{C^{m,\nu}}, \quad \bar{c}_m = \gamma_m^m c_m.$$

This form of the estimate enables a comparison of $(C, C^{m,\nu})$ fast solvers with different complexity parameters γ_m : to smaller $\bar{c}_m$ there corresponds more effective solver.

Remark 3.2.2. Consider also an alternative definition of a $(C, C^{m,\nu})$ quasi-fast solver requiring accuracy (3.2.1) but allowing $\bar{\gamma}_m n_\star \log n_\star$ arithmetical operations. Then with respect to the complexity

$$n_{\star\star} = \bar{\gamma}_m n_\star \log n_\star$$

estimate (3.2.1) takes the form

$$\begin{aligned} \|u - u_n\|_\infty &\leq c_m (\omega^{-1}(n_{\star\star}/\bar{\gamma}_m))^{-m} \|f\|_{C^{m,\nu}} \\ &\sim c_m \bar{\gamma}_m^m n_{\star\star}^{-m} (\log(n_{\star\star}/\bar{\gamma}_m))^m \|f\|_{C^{m,\nu}} \end{aligned}$$

where

$$\omega(x) = x \log x, \;\; \omega^{-1}(x) \sim x/\log x \quad \text{as} \quad x \longrightarrow \infty,$$

i.e,

$$(\omega^{-1}(x) \log x)/x \longrightarrow 1 \quad \text{as} \quad x \to \infty.$$

We see that for $m \geq 2$ Definition 3.2.2 is more restrictive (leads to a more high accuracy) than the alternative definition.

Below we will construct the solvers which under the conditions $(A1)$ – $(A3)$ are $(C, C^{m,\nu})$ quasi-fast and $(L^p, C^{m,\nu})$ fast, $1 \leq p < \infty$. Moreover, these solvers are (C, C^m) fast under the conditions $(A1), (A2'), (A3)$. Since we must discretize functions a and b of two variables, we use in our constructions $n_\star = O(n^2)$ that is more convenient rather than $n_\star = n$. Since the obtained result will characterize the smallest possible complexity of discretization methods for weakly singular integral equations of the second kind, we obtain a possibility to estimate how "good" an arbitrary discretization method is from the point of view of complexity.

3.3 Periodization of the integral equation

It follows from Section 3.1 that the derivatives of a solution to a weakly singular integral equation such as (3.1.1), in general , have certain boundary singularities. With the help of a suitable change of variables in the equation, these boundary singularities can be suppressed, see Theorem 2.4.1, see also [19, 37, 45, 48, 51, 72]. Below we use a change integration variables and a change of the unknown functions using function φ defined by the formula (2.4.1) in order to transform (3.1.1) into a 1-periodic problem.

To reduce (3.1.1) to a periodic problem with a smooth exact solution we need a smooth function $g(t)$, $0 < t < 1$. Actually, for given $\nu \in (0,1)$ and $m \in \mathbb{N}$, we introduce the function $g = g(t)$, as follows:

$$g(t) = \begin{cases} t^{-\nu}, & 0 < t \le 1/3, \\ (1-t)^{-\nu}, & 2/3 \le t < 1, \\ \gamma(t), & 1/3 < t < 2/3, \end{cases} \tag{3.3.1}$$

where $\gamma(t)$ is a polynomial of degree $4m+1$ determined by the conditions

$$\gamma(t)^{(j)}(1/3) = \left(\frac{d}{dt}\right)^j t^{-\nu}\bigg|_{t=1/3}, \quad \gamma(t)^{(j)}(2/3) = \left(\frac{d}{dt}\right)^j (1-t)^{-\nu}\bigg|_{t=2/3},$$

with $j = 0, 1, \ldots, 2m$. For example, in the case $\nu = 1/2$ and $m = 2$ we have

$$g(t) = \begin{cases} t^{-1/2}, & 0 < t \le 1/3, \\ (1-t)^{-1/2}, & 2/3 \le t < 1, \\ \gamma(t), & 1/3 < t < 2/3, \end{cases} \tag{3.3.2}$$

with

$$\gamma(t) = a_0 + a_1 t + a_2 t^2 + a_3 t^3 + a_4 t^4 + a_5 t^5 + a_6 t^6 + a_7 t^7 + a_8 t^8 + a_9 t^9$$

where

$$a_0 = -25.33, \qquad a_1 = 537.81, \qquad a_2 = 422.4,$$

$$a_3 = 20.08, \qquad a_4 = 5.47, \qquad a_5 = 95.53,$$

$$a_6 = 210.00, \qquad a_7 = 58.59, \qquad a_8 = 4.65,$$

$$a_9 = 0.00000055.$$

For the behavior of the graph of the function (3.3.2) see Figure 3.0.1.

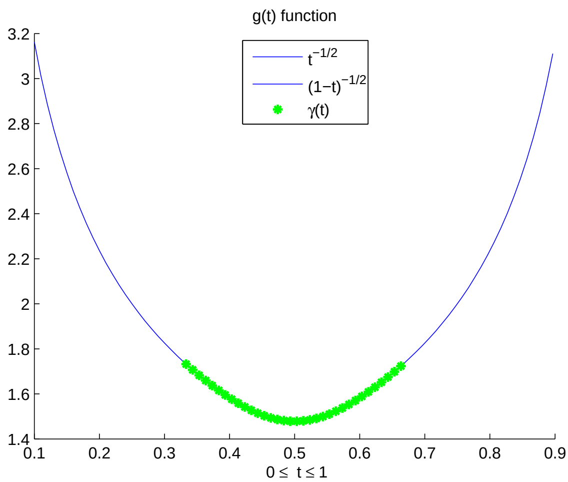

Figure 3.0.1: Graph of the function (3.3.2) for $t \in [0.1, 0.9]$.

Next we extend g from $(0, 1)$ into a $1-$periodic function $\widetilde{g} \in C^{2m}(\mathbb{R} \backslash \mathbb{Z})$:

$$\widetilde{g}(t) = g(t - j), \quad t \in (j, j + 1), \quad j \in \mathbb{Z}.$$

For the graph of the function $\widetilde{g}$ see Figure 3.0.2. Note that

$$|\widetilde{g}^{(j)}(t)| \leq c|t|^{-\nu-j} \quad \text{for} \quad 0 < |t| \leq 1/2, \;\; j = 0, 1, \ldots, 2m, \tag{3.3.3}$$

$$\widetilde{g}(t) - |t|^{-\nu} = 0 \quad \text{for} \quad |t| \leq 1/3. \tag{3.3.4}$$

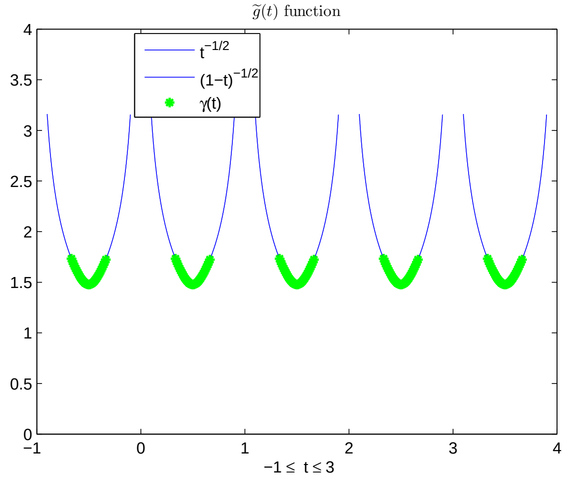

Figure 3.0.2: Graph of the periodic function $\widetilde{g}$ for $t \in (-1,0) \cup (0,1) \cup (1,2) \cup (2,3) \cup (3,4)$.

Next we perform in equation (3.1.1) the change the variables

$$x = \varphi(t),\ 0 \le t \le 1; \quad y = \varphi(s),\ 0 \le s \le 1, \tag{3.3.5}$$

where $\varphi : [0,1] \longrightarrow [0,1]$ is given by the formula (2.4.1):

$$\varphi(t) = \frac{1}{c_r}\int_0^t \sigma^{r-1}(1-\sigma)^{r-1}d\sigma, \quad c_r = \int_0^1 \sigma^{r-1}(1-\sigma)^{r-1}d\sigma = \frac{((r-1)!)^2}{(2r-1)!}; \tag{3.3.6}$$

condition on the smoothing parameter $r \in \mathbb{N}$ will be set later.

We have

$$\Big(\varphi(t) - \varphi(s)\Big)/(t-s) > 0 \ \text{ for } \ t, s \in [0,1],\ t \neq s.$$

Taking $x = \varphi(t)$ and $y = \varphi(s)$, we get $dy = \varphi'(s)ds$ and therefore

$$\begin{aligned} & \int_0^1 a(x,y)|\, x - y\,|^{-\nu} u(y)dy \\ = & \int_0^1 a(\varphi(t),\varphi(s))\left|\frac{\varphi(t)-\varphi(s)}{t-s}\right|^{-\nu} |\, t-s\,|^{-\nu} u(\varphi(s))\varphi'(s)ds \\ = & \int_0^1 a(\varphi(t),\varphi(s))\left(\frac{\varphi(t)-\varphi(s)}{t-s}\right)^{-\nu} |\, t-s\,|^{-\nu} u(\varphi(s))\varphi'(s)ds. \end{aligned}$$

Thus, on the basis of the (3.3.5) change of variables equation (3.1.1) takes the form

$$\begin{aligned} u(\varphi(t)) &= \int_0^1 \Big[a\Big(\varphi(t),\varphi(s)\Big)\left(\frac{\varphi(t)-\varphi(s)}{t-s}\right)^{-\nu}|t-s|^{-\nu} \\ &+ b\Big(\varphi(t),\varphi(s)\Big)\Big]u(\varphi(s))\varphi'(s)ds + f(\varphi(t)), \quad t\in[0,1]. \end{aligned} \tag{3.3.7}$$

For $(t,s)\in[0,1]\times[0,1]$ denote

$$\Phi(t,s) = \begin{cases} \Big(\varphi(t)-\varphi(s)\Big)/(t-s), & t\neq s, \\ \varphi'(t), & t=s. \end{cases} \tag{3.3.8}$$

Using $\widetilde{g}$ and Φ we rewrite (3.3.7) in the form

$$\begin{aligned} u(\varphi(t)) &= \int_0^1 \Big[a\Big(\varphi(t),\varphi(s)\Big)\Phi(t,s)^{-\nu}\Big(|t-s|^{-\nu}-\widetilde{g}(t-s)+\widetilde{g}(t-s)\Big) \\ &+ b\Big(\varphi(t),\varphi(s)\Big)\Big]u(\varphi(s))\varphi'(s)ds + f(\varphi(t)), \end{aligned}$$

or, after some reorganizing,

$$\begin{aligned} u(\varphi(t)) &= \int_0^1 \Big[a\Big(\varphi(t),\varphi(s)\Big)\Phi(t,s)^{-\nu}\widetilde{g}(t-s)\varphi'(s) + \Big\{b\Big(\varphi(t),\varphi(s)\Big) \\ &+ a\Big(\varphi(t),\varphi(s)\Big)\Phi(t,s)^{-\nu}\Big(|t-s|^{-\nu}-\widetilde{g}(t-s)\Big)\Big\}\varphi'(s)\Big]u(\varphi(s))ds \\ &+ f(\varphi(t)), \end{aligned}$$

that is, in the form

$$\bar{u}(t) = \int_0^1 \Big(\bar{a}(t,s)\widetilde{g}(t-s) + \bar{b}(t,s)\Big)\bar{u}(s)ds + \bar{f}(t), \quad 0\le t\le 1, \tag{3.3.9}$$

where

$$\bar{u}(t) = u(\varphi(t)), \quad \bar{f}(t) = f(\varphi(t)), \quad t\in[0,1], \tag{3.3.10}$$

$$\bar{a}(t,s) = a\Big(\varphi(t),\varphi(s)\Big)\Phi(t,s)^{-\nu}\varphi'(s), \quad (t,s)\in[0,1]\times[0,1],$$

$$
\begin{aligned}
\bar{b}(t,s) &= \Big[b\Big(\varphi(t),\varphi(s)\Big) \\
&+ a\Big(\varphi(t),\varphi(s)\Big)\Phi(t,s)^{-\nu}\Big(|t-s|^{-\nu}-\widetilde{g}(t-s)\Big)\Big]\varphi'(s),
\end{aligned}
$$

where $t \in [0,1]\times[0,1]$.

Next we introduce in equation (3.3.10) a change of the unknown function $\bar{u}$. With respect to

$$
\widetilde{u}(t) := \varphi'(t)^{(1-\nu)/2}\bar{u}(t) = \varphi'(t)^{(1-\nu)/2}u(\varphi(t))
$$

equation (3.3.9) reads as

$$
\widetilde{u}(t) = \int_0^1 \Big(\widetilde{a}(t,s)\widetilde{g}(t-s) + \widetilde{b}(t,s)\Big)\,\widetilde{u}(s)ds + \widetilde{f}(t),\ 0 \le t \le 1, \tag{3.3.11}
$$

or

$$
\widetilde{u} = \widetilde{T}\widetilde{u} + \widetilde{f}, \tag{3.3.12}
$$

where

$$
\widetilde{f}(t) = \varphi'(t)^{(1-\nu)/2} f(\varphi(t)) \quad (0 \le t \le 1) \tag{3.3.13}
$$

and

$$
\widetilde{T} = \widetilde{A} + \widetilde{B}, \tag{3.3.14}
$$

with $\widetilde{A}$ and $\widetilde{B}$ defined by

$$
(\widetilde{A}u)(t) = \int_0^1 \widetilde{a}(t,s)\widetilde{g}(t-s)u(s)ds, \quad (\widetilde{B}u)(t) = \int_0^1 \widetilde{b}(t,s)u(s)ds. \tag{3.3.15}
$$

Here $t,s \in [0,1]$,

$$
\begin{aligned}
\widetilde{a}(t,s) &= \varphi'(t)^{(1-\nu)/2}\bar{a}(t,s)\varphi'(s)^{-(1-\nu)/2} \\
&= \varphi'(t)^{(1-\nu)/2}a\Big(\varphi(t),\varphi(s)\Big)\varphi'(s)^{(1+\nu)/2}\Phi(t,s)^{-\nu}
\end{aligned}
$$

and

$$
\begin{aligned}
\widetilde{b}(t,s) &= \varphi'(t)^{(1-\nu)/2}\bar{b}(t,s)\varphi'(s)^{-(1-\nu)/2} \\
&= \varphi'(t)^{(1-\nu)/2}\Big[b\Big(\varphi(t),\varphi(s)\Big) \\
&+ a\Big(\varphi(t),\varphi(s)\Big)\Phi(t,s)^{-\nu}\Big(|t-s|^{-\nu}-\widetilde{g}(t-s)\Big)\Big]\varphi'(s)^{(1+\nu)/2}.
\end{aligned}
$$

Besides the equality

$$\widetilde{u}(t) = \varphi'(t)^{(1-\nu)/2}\bar{u}(t),$$

the solutions of equations (3.3.9) and (3.3.11) satisfy the integral relation

$$\bar{u}(t) = \int_0^1 \Big(a_\star(t,s)\widetilde{g}(t-s) + b_\star(t,s) \Big) \widetilde{u}(s)ds + \bar{f}(t), \quad 0 \le t \le 1, \tag{3.3.16}$$

where

$$a_\star(t,s) = a\Big(\varphi(t), \varphi(s)\Big)\Phi(t,s)^{-\nu}\varphi'(s)^{(1+\nu)/2}, \quad (t,s) \in [0,1]\times[0,1], \tag{3.3.17}$$

$$\begin{aligned} b_\star(t,s) &= \Big[b\Big(\varphi(t),\varphi(s)\Big) + a\Big(\varphi(t),\varphi(s)\Big)\Phi(t,s)^{-\nu}\Big(|t-s|^{-\nu} \\ &- \widetilde{g}(t-s)\Big)\Big]\varphi'(s)^{(1+\nu)/2}, \quad (t,s) \in [0,1]\times[0,1]. \end{aligned} \tag{3.3.18}$$

It is easy to see that $\varphi(t)$ is a polynomial of degree $2r-1$:

$$\varphi(t) = \frac{1}{c_*}\sum_{j=0}^{r-1}(-1)^j \binom{r-1}{j}\frac{1}{r+j}t^{r+j}.$$

This together with (3.3.8) yields that $\Phi(t,s)$ is a polynomial of degree $2r-2$.

The values of polynomials $\varphi(t)$ and $\Phi(t,s)$ at one point can be computed in $O(1)$ arithmetical operations, see [72] for the corresponding procedures. Hence the values of $\bar{a}(t,s)$, $\widetilde{a}(t,s)$, $a_\star(t,s)$, $\bar{b}(t,s)$, $\widetilde{b}(t,s)$, $b_\star(t,s)$ at one point can be computed at the cost of $O(1)$ arithmetical operations provided that the values $a(\varphi(t),\varphi(s))$ and $b(\varphi(t),\varphi(s))$ are given. Using a uniform grid for t and s, $a(x,y)$ and $b(x,y)$ must be given or evaluated on a non-uniform grid with respect to x and y.

Due to (3.3.4),

$$|t-s|^{-\nu} - \widetilde{g}(t-s) = 0$$

in the vicinity of the diagonal $t = s$. Thus $\bar{b}(t,s)$, $\widetilde{b}(t,s)$ and $b_\star(t,s)$ are regular on the diagonal $t = s$ of the square $[0,1]\times[0,1]$, but there appear point singularities at $(t,s) = (0,1)$ and $(t,s) = (1,0)$ of the term $\widetilde{g}(t-s)$.

Further, it follows from [72] that $\Phi(t,s)$ vanishes at $(t,s) = (0,0)$ and $(t,s) = (1,1)$ causing there point singularities of $\Phi(t,s)^{-\nu}$ of the order

$(t+s)^{-(r-1)\nu}$ and $\Big((1-t)+(1-s)\Big)^{-(r-1)\nu}$, respectively:

$$\Big|\left(\frac{\partial}{\partial s}\right)^j \Big(\Phi(t,s)\Big)^{-\nu}\Big| \le c(t+s)^{-\nu(r-1)-j}\Big((1-t)+(1-s)\Big)^{-\nu(r-1)-j}, \tag{3.3.19}$$

where $j = 0, 1 \ldots, 2m$; for $r > 2$ these singularities are more strong than the singularities of $\widetilde{g}^{(j)}(t-s)$ at $(t,s) = (0,0)$ and $(t,s) = (1,1)$, see (3.3.3). All these four point singularities can be suppressed by the factors $\varphi'(t)^{(1-\nu)/2}$ and $\varphi'(s)^{(1+\nu)/2}$ in the expressions for $\widetilde{a}, \widetilde{b}$ and by the factor $\varphi'(s)^{(1+\nu)/2}$ in the expressions for $a_\star, b_\star$.

Actually, an analysis shows that the functions $\widetilde{a}, \widetilde{b}, a_\star$ and $b_\star$ can be extended so that these extensions (which we denote again by $\widetilde{a}, \widetilde{b}, a_\star, b_\star$) belong to the space $C^{2m}([0,1]\times[0,1])$ provided that the smoothing parameter r in the definition (3.3.6) is sufficiently large: the condition

$$r - 1 > 4m/(1-\nu) \tag{3.3.20}$$

is sufficient.

Namely, under condition (3.3.20) it holds that

$$\Big|\left(\frac{\partial}{\partial s}\right)^j \Big(\Phi(t,s)^{-\nu}\Big)\Big|\Big|\left(\frac{d}{ds}\right)^{2m-j} \Big(\varphi'(s)^{(1+\nu)/2}\Big)\Big| \to 0 \text{ as } s \to 0, \tag{3.3.21}$$

for any $t \in [0,1]$ and $j = 0, 1, \ldots, 2m$.

Indeed, let $t \in [0,1]$, $0 < s \le 1/2$ and $j = 0, 1, \ldots, 2m$. Then we obtain from (3.3.19) the estimate

$$\Big|\left(\frac{\partial}{\partial s}\right)^j \Big(\Phi(t,s)^{-\nu}\Big)\Big| \le cs^{-\nu(r-1)-j},$$

and from (3.3.6) that

$$\begin{aligned}\left(\frac{d}{ds}\right)^{2m-j} \Big(\varphi'(s)^{(1+\nu)/2}\Big) &= \left(\frac{d}{ds}\right)^{2m-j} \Big[\Big(s^{r-1}(1-s)^{r-1}\Big)/c_*\Big]^{(1+\nu)/2} \\ &\le c_1 s^{(r-1)((1+\nu)/2)-(2m-j)}.\end{aligned}$$

Therefore,

$$\Big|\left(\frac{\partial}{\partial s}\right)^j\left(\Phi(t,s)^{-\nu}\right)\Big|\Big|\left(\frac{d}{ds}\right)^{2m-j}\left(\varphi'(s)^{(1+\nu)/2}\right)\Big| \leq c_2 s^{(r-1)((1-\nu)/2)-2m}, \tag{3.3.22}$$

with a constant $c_2 > 0$ which is independent of $s \in (0, \frac{1}{2}]$. If $r - 1 > 4m/(1-\nu)$, then

$$(r-1)((1-\nu)/2) - 2m > 0$$

and (3.3.21) follows from (3.3.22).

Similarly, by symmetry,

$$\Big|\left(\frac{\partial}{\partial s}\right)^j\left(\Phi(t,s)^{-\nu}\right)\Big|\Big|\left(\frac{d}{ds}\right)^{2m-j}\left(\varphi'(s)^{(1+\nu)/2)}\right)\Big| \to 0 \text{ as } s \to 1,$$

for any $t \in [0,1]$ and $j = 0, 1, \ldots, 2m$. Thus, the derivatives

$$\left(\frac{\partial}{\partial s}\right)^j\left(\Phi(t,s)^{-\nu}\right)\left(\frac{d}{ds}\right)^{2m-j}\left(\varphi'(s)^{(1+\nu)/2)}\right) \quad (j = 0, 1, \ldots, 2m)$$

have under the condition (3.3.20) no singularities at the points $(t,s) = (0,0)$ and $(t,s) = (1,1)$ which, in turn yields that $\widetilde{a}, \widetilde{b}, a_\star, b_\star \in C^{2m}([0,1] \times [0,1])$.

Moreover, under the condition (3.3.20),

$$\left(\frac{\partial}{\partial t}\right)^j\left(\frac{\partial}{\partial s}\right)^k \widetilde{a}(t,s)\Big|_{(t,s)\in\Gamma} = 0, \quad \left(\frac{\partial}{\partial t}\right)^j\left(\frac{\partial}{\partial s}\right)^k \widetilde{b}(t,s)\Big|_{(t,s)\in\Gamma} = 0, \tag{3.3.23}$$

with $0 \leq j + k \leq 2m$, and

$$\begin{aligned} \left(\frac{\partial}{\partial s}\right)^k a_\star(t,s)\Big|_{s=0} &= \left(\frac{\partial}{\partial s}\right)^k a_\star(t,s)\Big|_{s=1} = 0, \\ \left(\frac{\partial}{\partial s}\right)^k b_\star(t,s)\Big|_{s=0} &= \left(\frac{\partial}{\partial s}\right)^k b_\star(t,s)\Big|_{s=1} = 0,\ k = 0, 1, \ldots, 2m, \end{aligned} \tag{3.3.24}$$

where Γ is the boundary of the square $[0,1] \times [0,1]$.

Relations (3.3.23) allow to treat $\widetilde{a}$ and $\widetilde{b}$ as $C^{2m}(\mathbb{R} \times \mathbb{R})$-smooth 1-biperiodic functions, i.e., $\widetilde{a}, \widetilde{b} \in \widetilde{C}^{2m}(\mathbb{R} \times \mathbb{R})$. According to (3.3.24) $a_\star(t,s)$ and $b_\star(t,s)$ can be treated as $C^{2m}([0,1] \times \mathbb{R})$-smooth functions which are

1-periodic with respect to s.

Further, (3.3.20) and $f \in C^{m,\nu}(0,1)$ together with (3.3.10), (3.3.13) and Theorem 2.4.1 imply that $\bar{f}, \widetilde{f} \in C^m[0,1]$,

$$\bar{f}^{(j)}(0) = \bar{f}^{(j)}(1) = 0, \quad j = 1, \dots, m,$$

$$\widetilde{f}^{(j)}(0) = \widetilde{f}^{(j)}(1) = 0, \quad j = 0, \dots, m,$$

$$\|\widetilde{f}\|_{C^m} \leq c \|\bar{f}\|_{C^m} \leq c' \|f\|_{C^{m,\nu}}, \tag{3.3.25}$$

where the constants c and c' are independent of f; in particular, we can treat $\widetilde{f}$ as a $C^m(\mathbb{R})$-smooth 1-periodic function, i.e., $\widetilde{f} \in \widetilde{C}^m(\mathbb{R})$.

3.4 An approximate method for the periodized problem

In Section 3.3 we obtained an 1-periodic problem (3.3.11) with $m \in \mathbb{N}, \widetilde{f} \in \widetilde{C}^m(\mathbb{R})$, $\widetilde{a}, \widetilde{b} \in \widetilde{C}^{2m}(\mathbb{R} \times \mathbb{R})$, and $\widetilde{g} \in \widetilde{C}^{2m}(\mathbb{R} \setminus \mathbb{Z})$ satisfying (3.3.3) and (3.3.4). Following [61], in the present section we introduce an approximation method for the numerical solution of the periodized problem (3.3.11).

For $n \in \mathbb{N}$, let $n' \in \mathbb{N}$ be such that

$$n' \geq 2n, \quad n' \sim n^\tau, \; 2m/(m+1-\nu) < \tau < 2. \tag{3.4.1}$$

where $n' \sim n^\tau$ means that $n'/n^\tau \to 1$ as $n \to \infty$.

We approximate equation (3.3.12) (equation (3.3.11)) by the equation

$$\widetilde{u}_{n,n'} = \widetilde{T}_{n,n'} \widetilde{u}_{n,n'} + P_{n'} \widetilde{f}, \tag{3.4.2}$$

where $P_{n'} \widetilde{f}$ is an approximation for the free term $\widetilde{f}$ in equation (3.3.12) (see (3.3.13)) and

$$\widetilde{T}_{n,n'} = Q_{2n}(\widetilde{A}_{2n} + \widetilde{B}_{2n}) P_n + Q_{n'} \widetilde{A}_n^{(m)} (I - P_n) \tag{3.4.3}$$

is an approximation for the integral operator $\widetilde{T} = \widetilde{A} + \widetilde{B}$ of equation (3.3.11) (see (3.3.14) and (3.3.15)). Here P_n and Q_n are respectively the orthogonal and interpolation projection operators introduced in Section 2.6.3,

$$(\widetilde{A}_n v)(t) = \int_0^1 \widetilde{g}(t-s) Q_{n;s}\Big(\widetilde{a}(t,s)v(s)\Big)ds,$$
$$(\widetilde{B}_n v)(t) = \int_0^1 Q_{n;s}\Big(\widetilde{b}(t,s)v(s)\Big)ds$$

(s in the index of $Q_{n;s}$ refers that Q_n is applied w.r.t. the argument s),

$$\widetilde{A}_n^{(m)} = \sum_{j=0}^{m-1} M_{\alpha_j,n} G_j,$$

$$(G_j v)(t) = \int_0^1 \widetilde{g}_j(t-s)v(s)ds, \;\; \widetilde{g}_j(t) = (e^{-\mathrm{i}2\pi t} - 1)^j \widetilde{g}(t),$$

$$(M_{\alpha_j,n} v)(t) = \alpha_{j,n}(t)v(t), \quad \alpha_{j,n}(i/n) = \frac{1}{j!} L_{j,n} \widetilde{a}(i/n, s) \mid_{s=i/n},$$

where $i = 0, \ldots, n$ and $L_{j,n}$ ($j \geq 1$) is some difference approximation on the grid i/n $(i \in \mathbb{Z})$ of the differential operator

$$L_j = \prod_{l=0}^{j-1} ((2\pi \mathrm{i})^{-1} \frac{\partial}{\partial s} - lI)$$

of the accuracy

$$\mid (L_j v)(s) - (L_{j,n} v)(s) \mid \leq c n^{-2m+j} \parallel v^{(2m)} \parallel_\infty, \; s \in \mathbb{R};$$

for $j = 0$ we put $L_0 = L_{0,n} = I$.

The matrix form of $\widetilde{A}_n^{(m)}$ needs also the values $\alpha_{j,n}(j/n')$, $j = 0, \ldots, n'-1$, which we approximate via interpolation of $\alpha_{j,n}(i/n)$, $i = 0, \ldots, n$, by splines of degree $2m$. See [52, 63, 75] for more detailed comments on the listed operators.

Lemma 3.4.1. *Assume* (A2) (*which implies that* $\widetilde{a}, \widetilde{b} \in \widetilde{C}^{2m}(\mathbb{R} \times \mathbb{R})$). *Let* $n, n' \in \mathbb{N}$, *and let* $n' \sim n^\tau$, $\tau > 1$. *Then for* $1/(1-\nu) < p < \infty$,

$$\parallel \widetilde{T}(I - P_{n'}) \parallel_{\widetilde{L}^p \to \widetilde{C}} \leq c_p n^{-\tau(1-\nu-1/p)}; \tag{3.4.4}$$

for $1/(1-\nu) < p \leq \infty$, $0 < \lambda < 1 - \nu - p^{-1}$,

$$\parallel \widetilde{T} - \widetilde{T}_{n,n'} \parallel_{\widetilde{L}^p \to \widetilde{C}} \leq c_{p,\lambda}(n^{-m} + n^{-\tau\lambda}) \log n; \tag{3.4.5}$$

for $1/(1-\nu) < p < \infty$, $0 < \lambda < 1 - \nu - p^{-1}$, $v \in \widetilde{W}^{m,p}(\mathbb{R})$,

$$\| \widetilde{T}v - \widetilde{T}_{n,n'}v \|_p \leq c_{p,\lambda}(n^{-2m} + n^{-\tau(m+\lambda)}(\log n)) \| v \|_{W^{m,p}}, \qquad (3.4.6)$$

$$\| \widetilde{T}v - \widetilde{T}_{n,n'}v \|_\infty \leq c_{p,\lambda}(n^{-2m} + n^{-\tau(m+\lambda)})(\log n) \| v \|_{W^{m,p}}, \qquad (3.4.7)$$

and, strengthening $(A2)$ *to the assumption* $(A2')$,

$$\| \widetilde{T}v - \widetilde{T}_{n,n'}v \|_\infty \leq c_{p,\lambda}(n^{-2m-\min\{\mu,\, 1-\nu-1/p\}} + n^{-\tau(m+\lambda)})(\log n) \| v \|_{W^{m,p}} . \qquad (3.4.8)$$

Proof. Let $1/(1-\nu) < p < \infty$. Since $I - P_n$ as an orthogonal projector in $L^2(0,1)$ is self-adjoint, it holds for $t \in [0,1]$ that

$$\begin{aligned} (\widetilde{A}(I-P_n)v)(t) &= \int_0^1 \widetilde{a}(t,s)\widetilde{g}(t-s)((I-P_n)v)(s)ds \\ &= \int_0^1 (I-P_{n;s})\Big(\widetilde{a}(t,s)\widetilde{g}(t-s)\Big)v(s)ds, \end{aligned}$$

$$| (\widetilde{A}(I-P_n)v)(t) | \leq \| (I-P_n)\Big(\widetilde{a}(t,\cdot)\widetilde{g}(t-\cdot)\Big) \|_q \| v \|_p, \quad p^{-1} + q^{-1} = 1.$$

Using (3.3.3) for $j = 0$ and $j = 1$, it is easy to check that for $| \sigma | \leq \delta$

$$\left(\int_0^1 | \widetilde{a}(t,s+\sigma)\widetilde{g}(t-s-\sigma) - \widetilde{a}(t,s)\widetilde{g}(t-s) |^q \, ds \right)^{1/q} \leq c \,| \sigma |^{1-\nu-1/p} . \qquad (3.4.9)$$

By Nikolski's inequality (formulated in [52] as Lemma 2.2), estimate (3.4.9) implies that

$$\| (I-P_n)\Big(\widetilde{a}(t,\cdot)\widetilde{g}(t-\cdot)\Big) \|_q \leq cn^{-(1-\nu-1/p)}, \quad t \in [0,1],$$

$$\| \widetilde{T}(I-P_n) \|_{\widetilde{L}^p \to \widetilde{C}} \leq c_p n^{-(1-\nu-1/p)},$$

which for $n' \sim n^\tau$ yields (3.4.4). Estimates (3.4.5)–(3.4.8) are established in Lemma 3.5 of [52]; in the estimate (3.4.5) we corrected a misprint of [52]. □

Theorem 3.4.1. *Assume* $(A1)-(A3)$. *Moreover, assume that the smoothing parameter* r *in* (3.3.6) *satisfies* (3.3.20) *and the dimension parameter* n' *satisfies* (3.4.1).

Then equations (3.1.1) *and* (3.3.11) *have unique solutions* $u \in C^{m,\nu}(0,1)$ *and* $\widetilde{u} \in \widetilde{C}^m(\mathbb{R})$, *respectively. These solutions are connected by the relation* $\widetilde{u}(t) = \varphi'(t)^{(1-\nu)/2} u(\varphi(t))$; *for sufficiently large* n, *say* $n \geq n_0$, *equation* (3.4.2) *has a unique solution* $\widetilde{u}_{n,n'} \in \mathcal{T}_{n'}$, *and*

$$\| \widetilde{u} - \widetilde{u}_{n,n'} \|_\infty \leq c n^{-\tau m} (\log n) \| \widetilde{f} \|_{C^m}, \tag{3.4.10}$$

$$\| \widetilde{u} - \widetilde{u}_{n,n'} \|_p \leq c' n^{-\tau m} \| \widetilde{f} \|_{W^{m,p}}, \quad 1/(1-\nu) < p < \infty, \tag{3.4.11}$$

with some positive constants c *and* c' *which are independent of* n *and* $\widetilde{f}$.

Proof. With the help of Theorem 2.1.1 we get that, for $1/(1-\nu) < p \leq \infty$, the operator $\widetilde{T} : \widetilde{L}^p \to \widetilde{C}(\mathbb{R})$ is compact, and since by the assumption $(A3)$, the nullspace of $I - T$ is trivial in $C[0,1]$ implying that the null space of $I - \widetilde{T}$ is trivial in $\widetilde{C}(\mathbb{R})$, the bounded inverse $(I - \widetilde{T})^{-1} : \widetilde{L}^p \to \widetilde{L}^p$ exists. Hence equation (3.3.11) and with it also (3.1.1) are uniquely solvable. Further, due to (3.4.5),

$$\| \widetilde{T} - \widetilde{T}_{n,n'} \|_{\widetilde{L}^p \to \widetilde{L}^p} \to 0 \quad \text{as} \quad n \to \infty \quad \text{for} \quad 1/(1-\nu) < p \leq \infty,$$

hence, due to Theorem 2.2.4, for sufficiently large n, say $n \geq n_0$, also $(I - \widetilde{T}_{n,n'})^{-1} : \widetilde{L}^p \to \widetilde{L}^p$ exists and is uniformly bounded in n:

$$\| (I - \widetilde{T}_{n,n'})^{-1} \|_{\widetilde{L}^p \to \widetilde{L}^p} \leq c_p, \ n \geq n_0, \ 1/(1-\nu) < p \leq \infty. \tag{3.4.12}$$

Thus equation (3.4.2) is uniquely solvable for $n \geq n_0$.

Clearly,

$$(I - \widetilde{T}_{n,n'})(\widetilde{u} - \widetilde{u}_{n,n'}) = \widetilde{f} - P_{n'} \widetilde{f} + (\widetilde{T} - \widetilde{T}_{n,n'}) \widetilde{u}.$$

Therefore,

$$\widetilde{u} - \widetilde{u}_{n,n'} = (I - \widetilde{T}_{n,n'})^{-1} \left(\widetilde{f} - P_{n'} \widetilde{f} + (\widetilde{T} - \widetilde{T}_{n,n'}) \widetilde{u} \right), \quad n \geq n_0. \tag{3.4.13}$$

Due to (3.4.12),

$$\| \widetilde{u} - \widetilde{u}_{n,n'} \|_p \leq c_p \left(\| \widetilde{f} - P_{n'}\widetilde{f} \|_p + \| (\widetilde{T} - \widetilde{T}_{n,n'})\widetilde{u} \|_p \right)$$

with $n \geq n_0$, $1/(1-\nu) < p \leq \infty$. Since $\widetilde{f} \in C^m(\mathbb{R})$, it follows from (2.6.3.6) and (2.6.3.7) that

$$\| \widetilde{f} - P_n\widetilde{f} \|_\infty \leq cn^{-m}(\log n) \| \widetilde{f}^{(m)} \|_\infty,$$

$$\| \widetilde{f} - P_n\widetilde{f} \|_p \leq c_p n^{-m} \| \widetilde{f}^{(m)} \|_p, \; 1 < p < \infty.$$

This in view of (3.4.1) yields

$$\begin{aligned} \| \widetilde{f} - P_{n'}\widetilde{f} \|_\infty &\leq cn^{-\tau m}(\log n) \| \widetilde{f} \|_{C^m}, \\ \| \widetilde{f} - P_{n'}\widetilde{f} \|_p &\leq c_p n^{-\tau m} \| \widetilde{f} \|_{W^{m,p}} . \end{aligned} \tag{3.4.14}$$

Applying also (3.4.7) and (3.4.6) with any $\lambda \in (0, 1-\nu-p^{-1})$ we arrive at the estimates (3.4.10) and (3.4.11). □

Following the notation of [61], the matrix form of equation (3.4.2) reads as

$$\underline{u}_{n'} = \underline{T}_{n'}\underline{u}_{n'} + \mathcal{F}_{n'}^{-1}\widehat{\widetilde{f}}_{n'}, \tag{3.4.15}$$

where

$$\underline{u}_{n'} = \begin{pmatrix} \widetilde{u}_{n,n'}(0) \\ \widetilde{u}_{n,n'}(1/n') \\ \vdots \\ \widetilde{u}_{n,n'}(n'-1)/n' \end{pmatrix}$$

is the vector of values of $\widetilde{u}_{n,n'}$ at the points $0, \frac{1}{n'}, \ldots, \frac{n'-1}{n'}$, $\widehat{\widetilde{f}}_{n'}$ is the vector of the Fourier coefficients $\widehat{\widetilde{f}}(k) = \int_0^1 \widetilde{f}(t)e^{-\mathrm{i}k2\pi t}dt$ $(k \in \mathbb{Z}_{n'})$ of the function $\widetilde{f}$, and

$$\underline{T}_{n'} = \mathcal{F}_{n'}^{-1}\mathcal{E}_{n',2n}\mathcal{F}_{2n}(\mathcal{A}_{2n} + \mathcal{B}_{2n})\mathcal{F}_{2n}^{-1}\mathcal{E}_{2n,n}\mathcal{P}_{n,n'}\mathcal{F}_{n'} + \sum_{j=0}^{m-1} \mathcal{M}_{n',j}\mathcal{F}_{n'}^{-1}\mathcal{G}_{n',j}\mathcal{F}_{n'} \tag{3.4.16}$$

is an $n' \times n'$ matrix. Here $\mathcal{F}_{n'}, \mathcal{F}_{n'}^{-1}$ and $\mathcal{F}_{2n}, \mathcal{F}_{2n}^{-1}$ are the Fourier transform matrices of dimensions $n' \times n'$ and $2n \times 2n$, respectively, changing the representation form of trigonometric polynomials (see Section 2.6.2); the projection-convolution

$$G_j(I - P_n) : \mathcal{T}_{n'} \to \mathcal{T}_{n'} \quad (j = 0, \ldots, m-1)$$

is realized by the diagonal $n' \times n'$ matrix $\mathcal{G}_{n',j}$ with the diagonal elements

$$\mathcal{G}_{n',j}(k,k) = 0 \quad \text{for } k \in \mathbb{Z}_n, \quad \mathcal{G}_{n',j}(k,k) = \widehat{\widetilde{g}}_j(k) \quad \text{for } k \in \mathbb{Z}_{n'} \diagdown \mathbb{Z}_n;$$

the multiplication operator $M_{\alpha_j,n}$ ($j = 0, \ldots, m-1$) is realized by the diagonal $n' \times n'$ matrix $\mathcal{M}_{n',j}$ with the diagonal elements

$$\mathcal{M}_{n',j}(k,k) = \alpha_{j,n}\left(\frac{k}{n'}\right), \quad k = 0, 1, \ldots, n'-1;$$

the projection $P_n : \mathcal{T}_{n'} \to \mathcal{T}_n$ is realized by $n \times n'$ matrix $\mathcal{P}_{n,n'}$ consisting of the $n \times n$ identity matrix completed from left and right by $n \times \frac{n'-n}{2}$ zero matrix; the embedding $\mathcal{T}_n \subset \mathcal{T}_{2n}$ is realized by the $2n \times n$ matrix $\mathcal{E}_{2n,n}$ consisting of the $n \times n$ identity matrix complemented by $\frac{n}{2} \times n$ zero matrix from above and below (for simplicity we assume that n and n' are even); the embedding $\mathcal{T}_{2n} \subset \mathcal{T}_{n'}$ is realized by the $n' \times 2n$ matrix $\mathcal{E}_{n',2n}$ consisting of the $2n \times 2n$ identity matrix completed by $\frac{n'-2n}{2} \times 2n$ zero matrix from above and below; finally, $\mathcal{A}_{2n} = (a_{j,j'})$ and $\mathcal{B}_{2n} = (b_{j,j'})$ are $2n \times 2n$ matrices with entries defined by the formulas

$$a_{j,j'} = \frac{1}{2n}\widetilde{a}\left(\frac{j}{2n}, \frac{j'}{2n}\right)\sigma_{j-j'}, \; b_{j,j'} = \frac{1}{2n}\widetilde{b}\left(\frac{j}{2n}, \frac{j'}{2n}\right), \; j, j' = 0, \ldots, 2n-1,$$

$$\sigma_l = \sum_{k \in \mathbb{Z}_{2n}} e^{\mathrm{i}k2\pi l/(2n)}\widehat{\widetilde{g}}(k), \quad l = -2n+1, \ldots, 2n-1.$$

In (3.4.15) and (3.4.16) only the $2n \times 2n$ matrix $\mathcal{A}_{2n} + \mathcal{B}_{2n}$ and the diagonal $n' \times n'$ matrices $\mathcal{M}_{n',j}$ and $\mathcal{G}_{n',j}$, $j = 0, \ldots, m-1$, and vector $\widehat{\widetilde{f}}_{n'}$ depend on the data of problem (3.1.1), namely on $O(n^2)$ values

$$\widetilde{a}\left(\frac{j}{2n}, \frac{j'}{2n}\right), \quad \widetilde{b}\left(\frac{j}{2n}, \frac{j'}{2n}\right), \quad j, j' = 0, \ldots, 2n-1,$$

and on the Fourier coefficients $\widehat{\widetilde{g}}(k)$ ($k \in \mathbb{Z}_{n'+m-1}$) of the function $\widetilde{g} \in \widetilde{C}^{2m}(\mathbb{R} \setminus \mathbb{Z})$ defined due to (3.3.1), which can be computed in $O(n^2)$ arithmetical operations with a very high accuracy $O(n^{-4m})$ from the values on a suitable graded grid consisting of n^2 points; see [52] for details. The application of $\mathcal{A}_{2n} + \mathcal{B}_{2n}$ to an $2n$-vector costs $4n^2$, the application of FFT transformation matrices $\mathcal{F}_{n'}$ and $\mathcal{F}_{n'}^{-1}$ of dimension $n' \times n'$ to an n'-vector costs $O(n' \log n)$, and the application of other matrices in $\underline{T}_{n'}$ is cheaper. Thus the application of $\underline{T}_{n'}$ to a n'-vector costs $4n^2 + O(n' \log n)$ arithmetical operations. This enables to solve system (3.4.15) in $O(n^2)$ arithmetical operations by the two grid iteration method combined with the GMRES [47, 63, 71, 76] on the coarse level.

It remains to comment on the computation of the Fourier coefficients $\widehat{\widetilde{f}}(k)$, $k \in \mathbb{Z}_{n'}$. In [52], a more complicated approximation of the free term is used that restricts the computation of Fourier coefficients to $k \in \mathbb{Z}_n$; this approximation is not suitable for goals of the present paper. Nevertheless, the idea of [52] can be used to compute also the Fourier coefficients $\widehat{\widetilde{f}}(k)$, $k \in \mathbb{Z}_{n'}$: in a special way described below, we approximate $\widetilde{f}$ by a function $\widetilde{f}_{n,n''}$ depending on the parameters n and $n'' \sim n^\sigma$, $\tau < \sigma < 2$ (with τ from (3.4.1)) so that the Fourier coefficients $\widehat{\widetilde{f}}_{n,n''}$ of $\widetilde{f}_{n,n''}$ are for $k \in \mathbb{Z}_{n'}$ (exactly) computable in $O(n^2)$ arithmetical operations and, with a $p \in (1/(1-\nu), \infty)$,

$$\| P_{n'}(\widetilde{f} - \widetilde{f}_{n,n''}) \|_p \leq cn^{-2m} \| \widetilde{f} \|_{C^m} . \tag{3.4.17}$$

The following remark reveals that this accuracy occurs to be sufficient for our purposes.

Remark 3.4.1. Let $\widetilde{u}_{n,n'}$ be the solution of equation (3.4.2), and let $\widetilde{u}_{n,n',n''}$ be the solution of the perturbed equation $\widetilde{u}_{n,n',n''} = \widetilde{T}_{n,n'}\widetilde{u}_{n,n',n''} + P_{n'}\widetilde{f}_{n,n''}$ where $\widetilde{f}_{n,n''}$ satisfies (3.4.17) with a $p \in (1/(1-\nu), \infty)$. Then, for sufficiently large n,

$$\| \widetilde{u}_{n,n'} - \widetilde{u}_{n,n',n''} \|_p \leq cn^{-2m} \| \widetilde{f} \|_{C^m}, \tag{3.4.18}$$

where the constant $c > 0$ is independent of $n_\star$ and $\widetilde{f}$.

Indeed, the estimate (3.4.18) immediately follows from the equality

$$\widetilde{u}_{n,n'} - \widetilde{u}_{n,n',n''} = (I - \widetilde{T}_{n,n'})^{-1} P_{n'}(\widetilde{f} - \widetilde{f}_{n,n''})$$

and (3.4.12).

Let us describe how a suitable $\widetilde{f}_{n,n''}$ can be constructed. Using (smooth) splines of degree m with knots i/n^2, $i \in \mathbb{Z}$, we construct in $O(n^2)$ arithmetical operations the interpolant or quasi-interpolant $\widetilde{f}_n$ of $\widetilde{f}$ which satisfies (cf. [72]) with a $p \in (1/(1-\nu), \infty)$ the following inequalities:

$$\| P_{n'}(\widetilde{f} - \widetilde{f}_n) \|_p \leq c \| \widetilde{f} - \widetilde{f}_n \|_\infty \leq c' n^{-2m} \| \widetilde{f}^{(m)} \|_\infty,$$

$$\| \widetilde{f}_n^{(m)} \|_\infty \leq c \| \widetilde{f} \|_{C^m} .$$

Take a number $n'' \sim n^\sigma$, $\tau < \sigma < 2$ with τ from condition (3.4.1). Starting from $\widetilde{f}_n^{(0)} = \widetilde{f}_n$, compute the 1-periodic antiderivatives

$$\widetilde{f}_n^{(-j)}(t) = \int_0^t \left(\widetilde{f}_n^{(-j+1)}(s) - \int_0^1 \widetilde{f}_n^{(-j+1)}(s)ds \right) ds, \quad j = 1, \ldots, l,$$

where $l \geq \frac{2m+\frac{1}{2}\tau}{\sigma-\tau}$, and put

$$\widetilde{f}_{n,n''} = \int_0^1 \widetilde{f}_n(z)dz + \left(Q_{n''} \widetilde{f}_n^{(-l)} \right)^{(l)} .$$

The values

$$\widetilde{f}_n^{(-l)}(i/n''), \, i = 0, \ldots, n'' - 1,$$

determining the interpolant $Q_{n''} \widetilde{f}_n^{(-l)}$, are available at the cost of $O(n^2)$ arithmetical operations. By FFT we can compute the Fourier coefficients

$$(\widehat{Q_{n''} \widetilde{f}_n^{(-l)}})(k), \, k \in \mathbb{Z}_{n''},$$

at the cost of $O(n'' \log n)$ arithmetical operations. The Fourier coefficients of $P_{n'} \widetilde{f}_{n,n''}$ can be picked from the equality

$$(P_{n'} \widetilde{f}_{n,n''})(x) = \int_0^1 \widetilde{f}_n(z)dz + \sum_{0 \neq k \in \mathbb{Z}_{n'}} (\widehat{Q_{n''} \widetilde{f}_n^{(-l)}})(k)(\mathrm{i}k2\pi)^l e^{\mathrm{i}k2\pi x}.$$

Repeating the argument of [52] we get the estimate

$$\| P_{n'}(\widetilde{f}_n - \widetilde{f}_{n,n''}) \|_\infty \leq \pi^l (n')^{l+\frac{1}{2}} \| \widetilde{f}_n^{(-l)} - Q_{n''} \widetilde{f}_n^{(-l)} \|_2 .$$

Since

$$\| \widetilde{f}_n^{(-l)} - Q_{n''} \widetilde{f}_n^{(-l)} \|_2 \leq c(n'')^{-(m+l)} \| \widetilde{f}_n \|_{W^{m,2}} \leq c' n^{-2m} \| \widetilde{f} \|_{C^m}$$

the estimate (3.4.17) follows:

$$\begin{aligned} \| P_{n'}(\widetilde{f} - \widetilde{f}_{n,n''}) \|_p &\leq \| P_{n'}(\widetilde{f} - \widetilde{f}_n) \|_p + \| P_{n'}(\widetilde{f}_n - \widetilde{f}_{n,n''}) \|_p \\ &\leq cn^{-2m} \| \widetilde{f} \|_{C^m} . \end{aligned}$$

Our next step is to compute the approximate solution of equation (3.3.9) using integral relation (3.3.16) in which we replace $\widetilde{u}$, the solution of (3.3.11), by $\widetilde{u}_{n,n'}$, the solution of (3.4.2). Recall that the kernel of integral operator in (3.3.16) is periodic only w.r.t. argument s, so we cannot immediately use the techniques of the present section. To make this possible, we first decompose the kernel extracting from it a periodic part and a polynomial part. This is done in Section 3.6 after some preliminaries given in Section 3.5 .

3.5 Extracting periodic and polynomial parts of a function

Denote by $\mathbb{P}_i$ the space of polynomials z of degree not exceeding $i+1$ and satisfying

$$\int_0^1 z(t)dt = 0.$$

There exist unique polynomials $z_i \in \mathbb{P}_i$, $i = 0, 1, \ldots$, such that

$$z_i^{(k)}(1) - z_i^{(k)}(0) = \delta_{i,k} \text{ (Kronecker symbol)}, \ \ k = 0, 1, 2, \ldots \qquad (3.5.1)$$

Indeed, fixing i, (3.5.1) is trivially fulfilled for $k \geq i+1$, so we have to determine the coefficients c_{ij}, $j = 0, \ldots, i+1$, of

$$z_i(t) = \sum_{j=0}^{i+1} c_{ij} t^j$$

so that (3.5.1) holds for $k = 0, \ldots, i$ and $\int_0^1 z_i(t)dt = 0$. These conditions yield an $i+2$ dimensional triangular system of algebraic equations with respect to the coefficients c_{ij}, $j = 0, 1, \ldots, i+1$, with nonzeroes on the main diagonal uniquely determining c_{ij}, $j = 0, \ldots, i+1$.

Namely, the requirement $\int_0^1 z_i(t)dt = 0$ yields that

$$c_{i0} + \frac{1}{2}c_{i1} + \frac{1}{3}c_{i2} + \cdots + \frac{1}{i+2}c_{i,i+1} = 0,$$

and the conditions (3.5.1) for $k = 0, \dots, i$ are equivalent to the equations

$$c_{i1} + c_{i2} + \cdots + c_{i,i+1} = 0,$$

$$2c_{i2} + \cdots + (i+1)c_{i,i+1} = 0,$$

$$\vdots \qquad \ddots \qquad \vdots$$

$$(i+1)!\, c_{i,i+1} = 1.$$

In particular, for $i = 0$ we have $c_{00} + \frac{1}{2}c_{01} = 0$, $c_{01} = 1$, implying $c_{01} = 1$ and $c_{00} = -\frac{1}{2}$ and hence $z_0(t) = t - \frac{1}{2}$.

Note also that having z_i in hand and knowing that z_{i+1} exists and is unique, it is easy to check that

$$z_{i+1}(t) = \int_0^t z_i(s)ds + \int_0^1 s z_i(s)ds, \quad i = 0, 1, \dots$$

Thus, the first three polynomials $z_i \in \mathbb{P}_i$ $(i = 0, 1, 2)$ satisfying $\int_0^1 z_i(t) = 0$ and (3.5.1) are as follows:

$$z_0(t) = t - \frac{1}{2},$$
$$z_1(t) = \frac{1}{2}t^2 - \frac{1}{2}t + \frac{1}{12},$$
$$z_2(t) = \frac{1}{6}t^3 - \frac{1}{4}t^2 + \frac{1}{12}t.$$

Further, denote by $\widetilde{C}^m[0,1]$ the subspace of functions $\widetilde{w} \in C^m[0,1]$ satisfying the periodic boundary conditions

$$\widetilde{w}^{(i)}(1) - \widetilde{w}^{(i)}(0) = 0,\ i = 0, \dots, m.$$

This yields that every function $w \in C^m[0,1]$ has the representation

$$w = \widetilde{w} + w_m, \tag{3.5.2}$$

with

$$w_m = \sum_{i=0}^{m} [w^{(i)}(1) - w^{(i)}(0)] z_i \in \mathbb{P}_m, \quad \widetilde{w} = w - w_m \in \widetilde{C}^m[0,1].$$

Since an 1-periodic extension of a function $\widetilde{w} \in \widetilde{C}^m[0,1]$ is $C^m(\mathbb{R})$-smooth, we can identify $\widetilde{C}^m[0,1]$ with $\widetilde{C}^m(\mathbb{R})$.

3.6 An approximate method for equation (3.3.9)

Using polynomials $z_i \in \mathbb{P}_i$ of degree not exceeding $i+1$ and satisfying (3.5.1) and $\int_0^1 z_i(t)dt = 0$, we apply decomposition (3.5.2) for $a_\star(t,s)$ and $b_\star(t,s)$ (see (3.3.17) and (3.3.18)) as functions of t:

$$a_\star(t,s) = \widetilde{a}_\star(t,s) + \sum_{i=0}^{2m} z_i(t)\widetilde{a}_i(s), \tag{3.6.1}$$

where

$$\widetilde{a}_i(s) = \left(\frac{\partial}{\partial t}\right)^i a_\star(t,s)\Big|_{t=1} - \left(\frac{\partial}{\partial t}\right)^i a_\star(t,s)\Big|_{t=0}$$

and

$$b_\star(t,s) = \widetilde{b}_\star(t,s) + \sum_{i=0}^{2m} z_i(t)\widetilde{b}_i(s), \tag{3.6.2}$$

where

$$\widetilde{b}_i(s) = \left(\frac{\partial}{\partial t}\right)^i b_\star(t,s)\Big|_{t=1} - \left(\frac{\partial}{\partial t}\right)^i b_\star(t,s)\Big|_{t=0}.$$

In accordance to (3.5.2), the functions $\widetilde{a}_\star(t,s)$ and $\widetilde{b}_\star(t,s)$ are 1-periodic in t. Using (3.3.17) we get

$$\begin{aligned} \widetilde{a}_i(s) &= \Big[a(1,\varphi(s))\left(\frac{\partial}{\partial t}\right)^i \Phi(t,s)^{-\nu}\Big|_{t=1} \\ &- a(0,\varphi(s))\left(\frac{\partial}{\partial t}\right)^i \Phi(t,s)^{-\nu}\Big|_{t=0}\Big]\varphi'(s)^{(1+\nu)/2}, \quad i = 0,\ldots,2m. \end{aligned}$$

Due to (3.3.20) it holds $\varphi^{(j)}(0) = \varphi^{(j)}(1) = 0$ for $j = 1, \ldots, 2m+1$ implying

$$\widetilde{a}_i^{(j)}(0) = \widetilde{a}_i^{(j)}(1) = 0 \quad \text{for} \quad j = 0, \ldots, 2m.$$

After 1-periodic extension, we have $\widetilde{a}_i \in \widetilde{C}^{2m}(\mathbb{R})$, $i = 0, \ldots, 2m$. In a similar way we obtain that $\widetilde{b}_i \in \widetilde{C}^{2m}(\mathbb{R})$, $i = 0, \ldots, 2m$. Hence $\widetilde{a}_\star, \widetilde{b}_\star \in \widetilde{C}^{2m}(\mathbb{R} \times \mathbb{R})$ (remember that $a_\star(t,s)$ and $b_\star(t,s)$ are 1-periodic in s).

Denoting by

$$T_\star = A_\star + B_\star$$

the integral operator in (3.3.16), with $A_\star$ and $B_\star$ determined by

$$(A_\star v)(t) = \int_0^1 a_\star(t,s)\widetilde{g}(t-s)v(s)ds, \;\; (B_\star v)(t) = \int_0^1 b_\star(t,s)v(s)ds,$$

we obtain the decomposition

$$T_\star = \widetilde{A}_\star + \widetilde{B}_\star + \sum_{i=0}^{2m} M_{z_i}(\widetilde{A}_i + \widetilde{B}_i) \tag{3.6.3}$$

where

$$(\widetilde{A}_\star v)(t) = \int_0^1 \widetilde{a}_\star(t,s)\widetilde{g}(t-s)v(s)ds, \;\; (\widetilde{B}_\star v)(t) = \int_0^1 \widetilde{b}_\star(t,s)v(s)ds,$$

$$(\widetilde{A}_i v)(t) = \int_0^1 \widetilde{g}(t-s)\widetilde{a}_i(s)v(s)ds, \;\; (\widetilde{B}_i v)(t) = \int_0^1 \widetilde{b}_i(s)v(s)ds. \tag{3.6.4}$$

Clearly, $\widetilde{A}_\star$, $\widetilde{B}_\star$, $\widetilde{A}_i$, $\widetilde{B}_i$ ($i = 0, \ldots, 2m$) are periodic integral operators with biperiodic kernels, and the only nonperiodic operators in the decomposition (3.6.3) are M_{z_i} :

$$(M_{z_i} v)(t) = z_i(t)v(t), \;\; i = 0, \ldots, 2m,$$

realizing the multiplication with the polynomials z_i. We approximate the periodic integral operator

$$\widetilde{A}_\star + \widetilde{B}_\star$$

in (3.6.3) in analogy to (3.4.3) by the operator

$$Q_{2n}(\widetilde{A}_{\star,2n} + \widetilde{B}_{\star,2n})P_n + Q_{n'}\widetilde{A}_{\star,n}^{(m)}(I - P_n).$$

Also $\widetilde{A}_i + \widetilde{B}_i$ in (3.6.3) could be approximated similarly as in (3.4.3) but since the coefficient functions $\widetilde{a}_i(s)$ and $\widetilde{b}_i(s)$ in (3.6.4) are independent of t, more simple approximations are available: we introduce for

$$\widetilde{A}_i + \widetilde{B}_i \quad (i = 0, \ldots, 2m)$$

the approximation

$$(\widetilde{A}_{i,n'} + \widetilde{B}_{i,n'})P_{[n'/2]}$$

where $[n'/2]$ is the integer part of $n'/2$ and

$$(\widetilde{A}_{i,n'}v)(t) = \int_0^1 \widetilde{g}(t-s)Q_{n'}\Big(\widetilde{a}_i(s)v(s)\Big)ds, \quad \widetilde{B}_{i,n'}v = \int_0^1 Q_{n'}\left(\widetilde{b}_i(s)v(s)\right) ds$$

Thus we approximate $T_\star$ (see (3.6.3)) by

$$\begin{aligned} T_{\star,n,n'} = Q_{2n}(\widetilde{A}_{\star,2n} + \widetilde{B}_{\star,2n})P_n \quad &+ \quad Q_{n'}\widetilde{A}^{(m)}_{\star,n}(I - P_n) \\ &+ \quad \sum_{i=0}^{2m} M_{z_i}(\widetilde{A}_{i,n'} + \widetilde{B}_{i,n'})P_{[n'/2]}. \end{aligned} \tag{3.6.5}$$

Given the sample values $v(i/n')$, $i = 0, \ldots, n'-1$, of a function $v \in \mathcal{T}_{n'}$, the computation of

$$(T_{\star,n,n'}v)(i/n'),\ i = 0, \ldots, n'-1,$$

costs $O(n^2)$ arithmetical operations (including the assembling of the matrix form of the operator), cf. Section 3.4 .

The following lemma reveals that for the operators $T_\star$ and $T_{\star,n,n'}$, similar estimates as in Lemma 3.4.1 hold true.

Lemma 3.6.1. *Assume (A2), and let $n' \sim n^\tau$, $\tau > 1$. Then for $1/(1-\nu) < p < \infty$,*

$$\| T_\star(I - P_{n'}) \|_{\widetilde{L}^p \to C} \leq cn^{-\tau(1-\nu-1/p)}; \tag{3.6.6}$$

for $1/(1-\nu) < p \leq \infty$, $0 < \lambda < 1 - \nu - p^{-1}$,

$$\| T_\star - T_{\star,n.n'} \|_{\widetilde{L}^p \to C} \leq c_{p,\lambda}(n^{-m} + n^{-\tau\lambda}) \log n; \tag{3.6.7}$$

for $1/(1-\nu) < p < \infty$, $0 < \lambda < 1 - \nu - p^{-1}$, $v \in \widetilde{W}^{m,p}(\mathbb{R})$,

$$\| T_\star v - T_{\star,n.n'} v \|_p \leq c_{p,\lambda}(n^{-2m} + n^{-\tau(m+\lambda)}(\log n)) \| v \|_{W^{m,p}}, \qquad (3.6.8)$$

$$\| T_\star v - T_{\star,n.n'} v \|_\infty \leq c_{p,\lambda}(n^{-2m} + n^{-\tau(m+\lambda)})(\log n) \| v \|_{W^{m,p}}, \qquad (3.6.9)$$

and, strengthening $(A2)$ *to the assumption* $(A2')$,

$$\begin{aligned} \| T_\star v - T_{\star,n.n'} v \|_\infty &\leq c_{p,\lambda}(n^{-2m-\min\{\mu,\,1-\nu-1/p\}} \\ &+ n^{-\tau(m+\lambda)})(\log n) \| v \|_{W^{m,p}} . \qquad (3.6.10) \end{aligned}$$

Proof. By Lemma 3.4.1 the counterparts of the estimates (3.6.6)–(3.6.10) hold for $A_\star + B_\star$ and its approximation

$$Q_{2n}(\widetilde{A}_{\star,2n} + \widetilde{B}_{\star,2n})P_n + Q_{n'}\widetilde{A}^{(m)}_{\star,n}(I - P_n),$$

and the counterpart of (3.6.6) holds also for $\widetilde{A}_i + \widetilde{B}_i$. So we only have to check that the counterparts of estimates (3.6.7)–(3.6.10) hold for $\widetilde{A}_i + \widetilde{B}_i$ and their approximation

$$(\widetilde{A}_{i,n'} + \widetilde{B}_{i,n'})P_{[n'/2]}\,, \quad i = 0, \ldots, 2m.$$

Split

$$\widetilde{A}_i - \widetilde{A}_{i,n'}P_{[n'/2]} = \widetilde{A}_i(I - P_{[n'/2]}) + (\widetilde{A}_i - \widetilde{A}_{i,n'})P_{[n'/2]}. \qquad (3.6.11)$$

Consider the first addend in the r.h.s. of (3.6.11). Let p and λ satisfy the conditions

$$1/(1-\nu) < p < \infty,\ 0 < \lambda < 1 - \nu - p^{-1}.$$

By (3.4.4), we obtain in support of (3.6.7)

$$\| \widetilde{A}_i(I - P_{[n'/2]}) \|_{\widetilde{L}^p \to C} \leq c n^{-\tau(1-\nu-1/p)},$$

that is we can write

$$\| \widetilde{A}_i(I - P_{[n'/2]}) \|_{\widetilde{L}^p \to C} = o(n^{-\tau\lambda});$$

for $p=\infty$, according to [52] we again obtain

$$\| \widetilde{A}_i(I-P_{[n'/2]}) \|_{\widetilde{L}^\infty\to C} \leq cn^{-\tau(1-\nu)}\log n$$

allowing to write

$$\| \widetilde{A}_i(I-P_{[n'/2]}) \|_{\widetilde{L}^\infty\to C} = o(n^{-\tau\lambda}).$$

For $v\in\widetilde{W}^{m,p}(\mathbb{R})$, due to (3.4.14) we obtain in support of (3.6.8)–(3.6.10)

$$\| \widetilde{A}_i(I-P_{[n'/2]})v \|_\infty \leq c \| \widetilde{A}_i(I-P_{[n'/2]}) \|_{\widetilde{L}^p\to C} \| (I-P_{[n'/2]})v \|_p$$

$$\leq c'n^{-\tau(1-\nu-1/p)}n^{-\tau m} \| v \|_{W^{m,p}},$$

and thus,

$$\| \widetilde{A}_i(I-P_{[n'/2]})v \|_\infty = o(n^{-\tau(m+\lambda)} \| v \|_{W^{m,p}}, \quad i=0,\dots,2m.$$

Consider the second addend in the r.h.s of (3.6.11). We have with any $w\in\mathfrak{T}_{[n'/2]}$ and $t\in[0,1]$ that

$$((\widetilde{A}_i-\widetilde{A}_{i,n'})P_{[n'/2]}v)(t)=\int_0^1\widetilde{g}(t-s)(I-Q_{n'})\Big((\widetilde{a}_i(s)-w(s))(P_{[n'/2]}v)(s)\Big)ds,$$

which together with $\frac{1}{p}+\frac{1}{q}=1$ yields

$$\begin{aligned} \| (\widetilde{A}_i-\widetilde{A}_{i,n'})P_{[n'/2]}v \|_\infty &\leq \| \widetilde{g} \|_q \| (I-Q_{n'})(\widetilde{a}_i-w)P_{[n'/2]}v \|_p \\ &\leq c \| \widetilde{a}_i-w \|_\infty \| P_{[n'/2]}v \|_\infty, \end{aligned} \tag{3.6.12}$$

where $i=0,\dots,2m$. This supports estimates (3.6.6)–(3.6.10) since

$$\| P_{[n'/2]}v \|_\infty \leq n \| v \|_1, \quad \| P_{[n'/2]}v \|_\infty \leq c_p \| v \|_{W^{m,p}},$$

and for $\widetilde{a}_i\in\widetilde{C}^{2m}(\mathbb{R})$, there exists a $w\in\mathfrak{T}_{[n'/2]}$ such that

$$\| \widetilde{a}_i-w \|_\infty \leq cn^{-2\tau m}, \quad i=0,\dots,2m.$$

□

In the equality $\bar{u} = T_\star \widetilde{u} + \bar{f}$ which is a short writing of the integral relation (3.3.16), we approximate the solution $\widetilde{u}$ of (3.3.11) by the solution $\widetilde{u}_{n,n'}$ of (3.4.2), and we approximate the integral operator (3.6.3) by (3.6.5), obtaining

$$\bar{u}_{n,n'} := T_{\star,n.n'} \widetilde{u}_{n,n'} + \bar{f} \tag{3.6.13}$$

which we treat as an approximate solution of equation (3.3.9). The approximation $\bar{u}_{n,n'}$ is not discrete. Its discrete counterpart will be introduced and discussed in Section 7.

Theorem 3.6.1. *Assume the conditions of Theorem 3.4.1. Then*

$$\| \bar{u} - \bar{u}_{n,n'} \|_\infty \leq c n^{-2m} (\log n) \| \widetilde{f} \|_{C^m}, \tag{3.6.14}$$

$$\| \bar{u} - \bar{u}_{n,n'} \|_p \leq c_p n^{-2m} \| \widetilde{f} \|_{C^m}, \;\; 1 \leq p < \infty, \tag{3.6.15}$$

where $\bar{u}(t) = u(\varphi(t))$ *is the solution of equation* (3.3.9)*,* u *is the solution of equation* (3.1.1)*, and* $\bar{u}_{n,n'}$ *is defined by* (3.6.13) *in which* $\widetilde{u}_{n,n'}$ *is the solution of equation* (3.4.2)*. If* a *and* b *satisfy* $(A2')$ *then*

$$\| \bar{u} - \bar{u}_{n,n'} \|_\infty = o(n^{-2m}) \| \widetilde{f} \|_{C^m}, \tag{3.6.16}$$

i.e. $\| \bar{u} - \bar{u}_{n,n'} \|_\infty n^{2m} \to 0 \quad$ *as* $n \to \infty$.

Proof. Since

$$\bar{u} - \bar{u}_{n,n'} = T_\star \widetilde{u} - T_{\star,n.n'} \widetilde{u}_{n,n'}$$

$$= T_\star(\widetilde{u} - \widetilde{u}_{n,n'}) + (T_{\star,n.n'} - T_\star)(\widetilde{u} - \widetilde{u}_{n,n'}) + (T_\star \widetilde{u} - T_{\star,n.n'} \widetilde{u}),$$

we have

$$\begin{aligned} \| \bar{u} - \bar{u}_{n,n'} \|_\infty \leq \| T_\star(\widetilde{u} - \widetilde{u}_{n,n'}) \|_\infty \quad &+ \quad \| T_\star - T_{\star,n.n'} \|_{\widetilde{L}^p \to C} \| \widetilde{u} - \widetilde{u}_{n,n'} \|_p \\ &+ \quad \| T_\star \widetilde{u} - T_{\star,n.n'} \widetilde{u} \|_\infty \, . \end{aligned} \tag{3.6.17}$$

Due to (3.4.1), $(1+\tau)m > 2m$; taking $p < \infty$ sufficiently large and $\lambda > 0$ sufficiently close to $1 - \nu - p^{-1}$, we have $\tau(m + \lambda) > 2m$, and the estimates (3.6.7), (3.4.11) imply for $1/(1-\nu) < p < \infty$ that

$$\| T_\star - T_{\star,n.n'} \|_{\widetilde{L}^p \to C} \| \widetilde{u} - \widetilde{u}_{n,n'} \|_p$$

$$\leq c(n^{-m} + n^{-\tau\lambda}) n^{-\tau m} (\log n)^2 \| \widetilde{f} \|_{W^{m,p}},$$

which allows to write

$$\| T_\star - T_{\star,n.n'} \|_{\widetilde{L}^p\to C} \| \widetilde{u} - \widetilde{u}_{n,n'} \|_p = o(n^{-2m}) \| \widetilde{f} \|_{W^{m,p}}, \qquad (3.6.18)$$

whereas (3.6.8)–(3.6.10) yield

$$\| T_\star\widetilde{u} - T_{\star,n.n'}\widetilde{u} \|_p \leq c_p n^{-2m} \| \widetilde{f} \|_{W^{m,p}},\ 1/(1-\nu) < p < \infty, \qquad (3.6.19)$$

$$\| T_\star\widetilde{u} - T_{\star,n.n'}\widetilde{u} \|_\infty \leq cn^{-2m}(\log n) \| \widetilde{f} \|_{W^{m,p}},$$

and in the case of $(A2')$,

$$\| T_\star\widetilde{u} - T_{\star,n.n'}\widetilde{u} \|_\infty = o(n^{-2m}) \| \widetilde{f} \|_{W^{m,p}} . \qquad (3.6.20)$$

Below we show that

$$\| T_\star(\widetilde{u} - \widetilde{u}_{n,n'}) \|_\infty \leq cn^{-2m} \| \widetilde{f} \|_{C^m} \text{ if } a, b \in C^{2m}([0,1]\times[0,1]), \qquad (3.6.21)$$

$$\| T_\star(\widetilde{u} - \widetilde{u}_{n,n'}) \|_\infty = o(n^{-2m}) \| \widetilde{f} \|_{C^m} \text{ if } a, b \in \mathcal{H}^{2m,\mu}([0,1]\times[0,1]). \qquad (3.6.22)$$

Now the estimates (3.6.14)–(3.6.16) immediately follow from (3.6.17)–(3.6.21); actually, we obtain (3.6.15) only for $1/(1-\nu) < p < \infty$; to obtain (3.6.15) for smaller $p \in [1, 1/(1-\nu)]$ we exploit the monotonicity in p of the norm $\| \cdot \|_p$.

It remains to prove (3.6.21) and (3.6.22). Applying to both sides of equality (3.4.13) the operator $T_\star$ and using the equality

$$(I - \widetilde{T}_{n,n'})^{-1} = I + (I - \widetilde{T}_{n,n'})^{-1}\widetilde{T}_{n,n'},$$

we obtain

$$\begin{aligned} T_\star(\widetilde{u} - \widetilde{u}_{n,n'}) &= T_\star(\widetilde{f} - P_{n'}\widetilde{f}) + T_\star(\widetilde{T} - \widetilde{T}_{n,n'})\widetilde{u} \\ &+ T_\star(I - \widetilde{T}_{n,n'})^{-1}\Big(\widetilde{T}_{n,n'}(\widetilde{f} - P_{n'}\widetilde{f}) + \widetilde{T}_{n,n'}(\widetilde{T} - \widetilde{T}_{n,n'})\widetilde{u}\Big) \end{aligned} \qquad (3.6.23)$$

and

$$\| T_\star(\widetilde{u} - \widetilde{u}_{n,n'}) \|_\infty \leq \| T_\star(\widetilde{f} - P_{n'}\widetilde{f}) \|_\infty \quad + \quad c_1 \| \widetilde{T}_{n,n'}(\widetilde{f} - P_{n'}\widetilde{f}) \|_\infty$$
$$+ \quad c_2 \| (\widetilde{T} - \widetilde{T}_{n,n'})\widetilde{u} \|_p, \tag{3.6.24}$$

where we took into account (3.4.12), the boundedness of $T_\star : \widetilde{L}^p \to C$, and the uniform boundedness of operators $\widetilde{T}_{n,n'} : \widetilde{L}^p \to \widetilde{C}$, $n \in \mathbb{N}$ (see (3.4.5)).

Using the equality $I - P_{n'} = (I - P_{n'})^2$, (3.6.6) and (3.4.14) , we estimate

$$\begin{aligned} \| T_\star(\widetilde{f} - P_{n'}\widetilde{f}) \|_\infty &\leq \| T_\star(I - P_{n'}) \|_{\widetilde{L}^p \to C} \| \widetilde{f} - P_{n'}\widetilde{f}) \|_p \\ &\leq cn^{-\tau(1-\nu-1/p)} \cdot n^{-\tau m} \| \widetilde{f} \|_{W^{m,p}}, \end{aligned} \tag{3.6.25}$$

that is,

$$\| T_\star(\widetilde{f} - P_{n'}\widetilde{f}) \|_\infty \leq cn^{-\tau(m+1-\nu-1/p)} \| \widetilde{f} \|_{W^{m,p}} . \tag{3.6.26}$$

According to (3.4.1),

$$\tau(m + 1 - \nu) > 2m.$$

Then for sufficiently large p also

$$\tau(m + 1 - \nu - 1/p) > 2m.$$

This together with (3.6.26) yields

$$\| T_\star(\widetilde{f} - P_{n'}\widetilde{f}) \|_\infty = o(n^{-2m}) \| \widetilde{f} \|_{C^m} . \tag{3.6.27}$$

With the help of (3.6.7) and (3.4.14) we get similarly as in (3.6.18), (3.6.27) that

$$\begin{aligned} &\| \widetilde{T}_{n,n'}(\widetilde{f} - P_{n'}\widetilde{f}) \|_\infty \\ &= \| \widetilde{T}_{n,n'}\widetilde{f} - \widetilde{T}_{n,n'}P_{n'}\widetilde{f} + \widetilde{T}\widetilde{f} - \widetilde{T}\widetilde{f} - \widetilde{T}P_{n'}\widetilde{f} + \widetilde{T}P_{n'}\widetilde{f} \|_\infty \\ &\leq \| \widetilde{T}_{n,n'} - \widetilde{T} \|_{\widetilde{L}^p \to C} \| \widetilde{f} - P_{n'}\widetilde{f} \|_p + \| \widetilde{T}(\widetilde{f} - P_{n'}\widetilde{f}) \|_\infty, \end{aligned}$$

and thus

$$\| \widetilde{T}_{n,n'}(\widetilde{f} - P_{n'}\widetilde{f}) \|_\infty = o(n^{-2m}) \| \widetilde{f} \|_{C^m} . \tag{3.6.28}$$

Similarly to (3.6.19), (3.6.20) we have

$$\| \widetilde{T}\widetilde{u} - \widetilde{T}_{n.n'}\widetilde{u} \|_p \leq c_p n^{-2m} \| \widetilde{f} \|_{W^{m,p}}, \; 1/(1-\nu) < p < \infty,$$
$$\| \widetilde{T}\widetilde{u} - \widetilde{T}_{n.n'}\widetilde{u} \|_\infty = o(n^{-2m}) \| \widetilde{f} \|_{C^m} \quad \text{if } a, b \in \mathcal{H}^{2m,\mu}([0,1] \times [0,1]). \tag{3.6.29}$$

Plugging (3.6.27)–(3.6.29) into (3.6.24) we obtain (3.6.21) and (3.6.22). □

Remark 3.6.1. Consider the case where we use some approximation $\widetilde{f}_{n,n''}$ of $\widetilde{f}$ to compute the Fourier coefficients of $\widetilde{f}$ as in Remark 3.4.1 so that (3.4.17) and hence also (3.4.18) hold true. Then instead of (3.6.13) we obtain the approximation

$$\bar{u}_{n,n',n''} = T_{\star,n.n'}\widetilde{u}_{n,n',n''} + \bar{f}.$$

Following the proof scheme of Theorem 3.6.1 we can see that, under conditions of Theorem 3.4.1.

$$\| \bar{u}_{n,n'} - \bar{u}_{n,n',n''} \|_\infty \leq cn^{-2m} \| \widetilde{f} \|_{C^m}$$

and

$$\| \bar{u}_{n,n'} - \bar{u}_{n,n',n''} \|_\infty \leq cn^{-2m} \| \widetilde{f} \|_{C^m} \;\; if \;\, a, b \text{ satisty } (A2').$$

Hence estimates (3.6.14)–(3.6.16) remain to be valid also for $\bar{u}_{n,n',n''}$.

Remark 3.6.2. The operator $\widetilde{T}$ has the property (compare (3.2) in [52]) that for $\frac{1}{1-\nu} < p \leq \infty$, $0 < \lambda < 1 - \nu - \frac{1}{p}$,

$$\widetilde{T} : \widetilde{W}^{m,p}(\mathbb{R}) \to \widetilde{\mathcal{H}}^{m,\lambda}(\mathbb{R}) \text{ is bounded;}$$

moreover, for those p and λ, it can be proved by the techniques of [52] that in a qualified manner

$$\| \widetilde{T} - \widetilde{T}_{n,n'} \|_{\widetilde{W}^{m,p}(\mathbb{R}) \to \widetilde{\mathcal{H}}^{m,\lambda}(\mathbb{R})} \to 0 \text{ as } n \to \infty.$$

The definitions (3.6.3) and (3.6.5) of $T_\star$ and $T_{\star,n,n'}$ enable to extend these relations: for $\frac{1}{1-\nu} < p \leq \infty, \; 0 < \lambda < 1 - \nu - \frac{1}{p}$,

$$T_\star : \widetilde{W}^{m,p}(\mathbb{R}) \to \mathcal{H}^{m,\lambda}[0,1] \text{ is bounded,}$$

$$\| T_\star - T_{\star,n,n'} \|_{\widetilde{W}^{m,p}(\mathbb{R}) \to \mathcal{H}^{m,\lambda}} \to 0 \text{ as } n \to \infty.$$

From here we conclude for the solution $\widetilde{u}_{n,n'}$ of equation (3.4.2) that

$$\| \widetilde{u}_{n,n'} \|_{W^{m,p}} \leq c_p \| P_n \widetilde{f} \|_{W^{m,p}} \leq c'_p \| \widetilde{f} \|_{W^{m,p}}, \quad \frac{1}{1-\nu} < p < \infty, \quad (3.6.30)$$

$$\| T_{\star,n,n'} \widetilde{u}_{n,n'} \|_{\mathcal{H}^{m,\lambda}} \leq c_{p,\lambda} \| \widetilde{f} \|_{W^{m,p}}, \quad \frac{1}{1-\nu} < p < \infty, \; 0 < \lambda < 1 - \nu - \frac{1}{p}. \quad (3.6.31)$$

Deriving (3.6.30) we exploited the fact that $\| P_n \|_{\widetilde{W}^{m,p}(\mathbb{R}) \to \widetilde{W}^{m,p}(\mathbb{R})} \leq c_p$ for $n \in \mathbb{N}$, $1 < p < \infty$.

Inequality (3.6.31) will be helpful in Section 3.7. It is valid also if we replace $\widetilde{u}_{n,n'}$ by $\widetilde{u}_{n,n',n''}$ assuming the conditions of Remark 3.4.1.

3.7 Fast and quasi-fast solvers for equation (3.1.1)

According to Section 3.4, the sample values $\widetilde{u}_{n,n'}(i/n')$, $i = 0, \ldots, n'-1$, of the solution $\widetilde{u}_{n,n'} \in \mathcal{T}_{n'}$ to (3.4.2) are available at the cost of $O(n^2)$ arithmetical operations. After that also the values $(T_{\star,n,n'} \widetilde{u}_{n,n'})(i/n')$, $i = 0, \ldots, n'$, of $T_{\star,n,n'} \widetilde{u}_{n,n'}$ can be computed in $O(n^2)$ arithmetical operations, and we can use the local (algebraic) polynomial interpolation to approximate $T_{\star,n,n'} \widetilde{u}_{n,n'}$ between the grid points i/n', $i = 0, \ldots, n'$. Define the polynomially interpolated version $\bar{v}_{n,n'}(t)$ of $\bar{u}_{n,n'} = T_{\star,n,n'} \widetilde{u}_{n,n'} + \bar{f}$ (see (3.6.13) and Section 2.5) as

$$\bar{v}_{n,n'} = \Pi_{n',m+1} T_{\star,n,n'} \widetilde{u}_{n,n'} + \Pi_{n^2,m} \bar{f} \quad (3.7.1)$$

where $\Pi_{n',m+1}$ is an operator of local interpolation of functions given on the grid $\{i/n' : \; i = 0, \ldots, n'\}$ by polynomials of degree m (or of order $m+1$) and $\Pi_{n^2,m}$ is an operator of local interpolation by polynomials of degree $m-1$ from the grid values on the net $\left\{i/n^2 : i = 0, \ldots, n^2\right\}$. Thus, the unknown parameters $(T_{\star,n,n'} \widetilde{u}_{n,n'})(i/n')$, $i = 0, \ldots, n'$, of $\bar{v}_{n,n'}$ are available at the cost of $O(n^2)$ arithmetical operations, and having them in hand, the value of $\bar{v}_{n,n'}$ at any point $t \in [0,1]$ is available at the cost of $O(1)$ arithmetical operations.

We could define the approximate solution of equation (3.1.1) by the formula

$$v_{n,n'}(x) = \bar{v}_{n,n'}(\varphi^{-1}(x)), \ \ 0 \le x \le 1, \tag{3.7.2}$$

with φ^{-1}, the inverse of φ (see (2.4.1)), but since we cannot present a closed formula for $\varphi^{-1}(x)$ we first approximate it by a suitable local interpolation. Let us precompute $x_{i,n} = \varphi(i/n)$ for $i = 0, \ldots, n$ (this costs altogether only $O(n)$ arithmetical operations) and use them to approximate $t = \varphi^{-1}(x)$ for any given $x \in [0,1]$ via a local interpolation by polynomials of degree $2m-1$. Note that $\varphi^{-1} \in C^{2m,1/r}(0,1)$ where r is the (smoothing) parameter in the definition of φ (actually $\varphi^{-1} \in C^{l,1/r}(0,1)$ for any $l \in \mathbb{N}$), and $\{x_{i,n} : i = 0, \ldots, n\}$ is a suitable graded grid for the interpolation of such functions – denoting by ψ_n the interpolation approximation of $\psi = \varphi^{-1}$, we have

$$|\varphi^{-1}(x) - \psi_n(x)| \le cn^{-2m} \parallel \varphi^{-1} \parallel_{C^{2m,1/r}}, \ \ 0 \le x \le 1, \tag{3.7.3}$$

see, e.g., [74]. Instead of (3.7.2), we define the final approximation to the solution of equation (3.1.1) via the formula

$$u_{n,n'}(x) = \bar{v}_{n,n'}(\psi_n(x)), \ \ 0 \le x \le 1. \tag{3.7.4}$$

The grid values

$$(T_{\star,n,n'}\widetilde{u}_{n,n'})(i/n') \quad (i = 0, \ldots, n')$$

and

$$\bar{f}(i/n^2) = f(\varphi(i/n^2)) \quad (i = 0, \ldots, n^2)$$

can be considered as the parameters of the approximate solution (3.7.4). The grid values of f belong to a given information whereas the grid values of $T_{\star,n,n'}\widetilde{u}_{n,n'}$ are available at the cost of $O(n^2)$ arithmetical operations using $n_\star = O(n^2)$ sample values of a and b and the $n^2 + 1$ sample values of f just listed. To see that with (3.7.4) we have designed a fast/quasi-fast solver of equation (3.1.1), it remains to establish for $u_{n,n'}$ the estimates of type (3.2.1)/(3.2.2). Estimates (3.7.5)–(3.7.7) (see Theorem 3.7.1 below) mean that under condition $a, b \in C^{2m}([0,1] \times [0,1])$ approximation (3.7.4) defines a $(C, C^{m,\nu})$ quasi-fast solver of equation (3.1.1) which is $(L^p, C^{m,\nu})$ fast for $1 \le p < \infty$ and, moreover, this solver is even $(C, C^{m,\nu})$ fast if the mth derivatives of a, b are Hoelder continuous.

Theorem 3.7.1. *Assume* $(A1)-(A3)$*. Moreover, assume that the smoothing parameter* r *in* (3.3.6) *satisfies* (3.3.20) *and the dimension parameter* n' *satisfies* (3.4.1).

Then we have for all sufficiently large $n \in \mathbb{N}$ *that*

$$\| u - u_{n,n'} \|_\infty \le cn^{-2m}(\log n) \| f \|_{C^{m,\nu}}, \tag{3.7.5}$$

$$\| u - u_{n,n'} \|_p \le c_p n^{-2m} \| f \|_{C^{m,\nu}}, \ \ 1 \le p < \infty, \tag{3.7.6}$$

where u *is the solution of* (3.1.1) *and* $u_{n,n'}$ *is defined by* $\{(3.7.1), (3.7.4)\}$ *with the solution* $\widetilde{u}_{n,n'}$ *of equation* (3.4.2)*. The full computation cost of* $u_{n,n'}$ *is* $O(n^2)$ *flops. So we have constructed a* $(C, C^{m,\nu})$ *quasi-fast solver of equation* (3.1.1)*; this solver is* $(L^p, C^{m,\nu})$ *fast for* $1 \le p < \infty$*. Moreover under the condition* $(A2')$ *this solver is* $(C, C^{m,\nu})$ *fast:*

$$\| u - u_{n,n'} \|_\infty \le cn^{-2m} \| f \|_{C^{m,\nu}} . \tag{3.7.7}$$

Proof. Let the conditions of Theorem 3.7.1 be fulfilled. With $x = \varphi(t)$, $0 \le t \le 1$, we have for u, the solution of equation (3.1.1), and for its approximation $u_{n,n'}$, defined by (3.7.1) and (3.7.4), that

$$u(x) - u_{n,n'}(x) = u(\varphi(t)) - \bar{v}_{n,n'}(\psi_n(x)) =$$

$$= \left[\bar{u}(t) - \bar{u}_{n,n'}(t)\right] + \left[\bar{u}_{n,n'}(t) - \bar{v}_{n,n'}(t)\right] + \left[\bar{v}_{n,n'}(\varphi^{-1}(x)) - \bar{v}_{n,n'}(\psi_n(x))\right].$$

We obtain the claims (3.7.5)–(3.7.7) estimating $\bar{u}-\bar{u}_{n,n'}$ by (3.6.14)–(3.6.16), respectively, and noticing that in accordance to (3.7.1), (3.7.3) and (3.6.31)

$$\max_{0\le x\le 1} | \bar{v}_{n,n'}(\varphi^{-1}(x)) - \bar{v}_{n,n'}(\psi_n(x)) | \le \sup_{0\le t\le 1} | \bar{v}'_{n,n'}(t) | \| \varphi^{-1} - \psi_n \|_\infty$$

$$\le c(\| T_{\star,n,n'}\widetilde{u}_{n,n'} \|_{C^1} + \| \bar{f} \|_{C^1})n^{-2m} \le c'n^{-2m} \| f \|_{C^{m,\nu}},$$

and that in accordance to (3.3.25), (3.6.13), (3.6.31), (3.7.1) and Theorem 2.5.1, with sufficiently large $p < \infty$ and λ close to $1-\nu$,

$$\begin{aligned} \| \bar{u}_{n,n'} - \bar{v}_{n,n'} \|_\infty &\le \| (I - \Pi_{n^2,m})\bar{f} \|_\infty + \| (I - \Pi_{n',m+1})T_{\star,n,n'}\widetilde{u}_{n,n'} \|_\infty \\ &\le c(n^2)^{-m} \| \bar{f}^{(m)} \|_\infty + c(n')^{-(m+\lambda)} \| T_{\star,n,n'}\widetilde{u}_{n,n'} \|_{\mathcal{H}^{m,\lambda}} \\ &\le c'n^{-2m} \| f \|_{C^{m,\nu}} . \end{aligned}$$

□

Introducing a more dense basic interpolation set rather than

$$\{x_{i,n} : i = 0, \dots, n\}$$

exploited above we obtain more accurate approximation of φ^{-1} rather than (3.7.3). For instance, the node set $\left\{x_{i,n} = \varphi(i/n^2) : i = 0, \dots, n^2\right\}$ is still computable in $O(n^2)$ arithmetical operations and leads to the estimate (cf. (3.7.3))

$$\| \varphi^{-1} - \psi_n \|_\infty \leq cn^{-4m} \| \varphi^{-1} \|_{C^{2m,1/r}} .$$

In the following remark we assume quite moderate strengthening of (3.7.3).

Remark 3.7.1. Assume $(A2')$ and let $\| \varphi^{-1} - \psi_n \|_\infty = o(n^{-2m})$. Then (cf. (3.7.7))

$$\| u - u_{n,n'} \|_\infty \leq c \| (I - \Pi_{n^2,m})\bar{f} \|_\infty + o(n^{-2m}) \| f \|_{C^{m,\nu}} .$$

Thus, the main part of the error $u - u_{n,n'}$ is caused by the interpolation of $\bar{f}$.

Chapter 4

Modified Fast and Quasi-Fast Solvers

In this chapter we slightly change the approach used in Chapter 3. Here we will follow the paper [62].

4.1 A modified approach

In Chapter 3 we saw how to construct fast or at least quasi-fast solvers for a class of Fredholm integral equations of the second kind with weakly singular kernels. As a matter of fact, our approached was based on the idea to reduce equation (3.1.1), that is the equation

$$u(x) = \int_0^1 \left[a(x,y)|x-y|^{-\nu} + b(x,y)\right] u(y)dy + f(x),\ x \in [0,1],\ \nu \in (0,1), \tag{4.1.1}$$

or, in short,

$$u = Tu + f,$$

to a periodic integral equation (3.3.11), or, in short, to equation

$$\widetilde{u} = \widetilde{T}\widetilde{u} + \widetilde{f}, \tag{4.1.2}$$

with a smooth 1-periodic solution $\widetilde{u} \in \widetilde{C}^m(\mathbb{R})$, and to approximate $\widetilde{u}$ by an 1-periodic trigonometric polynomial $\widetilde{u}_{n,n'} \in \mathcal{T}_{n'}$, the solution of equation (see (3.4.2))

$$\widetilde{u}_{n,n'} = \widetilde{T}_{n,n'}\widetilde{u}_{n,n'} + P_{n'}\widetilde{f}. \tag{4.1.3}$$

Here $n \in \mathbb{N}$, and the dimension parameter $n' \in \mathbb{N}$ satisfies (3.4.1), that is,

$$n' \geq 2n, \quad n' \sim n^{\tau}, \ 2m/(m+1-\nu) < \tau < 2, \tag{4.1.4}$$

the term $P_{n'}\widetilde{f}$ is an approximation for the free term $\widetilde{f}$ in equation (4.1.2), the operator $\widetilde{T}_{n,n'}$, defined by the formula (see (3.4.3))

$$\widetilde{T}_{n,n'} = Q_{2n}(\widetilde{A}_{2n} + \widetilde{B}_{2n})P_n + Q_{n'}\widetilde{A}_n^{(m)}(I - P_n), \tag{4.1.5}$$

is an approximation for the integral operator $\widetilde{T} = \widetilde{A} + \widetilde{B}$ of equation (4.1.2), and P_n and Q_n are respectively the trigonometric orthogonal and interpolation projection operators introduced in Section 2.6.3.

In the present chapter we shall study a simple modification of this approach introducing in (4.1.3) instead of the orthogonal projection $P_{n'}\widetilde{f}$ the interpolation projection $Q_{n'}\widetilde{f}$. More precisely, now we shall approximate the solution $\widetilde{u}$ of equation (4.1.2) by $\widetilde{u}^{(Q)}_{n,n'}$, the solution of equation

$$\widetilde{u}^{(Q)}_{n,n'} = \widetilde{T}_{n,n'}\widetilde{u}^{(Q)}_{n,n'} + Q_{n'}\widetilde{f}, \tag{4.1.6}$$

where $\widetilde{T}_{n,n'}$ is the same operator (4.1.5) as in equation (4.1.3). It turns out that the modified method (4.1.6) has an advantage compared with the method (4.1.3), since in the present case it will be much easier to compute the values of the free term in the matrix form of (4.1.6).

Let T be an integral operator of the equation (4.1.1), that is, T is defined by the formula

$$(Tu)(x) = \int_0^1 [a(x,y)| \, x - y \, |^{-\nu} + b(x,y)]u(y)dy, \; 0 \leq x \leq 1, \; 0 < \nu < 1. \tag{4.1.7}$$

Let

$$N(I - T) = \{u \in C[0,1] : u = Tu\}$$

be the null-space of the operator $I - T$. Observe that the assumption $(A3)$ from Section 3.1 is equivalent to the condition $N(I - T) = \{0\}$.

4.2 Error estimates for the modified approximations

Theorem 4.2.1. *Assume that* $f \in C^{m,\nu}(0,1)$, $a, b \in C^{2m}([0,1]\times[0,1])$, $m \in \mathbb{N}$, $\nu \in (0,1)$ *and* $N(I-T) = \{0\}$. *Further, assume that the smoothing parameter* r *in the definition of the transformation* φ (see (3.3.6)) *satisfies* (3.3.20), *that is,*

$$r - 1 > 4m/(1-\nu), \tag{4.2.1}$$

and the dimension parameter n' *satisfies* (4.1.4).

Then equations (4.1.1) *and* (4.1.2) (*equation* (3.3.11)) *have unique solutions* $u \in C^{m,\nu}(0,1)$ *and* $\widetilde{u} \in \widetilde{C}^m(\mathbb{R})$, *respectively. They are connected by the relation*

$$\widetilde{u}(t) = \varphi'(t)^{(1-\nu)/2} u(\varphi(t)).$$

For sufficiently large n, *say* $n \geq n_1 \geq 2$, *equation* (4.1.6) *has a unique solution* $\widetilde{u}^{(Q)}_{n,n'} \in \mathcal{T}_{n'}$, *and*

$$\| \widetilde{u} - \widetilde{u}^{(Q)}_{n,n'} \|_\infty \leq c n^{-\tau m} (\log n) \| \widetilde{f} \|_{C^m}, \tag{4.2.2}$$

$$\| \widetilde{u} - \widetilde{u}^{(Q)}_{n,n'} \|_p \leq c_p n^{-\tau m} \| \widetilde{f} \|_{W^{m,p}}, \quad 1/(1-\nu) < p < \infty, \tag{4.2.3}$$

with some positive constants c *and* c_p *which are independent of* n *and* $\widetilde{f}$.

Proof. From the proof of Theorem 3.4.1 we know that equations (4.1.1) and (4.1.2) are uniquely solvable. Further, let $1/(1-\nu) < p \leq \infty$. Then for sufficiently large n, say $n \geq n_1$, we get that an inverse $(I - \widetilde{T}_{n,n'})^{-1} : \widetilde{L}^p \to \widetilde{L}^p$ exists and is uniformly bounded in n:

$$\| (I - \widetilde{T}_{n,n'})^{-1} \|_{\widetilde{L}^p \to \widetilde{L}^p} \leq c_p, \; n \geq n_1, \; 1/(1-\nu) < p \leq \infty. \tag{4.2.4}$$

Thus equation (4.1.6) is uniquely solvable for $n \geq n_1$. We have for $\widetilde{u}$, the solution of (4.1.2), and $\widetilde{u}^{(Q)}_{n,n'}$, the solution of (4.1.6), that

$$\widetilde{u} - \widetilde{u}^{(Q)}_{n,n'} = (I - \widetilde{T}_{n,n'})^{-1} \left(\widetilde{f} - Q_{n'} \widetilde{f} + (\widetilde{T} - \widetilde{T}_{n,n'}) \widetilde{u} \right), \quad n \geq n_1. \tag{4.2.5}$$

This together with (2.6.3.8), (2.6.3.10), (4.2.4) and Lemma 3.4.1 yields (4.2.2) and (4.2.3). □

Following Section 3.4, the matrix form of equation (4.1.6) reads as

$$\underline{u}_{n'}^{(Q)} = \underline{T}_{n'}\underline{u}_{n'}^{(Q)} + \underline{Q}_{n'}\widetilde{f} \tag{4.2.6}$$

where

$$\underline{u}_{n'}^{(Q)} = \begin{pmatrix} \widetilde{u}_{n,n'}^{(Q)}(0) \\ \widetilde{u}_{n,n'}^{(Q)}(1/n') \\ \vdots \\ \widetilde{u}_{n,n'}^{(Q)}((n'-1)/n') \end{pmatrix}$$

is the vector of the sample values $\widetilde{u}_{n,n'}^{(Q)}(j/n'), \quad j = 0, \ldots, n'$,

$$\underline{Q}_{n'}\widetilde{f} = \begin{pmatrix} \widetilde{f}(0) \\ \widetilde{f}(1/n') \\ \vdots \\ \widetilde{f}((n'-1)/n') \end{pmatrix} \tag{4.2.7}$$

is the vector of the sample values $\widetilde{f}(j/n'), \quad j = 0, \ldots, n'-1$ and $\underline{T}_{n'}$ is an $n' \times n'$ matrix given by the formula (3.4.16). For more details see Section 3.4.

Remark 4.2.1. The matrix form of equation (4.1.3) is given by the formula (3.4.15), with the free term $\mathcal{F}_{n'}^{-1}\widehat{\widetilde{f}}_{n'}$, where $\widehat{\widetilde{f}}_{n'}$ is the vector of Fourier coefficients

$$\widehat{\widetilde{f}}(k) = \int_0^1 \widetilde{f}(t)e^{-\mathrm{i}k2\pi t}dt \quad (k \in \mathbb{Z}_{n'})$$

of the function $\widetilde{f}$. However, it is not easy to see that the Fourier coefficients $\widehat{\widetilde{f}}(k)\,(k \in \mathbb{Z}_{n'})$ are computable in $O(n^2)$ arithmetical operations. For their calculation in Section 3.4 a special way is proposed (see the discussion before and after Remark 3.4.1). This is a serious computational work which is not needed for $\underline{Q}_{n'}\widetilde{f}$ in (4.2.6). Since the values of $\underline{Q}_{n'}\widetilde{f}$ are given by the grid values of $\widetilde{f}$ (see (4.2.7)) which are easily available by the sample values of f (see (3.3.13)), the realization of the method (4.1.6) is essential easier than the realization of the method (4.1.3).

Remark 4.2.2. We see that the error estimates (4.2.2) and (4.2.3) in Theorem 4.2.1 hold on the same assumptions as those imposed for the method (4.1.3) in Theorem 3.4.1 of Chapter 3. It turns out that we can obtain similar results as those in Theorems 3.6.1 and 3.7.1 also for the method (4.1.6) if we impose a stronger smoothness condition on f. Actually, below we shall assume that $f \in C^{m+1}[0,1]$ (instead of $f \in C^{m,\nu}(0,1)$).

Let us consider the equality

$$\bar{u} = T_\star \widetilde{u} + \bar{f}, \tag{4.2.8}$$

which is a short writing of the integral relation (3.3.16), with $\widetilde{u}$, the solution of equation (4.1.2), and $T_\star$, given by the formula (3.6.3). Following the discussion before Theorem 3.6.1 in Chapter 3, we approximate $\widetilde{u}$ in (4.2.8) by the solution $\widetilde{u}^{(Q)}_{n,n'}$ of (4.1.6) and the integral operator $T_\star$ by the operator $T_{\star,n,n'}$, given by (3.6.5), obtaining (compare with (3.6.13))

$$\bar{u}^{(Q)}_{n,n'} = T_{\star,n,n'} \widetilde{u}^{(Q)}_{n,n'} + \bar{f}, \tag{4.2.9}$$

which we treat as an approximate solution of equation (3.3.9).

Theorem 4.2.2. *Assume that* $f \in C^{m+1}[0,1]$, $a, b \in C^{2m}([0,1]\times[0,1])$, $m \in \mathbb{N}$ *and* $N(I - T) = \{0\}$, *where* T *be defined by formula* (4.1.7). *Further, assume that the smoothing parameter* r *in the definition of the transformation* φ (see (3.3.6)) *satisfies* (4.2.1) *and the dimension parameter* n' *satisfies* (4.1.4).

Then, for sufficiently large n ($n \geq n_1 \geq 2$, *with* n_1 *from Theorem 4.2.1*),

$$\| \bar{u} - \bar{u}^{(Q)}_{n,n'} \|_\infty \leq c n^{-2m} (\log n) \| \widetilde{f} \|_{C^{m+1}}, \tag{4.2.10}$$

$$\| \bar{u} - \bar{u}^{(Q)}_{n,n'} \|_p \leq c_p n^{-2m} \| \widetilde{f} \|_{C^{m+1}}, \;\; 1 \leq p < \infty, \tag{4.2.11}$$

where $\bar{u}(t) = u(\varphi(t))$ *is the solution of equation* (3.3.9), u *is the solution of equation* (4.1.1), $\bar{u}^{(Q)}_{n,n'}$ *is defined by* (4.2.9) *in which* $T_{\star,n,n'}$ *is determined by the formula* (3.6.5), $\widetilde{u}^{(Q)}_{n,n'}$ *is the solution of equation* (4.1.6) *and* $\bar{f}(t) = f(\varphi(t))$, $0 \leq t \leq 1$.

Proof. Since $\bar{u} = T_\star \widetilde{u} + \bar{f}$ and $\bar{u}^{(Q)}_{n,n'}$ is given by the formula (4.2.9), we have

$$\bar{u} - \bar{u}^{(Q)}_{n,n'} = T_\star \widetilde{u} - T_{\star,n.n'} \widetilde{u}^{(Q)}_{n,n'}. \tag{4.2.12}$$

Adding and subtracting $T_\star \widetilde{u}^{(Q)}_{n,n'}$, $T_{\star,n.n'} \widetilde{u}$ and $T_\star \widetilde{u}$ to the r.h.s of (4.2.12), we get

$$\bar{u} - \bar{u}^{(Q)}_{n,n'} = T_\star(\widetilde{u} - \widetilde{u}^{(Q)}_{n,n'}) + (T_\star - T_{\star,n.n'})(\widetilde{u} - \widetilde{u}^{(Q)}_{n,n'}) + T_\star\widetilde{u} - T_{\star,n.n'}\widetilde{u}.$$

It follows from this that

$$\begin{aligned} \| \bar{u} - \bar{u}^{(Q)}_{n,n'} \|_\infty \leq \| T_\star(\widetilde{u} - \widetilde{u}^{(Q)}_{n,n'}) \|_\infty &+ \| T_\star - T_{\star,n.n'} \|_{\widetilde{L}^p \to C} \| \widetilde{u} - \widetilde{u}^{(Q)}_{n,n'} \|_p \\ &+ \| T_\star\widetilde{u} - T_{\star,n.n'}\widetilde{u} \|_\infty . \end{aligned} \tag{4.2.13}$$

For the third term in the *r.h.s* of (4.2.13) it follows from the proof of Theorem 3.6.1 in Chapter 3 for $\frac{1}{1-\nu} < p < \infty$ that

$$\| T_\star\widetilde{u} - T_{\star,n.n'}\widetilde{u} \|_p \leq c_p n^{-2m} \| \widetilde{f} \|_{W^{m,p}}, \tag{4.2.14}$$

$$\| T_\star\widetilde{u} - T_{\star,n.n'}\widetilde{u} \|_\infty \leq c n^{-2m} (\log n) \| \widetilde{f} \|_{W^{m,p}} . \tag{4.2.15}$$

In a similar way we obtain for $\frac{1}{1-\nu} < p < \infty$ that

$$\| T_\star - T_{\star,n.n'} \|_{\widetilde{L}^p \to C} \| \widetilde{u} - \widetilde{u}^{(Q)}_{n,n'} \|_p = o(n^{-2m}) \| \widetilde{f} \|_{W^{m,p}} . \tag{4.2.16}$$

Below we show that, for $n \geq n_1$,

$$\| T_\star(\widetilde{u} - \widetilde{u}^{(Q)}_{n,n'}) \|_\infty \leq c n^{-2m} \| \widetilde{f} \|_{C^{m+1}}, \quad 1/(1-\nu) < p < \infty. \tag{4.2.17}$$

Now the estimates (4.2.10)–(4.2.11) immediately follow from the estimates (4.2.13)–(4.2.17); actually, we obtain (4.2.11) only for $1/(1-\nu) < p < \infty$; to obtain (4.2.11) for smaller $p \in [1, 1/(1-\nu)]$ we exploit the monotonicity in p of the norm $\| \cdot \|_p$.

It remains to prove (4.2.17). We have for $\widetilde{u} - \widetilde{u}^{(Q)}_{n,n'}$ the following expansion (see (4.2.5)):

$$\widetilde{u} - \widetilde{u}^{(Q)}_{n,n'} = (I - \widetilde{T}_{n,n'})^{-1} \left(\widetilde{f} - Q_{n'}\widetilde{f} + (\widetilde{T} - \widetilde{T}_{n,n'})\widetilde{u} \right), \quad n \geq n_1.$$

Since

$$(I - \widetilde{T}_{n,n'})^{-1} = I + (I - \widetilde{T}_{n,n'})^{-1}\widetilde{T}_{n,n'},$$

we obtain

$$\widetilde{u} - \widetilde{u}^{(Q)}_{n,n'} = \Big[I + (I - \widetilde{T}_{n,n'})^{-1}\widetilde{T}_{n,n'} \Big] \Big[\widetilde{f} - Q_{n'}\widetilde{f} + (\widetilde{T} - \widetilde{T}_{n,n'})\widetilde{u} \Big],$$

and after applying an operator $T_\star$,

$$\begin{aligned} T_\star(\widetilde{u}-\widetilde{u}^{(Q)}_{n,n'}) &= T_\star(\widetilde{f}-Q_{n'}\widetilde{f}) + T_\star(\widetilde{T}-\widetilde{T}_{n,n'})\widetilde{u} \\ &+ T_\star(I-\widetilde{T}_{n,n'})^{-1}\left(\widetilde{T}_{n,n'}(\widetilde{f}-Q_{n'}\widetilde{f}) + \widetilde{T}_{n,n'}(\widetilde{T}-\widetilde{T}_{n,n'})\widetilde{u}\right), \end{aligned}$$

where $n \geq n_1$. Thus, for $n \geq n_1$,

$$\begin{aligned} \| T_\star(\widetilde{u}-\widetilde{u}^{(Q)}_{n,n'}) \|_\infty \leq \| T_\star(\widetilde{f}-Q_{n'}\widetilde{f}) \|_\infty &+ c_1 \| \widetilde{T}_{n,n'}(\widetilde{f}-Q_{n'}\widetilde{f}) \|_\infty \\ &+ c_2 \| (\widetilde{T}-\widetilde{T}_{n,n'})\widetilde{u} \|_p \end{aligned} \quad (4.2.18)$$

where we took into account the estimate (4.2.4), the boundedness of $T_\star : \widetilde{L}^p \to C$ and the uniform boundedness of operators $\widetilde{T}_{n,n'} : \widetilde{L}^p \to \widetilde{C}$, $n \in \mathbb{N}$ (see (3.4.6) and (3.4.7)).

According to (4.1.4), $\tau(m+1-\nu) > 2m$. Therefore we have $\tau(m+1) > 2m$. Since $T_\star : \widetilde{C} \to C$, is bounded, we obtain with the help of (2.6.3.8) for $n \geq n_1$ that

$$\begin{aligned} \| T_\star(\widetilde{f}-Q_{n'}\widetilde{f}) \|_\infty &\leq \| T_\star \|_{\widetilde{C}\to C} \| \widetilde{f}-Q_{n'}\widetilde{f} \|_\infty \\ &\leq cn^{-\tau(m+1)} (\log n^\tau) \| \widetilde{f}^{(m+1)} \|_\infty \\ &\leq c_1 n^{-2m}\, n^{2m-\tau(m+1)} (\log n) \| \widetilde{f}^{(m+1)} \|_\infty . \end{aligned}$$

Thus, we have

$$\| T_\star(\widetilde{f}-Q_{n'}\widetilde{f}) \|_\infty = o(n^{-2m}) \| \widetilde{f}^{(m+1)} \|_\infty . \quad (4.2.19)$$

In a similar way we obtain that

$$\| \widetilde{T}_{n,n'}(\widetilde{f}-Q_{n'}\widetilde{f}) \|_\infty = o(n^{-2m}) \| \widetilde{f}^{(m+1)} \|_\infty . \quad (4.2.20)$$

Indeed, due to the uniform boundedness of operators $\widetilde{T}_{n,n'} : \widetilde{L}^p \to \widetilde{C}$ in $n \in \mathbb{N}$ (see (3.4.7)) and (2.6.3.10) we have

$$\begin{aligned} \| \widetilde{T}_{n,n'}(\widetilde{f}-Q_{n'}\widetilde{f}) \|_\infty &\leq \| \widetilde{T}_{n,n'} \|_{\widetilde{L}^p\to C} \| \widetilde{f}-Q_{n'}\widetilde{f}) \|_p \\ &\leq cn^{-\tau(m+1)} \log n^\tau \| \widetilde{f}^{(m+1)} \|_\infty . \end{aligned}$$

This together with $\tau(m+1) > 2m$ yields (4.2.20).

Finally, it follows from the proof of Theorem 3.6.1 (see (3.6.29)) that, for $n \geq n_1$,

$$\| \widetilde{T}\widetilde{u} - \widetilde{T}_{n.n'}\widetilde{u} \|_p \leq c_p n^{-2m} \| \widetilde{f} \|_{W^{m,p}}, \ 1/(1-\nu) < p < \infty. \qquad (4.2.21)$$

Plugging (4.2.19)–(4.2.21) into (4.2.18) we obtain (4.2.17). □

Remark 4.2.3. Although our overall proof scheme in the proof of Theorem 4.2.2 is very similar to the one of Theorem 3.6.1 in Chapter 3, the replacment $P_{n'}\widetilde{f}$ by $Q_{n'}\widetilde{f}$ in (4.1.3) causes essential differences in the proof argument, and thus the argument that has been used in the proof of Theorem 3.6.1 does not apply. In particular, a part of the proof of Theorem 3.6.1 is based on the estimates (3.4.4) and (3.6.6) for the norms of the operators $\widetilde{T}(I - P_{n'})$ and $T_\star(I - P_{n'})$, respectively. However, similar estimates do not hold for the operators $\widetilde{T}(I - Q_{n'})$ and $T_\star(I - Q_{n'})$ since $I - Q_{n'}$ is not an orthogonal projection operator in $L^2(0, 1)$.

4.3 Modified fast and quasi-fast solvers

According to the discussion in Section 3.4, the sample values

$$\widetilde{u}^{(Q)}_{n,n'}(0), \widetilde{u}^{(Q)}_{n,n'}\Big(\frac{1}{n'}\Big), \dots, \widetilde{u}^{(Q)}_{n,n'}\Big(\frac{n'-1}{n'}\Big)$$

of the solution $\widetilde{u}^{(Q)}_{n,n'} \in \mathcal{T}_{n'}$ of equation (4.1.6) are available at the cost of $O(n^2)$ arithmetical operations. After that also the values

$$\Big(T_{\star,n,n'}\widetilde{u}^{(Q)}_{n,n'}\Big)(0), \Big(T_{\star,n,n'}\widetilde{u}^{(Q)}_{n,n'}\Big)\Big(\frac{1}{n'}\Big), \dots, \Big(T_{\star,n,n'}\widetilde{u}^{(Q)}_{n,n'}\Big)\Big(\frac{n'-1}{n'}\Big) \qquad (4.3.1)$$

of $T_{\star,n,n'}\widetilde{u}^{(Q)}_{n,n'}$ can be computed in $O(n^2)$ arithmetical operations, and we can use the local (algebraic) polynomial interpolation to approximate the $T_{\star,n,n'}\widetilde{u}^{(Q)}_{n,n'}$ between the grid points $\frac{i}{n'}$, $i = 0, \dots, n'$.

Following ideas and notations used in Section 3.7, we define the polynomially interpolated version $\bar{v}^{(Q)}_{n,n'} = \bar{v}^{(Q)}_{n,n'}(t)$ with $t \in [0, 1]$ of $\bar{u}^{(Q)}_{n,n'}$ determined by (4.2.9) as

$$\bar{v}^{(Q)}_{n,n'} = \Pi_{n',m+1}T_{\star,n,n'}\widetilde{u}^{(Q)}_{n,n'} + \Pi_{n^2,m}\bar{f} \qquad (4.3.2)$$

where $\Pi_{n',m+1}$ is an operator of local interpolation of functions given on the grid

$$\Big\{\frac{i}{n'} : \ i = 0, \dots, n'\Big\}$$

by polynomials of degree m (see Section 2.5) and $\Pi_{n^2,m}$ is an operator of local interpolation by polynomials of degree $m-1$ from the grid values on the grid

$$\left\{\frac{i}{n^2} : i = 0, \ldots, n^2\right\}.$$

Thus, the unknown parameters (4.3.1) of $\bar{v}^{(Q)}_{n,n'}$ in (4.3.2) are available at the cost of $O(n^2)$ arithmetical operations, and having them in hand, the value of $\bar{v}^{(Q)}_{n,n'}(t)$ at any point $t \in [0,1]$ is available at the cost of $O(1)$ arithmetical operations.

We know from our earlier examination of φ that φ has a continuous inverse φ^{-1}, see Section 2.4. However, we cannot present a closed form for it, and a definition for an approximate solution of equation (4.1.1) in the form

$$\bar{v}^{(Q)}_{n,n'}(\varphi^{-1}(x)), \;\; 0 \leq x \leq 1, \tag{4.3.3}$$

with (4.3.2) and φ^{-1}, the inverse of φ, is not acceptable. Therefore, following the same approach as in Section 3.7, we first approximate $\varphi^{-1}(x)$ by a suitable interpolation.

Let us precompute $x_{i,n} = \varphi(\frac{i}{n})$ for $i = 0, \ldots, n$ (this costs altogether only $O(n)$ arithmetical operations) and use them to approximate $t = \varphi^{-1}(x)$ for any given $x \in [0,1]$ via an interpolation by polynomials of degree $2m-1$. Note that $\varphi^{-1} \in C^{2m,1/r}(0,1)$ where r is the (smoothing) parameter in the definition of φ and $\{x_{i,n} : i = 0, \ldots, n\}$ is a suitable graded grid for the interpolation of such functions. Let ψ_n be the corresponding interpolation approximation of φ^{-1}. As pointed out in Section 3.7, then

$$|\varphi^{-1}(x) - \psi_n(x)| \leq c n^{-2m} \parallel \varphi^{-1} \parallel_{C^{2m,1/r}}, \;\; 0 \leq x \leq 1. \tag{4.3.4}$$

Thus, instead of (4.3.3), we can define the final approximation to the solution of equation (4.1.1) via the formula

$$u^{(Q)}_{n,n'}(x) = \bar{v}^{(Q)}_{n,n'}(\psi_n(x)), \;\; 0 \leq x \leq 1. \tag{4.3.5}$$

The grid values (4.3.1) and

$$\bar{f}\Big(\frac{i}{n^2}\Big) = f\Big(\varphi\Big(\frac{i}{n^2}\Big)\Big) \; (i = 0, \ldots, n^2)$$

can be considered as the parameters of the approximate solution (4.3.5).

The grid values of f belong to a given information whereas the grid values of $T_{\star,n,n'}\widetilde{u}^{(Q)}_{n,n'}$ are available at the cost of $O(n^2)$ arithmetical operations using $n_\star = O(n^2)$ sample values of a and b and the n^2+1 sample values of f just listed.

The following theorem below shows that with the definition (4.3.5) we have designed a (C, C^{m+1}) quasi-fast solver for equation (4.1.1) which is (L^p, C^{m+1}) fast for $1 \le p < \infty$.

Theorem 4.3.1. *Assume that* $f \in C^{m+1}[0,1]$, $a, b \in C^{2m}([0,1]\times[0,1])$, $m \in \mathbb{N}$ *and* $N(I-T) = \{0\}$, *where* T *be defined by formula* (4.1.7). *Further, assume that the smoothing parameter* r *in the definition of the transformation* φ (see (3.3.6)) *satisfies* (4.2.1) *and the dimension parameter* n' *satisfies* (4.1.4).

Then we have for all sufficiently large $n \in \mathbb{N}$ *that*

$$\| u - u^{(Q)}_{n,n'} \|_\infty \le c n^{-2m} (\log n) \| f \|_{C^{m+1}}, \tag{4.3.6}$$

$$\| u - u^{(Q)}_{n,n'} \|_p \le c_p n^{-2m} \| f \|_{C^{m+1}}, \ \ 1 \le p < \infty, \tag{4.3.7}$$

where u *is the solution of* (4.1.1) *and* $u^{(Q)}_{n,n'}$ *is defined by* (4.3.5) *with the help of* (4.3.2), *where* $T_{\star,n,n'}$ *is determined by the formula* (3.6.5), $\widetilde{u}^{(Q)}_{n,n'}$ *is the solution of equation* (4.1.6) *and* $\bar{f}(t) = f(\varphi(t))$, $0 \le t \le 1$. *The full computation cost of* $u^{(Q)}_{n,n'}$ *is* $O(n^2)$ *flops. So we obtain a* (C, C^{m+1}) *quasi-fast solver for equation* (4.1.1) *which is* (L^p, C^{m+1}) *fast,* $1 \le p < \infty$.

Proof. We have for u, the solution of equation (4.1.1), and for its approximation $u^{(Q)}_{n,n'}$, defined by (4.3.5) and (4.2.9), that

$$u(x) - u^{(Q)}_{n,n'}(x) = u(\varphi(t)) - \bar{v}^{(Q)}_{n,n'}(\psi_n(x))$$

$$= \bar{u}(t) - \bar{u}^{(Q)}_{n,n'}(t) + \bar{u}^{(Q)}_{n,n'}(t) - \bar{v}^{(Q)}_{n,n'}(t) + \bar{v}^{(Q)}_{n,n'}(\varphi^{-1}(x)) - \bar{v}^{(Q)}_{n,n'}(\psi_n(x)),$$

with

$$x = \varphi(t), \quad t \in [0,1]; \quad t = \varphi^{-1}(x), \quad x \in [0,1].$$

It follows from Theorem 4.2.2

$$\| \bar{u} - \bar{u}^{(Q)}_{n,n'} \|_\infty \leq cn^{-2m}(\log n) \| \widetilde{f} \|_{C^{m+1}} \tag{4.3.8}$$

and

$$\| \bar{u} - \bar{u}^{(Q)}_{n,n'} \|_p \leq c_p n^{-2m} \| \widetilde{f} \|_{C^{m+1}}, \ \ 1 \leq p < \infty. \tag{4.3.9}$$

In accordance to (4.3.2), (4.3.4) and (3.6.31) we obtain

$$\begin{aligned} \max_{0\leq x\leq 1} | \bar{v}^{(Q)}_{n,n'}(\varphi^{-1}(x)) & - \bar{v}^{(Q)}_{n,n'}(\psi_n(x)) | \leq \sup_{0\leq t\leq 1} | \bar{v}'_{n,n'}(t) | \| \varphi^{-1} - \psi_n \|_\infty \\ & \leq c(\| T_{\star,n,n'} \widetilde{u}^{(Q)}_{n,n'} \|_{C^1} + \| \bar{f} \|_{C^1}) n^{-2m} \\ & \leq c' n^{-2m} \| f \|_{C^{m+1}} . \end{aligned} \tag{4.3.10}$$

In accordance to (4.2.9), (4.3.2), (3.6.31) and Theorem 2.5.1, with sufficiently large $p < \infty$ and λ close to $1 - \nu$,

$$\begin{aligned} \| \bar{u}^{(Q)}_{n,n'} & - \bar{v}^{(Q)}_{n,n'} \|_\infty \leq \| (I - \Pi_{n^2,m}) \bar{f} \|_\infty + \| (I - \Pi_{n',m+1}) T_{\star,n,n'} \widetilde{u}^{(Q)}_{n,n'} \|_\infty \\ & \leq c(n^2)^{-m} \| \bar{f}^{(m)} \|_\infty + c(n')^{-(m+\lambda)} \| T_{\star,n,n'} \widetilde{u}^{(Q)}_{n,n'} \|_{\mathcal{H}^{m,\lambda}} \\ & \leq c' n^{-2m} \| f \|_{C^{m+1}} . \end{aligned} \tag{4.3.11}$$

The estimates (4.3.6)–(4.3.7) follows from (4.3.8)–(4.3.11). □

Thus, we have constructed a (C, C^{m+1}) quasi-fast solver for equation (4.1.1) which is (L^p, C^{m+1}) fast, $1 \leq p < \infty$. This new solver has an advantage compared with the method considered in Chapter 3, since the grid values of the free term in the matrix form (4.2.6) of the present method are easily available from the values of f, which is not the case in the matrix form (3.4.15) for method (4.1.3).

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Summary in Estonian

Kiired ja kvaasikiired lahendusmeetodid nõrgalt singulaarsete Fredholmi teist liiki integraalvõrrandite jaoks

Paljud matemaatika, füüsika, mehaanika, bioloogia ja teiste teadusalade probleemid on formuleeritavad integraalvõrrandite kujul. Käesolevas doktoritöös vaadeldakse lineaarset Fredholmi teist liiki integraalvõrrandit

$$u(x) = \int_0^1 K(x,y)u(y)dy + f(x), \quad 0 \leq x \leq 1, \tag{1}$$

kus tuum K ja vabaliige f on antud funktsioonid ning otsitavaks on funktsioon u. Nimetus "Fredholmi teist liiki võrrand" tuleneb asjaolust, et otsitav funktsioon u asub nii integraali märgi all kui ka väljaspool integraali märki võrrandi (1) vasakul poolel ning lõplike rajadaga teist liiki li neaarsete integraalvõrrandite süstemaatilisele käsitlusele pani aluse Rootsi matemaatik Ivar Igor Fredholm (1866 - 1927) oma töödega eelmise sajandi algusaastatel.

Sileda tuumaga võrrandi (1) korral tagab tuuma K ja vabaliikme f siledus lahendi u sileduse kogu integreerimislõigul $[0,1]$: kui K ja f on m korda ($m \geq 0$) pidevalt diferentseeruvad funktsioonid vastavalt ruudul $[0,1] \times [0,1]$ ja lõigul $[0,1]$, ning kui eeldada, et võrrandil (1) on olemas lahend u, siis u on m korda pidevalt diferentseeruv lõigul $[0,1]$. Kahjuks ei ole see üldiselt enam nii praktikas sageli esinevate võrrandite puhul, mille tuum $K(x,y)$ omab kuju

$$K(x,y) = a(x,y)|\, x - y\,|^{-\nu} + b(x,y), \quad 0 < \nu < 1, \tag{2}$$

kus a ja b on mingid pidevad funktsioonid ruudul $[0,1] \times [0,1]$. Sel korral $K(x,y)$ võib tõkestamatult kasvada, kui $y \to x$; eelduse $\nu < 1$ tõttu on siiski tagatud, et $K(x,y)$ singulaarsus $y = x$ korral on integreeruv:

$$\sup_{0 \le x \le 1} \int_0^1 \mid K(x,y) \mid dy < \infty.$$

Tuuma $K(x,y)$ iseärasus $y = x$ korral toob reeglina kaasa võrrandi (1) lahendi u iseärase käitumise integreerimislõigu $[0,1]$ rajapunktide lähedal: tuuma (2) puhul võivad võrrandi (1) lahendi tuletised puntide 0 ja 1 lähedal tõkestamatult kasvada isegi siis, kui võrrandi vabaliige f on lõpmata arv kordi pidevalt diferentseeruv lõigul $[0,1]$. Seetõttu vabaliikme f kohta on käesolevas doktoritöös seatud tingimused, mis on täidetud kõigi lõigul $[0,1]$ m korda pidevalt diferentseeruvate funktsioonide korral ning võimaldavad vaadelda ka selliseid funktsioone, mille tuletised mingist järgust alates võivad olla tõkestamata lõigu $[0,1]$ rajapunktide 0 ja 1 lähedal.

Täpsemalt, töös eeldatakse, et on täidetud järgmised tingimused $(A1)-(A3)$:

$(A1)$ $f \in C^{m,\nu}(0,1), m \in \mathbb{N} = \{1,2,\dots\}, \nu \in (0,1)$, see tähendab, et $f(x)$ on pidev, kui $x \in [0,1]$ ja m korda pidevalt diferentseeruv, kui $x \in (0,1)$ ning selline, et

$$\sum_{k=1}^{m} \sup_{0<x<1} [x(1-x)]^{k-1+\nu} \mid f^{(k)}(x) \mid < \infty;$$

$(A2)$ tuum K avaldub kujul (2) ning funktsioonid a ja b avaldises (2) on $2m$ korda pidevalt diferentseeruvad ruudul $[0,1] \times [0,1]$;

$(A3)$ võrrandile (1) vastaval homogeensel integraalvõrrandil

$$u(x) = \int_0^1 K(x,y)u(y)dy \quad (0 \le x \le 1)$$

on lõigul $[0,1]$ pidevate funksioonide klassis olemas vaid triviaalne lahend $u = 0$.

Doktoritöös on tingimustega $(A1)-(A2)$ määratud ülesannete klassi korral välja töötatud kiired ja kvaasikiired lahendusmeetodid võrrandi (1) ligikaudeks lahendamiseks.

Siin “kiire meetod” (inglise keeles “fast solver”) tähendab meetodit, mis antud ülesannete klassi korral annab lähislahenditele optimaalset järku täpsuse ning mille realisatsioon on “odavaim” võrreldes teiste meetoditega (kiire meetodi puhul on meetodi rakendamiseks vajaminevate aritmeetiliste tehete arv minimaalne võrreldes kõigi teiste sama informatsiooni ja võrgu punktide arvu kasutatavate meetoditega).

Kiirete ja kvaasikiirete meetodite täpsed definitsioonid on esitatud doktoritöö kolmanda peatüki alapunktis 3.2. Seejärel on kolmandas peatükis välja töötatud metoodika, mis annab võimaluse kiirete ja kvaasikiirete meetodite konstrueerimiseks võrrandi (1) jaoks eeldustel $(A1) - (A2)$. Sama metoodikat on kasutatud ka peatükis 4 modifitseeritud kiirete ja kvaasikiirete meetodite konstrueerimisel.

Doktoritöös saadud põhitulemusi kajastavad teoreemid 3.4.1, 3.6.1, 3.7.1, 4.2.1, 4.2.2 ja 4.3.1. Töös saadud tulemused tuginevad artiklitele [61], [62].

Acknowledgments

First of all, I am most grateful to Allah, the almighty, for providing me with the chance, energy, and capability to complete my PhD under the best supervision.

Secondly, I would like to express great gratitude and appreciation to my supervisor, Prof. Arvet Pedas, for his patient guidance, kindness, encouragement, and advice he has provided throughout my 4-year studies. Without his valuable suggestions and kind support, this thesis would have taken so much longer to complete.

I have been lucky to have Prof. Gennadi Vainikko as my second adviser, whose insight and knowledge into the subject matter steered me through this research. His valuable remarks, suggestions throughout my doctoral studies and proofreading of my thesis made a great difference.

I would like to express my thankfulness to all faculty members of the Institute of Mathematics and Statistics, especially Peeter Oja for being a great mentor with his useful discussions that have lightened the core definitions of functional analysis. I appreciate Indrek Zolk and Mikk Vikerpuur for their assistance and suggestions.

I am grateful to secretaries and all other members of management committees for their encouragement and also for their support by providing facilities in power excellence with the university to complete this thesis.

Eventually, I am very grateful to all of my friends and relatives, especially my husband, for their moral support throughout the years.

Finally, I gratefully acknowledge the funding received by the Estonian Research Council grant PRG864 and institutional research funding IUT20-57 of the Estonian Ministry of Education and Research. I also give my gratitude to the Estonian Doctoral School of Mathematics and Statistics who financially support my conferences and summer schools.

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Publikatsioonid

M. Shahzad, S. Rehman, et.al. Measuring complex behavior of SO2 oxidation reaction. Computational Ecology and Software, 5(3): 254-270, 2015.

S. Rehman, A. Pedas and G. Vainikko. Fast solvers of weakly singular integral equations of the second kind. Math. Model. Anal., 23(4):639–664, 2018.

S. Rehman, A. Pedas and G. Vainikko. A quasi-fast method for weakly singular integral equations of the second kind. Numerical Functional Analysis and Optimization, 41(7):850–870, 2019.

F. Sultan, et.al, S. Rehman. A Numerical treatment on Rheology of mixed convective Carreau nanofluid with variable viscosity and thermal conductivity . Applied Nano science, 2020. https://doi 10.1007/s13204-020-01294-1

Teaduslikud huvialad

Peamised huvialad on:

kõrgemat järku numbrilised meetodid nõgalt singulaarsete integraalvõrandite lahendaniseks ja nende optimaalsus;

keemilise kineetika matemaatiline modelleerimine ja arvuline analüüs.

List of Publications

This thesis is based on the following publications:

1) S. Rehman, A. Pedas and G. Vainikko. Fast solvers of weakly singular integral equations of the second kind. Math. Model. Anal., 23(4):639–664, 2018.

2) S. Rehman, A. Pedas and G. Vainikko. A quasi-fast method for weakly singular integral equations of the second kind. Numerical Functional Analysis and Optimization, 41(7):850–870, 2019.

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