DISSERTATIONES
MATHEMATICAE
UNIVERSITATIS
TARTUENSIS
133

KATRIIN PIRK

Diametral diameter two properties, Daugavet-, and Δ-points in Banach spaces

DISSERTATIONES MATHEMATICAE UNIVERSITATIS TARTUENSIS
133

DISSERTATIONES MATHEMATICAE UNIVERSITATIS TARTUENSIS
133

KATRIIN PIRK

Diametral diameter two properties, Daugavet-, and Δ-points in Banach spaces

UNIVERSITY OF TARTU
Press

Institute of Mathematics and Statistics, Faculty of Science and Technology, University of Tartu, Estonia.

Dissertation has been accepted for the commencement of the degree of Doctor of Philosophy (PhD) in mathematics on 19 June 2020, by the Council of the Institute of Mathematics and Statistics, Faculty of Science and Technology, University of Tartu.

Supervisors:

Associate Professor Trond Arnold Abrahamsen, PhD
University of Agder
Kristiansand, Norway

Senior Research Fellow Rainis Haller, PhD
University of Tartu
Tartu, Estonia

Research Fellow Johann Langemets, PhD
University of Tartu
Tartu, Estonia

Opponents:

Professor Anna Helena Kamińska, Dr habil
University of Memphis
Memphis, USA

Full Professor Miguel Martín Suárez, PhD
University of Granada
Granada, Spain

Commencement will take place on 28 August 2020 at 17.00 in Narva mnt 18–1020.

Publication of this dissertation has been granted by the Estonian Research Council grants PRG877 and PSG487.

ISSN 1024-4212
ISBN 978-9949-03-406-2 (print)
ISBN 978-9949-03-407-9 (pdf)

University of Tartu Press
http://www.tyk.ee

Contents

Acknowledgements

First of all, I would like to express sincere gratitude to my supervisors Trond Arnold Abrahamsen, Rainis Haller, and Johann Langemets for their continuous and kind support throughout these years, and their patience with me.

I am thankful to my co-authors Vegard Lima and Triinu Veeorg who inspired me to work harder and enlightened me with their different approaches.

I would also like to express my heartfelt gratitude to all lecturers whose work has enriched me both mathematically and otherwise. I am the most indebted to late Professor Eve Oja, whose deep devotion to mathematics and incredible warmth towards students, along with her endless energy and amazing lectures was the main reason I ended up on the path of mathematics and more precisely functional analysis.

Last but not least, I would like to thank the friends among amazing mathematicians I have had pleasure to meet during all these years. Quite many of them have my thanks for encouraging me to look forward and believe in myself.

The research for the thesis was partially supported by the Estonian Research Council grants PRG877 and PSG487, the institutional research funding IUT20-57 of the Estonian Ministry of Education and Research, Estonian Doctoral School in Mathematics and Statistics, and the Tartu University Foundation's CWT Estonia travel scholarship.

Chapter 1

Introduction

1.1 Background

In [Dau63], I. K. Daugavet discovered that all compact operators T on $C[0,1]$ satisfy the following norm identity

$$\| \operatorname{Id} + T\| = 1 + \|T\|,$$

now known as the *Daugavet equation.* Not long after, other examples of Banach spaces on which all compact operators satisfy the Daugavet equation followed, e.g. $L_1[0,1]$ (see [Loz66]). A Banach space on which every compact operator satisfies the Daugavet equation is said to have the *Daugavet property.*

It is known that a Banach space with the Daugavet property satisfies that all convex combinations of slices of the unit ball have diameter 2 (see [Shv00]). A Banach space with such a property is said to have the *strong diameter two property (SD2P).* From Bourgain's Lemma (see Lemma 1.4.10 below) a Banach space with the SD2P satisfies that all relatively weakly open subsets of the unit ball, in particular all slices of the unit ball, have diameter two. Banach spaces with such properties are respectively said to have the *diameter two property (D2P)* and the *local diameter two property (LD2P).*

One of the first papers that studies the D2P is [NW01]. A few years later a more systematic study of all the diameter two properties mentioned above, started to develop (see [ABL15], [ALN13], [BLR15], etc.). From the discussion above we have

$$\text{Daugavet property} \Rightarrow \text{SD2P} \Rightarrow \text{D2P} \Rightarrow \text{LD2P},$$

however, none of the reverse implications hold. That LD2P $\not\Rightarrow$ D2P was proved in [BLR15] and that D2P $\not\Rightarrow$ SD2P was proved in [ABL15] and [HL14] independently. For SD2P $\not\Rightarrow$ Daugavet property one can take, e.g. c_0 or ℓ_∞ (see [ALN13]).

In [IK04], Y. Ivakhno and V. Kadets introduced the notion of a *space with bad projections (SBP)*. This notion is a natural weakening of the Daugavet property in that compact operators are replaced by rank-1 projections in the Daugavet equation. They obtained the following geometrical characterisation of an SBP space.

Theorem 1.1.1 (see [IK04, Theorem 1.4]). *A Banach space X is SBP if and only if for every slice S of B_X, every unit sphere element $x \in S$, and every $\varepsilon > 0$, there exists $y \in S$ such that $\|x - y\| \geq 2 - \varepsilon$.*

It is clear that an SBP space has the LD2P in a strong sense since for any slice S of the unit ball and any unit sphere element x in S there exists a point y in S almost diametral to x. Inspired by this the authors of [BLR18] called the equivalent formulation of an SBP space appearing in Theorem 1.1.1 the *diametral local diameter two property (DLD2P)*. In the same paper also the diametral analogous of the D2P and the SD2P, were introduced and systematically studied: a Banach space X is said to have the *diametral diameter two property (DD2P)* if for every non-empty relatively weakly open subset U of B_X, every unit sphere element $x \in U$, and every $\varepsilon > 0$ there exists $y \in U$ such that $\|x - y\| \geq 2 - \varepsilon$, and it is said to have the *diametral strong diameter two property (DSD2P)* if for every $n \in \mathbb{N}$, non-empty relatively weakly open subsets $U_1, \ldots, U_n$ of B_X, $\lambda_1, \ldots, \lambda_n \in [0, 1]$ such that $\sum_{i=1}^n \lambda_i = 1$, every $x \in \sum_{i=1}^n \lambda_i U_i$, and every $\varepsilon > 0$ there exists $y \in \sum_{i=1}^n \lambda_i U_i$ satisfying $\|x - y\| \geq 1 + \|x\| - \varepsilon$.

In [BLR18], among others the following implications were proved to hold

$$\text{Daugavet property} \Rightarrow \text{DSD2P} \Rightarrow \text{DD2P} \Rightarrow \text{DLD2P}.$$

An example of a Banach space with the DD2P and failing the DSD2P was also given (see [BLR18, Example 2.2]), but the questions whether the converse of the other two implications in the above diagram hold, were left open (see [BLR18, Question 4.1]). In fact, these questions are still open.

There are many results regarding the diametral diameter two properties. In [IK04, Theorem 3.2], it was established that the DLD2P is stable under unconditional sums of Banach spaces. This investigation was continued in [BLR18] with the properties DD2P and DSD2P. It was shown that the DD2P is stable under all ℓ_p-sums for $1 \leq p \leq \infty$, (see [BLR18, Theorem 2.12 and Proposition 2.13]). This result was extended to absolute sums in [Pir16]. For the DSD2P it was proved in [BLR18, Proposition 3.6 and Theorem 3.7] that the property is stable under ℓ_∞-sums and, in [BLR18, Proposition 3.6] for the case of an ℓ_1-sum, that there is stability in the direction from the sum to the summands. The other direction, from the summands to the ℓ_1-sum, was proved in [HPP16]. It is known that no other ℓ_p-sum, and even no other absolute sum, can provide positive stability results (see [HLN18, Corollary 3.8]). This means that in the setting of sums of Banach spaces, the DSD2P behaves the same way as the Daugavet property, which is also stable only under ℓ_1-sums and ℓ_∞-sums (see [BKSW05, Theorem 5.1 and Corollary 5.4]). Despite these and other known results, it is, as mentioned above, still unknown whether the Daugavet property and the DSD2P differ from each other.

Given a unit sphere element x in a Banach space X and an $\varepsilon > 0$ we define

$$\Delta_\varepsilon(x) = \{y \in B_X : \|x - y\| \geq 2 - \varepsilon\}.$$

In [Wer01], the following geometrical characterisations of the Daugavet property appeared.

Proposition 1.1.2. *Let X be a Banach space. The following assertions are equivalent:*

(i) X has the Daugavet property;

(ii) for every slice S of B_X, every unit sphere element x and every $\varepsilon > 0$, there exists $y \in S$ such that $\|x - y\| \geq 2 - \varepsilon$;

(iii) for every unit sphere element x we have $B_X = \overline{\operatorname{conv}}\, \Delta_\varepsilon(x)$ for every $\varepsilon > 0$.

From [IK04, Theorem 1.4] and [Wer01, Problem (7)] we have a similar characterisation of the DLD2P.

Proposition 1.1.3. *Let X be a Banach space. The following assertions are equivalent:*

(i) X has the DLD2P;

(ii) for every unit sphere element x we have $x \in \overline{\operatorname{conv}}\, \Delta_\varepsilon(x)$ for every $\varepsilon > 0$.

From the Hahn–Banach Separation Theorem it is straightforward that pointwise versions of the equivalence *(ii)* $\Leftrightarrow$ *(iii)* in Proposition 1.1.2 and the equivalence *(i)* $\Leftrightarrow$ *(ii)* in Proposition 1.1.3, hold. A point satisfying the statement in Proposition 1.1.2 part *(iii)* (respectively, satisfying the statement in Proposition 1.1.3 part *(ii)*) is naturally called a *Daugavet-point* (respectively, *Δ-point*). In a Banach space with the Daugavet property (respectively, DLD2P) every unit sphere element is a Daugavet-point (respectively,

Δ-point). If the set Δ of all Δ-points on sphere of Banach space X is such that B_X can be realised as the closed convex hull of Δ (which trivially is the case if X has the DLD2P), then it follows again from Hahn–Banach Separation Theorem that X has the LD2P and actually more. Indeed, one can observe that c has this property, but that c_0 fails it (see Corollary 2.4.4 and Example 2.4.5). Since c does not have the DLD2P (see Example 2.1.4), this gives rise to a new diameter two property, naturally named the *convex diametral local diameter two property (convex DLD2P)* strictly between the DLD2P and the LD2P. This observation was the starting point for [AHLP20] and [HPV] on which this thesis is partly based.

1.2 Summary of the thesis

The main aim of this thesis is to investigate the diametral diameter two properties in Banach spaces and the related Banach space notions Daugavet-point and Δ-point. Stability results for the diametral diameter two properties by taking absolute sums will be presented. These results resemble those for the diameter two properties. Stability results for Daugavet- and Δ-points are also obtained. A consequence of these results is that the notions Daugavet-point and Δ-point in general are different. It turns out, however, that in some well-known classes of spaces these notions are equivalent. We will show that this is the case for $C(K)$ spaces for infinite compact Hausdorff space K, $L_1(\mu)$ spaces and their preduals, and for some specific Müntz spaces. Moreover, simple and easy to check characterisations of Daugavet-points are obtained for these classes of spaces. We will also show that $C(K)$ spaces, K infinite compact Hausdorff space, as well as Müntz spaces, have a diameter two property called the convex diametral local diameter two property.

The thesis consists of three chapters and one appendix. The content of the thesis is organised as follows.

In Chapter 1, we start by briefly introducing the historical background of the topics addressed in this thesis. Subsequently the summary of the thesis is presented and then the notation used throughout the thesis is clarified. The chapter ends with a section containing basic definitions and results about the Daugavet property and the diameter two properties that are needed throughout the thesis.

Chapter 2 is based on [HPP16] and [AHLP20]. It focuses on the diametral diameter two properties. We start with an overview of the three diametral diameter two properties and their relations to other well-known Banach space properties. A section is dedicated to the stability results of these properties under absolute sums. It is proved that the DLD2P is stable under all absolute sums and that the DSD2P is stable only under ℓ_1-sums and ℓ_∞-sums. We also present some results about subspaces with these properties and a few, yet unpublished, results showing that the DLD2P and the DD2P are inherited by M-ideals. Finally, in Chapter 2, a new diametral diameter two property, the convex diametral local diameter two property, is introduced. We show that this new property falls strictly between the DLD2P and the LD2P. We show that $C(K)$ spaces for infinite compact Hausdorff space K, have the convex diametral local diameter two property. Also it is shown that the convex diametral local diameter two property is stable under absolute sums for all absolute normalised norms. The chapter ends with some open questions.

Chapter 3 is based on [AHLP20] and [HPV]. We start with the definitions of Daugavet- and Δ-points and some useful general characterisations of these notions. We then obtain characterisations of these notions in $L_1(\mu)$ spaces, $C(K)$ spaces, for infinite compact Hausdorff space K, and a wide class of Müntz spaces. It is proved that in all the aforementioned classes of Banach spaces, as well as in the preduals of $L_1(\mu)$, the notions Daugavet-point and Δ-point coincide. The last two sections of this chapter are dedicated to stability results of Daugavet- and Δ-points under absolute sums. It is shown that absolute sums can be divided into two classes: absolute sums equipped with so-called A-octahedral norms and absolute sums equipped with norms with property (α); the former absolute sums provide positive stability results whereas the latter absolute sums can have no Daugavet-points. The behaviour of Δ-points is in general easier to describe. In any absolute sum where the norm is normalised, and different from the ℓ_∞-norm, the absolute sum has Δ-points if and only if the summands do. In the case of ℓ_∞-sums, however, the absolute sum can have Δ-points even if the summands fail to have any. The chapter ends with a short section of open problems.

In Appendix, we capture all diameter two properties in a diagram along with explanatory comments. We also recapitulate the stability results for the diameter two properties and for Daugavet- and Δ-points in three tables.

1.3 Notation

We use standard notation. In this thesis we consider only nontrivial Banach spaces over the field of real numbers. In general, we assume that the Banach spaces we deal with are infinite-dimensional.

In a Banach space X we denote the unit sphere by S_X and the closed unit ball by B_X. The dual space of a Banach space X is denoted by X^*. For a subset A of X, its diameter is denoted by $\operatorname{diam} A$, its linear span by $\operatorname{span} A$, and its convex hull by $\operatorname{conv} A$. The closed convex hull and the closed linear span are denoted by $\overline{\operatorname{conv}}\, A$ and $\overline{\operatorname{span}}\, A$, respectively. The cardinality of the set A is denoted by $|A|$ and the complement of the set A by A^C. The quotient space of a Banach space X with respect to a subspace $Y \subset X$ is denoted by X/Y. The characteristic function of the subset A is denoted by χ_A.

For Banach spaces X and Y we denote all bounded linear maps from X to Y by $\mathcal{L}(X,Y)$. By an *operator* we always mean a bounded linear map. For an operator $T \in \mathcal{L}(X,Y)$ we denote its kernel by $\ker T$ and its range by $\operatorname{ran} T$. An operator $P\colon X \to X$ is considered as a linear projection, provided $P^2 = P$. For a functional f, we use the notation $\operatorname{supp} f$, to mark the support of f.

It is expected that the reader is familiar with the well-known basic notions and results in the theory of Banach spaces and topological vector spaces. Results as the Hahn–Banach Separation Theorem, the Krein–Milman Theorem, Choquet's Lemma, the Principle of Local Reflexivity, Urysohn's Lemma, will not be presented in the thesis, and some of these basic results are sometimes used without proper references.

1.4 Preliminaries

The aim of this section is to introduce basic concepts and results used throughout the thesis. Our goal is to get the reader familiar with some fundamental results about the Daugavet property and the diameter two properties. At the end of the section we explain the concept of an absolute sum which is one of the key concepts in this thesis. We will also briefly discuss the behaviour of the Daugavet property as well as the behaviour of the diameter two properties when taking absolute sums.

Let us begin with some basic definitions and results.

Definition 1.4.1. Let X be a Banach space and B a non-empty bounded subset of X. A *slice* of B is the set of the following form

$$S(B, x^*, \alpha) = \{x \in B \colon x^*(x) > \sup_{y \in B} x^*(y) - \alpha\},$$

where $x^* \in X^*$ and $\alpha > 0$.

In this thesis we consider almost always the case where $x^* \in S_{X^*}$ and where the bounded set B is the unit ball B_X.

If X is a dual space, we can similarly define w^*-slices, with the defining functional coming from the predual of X.

A slice $S(B, x^*, \alpha)$ is a relatively weakly open subset of B because it is an intersection of a weakly open half-space and the set B. Analogically, w^*-slices are always relatively w^*-open subsets.

We will later make use of the following lemma which tells us that, for a given slice of the unit ball and a prescribed unit sphere element in that slice, there exists a subslice of the given slice containing the prescribed unit sphere element.

Lemma 1.4.2 (see [IK04, Lemma 2.1])**.** *Let X be a Banach space and $S(B_X, x^*, \alpha)$ a slice of the unit ball B_X. Then for every $x \in S_X \cap S(B_X, x^*, \alpha)$ and every $\beta \in (0, \alpha)$ there exists $y^* \in S_{X^*}$ such that*

$$x \in S(B_X, y^*, \beta) \subset S(B_X, x^*, \alpha).$$

In order to define the strong diameter two property and the diametral strong diameter two property, as well as the convex diametral local diameter two property later on, we need the concept of convex combinations of sets. (In this thesis the sets will typically be slices and relatively weakly open subsets of the unit ball.)

Definition 1.4.3. Let X be a Banach space, $n \in \mathbb{N}$, $A_1, \dots, A_n$ subsets of X, and $\lambda_1, \dots, \lambda_n \in [0, 1]$ such that $\sum_{i=1}^n \lambda_i = 1$. The set of the form

$$\sum_{i=1}^n \lambda_i A_i$$

is called a *convex combination* of the sets $A_1, \dots, A_n$.

If we substitute the subsets $A_1, \dots, A_n$ with elements $x_1, \dots, x_n$ of X then the element of the form $\sum_{i=1}^n \lambda_i x_i$ is called a *convex combination* of the elements $x_1, \dots, x_n$.

It is useful to notice that every convex combination of elements in a normed vector space can be approximated in norm with an average of the same set of elements if repetitions of the elements are allowed. This assertion is an immediate consequence of the following elementary lemma.

Lemma 1.4.4. *Let $m \in \mathbb{N}$. Then for every $\varepsilon > 0$ and every $\lambda_i > 0$ with $\sum_{i=1}^m \lambda_i = 1$, there exist $n \in \mathbb{N}$, $k_1, \dots, k_m \in \mathbb{N}$ such that*

$$\sum_{i=1}^m \left| \lambda_i - \frac{k_i}{n} \right| < \varepsilon \qquad \textit{and} \qquad \sum_{i=1}^m k_i = n.$$

Proof. By Dirichlet's Approximation Theorem, given $N \in \mathbb{N}$ there exist integers $k_1, \dots, k_m$ and $1 \le n \le N$ such that

$$\left| \lambda_i - \frac{k_i}{n} \right| \le \frac{1}{nN^{1/m}}.$$

Then

$$\left| n - \sum_{i=1}^{m} k_i \right| = n \left| \sum_{i=1}^{m} \lambda_i - \sum_{i=1}^{m} \frac{k_i}{n} \right| \le n \sum_{i=1}^{m} \frac{1}{nN^{1/m}} = \frac{m}{N^{1/m}}.$$

By just choosing N so large that $N^{-1/m} < \varepsilon$ and $mN^{-1/m} < 1$, we get the desired conclusion. By choosing $\varepsilon > 0$ smaller if necessary, we can make sure that $k_i \ge 0$ for $i = 1, \dots, m$. $\square$

Now, if $x = \sum_{i=1}^{m} \lambda_i x_i$ is a convex combination of elements $x_1, \dots, x_n$ of a normed vector space X, and we choose $k_1, \dots, k_m$ and n to be as in Lemma 1.4.4, then $\sum_{i=1}^{m} (k_i/n) x_i$ is an average of the elements

$$\underbrace{x_1, \dots, x_1}_{k_1 \text{ times}}, \quad \dots \quad , \underbrace{x_m, \dots, x_m}_{k_m \text{ times}}$$

that approximates x. Furthermore, given two different convex combinations, we can approximate them both by averages of the same number of elements. This observation will be used repeatedly throughout the thesis without reference.

Now we are ready to introduce the Daugavet property, which is a well-known and extensively researched property of Banach spaces. The following characterisation of the Daugavet property will be of great importance throughout the thesis.

Proposition 1.4.5 (see [KSSW00, Lemma 2.2], [Shv00, Lemma 3] and [Wer01, Lemma 2.2, Corollary 2.3, and Theorem 2.7])**.** *Let X be a Banach space. The following assertions are equivalent:*

(*i*) *X has the* Daugavet property*, i.e.*

$$\|\operatorname{Id} - T\| = 1 + \|T\|$$

for every compact operator $T \colon X \to X$;

(*i'*)

$$\|\operatorname{Id} - T\| = 1 + \|T\|$$

for every rank-1, norm-1 operator $T \colon X \to X$;

(*ii*) *for every slice S of B_X, every $x \in S_X$, and every $\varepsilon > 0$ there exists $y \in S$ such that $\|x - y\| \geq 2 - \varepsilon$;*

(*iii*) $B_X = \overline{\operatorname{conv}}\, \Delta_\varepsilon(x)$ *for every $x \in S_X$ and every $\varepsilon > 0$, where*

$$\Delta_\varepsilon(x) = \{y \in B_X \colon \|x - y\| \geq 2 - \varepsilon\};$$

(*iv*) *for every convex combination C of relatively weakly open subsets of B_X, every $x \in S_X$, and every $\varepsilon > 0$ there exists $y \in C$ such that $\|x - y\| \geq 2 - \varepsilon$.*

Because of the central importance of the descriptions (*ii*) and (*iii*) in the previous result, we will also provide a proof for their equivalence to the Daugavet property. This, however, calls for some preliminary work.

It is useful to note that in the definition of the Daugavet property, i.e. Proposition 1.4.5 part (*i*), the operators may, indeed, be assumed to be of norm 1. In order to prove that, we need the following basic, yet helpful fact.

Lemma 1.4.6. *Let X be a normed space and $x, y \in X$ such that $\|x + y\| = \|x\| + \|y\|$. Then for every $r, s > 0$ we have*

$$\|rx + sy\| = r\|x\| + s\|y\|.$$

Proof. On the one hand, we have trivially that $\|rx + sy\| \leq r\|x\| + s\|y\|$. We complete the proof by showing the other inequality. We may assume without loss of generality that $r \geq s$. Therefore,

$$\begin{aligned}
\|rx + sy\| &= \|r(x + y) + (s - r)y\| \\
&\geq \|r(x + y)\| - \|(s - r)y\| \\
&= r\|x + y\| + (s - r)\|y\| \\
&= r\|x\| + r\|y\| + s\|y\| - r\|y\| \\
&= r\|x\| + s\|y\|.
\end{aligned}$$

□

Corollary 1.4.7. *In the definition of the Daugavet property, it is enough to consider the case $\|T\| = 1$.*

Proof. Assume that the Daugavet equation holds for all compact norm-1 operators. Let T be an arbitrary compact operator. Then by Lemma 1.4.6, we get

$$\begin{aligned}
\|\operatorname{Id} - T\| &= \left\| \operatorname{Id} - \|T\| \frac{T}{\|T\|} \right\| \\
&= \|\operatorname{Id}\| + \left\| \frac{T}{\|T\|} \right\| \|T\| \\
&= 1 + \|T\|.
\end{aligned}$$

□

Now we are ready to present the proof of the equivalence of the different characterisations of the Daugavet property.

Proof of Proposition 1.4.5. $(i) \Rightarrow (i')$. This is trivial as rank-1 operators are compact.

$(i') \Rightarrow (ii)$. Let $x^* \in S_{X^*}$ and consider a slice $S(B_X, x^*, \alpha)$ of B_X, an $x \in S_X$, and an $\varepsilon > 0$. Assume $\alpha \leq \varepsilon$. Define a rank-1 operator $T : X \to X$ by $Ty = x^*(y)x$. Since $\|\operatorname{Id} - T\| = 2$, there exists $y \in B_X$ such that

$$\|y - x^*(y)x\| \geq 2 - \alpha/2.$$

Thus $x^*(y) \geq 1 - \alpha/2$, and this in turn gives

$$\|x - y\| \geq \|y - x^*(y)x\| - \|x^*(y)x - x\| \geq 2 - \varepsilon\,.$$

$(ii) \Rightarrow (i)$. Let $T : X \to X$ be a compact operator. From Corollary 1.4.7 we can assume that $\|T\| = 1$. Since T is compact, the restriction of the adjoint T^* to the dual unit ball is $w^* - \|\cdot\|$ continuous. By the Krein–Milman Theorem, there exists an extreme point p^* of B_{X^*} such that $\|T^*p^*\| = \|T^*\| = 1$. From Choquet's Lemma (see [FHHMZ11, Lemma 3.69]) we get that p^* has a neighbourhood base for the w^*-topology of B_{X^*} consisting of w^*-closed slices S_α. By the assumptions every slice S_α contains some x^*_α such that $\|x^*_\alpha - T^*p^*\| \geq 2 - \varepsilon_\alpha$ where (ε_α) is a net of reals converging to 0. As x^*_α converges to p^* in the w^*-topology, we get that Tx^*_α converges to T^*p^* in norm. Hence $\|x^*_\alpha - T^*x^*_\alpha\|$ converges to 2 in norm, and thus $\|\operatorname{Id} - T\| = 2$.

$(ii) \Rightarrow (iii)$. Assume by contradiction that (ii) holds but (iii) does not hold. Now, let $x \in S_X$ and $\varepsilon > 0$ be such that $B_X \neq \overline{\operatorname{conv}}\,\Delta_\varepsilon(x)$. Therefore, $S_X \not\subset \overline{\operatorname{conv}}\,\Delta_\varepsilon(x)$. Let $y \in S_X \setminus \overline{\operatorname{conv}}\,\Delta_\varepsilon(x)$. It is evident that $\overline{\operatorname{conv}}\,\Delta_\varepsilon(x)$ and $\{y\}$ are closed convex disjoint sets and the singleton $\{y\}$ is compact. By Hahn–Banach Separation Theorem there exist $x^* \in S_{X^*}$ and $\alpha > 0$ such that for every $z \in \overline{\operatorname{conv}}\,\Delta_\varepsilon(x)$ we have that

$$x^*(z) < \alpha < x^*(y) \leq 1.$$

It easily follows that $z \notin S(B_X, x^*, 1 - \alpha)$ for every $z \in \overline{\operatorname{conv}}\,\Delta_\varepsilon(x)$. This however means that there exists a slice which does not intersect the set $\overline{\operatorname{conv}}\,\Delta_\varepsilon(x)$, hence we have a contradiction with (ii).

$(iii) \Rightarrow (ii)$. Assume by contradiction that (iii) holds but (ii) does not hold. Now, let $S(B_X, x^*, \alpha)$, $x \in S_X$, and $\varepsilon > 0$ be such that

$$S(B_X, x^*, \alpha) \cap \Delta_\varepsilon(x) = \emptyset.$$

Find $y \in S(B_X, x^*, \alpha)$ and $\delta > 0$ such that $x^*(y) > 1 - \alpha + \delta$. Since (iii) holds, we get that $y \in S(B_X, x^*, \alpha) \subset \overline{\operatorname{conv}}\, \Delta_\varepsilon(x)$. Therefore, there exist $y_1, \ldots, y_n \in \Delta_\varepsilon(x)$ such that

$$\left\| y - \sum_{i=1}^{n} \frac{1}{n} y_i \right\| < \delta.$$

Then

$$x^*(y) - \sum_{i=1}^{n} \frac{1}{n} x^*(y_i) = x^*\Big(y - \sum_{i=1}^{n} \frac{1}{n} y_i\Big) < \delta.$$

Since $y_i \in \Delta_\varepsilon(x)$, we have that y_i cannot be in the slice $S(B_X, x^*, \alpha)$ for $i \in \{1, \ldots, n\}$. Therefore,

$$1 - \alpha < x^*(y) - \delta < \sum_{i=1}^{n} \frac{1}{n} x^*(y_i) < \sum_{i=1}^{n} \frac{1}{n} (1 - \alpha) = 1 - \alpha,$$

which gives us a contradiction.

For the proof of $(i) \Leftrightarrow (iv)$ see [Shv00]. □

Definition 1.4.8 (see [ALN13])**.** Let X be a Banach space. We say that X has the

(a) *local diameter two property (LD2P)* if every slice of B_X has diameter 2;

(b) *diameter two property (D2P)* if every non-empty relatively weakly open subset of B_X has diameter 2;

(c) *strong diameter two property (SD2P)* if every convex combination of slices of B_X has diameter 2.

Remark 1.4.9. Provided X is a dual Banach space the w^*-versions of these properties are also of interest. To get the respective properties w^*-LD2P, w^*-D2P, and w^*-SD2P, we consider w^*-slices in Definition 1.4.8 parts (a) and (c) and relatively w^*-open subsets in part (b), respectively.

It is clear from Proposition 1.4.5 that we have the following chain of implications

$$\text{Daugavet property} \Rightarrow \text{SD2P} \Rightarrow \text{D2P} \Rightarrow \text{LD2P}.$$

The first implication follows from Proposition 1.4.5 $(i) \Rightarrow (iv)$. The second implication is due to the following lemma of Bourgain (see, e.g. [GGMS87, Lemma II.1]).

Lemma 1.4.10 (Bourgain's Lemma)**.** *Let X be a Banach space and U a non-empty relatively weakly open subset in B_X. There exist $n \in \mathbb{N}$, slices $S_1, \ldots, S_n$ of the unit ball, and scalars $\lambda_1, \ldots, \lambda_n \in [0,1]$ with $\sum_{i=1}^n \lambda_i = 1$ such that*

$$\sum_{i=1}^n \lambda_i S_i \subset U.$$

The third implication is trivial since every slice is a relatively weakly open subset of the unit ball. None of the reverse implications in the diagram above hold, in particular all three diameter two properties from Definition 1.4.8 are different. Indeed, it was shown in [BLR15] that c_0 can be renormed in such a way that it has the LD2P, but not the D2P. In particular LD2P $\not\Rightarrow$ D2P. In [ABL15] and [HL14], it was independently proved that ℓ_p-sums fail to have the SD2P for all $1 < p < \infty$, so the D2P $\not\Rightarrow$ SD2P since such sums preserve the D2P (see [ALN13]). Also c_0 is an example of a space with the SD2P which fails the Daugavet property (see [ALN13]), so SD2P $\not\Rightarrow$ Daugavet property.

It is known that a Banach space X has the LD2P (respectively, D2P, SD2P) if and only if X^{**} has the w^*-LD2P (respectively, w^*-D2P, w^*-SD2P) (see [HLP15]). Additionally, it follows from Definition 1.4.8 and Remark 1.4.9 that if a dual space has any of the diameter two properties, then it also has the corresponding w^*-version. Thus, if the second dual X^{**} of a Banach space X has the LD2P (respectively, D2P, SD2P), then X has the LD2P (respectively, D2P, SD2P). The converse of this is not true for any diameter two property as can be seen from the following example.

Example 1.4.11 (see [Lan15, Example 2.16]). The Banach space $L[0,1]$ has the SD2P, but its second dual contains slices of arbitrarily small diameter, hence fails the LD2P.

The phenomenon that all slices of the unit ball of a Banach space have diameter two cannot be observed among the finite-dimensional ones. In finite-dimensional spaces there always exist slices of the unit ball with arbitrarily small diameter. This is even true for the class of Banach spaces with the well-known *Radon–Nikodým property* which contains the reflexive spaces, in particular the finite-dimensional spaces. The Radon–Nikodým property has many equivalent formulations. We will use the one with slices.

Definition 1.4.12 (see, e.g. [FHHMZ11, Theorem 11.15])**.** Let X be a Banach space. We say that X has the *Radon–Nikodým property* if every non-empty bounded subset of X has slices of arbitrarily small diameter, that is, for every bounded subset B of X and for every $\varepsilon > 0$, there is a slice S of B such that

$$\operatorname{diam} S < \varepsilon.$$

Throughout the thesis one of the main methods we use to explore the properties that we study, is to see to what extent they are affected by taking direct sums equipped with absolute normalised norms. We recall that a norm N on $\mathbb{R}^2$ is called *absolute* if

$$N(a,b) = N(|a|,|b|) \qquad \text{for all } (a,b) \in \mathbb{R}^2$$

and *normalised* if

$$N(1,0) = N(0,1) = 1$$

(see [GGMS87]).

For example, the ℓ_p-norm $\|\cdot\|_p$ on $\mathbb{R}^2$ is absolute and normalised for every $p \in [1,\infty]$. If N is an absolute normalised norm on $\mathbb{R}^2$ (see [GGMS87, Lemmata 21.1 and 21.2]), then

(*a*) $\|(a,b)\|_\infty \le N(a,b) \le \|(a,b)\|_1$ for all $(a,b) \in \mathbb{R}^2$;

(*b*) if $(a,b),(c,d) \in \mathbb{R}^2$ with $|a| \le |c|$ and $|b| \le |d|$, then

$$N(a,b) \le N(c,d);$$

(*c*) the dual norm N^* on $\mathbb{R}^2$ defined by

$$N^*(c,d) = \max_{N(a,b)\le 1} (|ac| + |bd|) \qquad \text{for all } (c,d) \in \mathbb{R}^2$$

is also absolute and normalised. Note that $(N^*)^* = N$.

If X and Y are Banach spaces and N is an absolute normalised norm on $\mathbb{R}^2$, then we denote by $X \oplus_N Y$ the product space $X \times Y$ with respect to the norm

$$\|(x,y)\|_N = N(\|x\|,\|y\|) \qquad \text{for all } x \in X \text{ and } y \in Y,$$

and we call this Banach space the *absolute sum* of X and Y. In the special case where N is the ℓ_p-norm, we write $X \oplus_p Y$. Note that $(X \oplus_N Y)^* = X^* \oplus_{N^*} Y^*$. The following example shows that there are plenty of absolute normalised norms that are not ℓ_p-norms.

Example 1.4.13 (see Figure 1.1). Let X and Y be Banach spaces and $\lambda \in (\frac{1}{2}, 1)$. Let the product space $X \times Y$ be equipped with the following norm

$$|||(x,y)||| = \max\Big\{\|(x,y)\|_\infty, \lambda\|(x,y)\|_1\Big\}.$$

The norm $|||\cdot|||$ is an absolute normalised norm which differs from the ℓ_p-norms (see Figure 1.1). Note that if $\lambda = 1$, then this norm coincides with the ℓ_1-norm and if $\lambda = \frac{1}{2}$, then this norm coincides with the ℓ_∞-norm.

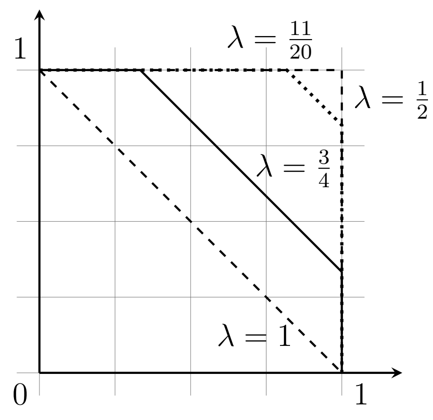

Figure 1.1: First quadrant of the unit sphere of $\mathbb{R}^2$ with the norm from Example 1.4.13 for different values of λ.

In Chapter 3, we make use of the following property of an absolute normalised norm on $\mathbb{R}^2$.

Lemma 1.4.14. *Let N be an absolute normalised norm on $\mathbb{R}^2$. For every $\varepsilon > 0$ there exists $\delta \in (0, \varepsilon)$ such that for every $p, q, r \geq 0$, if*

$$2 - \delta \leq N(p, q) \leq N(r, q) \leq 2 \quad \text{and} \quad q < 2 - \delta,$$

then $|p - r| < \varepsilon$.

Proof. Fix $\varepsilon \in (0, 2)$ and set $c = \max_{N(p,1)=1} p$. To every $q \in [0, 2)$ there corresponds a unique $p_q \geq 0$ such that $N(p_q, q) = 2$. Choose $s \in (2 - \varepsilon, 2)$ such that $p_s - cs < \varepsilon$. It is easy to see that $\delta = 2 - s$ satisfies the conditions of the lemma, since the function $[0, s] \to \mathbb{R}$, $q \mapsto p_q - r_q$, where $r_q \geq 0$ and $N(r_q, q) = s$, is non-decreasing and $p_s - r_s < \varepsilon$ (see Figure 1.2). $\square$

Investigating properties of absolute sums of Banach spaces can often lead to fruitful discoveries. For example, the first Banach space discovered with the D2P, but without the SD2P, was an ℓ_p-sum, $1 < p < \infty$, as mentioned above. We will see in Chapter 3 that absolute sums can also be used to prove that the notions Daugavet-point and Δ-point are different.

Among all absolute sums the Daugavet property is stable only by taking ℓ_1-sums and ℓ_∞-sums.

Theorem 1.4.15 (see [BKSW05, Theorem 5.1 and Corollary 5.4]). *Let X and Y be Banach spaces.*

(*a*) *If N is either ℓ_1- or ℓ_∞-norm, then $X \oplus_N Y$ has the Daugavet property if and only if X and Y have the Daugavet property.*

(*b*) *If N is any other absolute norm, then $X \oplus_N Y$ does not have the Daugavet property.*

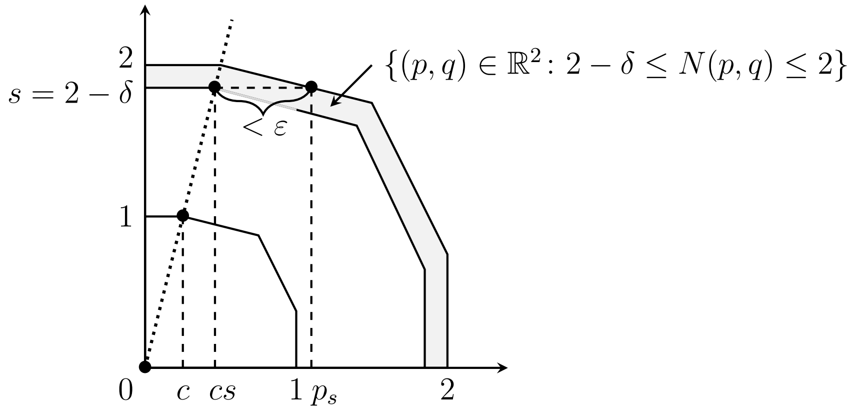

Figure 1.2: The proof of Lemma 1.4.14.

In [ALN13] and [ABL15], it was proved that both the LD2P and the D2P are stable by taking ℓ_p-sums for every $1 \le p \le \infty$. This is far from being true for the SD2P which behaves more like the Daugavet property in this setting.

Theorem 1.4.16 (see [ABL15, Proposition 3.1], [ALN13, Theorem 2.7 (*iii*) and Proposition 4.6], and [BL06, Lemma 2.1])**.** *Let X and Y be Banach spaces.*

(*a*) *The absolute sum $X \oplus_1 Y$ has the SD2P if and only if X and Y have the SD2P.*

(*b*) *If X has the SD2P, then $X \oplus_\infty Y$ has the SD2P.*

Theorem 1.4.17 (see, e.g. [ABL15, Theorem 3.2] or [HL14, Theorem 1])**.** *Let X and Y be Banach spaces and let p be such that $1 < p < \infty$. Then $X \oplus_p Y$ does not have the SD2P.*

Chapter 2

Diametral diameter two properties

The first systematic treatment of the diametral diameter two properties was given in [BLR18]. Some preliminary work was done in [IK04], [AHNTT16], and [ALNT16]. We give a brief overview of the latest research and study stability properties of different types of diametral diameter two properties. These results were obtained in [AHLP20] and [HPP16], except the results regarding M-ideals that are new.

2.1 Definitions and examples

In this section we introduce the notions of the diametral diameter two properties which were considered in [BLR18].

Definition 2.1.1 (see [BLR18])**.** Let X be a Banach space. We say that X has the

(a) *diametral local diameter two property (DLD2P)* if for every slice S of B_X, every $x \in S_X \cap S$, and every $\varepsilon > 0$ there exists $y \in S$ such that $\|x - y\| \geq 2 - \varepsilon$;

(b) *diametral diameter two property (DD2P)* if for every non-empty relatively weakly open subset U of B_X, every $x \in S_X \cap U$, and every $\varepsilon > 0$ there exists $y \in U$ such that $\|x - y\| \geq 2 - \varepsilon$;

(c) *diametral strong diameter two property (DSD2P)* if for every $n \in \mathbb{N}$, non-empty relatively weakly open subsets $U_1, \ldots, U_n$ of B_X, $\lambda_1, \ldots, \lambda_n \in [0, 1]$ such that $\sum_{i=1}^{n} \lambda_i = 1$, every $x \in \sum_{i=1}^{n} \lambda_i U_i$, and every $\varepsilon > 0$ there exists $y \in \sum_{i=1}^{n} \lambda_i U_i$ satisfying $\|x - y\| \geq 1 + \|x\| - \varepsilon$.

Remark 2.1.2. The DLD2P was originally introduced in [IK04] under the name *space with bad projections*, and was also studied in [ALNT16] and [AHNTT16] under the name the *local diameter two property +*. It is known that the DLD2P is different from the Daugavet property (see [IK04]). The formal difference between the DLD2P and the Daugavet property appears by comparing Definition 2.1.1 part (a) and Proposition 1.4.5 part (ii), an equivalent condition to the Daugavet property. Note that in the first case the $x \in S_X$ is taken from the slice S of B_X, whereas in the other case it is arbitrary.

Remark 2.1.3. For dual Banach spaces one defines w^*-versions of the diametral diameter two properties (w^*-DLD2P, w^*-DD2P, and w^*-DSD2P), similarly to w^*-versions of the regular diameter two properties.

It is clear from the definitions that the DSD2P implies the DD2P, and the DD2P implies the DLD2P. The Daugavet property implies the DSD2P, as it was shown in [BLR18]. Moreover, it is easy to see that each diametral diameter two property implies its counterpart in the regular diameter two properties, that is, the DSD2P implies the SD2P, the DD2P implies the D2P, and the DLD2P implies the LD2P. These observations are captured in the following diagram.

$$\begin{array}{ccccccc} \text{Daugavet property} & \Rightarrow & \text{DSD2P} & \Rightarrow & \text{DD2P} & \Rightarrow & \text{DLD2P} \\ & & \Downarrow & & \Downarrow & & \Downarrow \\ & & \text{SD2P} & \Longrightarrow & \text{D2P} & \Longrightarrow & \text{LD2P} \end{array}$$

Let us now look at the reverse implications. An example of a Banach space with the DD2P and failing the DSD2P was given in [BLR18, Example 2.2], hence DD2P $\not\Rightarrow$ DSD2P. The converse of both Daugavet property $\Rightarrow$ DSD2P and DD2P $\Rightarrow$ DLD2P, are still open questions (see [BLR18, Question 4.1]). That SD2P $\not\Rightarrow$ DSD2P, D2P $\not\Rightarrow$ DD2P, and LD2P $\not\Rightarrow$ DLD2P, follows, for example, from the fact that c_0 has the SD2P, but fails the DLD2P (see also [AHNTT16]).

Example 2.1.4. The sequence space c_0 has the SD2P, but fails to have the DLD2P. Indeed, it is known that the sequence space c_0 has the SD2P (see [ALN13]). Let us show that c_0 does not have the DLD2P. Consider a slice of the unit ball B_{c_0} of the form $S = \{x = (\xi_k) \in B_{c_0} \colon e_1(x) = \xi_1 > 0\}$. Then $e_1 \in S_{c_0} \cap S$. However, for any $y = (\eta_k) \in S$ we have that

$$\|e_1 - y\| = \max\{|1 - \eta_1|, |\eta_2|, |\eta_3|, \dots\} \leq 1.$$

Therefore for $\varepsilon \in (0, 1)$ there is no $y \in S$ such that $\|e_1 - y\| \geq 2 - \varepsilon$, hence c_0 does not have the DLD2P. We remark that one can show similarly that sequence spaces c and ℓ_∞ do not have the DLD2P.

Let us briefly consider w^*-versions of the diametral diameter two properties. It is known that there is a strong connection between the DLD2P and the w^*-DLD2P in the following sense.

Proposition 2.1.5 (see [AHNTT16, Theorem 3.6])**.** *Let X be a Banach space. Then X has the DLD2P if and only if X^* has the w^*-DLD2P.*

Whether the similar connection holds between the DD2P and the w^*-DD2P or between the DSD2P and the w^*-DSD2P were posed as open problems in [BLR18] and have remained unanswered.

Recall that all regular diameter two properties have a convenient connection to their corresponding w^*-versions in bidual, e.g. a Banach space X has the LD2P if and only if X^{**} has the w^*-LD2P. Similar equivalences do not hold for the diametral diameter two properties, since $X = C[0,1]$ has the Daugavet property (hence also the DSD2P, the DD2P, the DLD2P) but its dual has the Radon–Nikodým property, so X^* fails the DLD2P, and consequently, by Proposition 2.1.5, X^{**} fails the w^*-DLD2P (cf. [BLR18, Remark 2.4]). Nonetheless, for all diametral diameter two properties the implication in the other direction holds, i.e. a Banach space whose bidual has a w^*-diametral diameter two property has the corresponding diametral diameter two property. In the case of the DLD2P this result is a simple corollary of Proposition 2.1.5, the cases regarding the DD2P and the DSD2P were proved in [BLR18].

Proposition 2.1.6 (see [AHNTT16, Theorem 3.6] and [BLR18, Proposition 2.3])**.** *Let X be a Banach space. If X^{**} has the w^*-DLD2P (respectively, w^*-DD2P, w^*-DSD2P), then X has the DLD2P (respectively, DD2P, DSD2P).*

To end this section, we observe different characterisations of the diametral diameter two properties. For the DLD2P we have the following equivalent descriptions (cf. Proposition 1.4.5).

Proposition 2.1.7 (see [IK04, Theorem 1.4] and [Wer01, Corollary 2.3])**.** *Let X be a Banach space. The following assertions are equivalent:*

(*i*) *X has the DLD2P, i.e. for every slice S of B_X, every $x \in S_X \cap S$ and every $\varepsilon > 0$ there exists $y \in S$ such that $\|x - y\| \geq 2 - \varepsilon$;*

(*ii*) *$\|\operatorname{Id} - P\| \geq 2$ for every rank-1 projection $P\colon X \to X$;*

(*iii*) *$x \in \overline{\operatorname{conv}}\, \Delta_\varepsilon(x)$ for every $x \in S_X$ and every $\varepsilon > 0$, where*

$$\Delta_\varepsilon(x) = \{y \in B_X \colon \|x - y\| \geq 2 - \varepsilon\}.$$

The DD2P can be equivalently formulated using special kind of rank-1 projections.

Proposition 2.1.8 (see [BLR18, Proposition 2.9]). *Let X be a Banach space. The following assertions are equivalent:*

(i) *X has the DD2P;*

(ii) *for every $n \in \mathbb{N}$, $x_1^*, \ldots, x_n^* \in S_{X^*}$, every $x \in X$ with $x_i^*(x) \neq 0$ for every $i \in \{1, \ldots, n\}$, and every $\varepsilon > 0$ there exists $y \in B_X$ such that for every $i \in \{1, \ldots, n\}$ we have*

$$\frac{x_i^*(y)}{x_i^*(x)} \geq 0$$

and

$$\|y - P_i y\| > 2 - \varepsilon,$$

where

$$P_i = \frac{1}{x_i^*(x)} x_i^* \otimes x.$$

There is no known equivalent characterisation by the set of almost diametral points for the DD2P. For the DSD2P, however, there is no known characterisation with projections either.

The following lemma suggests that, in the definition of DSD2P (see Definition 2.1.1 part (c)), we can replace the elements of the unit ball in the convex combination with elements of the unit sphere. This lemma plays the key role in the proof of Theorem 2.2.6, where it is proved that given two Banach spaces with the DSD2P, their ℓ_1-sum also has the DSD2P.

Lemma 2.1.9. *Let X be a Banach space. Then X has the DSD2P if and only if for every $n \in \mathbb{N}$, non-empty relatively weakly open subsets $U_1, \ldots, U_n$ of B_X, $\lambda_1, \ldots, \lambda_n \in [0, 1]$ such that $\sum_{i=1}^n \lambda_i = 1$, every $x \in \sum_{i=1}^n \lambda_i \big(S_X \cap U_i\big)$, and every $\varepsilon > 0$ there exists $y \in \sum_{i=1}^n \lambda_i U_i$ satisfying*

$$\|x - y\| \geq 1 + \|x\| - \varepsilon.$$

Proof. Firstly, notice that the space X may be assumed to be infinite-dimensional (because clearly no finite-dimensional space can have the DSD2P) and the sets $U_1, \ldots, U_n$ to be convex (because, since $x = \sum_{i=1}^n \lambda_i u_i$ where $u_i \in U_i$, for every $i \in \{1, \ldots, n\}$, it suffices to consider in the role of U_i a convex relatively weakly open neighbourhood V_i of u_i satisfying $V_i \subset U_i$). Now, to prove the lemma, it suffices to observe that

($*$) every $a \in U_i$ can be written in the form $a = (1-\mu_i)y_i + \mu_i z_i$ where $\mu_i \in [0,1]$ and $y_i, z_i \in S_X \cap U_i$,

because, if ($*$) holds, then the element x can be written as

$$x = \sum_{i=1}^{n} \lambda_i(1-\mu_i)y_i + \sum_{i=1}^{n} \lambda_i\mu_i z_i$$

and (by the convexity of $U_1, \dots, U_n$)

$$\sum_{i=1}^{n} \lambda_i(1-\mu_i)U_i + \sum_{i=1}^{n} \lambda_i\mu_i U_i \subset \sum_{i=1}^{n} \lambda_i U_i.$$

It remains to prove ($*$). Let $i \in \{1, \dots, n\}$ and let $a \in U_i$, $\|a\| < 1$. Let $m \in \mathbb{N}$, $x_1^*, \dots, x_m^* \in X^*$, and $\delta > 0$ be such that

$$U_i \supset \{b \in B_X \colon |x_j^*(b) - x_j^*(a)| < \delta, j = 1, \dots, m\}.$$

Choose a non-zero $c \in \bigcap_{j=1}^{m} \ker x_j^*$ (such a c exists when the space X is infinite-dimensional), and consider the function $f(t) = \|a + tc\|$, $t \in \mathbb{R}$. Since $f(0) = \|a\| < 1$ and $f(t) \to \infty$ in the process $t \to \pm\infty$, there are $s, t \in (0, \infty)$ such that $f(-s) = f(t) = 1$, but now $y_i = a - sc$, $z_i = a + tc$, and $\mu_i = s/(s+t)$ do the job. $\square$

2.2 Diametral diameter two properties in absolute sums

Y. Ivakhno and V. Kadets proved that an absolute sum of Banach spaces has the DLD2P if and only if each component space has the DLD2P (cf. [IK04, Theorem 3.2]). We give the result with a proof to present the basic techniques used in dealing with the DLD2P.

Theorem 2.2.1 (cf. [IK04, Theorem 3.2])**.** *Let X and Y be Banach spaces and N an absolute normalised norm on $\mathbb{R}^2$. Then $X \oplus_N Y$ has the DLD2P if and only if both X and Y have the DLD2P.*

Proof. ($\Leftarrow$) Assume $Z = X \oplus_N Y$ has the DLD2P. We show that X has the DLD2P (the case Y has the DLD2P is similar). Assume by contradiction that X does not have the DLD2P. In that case, there exist a slice $S(B_X, x^*, \alpha)$, $x \in S_X \cap S(B_X, x^*, \alpha)$, and $\delta > 0$ such that for every $u \in S(B_X, x^*, \alpha)$ we have that

$$\|x - u\| < 2 - \delta.$$

By Lemma 1.4.2, we may assume that $\alpha < \delta$. Denote $z^* = (x^*, 0)$ and $z = (x, 0)$. Clearly $z^* \in S_{Z^*}$ and $z \in S_Z$. Note that $z \in S(B_Z, z^*, \alpha)$, because $z^*(z) = x^*(x) > 1 - \alpha$. An arbitrary $w = (u, v) \in S(B_Z, z^*, \alpha)$ satisfies that $u \in S(B_X, x^*, \alpha)$, since $x^*(u) = z^*(w) > 1 - \alpha$. Therefore, $\|x - u\| < 2 - \delta$ and $\|u\| > 1 - \alpha$. In conclusion we get that

$$\begin{aligned}
\|z - w\|_N = \|(x, 0) - (u, v)\|_N &= N(\|x - u\|, \|v\|) \\
&< N(2 - \delta, \|v\|) \\
&= N\Big((2 - \delta - \|u\|, 0) + (\|u\|, \|v\|)\Big) \\
&\leq (2 - \delta - \|u\|) + 1 \\
&< \Big(2 - \delta - (1 - \alpha)\Big) + 1 \\
&= 2 - \delta + \alpha,
\end{aligned}$$

which contradicts the fact that Z has the DLD2P.

($\Rightarrow$) Assume X and Y have the DLD2P. Set $Z = X \oplus_N Y$. Let $\varepsilon > 0$. Consider a slice $S(B_Z, z^*, \alpha)$ and an element $z \in S_X \cap S(B_Z, z^*, \alpha)$, where $z^* = (x^*, y^*)$ and $z = (x, y)$. Let

$$S_1 = \{u \in B_X \colon \|x\| x^*(u) > x^*(x) - \alpha_1\}$$

and

$$S_2 = \{v \in B_Y \colon \|y\| y^*(v) > y^*(y) - \alpha_2\},$$

where $\alpha_1, \alpha_2 > 0$ satisfy the condition $\alpha_1 + \alpha_2 \leq z^*(z) - (1 - \alpha)$.

The sets S_1 and S_2 are slices of the unit ball. For example, S_1 is a slice of B_X, since for $x = 0$ or $x^* = 0$ we have $S_1 = B_X$ and if $x \neq 0$ and $x^* \neq 0$ we have

$$S_1 = S\Big(B_X, \frac{x^*}{\|x^*\|}, 1 - \frac{x^*(x) - \alpha_1}{\|x^*\| \|x\|}\Big).$$

Take $u \in S_1$ and $v \in S_2$ such that

$$\Big\| x - \|x\| u \Big\| \geq (2 - \varepsilon)\|x\|$$

and

$$\Big\| y - \|y\| v \Big\| \geq (2 - \varepsilon)\|y\|.$$

Such an element u exists for the following reason (the existence of v can be shown similarly): if $x = 0$, then any $u \in S_1$ will be suitable; if $x \neq 0$, then $x/\|x\| \in S_X \cap S_1$ and since X has the DLD2P, there exists $u \in S_1$ such that $\|x/\|x\| - u\| \geq 2 - \varepsilon$, i.e. $\|x - \|x\| u\| \geq (2 - \varepsilon)\|x\|$.

Note that $w = \Big(\|x\|u, \|y\|v\Big) \in S(B_Z, z^*, \alpha)$, because

$$\begin{aligned} z^*(w) &= \|x\|x^*(u) + \|y\|y^*(v) \\ &> x^*(x) + y^*(y) - (\alpha_1 + \alpha_2) \\ &= z^*(z) - (\alpha_1 + \alpha_2) \\ &\geq 1 - \alpha. \end{aligned}$$

In addition we have that $\|z - w\|_N \geq 2 - \varepsilon$, since

$$\begin{aligned} \|z - w\|_N &= \Big\|(x, y) - (\|x\|u, \|y\|v)\Big\|_N \\ &= N\Big(\Big\|x - \|x\|u\Big\|, \Big\|y - \|y\|v\Big\|\Big) \\ &\geq N\Big((2 - \varepsilon)\|x\|, (2 - \varepsilon)\|y\|\Big) \\ &= 2 - \varepsilon. \end{aligned}$$

In conclusion, Z has the DLD2P. □

The stability of the DD2P and the DSD2P under ℓ_p-sums was explored in [BLR18]. We consider the stability of the DD2P in a more general setting of absolute sums. Our proof slightly differs from the original proof for ℓ_p-sums in [BLR18].

Proposition 2.2.2 (cf. [BLR18, Theorem 2.12 and Proposition 2.13])**.** *Let X and Y be Banach spaces and N an absolute normalised norm on $\mathbb{R}^2$. Then $X \oplus_N Y$ has the DD2P if and only if both X and Y have the DD2P.*

Proof. ($\Rightarrow$) Assume $Z = X \oplus_N Y$ has the DD2P. We show that X has the DD2P, the proof for the case Y has the DD2P is analogical. Let U be a relatively weakly open subset of B_X, $x \in S_X \cap U$, and $\varepsilon \in (0, 1)$. Let $\delta \in (0, \varepsilon/2)$. Then $1 < (2 - \varepsilon)/(1 - \delta) < 2$. Consider a relatively weakly open subset of B_Z

$$W = \Big\{(u, v) \in B_Z \colon u \in U, \|u\| > 1 - \delta\Big\}.$$

Obviously, $(x, 0) \in S_Z \cap W$. Since $X \oplus_N Y$ has the DD2P, there exists $(u, v) \in W$ such that

$$\|(x, 0) - (u, v)\|_N \geq \frac{2 - \varepsilon}{1 - \delta}.$$

Since

$$\begin{aligned}\|(x-u,v)\|_N &\leq \left\|\left(\frac{\|x-u\|}{1-\delta}u, v\right)\right\|_N \\ &\leq \max\left\{\frac{\|x-u\|}{1-\delta}, 1\right\}\|(u,v)\|_N \\ &\leq \max\left\{\frac{\|x-u\|}{1-\delta}, 1\right\},\end{aligned}$$

we can conclude that

$$\frac{2-\varepsilon}{1-\delta} \leq \max\left\{\frac{\|x-u\|}{1-\delta}, 1\right\},$$

i.e. $\|x-u\| \geq 2-\varepsilon$, hence X has the DD2P.

($\Leftarrow$) Assume X and Y have the DD2P. In the following we prove that $Z = X \oplus_N Y$ has the DD2P. Let W be a relatively weakly open subset of B_Z, $z = (x, y) \in S_Z \cap W$, and $\varepsilon > 0$. We show the existence of $w \in W$ such that

$$\|z - w\|_N \geq 2 - \varepsilon.$$

It is sufficient to look at the case

$$W = \Big\{w \in B_Z \colon |z_i^*(z-w)| < 1 \quad \forall i \in \{1, \dots, n\}\Big\},$$

where $n \in \mathbb{N}$, $z_i^* = (x_i^*, y_i^*) \in Z^*$. Let us first consider the case $x \neq 0$ and $y \neq 0$. The sets

$$U = \left\{u \in B_X \colon \left|x_i^*\left(\frac{x}{\|x\|} - u\right)\right| < \frac{1}{2\|x\|} \quad \forall i \in \{1, \dots, n\}\right\}$$

and

$$V = \left\{v \in B_Y \colon \left|y_i^*\left(\frac{y}{\|y\|} - v\right)\right| < \frac{1}{2\|y\|} \quad \forall i \in \{1, \dots, n\}\right\}$$

are obviously relatively weakly open. Since $x/\|x\| \in S_X \cap U$ and $y/\|y\| \in S_Y \cap V$, then there exist $u \in U$ and $v \in V$ such that

$$\left\|\frac{x}{\|x\|} - u\right\| > 2 - \varepsilon$$

and

$$\left\|\frac{y}{\|y\|} - v\right\| > 2 - \varepsilon.$$

Take $w = (\|x\|u, \|y\|v)$. Then $w \in W$, because

$$\begin{aligned}\|w\|_N &= \Big\|(\|x\|u, \|y\|v)\Big\|_N \\ &= N(\|x\|\|u\|, \|y\|\|v\|) \\ &\leq N(\|x\|, \|y\|) = 1,\end{aligned}$$

which gives us that $w \in B_Z$, and for every $i \in \{1, \ldots, n\}$

$$\begin{aligned}|z_i^*(z-w)| &= \Big|x_i^*(x - \|x\|u) + y_i^*(y - \|y\|v)\Big| \\ &\leq \|x\|\Big|x_i^*\Big(\frac{x}{\|x\|} - u\Big)\Big| + \|y\|\Big|y_i^*\Big(\frac{y}{\|y\|} - v\Big)\Big| \\ &< \|x\|\frac{1}{2\|x\|} + \|y\|\frac{1}{2\|y\|} \\ &= 1.\end{aligned}$$

Notice that

$$\begin{aligned}\|z-w\|_N &= \Big\|(x - \|x\|u, y - \|y\|v)\Big\|_N \\ &= N\Big(\Big\|x - \|x\|u\Big\|, \Big\|y - \|y\|v\Big\|\Big) \\ &= N\Big(\|x\|\Big\|\frac{x}{\|x\|} - u\Big\|, \|y\|\Big\|\frac{y}{\|y\|} - v\Big\|\Big) \\ &\geq N\Big((2-\varepsilon)\|x\|, (2-\varepsilon)\|y\|\Big) \\ &= 2 - \varepsilon.\end{aligned}$$

Now, let us consider the case $x = 0$ or $y = 0$. Assume $y = 0$. The set

$$U = \Big\{u \in B_X : |x_i^*(x-u)| < 1 \quad \forall i \in \{1, \ldots, n\}\Big\}$$

is obviously relatively weakly open. Since $x \in S_X \cap U$ and X has the DD2P, there is $u \in U$ such that $\|x - u\| \geq 2 - \varepsilon$. Take $w = (u, 0)$. Then $w \in W$, because

$$\|w\|_N = \|(u,0)\|_N = N(\|u\|, 0) = \|x\| \leq 1,$$

hence $w \in B_Z$, and for every $i \in \{1, \ldots, n\}$

$$|z_i^*(z-w)| = |x_i^*(x-u)| < 1.$$

Notice that

$$\|z-w\|_N = \|(x-u, 0)\|_N = N(\|x-u\|, 0) = \|x-u\| \geq 2 - \varepsilon.$$

$\square$

Recall that no ℓ_p-sum for $1 < p < \infty$ can have the SD2P (see Theorem 1.4.17). Since the DSD2P implies the SD2P, it is evident that such ℓ_p-sums cannot have the DSD2P either.

Theorem 2.2.3. *Let X and Y be Banach spaces. The Banach space $X \oplus_p Y$ fails the DSD2P for every $1 < p < \infty$.*

The behaviour of the DSD2P by taking ℓ_1- and ℓ_∞-sums, was studied in [BLR18]. In the ℓ_∞-case the DSD2P was shown to be stable.

Theorem 2.2.4 (see [BLR18, Proposition 3.6 and Theorem 3.7])**.** *Let X and Y be Banach spaces. Then $X \oplus_\infty Y$ has the DSD2P if and only if X and Y have the DSD2P.*

For the ℓ_1-sum, however, the behaviour was described in the direction from the sum to the summands.

Proposition 2.2.5 (see [BLR18, Proposition 3.6])**.** *Let X and Y be Banach spaces. If $X \oplus_1 Y$ has the DSD2P, then X and Y have the DSD2P.*

We completed this research in [HPP16] with the other direction.

Theorem 2.2.6. *Let X and Y be Banach spaces with the DSD2P. Then $X \oplus_1 Y$ has the DSD2P.*

Proof. Set $Z = X \oplus_1 Y$, and let $n \in \mathbb{N}$, let $W_1, \dots, W_n$ be relatively weakly open subsets of B_Z, let $\lambda_1, \dots, \lambda_n \in [0,1]$ satisfy $\sum_{i=1}^n \lambda_i = 1$, and let $z = \sum_{i=1}^n \lambda_i z_i$ where $z_i = (x_i, y_i) \in S_Z \cap W_i$. (The last inclusion holds because of Lemma 2.1.9.) We must find a $w = (u, v) \in \sum_{i=1}^n \lambda_i W_i$ so that $\|z - w\| \geq \|z\| + 1 - \varepsilon$, i.e. putting $x = \sum_{i=1}^n \lambda_i x_i$ and $y = \sum_{i=1}^n \lambda_i y_i$ (now one has $z = (x, y)$),

$$\|x - u\| + \|y - v\| \geq \|x\| + \|y\| + 1 - \varepsilon.$$

For every $i \in \{1, \dots, n\}$, putting

$$\hat{x}_i = \begin{cases} x_i/\|x_i\|, & \text{if} \quad x_i \neq 0, \\ 0, & \text{if} \quad x_i = 0, \end{cases} \quad \text{and} \quad \hat{y}_i = \begin{cases} y_i/\|y_i\|, & \text{if} \quad y_i \neq 0, \\ 0, & \text{if} \quad y_i = 0, \end{cases}$$

there are relatively weakly open neighbourhoods $U_i \subset B_X$ and $V_i \subset B_Y$ of $\hat{x}_i$ and $\hat{y}_i$, respectively, such that $\big(\|x_i\| U_i\big) \times \big(\|y_i\| V_i\big) \subset W_i$. Indeed, letting $m \in \mathbb{N}$, $z_j^* = (x_j^*, y_j^*) \in S_{Z^*}$, $j = 1, \dots, m$, and $\delta > 0$ such that

$$W_i \supset \{w \in B_Z \colon |z_j^*(w) - z_j^*(z_i)| < \delta, j = 1, \dots, m\},$$

and defining

$$U_i = \{u \in B_X \colon |x_j^*(u) - x_j^*(\hat{x}_i)| < \delta, j = 1, \ldots, m\},$$

$$V_i = \{v \in B_Y \colon |y_j^*(v) - y_j^*(\hat{y}_i)| < \delta, j = 1, \ldots, m\},$$

one has, whenever $u \in U_i$ and $v \in V_i$, for every $j \in \{1, \ldots, m\}$,

$$\begin{aligned}
\Big|z_j^*\Big(\|x_i\|u, \|y_i\|v\Big) - z_j^*(z_i)\Big| &= \Big|z_j^*\Big(\|x_i\|u, \|y_i\|v\Big) - z_j^*(x_i, y_i)\Big| \\
&= \Big|x_j^*\Big(\|x_i\|u\Big) + y_j^*\Big(\|y_i\|v\Big) - x_j^*(x_i) - y_j^*(y_i)\Big| \\
&= \Big|x_j^*\Big(\|x_i\|u\Big) + y_j^*\Big(\|y_i\|v\Big) - x_j^*(\|x_i\|\hat{x}_i) - y_j^*(\|y_i\|\hat{y}_i)\Big| \\
&= \Big|\|x_i\|x_j^*(u - \hat{x}_i) + \|y_i\|y_j^*(v - \hat{y}_i)\Big| \\
&\le \|x_i\||x_j^*(u - \hat{x}_i)| + \|y_i\||y_j^*(v - \hat{y}_i)| \\
&< \Big(\|x_i\| + \|y_i\|\Big)\delta \\
&= \|z_i\|\delta = \delta.
\end{aligned}$$

Put

$$\alpha = \sum_{i=1}^n \lambda_i \|x_i\| \quad \text{and} \quad \beta = \sum_{i=1}^n \lambda_i \|y_i\|.$$

Notice that

$$\alpha + \beta = \sum_{i=1}^n \lambda_i \Big(\|x_i\| + \|y_i\|\Big) = \sum_{i=1}^n \lambda_i \|z_i\| = \sum_{i=1}^n \lambda_i = 1.$$

We only consider the case when both $\alpha \neq 0$ and $\beta \neq 0$. (The case when $\alpha = 0$ or $\beta = 0$ can be handled similarly and is, in fact, simpler.)

For every $i \in \{1, \ldots, n\}$, letting

$$\alpha_i = \frac{\lambda_i \|x_i\|}{\alpha} \quad \text{and} \quad \beta_i = \frac{\lambda_i \|y_i\|}{\beta},$$

one has $\alpha_i, \beta_i \in [0, 1]$, and $\sum_{i=1}^n \alpha_i = \sum_{i=1}^n \beta_i = 1$. Since X and Y have the DSD2P, observing that

$$\frac{x}{\alpha} = \sum_{i=1}^n \frac{\lambda_i \|x_i\|}{\alpha} \hat{x}_i \in \sum_{i=1}^n \alpha_i U_i \quad \text{and} \quad \frac{y}{\beta} = \sum_{i=1}^n \frac{\lambda_i \|y_i\|}{\beta} \hat{y}_i \in \sum_{i=1}^n \beta_i V_i,$$

there are $u_0 \in \sum_{i=1}^n \alpha_i U_i$ and $v_0 \in \sum_{i=1}^n \beta_i U_i$ such that

$$\left\|\frac{x}{\alpha} - u_0\right\| \ge \frac{1}{\alpha}\|x\| + 1 - \varepsilon \quad \text{and} \quad \left\|\frac{y}{\beta} - v_0\right\| \ge \frac{1}{\beta}\|y\| + 1 - \varepsilon.$$

Finally, putting

$$u = \alpha u_0 \in \sum_{i=1}^{n} \alpha\alpha_i U_i = \sum_{i=1}^{n} \lambda_i \|x_i\| U_i,$$

$$v = \beta v_0 \in \sum_{i=1}^{n} \beta\beta_i V_i = \sum_{i=1}^{n} \lambda_i \|y_i\| V_i,$$

one has

$$(u, v) \in \sum_{i=1}^{n} \lambda_i \Big(\big(\|x_i\| U_i \big) \times \big(\|y_i\| V_i \big) \Big) \subset \sum_{i=1}^{n} \lambda_i W_i$$

and

$$\|x - u\| + \|y - v\| \geq \|x\| + \|y\| + (\alpha + \beta)(1 - \varepsilon) = \|x\| + \|y\| + 1 - \varepsilon,$$

as desired. □

Theorems 2.2.4, 2.2.6, and Proposition 2.2.5 provide a complete picture regarding the stability of the DSD2P taking ℓ_p-sums. By [HLN18] no other absolute sum can have the DSD2P.

Proposition 2.2.7 (see [HLN18, Corollary 3.8]). *Let X and Y be Banach spaces and N an absolute normalised norm on $\mathbb{R}^2$. If $Z = X \oplus_N Y$ has the DSD2P, then either $Z = X \oplus_1 Y$ or $Z = X \oplus_\infty Y$.*

2.3 Diametral diameter two properties in subspaces and M-ideals

The aim of this section is to complement the existing inheritance results for the diametral diameter two properties by showing that the DLD2P and the DD2P pass from the space to its M-ideals. These results are new to the knowledge of the author.

In [BLR18], it was shown that the DD2P and the DLD2P pass down from the space to finite co-dimensional subspaces, similarly to the regular diameter two properties (see [BLR16]).

Proposition 2.3.1 (see [BLR18, Theorems 2.14 and 3.8]). *Let X be a Banach space.*

(a) *Assume that X has the DD2P. If Y is a closed subspace of X such that X/Y is finite-dimensional, then Y has the DD2P.*

(b) *Assume that X has the DSD2P. If X has the DSD2P and X/Y is strongly regular, then Y also has the DSD2P.*

Remark 2.3.2. Recall that a Banach space X is *strongly regular* if every non-empty closed convex bounded subset of X has finite convex combinations of slices (equivalently, relatively weakly open subsets) with arbitrarily small diameter (see, e.g. [BLR18]). Obviously, every finite-dimensional subspace is strongly regular.

It is known that all diametral diameter two properties pass down from the second dual (see Proposition 2.1.6). More generally, all three diametral diameter two properties and the Daugavet property are inherited by certain subspaces called almost isometric ideals (see [ALN14], [ALNT16], and [BLR18]).

Furthermore, the Daugavet property is inherited by M-ideals, and assuming that both the M-ideal Y in a Banach space X and the quotient space X/Y share the Daugavet property, we obtain the Daugavet property for X as well (see [KSSW00]). Note that if Y is an M-ideal in X, then $X\backslash Y$ need not have the Daugavet property. For example, $Y = \{f \in C[0,1]\colon f(0) = 0\}$ is an M-ideal in $X = C[0,1]$ but in this case $X\backslash Y$ is one dimensional, therefore, fails even the LD2P.

It is known that the regular diameter two properties lift from an M-ideal to the superspace (see [HL14]). We will now show that the DLD2P and the DD2P behave differently, they pass down from the space to its M-ideals, similarly to the Daugavet property.

Definition 2.3.3 (see, e.g. [HWW93])**.** Let X be a Banach space and Y a closed subspace of X. Then Y is said to be an *M-ideal* in X if there exists a linear projection $P\colon X^* \to X^*$ such that $\|x^*\| = \|Px^*\| + \|x^* - Px^*\|$ for every $x^* \in X^*$ and $\ker P = Y^\perp$, where

$$Y^\perp = \{x^* \in X^*\colon x^*(y) = 0 \quad \text{for all} \quad y \in Y\}.$$

If Y is an M-ideal in X and P is a linear projection from Definition 2.3.3, then $\operatorname{ran} P$ is isometric to Y^* and we shall identify them by identifying Px^* and $x^*|_Y$ for every $x^* \in X^*$.

Proposition 2.3.4. *Let X be a Banach space. If X has the DLD2P, then every M-ideal in X also has the DLD2P.*

Proof. Assume that X has the DLD2P. Let Y be an M-ideal in X and let $P\colon X^* \to X^*$ be a linear projection as in Definition 2.3.3. We will show that Y has the DLD2P.

Fix $\varepsilon > 0$ and $y^* \in S_{Y^*}$. Let $y \in S_Y \cap S(B_Y, y^*, \varepsilon)$. Pick $\delta > 0$ such that $y^*(y) > 1 - \varepsilon + \delta$. Consider the slice $S(B_X, Px^*, \varepsilon - \delta)$, where $x^* \in X^*$

is a norm-preserving extension of y^*. Since X has the DLD2P and $y \in S(B_X, Px^*, \varepsilon - \delta)$, we can find $x_0 \in S(B_X, Px^*, \varepsilon - \delta)$ such that

$$\|y - x_0\| \geq 2 - \frac{\varepsilon}{3}.$$

For some $z^* \in S_{X^*}$ we have $z^*(y - x_0) \geq 2 - \varepsilon/3$, which gives $z^*(y) \geq 1 - \varepsilon/3$, and so

$$\|z^*|_Y\| \geq 1 - \frac{\varepsilon}{3} \quad \text{and} \quad \|z^* - Pz^*\| = \|z^*\| - \|z^*|_Y\| \leq \frac{\varepsilon}{3}.$$

According to [Wer94, Propostion 2.3], since Y is an M-ideal in X, there exists $y_0 \in B_Y$ such that

$$|Px^*(x_0 - y_0)| < \delta \quad \text{and} \quad |Pz^*(x_0 - y_0)| < \frac{\varepsilon}{3}.$$

Then $y_0 \in S(B_Y, y^*, \varepsilon)$, because

$$\begin{aligned} y^*(y_0) &= (Px^*)(y_0) \\ &= (Px^*)(x_0) - (Px^*)(x_0 - y_0) \\ &> 1 - (\varepsilon - \delta) - \delta \\ &= 1 - \varepsilon. \end{aligned}$$

Notice that

$$\begin{aligned} \|y - y_0\| &\geq z^*(y - y_0) \\ &= z^*(y - x_0) + Pz^*(x_0 - y_0) + (z^* - Pz^*)(x_0) \\ &\geq 2 - \frac{\varepsilon}{3} - \frac{\varepsilon}{3} - \frac{\varepsilon}{3} \\ &= 2 - \varepsilon. \end{aligned}$$

Therefore, Y has the DLD2P. $\square$

Proposition 2.3.5. *Let X be a Banach space. If X has the DD2P, then every M-ideal in X also has the DD2P.*

Proof. Assume that X has the DD2P. Let Y be an M-ideal in X and let $P\colon X^* \to X^*$ be a linear projection as in Definition 2.3.3. We will show that Y has the DD2P.

Fix $\varepsilon > 0$, $y \in S_Y$, $n \in \mathbb{N}$, and $y_1^*, \ldots, y_n^* \in S_{Y^*}$. Consider relatively weakly open subsets

$$U = \Big\{z \in B_Y \colon |y_i^*(y - z)| < \varepsilon \quad \forall i \in \{1, \ldots, n\}\Big\},$$

$$V = \Big\{z \in B_X \colon |Px_i^*(y-z)| < \frac{\varepsilon}{2} \quad \forall i \in \{1,\dots,n\}\Big\},$$

where $x_i^* \in X^*$ is a norm-preserving extension of y_i^* for every $i \in \{1,\dots,n\}$. Since X has the DD2P, we can find $x_0 \in V$ such that

$$\|y - x_0\| \geq 2 - \varepsilon/3.$$

For some $x^* \in S_{X^*}$ we have $x^*(y-x_0) \geq 2-\varepsilon/3$, which gives $x^*(y) \geq 1-\varepsilon/3$, and so

$$\|x^*|_Y\| \geq 1 - \frac{\varepsilon}{3} \quad \text{and} \quad \|x^* - Px^*\| = \|x^*\| - \|x^*|_Y\| \leq \frac{\varepsilon}{3}.$$

According to [Wer94, Propostion 2.3], since Y is an M-ideal in X, there exists $y_0 \in B_Y$ such that for every $i \in \{1,\dots,n\}$

$$|Px^*(x_0 - y_0)| < \frac{\varepsilon}{3} \quad \text{and} \quad |Px_i^*(x_0 - y_0)| < \frac{\varepsilon}{2}.$$

Then $y_0 \in U$ because for every $i \in \{1,\dots,n\}$ we have

$$\begin{aligned} |y_i^*(y-y_0)| &\leq |(Px_i^*)(y-x_0)| + |(Px_i^*)(x_0-y_0)| \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \end{aligned}$$

Notice that

$$\begin{aligned} \|y - y_0\| &\geq x^*(y-y_0) \\ &= x^*(y-x_0) + Px^*(x_0-y_0) + (x^* - Px^*)(x_0) \\ &\geq 2 - \frac{\varepsilon}{3} - \frac{\varepsilon}{3} - \frac{\varepsilon}{3} \\ &= 2 - \varepsilon. \end{aligned}$$

Therefore, Y has the DD2P. $\square$

So far, it is not clear whether the DSD2P is inherited by M-ideals. It is also not known whether any of the diametral diameter two properties lifts from an M-ideal to the whole space.

2.4 Convex diametral local diameter two property

In [AHLP20], we introduced and studied the property that was inspired by the description of the DLD2P via the set of almost diametral points (see

Proposition 2.1.7): it is known from that characterisation that a Banach space has the DLD2P if and only if every element of the unit sphere is contained in the closed convex hull of its almost diametral points. In this section, we look at the Banach spaces where there are enough elements of the unit sphere satisfying this condition so that the closed convex hull of their collection equals to the whole unit ball.

For a Banach space X set Δ_X to be the collection of all those unit sphere elements x such that $x \in \overline{\operatorname{conv}}\, \Delta_\varepsilon^X(x)$ for every $\varepsilon > 0$, where

$$\Delta_\varepsilon^X(x) = \{y \in B_X \colon \|x - y\| \geq 2 - \varepsilon\}.$$

We write Δ and $\Delta_\varepsilon(x)$ instead of Δ_X and $\Delta_\varepsilon^X(x)$, respectively, when no confusion can arise.

Definition 2.4.1. Let X be a Banach space. If $B_X = \overline{\operatorname{conv}}(\Delta)$, then we say that X has the *convex diametral local diameter two property (convex DLD2P)*.

According to Proposition 2.1.7 part (c), a Banach space X has the DLD2P if and only if $\Delta = S_X$. It immediately follows that the DLD2P implies the convex DLD2P. It turns out that the convex DLD2P implies the LD2P.

Proposition 2.4.2. *Let X be a Banach space. If X has the convex DLD2P, then X has the LD2P.*

Proof. Let $x^* \in S_{X^*}$, $\varepsilon > 0$, and consider the slice $S(B_X, x^*, \varepsilon)$. Pick some $\hat{x} \in S(B_X, x^*, \varepsilon/4)$. Choose $(x_i)_{i=1}^n \subset \Delta$ and a convex combination $x = \sum_{i=1}^n \lambda_i x_i$ with $\|x - \hat{x}\| < \varepsilon/4$. Now at least one of the x_i-s must be in $S(B_X, x^*, \varepsilon/2)$ otherwise

$$x^*(x) = \sum_{i=1}^n \lambda_i x^*(x_i) < \sum_{i=1}^n \lambda_i (1 - \varepsilon/2) < 1 - \varepsilon/2,$$

which contradicts the fact that $\hat{x} \in S(B_X, x^*, \varepsilon/4)$ and $\|\hat{x} - x\| < \varepsilon/4$. Now let x_k be one of the x_i-s which are in $S(B_X, x^*, \varepsilon/2)$ and use the same idea as above to produce some $y \in \Delta_\varepsilon(x_k)$ such that $y \in S(B_X, x^*, \varepsilon)$. Since $x_k \in S(B_X, x^*, \varepsilon/2) \subset S(B_X, x^*, \varepsilon)$ and $\|x_k - y\| > 2 - \varepsilon$ we are done. $\square$

The following result provides a class of Banach spaces with the convex DLD2P.

Proposition 2.4.3. *If K is an infinite compact Hausdorff space, then $C(K)$ has the convex DLD2P.*

Proof. We only need to show that $S_{C(K)} \subset \overline{\operatorname{conv}}\,\Delta$. Let $f \in C(K)$ with $\|f\| = 1$. If $|f(x)| = 1$ for some limit point of K, then $f \in \Delta$ by Theorem 3.2.4. Assume that $|f(x)| < 1$ for every limit point of K and let x_0 be a limit point of K.

Let $\varepsilon > 0$ and choose a neighbourhood U of x_0 such that $|f(x)-f(x_0)| < \varepsilon$ for every $x \in U$. We use Urysohn's Lemma to find a function $\eta\colon K \to [0,1]$ such that $\eta(x_0) = 1$ and $\eta = 0$ on $K \setminus U$. Define

$$\begin{aligned} f^+(x) &= (1-\eta(x))f(x) + \eta(x)(1), \\ f^-(x) &= (1-\eta(x))f(x) + \eta(x)(-1). \end{aligned}$$

Then $f^\pm \in B_{C(K)}$ and both are in Δ by Theorem 3.2.4. Let $\lambda = \big(1+f(x_0)\big)/2$ and consider

$$g(x) = \lambda f^+(x) + (1-\lambda)f^-(x).$$

Then

$$g(x) = \begin{cases} f(x), & x \in K \setminus U, \\ (1-\eta(x))f(x) + \eta(x)f(x_0), & x \in U. \end{cases}$$

We get

$$\|g - f\| \le \max_{x\in U} |\eta(x)(f(x) - f(x_0))| < \varepsilon.$$

Since $\varepsilon > 0$ was arbitrary we get that $f \in \overline{\operatorname{conv}}\,\Delta$. □

Corollary 2.4.4. *Both $c = C([0,\omega])$ and $\ell_\infty = C(\beta\mathbb{N})$ have the convex DLD2P.*

Recall that due to Example 2.1.4 the spaces c_0, c, and ℓ_∞ fail to have the DLD2P. However, c_0 even fails the convex DLD2P, because of the following observation.

Example 2.4.5. For c_0 we have $\Delta = \emptyset$. Indeed, assume to the contrary that $\Delta \ne \emptyset$. Let $x = (\xi_i)_{i=1}^\infty \in S_{c_0}$ be such that $x \in \overline{\operatorname{conv}}\,\Delta_\varepsilon(x)$ for every $\varepsilon > 0$. Since $x \in c_0$, there are only finite number of indices such that $|\xi_i| = 1$, say $n \in \mathbb{N}$. We may assume without loss of generality that these are the first n terms of x, and find $\varepsilon > 0$ such that $|\xi_i| < 1-\varepsilon$ for every $i > n$. Since $x \in \Delta$, there exist $y_k = (\eta_i^k) \in B_{c_0}$, $k \in \{1,\ldots,m\}$ such that

$$\left\| x - \frac{1}{m}\sum_{k=1}^m y_k \right\| \le \frac{\varepsilon}{2} \tag{2.4.1}$$

and

$$\|x - y_k\| \ge 2 - \frac{\varepsilon}{2} \quad \text{for every} \quad k \in \{1,\ldots,m\}. \tag{2.4.2}$$

It follows from (2.4.1) that that $\operatorname{sgn}\xi_i = \operatorname{sgn}\eta_i^k$, for every $i \in \{1,\dots,n\}$ and every $k \in \{1,\dots,m\}$. Consequently, $|\xi_i - \eta_i^k| \leq 1$ for every $i \in \{1,\dots,n\}$ and every $k \in \{1,\dots,m\}$. However, from (2.4.2) we get that for every $k \in \{1,\dots,m\}$ there is $i_k \in \{1,\dots,n\}$ such that $|\xi_{i_k} - \eta_{i_k}^k| \geq 2 - \varepsilon/2$, a contradiction.

Remark 2.4.6. Unlike the DLD2P and the DD2P, the convex DLD2P is not inherited by M-ideals, since c_0 is an M-ideal in ℓ_∞.

Remark 2.4.7. It is known that the LD2P is inherited from the bidual (see [Lan15, Corollary 2.15]). In contrast, it can be said by the previous observations, that the convex DLD2P is not inherited from the bidual. Moreover, the convex DLD2P is not inherited by subspaces of codimension 1, since c_0 is of codimension 1 in c.

In conclusion we have that the convex DLD2P is a new diameter two property, different from the ones observed so far, i.e. the following implications hold whereas the reverse implications do not

$$\text{DLD2P} \Rightarrow \text{convex DLD2P} \Rightarrow \text{LD2P}.$$

We continue with another example of a class of Banach spaces that has the convex DLD2P, more precisely, we will show that Müntz spaces have the convex DLD2P.

Definition 2.4.8. Let $\Lambda = (\lambda_n)_{n=0}^\infty$ be an increasing sequence of non-negative real numbers

$$0 = \lambda_0 < \lambda_1 < \cdots < \lambda_n < \cdots$$

such that $\sum_{i=1}^\infty 1/\lambda_i < \infty$. Then $M(\Lambda) := \overline{\operatorname{span}}\{t^{\lambda_n}\}_{n=0}^\infty \subset C[0,1]$ is called the Müntz space associated with Λ.

We will sometimes need to exclude the constants and consider the subspace $M_0(\Lambda) := \overline{\operatorname{span}}\{t^{\lambda_n}\}_{n=1}^\infty$ of $M(\Lambda)$.

Theorem 2.4.9. *Let $X = M(\Lambda)$ or $X = M_0(\Lambda)$ be a Müntz space. Then X has the convex DLD2P.*

Proof. It is enough to show that $S_X \subset \overline{\operatorname{conv}}\,\Delta$. Since $P := \operatorname{span}\{t^{\lambda_n}\}$ is dense in X, it is enough to show that if $f \in B_P$ with $\|f\| = 1 - s$ for some $0 < s < 1$, then $f \in \operatorname{conv}\Delta$. To this end, given $n \in \mathbb{N}$ we define

$$f_n^+(x) = f(x) + (1 - f(1))x^{\lambda_n}$$

and

$$f_n^-(x) = f(x) - (1 + f(1))x^{\lambda_n}.$$

From Proposition 3.2.10 we see that $f_n^{\pm}$ are candidates for being elements of the set Δ since

$$f_n^{\pm}(1) = f(1) \pm (1 \mp f(1)) = \pm 1.$$

If we define $\mu = (f(1)+1)/2$, that is, $2\mu - 1 = f(1)$, we have a convex combination

$$\mu f_n^+(x) + (1-\mu) f_n^-(x) = f(x) + \Big(2\mu - 1 - f(1)\Big)x^{\lambda_n} = f(x).$$

We need to show that when n is large enough we have $f_n^{\pm} \in S_P$.

Since $f \in P$ we can write

$$f(x) = \sum_{k=0}^{m} a_k x^{\lambda_k}.$$

Now, f, f', and f'' are all generalised polynomials so by Descartes' rule of signs, see, e.g. [Jam06, Theorem 3.1], they only have a finite number of zeros on $(0,1]$. Hence, there exists $t_0 \in (0,1)$ such that neither f' nor f'' changes sign on $(t_0,1)$. Without loss of generality we may assume that $f' < 0$ on $(t_0,1)$. (If $f' > 0$ on $(t_0,1)$ we consider $-f$.)

There exists N such that

$$t_0^{\lambda_n} < s/2 \quad \text{for } n > N. \tag{2.4.3}$$

For $n > N$ we get

$$|f_n^-(x)| \leq 1 - s + (1 + f(1))s/2 \leq 1$$

on $[0,t_0]$ and on $[t_0,1]$ we have

$$\frac{d}{dx}(f_n^-(x)) = f'(x) - \lambda_n(1+f(1))x^{\lambda_n - 1} < 0.$$

We have $|f_n^-(x)| \leq 1$ in both endpoints of $[t_0,1]$. Hence $\|f_n^-\| \leq 1$.

It remains to find $n > N$ such that also $f_n^+ \in S_P$. We consider two cases.

Case I: Assume there exists $0 < t_0 < 1$ such that $f' < 0$ and $f'' > 0$ on $(t_0,1)$. For $n > N$ we have $d^2/dx^2(f_n^+) > 0$ on $(t_0,1)$, hence f_n^+ is convex on $[t_0,1]$ and (by using (2.4.3))

$$\|f_n^+\| \leq \max(f_n^+(t_0), f_n^+(1)) \leq \max(1 - s + (1 - f(1))t_0^{\lambda_n}, 1) \leq 1$$

since also $f_n^+(x) > f(x) \geq -1$ for all $x \in [0,1]$.

Case II: Assume there exists $0 < t_0 < 1$ such that $f' < 0$ and $f'' < 0$ on $[t_0,1]$.

Let $\delta = f(t_0) - f(1) > 0$. Define

$$t_n = \sqrt[\lambda_n]{1 - \frac{\delta}{1 - f(1)}},$$

that is

$$t_n^{\lambda_n} = \frac{1 - f(1) - \delta}{1 - f(1)}$$

Note that $t_n \to 1$.

Write $g_n(x) = (1 - f(1))x^{\lambda_n}$. Then $g_n'(x) = (1 - f(1))\lambda_n x^{\lambda_n - 1}$ and

$$\begin{aligned} g_n'(t_n) &= (1 - f(1))\lambda_n \frac{1 - f(1) - \delta}{1 - f(1)} \left(\frac{1 - f(1) - \delta}{1 - f(1)} \right)^{-1/\lambda_n} \\ &= \lambda_n (1 - f(1) - \delta) \left(\frac{1 - f(1) - \delta}{1 - f(1)} \right)^{-1/\lambda_n}. \end{aligned}$$

Note that $g_n'(t_n) \to \infty$ (since we assume that $\sum_{n=1}^{\infty} \lambda_n^{-1} < \infty$). Let $M = \max_{x \in [t_0, 1]} |f'(x)|$. Choose $n > N$ such that $t_0 < t_n < 1$ and

$$g_n'(t_n) > M.$$

Then for $x \in [t_n, 1]$ we have

$$\frac{d}{dx}(f_n^+(x)) = f'(x) + \lambda_n (1 - f(1)) x^{\lambda_n - 1} > -M + g_n'(t_n) > 0$$

hence $f_n^+(x) \leq f_n^+(1)$ on $[t_n, 1]$.

For $x \in [t_0, t_n]$ we get

$$\begin{aligned} f_n^+(x) &= f(x) + g_n(x) \\ &\leq f(1) + \delta + (1 - f(1)) t_n^{\lambda_n} \\ &= f(1) + \delta + (1 - f(1) - \delta) \\ &\leq 1. \end{aligned}$$

While on $[0, t_0]$ we have, by using (2.4.3),

$$|f_n^+(x)| \leq \|f\| + 2 \cdot s/2 \leq 1.$$

Hence, $\|f_n^+\| \leq 1$. $\square$

Recall that the convex DLD2P lies in between the DLD2P and the LD2P which are both stable by taking absolute sums. It turns out that the convex DLD2P moving from summands to the absolute sum is also stable by taking absolute sums. In order to have ℓ_∞-sum with the convex DLD2P, it is enough to assume that only one summand has the convex DLD2P. The stability in the direction from the absolute sum to summands is not known for any absolute sum.

Theorem 2.4.10. *Let N be an absolute normalised norm on $\mathbb{R}^2$. If X and Y have the convex DLD2P, then $X \oplus_N Y$ has the convex DLD2P.*

Proof. Assume that X and Y are Banach spaces with the convex DLD2P. Denote $Z = X \oplus_N Y$. Now let $(x, y) \in S_Z$. We will show that $(x, y) \in \overline{\operatorname{conv}}\, \Delta_Z$.

Let $\delta > 0$. First consider the case $x \neq 0$ and $y \neq 0$. Then $x/\|x\| \in \overline{\operatorname{conv}}\, \Delta_X$ and $y/\|y\| \in \overline{\operatorname{conv}}\, \Delta_Y$ by the assumption, hence there are $x_1, \ldots, x_n \in \Delta_X$ and $y_1, \ldots, y_n \in \Delta_Y$ such that

$$\left\| \frac{x}{\|x\|} - \frac{1}{n} \sum_{i=1}^{n} x_i \right\| < \delta \quad \text{and} \quad \left\| \frac{y}{\|y\|} - \frac{1}{n} \sum_{i=1}^{n} y_i \right\| < \delta.$$

By Proposition 3.4.1 we have $(\|x\| x_i, \|y\| y_i) \in \Delta_Z$. All that remains is to note that

$$\begin{aligned}
&\left\| (x, y) - \frac{1}{n} \sum_{i=1}^{n} \Big(\|x\| x_i, \|y\| y_i \Big) \right\|_N \\
&= N\left(\|x\| \left\| \frac{x}{\|x\|} - \frac{1}{n} \sum_{i=1}^{n} x_i \right\|, \|y\| \left\| \frac{y}{\|y\|} - \frac{1}{n} \sum_{i=1}^{n} y_i \right\| \right) \\
&\leq N\Big(\delta \|x\|, \delta \|y\| \Big) \\
&= \delta N\Big(\|x\|, \|y\| \Big) = \delta.
\end{aligned}$$

Now consider the case where $y = 0$ (a similar argument holds for the case $x = 0$). We have

$$\|(x, 0)\|_N = N(\|x\|, 0) = \|x\|,$$

so that $(x, 0) \in \overline{\operatorname{conv}}\, \Delta_Z$ follows from $x \in \overline{\operatorname{conv}}\, \Delta_X$ since the claim above shows that $(x_i, 0) \in \Delta_Z$ when $x_i \in \Delta_X$. □

Proposition 2.4.11. *Let X and Y be Banach spaces. If X has the convex DLD2P, then $X \oplus_\infty Y$ has the convex DLD2P.*

Proof. Assume that X has the convex DLD2P. Denote $Z = X \oplus_\infty Y$. Let $(x, y) \in S_Z$. We will show that $(x, y) \in \overline{\operatorname{conv}}\, \Delta_Z$.

Let $\delta > 0$. Since X has the convex DLD2P, we have that $x \in \overline{\operatorname{conv}}\, \Delta_X = B_X$. Hence, there are $x_1, \ldots, x_n \in \Delta_X$ such that $\|x - 1/n \sum_{i=1}^{n} x_i\| < \delta$.

Let $\varepsilon, \gamma > 0$. Since $x_i \in \overline{\operatorname{conv}}\, \Delta_\varepsilon(x_i)$ for every $i \in \{1, \ldots, n\}$, there are $x_i^j \in \Delta_\varepsilon(x_i)$ such that

$$\left\| x_i - \frac{1}{m} \sum_{j=1}^{m} x_i^j \right\| < \gamma \quad \text{and} \quad \|x_i - x_i^j\| \geq 2 - \varepsilon.$$

Therefore, we have firstly, that

$$\|(x_i, y) - (x_i^j, y)\|_\infty = \max\{\|x_i - x_i^j\|, 0\} \geq 2 - \varepsilon,$$

and secondly, that

$$\left\|(x_i, y) - \frac{1}{m}\sum_{j=1}^{m}(x_i^j, y)\right\|_\infty = \max\left\{\left\|x_i - \frac{1}{m}\sum_{j=1}^{m} x_i^j\right\|, 0\right\} < \gamma.$$

As a result, $(x_i, y) \in \Delta_Z$ for every $i \in \{1, \ldots, n\}$. Finally, it is now obvious that

$$\left\|(x, y) - \frac{1}{n}\sum_{i=1}^{n}(x_i, y)\right\|_\infty = \max\left\{\left\|x - \frac{1}{n}\sum_{j=1}^{n} x_i\right\|, 0\right\} < \delta,$$

hence $(x, y) \in \overline{\operatorname{conv}}\, \Delta_Z$. □

2.5 Perspectives

In the following we list a few possible perspectives for the future research. The main open problems in the world of the diametral diameter two properties are of course the following two.

PROBLEM 1. *Does there exist a Banach space with the DSD2P that fails the Daugavet property?*

PROBLEM 2. *Does there exist a Banach space with the DLD2P that fails the DD2P?*

Regarding the w^*-versions of the diametral diameter two properties, it is known that a Banach space X has the DLD2P if and only if the dual space X^* has the w^*-DLD2P. Similar results for the stronger diametral diameter two properties are still unknown (see [BLR18, Questions 4.2 and 4.3]).

PROBLEM 3. *Let X be a Banach space. Is it true that X has the DD2P (respectively, DSD2P) if and only if the dual X^* has the w^*-DD2P (respectively, w^*-DSD2P)?*

It is known that the DLD2P and the DD2P have a full characterisation with rank-1 projections.

PROBLEM 4. *Does the DSD2P admit to a characterisation with finite-rank projections?*

It is known that the convex DLD2P falls strictly between the DLD2P and the LD2P, which have characterisations with slices.

Problem 5. *Is there a characterisation for the convex DLD2P with slices?*

The Daugavet property and the DLD2P have a characterisation given by describing all elements of the unit sphere (see Propositions 1.4.5 (*iii*) and 2.1.7 (*iii*), respectively).

Problem 6. *Do the DD2P and the DSD2P have a characterisation by describing all elements of the unit sphere?*

The Daugavet property, the DD2P and the DLD2P have been proved to pass on to M-ideals.

Problem 7. *Does the DSD2P also pass to the M-ideals?*

Additionally, if Y is an M-ideal in a Banach space X such that Y and the quotient space X/Y share the Daugavet property, then X also has the Daugavet property.

Problem 8. *Let X be a Banach space and Y an M-ideal in X. Assume Y and X/Y share the DSD2P (respectively, DD2P, DLD2P, convex DLD2P). Then does X have the DSD2P (respectively, DD2P, DLD2P, convex DLD2P)?*

It is known that the convex DLD2P passes on from component spaces to every absolute sum. How about the reverse?

Problem 9. *Provided the absolute sum has the convex DLD2P, do the summands have the convex DLD2P?*

Chapter 3

Daugavet- and Δ-points

Inspired by the characterisations of the Daugavet property and the DLD2P due to [Wer01] and [IK04], we introduce Daugavet- and Δ-points. We explore these notions in different classes of spaces and show that, despite the general difference, Daugavet- and Δ-points coincide in $L_1(\mu)$ spaces and their preduals, $C(K)$ spaces for infinite compact Hausdorff space K, and a wide class of Müntz spaces. A thorough treatment of the stability of Daugavet- and Δ-points by taking absolute sums is also presented. We show that all absolute sums can be divided into two classes: absolute sums equipped with A-octahedral norms (see Definition 3.3.3) and absolute sums equipped with norms with property (α) (see Definition 3.3.4); the former absolute sums provide positive stability results for Daugavet-points whereas the latter absolute sums can have no Daugavet-points. Δ-points, however, behave well under all absolute sums. Unexpectedly, the ℓ_∞-sum can have Δ-points even if the component spaces have no Δ-points. These results were obtained in [AHLP20] and [HPV].

3.1 Definitions and examples

Daugavet- and Δ-points are both relatively new notions. Their origin can be tracked back to [Wer01, Corollary 2.3], which states that a Banach space has the Daugavet property if and only if every element on the unit sphere satisfies that the closed convex hull of its almost diametral points equals the unit ball (see also Proposition 1.4.5 part (*iii*)). Recall that the DLD2P admits a similar characterisation, i.e. every element of the unit sphere is contained in the closed convex hull of its almost diametral points (see Proposition 2.1.7 part (*iii*)). The notions of Daugavet-point and Δ-point arise naturally from these characterisations.

Definition 3.1.1. Let X be a Banach space. We say that $x \in S_X$ is a

(*a*) *Daugavet-point* if $B_X = \overline{\operatorname{conv}}\, \Delta_\varepsilon(x)$ for every $\varepsilon > 0$;

(*b*) Δ-*point* if $x \in \overline{\operatorname{conv}}\, \Delta_\varepsilon(x)$ for every $\varepsilon > 0$,

where

$$\Delta_\varepsilon(x) = \{y \in B_X \colon \|x - y\| \geq 2 - \varepsilon\}.$$

Remark 3.1.2. Note that a Banach space has the DLD2P if and only if every element of the unit sphere is a Δ-point. The set Δ from Chapter 2 (see p. 40) can now be regarded as the set of all Δ-points. Consequently, a Banach space has the convex DLD2P if and only if its unit ball is equal to the closed convex hull of all of its Δ-points.

Every Daugavet-point is obviously a Δ-point. The converse is not true in general (see, e.g. Example 3.4.3). The difference between these notions was established by inspecting the behaviour of Daugavet- and Δ-points by taking absolute sums and these results are presented later in this chapter (see Sections 3.3 and 3.4). Despite these notions being different, there are examples of spaces where Daugavet- and Δ-points are the same (see Section 3.2).

Sometimes it is useful to take into account that in the previous definitions the approximating elements can be chosen to be of norm 1, i.e. the following result holds.

Lemma 3.1.3. *In the definition of Daugavet- and Δ-points one can equivalently use the set $\{y \in S_X \colon \|x - y\| \geq 2 - \varepsilon\}$ instead of $\Delta_\varepsilon(x)$.*

Proof. Let $x, y \in S_X$ be such that $y \in \overline{\operatorname{conv}}\, \Delta_\varepsilon(x)$ for all $\varepsilon > 0$. Fix $\varepsilon > 0$ and $\gamma > 0$. Assume without loss of generality that $\gamma \leq \varepsilon \leq 1$.

Let $x_1, \ldots, x_n \in \Delta_{\gamma/2}(x)$ be such that $\|x - 1/n \sum_{i=1}^n x_i\| < \gamma/2$. Then $\|x_i\| > 1 - \gamma/2$. Therefore,

$$\begin{aligned}\left\| x - \frac{x_i}{\|x_i\|} \right\| &\geq \|x - x_i\| - (1 - \|x_i\|) \\ &> 2 - \frac{\gamma}{2} - \frac{\gamma}{2} \\ &\geq 2 - \varepsilon\end{aligned}$$

and

$$\begin{aligned}\left\| x - \frac{1}{n} \sum_{i=1}^n \frac{x_i}{\|x_i\|} \right\| &\leq \left\| x - \frac{1}{n} \sum_{i=1}^n x_i \right\| + \left\| \frac{1}{n} \sum_{i=1}^n \left(x_i - \frac{x_i}{\|x_i\|} \right) \right\| \\ &\leq \frac{\gamma}{2} + \frac{\gamma}{2} = \gamma.\end{aligned}$$

Hence, $y \in \overline{\operatorname{conv}}\{y \in S_X \colon \|x - y\| > 2 - \varepsilon\}$ for all $\varepsilon > 0$. $\square$

Inspecting the proofs of the characterisations of the Daugavet property and the DLD2P appearing in Propositions 1.4.5 and 2.1.7, respectively, one observes that pointwise versions of some of these hold as well.

Lemma 3.1.4. *Let X be a Banach space and $x \in S_X$. The following assertions are equivalent:*

(*i*) *x is a Daugavet-point, that is, $B_X = \overline{\operatorname{conv}}\,\Delta_\varepsilon(x)$ for every $\varepsilon > 0$;*

(*ii*) *for every slice S of B_X and every $\varepsilon > 0$ there exists $y \in S$ such that $\|x - y\| \geq 2 - \varepsilon$;*

(*iii*) *for every non-zero $x^* \in X^*$, the rank-1 operator $T = x^* \otimes x$ satisfies $\|\operatorname{Id} - T\| = 1 + \|T\|$;*

(*iv*) *for every $x^* \in S_{X^*}$ the rank-1 norm-1 operator $T = x^* \otimes x$ satisfies $\|\operatorname{Id} - T\| = 2$.*

Lemma 3.1.5. *Let X be a Banach space and $x \in S_X$. The following assertions are equivalent:*

(*i*) *x is a Δ-point, that is, $x \in \overline{\operatorname{conv}}\,\Delta_\varepsilon(x)$ for every $\varepsilon > 0$;*

(*ii*) *for every slice S of B_X with $x \in S$ and every $\varepsilon > 0$ there exists $y \in S$ such that $\|x - y\| \geq 2 - \varepsilon$;*

(*iii*) *for every $x^* \in X^*$ with $x^*(x) = 1$ the projection $P = x^* \otimes x$ satisfies $\|\operatorname{Id} - P\| \geq 2$.*

Note that the version (*iv*) of Lemma 3.1.4 cannot be added to Lemma 3.1.5.

Proposition 3.1.6. *Let $X = \ell_1$ and let $x = (x_i)_{i=1}^\infty \in S_X$ be a smooth point with $|x_1| > 1/3$. Then the following statements hold:*

(*a*) *for $x^* \in S_{X^*}$ with $x^*(x) = 1$, the projection $P = x^* \otimes x$ satisfies $\|\operatorname{Id} - P\| = 2$;*

(*b*) *the projection $P = x_1^{-1} e_1^* \otimes x$ satisfies $\|\operatorname{Id} - P\| < 2$.*

Proof. (*a*) Write $x = (x_i)_{i=1}^\infty$. Let $x^* = (\operatorname{sgn} x_i)_{i=1}^\infty \in S_{X^*}$ and $P = x^* \otimes x$. Observe that $x^*(x) = 1$. If e_n is the n'th standard basis vector in X, then

$$\begin{aligned}
\|(\operatorname{Id} - P)(e_n)\| &= \|e_n - \operatorname{sgn} x_n x\| \\
&= |1 - (\operatorname{sgn} x_n) x_n| + \sum_{i \neq n} |x_i| \\
&= 1 - |x_n| + \|x\| - |x_n| \\
&= 2 - 2|x_n|,
\end{aligned}$$

and since this holds for all n, we get $\|\operatorname{Id} - P\| = 2$.

(*b*) Let $P = x_1^{-1} e_1^* \otimes x$, where e_1^* is the first coordinate vector in $X^* = \ell_\infty$. Observe that $x_1^{-1} e_1^*(x) = 1$, so that P is a projection. If $y \in S_X$ we get

$$\begin{aligned}
\|(\operatorname{Id} - P)y\| &= \|y - x_1^{-1} y_1 x\| \\
&= \sum_{i>1} |y_i - x_1^{-1} y_1 x_i| \\
&\leq \sum_{i>1} |y_i| + |x_1|^{-1} |y_1| \sum_{i>1} |x_i| \\
&= 1 - 2|y_1| + |x_1|^{-1} |y_1| \\
&\leq 1 + \left|2 - |x_1|^{-1}\right| < 2,
\end{aligned}$$

so $\|\operatorname{Id} - P\| < 2$, and we are done. □

Remark 3.1.7. It follows from Theorem 3.2.1 below that, in fact, no point on the unit sphere in ℓ_1 is a Δ-point.

In our language it is stated in [Wer01, Problem (7)] without a proof that the DLD2P is equivalent to the following property.

Definition 3.1.8. Let X be a Banach space. We say that X has *property* $\mathcal{D}$ if every rank-1 norm-1 projection $P\colon X \to X$ satisfies that $\|\operatorname{Id} - P\| = 2$.

Looking at the Daugavet property and the fact that in this case it is enough to only consider rank-1 norm-1 operators T, one would at first glance maybe expect the equivalence of the DLD2P and property $\mathcal{D}$ to be true. The problem is, however, that a scaled projection is not a projection, so the argument in the Daugavet setting does not apply in the DLD2P setting. As remarked in the Introduction of [AHLP20], neither the authors of that paper nor the author of [Wer01] have been able to give a correct proof of the equivalence of the DLD2P and property $\mathcal{D}$. Whether this equivalence is true or not is therefore still an open question. In the following we will see that there are similarities between the DLD2P and property $\mathcal{D}$. Recall from Proposition 2.1.5 that a Banach space has the DLD2P if and only if its dual has the w^*-DLD2P. In particular, if the dual of a space has the DLDP2, then so does the space. The same is true for property $\mathcal{D}$.

Proposition 3.1.9. *Let X be a Banach space. If $\|\operatorname{Id}_{X^*} - P\| = 2$ for all rank-1 norm-1 projections P on X^*, then $\|\operatorname{Id}_X - Q\| = 2$ for all rank-1 norm-1 projections Q on X.*

Proof. If Q is a rank-1 projection on X, then $Q = x^* \otimes x$ with $x^* \in X^*$, $x \in S_X$, and $x^*(x) = 1$. Then

$$P = Q^* = x \otimes x^* = (\|x^*\| x) \otimes \frac{x^*}{\|x^*\|}$$

is a rank-1 projection on X^* and by assumption $\|\operatorname{Id}_{X^*} - P\| = \|\operatorname{Id}_X - Q\| \geq 2$. $\square$

If a Banach space has the DLD2P, then the space has the LD2P and its dual has the w^*-LD2P. The same is true for spaces with property $\mathcal{D}$.

Proposition 3.1.10. *Let X be a Banach space. If $\|\operatorname{Id} - P\| = 2$ for all rank-1 norm-1 projections P on X, then X has the LD2P and X^* has the w^*-LD2P.*

Proof. Let $x^* \in S_{X^*}$ and $\alpha > 0$ define a slice $S(B_X, x^*, \alpha)$. Let $\varepsilon > 0$ such that $\varepsilon < \alpha/2$. Find $y^* \in S_{X^*}$ such that y^* attains its norm on B_X and $\|x^* - y^*\| < \alpha/2$. Let $y \in B_X$ be such that $y^*(y) = 1$ and define $P = y^* \otimes y$. Then $\|\operatorname{Id} - P\| = 2$, by assumption, and we can find $z \in S_X$ such that

$$\|z - P(z)\| = \|z - y^*(z)y\| > 2 - \varepsilon.$$

We may assume that $y^*(z) > 0$. We have

$$y^*(z) = |y^*(z)| = \|P(z)\| \geq \|P(z) - z\| - \|z\| > 2 - \varepsilon - 1 > 1 - \frac{\alpha}{2}.$$

Hence,

$$x^*(z) = y^*(z) - (y^* - x^*)(z) > 1 - \frac{\alpha}{2} - \frac{\alpha}{2} = 1 - \alpha,$$

i.e. $z \in S(B_X, x^*, \alpha)$, and

$$\|z - y\| \geq \|z - y^*(z)y\| - \|y^*(z)y - y\| > 2 - \varepsilon - |y^*(z) - 1| > 2 - 2\varepsilon.$$

This proves that X has the LD2P.

To show that X^* has the w^*-LD2P we start with a w^*-slice $S(B_{X^*}, x, \alpha)$, where $x \in S_X$ and $\alpha > 0$. Then we find a $y^* \in S_{X^*}$ where $\|\operatorname{Id}^* - P^*\|$ almost attains its norm. The proof is similar to the LD2P case. $\square$

There are Banach spaces with the LD2P that fail to have the DLD2P. The same is true for the LD2P and property $\mathcal{D}$ as well.

Example 3.1.11. The sequence space ℓ_∞ has the LD2P but fails property $\mathcal{D}$. Indeed, since ℓ_∞ has the SD2P (see [ALN13]), it has the LD2P. To see that ℓ_∞ fails property $\mathcal{D}$, consider rank-1 norm-1 projection $P = e_1^* \otimes e_1$, where e_1^* and e_1 are the first coordinate vectors of ℓ_1 and ℓ_∞, respectively. Then for every $x = (\xi_i)_{i=1}^\infty \in B_{\ell_\infty}$ we have

$$\begin{aligned}\|(\operatorname{Id} - P)x\| &= \|x - e_1^*(x)e_1\| \\ &= \|x - \xi_1 e_1\| \\ &= \|(0, \xi_2, \xi_3, \dots)\| \leq 1.\end{aligned}$$

Hence, $\|\operatorname{Id} - P\| \leq 1$, and therefore ℓ_∞ fails property $\mathcal{D}$.

Note that, despite Proposition 3.1.6, Proposition 3.1.10 tells us that ℓ_1 is not a candidate for separating property $\mathcal{D}$ and the DLD2P since it fails the LD2P.

3.2 Daugavet- and Δ-points in some classes of spaces

In the first two parts of this section we present the results regarding Daugavet- and Δ-points in Banach spaces X of the type $L_1(\mu)$, $C(K)$, K infinite compact Hausdorff space, and $L_1(\mu)$-preduals. We prove that Daugavet- and Δ-points coincide in all these spaces. Moreover, we obtain characterisations of Daugavet- and Δ-points in terms of their support. These characterisations are easy to check, for example, if $X = C([0,\omega]) = c$ then the Daugavet-points are exactly the sequences with limits ± 1. The last part of the section contains results about Daugavet- and Δ-points in Müntz spaces X of the type $M_0(\Lambda) \subset M(\Lambda) \subset C[0,1]$ (see Subsection 3.2.3 for a definition of these spaces). We prove that in $M_0(\Lambda)$ Daugavet- and Δ-points are the same and that they are exactly the functions $f \in S_X$ for which $f(1) = \pm 1$.

3.2.1 $L_1(\mu)$ spaces

Let μ be a (countably additive, non-negative) measure on some σ-algebra Σ on a set Ω. We will assume that μ is σ-finite even though it is not strictly necessary in all the results. We use the usual notion that an *atom for* μ is a set $A \in \Sigma$ such that $0 < \mu(A) < \infty$, and if $B \in \Sigma$ with $B \subseteq A$ satisfies $\mu(B) < \mu(A)$, then $\mu(B) = 0$. In this section we consider the space $L_1(\mu) = L_1(\Omega, \Sigma, \mu)$.

Theorem 3.2.1. *The following assertions for $f \in S_{L_1(\mu)}$ are equivalent:*

(*i*) *f is a Daugavet-point;*

(*ii*) *f is a Δ-point;*

(*iii*) $\operatorname{supp}(f)$ *does not contain an atom for μ.*

Proof. The implication $(i) \Rightarrow (ii)$ is trivial.

$(ii) \Rightarrow (iii)$. Fix $f \in S_{L_1(\mu)}$. Let A be an atom in $\operatorname{supp}(f)$. Note that a measurable function is a.e. constant on an atom. We may assume that $f|_A = c$ a.e. for some positive constant c. Fix $0 < \varepsilon < 2c\mu(A)$.

Let $g \in B_{L_1(\mu)}$ be such that $\|f-g\| \geq 2-\varepsilon$. We have $g|_A = d$ for some constant d. Note that

$$\begin{aligned}2-\varepsilon &\leq \int_\Omega |f-g|d\mu = \int_{\Omega\setminus A} |f-g|d\mu + \int_A |f-g|d\mu\\ &\leq \int_{\Omega\setminus A}|f|d\mu + \int_{\Omega\setminus A}|g|d\mu + \int_A |f-g|d\mu\\ &\leq 1-\int_A |f|d\mu + 1 - \int_A |g|d\mu + \int_A |f-g|d\mu\\ &= 1-c\mu(A)+1-|d|\mu(A)+|c-d|\mu(A).\end{aligned}$$

Therefore

$$c\mu(A)+d\mu(A) \leq |c-d|\mu(A)+\varepsilon.$$

If $c \leq d$, then $|c-d| = d-c$ and we get $c \leq \varepsilon/(2\mu(A))$, and this contradicts our choice of ε. Thus we have $c \geq d$, and hence $|c-d| = c-d$ and $d \leq \varepsilon/(2\mu(A)) < c$.

If $g_1,\ldots,g_m \in \Delta_\varepsilon(f)$, then

$$\Big\|f-\sum_{i=1}^m \frac{1}{m}g_i\Big\| \geq \int_A \Big|f-\sum_{i=1}^m \frac{1}{m}g_i\Big|\,d\mu \geq \Big(c-\frac{\varepsilon}{2\mu(A)}\Big)\mu(A) > 0.$$

This shows that $f \notin \overline{\operatorname{conv}}\,\Delta_\varepsilon(f)$ for this choice of ε.

(iii) $\Rightarrow$ *(i)*. Let $f \in S_{L_1(\mu)}$ be such that $\operatorname{supp}(f)$ does not contain atoms. Let $\varepsilon > 0$, $\delta > 0$, and $x_0^* \in S_{L_1(\mu)^*}$. By Lemma 3.1.4 we need to find $g \in S_{L_1(\mu)}$ with $\|f-g\| \geq 2-\varepsilon$ such that $g \in S(B_{L_1(\mu)}, x_0^*, \delta)$.

Since μ is σ-finite (so that $L_1(\mu)^* = L_\infty(\mu)$) we can find a step-function $x^* = \sum_{i=1}^n a_i\chi_{E_i} \in S_{L_1(\mu)^*}$ such that $\|x^*-x_0^*\| < \delta$ (and the $E_i \cap E_j = \emptyset$ for $i \neq j$).

We may assume that $|a_1| = 1$. Find a subset A of E_1 such that $\int_A |f|d\mu < \varepsilon/2$. Define

$$g = \frac{\operatorname{sgn}(a_1)}{\mu(A)}\chi_A \in S_{L_1(\mu)}.$$

Then

$$x^*(g) = \sum_{i=1}^n \int_{E_i} a_i g\,d\mu = \frac{1}{\mu(A)}\int_A a_1\operatorname{sgn}(a_1)d\mu = 1,$$

$$\|f-g\| = \int_{A^c}|f|d\mu + \int_A |f-g|d\mu \geq |f|+|g|-2\int_A |f|d\mu \geq 2-\varepsilon,$$

and finally

$$x_0^*(g) = x^*(g)-(x^*-x_0^*)(g) > 1-\delta$$

as desired. □

We also discovered that in $L_1(\mu)$ spaces the Daugavet property and property $\mathcal{D}$ are equivalent. In order to prove that result, note that $L_1(\mu)$ can have the LD2P only if the measure μ does not have any atoms.

Lemma 3.2.2. *If μ is a measure with an atom, then $L_1(\mu)$ does not have the LD2P.*

Proof. Assume that A is an atom and consider $\chi_A \in L_1(\mu)^*$. We have $\|\chi_A\| = 1$. If $f \in S(B_{L_1(\mu)}, \chi_A, \varepsilon)$, then

$$f(t) > \frac{1-\varepsilon}{\mu(A)} \qquad \text{for a.e. } t \in A,$$

and

$$f(t) \leq \frac{1}{\mu(A)} \qquad \text{for a.e. } t \in A.$$

Hence $\|f|_A\| > 1-\varepsilon$ and $\|f|_{A^C}\| < \varepsilon$.

Thus for $f_1, f_2 \in S(B_{L_1(\mu)}, \chi_A, \varepsilon)$ we have

$$\begin{aligned}\|f_1 - f_2\| &\leq \int_{A^c} |f_1 - f_2| \, d\mu + \int_A |f_1 - f_2| \, d\mu \\ &\leq \|f_1|_{A^c}\| + \|f_2|_{A^c}\| + \int_A \frac{\varepsilon}{\mu(A)} \, d\mu \leq 3\varepsilon,\end{aligned}$$

so this slice does not have diameter 2. □

Theorem 3.2.3. *Consider $X = L_1(\mu)$. The following assertions are equivalent:*

(i) $\|\operatorname{Id} - P\| = 2$ *for all rank-1 norm-1 projections on X;*

(ii) *X has the Daugavet property.*

Proof. If (i) holds, then X has the LD2P, by Proposition 3.1.10. From Lemma 3.2.2 we see that X does not have atoms. By [BM05] (see also [BM06] for the explicit statement for $L_1(\mu)$ spaces) X has the Daugavet property.

The implication $(ii) \Rightarrow (i)$ is trivial. □

3.2.2 $C(K)$ spaces and $L_1(\mu)$-preduals

In the following we examine Daugavet- and Δ-points in $C(K)$ spaces and in $L_1(\mu)$-preduals. We start with a characterization in $C(K)$ spaces.

Theorem 3.2.4. *Let K be an infinite compact Hausdorff space. The following assertions for $f \in S_{C(K)}$ are equivalent:*

(*i*) *f is a Daugavet-point;*

(*ii*) *f is a Δ-point;*

(*iii*) *$\|f\| = |f(x_0)|$ for a limit point x_0 of K.*

Proof. The implication $(i) \Rightarrow (ii)$ is trivial.

$(iii) \Rightarrow (i)$. Let $f \in S_{C(K)}$ and assume that there is a limit point x_0 of K such that $|f(x_0)| = 1$. We will show that f is a Daugavet-point. Fix $g \in B_X$, $\varepsilon > 0$, and $m \in \mathbb{N}$. Consider a neighbourhood U of x_0 such that $|f(x_0) - f(x)| < \varepsilon$ for every $x \in U$. Since x_0 is a limit point, we can find m different points $x_1, \dots, x_m \in U$ and corresponding pairwise disjoint neighbourhoods $U_1, \dots, U_m \subset U$. For every $1 \leq i \leq m$ use Urysohn's Lemma to find a continuous function $\eta_i \colon K \to [0,1]$ with $\eta_i(x_i) = 1$ and $\eta_i = 0$ on $K \setminus U_i$. Define $g_i \in B_{C(K)}$ by

$$g_i(x) = \big(1 - \eta_i(x)\big)g(x) - \eta_i(x)f(x_0).$$

From $g_i(x_i) = -f(x_0)$ it follows that

$$\|f - g_i\| \geq |f(x_i) - g(x_i)| = |f(x_i) + f(x_0)| > 2 - \varepsilon.$$

Hence $g_i \in \Delta_\varepsilon(f)$. Note that $g - g_i = 0$ on $K \setminus U_i$, and consequently

$$\left\| g - \frac{1}{m}\sum_{i=1}^{m} g_i \right\| \leq \frac{1}{m} \max_{1 \leq i \leq m} \|g - g_i\| \leq \frac{2}{m}.$$

We thus get $g \in \overline{\operatorname{conv}}\, \Delta_\varepsilon(f)$, and so f is a Daugavet-point.

$(ii) \Rightarrow (iii)$. We assume that there is no limit point x of K such that $|f(x)| = 1$ and show that f is not a Δ-point. Define

$$H = \{x \in K \colon |f(x)| = 1\}.$$

Then H is a set of isolated points. By compactness, H is finite since otherwise it would contain a limit point. Note that H is (cl)open, hence $\delta = 1 - \max_{x \in K \setminus H} |f(x)| > 0$. Let $\varepsilon_h = \operatorname{sgn} f(h)$ for all $h \in H$. Since $H \neq \emptyset$ we can define

$$\mu = \frac{1}{|H|} \sum_{h \in H} \varepsilon_h \delta_h,$$

where $\delta_h \in S_{C(K)^*}$ is the point evaluation map at h. We have $\|\mu\| = 1$ and $\langle \mu, f \rangle = 1$, hence $P = \mu \otimes f$ is a norm-1 projection.

Let $g \in B_{C(K)}$ and consider $\|(\mathrm{Id} - P)g\| = \|g - Pg\| = \|g - \langle \mu, g\rangle f\|$. For $x \notin H$ we have

$$|g(x) - \langle \mu, g\rangle f(x)| \leq 1 + 1 - \delta = 2 - \delta.$$

While for $x \in H$ we use that

$$\langle \mu, g\rangle = \frac{1}{|H|} \sum_{h \in H} \varepsilon_h g(h)$$

and $\varepsilon_h f(h) = |f(h)| = 1$, so that

$$\begin{aligned}
|g(x) - \langle \mu, g\rangle f(x)| &= |g(x) - \frac{1}{|H|} \sum_{h \in H} \varepsilon_h g(h) f(x)| \\
&= \left| \left(1 - \frac{1}{|H|}\right) g(x) - \frac{1}{|H|} \sum_{h \in H \setminus \{x\}} \varepsilon_h g(h) f(x) \right| \\
&\leq \left(1 - \frac{1}{|H|}\right) + \frac{|H| - 1}{|H|} \\
&= 2 - \frac{2}{|H|}.
\end{aligned}$$

With $\varepsilon = \min\{\delta, 2/|H|\}$ we have $\|(\mathrm{Id} - P)g\| \leq 2 - \varepsilon < 2$ for all $g \in B_{C(K)}$, hence $\|\mathrm{Id} - P\| < 2$. □

Now we move on to the case of $L_1(\mu)$-preduals. A Banach space is called an $L_1(\mu)$-predual if its dual is isometric to $L_1(\mu)$. It is well known that the bidual of an $L_1(\mu)$-predual is isometric to a $C(K)$ space for some (extremally disconnected) compact Hausdorff space K (see [Lin64, Theorem 6.1]). We will now show that Daugavet- and Δ-points are the same in $L_1(\mu)$-preduals. Its proof relies on the following lemma.

Lemma 3.2.5. *Let X be a Banach space and let $x, y \in S_X$. The following assertions are equivalent:*

(i) *$y \in \overline{\mathrm{conv}}\, \Delta_\varepsilon^X(x)$ for all $\varepsilon > 0$;*

(ii) *$y \in \overline{\mathrm{conv}}\, \Delta_\varepsilon^{X^{**}}(x)$ for all $\varepsilon > 0$.*

Proof. The implication $(i) \Rightarrow (ii)$ is trivial as $\Delta_\varepsilon^X(x) \subset \Delta_\varepsilon^{X^{**}}(x)$.

$(ii) \Rightarrow (i)$. Let $\varepsilon > 0$ and $\delta > 0$. Find $y_n^{**} \in B_{X^{**}}$ such that $\|x - y_n^{**}\| \geq 2 - \varepsilon$ and $\|y - \sum_{n=1}^m \lambda_n y_n^{**}\| < \delta$.

Define $E = \mathrm{span}\{x, y, y_n^{**}\}$. Let $\eta > 0$ and use the Principle of Local Reflexivity to find $T : E \to X$ such that

(1) $T(e) = e$ for all $e \in E \cap X$;

(2) $(1-\eta)\|e\| \le \|Te\| \le (1+\eta)\|e\|$.

Then $\|x - Ty_n^{**}\| = \|T(x - y_n^{**})\| \ge (1-\eta)\|x - y_n^{**}\| > 2 - \varepsilon$ if η is small enough. Also, if η is small enough,

$$\|y - \sum_{n=1}^{m} \lambda_n T y_n^{**}\| \le (1+\eta)\|y - \sum_{n=1}^{m} \lambda_n y_n^{**}\| < \delta.$$

□

Remark 3.2.6. The argument shows that the conclusion in Lemma 3.2.5 also holds in the more general setting of X being an almost isometric ideal (see [ALN14] for the definition) in Z, replacing X^{**} with Z.

Theorem 3.2.7. *Let X be an (infinite-dimensional) $L_1(\mu)$-predual and $x \in S_X$. The following assertions are equivalent:*

(i) x is a Daugavet-point;

(ii) x is a Δ-point.

Proof. The implication $(i) \Rightarrow (ii)$ is trivial.

$(ii) \Rightarrow (i)$. By Lemma 3.2.5 we get $x \in \overline{\operatorname{conv}}\, \Delta_\varepsilon^{X^{**}}(x)$ for all $\varepsilon > 0$. Since X^{**} is isometric to a $C(K)$ space, we get from Theorem 3.2.4 that x is a Daugavet-point in X^{**}, that is, $B_{X^{**}} = \overline{\operatorname{conv}}\, \Delta_\varepsilon^{X^{**}}(x)$ for all $\varepsilon > 0$. Using Lemma 3.2.5 again we get the desired conclusion.

□

As for $L_1(\mu)$ spaces, the Daugavet property and property $\mathcal{D}$ coincide for $L_1(\mu)$-preduals.

Theorem 3.2.8. *Let X be an $L_1(\mu)$-predual. The following assertions are equivalent:*

(i) $\|\operatorname{Id} - P\| = 2$ for all rank-1 norm-1 projections P on X;

(ii) X has the Daugavet property.

Proof. $(i) \Rightarrow (ii)$. If $\|\operatorname{Id} - P\| = 2$ for all rank-1 norm-1 projections, then X^* has the w^*-LD2P by Proposition 3.1.10 which is equivalent to X having extremely rough norm (cf. [JZ78, Proposition 1]). By [BM06, Theorem 2.4] this implies the Daugavet property for $L_1(\mu)$-preduals.

The implication $(ii) \Rightarrow (i)$ is trivial. □

3.2.3 Müntz spaces

Recall that a Müntz space is a space of the form $M(\Lambda) := \overline{\operatorname{span}}\{t^{\lambda_n}\}_{n=0}^{\infty} \subset C[0,1]$, where $\Lambda = (\lambda_n)_{n=0}^{\infty}$ is an increasing sequence of non-negative reals and $\sum_{i=1}^{\infty} 1/\lambda_i < \infty$. In this section, we will also consider the subspace $M_0(\Lambda) := \overline{\operatorname{span}}\{t^{\lambda_n}\}_{n=1}^{\infty}$ of $M(\Lambda)$.

It is known that the spaces $X = M(\Lambda)$ are isomorphic to subspaces of c, even isometrically isomorphic in the case $X = M_0(\Lambda)$ (see [Wer08] and [Mar18]). Therefore it is no surprise that they share many properties with c. As for c, all Müntz spaces fail to be *locally octahedral* and *almost square*. Moreover, if $X = M_0(\Lambda)$, then X is not even *locally almost square* (see [Mar18] and the definitions of these properties within). Furthermore, it is known that, as for c, the dual of a Müntz space is octahedral and that a Müntz space contains asymptotically isometric copies of c_0 (see [ALMN17] and the definitions within).

Based on the preceding observations one can expect that the results regarding Daugavet- and Δ-points for Müntz spaces would also be similar to those for c. This expectation proved to be correct. Before we can prove the result about the equivalence of Daugavet- and Δ-points in most Müntz spaces, the following lemma is needed.

Lemma 3.2.9. *For every $\varepsilon > 0$ and every $\delta > 0$, there exist $k, l \in \mathbb{N}$ with $k < l$ such that for $f = (t^{\lambda_k} - t^{\lambda_l})/\|t^{\lambda_k} - t^{\lambda_l}\|$ one has $f \geq 0$ and $f|_{[0,1-\varepsilon]} < \delta$.*

Proof. Fix positive numbers ε and δ. Let k be such that

$$t^{\lambda_k}|_{[0,1-\varepsilon]} < \frac{\delta}{2}.$$

Choose $l > k$ such that $\|t^{\lambda_k} - t^{\lambda_l}\| > 1/2$. Then

$$\frac{t^{\lambda_k} - t^{\lambda_l}}{\|t^{\lambda_k} - t^{\lambda_l}\|} < \frac{\delta/2}{1/2} = \delta$$

for any $t \in [0, 1-\varepsilon]$. □

Now we are ready to prove that unit sphere elements f in Müntz spaces satisfying the condition $f(1) = \pm 1$ are Daugavet-points. It is worth pointing out that this result holds as well for Müntz spaces where constant functions are included.

Proposition 3.2.10. *Let $X = M(\Lambda)$ or $X = M_0(\Lambda)$. If $f \in S_X$ satisfies $f(1) = \pm 1$, then f is a Daugavet-point.*

Proof. Fix $f \in S_X$ with $f(1) = \pm 1$ and $\varepsilon > 0$. We show that any $g \in S_X$ can be approximated by the elements of $\operatorname{conv} \Delta_\varepsilon(f)$. For this purpose, fix $g \in S_X$, $\delta > 0$ and choose $m \in \mathbb{N}$ with $m \geq 2/\delta$.

Let $t_1 \in (0,1)$ be such that $|f(1) - f(t)| < \delta$ and $|g(1) - g(t)| < \delta$ for all $t \in [t_1, 1]$. We use Lemma 3.2.9 to obtain f_1 such that $f_1|_{[0,t_1]} < \delta/2$.

Let $t_2 \in (0,1)$ be such that $f_1|_{[t_2,1]} < \delta/2$. We use Lemma 3.2.9 again to obtain f_2 such that $f_2|_{[0,t_2]} < \delta/2$.

We continue finding $t_0 < t_1 < \cdots < t_m < t_{m+1} = 1$ and $f_1, \ldots, f_m$. Define $g_i = g - [g(1)+1]f_i$ for $i = 1, \ldots, m$. Then $\|g_i\| \leq 1 + \delta$. Indeed, for $t \in [0,1] \setminus [t_i, t_{i+1}]$ we have that $f_i(t) < \delta/2$ and therefore

$$|g_i(t)| \leq |g(t)| + (1 + g(1))f_i(t) < 1 + 2\frac{\delta}{2} = 1 + \delta,$$

while for $t \in [t_i, t_{i+1}]$ we have

$$\begin{aligned}
|g_i(t)| &\leq |g(1) - [g(1)+1]f_i(t)| + |g(t) - g(1)| \\
&\leq |g(1)|(1 - f_i(t)) + f_i(t) + \delta \\
&\leq 1 - f_i(t) + f_i(t) + \delta \\
&= 1 + \delta.
\end{aligned}$$

Denote by s_i the unique point in (t_i, t_{i+1}) where $f_i(s_i) = 1$. We have

$$\begin{aligned}
\|g_i - f\| &\geq |g_i(s_i) - f(s_i)| \\
&= |(g(s_i) - (g(1)+1)) - f(s_i)| \\
&\geq |1 + f(s_i)| - |g(1) - g(s_i)| \\
&\geq 2 - \delta - \delta \\
&= 2 - 2\delta.
\end{aligned}$$

Hence,

$$\|(1+\delta)^{-1}g_i - f\| \geq \|g_i - f\| - \|(1+\delta)^{-1}g_i - g_i\| \geq 2 - 3\delta$$

since

$$\|(1+\delta)^{-1}g_i - g_i\| = |(1+\delta)^{-1} - 1|\|g_i\| \leq |(1+\delta)^{-1} - 1|(1+\delta) \leq \delta.$$

We get that $(1+\delta)^{-1}g_i \in \Delta_\varepsilon(f)$ whenever $3\delta < \varepsilon$. Finally

$$\begin{aligned}\left\|g - \sum_{i=1}^m \frac{1}{m}(1+\delta)^{-1}g_i\right\| &= \left\|(1-(1+\delta)^{-1})g + (1+\delta)^{-1}[g(1)+1]\sum_{i=1}^m \frac{1}{m}f_i\right\| \\ &\leq \frac{\delta}{1+\delta}\|g\| + \frac{(g(1)+1)}{m(1+\delta)}\|\sum_{i=1}^m f_i\| \\ &\leq \frac{\delta}{1+\delta} + \frac{2}{m}\left(1+(m-1)\frac{\delta}{2}\right) \\ &\leq \delta+\delta+\delta \\ &= 3\delta.\end{aligned}$$

Hence $g \in \overline{\operatorname{conv}}\,\Delta_\varepsilon(f)$. □

The proof of the upcoming Proposition 3.2.13, that clarifies the rest of the cases, relies, in part, on the following results.

Theorem 3.2.11 (Full Clarkson–Erdös–Schwartz Theorem, see [Erd03, Theorem 1.1])**.** *Suppose $(\lambda_i)_{i=1}^\infty$ is a sequence of distinct positive numbers. Then* $\operatorname{span}\{1, x^{\lambda_1}, x^{\lambda_2}, \dots\}$ *is dense in $C[0,1]$ if and only if*

$$\sum_{i=1}^\infty \frac{\lambda_i}{\lambda_i^2+1} = \infty.$$

Moreover, if

$$\sum_{i=1}^\infty \frac{\lambda_i}{\lambda_i^2+1} < \infty,$$

then every function from the $C[0,1]$ closure of $\operatorname{span}\{1, x^{\lambda_1}, x^{\lambda_2}, \dots\}$ *can be represented as an analytic function on $\{z \in \mathbb{C}\backslash(-\infty,0] \colon |z| < 1\}$ restricted to* $(0,1)$.

Theorem 3.2.12 (Bernstein's inequality, see [BE97, Theorem 3.2])**.** *Assume that $1 \leq \lambda_1 < \lambda_2 < \lambda_3 < \dots$ and $\sum_{i=1}^\infty 1/\lambda_i < \infty$. Then for every $\delta > 0$ there is a constant $c = c(\Lambda, \delta)$ such that*

$$\|p'\|_{[0,1-\delta]} \leq c\|p\|_{[0,1]},$$

for all $p \in \operatorname{span}(t^{\lambda_i})_{i=1}^\infty$.

Our next aim is to show that in Müntz spaces $M_0(\Lambda)$ with $\lambda_1 \geq 1$ only the elements of the unit sphere that satisfy the condition in Proposition 3.2.10 can be Δ-points, which also implies that every Δ-point in such a space is necessarily a Daugavet-point.

Proposition 3.2.13. *Let X be a Müntz space $M_0(\Lambda)$ with $\lambda_1 \geq 1$. If $f \in S_X$ with $|f(1)| < 1$, then $f \notin \Delta$.*

Proof. First note that from the Theorem 3.2.11, f is the restriction to $(0,1)$ of an analytic function on $\Omega = \{x \in \mathbb{C} \setminus (-\infty, 0] \colon |z| < 1\}$. Let I be the set of points in $[0,1]$ where f attains its norm, and put $I^{\pm} = \{x \in I \colon f(x) = \pm 1\}$. From the assumptions we have $I \subset (0,1)$ since every $g \in M_0(\Lambda)$ satisfies $g(0) = 0$.

Suppose I is infinite. Then either I^+ or I^- is infinite. Suppose without loss of generality that I^+ is. Then I^+ must have an accumulation point a in $[0,1]$. By the continuity of f we must have $f(a) = 1$, so $0 < a < 1$. Since f is analytic on Ω and I^+, I^+ has an accumulation point in $(0,1) \subset \Omega$, we must have $1 - f = 0$ everywhere, which is a contradiction.

Suppose I is finite and that f attains its norm on $(y_k)_{k=1}^m \subset (0,1)$ with $0 < y_1 < y_2 < \cdots < y_m < 1$, i.e. $1 = \|f\| = |f(y_k)|$ for every $k = 1, \ldots, m$. By density it suffices to show that there is $\varepsilon > 0$ such that $f \notin \overline{\operatorname{conv}}(\Delta_\varepsilon(f) \cap P)$ where $P = \operatorname{span}(t^{\lambda_n})_{n=1}^\infty \subset X$. To this end, let s be a point satisfying $(1 + y_m)/2 < s < 1$. By Theorem 3.2.12, there exists a constant $c = c(\Lambda, s)$ such that for any $p \in P$

$$\|p'\|_{[0,s]} \leq c\|p\|_{[0,1]}.$$

Since $f \in C[0,1]$ there exists $\delta > 0$ such that for all $x, y \in [0,1]$

$$|x - y| < \delta \implies |f(x) - f(y)| < 1.$$

By choosing δ smaller if necessary we may assume that $c\delta < 1/2$ and that $y_m + \delta/2 < s$. Let $I_{k,\delta} = (y_k - \delta/2, y_k + \delta/2)$. Note that f does not change sign on any $I_{k,\delta}$.

Put $I_\delta = \cup_{k=1}^m I_{k,\delta}$, and $M = \sup\{|f(y)| \colon y \in [0,1] \setminus I_\delta\}$. Since $[0,1] \setminus I_\delta$ is compact and since f is continuous, the value M is attained and thus $M < 1$. Let $0 < \varepsilon < \min\{1/(2m), 1 - M, 1/4\}$. Then

$$|f(x)| \geq 1 - \varepsilon \Rightarrow x \in I_\delta.$$

Assume that $p \in \Delta_\varepsilon(f) \cap P$. Since $\|f - p\| \geq 2 - \varepsilon$ the norm is attained on I_δ. Therefore there exist k and $x \in I_{k,\delta}$ such that

$$|f(x) - p(x)| \geq 2 - \varepsilon.$$

Since $|f(x)| \geq 1 - \varepsilon$ and f does not change sign on $I_{k,\delta}$ we must have $|f(x) - f(y_k)| \leq \varepsilon$, hence

$$\begin{aligned} |f(y_k) - p(y_k)| &\geq |f(x) - p(x)| - |f(y_k) - f(x)| - |p(x) - p(y_k)| \\ &\geq 2 - 2\varepsilon - \|p_i'\|_{[0,s]}|x - y_k| > 3/2 - c\delta > 1. \end{aligned}$$

Now, let $n \in \mathbb{N}$ and $p_1, \ldots, p_n \in \Delta_\varepsilon(f) \cap P$. Find $r \in \mathbb{N}$ such that $(r-1)m < n \leq rm$. By the pigeonhole principle, there is an interval $I_{j,\delta}$ where at least r of the polynomials $(p_i)_{i=1}^n$ satisfy $|f(y_j) - p_i(y_j)| > 1$. Put

$$L = \{i \in \{1, \ldots, n\} \colon |f(y_j) - p_i(x)| > 2 - 2\,\varepsilon, x \in I_{j,\delta}\}.$$

We get that

$$\begin{aligned} \left|f(y_j) - \frac{1}{n}\sum_{i=1}^{n} p_i(y_j)\right| &\geq \left|f(y_j) - \frac{1}{n}\sum_{i\in L} p_i(y_j)\right| - \frac{1}{n}\sum_{i\notin L} |p_i(y_j)| \\ &> 1 - \frac{1}{n}\sum_{i\notin L} 1 \\ &\geq \frac{r}{n} \geq \frac{1}{m} > \varepsilon\,. \end{aligned}$$

Hence $f \notin \overline{\mathrm{conv}}(\Delta_\varepsilon(f) \cap P)$. □

In conclusion, we have established the equivalence of Daugavet- and Δ-points in a large class of Müntz spaces along with a simple characterisation for such elements.

Theorem 3.2.14. *Let X be a Müntz space $M_0(\Lambda)$ with $\lambda_1 \geq 1$. The following assertions for $f \in S_X$ are equivalent:*

(i) *f is a Daugavet-point;*

(ii) *f is a Δ-point;*

(iii) $\|f\| = |f(1)|$.

Proof. The implication $(i) \Rightarrow (ii)$ is trivial, $(ii) \Rightarrow (iii)$ follows from Proposition 3.2.13, and $(iii) \Rightarrow (i)$ is the result of Proposition 3.2.10. □

3.3 Daugavet-points in absolute sums

In this section we present the results regarding Daugavet-points in absolute sums. This research was started in [AHLP20] and completed in [HPV]. On the one hand, we explore how the existence of Daugavet-points in the absolute sum of Banach spaces depends on the existence of Daugavet-points in the summands. On the other hand, we investigate how the existence of Daugavet-points in the summands is influenced by the existence of Daugavet-points in the absolute sum. Since these stability results are dependent on the absolute normalised norm that the absolute sum is equipped with, we will start with an overview of the absolute normalised norms that are important in this context.

3.3.1 On some specific absolute normalised norms

Absolute normalised norms on $\mathbb{R}^2$ were already defined in Chapter 1. As mentioned, ℓ_p-norms for $1 \leq p \leq \infty$, are simple examples of such norms. We will now introduce some new types of normalised absolute norms that will turn out to be useful when analysing the existence of Daugavet-points in absolute sums of Banach spaces.

Definition 3.3.1 (see [HLN18] and Figure 3.1)**.** We say that an absolute normalised norm N on $\mathbb{R}^2$ is *positively octahedral (POH)* if there exist non-negative reals a and b with $N(a,b) = 1$ such that

$$N\Big((0,1) + (a,b)\Big) = 2 \quad \text{and} \quad N\Big((1,0) + (a,b)\Big) = 2.$$

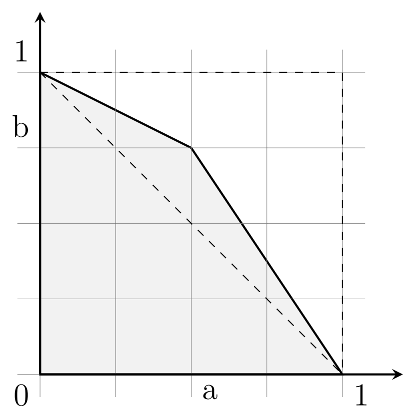

Figure 3.1: The first quadrant of the unit ball of a POH norm.

Example 3.3.2. Both ℓ_1-norm and ℓ_∞-norm, on $\mathbb{R}^2$ are POH. Indeed, in ℓ_1-norm case any element (a,b), $a,b \geq 0$, on the unit sphere will do. In the case of ℓ_∞-norm, the element $(1,1)$ satisfies the conditions.

POH norms have their own importance, for example, it is known that if X and Y are octahedral, then an absolute sum $X \oplus_N Y$ is octahedral if and only if the absolute normalised norm N is POH (see [HLN18, Theorem 3.2]). Recall that a norm on a Banach space X is called *octahedral* if, for every $n \in \mathbb{N}$, $x_1, \ldots, x_n \in S_X$, and every $\varepsilon > 0$, there is a $y \in S_X$ such that $\|x_i + y\| \geq 2 - \varepsilon$ for all $i \in \{1, \ldots, n\}$ (see, e.g. [HLP15]).

In this thesis POH norms are more important from the historical viewpoint, since the first stability results published regarding Daugavet-points were obtained for POH norms. It turns out that the same stability results that hold for POH norms extend to a wider class of absolute normalised norms.

Definition 3.3.3. Let X be a Banach space and $A \subset S_X$. We say that the norm on X is *A-octahedral (A-OH)* if for every $n \in \mathbb{N}$, $x_1, \ldots, x_n \in A$ and every $\varepsilon > 0$ there exists $y \in S_X$ such that $\|x_i + y\| \geq 2 - \varepsilon$ for every $i \in \{1, \ldots, n\}$.

It is obvious that every absolute normalised norm that is octahedral is, by Definition 3.3.3, S_X-OH, and a POH norm is exactly $\{(0,1),(1,0)\}$-OH. This justifies the name of this more general term, A-octahedrality, that we use to describe absolute sums, or more precisely absolute normalised norms, for which the absolute sums possess Daugavet-points

Consider an absolute normalised norm N on $\mathbb{R}^2$. We now define a specific set A. Set

$$c = \max_{N(e,1)=1} e \quad \text{and} \quad d = \max_{N(1,f)=1} f, \tag{3.3.1}$$

and define

$$A = \{(c,1),(1,d)\}.$$

Suppose that the norm N is A-OH. By Definition 3.3.3, there exists $(a,b) \in \mathbb{R}^2$ with the following property:

$$\begin{aligned} &a, b \geq 0, \quad N(a,b) = 1, \quad \text{and} \\ &N\big((a,b)+(c,1)\big) = 2 \quad \text{and} \quad N\big((a,b)+(1,d)\big) = 2. \end{aligned} \tag{3.3.2}$$

In the following, we will always consider an absolute normalised norm N to be A-OH for the specific set $A = \{(c,1),(1,d)\}$, where $c, d \geq 0$ are as in (3.3.1) (see also Figure 3.2).

It is not hard to see, that there are absolute normalised norms on $\mathbb{R}^2$ that are not A-OH. The absolute normalised norms with the following property, in fact, describe all absolute normalised norms on $\mathbb{R}^2$ that are not A-OH.

Definition 3.3.4. We will say that an absolute normalised norm N on $\mathbb{R}^2$ has *property* (α) if for all non-negative reals a and b with $N(a,b) = 1$, there exist $\varepsilon > 0$ and a neighbourhood W of (a,b) with

$$\sup_{(c,d)\in W} c < 1 \quad \text{or} \quad \sup_{(c,d)\in W} d < 1,$$

such that $(c,d) \in W$ for all non-negative reals c and d with

$$N(c,d) = 1 \quad \text{and} \quad N\big((a,b)+(c,d)\big) \geq 2 - \varepsilon.$$

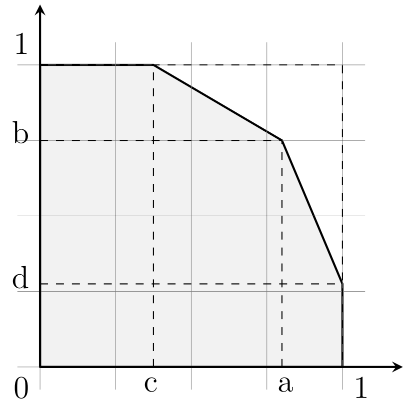

Figure 3.2: The first quadrant of the unit ball of an A-OH norm.

Example 3.3.5. The ℓ_p-norm, $1 < p < \infty$, on $\mathbb{R}^2$ has property (α). Indeed, given $c, d \geq 0$ with $\|(c,d)\|_p = 1$ for all $\delta > 0$ there exists $\varepsilon > 0$ such that for all (a,b) with $\|(a,b)\|_p \leq 1$ and $\|(a,b)+(c,d)\|_p \geq 2-\varepsilon$ we have $(a,b) \in B((c,d),\delta) =: W$. Choosing δ small enough we have either $\sup_{(a,b)\in W} a < 1$ or $\sup_{(a,b)\in W} b < 1$.

Similarly, any strictly convex absolute normalised norm N on $\mathbb{R}^2$ has property (α) (see also Figure 3.3).

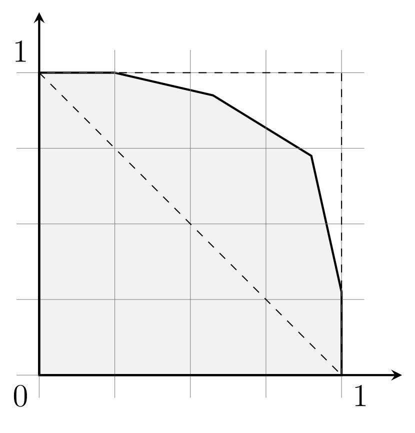

Figure 3.3: The first quadrant of the unit ball of a norm with property (α).

Our next aim is to show that every absolute normalised norm on $\mathbb{R}^2$ is either A-OH or has property (α). For that it is useful to notice that the $\varepsilon > 0$ in Definition 3.3.4 can be relaxed and we may let $\varepsilon = 0$.

Lemma 3.3.6. *Property (α) can also be formulated in the following way:*

(α) *for all non-negative reals a and b with $N(a,b)=1$, there exists a neighbourhood W of (a,b) with*

$$\sup_{(c,d)\in W} c<1 \quad \textit{or} \quad \sup_{(c,d)\in W} d<1,$$

such that $(c,d)\in W$ for all non-negative reals c and d with

$$N(c,d)=1 \quad \textit{and} \quad N\big((a,b)+(c,d)\big)=2.$$

Proof. On the one hand, it is obvious that if the statement holds for some $\varepsilon>0$, then it also holds for $\varepsilon=0$.

One the other hand, let us assume that the statement holds for the case $\varepsilon=0$. Choose a neighbourhood of (a,b) such that $\sup_{(e,f)\in W} e<1$ (the case $\sup_{(e,f)\in W} f<1$ is analogical). Fix $c\in(\sup_{(e,f)\in W} e, 1)$ and let $d\geq 0$ be such that $N(c,d)=1$. Then $N\big((a,b)+(c,d)\big)<2$. Set $\varepsilon=2-N\big((a,b)+(c,d)\big)$ and $\widetilde{W}=\{(x,y)\colon x\leq c\}$. Then for every $x,y\geq 0$ with $N(x,y)=1$ and $N\big((a,b)+(x,y)\big)\geq 2-\varepsilon$, we have that $x\leq c$ which means that $(x,y)\in\widetilde{W}$. □

Now we confirm that there is a dichotomy between absolute normalised norms with property (α) and absolute normalised norms that are A-OH for the set $A=\{(c,1),(1,d)\}$, where c,d are as in (3.3.1).

Proposition 3.3.7. *Every absolute normalised norm on $\mathbb{R}^2$ either has property (α) or is A-OH.*

Proof. Let N be an absolute normalised norm that does not have property (α). Then there exist $a,b\geq 0$ with $N(a,b)=1$ such that for every (a,b)-neighbourhood W which satisfies either $\sup_{(c,d)\in W} c<1$ or $\sup_{(c,d)\in W} d<1$, there exist $c,d\geq 0$ with $N(c,d)=1$ such that $(c,d)\notin W$ and $N\big((a,b)+(c,d)\big)=2$. To show that N is A-OH, we need to find $c,d\geq 0$ satisfying

$$N(c,1)=N(1,d)=1 \quad \text{and} \quad N\big((a,b)+(c,1)\big)=N\big((a,b)+(1,d)\big)=2.$$

If $a=1$ (respectively, $b=1$), then take $d=b$ (respectively, $c=a$). However, if $a\neq 1$, then for every $n\in\mathbb{N}$ large enough ($a<1-1/n$), by taking $W_n=\{(x,y)\colon x\leq 1-1/n\}$, we can find $c_n,d_n\geq 0$ with $N(c_n,d_n)=1$ such that $(c_n,d_n)\notin W_n$ and $N\big((a,b)+(c_n,d_n)\big)=2$. Passing to a subsequence if necessary, we can assume that $(c_n,d_n)\to(1,d)$ for some $d\geq 0$. Obviously $N(1,d)=1$ and $N\big((a,b)+(1,d)\big)=2$. It can be proved similarly that if $b\neq 1$, then there exists $c\geq 0$ with $N(c,1)=1$ and $N\big((a,b)+(c,1)\big)=2$. Combining these facts we have that N is A-OH. □

3.3.2 From summands to absolute sum

In the following we analyse the existence of Daugavet-points in the absolute sum of Banach spaces assuming the summands have Daugavet-points. It turns out that the existence of Daugavet-points in an absolute sum depends greatly on the absolute normalised norm. We start with absolute normalised norms with property (α) and then move on to the case of A-OH norms.

The following result shows that in the case where the absolute sum is equipped with a norm with property (α), the absolute sum cannot have any Daugavet-points.

Proposition 3.3.8. *Let X and Y be Banach spaces and N an absolute normalised norm on $\mathbb{R}^2$ with property (α). Then $X \oplus_N Y$ has no Daugavet-points.*

Proof. Let X and Y be Banach spaces and N an absolute normalised norm on $\mathbb{R}^2$ with property (α). Denote $Z = X \oplus_N Y$ and let $z = (x, y) \in S_Z$.

Let $(c, d) = (\|x\|, \|y\|)$. From the definition of property (α) there exists $\varepsilon > 0$ and a neighbourhood W of (c, d). Without loss of generality we may assume that $\sup_{(a,b)\in W} a < 1$ since the case $\sup_{(a,b)\in W} b < 1$ is similar. Choose $\delta > 0$ such that $\sup_{(a,b)\in W} \leq 1 - \delta$.

Assume that $(u, v) \in \Delta_\varepsilon(z)$. Then

$$2 - \varepsilon \leq N(\|u - x\|, \|v - y\|) \leq N(\|u\| + \|x\|, \|v\| + \|y\|),$$

hence, $(\|u\|, \|v\|) \in W$ from property (α). In particular, $\|u\| \leq 1 - \delta$.

Let $w \in S_X$ and consider $(w, 0) \in S_Z$. Given $(x_1, y_1), \ldots, (x_n, y_n) \in \Delta_\varepsilon(z)$ we have $\|x_i\| \leq 1 - \delta$ for each $i = 1, \ldots, n$ and we get that

$$\begin{aligned} \left\|(w, 0) - \frac{1}{n}\sum_{i=1}^{n}(x_i, y_i)\right\|_N &\geq \left\|w - \frac{1}{n}\sum_{i=1}^{n} x_i\right\| \\ &\geq \|w\| - \frac{1}{n}\sum_{i=1}^{n}\|x_i\| \\ &\geq 1 - \frac{1}{n}\sum_{i=1}^{n}(1 - \delta) = \delta, \end{aligned}$$

i.e $(w, 0) \notin \overline{\operatorname{conv}}\, \Delta_\varepsilon(z)$. Consequently, z is not a Daugavet-point. □

Despite the fact that all absolute sums with an absolute normalised norm with property (α) fail to have Daugavet-points, we also have positive stability results. Initially we managed to show that an absolute sum equipped with a POH norm has Daugavet-points provided the summands have (see [AHLP20, Proposition 4.3]). Not long after, this result was extended to all A-OH norms.

Theorem 3.3.9. *Let X and Y be Banach spaces, $x \in S_X$, $y \in S_Y$, and let N be an A-OH norm with (a,b) as in (3.3.2). If x and y are Daugavet-points in X and Y, respectively, then (ax, by) is a Daugavet-point in $X \oplus_N Y$.*

Proof. Assume that x and y are Daugavet-points. Set $Z = X \oplus_N Y$ and fix $f = (x^*, y^*) \in S_{Z^*}$, $\alpha > 0$, and $\varepsilon > 0$. Choose $\delta > 0$ to satisfy $\delta N(1,1) < \varepsilon$. By Lemma 3.1.4 we obtain $u \in B_X$ and $v \in B_Y$ such that

$$x^*(u) \geq \Big(1 - \frac{\alpha}{2}\Big)\|x^*\| \quad \text{and} \quad y^*(v) \geq \Big(1 - \frac{\alpha}{2}\Big)\|y^*\|$$

and

$$\|x - u\| \geq 2 - \delta \quad \text{and} \quad \|y - v\| \geq 2 - \delta.$$

By the properties of absolute normalised norms and A-OH norms, there exist $k, l \geq 0$ such that

$$N(k,l) = 1, \quad N\big((a,b) + (k,l)\big) = 2, \quad \text{and} \quad k\|x^*\| + l\|y^*\| = 1.$$

Therefore $(ku, lv) \in S(B_Z, f, \alpha)$, because

$$f(ku, lv) = kx^*(u) + ly^*(v) \geq \Big(1 - \frac{\alpha}{2}\Big)(k\|x^*\| + l\|y^*\|) > 1 - \alpha.$$

On the other hand, from $\|x-u\| \geq 2 - \delta$ and $\|y - v\| \geq 2 - \delta$, we get that

$$\Big\|ax - ku\Big\| \geq a + k - \delta \quad \text{and} \quad \Big\|by - lv\Big\| \geq b + l - \delta.$$

In conclusion we have that

$$\begin{aligned}
\|(ax, by) - (ku, lv)\|_N &= N(\|ax - ku\|, \|by - lv\|) \\
&\geq N(a + k - \delta, b + l - \delta) \\
&\geq N(a + k, b + l) - N(\delta, \delta) \\
&= N\big((a,b) + (k,l)\big) - \delta N(1,1) \\
&> 2 - \varepsilon,
\end{aligned}$$

which means that by Lemma 3.1.4 (ax, by) is a Daugavet-point. □

Since every absolute normalised norm either has property (α) or is A-OH, we now have from Proposition 3.3.8 and Theorem 3.3.9 a clear picture of the existence of Daugavet-points in absolute sums of Banach spaces, at least when both summands have Daugavet-points. In some cases, however, it is enough to assume that only one summand has Daugavet-points to guarantee that the absolute sum has Daugavet-points. The following two results describe these special occasions. The proofs are very similar to the proof of Theorem 3.3.9 and therefore omitted.

Proposition 3.3.10. *Let X and Y be Banach spaces, $x \in S_X$ and $y \in S_Y$, and let N be an A-OH norm with (a, b) as in (3.3.2).*

(a) *If $b = 0$ and x is a Daugavet-point in X, then $(ax, by) = (x, 0)$ is a Daugavet-point in $X \oplus_N Y$.*

(b) *If $a = 0$ and y is a Daugavet-point in Y, then $(ax, by) = (0, y)$ is a Daugavet-point in $X \oplus_N Y$.*

Proposition 3.3.11. *Let X and Y be Banach spaces, $x \in S_X$ and $y \in S_Y$.*

(a) *If x is a Daugavet-point in X, then (x, by) is a Daugavet-point in $X \oplus_\infty Y$ for every $b \in [0, 1]$.*

(b) *If y is a Daugavet-point in Y, then (ax, y) is a Daugavet-point in $X \oplus_\infty Y$ for every $a \in [0, 1]$.*

3.3.3 From absolute sum to summands

Now we consider the stability results of Daugavet-points in the other direction, i.e. the existence of Daugavet-points in the summands provided the absolute sum has Daugavet-points.

Due to Proposition 3.3.8 there is nothing to research in the case of absolute normalised norms with property (α), and according to Proposition 3.3.7 it is enough to consider A-OH norms. We show that the existence of Daugavet-points in an absolute sum assures the existence of Daugavet-points in the summands for all A-OH norms. For technical reasons we exclude the ℓ_∞-norm at first.

Theorem 3.3.12. *Let X and Y be Banach spaces, $x \in B_X$, $y \in B_Y$, and let N be an absolute normalised norm on $\mathbb{R}^2$, different from ℓ_∞-norm. Assume that (x, y) is a Daugavet-point in $X \oplus_N Y$.*

(a) *If $x \neq 0$, then $x/\|x\|$ is a Daugavet-point in X.*

(b) *If $y \neq 0$, then $y/\|y\|$ is a Daugavet-point in Y.*

Proof. We prove only the first statement, the second can be proved similarly. Suppose that $x \neq 0$. Fix $x^* \in S_{X^*}$, $\alpha > 0$, and $\varepsilon > 0$. We will find $u \in S(B_X, x^*, \alpha)$ such that $\|x/\|x\| - u\| \geq 2 - \varepsilon$. Set $f = (x^*, 0)$ and $Z = X \oplus_N Y$. Then $f \in S_{Z^*}$. Using Lemma 1.4.14 choose $\delta > 0$ such that, for every $p, q, r \geq 0$, if

$$2 - \delta \leq N(p, q) \leq N(r, q) \leq 2 \quad \text{and} \quad q < 2 - \delta,$$

then $|p-r| < \|x\|\varepsilon/2$. There is no loss of generality in assuming that $\delta \le \varepsilon/2$, $\delta \le \alpha$, and $(1-\delta)N(1,1) > 1$ (here we use the fact that $N(1,1) > 1$, i.e. N is not ℓ_∞-norm).

Since (x,y) is a Daugavet-point in Z, there exists $(u,v) \in S(B_Z, f, \delta)$ such that $\|(x,y)-(u,v)\|_N \ge 2-\delta$. Consequently,

$$x^*(u) = f(u,v) > 1-\delta \ge 1-\alpha,$$

which gives us that $u \in S(B_X, x^*, \alpha)$ and $\|u\| > 1-\delta$. We also conclude that $\|v\| < 1-\delta$, because otherwise

$$N(\|u\|,\|v\|) \ge N(1-\delta, 1-\delta) = (1-\delta)N(1,1) > 1,$$

a contradiction. In addition we have that

$$2-\delta \le N\Big(\|x-u\|, \|y-v\|\Big) \le N\Big(\|x\|+\|u\|, \|y-v\|\Big) \le 2$$

and

$$\|y-v\| \le \|y\|+\|v\| < 1+1-\delta = 2-\delta.$$

Hence, by the choice of δ, we have that

$$\Big|\|x-u\| - (\|x\|+\|u\|)\Big| < \|x\|\frac{\varepsilon}{2}.$$

Thus $\|x-u\| > \|x\|+\|u\| - \|x\|\varepsilon/2$, and therefore,

$$\begin{aligned}
\left\|\frac{x}{\|x\|} - u\right\| &= \left\|\frac{1}{\|x\|}(x-u) - \Big(u - \frac{1}{\|x\|}u\Big)\right\| \\
&\ge \frac{1}{\|x\|}\|x-u\| - \Big(\frac{1}{\|x\|} - 1\Big)\|u\| \\
&\ge \frac{1}{\|x\|}\Big(\|x\|+\|u\| - \|x\|\frac{\varepsilon}{2}\Big) - \frac{\|u\|}{\|x\|} + \|u\| \\
&= 1 + \|u\| - \frac{\varepsilon}{2} \\
&> 1 + 1 - \delta - \frac{\varepsilon}{2} \\
&\ge 2-\varepsilon.
\end{aligned}$$

According to Lemma 3.1.4, the element $x/\|x\|$ is a Daugavet-point in X. □

We now move to the case of ℓ_∞-sum. Recall that according to Proposition 3.3.11 ℓ_∞-sum can have Daugavet-points even if one of the summands fails to have Daugavet-points. However, assuming that ℓ_∞-sum has Daugavet-points, we will prove, that at least one of the summands has Daugavet-points.

Theorem 3.3.13. *Let X and Y be Banach spaces, $x \in B_X$ and $y \in B_Y$. Assume that (x, y) is a Daugavet-point in $X \oplus_\infty Y$. Then x is a Daugavet-point in X or y is a Daugavet-point in Y.*

Proof. Firstly, look at the case, where only one of x and y has norm 1, and the other has norm less than 1. Assume that $\|y\| < 1$ and let us prove that x is a Daugavet-point in X. (The statement, if $\|x\| < 1$, then y is a Daugavet-point in Y, can be proved similarly.) Fix $x^* \in S_{X^*}$, $\alpha > 0$ and $\varepsilon > 0$. Choose $\delta > 0$ such that $\delta \leq \varepsilon$ and $\|y\| < 1 - \delta$. Set $Z = X \oplus_\infty Y$ and $f = (x^*, 0) \in S_{Z^*}$. Since (x, y) is a Daugavet-point then there exists $(u, v) \in S(B_Z, f, \alpha)$ such that $\|(x, y) - (u, v)\|_\infty \geq 2 - \delta$. Therefore, $x^*(u) = f(u, v) > 1 - \alpha$, i.e. $u \in S(B_X, x^*, \alpha)$ and

$$\|y - v\| \leq \|y\| + \|v\| < 1 - \delta + 1 = 2 - \delta.$$

Combining this with the fact that

$$\|(x, y) - (u, v)\|_\infty = \max\{\|x - u\|, \|y - v\|\} \geq 2 - \delta,$$

we get that $\|x - u\| \geq 2 - \delta \geq 2 - \varepsilon$. Thus, by Lemma 3.1.4, x is a Daugavet-point in X.

Secondly, consider the case, where both x and y are of norm 1, and neither of them is a Daugavet-point. Then we can fix slices $S(B_X, x^*, \alpha)$ and $S(B_Y, y^*, \alpha)$, and $\varepsilon > 0$ such that $S(B_X, x^*, \alpha) \cap \Delta_\varepsilon(x) = \emptyset$ and $S(B_Y, y^*, \alpha) \cap \Delta_\varepsilon(y) = \emptyset$. There is no loss of generality in assuming that $\alpha < \varepsilon < 1$. Set $f = 1/2(x^*, y^*)$ and $Z = X \oplus_\infty Y$, and consider the slice $S(B_Z, f, \alpha/2)$. Note that

$$S(B_Z, f, \alpha/2) \subset S(B_X, x^*, \alpha) \times S(B_Y, y^*, \alpha).$$

Let $(u, v) \in S_Z \cap S(B_Z, f, \alpha/2)$ be arbitrary. Then

$$\|u\| > 1 - \alpha > 0 \quad \text{and} \quad \|v\| > 1 - \alpha > 0,$$

and

$$u/\|u\| \in S(B_X, x^*, \alpha) \quad \text{and} \quad v/\|v\| \in S(B_Y, y^*, \beta).$$

Therefore

$$\begin{aligned}
\|(u, v) - (x, y)\|_\infty &= \max\{\|u - x\|, \|v - y\|\} \\
&\leq \max\left\{\left\|u - \frac{u}{\|u\|}\right\| + \left\|\frac{u}{\|u\|} - x\right\|, \left\|v - \frac{v}{\|v\|}\right\| + \left\|\frac{v}{\|v\|} - y\right\|\right\} \\
&< \alpha + 2 - \varepsilon \\
&= 2 - (\varepsilon - \alpha).
\end{aligned}$$

As a result, $S(B_Z, f, \alpha/2) \cap \Delta_{\varepsilon - \alpha}(x, y) = \emptyset$, which by Lemma 3.1.4 implies that (x, y) is not a Daugavet-point. $\square$

3.4 Δ-points in absolute sums

This section is dedicated to results concerning the stability of Δ-points under absolute sums. We showed already in [AHLP20] that for any absolute normalised norm the existence of Δ-points in the summands easily ensures the existence of Δ-points in the absolute sum. This result was enough to separate the notions Daugavet-point and Δ-point. The stability results in the other direction were established in [HPV]. More precisely, in all absolute sums with the exception of ℓ_∞-sum, the existence of Δ-points in the absolute sum guarantees the existence of Δ-points in the summands. From here on, we mostly consider arbitrary absolute normalised norms N. We start with the result in the direction from the summands to the absolute sum.

Proposition 3.4.1. *Let X and Y be Banach spaces, x a Δ-point in X, and y a Δ-point in Y. If $a, b \geq 0$ with $N(a,b) = 1$, then (ax, by) is a Δ-point in Z.*

Proof. Let $\varepsilon > 0$ and $0 < \gamma < \varepsilon$. Since x is a Δ-point in X and y a Δ-point in Y, we have $x_1, \ldots, x_m \in \Delta_\varepsilon^X(x)$ and $y_1, \ldots, y_m \in \Delta_\varepsilon^Y(y)$ such that

$$\Big\|x - \frac{1}{m}\sum_{i=1}^m x_i\Big\| < \gamma \quad \text{and} \quad \Big\|y - \frac{1}{m}\sum_{i=1}^m y_i\Big\| < \gamma.$$

Note that

$$\begin{aligned}\Big\|(ax, by) - \frac{1}{m}\sum_{i=1}^m \Big(ax_i, by_i\Big)\Big\|_N &= N\Big(a\Big\|x - \frac{1}{m}\sum_{i=1}^m x_i\Big\|, b\Big\|y - \frac{1}{m}\sum_{i=1}^m y_i\Big\|\Big) \\ &\leq N(\gamma a, \gamma b) \\ &= \gamma N(a,b) = \gamma,\end{aligned}$$

and

$$\begin{aligned}\Big\|(ax, by) - (ax_i, by_i)\Big\|_N &= N(a\|x - x_i\|, b\|y - y_i\|) \\ &\geq N(a(2-\varepsilon), b(2-\varepsilon)) \\ &= (2-\varepsilon)N(a,b) \\ &= 2 - \varepsilon.\end{aligned}$$

□

Recall, that it was already proved in [IK04] using slices, that the DLD2P is stable under all absolute normalised norms (see [IK04, Theorem 3.2] or Theorem 2.2.1). The stability in one direction can also be seen as a simple consequence of Proposition 3.4.1.

Corollary 3.4.2. *Let X and Y be Banach spaces and N an absolute normalised norm on $\mathbb{R}^2$. If X and Y both have the DLD2P, then $X \oplus_N Y$ has the DLD2P.*

Proof. Recall that $Z = X \oplus_N Y$ has the DLD2P if and only if every element of the unit sphere is a Δ-point. Let $(x,y) \in S_Z$ be arbitrary. It is obvious that

$$\begin{aligned} 1 = \|(x,y)\|_N &= \left\| \left(\|x\| \frac{x}{\|x\|}, \|y\| \frac{y}{\|y\|} \right) \right\| \\ &= N\left(\|x\| \left\| \frac{x}{\|x\|} \right\|, \|y\| \left\| \frac{y}{\|y\|} \right\| \right) \\ &= N(\|x\|, \|y\|). \end{aligned}$$

Therefore, since by assumption $x/\|x\|$ is a Δ-point in X and $y/\|y\|$ is a Δ-point in Y, we have by Proposition 3.4.1 that (x,y) is a Δ-point in Z. $\square$

In contrast, as shown in the previous section, there are absolute normalised norms N for which the space $X \oplus_N Y$ has no Daugavet-points. There even exists a space where every point of the unit sphere is a Δ-point, but none of them are Daugavet-points.

Example 3.4.3. Consider the space $X = C[0,1] \oplus_2 C[0,1]$. Since $C[0,1]$ has the Daugavet property and in particular the DLD2P, then X has the DLD2P (see [IK04, Theorem 3.2]). But, by Proposition 3.3.8, X has no Daugavet-points even though every $x \in S_X$ is a Δ-point.

We now move on to the stability results in the other direction, i.e. from the absolute sum to summands. Firstly, we show that for most absolute sums the summands have Δ-points, provided the absolute sums do.

Theorem 3.4.4. *Let X and Y be Banach spaces, $x \in S_X$, $y \in S_Y$, N an absolute normalised norm on $\mathbb{R}^2$, and $a, b \geq 0$ such that $N(a,b) = 1$. Assume that (ax, by) is a Δ-point in $X \oplus_N Y$.*

(a) *If $b \neq 1$, then x is a Δ-point in X.*

(b) *If $a \neq 1$, then y is a Δ-point in Y.*

Proof. We prove only the first statement, the second can be proved similarly. Assume that $b \neq 1$. Note that then $a \neq 0$. Let $c, d \geq 0$ be such that $N^*(c,d) = 1$ and $ac + bd = 1$.

Suppose that x is not a Δ-point in X. Then, by Lemma 3.1.5, there exist $x^* \in S_{X^*}$, $\alpha > 0$, and $\varepsilon > 0$ such that

$$x \in S(B_X, x^*, \alpha) \quad \text{and} \quad S(B_X, x^*, \alpha) \cap \Delta_\varepsilon(x) = \emptyset.$$

Let $y^* \in S_{Y^*}$ be such that $y^*(y) = 1$ and let $f = (cx^*, (1-\alpha)dy^*)$. Then

$$f(ax, by) = acx^*(x) + (1-\alpha)bdy^*(y) > (1-\alpha)(ac + bd) = 1 - \alpha.$$

Choose $\beta, \gamma > 0$ such that $\beta < a\varepsilon$ and $\beta < \gamma\varepsilon$, and

$$f(ax, by) > 1 - (\alpha - \gamma).$$

Now, using Lemma 1.4.14, choose $\delta > 0$ such that, for every $p, q, r \geq 0$, if

$$2 - \delta \leq N(p, q) \leq N(r, q) \leq 2 \quad \text{and} \quad q < 2 - \delta,$$

then $|p - r| < \beta$. There is no loss of generality in assuming that $b < 1 - \delta$. There exists $(u, v) \in B_Z$, where $Z = X \oplus_N Y$, such that

$$f(u, v) > 1 - (\alpha - \gamma) \quad \text{and} \quad \|(ax, by) - (u, v)\|_N \geq 2 - \delta.$$

Then

$$\begin{aligned} cx^*(u) + (1-\alpha)d\|v\| &\geq cx^*(u) + (1-\alpha)dy^*(v) \\ &= f(u, v) \\ &> 1 - (\alpha - \gamma) \\ &> 1 - \alpha \\ &\geq (1-\alpha)(c\|u\| + d\|v\|), \end{aligned}$$

which yields

$$cx^*(u) > (1-\alpha)c\|u\|,$$

i.e. $x^*(u/\|u\|) > 1 - \alpha$. Since $S(B_X, x^*, \alpha) \cap \Delta_\varepsilon(x) = \emptyset$, we know now that $\|x - u/\|u\|\| < 2 - \varepsilon$. We now show that $\|ax - u\| < a + \|u\| - \beta$. Let us consider two cases. If $\|u\| \geq a$, then

$$\begin{aligned} \|ax - u\| &\leq \left\|ax - a\frac{u}{\|u\|}\right\| + \left\|a\frac{u}{\|u\|} - u\right\| \\ &\leq a(2 - \varepsilon) + \Big|a - \|u\|\Big| \\ &= a + \|u\| - a\varepsilon \\ &< a + \|u\| - \beta. \end{aligned}$$

On the other hand, if $a \geq \|u\|$, we have

$$\begin{aligned} c\|u\| + (1-\alpha)d\|v\| &\geq cx^*(u) + (1-\alpha)dy^*(v) \\ &= f(u, v) \\ &> 1 - \alpha + \gamma \\ &\geq (1-\alpha)d\|v\| + \gamma, \end{aligned}$$

from which we conclude $\|u\| \geq c\|u\| > \gamma$. Now we see that

$$\begin{aligned}\|ax - u\| &\leq \Big\|ax - \|u\|x\Big\| + \Big\|\|u\|x - u\Big\| \\ &\leq a - \|u\| + \|u\|(2-\varepsilon) \\ &= a + \|u\| - \|u\|\varepsilon \\ &< a + \|u\| - \gamma\varepsilon \\ &< a + \|u\| - \beta.\end{aligned}$$

That gives us the following:

$$\begin{aligned}2 - \delta \leq \|(ax, by) - (u, v)\|_N &= N(\|ax - u\|, \|by - v\|) \\ &\leq N(a + \|u\| - \beta, b + \|v\|)\end{aligned}$$

and therefore

$$2 - \delta \leq N(a + \|u\| - \beta, b + \|v\|) \leq N(a + \|u\|, b + \|v\|) \leq 2.$$

Since $b + \|v\| < 2 - \delta$, we have by the choice of δ that

$$\Big|(a + \|u\| - \beta) - (a + \|u\|)\Big| < \beta,$$

i.e. $\beta < \beta$, a contradiction. Hence x is a Δ-point in X. □

Theorem 3.4.4 does not cover the case $a = b = 1$ (for ℓ_∞-norm). Now we will show that in this case, if $x \in S_X$ and $y \in S_Y$, then (x, y) can be a Δ-point even if x and y are not Δ-points. Moreover, we introduce the conditions that $x \in S_X$ and $y \in S_Y$ must satisfy in order for (x, y) to be a Δ-point in $X \oplus_\infty Y$. The corresponding results rely heavily on the concept of another type of elements of the unit sphere similar to Δ-points (compare with Lemma 3.1.5).

Definition 3.4.5. Let X be a Banach space, $x \in S_X$, and $k > 1$. We say that x is a Δ_k*-point* in X, if for every $S(B_X, x^*, \alpha)$ with $x \in S(B_X, x^*, \alpha)$ and for every $\varepsilon > 0$ there exists $u \in S(B_X, x^*, k\alpha)$ such that $\|x - u\| \geq 2 - \varepsilon$.

It is obvious that every Δ-point is a Δ_k-point for every $k > 1$. The reverse, however, does not hold, since the upcoming example shows the existence of a Δ_k-point that is not a Δ-point, which proves that the concepts Δ-point and Δ_k-point are not the same.

Example 3.4.6. Let X and Y be Banach spaces, $x \in S_X$ and $y \in S_Y$, and let $k > 1$. Set $Z = X \oplus_1 Y$ and $z = \big((1 - 1/k)x, y/k\big)$. Assume that x is not a Δ-point in X and y is a Δ-point in Y. Then, according to Theorem 3.4.4, z is not a Δ-point in Z.

Fix $f = (x^*, y^*) \in S_{Z^*}$ and $\alpha > 0$, such that $f(z) > 1 - \alpha$, and fix $\varepsilon > 0$. Then

$$1 - \frac{1}{k} + \frac{1}{k} y^*(y) \geq \Big(1 - \frac{1}{k}\Big) x^*(x) + \frac{1}{k} y^*(y) = f(z) > 1 - \alpha.$$

It follows that $y^*(y) > 1 - \alpha k$. Since y is a Δ-point in Y, there exists $v \in B_Y$ such that

$$y^*(v) > 1 - \alpha k \quad \text{and} \quad \|y - v\| \geq 2 - \varepsilon.$$

Then $f(0, v) = y^*(v) > 1 - \alpha k$, i.e. $(0, v) \in S(B_Z, f, \alpha k)$, and

$$\begin{aligned} \Big\| \Big(\Big(1 - \frac{1}{k}\Big) x, \frac{1}{k} y \Big) - (0, v) \Big\|_1 &= \Big(1 - \frac{1}{k}\Big) \|x\| + \Big\| \frac{1}{k} y - v \Big\| \\ &\geq \Big(1 - \frac{1}{k}\Big) + \|y - v\| - \Big(1 - \frac{1}{k}\Big) \|y\| \\ &\geq 2 - \varepsilon. \end{aligned}$$

This proves that z is a Δ_k-point.

Remarkably, (x, y) with $x \in S_X$, $y \in S_Y$ can be a Δ-point in $X \oplus_\infty Y$ even if neither x nor y is a Δ-point in X and Y, respectively. This is a direct consequence of Example 3.4.6 and the following proposition.

Proposition 3.4.7. *Let X and Y be Banach spaces, $x \in S_X$ and $y \in S_Y$. Let $p, q > 1$ satisfy $1/p + 1/q = 1$.*

(a) If x is a Δ_p-point in X and y is a Δ_q-point in Y, then (x, y) is a Δ-point in $X \oplus_\infty Y$.

(b) If x is not a Δ_p-point in X and y is not a Δ_q-point in Y, then (x, y) is not a Δ-point in $X \oplus_\infty Y$.

Proof. (*a*) Assume that x is a Δ_p-point in X and y is Δ_q-point in Y. Set $Z = X \oplus_\infty Y$. Fix $f = (x^*, y^*) \in S_{Z^*}$ and $\alpha > 0$ such that $(x, y) \in S(B_Z, f, \alpha)$, and fix $\varepsilon > 0$. Then

$$x^*(x) + y^*(y) = f(x, y) > 1 - \alpha$$

from what we get

$$x^*(x) > 1 - (\alpha + y^*(y)) = \|x^*\| - \Big(\alpha + y^*(y) - \|y^*\|\Big).$$

We now show that there exists $(u,v) \in S(B_Z, f, \alpha)$ such that

$$\|(x,y)-(u,v)\|_\infty \geq 2-\varepsilon.$$

Let us consider two cases. If $\alpha + y^*(y) - \|y^*\| \leq \alpha/p$, then $x^*(x) > \|x^*\| - \alpha/p$ and therefore, since x is a Δ_p-point, there exists $u \in B_X$ such that $x^*(u) > \|x^*\| - \alpha$ and $\|x-u\| \geq 2-\varepsilon$. Let $v \in B_Y$ be such that

$$f(u,v) = x^*(u) + y^*(v) > \|x^*\| - \alpha + \|y^*\| = 1-\alpha.$$

Then also $\|(x,y)-(u,v)\|_\infty = \max\{\|x-u\|, \|y-v\|\} \geq 2-\varepsilon$.

If $\alpha + y^*(y) - \|y^*\| > \alpha/p$, then

$$y^*(y) > \|y^*\| - \alpha + \frac{\alpha}{p} = \|y^*\| - \frac{1}{q}\alpha$$

and analogically, using the fact that y is a Δ_q-point, we can find $(u,v) \in S(B_Z, f, \alpha)$ such that $\|(x,y)-(u,v)\|_\infty \geq 2-\varepsilon$. Therefore (x,y) is a Δ-point.

(b) Assume that x is not a Δ_p-point in X and y is not a Δ_q-point in Y. By definition there exist $x^* \in S_{X^*}$, $y^* \in S_{Y^*}$, and $\alpha_1, \alpha_2, \varepsilon > 0$, with $x \in S(B_X, x^*, \alpha_1)$ and $y \in S(B_X, y^*, \alpha_2)$ such that for every $u \in S(B_X, x^*, p\alpha_1)$ and for every $v \in S(B_Y, y^*, q\alpha_2)$ we have

$$\|x-u\| < 2-\varepsilon \quad \text{and} \quad \|y-v\| < 2-\varepsilon.$$

Set $Z = X \oplus_\infty Y$. Let $\lambda \in (0,1)$ satisfy $(1-\lambda)/\lambda = (p\alpha_1)/(q\alpha_2)$, let $\alpha = \lambda\alpha_1 + (1-\lambda)\alpha_2$ and let $f = (\lambda x^*, (1-\lambda)y^*) \in S_{Z^*}$. Then

$$\begin{aligned} f(x,y) &= \lambda x^*(x) + (1-\lambda)y^*(y) \\ &> \lambda(1-\alpha_1) + (1-\lambda)(1-\alpha_2) \\ &= 1-\alpha. \end{aligned}$$

Fix $(u,v) \in S(B_Z, f, \alpha)$. From

$$1-\alpha < f(u,v) = \lambda x^*(u) + (1-\lambda)y^*(v) \leq \lambda x^*(u) + 1 - \lambda$$

we get that

$$x^*(u) > 1 - \frac{\alpha}{\lambda} = 1 - \Big(\alpha_1 + \frac{1-\lambda}{\lambda}\alpha_2\Big) = 1 - p\Big(\frac{\alpha_1}{p} + \frac{\alpha_1}{q}\Big) = 1 - p\alpha_1.$$

Therefore $u \in S(B_X, x^*, p\alpha_1)$ and analogically $v \in S(B_Y, y^*, q\alpha_2)$. It follows that $\|x-u\| < 2-\varepsilon$ and $\|y-v\| < 2-\varepsilon$. Consequently,

$$\|(x,y)-(u,v)\|_\infty = \max\{\|x-u\|, \|y-v\|\} < 2-\varepsilon$$

and thus, (x,y) is not a Δ-point. □

In fact, as we can see from the following result, Proposition 3.4.7 gives an equivalent condition for (x, y) being a Δ-point in $X \oplus_\infty Y$ where neither x nor y is a Δ-point in X and Y, respectively.

Proposition 3.4.8. *Let X and Y be Banach spaces and $x \in S_X$ and $y \in S_Y$. Assume that neither x nor y is a Δ-point in X and Y, respectively. Then the following statements are equivalent:*

- (i) *there exist $p, q > 1$ with $1/p + 1/q = 1$ such that x is Δ_p-point in X and y is Δ_q-point in Y;*
- (ii) *for every $p, q > 1$ with $1/p + 1/q = 1$ either x is Δ_p-point in X or y is Δ_q-point in Y.*

Proof. $(i) \Rightarrow (ii)$. Assume that (i) holds. Let $p, q > 1$ be such that $1/p + 1/q = 1$. According to (i) x is $\Delta_{p'}$-point in X and y is $\Delta_{q'}$-point in Y for some $p', q' > 1$ with $1/p' + 1/q' = 1$. Then $p' \geq p$ or $q' \geq q$ and therefore x is Δ_p-point in X or y is Δ_q-point in Y, hence (ii) holds.

$(ii) \Rightarrow (i)$. Assume that (ii) holds. Define

$$A = \{k \in [1, \infty) \colon x \text{ is } \Delta_k\text{-point in } X\}$$

and

$$B = \{k \in [1, \infty) \colon y \text{ is } \Delta_k\text{-point in } Y\}.$$

Firstly, let us examine the case where set A is non-empty. Let $a = \inf A$. We show that $a \in A$. Fix $x^* \in S_{X^*}$, $\alpha > 0$ and $\varepsilon > 0$ such that $x \in S(B_X, x^*, \alpha)$. Let $\gamma > 0$ be such that $x^*(x) > 1 - (\alpha - \gamma)$ and let $k = a\alpha/(\alpha - \gamma)$. Then $k > a$ and therefore $k \in A$. Since $x \in S(B_X, x^*, \alpha - \gamma)$, there exists $u \in S(B_X, x^*, k(\alpha - \gamma)) = S(B_X, x^*, a\alpha)$ such that $\|x - u\| \geq 2 - \varepsilon$. From that we get $a \in A$. Analogically we can show that if B is non-empty, then $b = \inf B \in B$.

It is not hard to see that neither A nor B can be empty. Indeed, if $A = \emptyset$ (the case $B = \emptyset$ is analogical), then since (ii) holds, we have that $(1, \infty) \subset B$. However, according to the previous argumentation we now get that $1 \in B$, i.e. y is a Δ-point, which is a contradiction. Therefore, $A = [a, \infty)$ and $B = [b, \infty)$. From the assumption we can easily see that $1/a + 1/b \geq 1$, hence, there exist $p, q > 1$ that satisfy $1/p + 1/q = 1$ such that $p \in A$ and $q \in B$. $\square$

3.5 Perspectives

In the following we will list some of the relevant open problems regarding Daugavet- and Δ-points.

PROBLEM 10. *Does there exist a Banach space with property $\mathcal{D}$ that fails the DLD2P?*

PROBLEM 11. *Is there a simple characterisation of Daugavet- and Δ-points in the $L_1(\mu)$-preduals? Are there other well-known Banach spaces where Daugavet- and Δ-points coincide?*

Let X and Y be Banach spaces and N an absolute normalised norm. Then it is known what conditions $x \in S_X$ and $y \in S_Y$ that are not Δ-points need to satisfy, so that $(x, y) \in X \oplus_\infty Y$ would be a Δ-point.

PROBLEM 12. *What are the specific conditions for $x \in S_X$ and $y \in S_Y$ so that $(x, y) \in X \oplus_\infty Y$ would be a Δ-point (without the assumption that x and y are not Δ-points)?*

Appendix

In the following we present, firstly, a diagram with explanations about all the relations between the diameter two properties addressed in this thesis, and secondly, three tables that encapsulate the stability results of the diametral diameter two properties, Daugavet-, and Δ-points.

Diagram: Diameter two properties

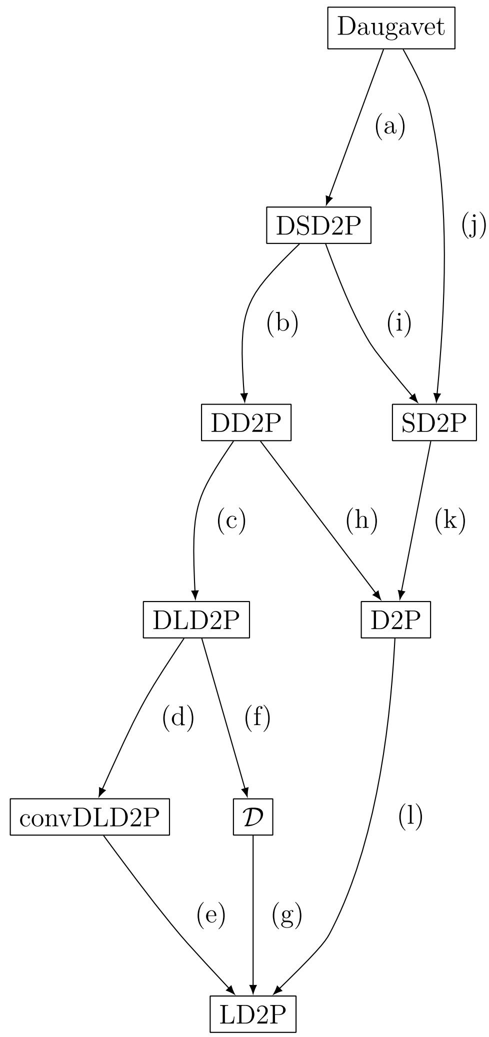

Figure 3.4: Diameter two properties

The reverse implications of (a), (c), and (f) are open problems. In the following we present a list of the counterexamples for all of other reverse implications (for detailed discussion see Sections 1.4, 2.1, 3.1, and 2.4).

(b) $C[0,1] \oplus_2 C[0,1]$;

(d), (g), (h), and (i) ℓ_∞;

(e), (j) c_0;

(k) $c_0 \oplus_2 c_0$;

(l) a renorming of c_0 (see [BLR15]).

Table 1: Overview of the stability results of the diametral diameter two properties

Absolute normalised norm N	X **and** Y		$X \oplus_N Y$	**Related reference(s)**
$\forall N$	DLD2P	$\Leftrightarrow$	DLD2P	[IK04]
$\forall N$	DD2P	$\Leftrightarrow$	DD2P	[BLR18], Thm 2.2.2
ℓ_1-norm ℓ_∞-norm	DSD2P	$\Leftrightarrow$	DSD2P	[BLR18], Thm 2.2.6 [BLR18]
$\forall N$	convex DLD2P	$\Rightarrow$	convex DLD2P	Thm 2.4.10
ℓ_1-norm ℓ_∞-norm	Daugavet property	$\Leftrightarrow$	Daugavet property	[BKSW05]

From the table above, for example, it reads: for every absolute normalised norm N both X and Y have the DLD2P if and only if the absolute sum $X \oplus_N Y$ has the DLD2P.

Table 2: Overview of the stability results of Daugavet-points

Let X and Y be Banach spaces, $x \in S_X$, and $y \in S_Y$. Let N be an absolute normalised norm.

Conditions on N, and $a, b \geq 0$	**Results**
$N \neq \|\cdot\|_\infty$, $a \neq 0$ and $b \neq 0$	x and y are Daugavet-points $\Leftrightarrow$ (ax, by) is a Daugavet-point
$N \neq \|\cdot\|_\infty$ and $a = 0$, $N\big((a,b) + (1,d)\big) = 2$	y is a Daugavet-point $\Leftrightarrow$ (ax, by) is a Daugavet-point
$N \neq \|\cdot\|_\infty$ and $b = 0$, $N\big((a,b) + (c,1)\big) = 2$	x is a Daugavet-point $\Leftrightarrow$ (ax, by) is a Daugavet-point
$b = 0$ and $N\big((a,b) + (1,d)\big) < 2$ or $a = 0$ and $N\big((a,b) + (c,1)\big) < 2$	(ax, by) is not a Daugavet-point
$N = \|\cdot\|_\infty$	x or y is a Daugavet-point $\Leftrightarrow$ (ax, by) is a Daugavet-point

From the table above, for example, it reads: let N be an absolute normalised norm that differs from ℓ_∞-norm, and let a and b be strictly positive; then x and y are Daugavet-points in Banach spaces X and Y, respectively, if and only if (ax, by) is a Daugavet-point in $X \oplus_N Y$.

Table 3: Overview of the stability results of Δ-points

Let X and Y be Banach spaces, $x \in S_X$, and $y \in S_Y$. Let N be an absolute normalised norm.

Conditions on $a, b \geq 0$	**Results**
$a \neq 1$ and $b \neq 1$	x and y are Δ-points $\Leftrightarrow$ (ax, by) is a Δ-point
$a = 1$ and $b \neq 1$	x is a Δ-point $\Leftrightarrow$ (ax, by) is a Δ-point
$a \neq 1$ and $b = 1$	y is a Δ-point $\Leftrightarrow$ (ax, by) is a Δ-point
$a = 1$ and $b = 1$	x or y is a Δ-point $\Rightarrow$ (ax, by) is a Δ-point

From the table above, for example, it reads: let $a \neq 1$ and $b \neq 1$; then x is a Δ-point in X and y is a Δ-point in Y if and only if (ax, by) is a Δ-point in $X \oplus_N Y$.

Bibliography

[ABL15] M. D. Acosta, J. Becerra-Guerrero, and G. López-Pérez, *Stability results of diameter two properties*, J. Convex Anal. **22** (2015), 1–17.

[AHLP20] T. A. Abrahamsen, R. Haller, V. Lima, and K. Pirk, *Delta- and Daugavet-points in Banach spaces*, Proc. Edinb. Math. Soc. **60** (2020), 475–496.

[AHNTT16] T. A. Abrahamsen, P. Hájek, O. Nygaard, J. Talponen, and S. Troyanski, *Diameter 2 properties and convexity*, Studia Math. **232** (2016), 227–242.

[ALMN17] T. A. Abrahamsen, A. Leerand, A. Martiny, and O. Nygaard, *Two properties of Müntz spaces*, Demonstr. Math. **50** (2017), 239–244.

[ALN13] T. A. Abrahamsen, V. Lima, and O. Nygaard, *Remarks on diameter 2 properties*, J. Convex Anal. **20** (2013), 439–452.

[ALN14] T. A. Abrahamsen, V. Lima, and O. Nygaard, *Almost isometric ideals in Banach spaces*, Glasgow Math. J. **56** (2014), 395–407.

[ALNT16] T. A. Abrahamsen, V. Lima, O. Nygaard, and S. Troyanski, *Diameter two properties, convexity and smoothness*, Milan J. Math. **84** (2016), 231–242.

[BE97] P. Borwein and T. Erdélyi, *Generalizations of Müntz's theorem via a Remez-type inequality for Müntz spaces*, J. Amer. Math. Soc. **10** (1997), 327–349.

[BKSW05] D. Bilik, V. M. Kadets, R. Shvidkoy, and D. Werner, *Narrow operators and the Daugavet property for ultraproducts*, Positivity **9** (2005), 45–62.

[BL06] J. Becerra Guerrero and G. López-Pérez, *Relatively weakly open subsets of the unit ball in functions spaces*, J. Math. Anal. Appl. **315** (2006), 544–554.

[BLR15] J. Becerra Guerrero, G. López-Pérez, and A. Rueda Zoca, *Big slices versus big relatively weakly open subsets in Banach spaces*, J. Math. Anal. Appl. **428** (2015), 855–865.

[BLR16] J. Becerra Guerrero, G. López-Pérez, and A. Rueda Zoca, *Subspaces of Banach spaces with big slices*, Banach J. Math. Analysis **10** (2016), 771–782.

[BLR18] J. Becerra Guerrero, G. López-Pérez, and A. Rueda Zoca, *Diametral Diameter Two Properties in Banach Spaces*, J. Convex Anal. **25** (2018), 817–840.

[BM05] J. Becerra Guerrero and M. Martín, *The Daugavet property of C^*-algebras, JB^*-triples, and of their isometric preduals*, J. Funct. Anal. **224** (2005), 316–337.

[BM06] J. Becerra Guerrero and M. Martín, *The Daugavet property for Lindenstrauss spaces*, Methods in Banach space theory, London Math. Soc. Lecture Note Ser. **337**, Cambridge Univ. Press, Cambridge, (2006), 91–96.

[Dau63] I. K. Daugavet, *On a property of completely continuous operators in the space C*, Uspekhi Mat. Nauk **18.5** (1963), 157–158 (in Russian).

[Erd03] T. Erdélyi, *The "full Clarkson-Erdös-Schwartz theorem" on the closure of non-dense Müntz spaces*, Studia Math. **155** (2003), 145–152.

[FHHMZ11] M. Fabian, P. Habala, P. Hájek, V. Montesinos, and V. Zizler, *Banach Space Theory. The Basis for Linear and Nonlinear Analysis*, Springer, Berlin, 2011.

[GGMS87] N. Ghoussoub, G. Godefroy, B. Maurey, and W. Schachermayer, *Some topological and geometrical structures in Banach spaces*, Mem. Amer. Math. Soc. **378**, 1987.

[HL14] R. Haller and J. Langemets, *Two remarks on diameter 2 properties*, Proc. Est. Acad. Sci. **63** (2014), 2–7.

[HLN18] R. Haller, J. Langemets, and R. Nadel, *Stability of average roughness, octahedrality, and strong diameter 2 properties of Banach spaces with respect to absolute sums*, Banach J. Math. Anal. **12** (2018), 222–239.

[HLP15] R. Haller, J. Langemets, and M. Põldvere, *On duality of diameter 2 properties*, J. Convex Anal. **22** (2015), 465–483.

[HPP16] R. Haller, K. Pirk, and M. Põldvere, *Diametral strong diameter two property of Banach spaces is stable under direct sums with 1-norm*, Acta Comment. Univ. Tartu Math. **20** (2016), 101–105.

[HPV] R. Haller, K. Pirk, and T. Veeorg, *Daugavet- and Delta-points in absolute sums of Banach spaces*, to appear in J. Conv. Anal. **27** (2020), preprint at arXiv:2001.06197 [math.FA].

[HWW93] P. Harmand, D. Werner, and W. Werner, *M-Ideals in Banach Spaces and Banach Algebras*, Lecture Notes in Math. 1547, Springer-Verlag, Berlin-Heidelberg-New York, 1993.

[IK04] Y. Ivakhno and V. M. Kadets, *Unconditional sums of spaces with bad projections*, Visn. Khark. Univ., Ser. Mat. Prykl. Mat. Mekh. **645** (2004), 30–35.

[Jam06] G. J. O. Jameson, *Counting zeros of generalised polynomials*, Math. Gazette July (2006), 223–234.

[JZ78] K. John and V. Zizler, *On rough norms on Banach spaces*, Commentat. Math. Univ. Carol. **19** (1978), 335–349.

[KSSW00] V. M. Kadets, R. V. Shvidkoy, G. G. Sirotkin, and D. Werner, *Banach spaces with the Daugavet property*, Trans. Amer. Math. Soc. **352** (2000), 855–873.

[Lan15] J. Langemets, *Geometrical structure in diameter 2 Banach spaces*, PhD thesis, University of Tartu, 2015, https://dspace.ut.ee/handle/10062/47446.

[Lin64] J. Lindenstrauss, *Extension of compact operators*, Mem. Amer. Math. Soc. **48** (1964).

[Loz66] G. Ya. Lozanovskii, *On almost integral operators in KB-spaces*, Vestnik Leningrad Univ. Mat. Mekh. Astr. **21.7** (1966), 35–44 (in Russian).

[Mar18] A. Martiny, *On octahedraliy and Müntz spaces*, preprint at arXiv:1811.02011 [math.FA].

[NW01] O. Nygaard and D. Werner, *Slices in the unit ball of a uniform algebra*, Arch. Math. **76** (2001), 441–444.

[Pir16] K. Pirk, *Diameetraalse diameeter 2 omadusega Banachi ruumid* (in Estonian), Master's thesis, University of Tartu, 2016, https://dspace.ut.ee/handle/10062/53813.

[Shv00] R. V. Shvydkoy, *Geometric aspects of the Daugavet property*, J. Funct. Anal. **176** (2000), 198–212.

[Wer94] D. Werner, *M-ideals and the 'basic inequality'*, J. Approx. Th. **76** (1994), 21–30.

[Wer01] D. Werner, *Recent progress on the Daugavet property*, Irish Math. Soc. Bull. **46** (2001), 77–97.

[Wer08] D. Werner, *A remark about Müntz spaces*, Preprint (2008).

Summary in Estonian

Diametraalsed diameeter-2 omadused, Daugaveti- ja Δ-punktid Banachi ruumides

Käesoleva väitekirja põhieesmärk on laiendada teadmisi Banachi ruumidel vaadeldavate diameeter-2 omaduste kohta, täpsemalt uuritakse töös diametraalseid diameeter-2 omadusi ning nendega seotud Daugaveti- ja Δ-punktide olemasolu. Ühelt poolt on diametraalsed diameeter-2 omadused tugevamad vastavatest klassikalistest diameeter-2 omadustest, teiselt poolt on nad tuntud Daugaveti omaduse loomulikud üldistused ja seega Daugaveti omadusest (formaalselt) nõrgemad. Uute omaduste sissetoomisel lähtuti Daugaveti omaduse geomeetrilistest kirjeldustest ühikkera viilude, suhteliselt nõrgalt lahtiste osahulkade ja nende kumerate kombinatsioonide abil. Diametraalse diameeter-2 omadusega Banachi ruumi ühikkera viilu (või omaduse tugevama versiooni korral suhteliselt nõrgalt lahtise osahulga või nende osahulkade kumera kombinatsiooni) elemendile leidub selles hulgas (viilus, suhteliselt nõrgalt lahtises osahulgas või nende hulkade kumeras kombinatsioonis) talle peaaegu diametraalne element. Kas Daugaveti omadus ja tugevaim vaadeldud uutest omadustest – tugev diametraalne diameeter-2 omadus – on üleüldse erinevad, pole siiski veel selge. Kokkulangevus oleks üllatav ja näitaks Daugaveti omadust uuest küljest.

Vastavalt tuntud geomeetrilisele kirjeldusele on Banachi ruumil X Daugaveti omadus parajasti siis, kui ruumi X iga normiga üks elemendi x korral: iga positiivse arvu ε korral on hulga

$$\Delta_\varepsilon(x) = \{y \in B_X \colon \|x - y\| \geq 2 - \varepsilon\}$$

kinnine kumer kate kogu ühikkera B_X. Sellist tingimust rahuldavat elementi x nimetatakse Daugaveti-punktiks. Element x on Δ-punkt, kui iga positiivse arvu ε korral x kuulub hulga $\Delta_\varepsilon(x)$ kinnisesse kumerasse kattesse. Ilmselt on iga Daugaveti-punkt Δ-punkt. Suvaline Δ-punkt ei tarvitse Daugaveti-punkt olla, kuigi intuitiivselt võib nii paista.

Töös esitatakse diametraalsete diameeter-2 omaduste stabiilsustulemused absoluutse normiga (nt ℓ_p-normiga) summaruumide ja vastavate komponentruumide vahel. Need tulemused näitavad, et osa diametraalseid diameeter-2 omadusi käituvad sarnaselt vastavale klassikalisele diameeter-2 omadusele, aga tugev diametraalne diameeter-2 omadus käitub nagu Daugaveti omadus. Väitekirjas uuritakse ka Daugaveti- ja Δ-punktide olemasolu summaruumides. Nendest tulemustest järeldub, et Daugaveti- ja Δ-punkti mõisted on erinevad, kuigi selgub, et mitmeski tuntud Banachi ruumide klassis

nagu näiteks $C(K)$ ruumides lõpmatu kompaktse Hausdorffi ruumi K korral, $L_1(\mu)$ ruumides ja teatud Müntzi ruumides langevad need punktid kokku. Lisaks sellele antakse nendes ruumides Daugaveti-punktide lihtne kirjeldus, millele vastavust on kerge kontrollida. Näidatakse, et $C(K)$ ruumidel ja Müntzi ruumidel on eriline diameeter-2 omadus, mida kutsutakse kumeraks diametraalseks lokaalseks diameeter-2 omaduseks.

Väitekiri koosneb kolmest peatükist ja ühest lisast. Esimeses peatükis antakse lühiülevaade väitekirjas vaadeldavate põhiomaduste ajaloolisest taustast ja senisest käsitlusest, esitatakse väitekirja kokkuvõte ning kirjeldatakse väitekirjas kasutatavaid tähistusi. Veel antakse läbi väitekirja töö põhiomadustega võrdlemisel vajalikud Daugaveti omaduse ja klassikaliste diameeter-2 omaduste mõisted ning töös kasutatud tulemused nende kohta.

Teine peatükk põhineb artiklitel [HPP16] ja [AHLP20]. Selles peatükis keskendutakse diametraalsete diameeter-2 omaduste uurimisele. Alustuseks antakse ülevaade kolmest diametraalsest diameeter-2 omadusest ja nende seostest Banachi ruumide teiste tuntud omadustega. Tõestatakse, et kaks diametraalset diameeter-2 omadust on stabiilsed kõikide absoluutsete summade puhul, kuid diametraalne tugev diameeter-2 omadus on stabiilne ainult ℓ_1- ja ℓ_∞-summa korral. Lisaks selgitame diametraalsete diameeter-2 omaduste pärandumist alamruumidesse, sh M-ideaalidele. Teise peatüki lõpus näidatakse, et kumer diametraalne lokaalne diameeter-2 omadus jääb rangelt diametraalse lokaalse diameeter-2 omaduse ja lokaalse diameeter-2 omaduse vahele, ning tõestatakse, et $C(K)$ ruumidel lõpmatu kompaktse Hausdorffi ruumi K korral ja kõigil Müntzi ruumidel on kumer diametraalne diameeter-2 omadus. Lisaks sellele osutub, et kumer diametraalne diameeter-2 omadus on stabiilne kõigi absoluutsete summade võtmisel. Peatükk lõpeb loeteluga lahtistest küsimustest.

Kolmas peatükk põhineb artiklitel [AHLP20] ja [HPV]. Peatükis selgitatakse Daugaveti- ja Δ-punktide olemasolu. Alguses antakse nende punktide üldised kriteeriumid. Järgnevas esitatakse nende punktide lihtsad ja praktilised kirjeldused $L_1(\mu)$ ruumides, $C(K)$ ruumides lõpmatu kompaktse Hausdorffi ruumi K korral ja laias klassis Müntzi ruumides. Näidatakse, et kõigis eelnimetatud ruumides langevad Daugaveti- ja Δ-punktid kokku. Kahes viimases alapeatükis keskendutakse Daugaveti- ja Δ-punktide olemasolu selgitamisele absoluutse normiga summaruumides. Daugaveti-punktide korral saab kõik absoluutsed summad jagada kahte klassi: absoluutseteks summadeks, mis on varustatud A-OH normiga ja absoluutseteks summadeks, mis on varustatud normiga, millel on omadus (α); millest esimestes võib olla Daugaveti punkte, teistes ei saa aga Daugaveti-punkte olla. Kõikides ℓ_∞-summast erinevates absoluutsetes summades leidub Δ-punkte parajasti siis, kui neid leidub mõlemas komponentruumis. Üllatuslikult võib ℓ_∞-summas olla Δ-punkte isegi siis, kui neid komponentruumides ei leidu. Peatükk lõpeb lahtiste küsimustega.

Lisas on esiteks selgitavate kommentaaridega diagramm, mis illustreerib kõikide väitekirjas vaadeldud diameeter-2 omaduste omavahelisi seoseid. Veel on lisatud kolm tabelit, mis võtavad kokku vastavalt diametraalsete diameeter-2 omaduste, Daugaveti- ja Δ-punktide stabiilsustulemused absoluutsete summade võtmisel.

Index

Curriculum vitae

NAME: Katriin Pirk
DATE AND PLACE OF BIRTH: 20.11.1992, Risti, Estonia
CITIZENSHIP: Estonian
ADDRESS: Institute of Mathematics and Statistics, Narva mnt 18, 51009 Tartu, Estonia
PHONE: +372 5665 4773
E-MAIL: katriin.pirk@ut.ee

EDUCATION:

2008–2011	Haapsalu Gymnasium
2011–2014	University of Tartu, bachelor studies in educational science (math and sciences), Bachelor of Arts in Education, 2011
2014–2016	University of Tartu, master studies in mathematics, Master of Science (Mathematics), cum laude, 2016
2016–	University of Tartu, doctoral studies in mathematics

LANGUAGES: Estonian, English, Russian, Japanese

EMPLOYMENT:

09/2016–06/2017	teacher of mathematics in Tartu Raatuse School (partial work load)

FIELDS OF SCIENTIFIC INTEREST: functional analysis, geometry of Banach spaces

Elulookirjeldus

Nimi: Katriin Pirk
Sünniaeg ja -koht: 20.11.1992, Risti, Eesti
Kodakondsus: Eesti
Aadress: TÜ matemaatika ja statistika instituut, Narva mnt 18, 51009 Tartu, Eesti
Telefon: +372 5665 4773
E-post: katriin.pirk@ut.ee

Haridus:

2008–2011	Haapsalu Gümnaasium
2011–2014	Tartu Ülikool, haridusteaduse (reaalained) bakalaureuseõpe, Haridusteaduse bakalaureus, 2014
2014–2016	Tartu Ülikool, matemaatika magistriõpe, Loodusteaduse magister (matemaatika), *cum laude*, 2016
2016–	Tartu Ülikool, matemaatika doktoriõpe

Keelte oskus: eesti, inglise, vene, jaapani

Teenistuskäik:

09/2016–06/2017	matemaatikaõpetaja Tartu Raatuse koolis (osaline koormus)

Teaduslikud huvid: funktsionaalanalüüs, Banachi ruumide geomeetria

List of original publications

1. T. A. Abrahamsen, R. Haller, V. Lima, and K. Pirk, *Delta- and Daugavet-points in Banach spaces*, Proc. Edinb. Math. Soc. **60** (2020), 475–496.
2. R. Haller, K. Pirk, and M. Põldvere, *Diametral strong diameter two property of Banach spaces is stable under direct sums with 1-norm*, Acta Comment. Univ. Tartu Math. **20** (2016), 101–105.
3. R. Haller, K. Pirk, and T. Veeorg, *Daugavet- and Delta-points in absolute sums of Banach spaces*, to appear in J. Conv. Anal. **27** (2020), preprint at arXiv:2001.06197 [math.FA].

DISSERTATIONES MATHEMATICAE UNIVERSITATIS TARTUENSIS

1. **Mati Heinloo.** The design of nonhomogeneous spherical vessels, cylindrical tubes and circular discs. Tartu, 1991, 23 p.
2. **Boris Komrakov.** Primitive actions and the Sophus Lie problem. Tartu, 1991, 14 p.
3. **Jaak Heinloo.** Phenomenological (continuum) theory of turbulence. Tartu, 1992, 47 p.
4. **Ants Tauts.** Infinite formulae in intuitionistic logic of higher order. Tartu, 1992, 15 p.
5. **Tarmo Soomere.** Kinetic theory of Rossby waves. Tartu, 1992, 32 p.
6. **Jüri Majak.** Optimization of plastic axisymmetric plates and shells in the case of Von Mises yield condition. Tartu, 1992, 32 p.
7. **Ants Aasma.** Matrix transformations of summability and absolute summability fields of matrix methods. Tartu, 1993, 32 p.
8. **Helle Hein.** Optimization of plastic axisymmetric plates and shells with piece-wise constant thickness. Tartu, 1993, 28 p.
9. **Toomas Kiho.** Study of optimality of iterated Lavrentiev method and its generalizations. Tartu, 1994, 23 p.
10. **Arne Kokk.** Joint spectral theory and extension of non-trivial multiplicative linear functionals. Tartu, 1995, 165 p.
11. **Toomas Lepikult.** Automated calculation of dynamically loaded rigid-plastic structures. Tartu, 1995, 93 p, (in Russian).
12. **Sander Hannus.** Parametrical optimization of the plastic cylindrical shells by taking into account geometrical and physical nonlinearities. Tartu, 1995, 74 p, (in Russian).
13. **Sergei Tupailo.** Hilbert's epsilon-symbol in predicative subsystems of analysis. Tartu, 1996, 134 p.
14. **Enno Saks.** Analysis and optimization of elastic-plastic shafts in torsion. Tartu, 1996, 96 p.
15. **Valdis Laan.** Pullbacks and flatness properties of acts. Tartu, 1999, 90 p.
16. **Märt Põldvere.** Subspaces of Banach spaces having Phelps' uniqueness property. Tartu, 1999, 74 p.
17. **Jelena Ausekle.** Compactness of operators in Lorentz and Orlicz sequence spaces. Tartu, 1999, 72 p.
18. **Krista Fischer.** Structural mean models for analyzing the effect of compliance in clinical trials. Tartu, 1999, 124 p.
19. **Helger Lipmaa.** Secure and efficient time-stamping systems. Tartu, 1999, 56 p.
20. **Jüri Lember.** Consistency of empirical k-centres. Tartu, 1999, 148 p.
21. **Ella Puman.** Optimization of plastic conical shells. Tartu, 2000, 102 p.
22. **Kaili Müürisep.** Eesti keele arvutigrammatika: süntaks. Tartu, 2000, 107 lk.

23. **Varmo Vene.** Categorical programming with inductive and coinductive types. Tartu, 2000, 116 p.
24. **Olga Sokratova.** Ω-rings, their flat and projective acts with some applications. Tartu, 2000, 120 p.
25. **Maria Zeltser.** Investigation of double sequence spaces by soft and hard analitical methods. Tartu, 2001, 154 p.
26. **Ernst Tungel.** Optimization of plastic spherical shells. Tartu, 2001, 90 p.
27. **Tiina Puolakainen.** Eesti keele arvutigrammatika: morfoloogiline ühestamine. Tartu, 2001, 138 p.
28. **Rainis Haller.** *M(r,s)*-inequalities. Tartu, 2002, 78 p.
29. **Jan Villemson.** Size-efficient interval time stamps. Tartu, 2002, 82 p.
30. Töö kaitsmata.
31. **Mart Abel.** Structure of Gelfand-Mazur algebras. Tartu, 2003. 94 p.
32. **Vladimir Kuchmei.** Affine completeness of some ockham algebras. Tartu, 2003. 100 p.
33. **Olga Dunajeva.** Asymptotic matrix methods in statistical inference problems. Tartu 2003. 78 p.
34. **Mare Tarang.** Stability of the spline collocation method for volterra integro-differential equations. Tartu 2004. 90 p.
35. **Tatjana Nahtman.** Permutation invariance and reparameterizations in linear models. Tartu 2004. 91 p.
36. **Märt Möls.** Linear mixed models with equivalent predictors. Tartu 2004. 70 p.
37. **Kristiina Hakk.** Approximation methods for weakly singular integral equations with discontinuous coefficients. Tartu 2004, 137 p.
38. **Meelis Käärik.** Fitting sets to probability distributions. Tartu 2005, 90 p.
39. **Inga Parts.** Piecewise polynomial collocation methods for solving weakly singular integro-differential equations. Tartu 2005, 140 p.
40. **Natalia Saealle.** Convergence and summability with speed of functional series. Tartu 2005, 91 p.
41. **Tanel Kaart.** The reliability of linear mixed models in genetic studies. Tartu 2006, 124 p.
42. **Kadre Torn.** Shear and bending response of inelastic structures to dynamic load. Tartu 2006, 142 p.
43. **Kristel Mikkor.** Uniform factorisation for compact subsets of Banach spaces of operators. Tartu 2006, 72 p.
44. **Darja Saveljeva.** Quadratic and cubic spline collocation for Volterra integral equations. Tartu 2006, 117 p.
45. **Kristo Heero.** Path planning and learning strategies for mobile robots in dynamic partially unknown environments. Tartu 2006, 123 p.
46. **Annely Mürk.** Optimization of inelastic plates with cracks. Tartu 2006. 137 p.
47. **Annemai Raidjõe.** Sequence spaces defined by modulus functions and superposition operators. Tartu 2006, 97 p.
48. **Olga Panova.** Real Gelfand-Mazur algebras. Tartu 2006, 82 p.

49. **Härmel Nestra.** Iteratively defined transfinite trace semantics and program slicing with respect to them. Tartu 2006, 116 p.
50. **Margus Pihlak.** Approximation of multivariate distribution functions. Tartu 2007, 82 p.
51. **Ene Käärik.** Handling dropouts in repeated measurements using copulas. Tartu 2007, 99 p.
52. **Artur Sepp.** Affine models in mathematical finance: an analytical approach. Tartu 2007, 147 p.
53. **Marina Issakova.** Solving of linear equations, linear inequalities and systems of linear equations in interactive learning environment. Tartu 2007, 170 p.
54. **Kaja Sõstra.** Restriction estimator for domains. Tartu 2007, 104 p.
55. **Kaarel Kaljurand.** Attempto controlled English as a Semantic Web language. Tartu 2007, 162 p.
56. **Mart Anton.** Mechanical modeling of IPMC actuators at large deformations. Tartu 2008, 123 p.
57. **Evely Leetma.** Solution of smoothing problems with obstacles. Tartu 2009, 81 p.
58. **Ants Kaasik.** Estimating ruin probabilities in the Cramér-Lundberg model with heavy-tailed claims. Tartu 2009, 139 p.
59. **Reimo Palm.** Numerical Comparison of Regularization Algorithms for Solving Ill-Posed Problems. Tartu 2010, 105 p.
60. **Indrek Zolk.** The commuting bounded approximation property of Banach spaces. Tartu 2010, 107 p.
61. **Jüri Reimand.** Functional analysis of gene lists, networks and regulatory systems. Tartu 2010, 153 p.
62. **Ahti Peder.** Superpositional Graphs and Finding the Description of Structure by Counting Method. Tartu 2010, 87 p.
63. **Marek Kolk.** Piecewise Polynomial Collocation for Volterra Integral Equations with Singularities. Tartu 2010, 134 p.
64. **Vesal Vojdani.** Static Data Race Analysis of Heap-Manipulating C Programs. Tartu 2010, 137 p.
65. **Larissa Roots.** Free vibrations of stepped cylindrical shells containing cracks. Tartu 2010, 94 p.
66. **Mark Fišel.** Optimizing Statistical Machine Translation via Input Modification. Tartu 2011, 104 p.
67. **Margus Niitsoo**. Black-box Oracle Separation Techniques with Applications in Time-stamping. Tartu 2011, 174 p.
68. **Olga Liivapuu.** Graded q-differential algebras and algebraic models in noncommutative geometry. Tartu 2011, 112 p.
69. **Aleksei Lissitsin.** Convex approximation properties of Banach spaces. Tartu 2011, 107 p.
70. **Lauri Tart.** Morita equivalence of partially ordered semigroups. Tartu 2011, 101 p.
71. **Siim Karus.** Maintainability of XML Transformations. Tartu 2011, 142 p.

72. **Margus Treumuth.** A Framework for Asynchronous Dialogue Systems: Concepts, Issues and Design Aspects. Tartu 2011, 95 p.
73. **Dmitri Lepp.** Solving simplification problems in the domain of exponents, monomials and polynomials in interactive learning environment T-algebra. Tartu 2011, 202 p.
74. **Meelis Kull.** Statistical enrichment analysis in algorithms for studying gene regulation. Tartu 2011, 151 p.
75. **Nadežda Bazunova.** Differential calculus $d^3 = 0$ on binary and ternary associative algebras. Tartu 2011, 99 p.
76. **Natalja Lepik.** Estimation of domains under restrictions built upon generalized regression and synthetic estimators. Tartu 2011, 133 p.
77. **Bingsheng Zhang.** Efficient cryptographic protocols for secure and private remote databases. Tartu 2011, 206 p.
78. **Reina Uba.** Merging business process models. Tartu 2011, 166 p.
79. **Uuno Puus.** Structural performance as a success factor in software development projects – Estonian experience. Tartu 2012, 106 p.
80. **Marje Johanson.** $M(r, s)$-ideals of compact operators. Tartu 2012, 103 p.
81. **Georg Singer.** Web search engines and complex information needs. Tartu 2012, 218 p.
82. **Vitali Retšnoi.** Vector fields and Lie group representations. Tartu 2012, 108 p.
83. **Dan Bogdanov.** Sharemind: programmable secure computations with practical applications. Tartu 2013, 191 p.
84. **Jevgeni Kabanov.** Towards a more productive Java EE ecosystem. Tartu 2013, 151 p.
85. **Erge Ideon**. Rational spline collocation for boundary value problems. Tartu, 2013, 111 p.
86. **Esta Kägo.** Natural vibrations of elastic stepped plates with cracks. Tartu, 2013, 114 p.
87. **Margus Freudenthal.** Simpl: A toolkit for Domain-Specific Language development in enterprise information systems. Tartu, 2013, 151 p.
88. **Boriss Vlassov.** Optimization of stepped plates in the case of smooth yield surfaces. Tartu, 2013, 104 p.
89. **Elina Safiulina.** Parallel and semiparallel space-like submanifolds of low dimension in pseudo-Euclidean space. Tartu, 2013, 85 p.
90. **Raivo Kolde.** Methods for re-using public gene expression data. Tartu, 2014, 121 p.
91. **Vladimir Šor.** Statistical Approach for Memory Leak Detection in Java Applications. Tartu, 2014, 155 p.
92. **Naved Ahmed.** Deriving Security Requirements from Business Process Models. Tartu, 2014, 171 p.
93. **Kerli Orav-Puurand.** Central Part Interpolation Schemes for Weakly Singular Integral Equations. Tartu, 2014, 109 p.
94. **Liina Kamm.** Privacy-preserving statistical analysis using secure multi-party computation. Tartu, 2015, 201 p.

95. **Kaido Lätt.** Singular fractional differential equations and cordial Volterra integral operators. Tartu, 2015, 93 p.
96. **Oleg Košik.** Categorical equivalence in algebra. Tartu, 2015, 84 p.
97. **Kati Ain.** Compactness and null sequences defined by ℓ_p spaces. Tartu, 2015, 90 p.
98. **Helle Hallik.** Rational spline histopolation. Tartu, 2015, 100 p.
99. **Johann Langemets.** Geometrical structure in diameter 2 Banach spaces. Tartu, 2015, 132 p.
100. **Abel Armas Cervantes.** Diagnosing Behavioral Differences between Business Process Models. Tartu, 2015, 193 p.
101. **Fredrik Milani.** On Sub-Processes, Process Variation and their Interplay: An Integrated Divide-and-Conquer Method for Modeling Business Processes with Variation. Tartu, 2015, 164 p.
102. **Huber Raul Flores Macario.** Service-Oriented and Evidence-aware Mobile Cloud Computing. Tartu, 2015, 163 p.
103. **Tauno Metsalu.** Statistical analysis of multivariate data in bioinformatics. Tartu, 2016, 197 p.
104. **Riivo Talviste.** Applying Secure Multi-party Computation in Practice. Tartu, 2016, 144 p.
105. **Md Raknuzzaman.** Noncommutative Galois Extension Approach to Ternary Grassmann Algebra and Graded q-Differential Algebra. Tartu, 2016, 110 p.
106. **Alexander Liyvapuu.** Natural vibrations of elastic stepped arches with cracks. Tartu, 2016, 110 p.
107. **Julia Polikarpus.** Elastic plastic analysis and optimization of axisymmetric plates. Tartu, 2016, 114 p.
108. **Siim Orasmaa.** Explorations of the Problem of Broad-coverage and General Domain Event Analysis: The Estonian Experience. Tartu, 2016, 186 p.
109. **Prastudy Mungkas Fauzi.** Efficient Non-interactive Zero-knowledge Protocols in the CRS Model. Tartu, 2017, 193 p.
110. **Pelle Jakovits.** Adapting Scientific Computing Algorithms to Distributed Computing Frameworks. Tartu, 2017, 168 p.
111. **Anna Leontjeva.** Using Generative Models to Combine Static and Sequential Features for Classification. Tartu, 2017, 167 p.
112. **Mozhgan Pourmoradnasseri.** Some Problems Related to Extensions of Polytopes. Tartu, 2017, 168 p.
113. **Jaak Randmets.** Programming Languages for Secure Multi-party Computation Application Development. Tartu, 2017, 172 p.
114. **Alisa Pankova.** Efficient Multiparty Computation Secure against Covert and Active Adversaries. Tartu, 2017, 316 p.
115. **Tiina Kraav.** Stability of elastic stepped beams with cracks. Tartu, 2017, 126 p.
116. **Toomas Saarsen.** On the Structure and Use of Process Models and Their Interplay. Tartu, 2017, 123 p.

117. **Silja Veidenberg.** Lifting bounded approximation properties from Banach spaces to their dual spaces. Tartu, 2017, 112 p.
118. **Liivika Tee.** Stochastic Chain-Ladder Methods in Non-Life Insurance. Tartu, 2017, 110 p.
119. **Ülo Reimaa.** Non-unital Morita equivalence in a bicategorical setting. Tartu, 2017, 86 p.
120. **Rauni Lillemets.** Generating Systems of Sets and Sequences. Tartu, 2017, 181 p.
121. **Kristjan Korjus.** Analyzing EEG Data and Improving Data Partitioning for Machine Learning Algorithms. Tartu, 2017, 106 p.
122. **Eno Tõnisson.** Differences between Expected Answers and the Answers Offered by Computer Algebra Systems to School Mathematics Equations. Tartu, 2017, 195 p.
123. **Kaur Lumiste.** Improving accuracy of survey estimators by using auxiliary information in data collection and estimation stages. Tartu, 2018, 112 p.
124. **Paul Tammo.** Closed maximal regular one-sided ideals in topological algebras. Tartu, 2018, 112 p.
125. **Mart Kals.** Computational and statistical methods for DNA sequencing data analysis and applications in the Estonian Biobank cohort. Tartu, 2018, 174 p.
126. **Annika Krutto.** Empirical Cumulant Function Based Parameter Estimation in Stable Distributions. Tartu, 2019, 140 p.
127. **Kristi Läll.** Risk scores and their predictive ability for common complex diseases. Tartu, 2019, 118 p.
128. **Gul Wali Shah.** Spline approximations. Tartu, 2019, 85 p.
129. **Mikk Vikerpuur.** Numerical solution of fractional differential equations. Tartu, 2019, 125 p.
130. **Priit Lätt.** Induced 3-Lie superalgebras and their applications in superspace. Tartu, 2020, 114 p.
131. **Sumaira Rehman.** Fast and quasi-fast solvers for weakly singular Fredholm integral equation of the second kind. Tartu, 2020, 105 p.
132. **Rihhard Nadel.** Big slices of the unit ball in Banach spaces. Tartu, 2020, 108 p.