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On the convex central configurations of the symmetric $(\ell+2)$–body problem

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Abstract For the 4–body problem there is the following conjecture: Given arbitrary positive masses the planar 4–body problem has a unique convex central configuration for each ordering of the masses on its convex hull. Until now this conjecture remains open. Our aim is to prove that this conjecture cannot be extended to the $(\ell+2)$–body problem with $\ell \geq 3$. In particular, we prove that the symmetric $(2n+1)$–body problem with masses $m_1 = \cdots = m_{2n-1} = 1$ and $m_{2n} = m_{2n+1} = m$ sufficiently small has at least two classes of convex central configuration when $n = 2$, five when $n = 3$, and four when $n = 4$. We conjecture that the $(2n+1)$-body problem has at least n classes of convex central configurations for $n > 4$ and we give some numerical evidences that the conjecture can be true. We also prove that the symmetric $(2n+2)$–body problem with masses $m_1 = \cdots = m_{2n} = 1$ and $m_{2n+1} = m_{2n+2} = m$ sufficiently small has at least three classes of convex central configuration when $n = 3$, two when $n = 4$, and three when $n = 5$. We also conjecture

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that the $(2n+2)$-body problem has at least $[(n+1)/2]$ classes of convex central configurations for $n > 5$ and we give some numerical evidences that the conjecture can be true.

Keywords Convex central configuration · $(\ell+2)$-body problem

Mathematics Subject Classification (2010) 70F10 · 70F15

1 Introduction and statement of the main result

The equations for the classical Newtonian N-body problem are given by

$$m_i\,\ddot{\mathbf{q}}_i = \sum_{\substack{j=1\\ j\neq i}}^{N} G\,m_i\,m_j\,\frac{\mathbf{q}_j-\mathbf{q}_i}{|\mathbf{q}_j-\mathbf{q}_i|^3},$$

$i = 1,\dots,N$, where $\mathbf{q}_i = (x_i, y_i) \in \mathbb{R}^2$ is the position vector of the punctual mass m_i in an inertial coordinate system, $|\mathbf{q}_j - \mathbf{q}_i|$ is the Euclidean distance between the masses m_j and m_i, and G is the gravitational constant which can be taken equal to one by choosing conveniently the unit of time. The *configuration space* of the planar N–body problem is

$$\mathcal{E} = \{(\mathbf{q}_1,\dots,\mathbf{q}_N) \in \mathbb{R}^{2N}\ :\ \mathbf{q}_i \neq \mathbf{q}_j,\ \text{for } i\neq j\}.$$

Given $m_1,\dots,m_N$ a configuration $(\mathbf{q}_1,\dots,\mathbf{q}_N) \in \mathcal{E}$ is *central* if there exists a positive constant λ such that

$$\ddot{\mathbf{q}}_i = -\lambda\,(\mathbf{q}_i - \mathbf{cm})$$

for $i = 1,\dots,N$, where

$$\mathbf{cm} = (cm_x, cm_y) = \left(\frac{\sum_{i=1}^N m_i x_i}{\sum_{i=1}^N m_i}, \frac{\sum_{i=1}^N m_i y_i}{\sum_{k=1}^N m_i}\right),$$

is the *center of mass* of the system. Thus a central configuration $(\mathbf{q}_1,\dots,\mathbf{q}_N) \in \mathcal{E}$ of the N–body problem with positive masses $m_1,\dots,m_N$ is a solution of the system of $2N$ equations

$$\begin{aligned} e_i &= \sum_{\substack{j=1\\ j\neq i}}^{N} m_j\,\frac{x_j-x_i}{|\mathbf{q}_j-\mathbf{q}_i|^3} + \lambda\,(x_i - cm_x) = 0,\\ e_{N+i} &= \sum_{\substack{j=1\\ j\neq i}}^{N} m_j\,\frac{y_j-y_i}{|\mathbf{q}_j-\mathbf{q}_i|^3} + \lambda\,(y_i - cm_y) = 0, \end{aligned} \tag{1}$$

with $i = 1,\dots,N$, for some λ.

A configuration is called *convex* if none of the masses is contained in the interior of the convex hull of the other remaining masses.

McMillan and Bartky in 1932 (see [8]) proved that the 4–body problem for any set of positive masses has a convex central configuration. This result was reproved in a simpler way by Xia in 2004 (see [11]).

In the paper of MacMillan and Bartky is implicit the following conjecture: *For the planar* 4*-body problem with positive masses there is a unique convex central configuration for each ordering of the masses on its convex hull.* This conjecture also appear explicitly in the papers of Pérez–Chavela and Santoprete [10], and of Albouy and Fu [1]. Until now this conjecture remains open and the opinion of the people that have thought in it is that its proof can be hard. Numerically it is known that the five body problem with equal masses has only one convex central configuration, see [7] and [9]. Chen and Hsiao [5] provided necessary conditions in order that a central configuration of the five body problem to be convex and also show some numerical convex central configurations for different values of the masses.

The objective of this paper is to show analytically that this conjecture cannot be extended to the $(\ell + 2)$–body problem with $\ell > 2$ except perhaps for $\ell = 4$. Here we consider *symmetric* $(\ell + 2)$*–body problem* with masses $m_1 = m_2 = \cdots = m_\ell = 1$ and $m_{\ell+1} = m_{\ell+2} = m$ having the following symmetries. When $\ell = 2n - 1$ is odd we consider the S_1*–symmetric* $(2n + 1)$*–body problem* having the S_1*–symmetry*:

$$\begin{aligned} &y_1 = 0, \\ &x_{2n+1-i} = x_i, \quad y_{2n+1-i} = -y_i, \quad y_i > 0, \quad 2 \le i \le n, \\ &x_{2n+1} = x_{2n}, \quad y_{2n+1} = -y_{2n}, \quad y_{2n} > 0. \end{aligned}$$

When $\ell = 2n$ is even we consider the S_2*–symmetric* $(2n + 2)$*–body problem* having the S_2*– symmetry*:

$$\begin{aligned} &y_1 = 0, \quad y_{n+1} = 0, \\ &x_{2n-i} = x_{i+2}, \quad y_{2n-i} = -y_{i+2}, \quad y_{i+2} > 0, \quad 0 \le i \le n-2, \\ &x_{2n+2} = x_{2n+1}, \quad y_{2n+2} = -y_{2n+1}, \quad y_{2n+1} > 0, \end{aligned}$$

and the S_3*–symmetric* $(2n + 2)$*–body problem* having the S_3*–symmetry*:

$$\begin{aligned} &x_{2n-i} = x_{i+1}, \quad y_{2n-i} = -y_{i+1}, \quad y_{i+1} > 0, \quad 0 \le i \le n-1, \\ &x_{2n+2} = x_{2n+1}, \quad y_{2n+2} = -y_{2n+1}, \quad y_{2n+1} > 0. \end{aligned}$$

Our results are the following.

Theorem 1 *For each $m > 0$ sufficiently small the S_1–symmetric $\ell + 2$–body problem with masses $m_i = 1$, $i = 1, \ldots, \ell = 2n - 1$ and $m_{\ell+1} = m_{\ell+2} = m$ has at least two, five, four convex central configurations for $\ell = 3, 5, 7$, respectively.*

Theorem 1 is proved in Section 3.

Conjecture 1 For each $m > 0$ sufficiently small and $n \geq 5$, the S_1–symmetric $\ell + 2$–body problem with masses $m_i = 1$, $i = 1, \ldots, \ell = 2n - 1$ and $m_{\ell+1} = m_{\ell+2} = m$ has at least n convex central configurations.

In Subsection 3.5 we give numerical evidences that Conjecture 1 holds.

Theorem 2 *For each $m > 0$ sufficiently small the S_2–symmetric $\ell + 2$–body problem with masses $m_i = 1$, $i = 1, \ldots, \ell = 2n$ and $m_{\ell+1} = m_{\ell+2} = m$ has at least three, two, three convex central configurations for $\ell = 6, 8, 10$, respectively. For $\ell = 4$ there are no convex S_2–symmetric central configurations that can be obtained from continuation of central configurations of the restricted S_2–symmetric $\ell + 2$–body problem.*

Theorem 2 is proved in Section 4.

Conjecture 2 For each $m > 0$ sufficiently small and for $n \geq 6$, the S_2–symmetric $\ell + 2$–body problem with masses $m_i = 1$, $i = 1, \ldots, \ell = 2n$ and $m_{\ell+1} = m_{\ell+2} = m$ has at least $[\frac{n+1}{2}]$ convex central configurations.

In Subsection 4.6 we give numerical evidences that Conjecture 2 holds.

Theorem 3 *For each $m > 0$ sufficiently small the S_3–symmetric $\ell + 2$–body problem with masses $m_i = 1$, $i = 1, \ldots, \ell = 2n$ and $m_{\ell+1} = m_{\ell+2} = m$ has no convex central configurations for $\ell = 4, 6, 8, 10$ that can be obtained from continuation of central configurations of the restricted S_2–symmetric $\ell+2$–body problem.*

Theorem 3 is proved in Section 5.

Conjecture 3 For each $m > 0$ sufficiently small and for $n \geq 6$, the S_3–symmetric $\ell + 2$–body problem with masses $m_i = 1$, $i = 1, \ldots, \ell = 2n$ and $m_{\ell+1} = m_{\ell+2} = m$ has no convex central configurations that can be obtained from continuation of central configurations of the restricted S_3–symmetric $\ell + 2$–body problem.

2 Symmetric restricted $\ell + 2$ body problem with ℓ equal masses at the vertices of a regular ℓ–gon

Now for $k = 1, 2, 3$ we consider the *S_k–symmetric restricted* $(\ell + 2)$*–body problem* with ℓ masses equal to one at the vertices of a regular ℓ–gon with positions $x_i = \cos \alpha_i$, $y_i = \sin \alpha_i, 1 \leq i \leq \ell$ where $\alpha_i = 2\pi(i - 1)/\ell + \alpha$ and $\alpha = 0$ when $k = 1, 2$ and $\alpha = \pi/\ell$ when $k = 3$, and two infinitesimal masses with $m = 0$ located at the points $(x_{\ell+1}, y_{\ell+1})$ and $(x_{\ell+1}, -y_{\ell+1})$. Here $(x_{\ell+1}, y_{\ell+1})$ corresponds to the position of the infinitesimal mass in a central configuration of the restricted $\ell + 1$–body problem with ℓ equal masses at the vertices of a regular ℓ-gon.

The restricted $(\ell + 1)$–body problem with ℓ equal masses at the vertices of a regular ℓ-gon have been studied by several authors. Arenstorf in [2] proved that for the central configurations of the restricted $(3+1)$–body problem with 3 equal masses at the vertices of an equilateral triangle, the infinitesimal mass must be on one of the three straight lines passing through barycenter and a vertex of the triangle. He also proved that on each straight line there are exactly four positions for the infinitesimal mass in a central configuration. Bang and Elmabsout in [3] and [4] and later on Fernandes et al. in [6] studied the problem for $\ell \geq 3$. The results of [3, 4, 6] are summarized in the following lemma.

Lemma 1 *For $\ell \geq 3$ consider the central configurations of the restricted $(\ell+1)$–body problem with equal masses at the vertices of a regular ℓ–gon of radius 1.*

(a) The infinitesimal mass is located on an axis of symmetry of the ℓ-gon (see Theorem 2 in [3]).
(b) Assuming that the bodies are arranged so that the positive x–axis is a semi-axis of symmetry containing one of the primaries, the possible positions for the infinitesimal mass are $(0,0)$ and $(\rho_1, 0)$ with $\rho_1 > 1$.
(c) Assuming that the bodies are arranged so that the positive x–axis is a semi-axis of symmetry that does not contain any of the primaries, the possible positions for the infinitesimal mass are $(0,0)$, $(\rho_2, 0)$ and $(\rho_3, 0)$ with $0 < \rho_2 < 1 < \rho_3$.

Neither [2], [4] nor [6] provide the explicit values of ρ_1, ρ_2 and ρ_3. In this paper these values are necessary and they would be given when we need them.

3 S_1–symmetric $(2n+1)$–body problem

3.1 The equations

Since the set of central configurations is invariant under rotations and dilations, and we have the first integral of the center of mass, without loss of generality we can assume that the center of masses is at the origin of coordinates and that $\mathbf{q}_1 = (1, 0)$. Since the center of masses is at the origin we have $(cm_x, cm_y) = (0, 0)$, so $\sum_{i=1}^{2n+1} m_i x_i = 0$, and $x_2 = -(1/2 + \sum_{i=3}^{n} x_i + m x_{2n})$. Notice that due to the symmetry $\sum_{i=1}^{2n+1} m_i y_i$ is identically zero.

Taking into account these conditions together with the S_1–symmetry the $4n+2$ equations $e_1 = 0, \ldots, e_{4n+2} = 0$ given by (1) with $N = 2n+1$ for the central configurations of the S_1–symmetric $(2n+1)$–body problem reduce to the following $2n$ equations

$$\begin{array}{lll} e_i = 0, & e_{2n+1+i} = 0, & i = 2, \ldots, n, \\ e_{2n} = 0, & e_{4n+1} = 0. & \end{array}$$

Indeed, the equations e_1 and e_{2n+2} are omitted because $\sum_{i=1}^{2n-1} e_i + m e_{2n} + m e_{2n+1} = 0$ and $\sum_{i=1}^{2n-1} e_{i+2n+1} + m e_{4n+1} + m e_{4n+2} = 0$. And from the S_1–symmetry, we have $e_i = e_{2n+1-i}$, $e_{i+2n+1} = -e_{4n+2-i}$, for $2 \leq i \leq n$, $e_{2n+1} = e_{2n}$ and $e_{4n+2} = -e_{4n+1}$.

From the first equation $e_2 = 0$ we isolate λ and we substitute it into the other $2n-1$ equations that we denote by

$$E_3 = 0, \cdots, E_n = 0, E_{2n} = 0, E_{2n+3}, \cdots, E_{3n+1} = 0, E_{4n+1} = 0. \qquad (2)$$

In short, we have $2n-1$ equations and $2n-1$ unknowns y_2, x_i, y_i, $3 \le i \le n$, x_{2n}, y_{2n}.

We are interested in the central configurations of the S_1–symmetric restricted $2n+1$–body problem, see Section 2. Since the number of primaries is odd, without loss of generality we can assume that the line of symmetry is the x–axis and that one of the primaries is in the positive x–axis. We assume also that the infinitesimal mass m_{2n} is in the x–axis, i.e. $y_{2n} = 0$. Straightforward computations show that when $m = 0$, $x_i = \cos\alpha_i$, $y_i = \sin\alpha_i$, with $\alpha_i = 2\pi(i-1)/(2n-1)$, $1 \le i \le 2n-1$ and $y_{2n} = 0$ all the equations in (2) are identically zero except the equation $E_{2n} = 0$ which becomes $f_n(x_{2n}) = 0$ with

$$f_n(x_{2n}) = \sum_{j=1}^{2n-1} \frac{\cos\alpha_j - x_{2n}}{(1 - 2\cos\alpha_j x_{2n} + x_{2n}^2)^{3/2}} + \lambda x_{2n},$$

where

$$\lambda = -\sum_{\substack{j=1\\ j\neq 2}}^{2n-1} \frac{\cos\alpha_j/\cos\alpha_2 - 1}{(2 - 2\cos\alpha_{j-1})^{3/2}}.$$

3.2 Proof of Theorem 1 for $\ell = 3$

For the central configurations of the S_1–symmetric 5–body problem (that is $n = 2$) with masses $m_1 = m_2 = m_3 = 1$ and $m_4 = m_5 = m$ equations (2) taking into account that $x_2 = -mx_4 - 1/2$ (by the fist integral of the center of masses) reduce to $E_4 = 0$, $E_7 = 0$ and $E_9 = 0$ where

$$\begin{aligned}
E_4 &= \lambda x_4 - \frac{(2mx_4+1)/2 + x_4}{r_{24}^3} - \frac{(2mx_4+1)/2 + x_4}{r_{34}^3} + \frac{1 - x_4}{r_{14}^3},\\
E_7 &= \lambda y_2 + \frac{m(y_4 - y_2)}{r_{24}^3} - \frac{m(y_2 + y_4)}{r_{34}^3} - \frac{y_2}{r_{12}^3} - \frac{1}{4y_2^2},\\
E_9 &= \lambda y_4 - \frac{m}{4y_4^2} - \frac{y_4}{r_{14}^3} + \frac{y_2 - y_4}{r_{24}^3} - \frac{y_2 + y_4}{r_{34}^3},
\end{aligned}$$

with

$$\begin{aligned}
r_{12} &= \sqrt{(2mx_4+3)^2/4 + y_2^2},\\
r_{14} &= \sqrt{(x_4-1)^2 + y_4^2},\\
r_{24} &= \sqrt{(2(m+1)x_4+1)^2/4 + (y_2 - y_4)^2},\\
r_{34} &= \sqrt{(2(m+1)x_4+1)^2/4 + (y_2 + y_4)^2},
\end{aligned}$$

and

$$\lambda = \frac{mr_{12}^3 r_{24}^3 + mr_{12}^3 r_{34}^3 + 3r_{24}^3 r_{34}^3 + 2m(r_{24}^3 r_{34}^3 + (1+m)r_{12}^3(r_{24}^3 + r_{34}^3))x_4}{r_{12}^3 r_{24}^3 r_{34}^3 (1+2mx_4)}.$$

In short, we have three equations and three unknowns y_2, x_4 and y_4.

In the next result we provide all the convex central configurations of the S_1–symmetric restricted (3+2)–body problem having the three primaries with masses equal to one located at the vertices of the equilateral triangle $(x_1, y_1) = (1, 0)$, $(x_2, y_2) = (-1/2, \sqrt{3}/2)$ and $(x_3, y_3) = (-1/2, -\sqrt{3}/2)$, and the two infinitesimal masses $m = 0$ m located at (x_4, y_4) and $(x_5, y_5) = (x_4, -y_4)$.

Proposition 1 *The S_1–symmetric restricted (3+2)–body problem has exactly two classes of convex central configurations which are given by*

(a) $(x_4,\, y_4) = (-a/2, -\sqrt{3}a/2)$*, see Figure* 1(*b*),

(b) $(x_4,\, y_4) = (a,\, 0)$*, see Figure* 1(*a*)*. Note that in this configuration the two infinitesimal masses are colliding.*

Here $a = -1.6197896088\ldots$ *is a root of the polynomial*

$$P(x) = x_4^{11} - x_4^{10} + \left(2\sqrt{3} - 3\right)x_4^8 + \left(3 + 2\sqrt{3}\right)x_4^7 + 2\sqrt{3}x_4^6 - 2\left(3 + 2\sqrt{3}\right)x_4^5 - \left(4\sqrt{3} - 42\right)x_4^4 - \left(9 + 4\sqrt{3}\right)x_4^3 + \left(8 + 2\sqrt{3}\right)x_4^2 + \left(37 + 2\sqrt{3}\right)x_4 + 2\sqrt{3} + 9.$$

Proof First we look for the central configurations of the S_1–symmetric restricted $(3+2)$–body problem with m_4 on the axis of symmetry containing m_1, i.e. with $y_4 = 0$. In the previous subsection we have seen that under this condition x_4 must satisfy the equation $f_2(x_4) = 0$.

Since we are only interested in convex central configurations, we can restrict to $x_4 < -1/2$. Under this hypothesis equation $f_2(x_4) = 0$ becomes

$$\frac{3(2x_4+1)}{\left(x_4^2 + x_4 + 1\right)^{3/2}} = \sqrt{3}x_4 + \frac{3}{(x_4-1)^2}. \tag{3}$$

Squaring both sides of this equation and dropping the denominators we get the polynomial equation $3x_4P(x) = 0$. Solving numerically this polynomial equation we get exactly two solutions with $x_4 < -1/2$, but only $x_4 = a$ is a solution for the initial equation (3). This proves statement (*b*).

Taking the axis of symmetry through the mass m_3, or equivalently rotating the solution of statement (b) an angle of $-2\pi/3$ we get statement (*a*).

Clearly the configuration with $(x_4, y_4) = (x_5, y_5) = (a, 0)$ and the one with $(x_4, y_4) = (-a/2, -\sqrt{3}a/2)$ and $(x_5, y_5) = (-a/2, \sqrt{3}a/2)$ are both convex. Moreover, if we take the axis of symmetry through the mass m_2, by symmetry we get again the configuration given in statement (*a*). Thus these are the unique convex central configurations of the restricted $(3+2)$–body problem.

Proposition 2 *The two convex central configurations of the symmetric restricted* $(3+2)$*–body problem given in Proposition 1 can be continued to two families of central configurations of the symmetric* 5*–body problem with masses* $m_i = 1$, $i = 1, 2, 3$ *and* $m_4 = m_5 = m > 0$ *sufficiently small.*

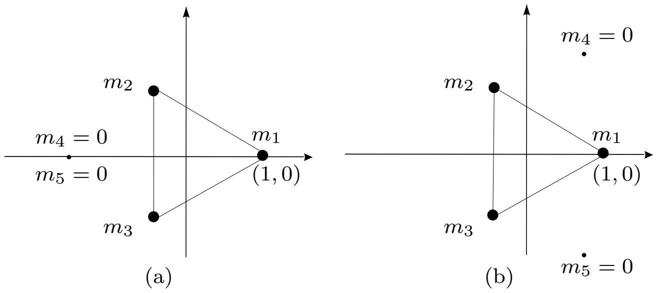

Fig. 1 Convex central configurations of the symmetric 5–body problem with $m_4 = m_5 = 0$.

Proof Using the Implicit Function Theorem we will see that the solutions of system $E_4 = 0$, $E_7 = 0$ and $E_9 = 0$ with $m = 0$ given in Proposition 1 can be continued to solutions of this system with $m > 0$ sufficiently small.

We start with the solution $(x_4, y_4) = (-a/2, -\sqrt{3}a/2)$. The determinant of the Jacobian of system $E_4 = 0$, $E_7 = 0$ and $E_9 = 0$ (with respect to the variables y_2, x_4, y_4), evaluated at $m = 0$, $y_2 = \sqrt{3}/2$, and $(x_4, y_4) = (-a/2, -\sqrt{3}a/2)$ takes the value

$$\frac{\sqrt{3}\left(-16a^2-16a+11\right)}{4(a-1)^3\left(a^2+a+1\right)^{5/2}}+\frac{\sqrt{3}\left(32a^4+64a^3-12a^2-44a+5\right)}{8\left(a^2+a+1\right)^5}$$
$$-\frac{1}{\left(a^2+a+1\right)^{3/2}}+\frac{1}{2(a-1)^3}+\frac{\sqrt{3}}{(a-1)^6}-\frac{1}{2\sqrt{3}}=-0.6885602058\cdots\neq 0.$$

Since this determinant is different from zero from the Implicit Function Theorem there exist an analytic family $y_2^1(m)$, $x_4^1(m)$ and $y_4^1(m)$ of solutions of system $E_4 = 0$, $E_7 = 0$ and $E_9 = 0$ with $y_2^1(0) = \sqrt{3}/2$, $x_4^1(0) = -a/2$ and $y_4^1(0) = -\sqrt{3}a/2$, defined for $m > 0$ sufficiently small.

Notice that system $E_4 = 0$, $E_7 = 0$ and $E_9 = 0$ is not analytic with respect to all its variables in a neighborhood of $m = 0$, $y_2 = \sqrt{3}/2$ and $(x_4, y_4) = (a, 0)$ because equation $E_9 = 0$ contains the term m/y_4^2. After doing the change of variables $y_4 = \mu Y_4/2$ with $\mu = m^{1/3}$ we obtain a new system of equations which is analytic with respect to all its variables in a neighborhood of $\mu = 0$, $y_2 = \sqrt{3}/2$, $x_4 = a$ and $Y_4 \neq 0$ and it is given by

$$\widetilde{E}_4 = -\frac{8(2x_4+1)}{\left((2x_4+1)^2+4y_2^2\right)^{3/2}}+\frac{24x_4}{\left(4y_2^2+9\right)^{3/2}}+\frac{1-x_4}{\left((x_4-1)^2\right)^{3/2}}+O(\mu^2),$$
$$\widetilde{E}_7 = \frac{16y_2}{\left(4y_2^2+9\right)^{3/2}}-\frac{y_2}{4\left(y_2^2\right)^{3/2}}+O(\mu^3),$$
$$\widetilde{E}_9 = \frac{1}{2}Y_4\left(-\frac{16\left((2x_4+1)^2-8y_2^2\right)}{\left((2x_4+1)^2+4y_2^2\right)^{5/2}}-\frac{1}{\left((x_4-1)^2\right)^{3/2}}+\frac{24}{\left(4y_2^2+9\right)^{3/2}}-\frac{2}{\left(Y_4^2\right)^{3/2}}\right)\mu+O(\mu^3).$$

Consider now the system of equations

$$\bar{E}_4 = \widetilde{E}_4, \qquad \bar{E}_7 = \widetilde{E}_7, \qquad \bar{E}_9 = \widetilde{E}_9/\mu,$$

which is analytic with respect to all its variables in a neighborhood of $\mu = 0$, $y_2 = \sqrt{3}/2$, $x_4 = a$ and $Y_4 \neq 0$. Substituting $\mu = 0$, $y_2 = \sqrt{3}/2$, $x_4 = a$ into system $\bar{E}_4 = 0$, $\bar{E}_7 = 0$ and $\bar{E}_9 = 0$ we get that $\bar{E}_4$ and $\bar{E}_7$ are identically zero and that equation $\bar{E}_9 = 0$ is equivalent to

$$\frac{1}{2} Y_4 \left(\frac{-4a^2 - 4a + 5}{2\left(a^2 + a + 1\right)^{5/2}} + \frac{1}{(a-1)^3} - \frac{2}{\left(Y_4^2\right)^{3/2}} + \frac{1}{\sqrt{3}} \right) = 0.$$

The solutions of this equation are $Y_4 = 0$ and

$$Y_4 = \pm b = \pm 1.4869700925\ldots.$$

Clearly, system $\bar{E}_4 = 0$, $\bar{E}_7 = 0$ and $\bar{E}_9 = 0$ is analytic with respect to all its variables in a neighborhood of the solution $\mu = 0$, $y_2 = \sqrt{3}/2$, $x_4 = a$ and $Y_4 = b$. Moreover the determinant of the Jacobian of the system (with respect to the variables y_2, x_4, Y_4) evaluated at this solution is

$$-\frac{1}{8\sqrt{3}} \left(\frac{3(2a+1)^2}{2\left(a^2+a+1\right)^{5/2}} - \frac{2}{\left(a^2+a+1\right)^{3/2}} - \frac{2}{(a-1)^3} + \frac{1}{\sqrt{3}} \right)$$
$$\left(-\frac{3\left(4a^2+4a-5\right)}{\left(a^2+a+1\right)^{5/2}} + \frac{6}{(a-1)^3} + \frac{24}{b^3} + 2\sqrt{3} \right) = -1.0328403087\cdots \neq 0.$$

Therefore, applying the Implicit Function Theorem, there exist a unique analytic family of solutions $y_2(\mu)$, $x_4(\mu)$ and $Y_4(\mu)$ of system $\bar{E}_4 = 0$, $\bar{E}_7 = 0$ and $\bar{E}_9 = 0$ with $y_2(0) = \sqrt{3}/2$, $x_4(0) = a$ and $Y_4(0) = b$ defined for $\mu > 0$ sufficiently small. This family provides the family of solutions of system $E_4 = 0$, $E_7 = 0$ and $E_9 = 0$ given by $y_2^2(m) = \sqrt{3}/2 + O(m^{1/3})$, $x_4^2(m) = a + O(m^{1/3})$ and $y_4^2(m) = b/2\, m^{1/3} + O(m^{2/3})$ defined for $m > 0$ sufficiently small. This completes the proof of the proposition.

Proof (Proof of Theorem 1 for $\ell = 3$) Let $x_2^1(m) = -m x_4^1(m) - 1/2$ and $x_2^2(m) = -m x_4^2(m) - 1/2$. Given $m > 0$ sufficiently small, Proposition 2 provides two central configurations: the one given by $(x_1, y_1) = (1, 0)$, $(x_2, y_2) = (x_2^1(m), y_2^1(m))$, $(x_3, y_3) = (x_2^1(m), -y_2^1(m))$, $(x_4, y_4) = (x_4^1(m), y_4^1(m))$, and $(x_5, y_5) = (x_4^1(m), -y_4^1(m))$, and the one given by $(x_1, y_1) = (1, 0)$, $(x_2, y_2) = (x_2^2(m), y_2^2(m))$, $(x_3, y_3) = (x_2^2(m), -y_2^2(m))$, $(x_4, y_4) = (x_4^2(m), y_4^2(m))$, and $(x_5, y_5) = (x_4^2(m), -y_4^2(m))$. Since for $m = 0$ they are two different convex central configurations, it follows that for $m > 0$ sufficiently small they continue being two different convex central configurations. This completes the proof of the theorem.

3.3 Proof of Theorem 1 for $\ell = 5$

For the symmetric 7-body problem (that is $n = 3$), the equations of central configuration (2) are reduced to 5 equations: $E_3 = 0$, $E_6 = 0$, $E_9 = 0$, $E_{10} = 0$, $E_{13} = 0$ where $x_2 = -mx_6 - 1/2 - x_3$ (by the fist integral of the center of masses), and the ones for the symmetric restricted $(5+2)$–body problem are reduced to the equation $f_3(x_6) = 0$, see Subsection 3.1.

In order to prove the convexity of our central configurations we will use the following result.

Lemma 2 *Let* $\mathbf{q}_i$ *for* $i = 1, \ldots, N$ *be the vertices of a* n*-gon ordered sequentially counterclockwise. If the signed areas of all the triangles formed by three consecutive vertices of the* n*-gon are positive, then the* n*-gon is convex. That is, if*

$$\overrightarrow{\mathbf{q}_i\mathbf{q}_{i+1}} \times \overrightarrow{\mathbf{q}_{i+1}\mathbf{q}_{i+2}} > 0,$$

for all $i = 1, \ldots, N$*, then the* n*-gon is convex. Here* $\mathbf{q}_{N+1} = \mathbf{q}_1$ *and* $\mathbf{q}_{N+2} = \mathbf{q}_2$*.*

Recall that the signed area of a triangle formed by the consecutive vertices in counterclockwise of a n-gon not necessarily regular $\mathbf{p}_1 = (x_1, y_1)$, $\mathbf{p}_2 = (x_2, y_2)$ and $\mathbf{p}_3 = (x_3, y_3)$ is given by

$$A = \frac{1}{2}\overrightarrow{\mathbf{p}_1\mathbf{p}_2} \times \overrightarrow{\mathbf{p}_2\mathbf{p}_3} = \frac{1}{2}\begin{vmatrix} x_2 - x_1 & x_3 - x_2 \\ y_2 - y_1 & y_3 - y_2 \end{vmatrix}.$$

Proposition 3 *Let* $a_1 = -1.5979217289\ldots$ *and* $a_2 = -0.8221828699\ldots$ *be the two roots of* $f_3(x_6) = 0$ *with* $x_6 < \cos(4\pi/5)$*. The symmetric restricted* $(5+2)$*–body problem has exactly five classes of convex central configurations which are given by*

(a) $(x_6,\, y_6) = (\cos(-2\pi/5)a_1,\, \sin(-2\pi/5)a_1)$*, see Figure 2 (b),*
(b) $(x_6,\, y_6) = (\cos(-2\pi/5)a_2,\, \sin(-2\pi/5)a_2)$*, see Figure 2 (d),*
(c) $(x_6,\, y_6) = (\cos(-4\pi/5)a_2,\, \sin(-4\pi/5)a_2)$*, see Figure 2 (e),*
(d) $(x_6,\, y_6) = (a_1,\, 0)$*, see Figure 2 (a). Note that in this configuration the two infinitesimal masses are colliding,*
(e) $(x_6,\, y_6) = (a_2,\, 0)$*, see Figure 2 (c). In this configuration the two infinitesimal masses are colliding.*

Proof Taking the mass m_6 on the axis of symmetry that passes through m_1, i.e. $y_6 = 0$, the equations for the central configurations of the S_1–symmetric restricted $(5+2)$–body problem are reduced to the equation $f_3(x_6) = 0$. Since we are only interested in convex central configurations we can assume that $x_6 < \cos(4\pi/5)$, under this assumption equation $f_3(x_6) = 0$ becomes

$$\begin{aligned} &\frac{1}{(x_6-1)^2} + \sqrt{1+\frac{2}{\sqrt{5}}}x_6 + \frac{\sqrt{2}(\sqrt{5}-1-4x_6)}{(2+x_6-\sqrt{5}x_6+2x_6^2)^{3/2}} \\ &-\frac{\sqrt{2}(\sqrt{5}+1+4x_6)}{(2+x_6+\sqrt{5}x_6+2x_6^2)^{3/2}} = 0 \end{aligned} \tag{4}$$

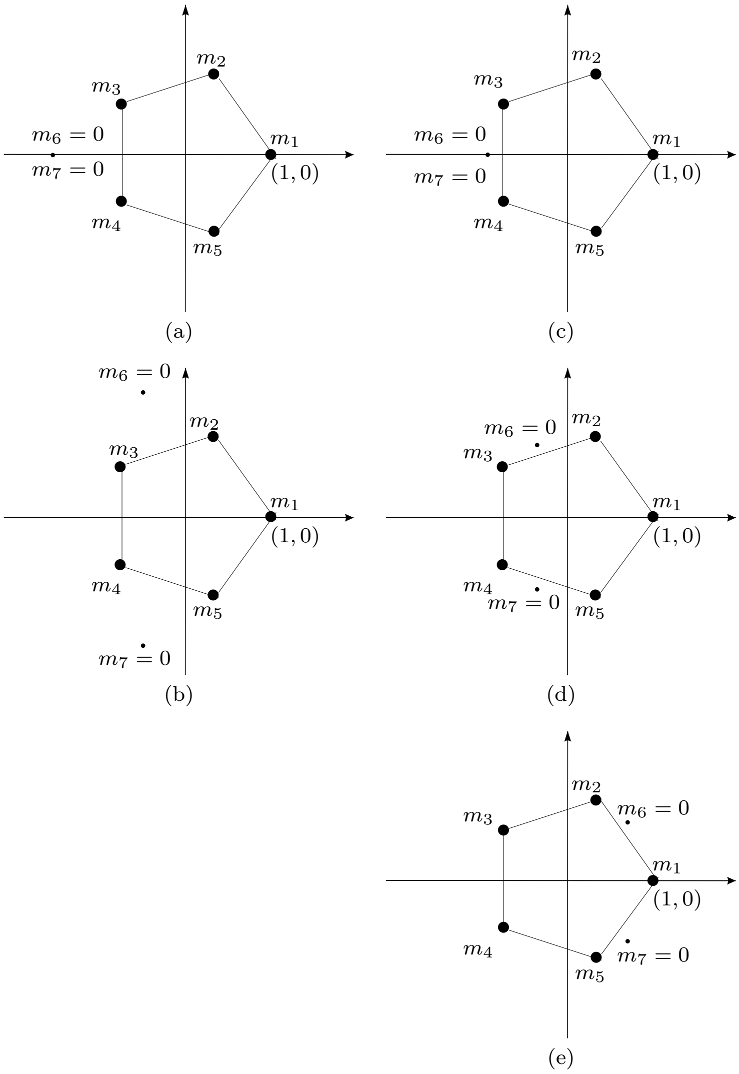

Fig. 2 Convex central configurations of the symmetric 7–body problem with $m_6 = m_7 = 0$
.

Squaring both sides of this equation and dropping the denominators we get the polynomial equation $-8192x_6P(x_6) = 0$ where $P(x_6)$ is a polynomial of degree 35 in the variable x_6. This polynomial equation has exactly four real roots with $x_6 < \cos(4\pi/5)$ of those only $x_6 = a_1$ and $x_6 = a_2$ are solutions of the initial equation (4).

It is easy to see that the configurations with $(x_6, y_6) = (a_1, 0)$ and the one with $(x_6, y_6) = (a_2, 0)$ are both convex because they satisfy the conditions of Lemma 2. This proves statements (d) and (e).

We also can take the mass m_6 on an axis of symmetry passing through m_4 and m_5. This corresponds to rotate the solutions $(x_6, y_6) = (a_1, 0)$ and $(x_6, y_6) = (a_2, 0)$ and angle of $-4\pi/5$ and $-2\pi/5$, respectively. Note that by symmetry if we take the mass m_6 on the axis of symmetry through m_2 (respectively, m_3) we obtain the same configuration than taking the axis of symmetry through m_5 (respectively, m_4).

By rotating the solution $(x_6, y_6) = (a_1, 0)$ with an angle of $-2\pi/5$ and rotating the solution $(x_6, y_6) = (a_2, 0)$ with an angle of $-2\pi/5$ and $-4\pi/5$ we get the configurations of statements (a), (b) and (c) respectively. Applying Lemma 2 again we see that all these configurations are convex. Note that the configuration given by rotating the solution $(x_6, y_6) = (a_1, 0)$ with an angle of $-4\pi/5$ is not convex because $a_1 \cos(-4\pi/5) > 1$. So these are the unique convex central configurations of the S_1–symmetric restricted $(5 + 2)$–body problem.

Proposition 4 *Each of the five convex central configurations of the symmetric restricted $(5+2)$–body problem can be continued to a family of central configurations of the symmetric 7–body problem with masses $m_i = 1, i = 1, \ldots, 5$ and $m_6 = m_7 = m > 0$ sufficiently small.*

Proof Using the Implicit Function Theorem we will see that the solutions of system $E_3 = 0$, $E_6 = 0$, $E_9 = 0$, $E_{10} = 0$, $E_{13} = 0$ with $m = 0$ given in Proposition 3 can be continued to solutions of this system with $m > 0$ sufficiently small.

We use the following notation: $U = (y_2, x_3, y_3)$, $V = (x_6, y_6)$, and $\tilde{U} = (\sin(2\pi/5), \cos(4\pi/5), \sin(4\pi/5))$, $\tilde{V}_1 = (\cos(-2\pi/5)a_1, \sin(-2\pi/5)a_1)$, $\tilde{V}_2 = (\cos(-2\pi/5)a_2, \sin(-2\pi/5)a_2)$, and $\tilde{V}_3 = (\cos(-4\pi/5)a_2, \sin(-4\pi/5)a_2)$.

System $E_3 = 0$, $E_6 = 0$, $E_9 = 0$, $E_{10} = 0$, $E_{13} = 0$ is analytic with respect to all its variables in a neighborhood of the solutions $m = 0$, $U = \tilde{U}$ and $V = \tilde{V}_i$ for $i = 1, 2, 3$. Let $\mathcal{J}(U, V, m)$ denote the Jacobian of the system $E_3 = 0$, $E_6 = 0$, $E_9 = 0$, $E_{10} = 0$, $E_{13} = 0$ with respect to the variables U, V. Straightforward computations show that

$$\det(\mathcal{J}(U, V, m))\,|_{U=\tilde{U}, V=\tilde{V}_1, m=0} = 61.9657422025\ldots,$$
$$\det(\mathcal{J}(U, V, m))\,|_{U=\tilde{U}, V=\tilde{V}_2, m=0} = -1928.5558212278\ldots,$$
$$\det(\mathcal{J}(U, V, m))\,|_{U=\tilde{U}, V=\tilde{V}_3, m=0} = -1928.5558212278\ldots.$$

Since these determinants are different from zero from the Implicit Function Theorem there exist three analytic families of solutions of system $E_3 = 0$, $E_6 = 0$, $E_9 = 0$, $E_{10} = 0$, $E_{13} = 0$ defined for $m > 0$ sufficiently small. They are given by $(U^i(m), V^i(m))$ for $i = 1, 2, 3$ with $U^i(m) = (y_2^i(m), x_3^i(m), y_3^i(m))$ and $V^i(m) = (x_6^i(m), y_6^i(m))$ satisfying $U^i(0) = \tilde{U}$, $V^i(0) = \tilde{V}_i$.

System $E_3 = 0$, $E_6 = 0$, $E_9 = 0$, $E_{10} = 0$, $E_{13} = 0$ is not analytic with respect to all its variables in a neighborhood of the solution $m = 0$, $X = \tilde{X}$ and $Y = (x_6, y_6) = (a, 0)$ with either $a = a_1$ or $a = a_2$ because equation $E_{13} = 0$ contains the term m/y_6^2. After doing the change of variables $y_6 = \mu Y_6/2$ with $\mu = m^{1/3}$ we obtain a new system of equations which is analytic with respect to all its variables in a neighborhood of $\mu=0$, $U=\tilde{U}$, $x_6=a$ and $Y_6\neq 0$ and it can be written as

$$\begin{aligned}
\widetilde{E}_3 &= \widetilde{E}_3(y_2,x_3,y_3)+O(\mu^3),\\
\widetilde{E}_6 &= \widetilde{E}_6(y_2,x_3,y_3,x_6)+O(\mu^2),\\
\widetilde{E}_9 &= \widetilde{E}_9(y_2,x_3,y_3)+O(\mu^3),\\
\widetilde{E}_{10} &= \widetilde{E}_{10}(y_2,x_3,y_3)+O(\mu^3),\\
\widetilde{E}_{13} &= \mu\widetilde{E}_{13}(y_2,x_3,y_3,x_6,Y_6)+O(\mu^3).
\end{aligned}$$

Now we consider the system of equations

$$\bar{E}_3=\widetilde{E}_3,\quad \bar{E}_6=\widetilde{E}_6,\quad \bar{E}_9=\widetilde{E}_9,\quad \bar{E}_{10}=\widetilde{E}_{10},\quad \bar{E}_{13}=\widetilde{E}_{13}/\mu,$$

which is also analytic with respect to all its variables in a neighborhood of $\mu=0$, $U=\tilde{U}$, $x_6=a$, and $Y_6\neq 0$. Substituting $\mu=0$, $U=\tilde{U}$ and $x_6=a$ into system $\bar{E}_3=0$, $\bar{E}_6=0$, $\bar{E}_9=0$, $\bar{E}_{10}=0$, $\bar{E}_{13}=0$ we get that $\bar{E}_3$, $\bar{E}_6$, $\bar{E}_9$, $\bar{E}_{10}$ are identically zero and that equation $\bar{E}_{13}=0$ is equivalent to

$$\bar{E}_{13}(Y_6)=\frac{1}{4}AY_6+\frac{\sqrt{Y_6^2}}{Y_6^3},$$

where

$$\begin{aligned}
A=&-\frac{2\sqrt{2}\left(-8a^2+4\left(\sqrt{5}-1\right)a+3\sqrt{5}+7\right)}{\left(2a^2-\sqrt{5}a+a+2\right)^{5/2}}+\\
&\frac{2\sqrt{2}\left(8a^2+4\left(1+\sqrt{5}\right)a+3\sqrt{5}-7\right)}{\left(2a^2+\sqrt{5}a+a+2\right)^{5/2}}-\frac{2}{(a-1)^3}-2\sqrt{1+\frac{2}{\sqrt{5}}}.
\end{aligned}$$

The solutions of this equation are

$$Y_6=\pm\sqrt[3]{-\frac{4}{A}}.$$

If $a=a_1$ then $Y_6=\pm b_1=\pm 1.1302790764\ldots$ and if $a=a_2$ then $Y_6=\pm b_2=\pm 0.4564622776\ldots$. By symmetry, we are only interested in the positive values of Y_6.

Clearly, system $\bar{E}_3=0$, $\bar{E}_6=0$, $\bar{E}_9=0$, $\bar{E}_{10}=0$, $\bar{E}_{13}=0$ is analytic with respect to all its variables in a neighborhood of the solutions $\mu=0$, $U=\tilde{U}$, $x_6=a_i$, $Y_6=b_i$ with $i=1,2$. Moreover the Jacobian of the system (with respect to the variables U, x_6, Y_6) evaluated at these solutions take the values $92.9486133\ldots$ when $i=1$ and $-2892.8337318417\ldots$ when $i=2$. Therefore, applying the Implicit Function Theorem, there exist two analytic families of solutions of system $\bar{E}_3=0$, $\bar{E}_6=0$, $\bar{E}_9=0$, $\bar{E}_{10}=0$, $\bar{E}_{13}=0$ defined for $\mu>0$ sufficiently small. They are given by $(U^{i+3}(\mu),x_6^{i+3}(\mu),Y_6^{i+3}(\mu))$ for $i=1,2$ with $U^{i+3}(\mu)=(y_2^{i+3}(\mu),x_3^{i+3}(\mu),y_3^{i+3}(\mu))$ and satisfying $U^{i+3}(0)=\tilde{U}$, $x_6^{i+3}(0)=a_i$ and $Y_6^{i+3}(0)=b_i$. Let $y_6^{i+3}(\mu)=\mu Y_6^{i+3}(\mu)/2$, then these two families provide the two families of solutions of system $E_3 = 0$, $E_6 = 0$, $E_9 = 0$, $E_{10} = 0$, $E_{13} = 0$ given by $(U^{i+3}(m^{1/3}), V^{i+3}(m^{1/3}))$ for $i = 1, 2$ where $V^{i+3}(\mu) = (x_6^{i+3}(\mu), y_6^{i+3}(\mu))$, defined for $m > 0$ sufficiently small. In particular, $y_2^{i+3}(m) = \sin(2\pi/5) + O(m^{1/3})$, $x_3^{i+3}(m) = \cos(4\pi/5) + O(m^{1/3})$, $y_3^{i+3}(m) = \sin(4\pi/5) + O(m^{1/3})$, $x_6^{i+3}(m) = a_i + O(m^{1/3})$ and $y_6^{i+3}(m) = b_i/2\, m^{1/3} + O(m^{1/3})$.

Proof (Proof of Theorem 1 *for* $\ell = 5$*)* Let $x_2^i(m) = -mx_6^i(m) - 1/2 - x_3^i(m)$ for $i = 1, \ldots, 5$. Given $m > 0$ sufficiently small, Proposition 4 provides the five central configurations given by $(x_1, y_1) = (1, 0)$, $(x_2, y_2) = (x_2^i(m), y_2^i(m))$, $(x_3, y_3) = (x_3^i(m), y_3^i(m))$, $(x_4, y_4) = (x_3^i(m), -y_3^i(m))$, $(x_5, y_5) = (x_2^i(m), -y_2^i(m))$, $(x_6, y_6) = (x_6^i(m), y_6^i(m))$, and $(x_7, y_7) = (x_6^i(m), -y_6^i(m))$ for $i = 1, \ldots, 5$.

Since for $m = 0$ we have five different central configurations that are convex, see Figure 2, it follows that for $m > 0$ sufficiently small they continue being five different convex central configurations. This completes the proof of the theorem.

3.4 Proof of Theorem 1 for $\ell = 7$

For the symmetric 9–body problem (that is $n = 4$), the equations of central configuration are reduced to 7 equations: $E_3 = 0$, $E_4 = 0$, $E_8 = 0$, $E_{11} = 0$, $E_{12} = 0$, $E_{13} = 0$, $E_{17} = 0$ where $x_2 = -m\, x_8 - 1/2 - x_3 - x_4$ and the ones for the symmetric restricted $(7 + 2)$–body problem are reduced to the equation $f_4(x_8) = 0$, see Subsection 3.1.

Proposition 5 *Let* $a_3 = -0.9189903637\ldots$ *be the biggest root of* $f_4(x_8) = 0$ *with* $x_8 < \cos(6\pi/7)$*. The symmetric restricted* (7+2)*–body problem has exactly four convex central configurations which are given by*

(a) $(x_8, y_8) = (\cos(-2\pi/7)a_3, \sin(-2\pi/7)a_3)$*, see Figure* 3 *(b),*

(b) $(x_8, y_8) = (\cos(-4\pi/7)a_3, \sin(-4\pi/7)a_3)$*, see Figure* 3 *(c),*

(c) $(x_8, y_8) = (\cos(-6\pi/7)a_3, \sin(-6\pi/7)a_3)$*, see Figure* 3 *(d),*

(d) $(x_8, y_8) = (a_3, 0)$*, see Figure* 3 *(a). Note that in this configuration the two infinitesimal masses are colliding.*

Proof Assuming that the mass m_8 is on the axis of symmetry that passes through m_1, i.e. $y_8 = 0$, the equations for the central configurations of the S_1–symmetric restricted $(7 + 2)$–body problem are reduced to the equation $f_4(x_8) = 0$ with

$$f_4(x_8) = \sum_{j=1}^{7} \frac{\cos(2(j-1)\pi/7) - x_8}{(1 - 2\cos(2(j-1)\pi/7)x_8 + x_8^2)^{3/2}} - x_8 \sum_{\substack{j=1\\ j\neq 2}}^{7} \frac{\cos(2(j-1)\pi/7)/\cos(2\pi/7) - 1}{(2 - 2\cos(2(j-2)\pi/7))^{3/2}},$$

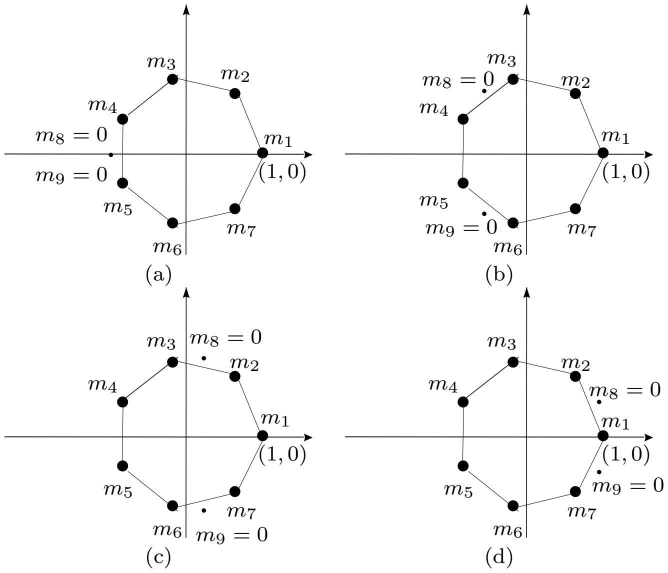

Fig. 3 Convex central configurations of the symmetric 9–body problem with $m_8 = m_9 = 0$.

see Subsection 3.1. Since we are only interested in convex central configurations we can assume that $x_8 < \cos(6\pi/7)$. Rearranging conveniently equation $f_4(x_8) = 0$, squaring both sides of the equation as many times as we need and dropping the denominators we get the polynomial equation $kx_8P(x_8) = 0$, where k is a constant and P is a polynomial of degree 95. This polynomial equation has exactly eight real roots with $x_8 < \cos(6\pi/7)$ of those only $x_8 = a_3$ and $x_8 = a_4 = -1.5841209012\ldots$ are solutions of the initial equation $f_4(x_8) = 0$.

We also can take the mass m_8 on the axis of symmetry passing through any other of the masses. By symmetry it is sufficient to take m_8 on the axes of symmetry passing through m_5, m_6, and m_7, this corresponds to rotate the solutions $(x_8, y_8) = (a_3, 0)$ and $(x_8, y_8) = (a_4, 0)$ an angle of $-6\pi/7$, $-4\pi/7$ and $-2\pi/7$, respectively. By taking the mass m_8 on the axes of symmetry passing through m_4, m_3 and m_2 respectively we do not obtain new configurations.

It is easy to see that the configuration with $(x_8,\, y_8) = (a_3,\, 0)$ is convex because it satisfies the conditions of Lemma 2. This proves statements (d). By rotating the solution $(x_8,\, y_8) = (a_3,\, 0)$ with an angle of $-2\pi/7$, $-4\pi/7$ and $-6\pi/7$ we get the configurations of statements (a), (b) and (c) respectively. Applying Lemma 2 again we see that all these configurations are convex. Note that the configurations with $(x_8, y_8) = (a_4, 0)$, $(x_8,\, y_8) = (\cos(-2\pi/7)a_4,\, \sin(-2\pi/7)a_4)$, $(x_8,\, y_8) = (\cos(-4\pi/7)a_4,\, \sin(-4\pi/7)a_4)$, and $(x_8,\, y_8) = (\cos(-6\pi/7)a_4,\, \sin(-6\pi/7)a_4)$ are not convex because they do not satisfy Lemma 2. So these are the unique convex central configurations of the S_1–symmetric restricted $(7 + 2)$–body problem.

Proposition 6 *Each of the four convex central configurations of the symmetric restricted* $(7+2)$*–body problem can be continued to a family of central configurations of the symmetric* 9*–body problem with masses* $m_i = 1, i = 1, \ldots, 7$ *and* $m_8 = m_9 = m > 0$ *sufficiently small.*

Proof The proof is similar to the proof for $\ell = 3$ and $\ell = 5$. Using the Implicit Function Theorem we will see that the solutions of system $E_3 = 0$, $E_4 = 0$, $E_8 = 0$, $E_{11} = 0$, $E_{12} = 0$, $E_{13} = 0$, $E_{17} = 0$ with $m = 0$ given in Proposition 5 can be continued to solutions of this system with $m > 0$ sufficiently small.

Let $U = (y_2, x_3, y_3, x_4, y_4)$, $V = (x_8, y_8)$ and

$$\tilde{U} = (\sin(2\pi/7), \cos(4\pi/7), \sin(4\pi/7), \cos(6\pi/7), \sin(6\pi/7)),$$
$$\tilde{V}_i = (\cos(-2\pi i/7)a_3, \sin(-2\pi i/7)a_3), \qquad i = 1, 2, 3.$$

Then E_3, E_4, E_8, E_{11}, E_{12}, E_{13}, E_{17} are functions with variables U, V. System $E_3 = 0$, $E_4 = 0$, $E_8 = 0$, $E_{11} = 0$, $E_{12} = 0$, $E_{13} = 0$, $E_{17} = 0$ is analytic with respect to U, V in a neighborhood of the solutions $m = 0$, $U = \tilde{U}$ and $V = \tilde{V}_i$ for $i = 1, 2, 3$. Let $\mathcal{J}(U, V, m)$ denote the Jacobian of the system with respect to the variables U, V. Straightforward computations show that

$$\det(\mathcal{J}(X, Y, m))\,|_{U=\tilde{U}, V=\tilde{V}_i, m=0} = -203091.2451410419\ldots, \quad i = 1, 2, 3.$$

Since these determinants are different from zero from the Implicit Function Theorem there exist three analytic families of solutions of system $E_3 = 0$, $E_4 = 0$, $E_8 = 0$, $E_{11} = 0$, $E_{12} = 0$, $E_{13} = 0$, $E_{17} = 0$ defined for $m > 0$ sufficiently small. They are given by by $(U^i(m), V^i(m))$ for $i = 1, 2, 3$ with $U^i(m) = (y_2^i(m), x_3^i(m), y_3^i(m)\, x_4^i(m), y_4^i(m))$ and $V^i(m) = (x_8^i(m), y_8^i(m))$ satisfying $U^i(0) = \tilde{U}$, $V^i(0) = \tilde{V}_i$.

System $E_3 = 0$, $E_4 = 0$, $E_8 = 0$, $E_{11} = 0$, $E_{12} = 0$, $E_{13} = 0$, $E_{17} = 0$ is not analytic with respect to all its variables in a neighborhood of the solution $m = 0$, $U = \tilde{U}$ and $(x_8, y_8) = (a_3, 0)$ because equation $E_{17} = 0$ contains the term m/y_8^2. After doing the change of variables $y_8 = \mu Y_8/2$ with $\mu = m^{1/3}$ we obtain a new equivalent system of equations which is analytic with respect to all its variables in a neighborhood of $\mu = 0$, $U = \tilde{U}$, $x_8 = a_3$ and $Y_8 \neq 0$ and it can be written as

$$\begin{aligned} \widetilde{E}_i &= \widetilde{E}_i(y_2, x_3, y_3, x_4, y_4) + O(\mu^3), \quad i = 3, 4, 11, 12, 13, \\ \widetilde{E}_8 &= \widetilde{E}_8(y_2, x_3, y_3, x_4, y_4, x_8) + O(\mu^2), \\ \widetilde{E}_{17} &= \mu \widetilde{E}_{17}(y_2, x_3, y_3, x_4, y_4, x_8, Y_8) + O(\mu^3). \end{aligned}$$

Now we consider the system of equations

$$\bar{E}_i = \widetilde{E}_i, \quad i = 3, 4, 8, 11, 12, 13, \qquad \bar{E}_{17} = \widetilde{E}_{17}/\mu,$$

which is also analytic with respect to all its variables in a neighborhood of $\mu = 0$, $U = \tilde{U}$, $x_8 = a_3$, and $Y_8 \neq 0$. Substituting $\mu = 0$, $U = \tilde{U}$, $x_8 = a_3$ and Y_8 into system $\bar{E}_3 = 0$, $\bar{E}_4 = 0$, $\bar{E}_8 = 0$, $\bar{E}_{11} = 0$, $\bar{E}_{12} = 0$, $\bar{E}_{13} = 0$, $\bar{E}_{17} = 0$

we get that all these equations are identically zero except the equation $\bar{E}_{17}=0$ which is equivalent to

$$\bar{E}_{17}(Y_8)=\bar{A}Y_8-\frac{\sqrt{Y_8^2}}{Y_8^3}=0,$$

where $\bar{A}=25.9395013227\ldots$. The solutions of this equation are

$$Y_8=\pm d=\pm 0.3378154118\ldots.$$

By symmetry, we are only interested in the positive values of Y_8.

Clearly, system $\bar{E}_3=0$, $\bar{E}_4=0$, $\bar{E}_8=0$, $\bar{E}_{11}=0$, $\bar{E}_{12}=0$, $\bar{E}_{13}=0$, $\bar{E}_{17}=0$ is analytic with respect to all its variables in a neighborhood of the solutions $\mu=0$, $U=\tilde{U}$, $x_8=a_3$, and $Y_8=d$. Moreover the Jacobian of the system (with respect to the variables U, x_8, Y_8) evaluated at this solution takes the value $-304636.8677115628\ldots$. Therefore, applying the Implicit Function Theorem, this solution can be continued to an analytical family of solutions of system $\bar{E}_3=0$, $\bar{E}_4=0$, $\bar{E}_8=0$, $\bar{E}_{11}=0$, $\bar{E}_{12}=0$, $\bar{E}_{13}=0$, $\bar{E}_{17}=0$ defined for $\mu>0$ sufficiently small. This family provides the family of solutions of system $E_3=0$, $E_4=0$, $E_8=0$, $E_{11}=0$, $E_{12}=0$, $E_{13}=0$, $E_{17}=0$ given by $y_2^4=\sin(2\pi/7)+O(m^{1/3})$, $x_3^4=\cos(4\pi/7)+O(m^{1/3})$, $y_3^4=\sin(4\pi/7)+O(m^{1/3})$, $x_4^4=\cos(6\pi/7)+O(m^{1/3})$, $y_4^4=\sin(6\pi/7)+O(m^{1/3})$, $x_8^4=a_3+O(m^{1/3})$, and $y_8^4=d/2\,m^{1/3}+O(m^{2/3})$.

Proof (Proof of Theorem 1 *for* $\ell=7$*)* Let $x_2^i(m)=-mx_8^i(m)-1/2-x_3^i(m)-x_4^i(m)$ for $i=1,\ldots,5$. Given $m>0$ sufficiently small, from Proposition 6, we have the four central configurations given by $(x_1,y_1)=(1,0)$, $(x_2,y_2)=(x_2^i(m),\,y_2^i(m))$, $(x_3,\,y_3)=(x_3^i(m),\,y_3^i(m))$, $(x_4,\,y_4)=(x_4^i(m),y_4^i(m))$, $(x_5,\,y_5)=(x_4^i(m),\,-y_4^i(m))$, $(x_6,\,y_6)=(x_3^i(m),\,-y_3^i(m))$, $(x_7,y_7)=(x_2^i(m),\,-y_2^i(m))$, $(x_8,\,y_8)=(x_8^i(m),\,y_8^i(m))$, $(x_9,\,y_9)=(x_8^i(m),\,-y_8^i(m))$ for $i=1,\ldots,4$.

Since for $m=0$ we have four different central configurations that are convex, see Figure 3, it follows that for $m>0$ sufficiently small they continue being four different convex central configurations. This completes the proof of the theorem.

3.5 Numerical evidences than Conjecture 1 is true

From Lemma 1 we know that for all $n\geq 3$ equation $f_n(x_{2n})=0$ has two solutions with $x_{2n}<0$ that we denote by $a_1(n)$ and $a_2(n)$, where $a_1(n)<a_2(n)$. We have computed numerically these two solutions for $n=4,\ldots,150$ (see Figure 3.5 for the plot of these solutions as a function of n). Each solution provides n central configurations of the S_1-symmetric restricted $(2n+1)$–body problem that are given by

$$(x_{2n},y_{2n})_{i,j}=\left(\cos\left(\frac{-2\pi(i-1)}{2n-1}\right)a_j(n),\,\sin\left(\frac{-2\pi(i-1)}{2n-1}\right)a_j(n)\right),$$

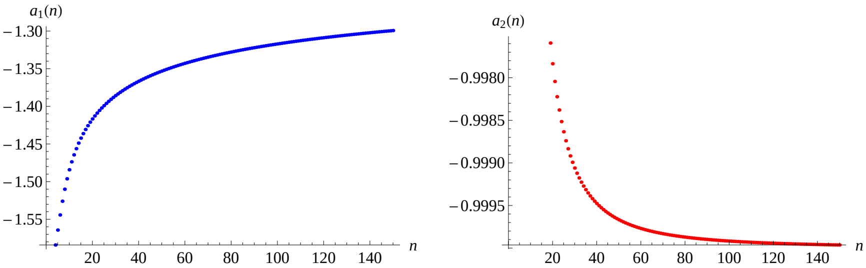

Fig. 4 The plot of $a_1(n)$ and $a_2(n)$ for $n = 4, \ldots, 150$.

with $j = 1, 2$ and $i = 1, \ldots, n-1$. Applying Lemma 2 we see that the central configurations $(x_{2n}, y_{2n})_{i,1}$ are concave for all $i = 1, \ldots, n-1$ whereas the central configurations $(x_{2n}, y_{2n})_{i,2}$ are convex for all $i = 1, \ldots, n-1$. In short for $n = 4, \ldots, 150$ the restricted $(2n+1)$–body problem with $2n-1$ masses at the vertices of a regular n–gon has exactly n classes of convex central configurations.

We have applied the Implicit function Theorem as in the proof of Theorem 1 for $\ell = 3, 5, 7$ to continue these central configurations to the symmetric $(2n+1)$–body problem with $m > 0$ sufficiently small for $n = 4, \ldots, 20$ and in all cases we have seen that the corresponding determinants are different form zero. In particular, we see that the absolute value of these determinants increases as n increases. This gives numerical evidence that Conjecture 1 is true.

4 S_2–symmetric $(2n+2)$–body problem

4.1 S_2–symmetric $(2n+2)$–body problem

We assume that $\mathbf{q}_1 = (1, 0)$. Since the center of masses is at the origin we have $(cm_x, cm_y) = (0, 0)$, so $\sum_{i=1}^{2n+2} m_i x_i = 0$, and $x_2 = -(\sum_{i=3}^{n} x_i + x_{n+1}/2 + 1/2 + m x_{2n+1})$. Notice that due to the symmetry $\sum_{i=1}^{2n+2} m_i y_i$ is identically zero.

Taking into account all these conditions the $4n+4$ equations $e_1 = 0, \ldots, e_{4n+4} = 0$ given by (1) with $N = 2n+2$ for the central configurations of the planar S_2–symmetric $(2n+2)$–body problem reduce to the following $2n+1$ equations

$$
\begin{array}{lll}
e_i = 0, & i = 1, \ldots, n, & e_{2n+1} = 0, \\
e_{2n+2+i} = 0 & i = 2, \ldots, n, & e_{4n+3} = 0.
\end{array}
$$

Indeed, the equation e_{n+1} are omitted because $\sum_{i=1}^{2n} e_i + m e_{2n+1} + m e_{2n+2} = 0$. And from the S_2–symmetry, we have $e_i = e_{2n+2-i}$, $e_{i+2n+2} = -e_{4n+4-i}$, for $2 \le i \le n$, $e_{2n+1} = e_{2n+2}$ and $e_{4n+3} = -e_{4n+4}$, $e_{2n+3} \equiv 0, e_{3n+3} \equiv 0$.

From the first equation $e_1 = 0$ we isolate λ and we substitute it into the other $2n$ equations that we denote by

$$
E_i = 0, \quad 2 \le i \le n,\ E_{2n+1} = 0,\ E_{2n+2+i} = 0,\ 2 \le i \le n,\ E_{4n+3} = 0.
$$

In short, we have $2n$ equations and $2n$ unknowns y_2, $x_i, y_i, 3 \leq i \leq n$, x_{n+1}, x_{2n+1}, y_{2n+1}.

Now we consider the symmetric restricted $(2n+2)$–body problem. From the results in Section 2 we know that the infinitesimal masses must be on one of the axis of symmetry of the $2n$–gon. Clearly when the infinitesimal masses are on an axis of symmetry containing one of the primaries the resulting central configuration cannot be convex, so we only consider the case where the primaries are on an axis of symmetry that does not contain any of the primaries. Without loss of generality we can assume that the $2n$ primaries with masses equal to one are at the vertices of the regular $2n$–gon $x_i = \cos\alpha_i$, $y_i = \sin\alpha_i, 1 \leq i \leq 2n$ with $\alpha_i = 2\pi(i-1)/2n$ and the two infinitesimal masses with $m = 0$ are at the points $(x_{2n+1}, y_{2n+1}) = (r\cos(\pi/2n), r\sin(\pi/2n))$ and $(x_{2n+2}, y_{2n+2}) = (r\cos(\pi/2n), -r\sin(\pi/2n))$ for some $r > 0$.

Straightforward computations show that if $m = 0$, $x_i = \cos\alpha_i$, $y_i = \sin\alpha_i$, $1 \leq i \leq 2n$, $(x_{2n+1}, y_{2n+1}) = (r\cos(\pi/2n), r\sin(\pi/2n))$ and $(x_{2n+2}, y_{2n+2}) = (r\cos(\pi/2n), -r\sin(\pi/2n))$, then all the equations in (2) are identically zero except the equations $E_{2n+1} = 0$ and $E_{4n+3} = 0$ which satisfy $E_{4n+3} = E_{2n+1}\tan(\pi/2n)$. So we only need to solve $E_{2n+1} = 0$, which becomes $f_n(r) = 0$ with

$$f_n(r) = \sum_{j=1}^{2n} \frac{\cos\alpha_j - r\cos(\pi/2n)}{(1 - 2r\cos(\alpha_j - \pi/2n) + r^2)^{3/2}} + \lambda r\cos(\pi/2n),$$

where

$$\lambda = \sum_{j=2}^{2n} \frac{1}{[2^3(1-\cos\alpha_j)]^{1/2}}.$$

4.2 Proof of Theorem 2 for $\ell = 4$

Next we see that there are no convex central configurations of the S_2–symmetric restricted $(4+2)$–body problem.

Indeed, in Subsection 4.1 we have seen that the equations for the central configurations of the symmetric restricted $(4+2)$–body problem with $(x_1, y_1) = (1, 0)$, $(x_2, y_2) = (0, 1)$, $(x_3, y_3) = (-1, 0)$, $(x_4, y_4) = (0, -1)$, $(x_5, y_5) = (r\sqrt{2}/2, r\sqrt{2}/2)$ and $(x_6, y_6) = (r\sqrt{2}/2, -r\sqrt{2}/2)$ are reduced to the equation $f_2(r) = 0$ with

$$f_2(r) = \frac{1 - \sqrt{2}r}{\left(r^2 - \sqrt{2}r + 1\right)^{3/2}} + \frac{-\sqrt{2}r - 1}{\left(r^2 + \sqrt{2}r + 1\right)^{3/2}} + \frac{1}{8}\left(4 + \sqrt{2}\right) r,$$

see Subsection 3.1. Dropping the square roots and the denominators of equation $f_2(r) = 0$ as above we get a polynomial equation $kr^2P(r) = 0$, where k is a constant and P is a polynomial of degree 26. This polynomial equation has exactly four positive real roots of those only $r = r_1 = 0.6973805098\ldots$

and $r = r_2 = 1.6024084862\ldots$ are solutions of the initial equation $f_2(r) = 0$. Applying Lemma 2 we see that neither $r = r_1$ nor $r = r_2$ provides convex central configurations of the S_2–symmetric restricted $(4+2)$–body problem.

Due to the symmetry if we take the position on m_5 on any other axis of symmetry that does not contain any primary we obtain the same central configurations. So there are no convex central configurations of the S_2–symmetric restricted $(4+2)$–body problem and consequently there are no convex central configurations of the S_2–symmetric 6–body problem with $m > 0$ sufficiently small coming from continuation of convex central configurations of the S_2–symmetric restricted $(4+2)$–body problem.

4.3 Proof of Theorem 2 for $\ell = 6$

For the S_2–symmetric 8–body problem (that is $n = 3$), the equations of central configuration are reduced to $E_2 = 0$, $E_3 = 0$, $E_7 = 0$, $E_{10} = 0$, $E_{11} = 0$, $E_{15} = 0$ where $x_2 = -m\,x_7 - 1/2 - x_3 - x_4/2$ and the ones for the symmetric restricted $(6+2)$–body problem are reduced to the equation $f_3(r) = 0$, see Subsection 4.1.

Proposition 7 *Let $a_1 = 1.5922353553\ldots$ and $a_2 = 0.8843211381\ldots$ be the two roots of $f_3(r) = 0$ with $r > 0$. The symmetric restricted $(6+2)$–body problem has exactly three classes of convex central configurations which are given by*

(a) $(x_7,\, y_7) = (0,\, a_1)$,
(b) $(x_7,\, y_7) = (a_2 \cos(\pi/6),\, a_2 \sin(\pi/6))$,
(c) $(x_7,\, y_7) = (0,\, a_2)$.

Proof Assuming that m_7 is on the axis of symmetry between m_1 and m_2, i.e. $(x_7, y_7) = (r\cos(\pi/6), r\sin(\pi/6))$, the equations of the S_2–symmetric restricted $(6+2)$–body reduce to the equation $f_3(r) = 0$ which becomes

$$-\frac{\sqrt{3}r}{\left(r^2+1\right)^{3/2}} + \frac{3-2\sqrt{3}r}{2\left(r^2-\sqrt{3}r+1\right)^{3/2}} + \frac{-2\sqrt{3}r-3}{2\left(r^2+\sqrt{3}r+1\right)^{3/2}} + \frac{1}{8}\left(4+5\sqrt{3}\right) r = 0.$$

By squaring both sides of this equation as many times as we need and dropping the denominators we get the polynomial equation $r^4 P(r) = 0$, where $P(r)$ is a polynomial of degree 76. Solving numerically this polynomial equation we get exactly ten real solutions with $r > 0$ of which only the solutions $r = a_1$, $r = a_2$ are solutions of the initial equation.

We also can take the mass m_7 on the axis of symmetry passing between of any other pair of consecutive masses. By symmetry the only axis of symmetry that provides different central configurations is the axis passing between m_2 and m_3 which correspond to a rotation with an angle of $2\pi/6$ of the solutions

$(x_7, y_7) = (a_1 \cos(\pi/6), a_1 \sin(\pi/6))$ and $(x_7, y_7) = (a_2 \cos(\pi/6), a_2 \sin(\pi/6))$. However, when $(x_7,\, y_7) = (a_1 \cos(\pi/6),\, a_1 \sin(\pi/6))$, the corresponding central configuration is not convex. So we have only the three convex central configurations given in statements (a), (b) and (c).

Proposition 8 *The three convex central configurations of the symmetric restricted* $(6+2)$*–body problem can be continued to three families of central configurations of the symmetric* 8*–body problem with masses* $m_i = 1$, $1 \le i \le 6$ *and* $m_7 = m_8 = m > 0$ *sufficiently small.*

Proof Using the Implicit Function Theorem we will see that the solutions of system $E_2 = 0$, $E_3 = 0$, $E_7 = 0$, $E_{10} = 0$, $E_{11} = 0$, $E_{15} = 0$ with $m = 0$ given in Proposition 7 can be continued to solutions of this system with $m > 0$ sufficiently small.

We use the notation: $U = (y_2,\, x_3,\, y_3,\, x_4)$, $V = (x_7,\, y_7)$, $\tilde{U} = (\sin(2\pi/8), \cos(4\pi/6), \sin(4\pi/6), -1)$, $\tilde{V}_1 = (0,\, a_1)$, $\tilde{V}_2 = (a_2 \cos(\pi/6),\, a_2 \sin(\pi/6))$, and $\tilde{V}_3 = (0,\, a_2)$.

The determinant of the Jacobian of system $E_2 = 0$, $E_3 = 0$, $E_7 = 0$, $E_{10} = 0$, $E_{11} = 0$, $E_{15} = 0$ (with respect to the variables U, V), evaluated at $m = 0$, $U = \tilde{U}$ and $V = \tilde{V}_1$ takes the value $1.0896102708\ldots$ Since this determinant is different from zero and the system is analytic with respect to the variables (U, V), from the Implicit Function Theorem there exist analytic families $U^1(m) = (y_2^1(m), x_3^1(m), y_3^1(m), x_4^1(m))$ and $V^1(m) = (x_7^1(m), y_7^1(m))$ of solutions of system $E_2 = 0$, $E_3 = 0$, $E_7 = 0$, $E_{10} = 0$, $E_{11} = 0$, $E_{15} = 0$ satisfying $U^1(0) = \tilde{U}$ and $V^1(0) = \tilde{V}_1$ defined for $m > 0$ sufficiently small.

The determinant of the Jacobian of system $E_2 = 0$, $E_3 = 0$, $E_7 = 0$, $E_{10} = 0$, $E_{11} = 0$, $E_{15} = 0$ (with respect to the variables U, V), evaluated at $m = 0$, $U = \tilde{U}$ and $V = \tilde{V}_i$ for $i = 2, 3$ takes the value $-57.55271347\ldots$ which is also different from zero. So these two solutions can be continued to the two families $U^1(m) = (y_2^i(m), x_3^1(m), y_3^i(m), x_4^1(m))$ and $V^1(m) = (x_7^i(m), y_7^i(m))$ for $i = 2, 3$ of solutions of system $E_2 = 0$, $E_3 = 0$, $E_7 = 0$, $E_{10} = 0$, $E_{11} = 0$, $E_{15} = 0$ satisfying $U^1(0) = \tilde{U}$ and $V^i(0) = \tilde{V}_i$ defined for $m > 0$ sufficiently small.

Since for $m = 0$ the three central configurations of the S_2–symmetric restricted $(8+2)$–body problem are different, the continued central configurations for $m > 0$ small continue being different. So if $x_2^i(m) = -m x_7^i(m) - 1/2 - x_3^i(m) - x_4^i(m)/2$ for $i = 1, 2, 3$, then for $m > 0$ sufficiently small Proposition 8 provides the three different families of central configurations given by $(x_1, y_1) = (1, 0)$, $(x_2, y_2) = (x_2^i(m),\, y_2^i(m))$, $(x_3,\, y_3) = (x_3^i(m),\, y_3^i(m))$, $(x_4,\, y_4) = (x_4^i(m), 0)$, $(x_5,\, y_5) = (x_3^i(m), -y_3^i(m))$, $(x_6,\, y_6) = (x_2^i(m), -y_2^i(m))$, $(x_7, y_7) = (x_7^i(m),\, y_7^i(m))$, $(x_8,\, y_8) = (x_7^i(m),\, -y_7^i(m))$ for $i = 1, 2, 3$. This proves Theorem 2 for $\ell = 6$.

4.4 Proof of Theorem 2 for $\ell = 8$

For the central configurations of the S_2–symmetric 10–body problem (that is $n = 4$) the equations of central configurations are reduced to $E_2 = 0$, $E_3 = 0$,$E_4 = 0$,$E_9 = 0$, $E_{12} = 0$, $E_{13} = 0$, $E_{14} = 0, E_{19} = 0$ where $x_2 = -mx_9 - 1/2 - x_3 - x_4 - x_5/2$ and the ones for the restricted S_2–symmetric $(8+2)$–body problem are reduce to $f_4(r) = 0$, see Subsection 4.1.

Proposition 9 *Let $a_3 = 0.9401381791\cdots$ be the smallest positive root of $f_4(r) = 0$. The symmetric restricted $(8 + 2)$–body problem has exactly two classes of convex central configurations which are given by*

(a) $(x_9, y_9) = (a_3 \cos(\pi/8), a_3 \sin(\pi/8))$,
(b) $(x_9, y_9) = (a_3 \cos(3\pi/8), a_3 \sin(3\pi/8))$.

Proof Let $(x_9, y_9) = (r\cos(\pi/8), r\sin(\pi/8))$ be the position of an infinitesimal mass in a central configuration of the restricted $(8 + 2)$–body problem. Under this hypothesis r satisfies equation $f_4(r) = 0$. Solving this equation we get exactly two solutions with $r > 0$, they are $r = a_4 = 1.5745151766\cdots$ and $r = a_3 = 0.9401381791\cdots$. By rotating the positions $(x_9,\, y_9) = (a_i \cos(\pi/8), a_i \sin(\pi/8))$ for $i = 3, 4$ with an angle of $2\pi/8$ we get another two solutions. However, when $(x_9,\, y_9) = (a_4 \cos(\pi/8), a_4 \sin(\pi/8))$, $(x_9,\, y_9) = (a_4 \cos(3\pi/8), a_4 \sin(3\pi/8))$ the corresponding central configurations are not convex. Any additional rotation of an angle of $2\pi/8$ does not provide any new central configuration. So we have only the two convex central configurations given in statements (a) and (b).

Proceeding as in the case $\ell = 6$, the proof of Theorem 2 for $\ell = 8$ follows from the following result.

Proposition 10 *The two convex central configurations of the symmetric restricted $(8 + 2)$–body problem can be continued to two families of central configurations of the symmetric 10–body problem with masses $m_i = 1$, $1 \le i \le 8$ and $m_9 = m_{10} = m > 0$ sufficiently small.*

Proof Using the Implicit Function Theorem we will see that the solutions of system $E_2 = 0$, $E_3 = 0$,$E_4 = 0$,$E_9 = 0$, $E_{12} = 0$, $E_{13} = 0$, $E_{14} = 0, E_{19} = 0$ with $m = 0$ given in Proposition 9 can be continued to solutions of this system with $m > 0$ sufficiently small.

Let $U = (y_2, x_3, y_3, x_4, y_4, x_5)$, $V = (x_9, y_9)$ and

$$\tilde{U} = (\sin(2\pi/8), \cos(4\pi/8), \sin(4\pi/8), \cos(6\pi/8), \sin(6\pi/8), -1),$$
$$\tilde{V}_1 = (a_3 \cos(\pi/8), a_3 \sin(\pi/8)),$$
$$\tilde{V}_2 = (a_3 \cos(3\pi/8), a_3 \sin(3\pi/8)).$$

The determinant of the Jacobian of system $E_2 = 0$, $E_3 = 0$,$E_4 = 0$, $E_9 = 0$, $E_{12} = 0$, $E_{13} = 0$, $E_{14} = 0, E_{19} = 0$ (with respect to the variables U, V), evaluated at $m = 0$, $U = \tilde{U}$ and $V = V_i$ for $i = 1, 2$ takes the value

$-2.9072711711 \times 10^6$. Since this determinant is different from zero from the Implicit Function Theorem the two solutions can be continued to solutions of $E_2 = 0$, $E_3 = 0$,$E_4 = 0$, $E_9 = 0$, $E_{12} = 0$, $E_{13} = 0$, $E_{14} = 0, E_{19} = 0$ for $m > 0$ sufficiently small.

4.5 Proof of Theorem 2 for $\ell = 10$

For the central configurations of the S_2–symmetric 12–body problem (that is $n = 5$) the equations of central configurations reduced to $E_2 = 0$, $E_3 = 0$, $E_4 = 0$, $E_5 = 0$, $E_{11} = 0$, $E_{14} = 0$, $E_{15} = 0$, $E_{16} = 0$, $E_{17} = 0$, $E_{23} = 0$ where $x_2 = -mx_{11} - 1/2 - x_3 - x_4 - x_5 - x_6/2$ and the ones for the restricted S_2–symmetric $(10+2)$–body problem reduce to $f_5(r) = 0$, see Subsection 4.1.

Proposition 11 *Let $a_4 = 0.9634598812\ldots$ be the smallest positive root of $f_5(r) = 0$. The S_2–symmetric restricted $(10+2)$–body problem has exactly three classes of convex central configurations which are given by*

(a) $(x_{11}, y_{11}) = (a_4 \cos(\pi/10), a_4 \sin(\pi/10))$,
(b) $(x_{11}, y_{11}) = (a_4 \cos(3\pi/10), a_4 \sin(3\pi/10))$,
(c) $(x_{11}, y_{11}) = (a_4 \cos(5\pi/10), a_4 \sin(5\pi/10))$.

Proof Let $(x_{11}, y_{11}) = (r\cos(\pi/10), r\sin(\pi/10))$ be the position of an infinitesimal mass in a central configuration of the S_2–symmetric restricted $(10+2)$–body problem. Under this hypothesis r satisfies equation $f_5(r) = 0$. Solving this equation we get exactly two solutions $r = a_5 = 1.5541234676\cdots$, $r = a_4 = 0.9634598812\cdots$ with $r > 0$. By rotating the positions $(x_9,\, y_9) = (a_i \cos(\pi/8),\, a_i \sin(\pi/8))$ for $i = 4, 5$ with angles of $2\pi/10$ and of $4\pi/10$ we get another four solutions. However if $(x_{11},\, y_{11}) = (a_5 \cos(\pi/10),\, a_5 \sin(\pi/10))$, $(x_{11},\, y_{11}) = (a_5 \cos(3\pi/10),\, a_5 \sin(3\pi/10))$, and $(x_{11},\, y_{11}) = (a_5 \cos(5\pi/10),\, a_5 \sin(5\pi/10))$, then the corresponding central configurations are not convex. Any additional rotation of an angle of $2\pi/10$ does not provide any new central configuration. So we have only the three convex central configurations given in statements (a), (b) and (c).

The proof of Theorem 2 for $\ell = 10$ follows from the following result.

Proposition 12 *The three convex central configurations of the S_2–symmetric restricted $(10+2)$–body problem can be continued to three families of central configurations of the S_2–symmetric 12–body problem with masses $m_i = 1$, $1 \le i \le 10$ and $m_{11} = m_{12} = m > 0$ sufficiently small.*

Proof Using the Implicit Function Theorem we will see that the solutions of system $E_2 = 0$, $E_3 = 0$,$E_4 = 0$, $E_5 = 0$, $E_{11} = 0$, $E_{14} = 0$, $E_{15} = 0$, $E_{16} = 0$, $E_{17} = 0$, $E_{23} = 0$ with $m = 0$ given in Proposition 11 can be continued to solutions of this system with $m > 0$ sufficiently small.

Let $U = (y_2, x_3, y_3, x_4, y_4, x_5, y_5, x_6)$, $V = (x_{11}, y_{11})$ and

$$\tilde{U} = (\sin(2\pi/10), \cos(4\pi/10), \sin(4\pi/10), \cos(6\pi/10), \sin(6\pi/10),$$

$$\cos(8\pi/10), \sin(8\pi/10), -1),$$
$$\tilde{V_i} = (a_4\cos(\pi/10 + 2\pi i/10), a_4\sin(\pi/10 + 2\pi i/10)), \qquad i = 1, 2, 3.$$

The determinant of the Jacobian of system $E_2 = 0$, $E_3 = 0$,$E_4 = 0$, $E_5 = 0$, $E_{11} = 0$, $E_{14} = 0$, $E_{15} = 0$, $E_{16} = 0$, $E_{17} = 0$, $E_{23} = 0$ (with respect to the variables U, V), evaluated at $m = 0$, $U = \tilde{U}$ and $V = \tilde{V_i}$ for $i = 1, 2, 3$ is $-5.3833737767 \times 10^8$ Since this determinant is different from zero from the Implicit Function Theorem the three central configurations of the S_2–symmetric restricted $(10 + 2)$–body problem given by Proposition 11 can be continued for $m > 0$ sufficiently small.

4.6 Some numerical evidences than Conjecture 2 is true

Proceeding as in Subsection 3.5 for the S_2–symmetric $(2n + 2)$–body problem we have computed the two solutions of $f_n(r)$ for $n = 6, \ldots, 151$. We denote these solutions by $r = a_1(n)$ and $r = a_2(n)$ with $a_1(n) < a_2(n)$. Then applying Lemma 2 we have seen that the solution $a_2(n)$ does not lead to convex central configurations of the S_2–symmetric rectricted $(2n+2)$–body problem whereas the solution $a_1(n)$ provides exactly $[\frac{n+1}{2}]$ convex central configurations.

We have also applied the Implicit Function Theorem as in the proof of Theorem 2 for $\ell = 6, 8, 10$ to continue these central configurations to the symmetric $(2n + 2)$–body problem with $m > 0$ sufficiently small for $n = 6, \ldots, 20$ and in all cases we have seen that the corresponding determinants are different form zero. Moreover the absolute value of these determinants increases as n increases. This gives numerical evidence that Conjecture 2 is true.

5 S_3–symmetric $(2n + 2)$–body problem

5.1 Equations of the S_3–symmetric $(2n + 2)$–body problem

Without loss of generality we can assume that the position of m_1 is fixed at the point $\mathbf{q}_1 = (\cos(\pi/(2n)), \sin(\pi/(2n)))$ and that the center of masses is at the origin of coordinates, so $\sum_{i=1}^{2n+2} m_i x_i = 0$, and using the S_3–symmetry we have $x_2 = -(\cos(\pi/(2n)) + \sum_{i=3}^{n} x_i + m x_{2n+1})$. Notice that using the S_2–symmetry again $\sum_{i=1}^{2n+2} m_i y_i$ is identically zero.

Taking into account all these conditions the $4n + 4$ equations $e_1 = 0, \ldots,$ $e_{4n+4} = 0$ given by (1) with $N = 2n + 2$ for the central configurations of the planar symmetric $(2n + 2)$–body problem reduce to the following $2n + 1$ equations

$$\begin{array}{lll} e_i = 0, & e_{2n+2+i} = 0, & i = 2, \ldots, n, \\ e_{2n+1} = 0, & e_{2n+3} = 0, & e_{4n+3} = 0. \end{array}$$

Indeed, from the S_3–symmetry we have $e_{i+1} = e_{2n-i}$, $e_{i+2n+3} = -e_{4n+2-i}$, for $0 \leq i \leq n - 1$, $e_{2n+2} = e_{2n+1}$ and $e_{4n+4} = -e_{4n+3}$. So equations e_{n+i},

e_{2n+2}, e_{3n+i+2} and e_{4n+4} can be omitted. Moreover equation e_1 can be omitted because $\sum_{i=1}^{2n} e_i + me_{2n+1} + me_{2n+2} = 2(\sum_{i=1}^{n} e_i + me_{2n+1}) = 0$. Nevertheless in this case equation e_{2n+3} cannot be omitted because $e_{2n+3} = -e_{4n+2}$ and we have already omitted e_{4n+2}.

From equation $e_{2n+3} = 0$ we isolate λ and we substitute it into the other $2n$ equations that we denote by

$$E_2 = 0, \cdots, E_n = 0;\ E_{2n+1} = 0;\ E_{2n+4} = 0, \cdots, E_{3n+2} = 0;\ E_{4n+3} = 0, \quad (5)$$

where E_k is e_k after the substitution. In short, we have $2n$ equations and $2n-1$ unknowns y_2, x_i, y_i, $3 \le i \le n$, x_{2n+1}, y_{2n+1}. Since we have more equations than unknowns this system could have no solution.

Now we consider the S_3–symmetric restricted $(2n+2)$–body problem where the $2n$ primaries with masses equal to one are at the vertices of the regular $2n$-gon $x_i = \cos(\alpha_i + \pi/(2n))$, $y_i = \sin(\alpha_i + \pi/(2n)), 1 \le i \le 2n$ with $\alpha_i = \pi(i-1)/n$ and the two infinitesimal masses with $m = 0$ are at the points (x_{2n+1}, y_{2n+1}) and $(x_{2n+1}, -y_{2n+1})$. Here (x_{2n+1}, y_{2n+1}) is the position of the infinitesimal mass in a central configuration of the restricted $(2n+1)$–body problem with $2n$ equal masses at the vertices of a regular $2n$-gon. From Lemma 1, the position of the infinitesimal mass is on an axis of symmetry of the $2n$–gon.

Without loss of generality we can assume that the line of symmetry is either $y_{2n+1} = 0$ (in this case none of the primaries are on the axis of symmetry) or $y_{2n+1} = \tan(\pi/(2n))x_{2n+1}$ (in this case two primaries are on the axis of symmetry). We start with the case $y_{2n+1} = 0$. Straightforward computations show that when $m = 0$, $x_i = \cos(\alpha_i + \pi/(2n))$, $y_i = \sin(\alpha_i + \pi/(2n)), 1 \le i \le 2n$ and $y_{2n+1} = 0$, all equations (5) are identically zero except the equation $E_{2n+1} = 0$. Using Lemma 1 again the solutions of $E_{2n+1} = 0$ are $(0, 0)$, $(\rho_2, 0)$ with $0 < \rho_2 < 1$ and $(\rho_3, 0)$ with $\rho_3 > 1$. Taking into account the symmetries of the problem, it is sufficient to consider the following positions for (x_{2n+1}, y_{2n+1}): $(0, 0)$, $\rho_2(\cos(\pi i/n), \sin(\pi i/n))$, and $\rho_3(\cos(\pi i/n), \sin(\pi i/n))$ for $i = 0, \ldots, [n/2]$ where $[x]$ denotes the integer part function of x.

When $m = 0$, $x_i = \cos(\alpha_i + \pi/(2n))$, $y_i = \sin(\alpha_i + \pi/(2n)), 1 \le i \le 2n$ and $y_{2n+1} = \tan(\pi/(2n))x_{2n+1}$, all equations (5) are identically zero except the equations $E_{2n+1} = 0$ and $E_{4n+3} = 0$, moreover $E_{4n+3} = \tan \pi/(2n) E_{2n+1}$. Using Lemma 1 the solutions of equation $E_{2n+1} = 0$ are $(0, 0)$, and $\rho_1(\cos(\pi/(2n))$, $\sin(\pi/(2n)))$ with $\rho_1 > 1$. Taking into account the symmetries of the problem, it is sufficient to consider the following positions for (x_{2n+1}, y_{2n}): $(0, 0)$ and $\rho_1(\cos(\pi i/n + \pi/(2n)), \sin(\pi i/n + \pi/(2n))$ for $i = 0, \ldots, [(n+1)/2] - 1$.

Since we are interested only in convex central configurations we do not consider $(x_{2n+1}, y_{2n+1}) = (0, 0)$ and $(x_{2n+1}, y_{2n+1}) = \rho_1(\cos(\pi i/n + \pi/(2n))$, $\sin(\pi i/n + \pi/(2n))$ for $i = 0, \ldots, [(n+1)/2] - 1$. Clearly $(x_{2n+1}, y_{2n+1}) = (0, 0)$ does not provide a convex central configuration. Consider now the three consecutive vertices of the configuration $\mathbf{p_1} = (\cos(\pi(i-1)/n + \pi/(2n)), \sin(\pi(i-1)/n + \pi/(2n))$, $\mathbf{p_2} = (\cos(\pi i/n + \pi/(2n)), \sin(\pi i/n + \pi/(2n))$ and $\mathbf{p_3} =$

$\rho_1(\cos(\pi i/n + \pi/(2n)), \sin(\pi i/n + \pi/(2n))$, it is easy to see that

$$\overrightarrow{\mathbf{p}_1\mathbf{p}_2} \times \overrightarrow{\mathbf{p}_2\mathbf{p}_3} = (1-\rho_1)\sin\left(\frac{\pi}{n}\right) < 0,$$

because $\rho_1 > 1$. Therefore the configuration is not convex because it does not satisfy the conditions of Lemma 2.

5.2 Proof of Theorem 3 for $\ell = 4$

For the central configurations of the S_3–symmetric 6–body problem (that is $n = 2$) equations (2) reduce to the four equations

$$E_2 = 0, \qquad E_5 = 0, \qquad E_8 = 0, \qquad E_{11} = 0, \tag{6}$$

and the three unknowns y_2, x_5 and y_5, plus the parameter m. Here $x_2 = mx_5 - \sqrt{2}/2$.

The equation $E_5 = 0$ for the S_2–symmetric restricted (4+2)–body problem when $y_5 = 0$ becomes

$$\frac{2\left(x_5 - \frac{1}{\sqrt{2}}\right)}{\left(x_5^2 - \sqrt{2}x_5 + 1\right)^{3/2}} + \sqrt{2}\left(-\frac{1}{2} - \frac{1}{4\sqrt{2}}\right)x_5 + \frac{2\left(x_5 + \frac{1}{\sqrt{2}}\right)}{\left(\left(x_5 + \frac{1}{\sqrt{2}}\right)^2 + \frac{1}{2}\right)^{3/2}} = 0. \tag{7}$$

We note that by the symmetry of the problem, we only are interested in solutions with $x_5 > 0$. Dropping off the denominators and the square roots in (7), the equation can be transformed to a polynomial equation of the form $x_5^2 P(x_5) = 0$, where $P(x_5)$ is a polynomial of degree 26 in the variable x_5 whose set of roots contains all solutions of (7). Solving numerically equation $P(x_5) = 0$ we get four positive real root of which only $x_5 = \rho_2 = 0.6973805098..$ and $x_5 = \rho_3 = 1.6024084862..$ are solutions of the initial equation (7). Since $\cos(\pi/4) = \sin(\pi/4) > \rho_2$ the solution $x_5 = \rho_2$ cannot give convex central configurations.

This proves the following result.

Proposition 13 *Up to symmetry the unique positions for the infinitesimal mass m_5 that can provide convex central configurations of the S_3–symmetric restricted $(4+2)$–body problem are $(x_5, y_5) = (\rho_3, 0)$ and $(x_5, y_5) = (0, \rho_3)$.*

Now we see that the central configuration of the S_3–symmetric restricted $(4+2)$–body problem with $(x_5, y_5) = (0, \rho_3)$ cannot be continued to a family of central configurations of the S_3–symmetric 6–body problem with $m > 0$ small. To do that we consider system (6) as a system of the four unknowns y_2, x_5, y_5 and m and we apply the Inverse Function Theorem. The Jacobian of the system with respect to the variables y_2, x_5, y_5 and m evaluated at the solution $y_2 = \sin(3\pi/4)$, $x_5 = 0$, $y_5 = \rho_3$ and $m = 0$ becomes $3.9841276914.. \neq 0$. Therefore the solution is isolated and it cannot be continued to a family of solutions of system (6) for $m > 0$ sufficiently small.

We consider now the central configuration of the S_3–symmetric restricted $(4+2)$–body problem with $(x_5, y_5) = (\rho_3, 0)$. We also will see that this central configuration cannot be continued to a family of central configurations of the S_3–symmetric restricted 6–body problem with $\mu > 0$ sufficiently small. Notice that in this case m_5 and m_6 collide, so system (6) is not analytic with respect to all its variables in a neighborhood of $y_2 = \sin(3\pi/4)$, $x_5 = \rho_3$, $y_5 = 0$ and $m = 0$. Proceeding as in the proofs of Theorems 1 and 2, after doing the change of variables $y_5 = \mu Y_5/2$ with $\mu = m^{1/3}$ we obtain a new system of equations

$$\begin{aligned} \tilde{E}_2 &= \tilde{E}_{20} + O(\mu^3) = 0, & \tilde{E}_5 &= \tilde{E}_{50} + O(\mu^2) = 0, \\ \tilde{E}_8 &= \tilde{E}_{80} + O(\mu^3) = 0, & \tilde{E}_{11} &= \tilde{E}_{110}\mu + O(\mu^3) = 0, \end{aligned}$$

which is analytic with respect to its variables in a neighborhood of

$$y_2 = \sin(3\pi/4), \quad x_5 = \rho_3, \quad Y_5 \neq 0, \quad \mu = 0. \tag{8}$$

Here $\tilde{E}_{20}$, $\tilde{E}_{50}$, $\tilde{E}_{80}$ and $\tilde{E}_{110}$ are functions that does not depend on μ. Finally we consider the system of equations

$$\bar{E}_2 = \tilde{E}_2 = 0, \quad \bar{E}_5 = \tilde{E}_5 = 0, \quad \bar{E}_8 = \tilde{E}_8 = 0, \quad \bar{E}_{11} = \tilde{E}_{11}/\mu = 0, \tag{9}$$

which is also analytic with respect to its variables in a neighborhood of (8). Substituting (8) into system (9) we get that $\bar{E}_2$, $\bar{E}_5$ and $\bar{E}_8$ are identically zero, and that equation $\bar{E}_{11} = 0$ is equivalent to equation

$$-A\,Y_5 + \frac{\sqrt{Y_5^2}}{Y_5^3} = 0,$$

with

$$\begin{aligned} A &= -\frac{\left(2\rho_3^2 + 2\sqrt{2}\rho_3 - 1\right)}{2\left(\rho_3^2 + \sqrt{2}\rho_3 + 1\right)^{5/2}} + \frac{\left(-2\rho_3^2 + 2\sqrt{2}\rho_3 + 1\right)}{2\left(\rho_3^2 - \sqrt{2}\rho_3 + 1\right)^{5/2}} + \frac{\left(4 + \sqrt{2}\right)}{8\sqrt{2}} \\ &= 0.5285045250... \end{aligned}$$

The solutions of this equation are $Y_5 = 0$ and $Y_5 = \pm b = \pm A^{-1/3}$.

Clearly the derivatives of $\bar{E}_2$, $\bar{E}_5$, $\bar{E}_8$, and $\bar{E}_{11}$ with respect to μ when $\mu = 0$ are zero, so the Jacobian of system (9) with respect to the variables y_2, x_5, Y_5 and μ evaluated at the solution (8) with $Y_5 = b$ is zero. In this case the Inverse Function Theorem is not sufficient to prove that the solution (8) with $Y_5 = b$ cannot be continued analytically to a solution with $\mu > 0$ sufficiently small. We assume that it can be continued; that is, that

$$\begin{aligned} y_2 &= \sin(3\pi/4) + y_{21}\mu + y_{22}\mu^2 + y_{23}\mu^3 + O(\mu^4), \\ x_5 &= \rho_3 + x_{51}\mu + x_{52}\mu^2 + x_{53}\mu^3 + O(\mu^4), \\ Y_5 &= b + Y_{51}\mu + Y_{52}\mu^2 + Y_{53}\mu^3 + O(\mu^4), \end{aligned}$$

is a solution of (9) and we shall arrive to a contradiction. Indeed, we substitute this solutions into (9), then expanding in power series of μ we get

$$\begin{aligned}
\bar{E}_2 =&\bar{E}_{21}(y_{21})\mu + \bar{E}_{22}(y_{21}, y_{22})\mu^2 + \bar{E}_{23}(y_{21}, y_{22}, y_{23})\mu^3 + O(\mu^4) = 0,\\
\bar{E}_5 =&\bar{E}_{51}(y_{21}, x_{51})\mu + \bar{E}_{52}(y_{21}, x_{51}, Y_{51}, y_{22}, x_{52})\mu^2+\\
&\bar{E}_{53}(y_{21}, x_{51}, Y_{51}, y_{22}, x_{52}, Y_{52}, y_{23}, x_{53})\mu^3 + O(\mu^4) = 0,\\
\bar{E}_8 =&\bar{E}_{81}(y_{21})\mu + \bar{E}_{82}(y_{21}, y_{22})\mu^2 + \bar{E}_{83}(y_{21}, y_{22}, y_{23})\mu^3 + O(\mu^4) = 0,\\
\bar{E}_{11} =&\bar{E}_{111}(y_{21}, x_{51}, Y_{51})\mu + \bar{E}_{112}(y_{21}, x_{51}, Y_{51}, y_{22}, x_{52}, Y_{52})\mu^2+\\
&\bar{E}_{113}(y_{21}, x_{51}, Y_{51}, y_{22}, x_{52}, Y_{52}, y_{23}, x_{53}, Y_{53})\mu^3 + O(\mu^4) = 0.
\end{aligned} \tag{10}$$

Solving system (10) at first order in μ, i.e.

$$\bar{E}_{21}(y_{21}) = 0, \quad \bar{E}_{51}(y_{21}, x_{51}) = 0, \quad \bar{E}_{81}(y_{21}) = 0, \quad \bar{E}_{111}(y_{21}, x_{51}, Y_{51}) = 0,$$

we get $y_{21} = 0$, $x_{51} = 0$, and $Y_{51} = 0$. Solving system (10) at second order in μ, i.e.

$$\begin{aligned}
&\bar{E}_{22}(0, y_{22}) = 0, \qquad \bar{E}_{52}(0, 0, 0, y_{22}, x_{52}) = 0, \qquad \bar{E}_{82}(0, y_{22}) = 0,\\
&\bar{E}_{112}(0, 0, 0, y_{22}, x_{52}, Y_{52}) = 0,
\end{aligned}$$

we get $y_{22} = 0$, $x_{52} = x_{52}(\rho_3, b) = 0.2007484218..$, and $Y_{52} = Y_{52}(\rho_3, b) = 0.4509017133...$ Finally solving equation $\bar{E}_2$ at third order in μ, i.e.

$$\bar{E}_{23}(0, 0, y_{23}) = 0,$$

we get

$$\begin{aligned}
y_{23} = y_{23}^* = -\frac{8}{7}\left(1 + 2\sqrt{2}\right)\Bigg(&-\frac{\sqrt{2}}{\left(\rho_3^2 - \sqrt{2}\rho_3 + 1\right)^{3/2}} - \frac{-2\rho_3 - \sqrt{2}}{\left(\rho_3\left(\rho_3 + \sqrt{2}\right) + 1\right)^{3/2}}\\
&- \frac{1}{8}\left(1 + 8\sqrt{2}\right)\rho_3\Bigg) = 13.5243491694...
\end{aligned}$$

We can see that $\bar{E}_{83}(0, 0, y_{23}^*) = -23.3594334957.. \neq 0$, Therefore the solution (8) with $Y_5 = b$ cannot be continued analytically to a solution of system (6) with $\mu > 0$ sufficiently small.

5.3 Proof of Theorem 3 for $\ell > 4$

Proceeding in a similar way than in the proof of Theorem 3 for $\ell = 4$ we get that when $n = 3$ the solutions of equation $E_7 = 0$ for the restricted $(6 + 2)$–body problem when $m = 0$ are

$$\rho_2 = 0.8843211381.., \qquad \rho_3 = 1.5922353553..$$

Applying Lemma 2 we see that, up to symmetry, the unique positions for the infinitesimal mass m_7 that can provide convex central configurations of the S_3–symmetric restricted $(6 + 2)$–body problem are $(x_7, y_7) = (\rho_2, 0)$, $(x_7, y_7) =$

$\rho_2(\cos(\pi/3), \sin(\pi/3))$, $(x_7, y_7) = (\rho_3, 0)$, $(x_7, y_7) = \rho_3(\cos(\pi/3), \sin(\pi/3))$. Using the Inverse Function Theorem as in the case $n = 2$ we can prove that the central configurations of the S_3–symmetric restricted $(6 + 2)$–body problem with $(x_7, y_7) = \rho_2(\cos(\pi/3), \sin(\pi/3))$ and $(x_7, y_7) = \rho_3(\cos(\pi/3), \sin(\pi/3))$ cannot be continued to convex central configurations of the S_3–symmetric 8–body problem with $m > 0$ sufficiently small. Moreover doing the change of variables $y_7 = \mu Y_7/2$ with $\mu = m^{1/3}$ and proceeding as in the case $\ell = 4$ we prove also that the central configurations of the S_3–symmetric restricted $(6 + 2)$–body problem with $(x_7, y_7) = (\rho_2, 0)$ and $(x_7, y_7) = (\rho_3, 0)$ cannot be continued to convex central configurations of the S_3–symmetric 8–body problem with $m > 0$ sufficiently small.

When $n = 4$ the solutions of equation $E_9 = 0$ for the restricted (8+2)–body problem when $m = 0$ are

$$\rho_2 = 0.9401381791.., \qquad \rho_3 = 1.574515766..$$

Applying Lemma 2 we can see that, up to symmetry, the unique positions for the infinitesimal mass m_7 that can provide convex central configurations of the S_3–symmetric restricted (8+2)–body problem are $(x_7, y_7) = (\rho_2, 0)$, $(x_7, y_7) = \rho_2(\cos(\pi/4), \sin(\pi/4))$, and $(x_7, y_7) = \rho_2(\cos(\pi/2), \sin(\pi/2))$. Proceeding as in the previous cases we prove that non of these central configurations can be continued to convex central configurations of the S_3–symmetric 10–body problem with $m > 0$ sufficiently small.

We can repeat these arguments for $n > 4$ arriving to similar results.

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