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WEIGHTED NORM INEQUALITIES FOR THE BILINEAR MAXIMAL OPERATOR ON VARIABLE LEBESGUE SPACES

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Abstract: We extend the theory of weighted norm inequalities on variable Lebesgue spaces to the case of bilinear operators. We introduce a bilinear version of the variable $\mathcal{A}_{p(\cdot)}$ condition and show that it is necessary and sufficient for the bilinear maximal operator to satisfy a weighted norm inequality. Our work generalizes the linear results of the first author, Fiorenza, and Neugebauer [**7**] in the variable Lebesgue spaces and the bilinear results of Lerner *et al.* [**22**] in the classical Lebesgue spaces. As an application we prove weighted norm inequalities for bilinear singular integral operators in the variable Lebesgue spaces.

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1. Introduction

In this paper we develop the theory of bilinear weighted norm inequalities in the variable Lebesgue spaces. To put our results in context we will first describe some previous results; for brevity, we will defer the majority of definitions until below. The Hardy–Littlewood maximal operator is defined by

$$Mf(x) = \sup_Q \fint_Q |f(y)|\,dy \cdot \chi_Q(x),$$

where the supremum is taken over all cubes in $\mathbb{R}^n$ with sides parallel to the coordinate axes. The now classical result of Muckenhoupt ([**23**]) is

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that a necessary and sufficient condition for M to be bounded on the weighted Lebesgue space $L^p(w)$, $1<p<\infty$, i.e., that

$$\int_{\mathbb{R}^n} (Mf)^p w\,dx \lesssim \int_{\mathbb{R}^n} |f|^p w\,dx,$$

is that $w \in A_p$:

$$\sup_Q \fint_Q w\,dx \left(\fint_Q w^{1-p'}\,dx\right)^{p-1} < \infty,$$

where again the supremum is taken over all cubes in $\mathbb{R}^n$ with sides parallel to the coordinate axes.

This result has been generalized in two directions. First, Lerner *et al.* [**22**], as part of the theory of weighted norm inequalities for bilinear Calderón–Zygmund singular integrals, introduced the bilinear (more properly, "bisublinear") maximal operator:

$$\mathcal{M}(f_1,f_2)(x) = \sup_Q \fint_Q |f_1(y)|\,dy \fint_Q |f_2(y)|\,dy \cdot \chi_Q(x).$$

It is immediate that $\mathcal{M}(f_1,f_2)(x) \leq Mf_1(x)Mf_2(x)$, and so by Hölder's inequality,

$$\mathcal{M}\colon L^{p_1}(w_1)\times L^{p_2}(w_2) \to L^p(w), \tag{1.1}$$

where $1<p_1,p_2<\infty$, $\frac{1}{p}=\frac{1}{p_1}+\frac{1}{p_2}$, $w_j\in A_{p_j}$, $j=1,2$, and $w = w_1^{\frac{p}{p_1}} w_2^{\frac{p}{p_2}}$.

However, while this condition is sufficient, it is not necessary. In [**22**] they introduced the class $A_{\vec{p}}$ of vector weights defined as follows. With the previous definitions, let $\vec{p}=(p_1,p_2,p)$ and let $\vec{w}=(w_1,w_2,w)$. Then $\vec{w}\in A_{\vec{p}}$ if

$$\sup_Q \left(\fint_Q w\,dx\right)^{\frac{1}{p}} \left(\fint_Q w_1^{1-p_1'}\,dx\right)^{\frac{1}{p_1'}} \left(\fint_Q w_1^{1-p_2'}\,dx\right)^{\frac{1}{p_2'}} < \infty.$$

They proved that a necessary and sufficient condition for inequality (1.1) to hold is that $\vec{w}\in A_{\vec{p}}$. If $w_j\in A_{p_j}$, then $\vec{w}\in A_{\vec{p}}$, but they gave examples to show that the class $A_{\vec{p}}$ is strictly larger than the weights gotten from $A_{p_1}\times A_{p_2}$.

A second generalization of Muckenhoupt's result is to the setting of the variable Lebesgue spaces. The first author, Fiorenza, and Neugebauer ([**6**]) proved that given an exponent function $p(\cdot)\colon \mathbb{R}^n \to [1,\infty)$ such that $1<p_-\leq p_+<\infty$ and $p(\cdot)$ is log-Hölder continuous, then the maximal operator is bounded on $L^{p(\cdot)}$. Given $p(\cdot)$ a log-Hölder continuous function, in [**7**] (see also [**4**]) they proved the corresponding weighted norm inequality: a necessary and sufficient condition for the maximal operator to be bounded on $L^{p(\cdot)}(w)$, i.e., that $\|(Mf)w\|_{p(\cdot)} \lesssim \|fw\|_{p(\cdot)}$, is that $w \in \mathcal{A}_{p(\cdot)}$,

$$\sup_Q |Q|^{-1}\|w\chi_Q\|_{p(\cdot)}\|w^{-1}\chi_Q\|_{p'(\cdot)} < \infty.$$

When $p(\cdot) = p$ is a constant function, then this reduces to the classical result of Muckenhoupt, since $L^{p(\cdot)}(w) = L^p(w^p)$ and $w \in \mathcal{A}_{p(\cdot)}$ is equivalent to $w^p \in A_p$.

The purpose of this paper is to extend both of these results and characterize the class of weights necessary and sufficient for the bilinear maximal operator to satisfy bilinear weighted norm inequalities over the variable Lebesgue spaces. The remainder of this paper is organized as follows. In Section 2 we make the necessary definitions to state our two main results. In particular, we introduce the class of vector weights $\mathcal{A}_{\vec{p}(\cdot)}$. Our first result, Theorem 2.4, is for the bilinear maximal operator. Our second, Theorem 2.8, shows that the weight condition $\mathcal{A}_{\vec{p}(\cdot)}$ is sufficient for bilinear Calderón–Zygmund singular integral operators to satisfy weighted norm inequalities over the variable Lebesgue spaces. This generalizes the main result of [**22**].

In Section 3 we gather some basic results about weights and the variable Lebesgue spaces that we need in our proof, and in Section 4 we prove some properties of $\mathcal{A}_{p(\cdot)}$ and $\mathcal{A}_{\vec{p}(\cdot)}$ weights. In Section 5 we give a characterization of vector weights $\mathcal{A}_{\vec{p}(\cdot)}$ in terms of averaging operators. In Section 6 we prove Theorem 2.4. The proof is broadly similar to the proof in the linear case given in [**7**], but there are many additional technical obstacles. Finally, in Section 7 we prove Theorem 2.8. The proof relies on an extrapolation theorem in the scale of weighted variable Lebesgue spaces proved in [**13**].

Remark 1.1 (Added in proof). One of the anonymous referees for this paper asked whether a shorter proof of Theorem 2.4 could be gotten by adapting the ideas of [**4**] to the bilinear case. We originally tried this approach, but were unsuccessful. This remains an open problem.

Throughout this paper, n will denote the dimension of the underlying space $\mathbb{R}^n$. A cube $Q \subset \mathbb{R}^n$ will always have its sides parallel to the coordinate axes. Let $\ell(Q)$ denote the side-length of Q. Given a cube Q and a function f, we will denote averages as follows:

$$\frac{1}{|Q|}\int_Q f\,dx = \fint_Q f\,dx = \langle f\rangle_Q.$$

Similarly, if σ is a non-negative measure, we denote weighted averages by

$$\frac{1}{\sigma(Q)}\int_Q f\sigma\,dx = \langle f\rangle_{\sigma,Q}.$$

Constants will be denoted by C, c, etc. and their value may change from line to line, even in the same computation. If we need to emphasize the dependence of a constant on some parameter we will write, for instance, $C(n)$. Given two positive quantities A and B, we will write $A \lesssim B$ if there is a constant c such that $A \leq cB$. If $A \lesssim B$ and $B \lesssim A$, then we write $A \approx B$.

2. Main results

We first recall the definition of variable Lebesgue spaces. For more information, see [**5**]. Let $\mathcal{P}$ denote the collection of measurable functions $p(\cdot)\colon \mathbb{R}^n \to [1,\infty]$ and $\mathcal{P}_0$ the collection of measurable functions $p(\cdot)\colon \mathbb{R}^n \to (0,\infty]$. Given $p(\cdot) \in \mathcal{P}_0$ and a set $E \subset \mathbb{R}^n$, define

$$p_-(E) = \operatorname*{ess\,inf}_{x\in E} p(x), \quad p_+(E) = \operatorname*{ess\,sup}_{x\in E} p(x).$$

For simplicity we will write $p_- = p_-(\mathbb{R}^n)$ and $p_+ = p_+(\mathbb{R}^n)$. Given $p(\cdot) \in \mathcal{P}$ we define the dual exponent $p'(\cdot)$ pointwise a.e. by

$$\frac{1}{p(x)} + \frac{1}{p'(x)} = 1,$$

with the convention that $\frac{1}{\infty} = 0$.

The space $L^{p(\cdot)}$ consists of all complex-valued, measurable functions f such that for some $\lambda > 0$,

$$\rho_{p(\cdot)}(f/\lambda) = \int_{\mathbb{R}^n\setminus\Omega_\infty} \left|\frac{f(x)}{\lambda}\right|^{p(x)} dx + \lambda^{-1}\|f\|_{L^\infty(\Omega_\infty)} < \infty,$$

where $\Omega_\infty = \{x \in \mathbb{R}^n : p(x) = \infty\}$. This becomes a quasi-Banach function space when equipped with the quasi-norm

$$\|f\|_{L^{p(\cdot)}} = \|f\|_{p(\cdot)} = \inf\{\lambda > 0 : \rho_{p(\cdot)}(f/\lambda) \leq 1\};$$

when $p_- \geq 1$ it is a Banach space. When $p(\cdot) = p$, $0 < p < \infty$, then $L^{p(\cdot)} = L^p$ with equality of quasi-norms.

An exponent $p(\cdot) \in \mathcal{P}_0$ is said to be locally log-Hölder continuous, denoted by $p(\cdot) \in LH_0$, if there exists a constant C_0 such that

$$\left|\frac{1}{p(x)} - \frac{1}{p(y)}\right| \leq \frac{C_0}{-\log(|x-y|)}, \quad |x-y| < \frac{1}{2};$$

$p(\cdot)$ is said to be log-Hölder continuous at infinity, denoted by $p(\cdot) \in LH_\infty$, if there exist $C_\infty, p_\infty > 0$ such that

$$\left| \frac{1}{p(x)} - \frac{1}{p_\infty} \right| \leq \frac{C_\infty}{\log(e + |x|)}.$$

If $p(\cdot) \in LH = LH_0 \cap LH_\infty$, we simply say that it is log-Hölder continuous.

Remark 2.1. For our main results we will assume $p_+ < \infty$. In this case Ω_∞ has measure zero and the definition of the norm is simpler. Moreover, in the definition of log-Hölder continuity, we can replace the left-hand sides by $|p(x) - p(y)|$ and $|p(x) - p_\infty|$, respectively, requiring only new constants that depend on p_+.

By a weight w we mean a non-negative function such that $0 < w(x) < \infty$ a.e. Given a weight w and $p(\cdot) \in \mathcal{P}_0$, we say $f \in L^{p(\cdot)}(w)$ if $fw \in L^{p(\cdot)}$.

Definition 2.2. Given exponents $p_1(\cdot), p_2(\cdot) \in \mathcal{P}(\mathbb{R}^n)$, define $p(\cdot)$ for a.e. x by

$$\frac{1}{p(x)} = \frac{1}{p_1(x)} + \frac{1}{p_2(x)}, \tag{2.1}$$

and let $\vec{p}(\cdot) = (p_1(\cdot), p_2(\cdot), p(\cdot))$. Given weights w_1, w_2, let $w = w_1 w_2$ and define $\vec{w} = (w_1, w_2, w)$. We say that $\vec{w} \in \mathcal{A}_{\vec{p}(\cdot)}$ if

$$\sup_Q |Q|^{-2} \|w\chi_Q\|_{p(\cdot)} \|w_1^{-1}\chi_Q\|_{p_1'(\cdot)} \|w_2^{-1}\chi_Q\|_{p_2'(\cdot)} < \infty.$$

Remark 2.3. If $p_1(\cdot)$ and $p_2(\cdot)$ are constant, then this condition reduces to the $A_{\vec{p}}$ condition for the triple $(w_1^{p_1}, w_2^{p_2}, (w_1 w_2)^p)$.

We can now state our first main result.

Theorem 2.4. *Given $p_1(\cdot), p_2(\cdot) \in \mathcal{P}$, suppose $1 < (p_j)_- \leq (p_j)_+ < \infty$ and $p_j(\cdot) \in LH$, $j = 1, 2$. Define $p(\cdot)$ by* (2.1). *Let w_1, w_2 be weights and define $w = w_1 w_2$. Then the bilinear maximal operator satisfies*

$$\mathcal{M}\colon L^{p_1(\cdot)}(w_1) \times L^{p_2(\cdot)}(w_2) \to L^{p(\cdot)}(w) \tag{2.2}$$

if and only if $\vec{w} \in \mathcal{A}_{\vec{p}(\cdot)}$.

Remark 2.5. We do not believe that the assumption $p(\cdot) \in LH$ is necessary in Theorem 2.4, but some additional hypothesis is. In the linear, unweighted case, while it is sufficient to assume that the exponent $p(\cdot)$ is log-Hölder continuous for the maximal operator to be bounded on $L^{p(\cdot)}$, it is not necessary: see [**5**, Section 4.4] for examples. Diening and Hästö ([**17**]) conjectured that in the weighted case, a necessary and sufficient condition for M to be bounded on $L^{p(\cdot)}(w)$ is that the maximal operator is bounded on $L^{p(\cdot)}$ and $w \in \mathcal{A}_{p(\cdot)}$. (The latter condition is given in Definition 4.1.) Unlike in the constant exponent case, when $w = 1$ these conditions are not the same since $1 \in \mathcal{A}_{p(\cdot)}$ is a necessary but not sufficient condition for M to be bounded on $L^{p(\cdot)}$ [**5**, Corollary 4.50, Example 4.51]. We conjecture that the analogous result holds in the bilinear case: $\mathcal{M}$ satisfies (2.2) if and only if $\mathcal{M}$ satisfies an unweighted bilinear estimate and $\vec{w} \in \mathcal{A}_{\vec{p}(\cdot)}$.

Remark 2.6. In the linear case, the maximal operator is bounded on $L^{p(\cdot)}$ if $p_- > 1$ and $1/p(\cdot) \in LH$: we can allow $p_+ = \infty$. (See [**5**] for details and references.) In [**7**] it was conjectured that M is bounded on $L^{p(\cdot)}(w)$ with the same hypotheses if $w \in \mathcal{A}_{p(\cdot)}$. This condition is well defined even if $p_- = 1$ and $p_+ = \infty$. Moreover, this conjecture is true if $p(\cdot) = \infty$ a.e. This is equivalent to a classical but often overlooked result of Muckenhoupt [**23**], that if $w^{-1} \in A_1$ and $fw \in L^\infty$, then $(Mf)w \in L^\infty$. Here we conjecture that we can remove the hypothesis $p_+ < \infty$ from Theorem 2.4. However, as in the linear case we believe that this will require a very different argument, as the fact that $p_+, (p_j)_+ < \infty$ plays an important role in our proof.

Remark 2.7. Though we have only proved our result in the bilinear case, an m-linear version of Theorem 2.4, $m \geq 3$, should be true with the obvious changes in the definition of $\mathcal{A}_{\vec{p}(\cdot)}$ and the statement of the theorem. But even in the bilinear case the proof is quite technical, and so we decided to avoid making our proof even more obscure by trying to prove the general result.

Our second main result is for bilinear Calderón–Zygmund singular integrals. These operators have been considered by a number of authors, and we refer the reader to [**22**] for details and further references.

Let $K(x, y, z)$ be a complex-valued, locally integrable function on $\mathbb{R}^{3n} \setminus \triangle$, where $\triangle = \{(x, x, x) : x \in \mathbb{R}^n\}$. K is a Calderón–Zygmund kernel if there exist $A > 0$ and $\delta > 0$ such that for all $(x, y, z) \in \mathbb{R}^{3n} \setminus \triangle$,

$$|K(x, y, z)| \leq \frac{A}{(|x-y| + |x-z| + |y-z|)^{2n}}$$

and

$$|K(x, y, z) - K(\tilde{x}, y, z)| \leq \frac{A|x-\tilde{x}|^\delta}{(|x-y| + |x-z| + |y-z|)^{2n+\delta}},$$

whenever $|x - \tilde{x}| \leq \frac{1}{2} \max(|x - z|, |x - y|)$. We also assume that the two analogous difference estimates with respect to the variables y and z hold.

An operator $T\colon \mathcal{S}\times\mathcal{S}\to\mathcal{S}'$ is a bilinear Calderón–Zygmund singular integral if:

(1) there exists a bilinear Calderón–Zygmund kernel K such that
$$T(f_1,f_2)(x)=\int_{\mathbb{R}^{2n}} K(x,y,z)f_1(y)f_2(z)\,dy\,dz$$
for all $f_1,f_2\in C_c^\infty(\mathbb{R}^n)$ and all $x\notin \operatorname{supp}(f_1)\cap\operatorname{supp}(f_2)$;

(2) there exist $1\le p,q<\infty$ and r such that $\frac{1}{r}=\frac{1}{p}+\frac{1}{q}$ and T can be extended to a bounded operator from $L^p\times L^q$ into L^r.

Theorem 2.8. *Given $p_1(\cdot),p_2(\cdot)\in\mathcal{P}$, suppose $1<(p_j)_-\le(p_j)_+<\infty$ and $p_j(\cdot)\in LH$, $j=1,2$. Define $p(\cdot)$ by (2.1). Let w_1, w_2 be weights, define $w=w_1w_2$, and assume $\vec{w}\in\mathcal{A}_{\vec{p}(\cdot)}$. If T is a bilinear Calderón–Zygmund singular integral, then*

$$T\colon L^{p_1(\cdot)}(w_1)\times L^{p_2(\cdot)}(w_2)\to L^{p(\cdot)}(w). \tag{2.3}$$

Remark 2.9. As for the bilinear maximal operator, we do not believe that the assumption that $p_1(\cdot),p_2(\cdot)\in LH$ is necessary for the conclusion in Theorem 2.8 to hold. In [**11**], the authors proved that in the unweighted case it was sufficient to assume that the (linear) maximal operator is bounded on $L_1^p(\cdot)$ and $L_2^p(\cdot)$. We conjecture that with this hypothesis, or even the weaker assumption that $\mathcal{M}$ satisfies the associated unweighted bilinear inequality, and $\vec{w}\in\mathcal{A}_{\vec{p}(\cdot)}$, then (2.3) holds.

Remark 2.10. Alongside the variable Lebesgue spaces there is a theory of variable Hardy spaces; see [**12**]. Very recently, the first author, Moen, and Nguyen ([**10**]) proved unweighted estimates on variable Hardy spaces for bilinear Calderón–Zygmund singular integrals. It would be interesting to extend these results to weighted variable Hardy spaces using the $\mathcal{A}_{\vec{p}(\cdot)}$ weights.

3. Preliminaries

In this section we gather some basic results about weights and about variable Lebesgue spaces that we will need in the subsequent sections.

Weights. First, we recall the definition of the class A_∞:
$$A_\infty=\bigcup_{p>1}A_p.$$

We will need the following property of A_∞ weights. For a proof, see [**18**].

Lemma 3.1. *Let $w\in A_\infty$. Then for each $0<\alpha<1$ there exists $0<\beta<1$ such that if Q is any cube and $E\subset Q$ is such that $\alpha|Q|\le|E|$, then $\beta w(Q)\le w(E)$. Similarly, for each $0<\gamma<1$ there exists $0<\delta<1$ such that if $|E|\le\gamma|Q|$, then $w(E)\le\delta w(Q)$.*

To state our next result, we introduce the weighted dyadic maximal operator. Given a weight σ,

$$M_\sigma^{\mathcal{D}_0} f(x) = \sup_{Q\in\mathcal{D}_0} \frac{1}{\sigma(Q)} \int_Q |f|\sigma\, dy \cdot \chi_Q(x) = \sup_{Q\in\mathcal{D}_0} \langle |f|\rangle_{\sigma,Q}\chi_Q(x),$$

where the supremum is taken over all cubes in the collection $\mathcal{D}_0$ of dyadic cubes:

$$\mathcal{D}_0 = \{2^{-k}([0,1)^n + j) : k \in \mathbb{Z},\, j \in \mathbb{Z}^n\}.$$

The following result is well-known but an explicit proof does not seem to have appeared in the literature. The proof is essentially the same as for the classical dyadic maximal operator; see Grafakos [**19**].

Lemma 3.2. *Given a weight σ, then for $1 < p < \infty$,*

$$\int_{\mathbb{R}^n} (M_\sigma^{\mathcal{D}_0} f)^p \sigma\, dx \lesssim \int_{\mathbb{R}^n} |f|^p \sigma\, dx$$

and the implicit constant depends only on p.

Variable Lebesgue spaces. Here we gather some basic results about variable Lebesgue spaces. All of these are found in the literature (with some minor variations). In some cases they were only proved for exponents $p(\cdot) \in \mathcal{P}$, but essentially the same proof works for $p(\cdot) \in \mathcal{P}_0$.

Lemma 3.3 ([**5**, Proposition 2.18])**.** *Given $p(\cdot) \in \mathcal{P}_0$, suppose $|\Omega_\infty| = 0$. If $s > 0$, then*

$$\| |f|^s \|_{p(\cdot)} = \|f\|_{sp(\cdot)}^s.$$

Lemma 3.4 ([**5**, Theorem 2.61])**.** *Given $p(\cdot) \in \mathcal{P}_0$, if $f \in L^{p(\cdot)}$ is such that $\{f_k\}$ converges to f pointwise a.e., then*

$$\|f\|_{p(\cdot)} \le \liminf_{k\to\infty} \|f_k\|_{p(\cdot)}.$$

Lemma 3.5 ([**5**, Corollary 2.23])**.** *Given $p(\cdot) \in \mathcal{P}_0$, suppose $0 < p_- \le p_+ < \infty$. If $\|f\|_{p(\cdot)} > 1$, then*

$$\rho_{p(\cdot)}(f)^{\frac{1}{p_+}} \le \|f\|_{p(\cdot)} \le \rho_{p(\cdot)}(f)^{\frac{1}{p_-}}.$$

If $\|f\|_{p(\cdot)} \le 1$, then

$$\rho_{p(\cdot)}(f)^{\frac{1}{p_-}} \le \|f\|_{p(\cdot)} \le \rho_{p(\cdot)}(f)^{\frac{1}{p_+}}.$$

Consequently, $\|f\|_{p(\cdot)} \lesssim 1$ if and only if $\rho_{p(\cdot)}(f) \lesssim 1$.

Lemma 3.6 ([**5**, Corollary 2.30])**.** *Fix $k \geq 2$ and let $p_j(\cdot) \in \mathcal{P}$ satisfy for a.e. x,*

$$\sum_{j=1}^{k} \frac{1}{p_j(x)} = 1.$$

Then, for all $f_j \in L^{p_j(\cdot)}$, $1 \leqslant j \leqslant k$,

$$\int_{\mathbb{R}^n} |f_1 \cdots f_k| \, dx \lesssim \prod_{j=1}^{k} \|f_j\|_{p_j(\cdot)}.$$

The implicit constant depends only on the $p_j(\cdot)$.

Lemma 3.7 ([**5**, Corollary 2.28])**.** *Given $p_1(\cdot), p_2(\cdot) \in \mathcal{P}_0$, define $p(\cdot) \in \mathcal{P}_0$ by* (2.1)*. Then*

$$\|fg\|_{p(\cdot)} \lesssim \|f\|_{p_1(\cdot)} \|g\|_{p_2(\cdot)}.$$

The implicit constant depends only on $p_1(\cdot)$ and $p_2(\cdot)$.

Lemma 3.8 ([**5**, Theorem 2.34])**.** *Given $p(\cdot) \in \mathcal{P}$, then for every $f \in L^{p(\cdot)}$,*

$$\|f\|_{p(\cdot)} \approx \sup_{\|g\|_{p'(\cdot)} \leq 1} \int_{\mathbb{R}^n} |fg| \, dx.$$

The implicit constants depend only on $p(\cdot)$.

Remark 3.9. It is immediate that in the weighted space $L^{p(\cdot)}(w)$, the same result is true if we take the supremum over all $g \in L^{p'(\cdot)}(w^{-1})$ with $\|gw^{-1}\|_{p'(\cdot)} \leq 1$.

Lemma 3.10 ([**16**, Corollary 4.5.9])**.** *Let $p(\cdot) \in \mathcal{P}$ be such that $p(\cdot) \in LH$. Then for every cube Q,*

$$\|\chi_Q\|_{p(\cdot)} \approx |Q|^{\frac{1}{p_Q}},$$

where p_Q is the harmonic mean of $p(\cdot)$ on Q:

$$\frac{1}{p_Q} = \fint_Q \frac{dx}{p(x)}.$$

The implicit constants depend only on $p(\cdot)$.

Lemma 3.11 ([**5**, Lemma 3.24])**.** *Given $p(\cdot) \in \mathcal{P}_0$, suppose $p(\cdot) \in LH_0$ and $0 < p_- \leq p_+ < \infty$. Then for every cube Q,*

$$|Q|^{p_-(Q) - p_+(Q)} \lesssim 1,$$

and the implicit constant depends only on $p(\cdot)$ and n. The same inequality holds if we replace one of $p_+(Q)$ or $p_-(Q)$ by $p(x)$ for any $x \in Q$.

Remark 3.12. Lemma 3.11 is sometimes referred to as Diening's condition, and it is the principal way in which we will apply the LH_0 condition.

Lemma 3.13 ([**2**, Lemmas 2.7, 2.8]). *Given two exponents $r(\cdot), s(\cdot) \in \mathcal{P}_0$, suppose*

$$|s(y) - r(y)| \leq \frac{C_0}{\log(e + |y|)}.$$

Then, given any set G and any non-negative measure μ, for every $t \geq 1$ there exists a constant $C = C(t, C_0)$ such that for all functions f such that $|f(y)| \leq 1$,

$$\int_Q |f(y)|^{s(y)}\, d\mu(y) \leq C \int_G |f(y)|^{r(y)}\, d\mu(y) + \int_G \frac{1}{(e+|y|)^{tns_-(G)}}\, d\mu(y).$$

If we instead assume that

$$0 \leq r(y) - s(y) \leq \frac{C_0}{\log(e + |y|)},$$

then the same inequality holds for any function f.

Remark 3.14. Lemma 3.13 is the principal way in which we will apply the LH_∞ condition.

4. Properties of $\mathcal{A}_{p(\cdot)}$ and $\mathcal{A}_{\vec{p}(\cdot)}$ weights

In this section we give some properties of the $\mathcal{A}_{p(\cdot)}$ and $\mathcal{A}_{\vec{p}(\cdot)}$ weights that will be used in the proof of Theorem 2.4. For simplicity, throughout this section assume that w_1, w_2 are weights and let $w = w_1 w_2$ and $\vec{w} = (w_1, w_2, w)$. Similarly, whenever we are given $p_1(\cdot), p_2(\cdot) \in \mathcal{P}$, define $p(\cdot)$ by (2.1) and let $\vec{p}(\cdot) = (p_1(\cdot), p_2(\cdot), p(\cdot))$. Note that in this case we always have that $p_- \geq \frac{1}{2}$.

We begin by recalling the definition of $\mathcal{A}_{p(\cdot)}$ weights and then state several results from [**7**] on their properties.

Definition 4.1. Given $p(\cdot) \in \mathcal{P}$ and a weight w, we say $w \in \mathcal{A}_{p(\cdot)}$ if

$$\sup_Q |Q|^{-1} \|w\chi_Q\|_{p(\cdot)} \|w^{-1}\chi_Q\|_{p'(\cdot)} < \infty.$$

The next two lemmas show the relationship between $\mathcal{A}_{p(\cdot)}$ and A_∞ weights.

Lemma 4.2 ([**7**, Lemma 3.4]). *Given $p(\cdot) \in \mathcal{P}$, suppose $p(\cdot) \in LH$ and $p_+ < \infty$. If $w \in \mathcal{A}_{p(\cdot)}$, then $u(\cdot) = w(\cdot)^{p(\cdot)} \in A_\infty$.*

Lemma 4.3 ([**7**, Lemmas 3.5, 3.6]). *Given $p(\cdot) \in \mathcal{P}$, suppose $p(\cdot) \in LH$ and $p_+ < \infty$. Let $w \in \mathcal{A}_{p(\cdot)}$ and let $u(x) = w(x)^{p(x)}$. Then, given any cube Q such that $\|w\chi_Q\|_{p(\cdot)} \geq 1$, we have $\|w\chi_Q\|_{p(\cdot)} \approx u(Q)^{\frac{1}{p_\infty}}$. Moreover, given any $E \subset Q$,*

$$\frac{|E|}{|Q|} \lesssim \left(\frac{u(E)}{u(Q)}\right)^{\frac{1}{p_\infty}}.$$

The implicit constants depend only on w and $p(\cdot)$.

Remark 4.4. To apply Lemma 4.3, note that by Lemma 3.5, $\|w\chi_Q\|_{p(\cdot)} \geq 1$ if and only if $u(Q) \geq 1$.

The next result is a weighted version of the Diening condition in Lemma 3.11 and will be used to apply the LH_0 condition.

Lemma 4.5 ([**7**, Lemma 3.3]). *Given $p(\cdot) \in \mathcal{P}$ such that $p(\cdot) \in LH$, if $w \in \mathcal{A}_{p(\cdot)}$, then for all cubes Q,*

$$\|w\chi_Q\|_{p(\cdot)}^{p_-(Q)-p_+(Q)} \lesssim 1.$$

The implicit constant depends only on $p(\cdot)$ and w.

The final lemma is an integral estimate that, in conjunction with Lemma 3.13, will be used to apply the LH_∞ condition.

Lemma 4.6 ([**7**, Inequality (3.3)]). *Given $p(\cdot) \in \mathcal{P}$, suppose $p(\cdot) \in LH$. If $w \in \mathcal{A}_{p(\cdot)}$, then there exists a constant $t > 1$, depending only on w, $p(\cdot)$, and n, such that*

$$\int_{\mathbb{R}^n} \frac{w(x)^{p(x)}}{(e + |x|)^{tnp_-}}\, dx \leq 1.$$

We now turn to the $\mathcal{A}_{\vec{p}(\cdot)}$ condition. If $w_1 \in \mathcal{A}_{p_1(\cdot)}$ and $w_2 \in \mathcal{A}_{p_2(\cdot)}$, then by Lemma 3.6 we have that $\vec{w} \in \mathcal{A}_{\vec{p}(\cdot)}$. However, this inclusion is proper, since it is in the constant exponent case. Nevertheless, we can characterize the bilinear $\mathcal{A}_{\vec{p}(\cdot)}$ weights in terms of the $\mathcal{A}_{p(\cdot)}$ condition. In the constant exponent case this is proved in [**22**], and our argument is adapted from theirs.

Proposition 4.7. *Given $\vec{w}$ and $\vec{p}(\cdot)$, $\vec{w} \in \mathcal{A}_{\vec{p}(\cdot)}$ if and only if*

$$\begin{cases} w_j^{-\frac{1}{2}} \in \mathcal{A}_{2p_j'(\cdot)}, & j = 1, 2, \\ w^{\frac{1}{2}} \in \mathcal{A}_{2p(\cdot)}. \end{cases} \tag{4.1}$$

Remark 4.8. Note that since $p_- \geq \frac{1}{2}$, we have $2p(\cdot) \in \mathcal{P}$ and $\mathcal{A}_{2p(\cdot)}$ is well defined.

Proof: First assume that $\vec{w} \in \mathcal{A}_{\vec{p}(\cdot)}$. Then for a.e. x,

$$\begin{aligned}\frac{1}{2p(x)} + \frac{1}{2p_2'(x)} &= 1 - \frac{1}{(2p)'(x)} + \frac{1}{2p_2'(x)} \\ &= 1 - \left(\frac{1}{2p_1'(x)} + \frac{1}{2p_2'(x)}\right) + \frac{1}{2p_2'(x)} \\ &= 1 - \frac{1}{2p_1'(x)} = \frac{1}{(2p_1')'(x)}.\end{aligned}$$

Therefore, by Lemmas 3.7 and 3.3, and by the definition of $\mathcal{A}_{\vec{p}(\cdot)}$,

$$\begin{aligned}\|w_1^{-\frac{1}{2}}\chi_Q\|_{2p_1'(\cdot)}\|w_1^{\frac{1}{2}}\chi_Q\|_{(2p_1'(\cdot))'} &= \|w_1^{-\frac{1}{2}}\chi_Q\|_{2p_1'(\cdot)}\|w_1^{\frac{1}{2}}w_2^{\frac{1}{2}}w_2^{-\frac{1}{2}}\chi_Q\|_{(2p_1')'(\cdot)} \\ &\lesssim \|w_1^{-\frac{1}{2}}\chi_Q\|_{2p_1'(\cdot)}\|w^{\frac{1}{2}}\chi_Q\|_{2p(\cdot)}\|w_2^{-\frac{1}{2}}\chi_Q\|_{2p_2'(\cdot)} \\ &= \left(\|w\chi_Q\|_{p(\cdot)}\prod_{j=1}^{2}\|w_j^{-1}\chi_Q\|_{p_j'(\cdot)}\right)^{\frac{1}{2}} \\ &\lesssim |Q|.\end{aligned}$$

Hence, $w_1^{-\frac{1}{2}} \in \mathcal{A}_{2p_1'(\cdot)}$. The same argument shows that $w_2^{-\frac{1}{2}} \in \mathcal{A}_{2p_2'(\cdot)}$. Finally, we have that

$$\begin{aligned}\|w^{\frac{1}{2}}\chi_Q\|_{2p(\cdot)}\|w^{-\frac{1}{2}}\chi_Q\|_{(2p)'(\cdot)} &\lesssim \|w^{\frac{1}{2}}\chi_Q\|_{2p(\cdot)}\prod_{j=1}^{2}\|w_j^{-\frac{1}{2}}\chi_Q\|_{2p_j'(\cdot)} \\ &= \left(\|w\chi_Q\|_{p(\cdot)}\prod_{j=1}^{2}\|w_j^{-1}\chi_Q\|_{p_j'(\cdot)}\right)^{\frac{1}{2}} \lesssim |Q|.\end{aligned}$$

Thus (4.1) holds.

Conversely, now suppose that (4.1) holds. Then for a.e. x,

$$\frac{1}{2(2p)'(x)} + \frac{1}{2(2p_1')'(x)} + \frac{1}{2(2p_2')'(x)} = 1,$$

so by Lemmas 3.6 and 3.3, for any cube Q,

$$\begin{aligned}1 = \langle w^{-\frac{1}{4}}w_1^{\frac{1}{4}}w_2^{\frac{1}{4}}\rangle_Q^2 &\lesssim |Q|^{-2}\|w^{-\frac{1}{4}}\chi_Q\|_{2(2p)'(\cdot)}^2\prod_{j=1}^{2}\|w_j^{\frac{1}{4}}\chi_Q\|_{2(2p_j')'(\cdot)}^2 \\ &= |Q|^{-2}\|w^{-\frac{1}{2}}\chi_Q\|_{(2p)'(\cdot)}\prod_{j=1}^{2}\|w_j^{\frac{1}{2}}\chi_Q\|_{(2p_j')'(\cdot)}.\end{aligned}$$

Therefore,

$$\|w\chi_Q\|_{p(\cdot)}^{\frac{1}{2}} \prod_{j=1}^{2} \|w_j^{-1}\chi_Q\|_{p_j'(\cdot)}^{\frac{1}{2}} = \|w^{\frac{1}{2}}\chi_Q\|_{2p(\cdot)} \prod_{j=1}^{2} \|w_j^{-\frac{1}{2}}\chi_Q\|_{2p_j'(\cdot)}$$

$$\lesssim |Q|^{-2}\|w^{\frac{1}{2}}\chi_Q\|_{2p(\cdot)}\|w^{-\frac{1}{2}}\chi_Q\|_{(2p)'(\cdot)} \prod_{j=1}^{2} \|w_j^{\frac{1}{2}}\chi_Q\|_{(2p_j')'(\cdot)}\|w_j^{-\frac{1}{2}}\chi_Q\|_{2p_j'(\cdot)}$$

$$\lesssim |Q|,$$

and so $w \in \mathcal{A}_{\vec{p}(\cdot)}$. □

Proposition 4.7 has the following corollary which will be used in our proof of Theorem 2.4.

Corollary 4.9. *Given $p_1(\cdot), p_2(\cdot) \in \mathcal{P}$, suppose $p_j(\cdot) \in LH$ and $(p_j)_+ < \infty$, $j = 1, 2$. If $\vec{w} \in \mathcal{A}_{\vec{p}(\cdot)}$, then $u(\cdot) = w(\cdot)^{p(\cdot)}$ and $\sigma_j(\cdot) = w_j(\cdot)^{-p_j'(\cdot)}$, $j = 1, 2$, are in A_∞.*

Proof: This follows immediately from Lemma 4.2 and Proposition 4.7. □

Our next lemma is a variant of Lemma 4.5 for $\mathcal{A}_{\vec{p}(\cdot)}$ weights. The proof is adapted from the proof in [**7**].

Proposition 4.10. *Given $p_1(\cdot), p_2(\cdot) \in \mathcal{P}$, suppose $p_j(\cdot) \in LH$, $j = 1, 2$. Define $p(\cdot)$ by* (2.1) *and suppose $p_+ < \infty$. For every cube Q, define $q(Q)$ by*

$$\frac{1}{q(Q)} = \frac{1}{(p_1)_-(Q)} + \frac{1}{(p_2)_-(Q)}.$$

Then, given $v \in \mathcal{A}_{p(\cdot)}$, for a.e. $x \in Q$,

$$\|v^{-1}\chi_Q\|_{p'(\cdot)}^{q(Q)-p(x)} \lesssim 1. \tag{4.2}$$

The implicit constant depends on $p_1(\cdot)$, $p_2(\cdot)$, n, and v, but is independent of Q.

Remark 4.11. When we apply Proposition 4.10 below, we will do so in conjunction with Proposition 4.7 to $w^{\frac{1}{2}} \in \mathcal{A}_{2p(\cdot)}$, so we will let $v^{-1} = w^{-\frac{1}{2}}$ and replace $p(\cdot)$ by $2p(\cdot)$ and q by $2q$. We could have stated this result in these terms, but for the purposes of the proof, it seemed easier to suppress the factor of 2. Recall that $\ell(Q)$ denotes the side-length of the cube Q (see Section 1).

Proof: Fix a cube $Q \subset \mathbb{R}^n$. It follows from the definition that for a.e. $x \in Q$, $q(Q) \leq p(x) \leq p_+$, so if $\|v^{-1}\chi_Q\|_{p'(\cdot)} > 1$, (4.2) holds immediately. Therefore, we may assume without loss of generality that $\|v^{-1}\chi_Q\|_{p'(\cdot)} \leq 1$.

Let Q_0 be the cube centered at the origin with $|Q_0| = 1$. Then either $|Q| \leq |Q_0|$ or $|Q| > |Q_0|$. We will prove (4.2) in the first case; the proof of the second case is the same, exchanging the roles of Q and Q_0. Suppose that $\operatorname{dist}(Q, Q_0) \leqslant \ell(Q_0)$. Then $Q \subset 5Q_0$. Define

$$q_- = \inf_Q q(Q) \leq \sup_Q q(Q) \leq p_+ < \infty. \tag{4.3}$$

Then

$$\frac{1}{q(Q)} - \frac{1}{p(x)} \leq \left(\frac{1}{(p_1)_-(Q)} - \frac{1}{(p_1)_+(Q)}\right) + \left(\frac{1}{(p_2)_-(Q)} - \frac{1}{(p_2)_+(Q)}\right),$$

so there exists a constant $C = C(p_1(\cdot), p_2(\cdot))$ such that

$$p(x) - q(Q) \leq C[(p_1)_+(Q) - (p_1)_-(Q)] + C[(p_2)_+(Q) - (p_2)_-(Q)]. \tag{4.4}$$

Therefore, by Lemma 3.6 and the $\mathcal{A}_{p'(\cdot)}$ condition we have that

$$\begin{aligned}|Q| &= \int_Q v^{-1}v\,dx \lesssim \|v^{-1}\chi_Q\|_{p'(\cdot)}\|v\chi_Q\|_{p(\cdot)} \\ &\leq 5^n\|v^{-1}\chi_Q\|_{p'(\cdot)}|5Q_0|^{-1}\|v\chi_{5Q_0}\|_{p(\cdot)} \\ &\lesssim \|v^{-1}\chi_Q\|_{p'(\cdot)}\|v^{-1}\chi_{5Q_0}\|_{p'(\cdot)}^{-1}.\end{aligned} \tag{4.5}$$

Hence, by (4.4) and Lemma 3.11,

$$\begin{aligned}\|v^{-1}\chi_Q\|_{p'(\cdot)}^{q(Q)-p(x)} &\lesssim \|v^{-1}\chi_{5Q_0}\|_{p'(\cdot)}^{q(Q)-p(x)}|Q|^{q(Q)-p(x)} \\ &\leq \left(1 + \|v^{-1}\chi_{5Q_0}\|_{p'(\cdot)}^{-1}\right)^{p_+-q_-}|Q|^{q(Q)-p(x)} \lesssim 1.\end{aligned}$$

Now assume that $\operatorname{dist}(Q, Q_0) \geqslant \ell(Q_0)$. Then there exists a cube $\hat{Q}$ such that $Q, Q_0 \subset \hat{Q}$ and $\ell(\hat{Q}) \approx \operatorname{dist}(Q, Q_0) \approx \operatorname{dist}(Q, 0) = d_Q$. Therefore, arguing as we did in inequality (4.5), replacing $5Q_0$ by $\hat{Q}$, we get

$$|Q| \lesssim |\hat{Q}|\|v^{-1}\chi_Q\|_{p'(\cdot)}\|v^{-1}\chi_{\hat{Q}}\|_{p'(\cdot)}^{-1}.$$

If we continue the above argument and use the fact that $\|v^{-1}\chi_{Q_0}\|_{p'(\cdot)} \leq \|v^{-1}\chi_{\hat{Q}}\|_{p'(\cdot)}$, we get

$$\|v^{-1}\chi_Q\|_{p'(\cdot)}^{q(Q)-p(x)} \lesssim |\hat{Q}|^{p(x)-q(Q)}.$$

To estimate this final term, note that since $p_j(\cdot) \in LH$, there exist $x_1, x_2 \in \overline{Q}$ such that $(p_1)_-(Q) = p_1(x_1)$ and $(p_2)_-(Q) = p_2(x_2)$. Moreover, $|x_1|, |x_2| \approx d_Q$. Therefore, again by log-Hölder continuity, and using that $\frac{1}{p_\infty} = \frac{1}{(p_1)_\infty} + \frac{1}{(p_2)_\infty}$,

$$\left|\frac{1}{q(Q)} - \frac{1}{p_\infty}\right| \leq \left|\frac{1}{p_1(x_1)} - \frac{1}{(p_1)_\infty}\right| + \left|\frac{1}{p_2(x_2)} - \frac{1}{(p_2)_\infty}\right| \lesssim \frac{1}{\log(e+d_Q)}.$$

Therefore, for $x \in Q$, since $|x| \approx d_Q$,

$$\left|\frac{1}{q(Q)} - \frac{1}{p(x)}\right| \leq \left|\frac{1}{q(Q)} - \frac{1}{p_\infty}\right| + \left|\frac{1}{p_\infty} - \frac{1}{p(x)}\right| \lesssim \frac{1}{\log(e+d_Q)}.$$

Given this, and since $|\hat{Q}| \lesssim (e+d_Q)^n$, we thus have that

$$|\hat{Q}|^{p(x)-q(Q)} \lesssim 1,$$

and so $\|v^{-1}\chi_Q\|_{p'(\cdot)}^{q(Q)-p(x)} \lesssim 1$. □

5. Characterization of $\mathcal{A}_{\vec{p}(\cdot)}$

In this section we give two characterizations of the $\mathcal{A}_{\vec{p}(\cdot)}$ condition in terms of averaging operators. The first is a very general condition that does not require assuming that the exponent functions are log-Hölder continuous. The second requires the additional assumption that $p_1(\cdot)$, $p_2(\cdot)$ are log-Hölder continuous.

Given a cube Q, define the multilinear averaging operator A_Q by

$$A_Q(f_1, f_2)(x) := \langle f_1\rangle_Q \langle f_2\rangle_Q \chi_Q(x).$$

More generally, given a family $\mathcal{Q} = \{Q\}$ of disjoint cubes, we define

$$T_{\mathcal{Q}}(f_1, f_2)(x) = \sum_{Q\in\mathcal{Q}} A_Q(f_1, f_2)(x)\chi_Q(x).$$

Theorem 5.1. *Given $p_1(\cdot), p_2(\cdot) \in \mathcal{P}$ and $\vec{w}$, then $\vec{w} \in \mathcal{A}_{\vec{p}(\cdot)}$ if and only if*

$$\sup_Q \|A_Q(f_1, f_2)w\|_{p(\cdot)} \lesssim \|f_1 w_1\|_{p_1(\cdot)} \|f_2 w_2\|_{p_2(\cdot)}, \tag{5.1}$$

where the supremum is taken over all cubes Q. If we assume further that $p_1(\cdot), p_2(\cdot) \in LH$, then $\vec{w} \in \mathcal{A}_{\vec{p}(\cdot)}$ if and only if

$$\sup_{\mathcal{Q}} \|T_{\mathcal{Q}}(f_1, f_2)w\|_{p(\cdot)} \lesssim \|f_1 w_1\|_{p_1(\cdot)} \|f_2 w_2\|_{p_2(\cdot)}, \tag{5.2}$$

where the supremum is taken over all collections $\mathcal{Q}$ of disjoint cubes.

Remark 5.2. When $p_- \geq 1$ (i.e., when $L^{p(\cdot)}$ is a Banach space) the characterization in terms of the operators T_Q is a consequence of a general result in the setting of Banach lattices due to Kokilashvili *et al.* [**21**]. However, even in this special case we would be required to show that condition $\vec{G}$ defined below holds in order to apply their result. In our case we can use the rescaling properties of variable Lebesgue spaces to prove it directly.

Remark 5.3. A very deep result in the theory of variable Lebesgue spaces is that the uniform boundedness of the linear version of the averaging operators T_Q is equivalent to the boundedness of the Hardy–Littlewood maximal operator, but the uniform boundedness of the (linear) operators A_Q is not. See [**5**, Section 4.4], [**15**], and [**16**, Section 5.2] for details and further references. We conjecture that the corresponding result holds in the bilinear case.

The proof of Theorem 5.1 is straightforward for A_Q, and so we give this proof separately.

Proof of Theorem 5.1 for A_Q: Let be $w \in \mathcal{A}_{\vec{p}(\cdot)}$. Then, given any cube Q, by Lemma 3.6 and the definition of $\mathcal{A}_{\vec{p}(\cdot)}$ we get

$$\begin{aligned} \|A_Q(f_1,f_2)w\|_{p(\cdot)} &= |Q|^{-2}\int_Q |f_1|w_1w_1^{-1}\,dy\int_Q |f_2|w_2w_2^{-1}\,dy\|w\chi_Q\|_{p(\cdot)} \\ &\lesssim |Q|^{-2}\|w_1^{-1}\chi_Q\|_{p_1'(\cdot)}\|w_2^{-1}\chi_Q\|_{p_2'(\cdot)}\|w\chi_Q\|_{p(\cdot)}\|f_1w_1\|_{p_1(\cdot)}\|f_2w_2\|_{p_2(\cdot)} \\ &\lesssim \|f_1w_1\|_{p_1(\cdot)}\|f_2w_2\|_{p_2(\cdot)}. \end{aligned}$$

Since the implicit constant depends only on the $\mathcal{A}_{\vec{p}(\cdot)}$ condition and is independent of Q, we get (5.1).

Now assume that (5.1) holds. By Lemma 3.8, there exist $h_jw_j \in L^{p_j(\cdot)}$, $\|h_jw_j\|_{p_j(\cdot)} \leq 1$, $j=1,2$, such that

$$\begin{aligned} \|w\chi_Q\|_{p(\cdot)}\prod_{j=1}^{2}\|w_j^{-1}\chi_Q\|_{p_j'(\cdot)} &\lesssim \|w\chi_Q\|_{p(\cdot)}\int_Q h_1\,dy\int_Q h_2\,dy \\ &= \|w\chi_Q\|_{p(\cdot)}\langle h_1\rangle_Q\langle h_2\rangle_Q|Q|^2 \\ &= \|A_Q(h_1,h_2)w\|_{p(\cdot)}|Q|^2 \\ &\lesssim \|h_1w_1\|_{p_1(\cdot)}\|h_2w_2\|_{p_2(\cdot)}|Q|^2 \\ &\lesssim |Q|^2. \end{aligned}$$

Again, the constant is independent of Q, so $\vec{w} \in \mathcal{A}_{\vec{p}(\cdot)}$. □

The proof of Theorem 5.1 for $T_{\mathcal{Q}}$ requires two ancillary tools. The first is a bilinear averaging operator that generalizes a linear operator introduced in [**16**]. Given $p(\cdot) \in \mathcal{P}$, define the $p(\cdot)$-average

$$\langle h\rangle_{p(\cdot),Q} := \frac{\|h\chi_Q\|_{p(\cdot)}}{\|\chi_Q\|_{p(\cdot)}},$$

and given a disjoint family of cubes $\mathcal{Q}$ define the $p(\cdot)$-averaging operator

$$T_{p(\cdot),\mathcal{Q}}f(x) = \sum_{Q\in\mathcal{Q}} \langle h\rangle_{p(\cdot),Q} \cdot \chi_Q(x).$$

In [**16**, Corollary 7.3.21] the authors showed that if $p(\cdot) \in LH$, then

$$\|T_{p(\cdot),\mathcal{Q}}f\|_{p(\cdot)} \lesssim \|f\|_{p(\cdot)}. \tag{5.3}$$

We define the bilinear $p(\cdot)$-average operator analogously: given $p_1(\cdot)$, $p_2(\cdot)$, and a family of disjoint cubes $\mathcal{Q}$, let

$$\vec{T}_{\vec{p}(\cdot),\mathcal{Q}}(f_1, f_2)(x) = \sum_{Q\in\mathcal{Q}} \frac{\|f_1\chi_Q\|_{p_1(\cdot)}\|g_1\chi_Q\|_{p_2(\cdot)}}{\|\chi_Q\|_{p(\cdot)}} \cdot \chi_Q(x).$$

Lemma 5.4. *Given $p_1(\cdot), p_2(\cdot) \in \mathcal{P}$, suppose $p_j(\cdot) \in LH$, $j = 1, 2$. Then*

$$\sup_{\mathcal{Q}} \|\vec{T}_{p(\cdot),\mathcal{Q}}(f_1, f_2)\|_{p(\cdot)} \lesssim \|f_1\|_{p_1(\cdot)}\|f_2\|_{p_2(\cdot)},$$

where the supremum is taken over all collections $\mathcal{Q}$ of disjoint cubes.

Proof: Since $p_1(\cdot), p_2(\cdot) \in LH$, $p(\cdot) \in LH$, and so by Lemma 3.10,

$$\|\chi_Q\|_{p(\cdot)} \approx |Q|^{\frac{1}{p_Q}} = |Q|^{\frac{1}{(p_1)_Q}}|Q|^{\frac{1}{(p_2)_Q}} \approx \|\chi_Q\|_{p_1(\cdot)}\|\chi_Q\|_{p_2(\cdot)}.$$

Therefore,

$$\vec{T}_{p(\cdot),\mathcal{Q}}(f_1, f_2)(x) \approx \sum_{Q\in\mathcal{Q}} \langle f_1\rangle_{p_1(\cdot),Q}\langle f_2\rangle_{p_2(\cdot),Q} \cdot \chi_Q,$$

and so by Lemma 3.6 and (5.3),

$$\begin{aligned}
\|\vec{T}_{p(\cdot),\mathcal{Q}}(f_1, f_2)\|_{p(\cdot)} &\lesssim \Big\|\sum_{Q\in\mathcal{Q}} \langle f_1\rangle_{p_1(\cdot),Q}\langle f_2\rangle_{p_2(\cdot),Q} \cdot \chi_Q\Big\|_{p(\cdot)} \\
&\lesssim \Big\|\sum_{Q\in\mathcal{Q}} \langle f_1\rangle_{p_1(\cdot),Q} \cdot \chi_Q \sum_{Q\in\mathcal{Q}} \langle f_2\rangle_{p_2(\cdot),Q} \cdot \chi_Q\Big\|_{p(\cdot)} \\
&\lesssim \|T_{p_1(\cdot),\mathcal{Q}}f_1\|_{p_1(\cdot)}\|T_{p_2(\cdot),\mathcal{Q}}f_2\|_{p_2(\cdot)} \\
&\lesssim \|f_1\|_{p_1(\cdot)}\|f_2\|_{p_2(\cdot)}.
\end{aligned}$$

□

The second tool is a summation property. Given $p_1(\cdot), p_2(\cdot) \in \mathcal{P}$, suppose $p(\cdot)$ is such that $p_- \geq 1$. Then we say that $\vec{p}(\cdot) \in \vec{G}$ if for every family of disjoint cubes $\mathcal{Q}$,

$$\sum_{Q\in\mathcal{Q}} \|f_1\chi_Q\|_{p_1(\cdot)}\|f_2\chi_Q\|_{p_2(\cdot)}\|h\chi_Q\|_{p'(\cdot)} \lesssim \|f_1\|_{p_1(\cdot)}\|f_2\|_{p_2(\cdot)}\|h\|_{p'(\cdot)},$$

where the implicit constant is independent of $\mathcal{Q}$.

Remark 5.5. The linear version of property G is due to Berezhnoĭ [**1**] in the setting of Banach function spaces. See also [**16**] where it is used to prove (5.3).

Lemma 5.6. *Given* $p_1(\cdot), p_2(\cdot) \in \mathcal{P}$*, suppose* $p_j(\cdot) \in LH$*,* $j = 1, 2$*. Then* $\vec{p}(\cdot) \in \vec{G}$*.*

Proof: Since both $p(\cdot), p'(\cdot) \in LH$, by Lemma 3.10, for any cube Q,

$$\|\chi_Q\|_{p(\cdot)}\|\chi_Q\|_{p'(\cdot)} \approx |Q|^{\frac{1}{p_Q}}|Q|^{\frac{1}{(p')_Q}} = |Q|.$$

Hence, by Lemma 3.6, (5.3), and Lemma 5.4,

$$\begin{aligned}
&\sum_{Q\in\mathcal{Q}} \|f_1\chi_Q\|_{p_1(\cdot)}\|f_2\chi_Q\|_{p_2(\cdot)}\|h\chi_Q\|_{p'(\cdot)} \\
&\qquad\approx \sum_{Q\in\mathcal{Q}} \int_{\mathbb{R}^n} \frac{\|f_1\chi_Q\|_{p_1(\cdot)}\|f_2\chi_Q\|_{p_2(\cdot)}}{\|\chi_Q\|_{p(\cdot)}} \frac{\|h\chi_Q\|_{p'(\cdot)}}{\|\chi_Q\|_{p'(\cdot)}} \cdot \chi_Q \, dx \\
&\qquad\lesssim \int_{\mathbb{R}^n} \sum_{Q\in\mathcal{Q}} \frac{\|f_1\chi_Q\|_{p_1(\cdot)}\|f_2\chi_Q\|_{p_2(\cdot)}}{\|\chi_Q\|_{p(\cdot)}} \cdot \chi_Q \sum_{Q\in\mathcal{Q}} \frac{\|h\chi_Q\|_{p'(\cdot)}}{\|\chi_Q\|_{p'(\cdot)}} \cdot \chi_Q \, dx \\
&\qquad= \int_{\mathbb{R}^n} \vec{T}_{\vec{p}(\cdot),\mathcal{Q}}(f_1, f_2)\, T_{p'(\cdot),\mathcal{Q}}h \, dx \\
&\qquad\lesssim \|\vec{T}_{\vec{p}(\cdot),\mathcal{Q}}(f_1, f_2)\|_{p(\cdot)}\|T_{p'(\cdot),\mathcal{Q}}h\|_{p'(\cdot)} \\
&\qquad\lesssim \|f_1\|_{p_1(\cdot)}\|f_2\|_{p_2(\cdot)}\|h\|_{p'(\cdot)}.
\end{aligned}$$

□

Proof of Theorem 5.1 for $T_{\mathcal{Q}}$*:* We first prove that the $\mathcal{A}_{\vec{p}(\cdot)}$ condition is sufficient. Since $|T_{\mathcal{Q}}(f_1, f_2)(x)| \leq T_{\mathcal{Q}}(|f_1|, |f_2|)(x)$, we may assume without loss of generality that f_1, f_2 are non-negative. Because $(p_j)_- \geq 1$,

$j=1,2$, we have $2p_- \geq 1$, and so $2p(\cdot) \in \mathcal{P}$. Therefore, by Lemmas 3.3 and 3.8, there exists $hw^{-\frac{1}{2}} \in L^{(2p)'(\cdot)}$, with $\|hw^{-\frac{1}{2}}\|_{(2p)'(\cdot)} \leq 1$, such that

$$\begin{aligned}
\|T_{\mathcal{Q}}(f_1,f_2)w\|_{p(\cdot)}^{\frac{1}{2}} &= \|T_{\mathcal{Q}}(f_1,f_2)^{\frac{1}{2}}w^{\frac{1}{2}}\|_{2p(\cdot)} \\
&\approx \int_{\mathbb{R}^n} T_{\mathcal{Q}}(f_1,f_2)^{\frac{1}{2}}w^{\frac{1}{2}}hw^{-\frac{1}{2}}\,dx \\
&\leq \sum_{Q\in\mathcal{Q}} \langle f_1\rangle_Q^{\frac{1}{2}}\langle f_2\rangle_Q^{\frac{1}{2}} \int_Q hw^{\frac{1}{2}}w^{-\frac{1}{2}}\,dx \\
&= \sum_{Q\in\mathcal{Q}} \|f_1^{\frac{1}{2}}w_1^{\frac{1}{2}}w_1^{-\frac{1}{2}}\chi_Q\|_2\|f_2^{\frac{1}{2}}w_2^{\frac{1}{2}}w_2^{-\frac{1}{2}}\chi_Q\|_2\|hw^{\frac{1}{2}}w^{-\frac{1}{2}}\chi_Q\|_1|Q|^{-1};
\end{aligned}$$

by Lemmas 3.6 and 3.7,

$$\begin{aligned}
&\leq \sum_{Q\in\mathcal{Q}} \Big[\|f_1^{\frac{1}{2}}w_1^{\frac{1}{2}}\chi_Q\|_{2p_1(\cdot)}\|w_1^{-\frac{1}{2}}\chi_Q\|_{2p_1'(\cdot)} \\
&\qquad \times\|f_2^{\frac{1}{2}}w_2^{\frac{1}{2}}\chi_Q\|_{2p_2(\cdot)}\|w_2^{-\frac{1}{2}}\chi_Q\|_{2p_2'(\cdot)}\|hw^{-\frac{1}{2}}\chi_Q\|_{(2p)'(\cdot)}\|w^{\frac{1}{2}}\chi_Q\|_{2p(\cdot)}|Q|^{-1}\Big] \\
&\leq \sum_{Q\in\mathcal{Q}} \Big[\|f_1^{\frac{1}{2}}w_1^{\frac{1}{2}}\chi_Q\|_{2p_1(\cdot)}\|f_2^{\frac{1}{2}}w_2^{\frac{1}{2}}\chi_Q\|_{2p_2(\cdot)}\|hw^{-\frac{1}{2}}\chi_Q\|_{(2p)'(\cdot)} \\
&\qquad \times\|w_1^{-\frac{1}{2}}\chi_Q\|_{2p_1'(\cdot)}\|w_2^{-\frac{1}{2}}\chi_Q\|_{2p_2'(\cdot)}\|w_1^{\frac{1}{2}}\chi_Q\|_{2p_1(\cdot)}\|w_2^{\frac{1}{2}}\chi_Q\|_{2p_2(\cdot)}|Q|^{-1}\Big];
\end{aligned}$$

by Proposition 4.7, Lemma 5.6 applied to the exponents $2p_1(\cdot)$, $2p_2(\cdot)$, and Lemma 3.3,

$$\begin{aligned}
&\lesssim \sum_{Q\in\mathcal{Q}} \|f_1^{\frac{1}{2}}w_1^{\frac{1}{2}}\chi_Q\|_{2p_1(\cdot)}\|f_2^{\frac{1}{2}}w_2^{\frac{1}{2}}\chi_Q\|_{2p_2(\cdot)}\|hw^{-\frac{1}{2}}\chi_Q\|_{(2p)'(\cdot)} \\
&\lesssim \|f_1^{\frac{1}{2}}w_1^{\frac{1}{2}}\|_{2p_1(\cdot)}\|g_1w_2^{\frac{1}{2}}\|_{2p_2(\cdot)}\|hw^{-\frac{1}{2}}\|_{(2p)'(\cdot)} \\
&\leq \|f_1w_1\|_{p_1(\cdot)}^{\frac{1}{2}}\|g_1w_2\|_{p_2(\cdot)}^{\frac{1}{2}}.
\end{aligned}$$

Since the implicit constants are independent of our choice of $\mathcal{Q}$, we conclude that $\vec{w} \in \mathcal{A}_{\vec{p}(\cdot)}$ implies (5.2).

The converse, that the $\mathcal{A}_{\vec{p}(\cdot)}$ condition is necessary, follows from the corresponding implication for A_Q proved above. □

6. Proof of Theorem 2.4

In this section we prove Theorem 2.4. As before, given weights w_1 and w_2 we define $w=w_1w_2$ and let $\vec{w}=(w_1,w_2,w)$. Given exponents $p_1(\cdot),p_2(\cdot) \in \mathcal{P}$, we define $p(\cdot)$ by (2.1) and let $\vec{p}(\cdot)=(p_1(\cdot),p_2(\cdot),p(\cdot))$.

We first prove the necessity of the $\mathcal{A}_{\vec{p}(\cdot)}$ condition. This is an immediate consequence of Theorem 5.1. Given any cube Q, we have

$$|A_Q(f_1, f_2)(x)| \leq \mathcal{M}(f_1, f_2)(x).$$

Therefore, given weights w_1, w_2 such that the boundedness condition (2.2) holds, we immediately have that (5.1) holds, and so $\vec{w} \in \mathcal{A}_{\vec{p}(\cdot)}$.

Remark 6.1. The proof that the $\mathcal{A}_{\vec{p}(\cdot)}$ condition is necessary does not require us to assume that the exponents are log-Hölder continuous.

The proof of the sufficiency of the $\mathcal{A}_{\vec{p}(\cdot)}$ condition is considerably more complicated. Fix $p_1(\cdot), p_2(\cdot) \in \mathcal{P}$ such that $(p_j)_- > 1$ and $p_j(\cdot) \in LH$, $j = 1, 2$. Let w_1, w_2 be such that $\vec{w} \in \mathcal{A}_{\vec{p}(\cdot)}$.

We begin with a series of reductions. First, for $t \in \{0, 1/3\}^n$, define

$$\mathcal{D}_t = \{2^{-k}([0,1)^n + j + (-1)^k t) : k \in \mathbb{Z},\ j \in \mathbb{Z}^n\}.$$

Each $\mathcal{D}_t$ is a "1/3" translate of the standard dyadic grid, and has exactly the same properties as $\mathcal{D}_0$ defined above. (Note that the two definitions agree when $t = 0$.) Define the dyadic bilinear maximal operator

$$\mathcal{M}^{\mathcal{D}_t}(f_1, f_2)(x) = \sup_{Q \in \mathcal{D}_t} \fint_Q |f_1(y)|\, dy \fint_Q |f_2(y)|\, dy \chi_Q(x).$$

Then we have the following remarkable inequality: there exists a constant $C(n)$ such that

$$\mathcal{M}(f_1, f_2)(x) \leq C(n) \sum_{t \in \{0,1/3\}^n} \mathcal{M}^{\mathcal{D}_t}(f_1, f_2)(x).$$

This was first proved in [**14**]. (For the linear case, see also [**3**].)

Therefore, to prove that inequality (2.2) holds, it suffices to prove it with $\mathcal{M}$ replaced by $\mathcal{M}^{\mathcal{D}_t}$, and in fact it suffices to prove it for $\mathcal{M}^d = \mathcal{M}^{\mathcal{D}_0}$, since the same proof holds for any dyadic grid $\mathcal{D}_t$ with different constants, where the difference only depends on t. (Below we will describe where this difference arises.)

Second, we may assume that f, g are non-negative, bounded functions with compact support. It is clear from the definition of $\mathcal{M}^d$ that we may take them non-negative. To show the approximation, it suffices to note that given f_1, f_2, there exists a sequence of non-negative, bounded functions of compact support, g_k, h_k, that increase pointwise to f and g and such that

$$\lim_{k \to \infty} \mathcal{M}^d(g_k, h_k)(x) = \mathcal{M}^d(f_1, f_2)(x).$$

In the linear case this is proved in [**5**, Lemma 3.30] and the same proof (with the obvious changes) works in the bilinear case. The desired result then follows by Lemma 3.4.

Third, we restate the desired inequality in an equivalent fashion. Given an exponent $p(\cdot) \in \mathcal{P}_0$ and a weight v, define $L^{p(\cdot)}_v$ to be the quasi-Banach function space with norm

$$\|g\|_{L^{p(\cdot)}_v} := \inf\left\{\lambda > 0 : \int_{\mathbb{R}^n} \left(\frac{|g(x)|}{\lambda}\right)^{p(x)} v(x)\,dx \le 1\right\}.$$

In other words, $L^{p(\cdot)}_v$ is defined exactly as $L^{p(\cdot)}$ with Lebesgue measure replaced by the measure $v\,dx$. This norm has many of the same basic properties as the $L^{p(\cdot)}$ norm.

Let $u(\cdot) = w(\cdot)^{p(\cdot)}$ and $\sigma_l(\cdot) = w_l(\cdot)^{-p_l'(\cdot)}$, $l = 1, 2$. Then

$$(\sigma_l(x)w_l(x))^{p_l(x)} = (w_l(x)^{1-p_l'(x)})^{p_l(x)} = w_l(x)^{-p_l'(x)} = \sigma_l(x).$$

Therefore,

$$\|\mathcal{M}^d(f_1\sigma_1, f_2\sigma_2)\|_{L^{p(\cdot)}_u} = \|\mathcal{M}^d(f_1\sigma_1, f_2\sigma_2)w\|_{L^{p(\cdot)}},$$

and for $l = 1, 2$,

$$\|f_l\|_{L^{p_l(\cdot)}_{\sigma_l}} = \|f_l\sigma_l w_l\|_{L^{p_l(\cdot)}}.$$

Hence, it will suffice to prove that

$$\|\mathcal{M}^d(f_1\sigma_1, f_2\sigma_2)\|_{L^{p(\cdot)}_u} \lesssim \|f_1\|_{L^{p_1(\cdot)}_{\sigma_1}} \|f_2\|_{L^{p_2(\cdot)}_{\sigma_2}},$$

since if we replace f_l by f_l/σ_l, $l = 1, 2$, we get (2.2).

Finally, by homogeneity we may assume without loss of generality that $\|f_l\|_{L^{p_l(\cdot)}_{\sigma_l}} = 1$, $l = 1, 2$, which by Lemma 3.5 (which holds in this setting) implies that

$$\int_{\mathbb{R}^n} |f_l(x)|^{p_l(x)}\sigma_l(x)\,dx \le 1.$$

Thus it will suffice to prove that

$$\|\mathcal{M}^d(f_1\sigma_1, f_2\sigma_2)\|_{L^{p(\cdot)}_u} \lesssim 1,$$

which, again by Lemma 3.5, is equivalent to proving that

$$\int_{\mathbb{R}^n} \mathcal{M}^d(f_1\sigma_1, f_2\sigma_2)^{p(x)} u(x)\,dx \lesssim 1 \tag{6.1}$$

with a constant independent of f_l, $l = 1, 2$.

We now begin our main estimate, which is to prove that (6.1) holds. Define the functions

$$h_1 = f_1\chi_{\{f_1>1\}}, \quad h_2 = f_1\chi_{\{f_1\le 1\}},$$
$$h_3 = f_2\chi_{\{f_2>1\}}, \quad h_4 = f_2\chi_{\{f_2\le 1\}},$$

and for brevity define

$$\rho(1) = 1, \quad \rho(2) = 1, \quad \rho(3) = 2, \quad \rho(4) = 2.$$

Then we can write

$$\begin{aligned}\int_{\mathbb{R}^n} \mathcal{M}^d(f_1\sigma_1, f_2\sigma_2)(x)^{p(x)}u(x)\,dx \le &\int_{\mathbb{R}^n} \mathcal{M}^d(h_1\sigma_1, h_3\sigma_2)(x)^{p(x)}u(x)\,dx \\ &+\int_{\mathbb{R}^n} \mathcal{M}^d(h_1\sigma_1, h_4\sigma_2)(x)^{p(x)}u(x)\,dx \\ &+\int_{\mathbb{R}^n} \mathcal{M}^d(h_2\sigma_1, h_3\sigma_2)(x)^{p(x)}u(x)\,dx \\ &+\int_{\mathbb{R}^n} \mathcal{M}^d(h_2\sigma_1, h_4\sigma_2)(x)^{p(x)}u(x)\,dx \\ &= I_1 + I_2 + I_3 + I_4.\end{aligned}$$

We will estimate each term on the right separately. The integral I_1 is the "local" term and the estimate will use the LH_0 condition. The integral I_4 is the "global" term and the estimate will use the LH_∞ condition. The estimates of I_2 and I_3 involve both local and global estimates and are the most complicated: this is where our proof diverges most significantly from the linear case. Note, however, that the estimates for these integrals are the same (making the obvious changes) so we will only estimate I_2.

The estimate for I_1. We begin by forming the bilinear Calderón–Zygmund cubes associated with $\mathcal{M}^d(h_1\sigma_1, h_3\sigma_2)$. For the details of this decomposition, see [**22**]. Fix $a > 2^{2n}$ and for each $k \in \mathbb{Z}$ define

$$\Omega_k = \{x \in \mathbb{R}^n : \mathcal{M}^d(h_1\sigma_1, h_3\sigma_2)(x) > a^k\}.$$

Then $\Omega_k = \bigcup_j Q_j^k$ where $\{Q_j^k\}_{k,j}$ is a family of maximal dyadic cubes contained in Ω_k with the property that

$$a^k < \langle h_1\sigma_1\rangle_{Q_j^k}\langle h_3\sigma_2\rangle_{Q_j^k} \le a^{k+1}.$$

Moreover, since $\Omega_{k+1} \subset \Omega_k$, the sets $E_j^k = Q_j^k \backslash \Omega_{k+1}$ are pairwise disjoint and there exists $0 < \alpha < 1$ such that

$$\alpha|Q_j^k| < |E_j^k|.$$

By Corollary 4.9, u, and σ_l, $l = 1, 2$, are A_∞ weights, so by Lemma 3.1 there exists $0 < \beta < 1$ such that

$$\beta u(Q_j^k) \le u(E_j^k), \quad \beta \sigma_l(Q_j^k) \le \sigma_l(E_j^k).$$

We will use this fact repeatedly throughout the proof without further comment.

We can now estimate I_1 as follows:

$$\begin{aligned} I_1 &= \int_{\mathbb{R}^n} \mathcal{M}^d(h_1\sigma_1, h_3\sigma_2)(x)^{p(x)} u(x)\,dx \\ &\le \sum_{k=0}^{\infty} \int_{\Omega_k \setminus \Omega_{k+1}} a^{(k+1)p(x)} u(x)\,dx \\ &\lesssim \sum_{k,j} \int_{E_j^k} \prod_{l=1,3} \langle h_l \sigma_{\rho(l)} \rangle_{Q_j^k}^{p(x)} u(x)\,dx \\ &= \sum_{k,j} \int_{E_j^k} \prod_{l=1,3} \left(\int_{Q_j^k} h_l \sigma_{\rho(l)}\,dy \right)^{p(x)} |Q_j^k|^{-2p(x)} u(x)\,dx. \end{aligned}$$

Since $h_1(x) \geqslant 1$ or $h_1(x) = 0$, we have that

$$\begin{aligned} \int_{Q_j^k} h_1(y)\sigma_1(y)\,dy &\le \int_{Q_j^k} h_1(y)^{p_1(y)} \sigma_1(y)\,dy \\ &\le \int_{\mathbb{R}^n} f_1(y)^{p_1(y)} \sigma_1(y)\,dy \le 1. \end{aligned} \tag{6.2}$$

The same estimate holds for h_3. For each j, k define

$$\frac{1}{q(Q_j^k)} = \frac{1}{(p_1)_-(Q_j^k)} + \frac{1}{(p_2)_-(Q_j^k)},$$

and note that for $x \in Q_j^k$, $q(Q_j^k) \le p_-(Q_j^k) \le p(x)$. Thus,

$$\begin{aligned} I_1 &\le \sum_{k,j} \int_{E_j^k} \prod_{l=1,3} \left(\int_{Q_j^k} h_l(y)\sigma_{\rho(l)}(y)\,dy \right)^{q(Q_j^k)} |Q_j^k|^{-2p(x)} u(x)\,dx \\ &\le \sum_{k,j} \int_{E_j^k} \prod_{l=1,3} \left(\frac{1}{\sigma_{\rho(l)}(Q_j^k)} \int_{Q_j^k} h_l(y)^{\frac{p_l(y)}{(p_l)_-(Q_j^k)}} \sigma_{\rho(l)}(y)\,dy \right)^{q(Q_j^k)} \\ &\qquad \times \sigma_{\rho(l)}(Q_j^k)^{q(Q_j^k)} |Q_j^k|^{-2p(x)} u(x)\,dx. \end{aligned}$$

By Hölder's inequality with measure $\sigma_l\,dx$, for $l=1,3$,

$$\begin{aligned}
&\left(\frac{1}{\sigma_{\rho(l)}(Q_j^k)}\int_{Q_j^k} h_l(y)^{\frac{p_l(y)}{(p_l)_-(Q_j^k)}}\sigma_{\rho(l)}(y)\,dy\right)^{q(Q_j^k)}\\
&\qquad\le\left(\frac{1}{\sigma_{\rho(l)}(Q_j^k)}\int_{Q_j^k} h_l(y)^{\frac{p_l(y)}{(p_l)_-}}\sigma_{\rho(l)}(y)\,dy\right)^{(p_l)_-\frac{q(Q_j^k)}{(p_l)_-(Q_j^k)}} \qquad (6.3)\\
&\qquad=\langle h_l^{\frac{p_l(\cdot)}{(p_l)_-}}\rangle_{\sigma_{\rho(l)},Q}^{(p_l)_-\frac{q(Q_j^k)}{(p_l)_-(Q_j^k)}}.
\end{aligned}$$

Further, we claim that

$$\begin{aligned}
&\int_{E_j^k}\prod_{l=1,3}\sigma_{\rho(l)}(Q_j^k)^{q(Q_j^k)}|Q_j^k|^{-2p(x)}u(x)\,dx \qquad (6.4)\\
&\qquad\qquad\lesssim \sigma_1(Q_j^k)^{\frac{q(Q_j^k)}{(p_1)_-(Q_j^k)}}\sigma_2(Q_j^k)^{\frac{q(Q_j^k)}{(p_2)_-(Q_j^k)}}.
\end{aligned}$$

If we assume this for the moment, then we can argue as follows: since

$$1=\frac{q(Q_j^k)}{(p_1)_-(Q_j^k)}+\frac{q(Q_j^k)}{(p_2)_-(Q_j^k)},$$

by (6.3) and Young's inequality,

$$\begin{aligned}
I_1&\lesssim\sum_{k,j}\prod_{l=1,3}\langle h_l^{\frac{p_l(\cdot)}{(p_l)_-}}\rangle_{\sigma_{\rho(l)},Q}^{(p_l)_-\frac{q(Q_j^k)}{(p_l)_-(Q_j^k)}}\sigma_{\rho(l)}(Q_j^k)^{\frac{q(Q_j^k)}{(p_l)_-(Q_j^k)}}\\
&\le\sum_{k,j}\sum_{l=1,3}\langle h_l^{\frac{p_l(\cdot)}{(p_l)_-}}\rangle_{\sigma_{\rho(l)},Q}^{(p_l)_-}\sigma_{\rho(l)}(Q_j^k) \qquad (6.5)\\
&\lesssim\sum_{k,j}\sum_{l=1,3}\langle h_l^{\frac{p_l(\cdot)}{(p_l)_-}}\rangle_{\sigma_{\rho(l)},Q}^{(p_l)_-}\sigma_{\rho(l)}(E_j^k).
\end{aligned}$$

By Lemma 3.2, since $(p_l)_->1$,

$$\begin{aligned}
&\le\sum_{l=1,3}\int_{\mathbb{R}^n} M_{\sigma_{\rho(l)}}^d(h_l^{\frac{p_l(\cdot)}{(p_l)_-}})(x)^{(p_l)_-}\sigma_{\rho(l)}(x)\,dx\\
&\lesssim\sum_{l=1,3}\int_{\mathbb{R}^n} h_l(x)^{p_l(x)}\sigma_{\rho(l)}(x)\,dx\\
&\lesssim 1.
\end{aligned}$$

Therefore, to complete the estimate of I_1 we will prove (6.4). First, rewrite the left-hand side as follows:

$$\int_{E_j^k} \prod_{l=1,3} \sigma_{\rho(l)}(Q_j^k)^{q(Q_j^k)} |Q_j^k|^{-2p(x)} u(x)\, dx \le \prod_{l=1,3} \left(\frac{\sigma_{\rho(l)}(Q_j^k)}{\|w_{\rho(l)}^{-1}\chi_{Q_j^k}\|_{p_l'(\cdot)}} \right)^{q(Q_j^k)}$$

$$\times \int_{Q_j^k} \left(\prod_{l=1,3} \|w_{\rho(l)}^{-1}\chi_{Q_j^k}\|_{p_l'(\cdot)}^{q(Q_j^k)-p(x)} \right) \left(\prod_{l=1,3} \|w_{\rho(l)}^{-1}\chi_{Q_j^k}\|_{p_l'(\cdot)}^{p(x)} |Q_j^k|^{-2p(x)} u(x) \right) dx.$$

By the $\mathcal{A}_{\vec{p}(\cdot)}$ condition we have that there is a constant c such that

$$\|c|Q_j^k|^{-2} \prod_{l=1}^{2} \|w_l^{-1}\chi_{Q_j^k}\|_{p_l'(\cdot)} w\chi_{Q_j^k}\|_{p(\cdot)} \le 1,$$

which by Lemma 3.5 implies that

$$\int_{Q_j^k} \prod_{l=1}^{2} \|w_l^{-1}\chi_{Q_j^k}\|_{p_l'(\cdot)}^{p(x)} |Q_j^k|^{-2p(x)} u(x)\, dx \lesssim 1. \tag{6.6}$$

Hence, to prove (6.4) it will suffice to show that for $l = 1, 2$,

$$\left(\frac{\sigma_l(Q_j^k)}{\|w_l^{-1}\chi_{Q_j^k}\|_{p_l'(\cdot)}} \right)^{q(Q_j^k)} \lesssim \sigma_l(Q_j^k)^{\frac{q(Q_j^k)}{(p_l)_-(Q_j^k)}} \tag{6.7}$$

and

$$\|w_l^{-1}\chi_{Q_j^k}\|_{p_l'(\cdot)}^{q(Q_j^k)-p(x)} \lesssim 1. \tag{6.8}$$

We first prove (6.7). Suppose that $\|w_l^{-1}\chi_{Q_j^k}\|_{p_l'(\cdot)} > 1$. Then by Lemma 3.5, since $(p_l')_\pm(Q_j^k) = (p_l)_\mp(Q_j^k)'$, we have that

$$\left(\frac{\sigma_l(Q_j^k)}{\|w_l^{-1}\chi_{Q_j^k}\|_{p_l'(\cdot)}} \right)^{q(Q_j^k)} \le \left(\sigma_l(Q_j^k)^{1-\frac{1}{(p_l)_-(Q_j^k)'}} \right)^{q(Q_j^k)} = \sigma_l(Q_j^k)^{\frac{q(Q_j^k)}{(p_l)_-(Q_j^k)}}.$$

On the other hand, if $\|w_l^{-1}\chi_{Q_j^k}\|_{p_l'(\cdot)} \leq 1$,

$$\frac{\sigma_l(Q_j^k)}{\|w_l^{-1}\chi_{Q_j^k}\|_{p_l'(\cdot)}} \leq \sigma_l(Q_j^k)^{1-\frac{1}{(p_l)_+(Q_j^k)'}} = \sigma_l(Q_j^k)^{\frac{1}{(p_l)_+(Q_j^k)}}$$

$$= \sigma_l(Q_j^k)^{\frac{1}{(p_l)_-(Q_j^k)}} \sigma_l(Q_j^k)^{\frac{1}{(p_l)_+(Q_j^k)}-\frac{1}{(p_l)_-(Q_j^k)}}.$$

Again by Lemma 3.5, and then by Lemma 3.3 and Lemma 4.5,

$$\begin{aligned}
\sigma_l(Q_j^k)^{\frac{1}{(p_l)_+(Q_j^k)}-\frac{1}{(p_l)_-(Q_j^k)}} &\leq \|w_l^{-\frac{1}{2}}\chi_{Q_j^k}\|_{2p_l'(\cdot)}^{[2(p_l')_-]\left(\frac{1}{(p_l)_+(Q_j^k)}-\frac{1}{(p_l)_-(Q_j^k)}\right)} \\
&= \|w_l^{-\frac{1}{2}}\chi_{Q_j^k}\|_{2p_l'(\cdot)}^{[2(p_l')_-]\left(1-\frac{1}{(p_l)_+(Q_j^k)'}-1+\frac{1}{(p_l)_-(Q_j^k)'}\right)} \\
&= \|w_l^{-\frac{1}{2}}\chi_{Q_j^k}\|_{2p_l'(\cdot)}^{[2(p_l')_-]\left(\frac{1}{(p_l')_+(Q_j^k)}-\frac{1}{(p_1')_-(Q_j^k)}\right)} \\
&\leqslant \|w_l^{-\frac{1}{2}}\chi_{Q_j^k}\|_{2p_l'(\cdot)}^{c[(2p_l')_-(Q_j^k)-(2p_l')_+(Q_j^k)]} \\
&\lesssim 1.
\end{aligned}$$

Hence,

$$\left(\frac{\sigma_1(Q_j^k)}{\|w_1^{-1}\chi_{Q_j^k}\|_{p_1'(\cdot)}}\right)^{q(Q_j^k)} \lesssim \sigma_1(Q_j^k)^{\frac{q(Q_j^k)}{(p_1)_-(Q_j^k)}}. \tag{6.9}$$

We now prove (6.8). If $\|w_l^{-1}\chi_{Q_j^k}\|_{p_l'(\cdot)} \geq 1$, then this is immediate. If $\|w_l^{-1}\chi_{Q_j^k}\|_{p_l'(\cdot)} < 1$, then by Lemma 3.3, and then by Propositions 4.7 and 4.10,

$$\|w_l^{-1}\chi_{Q_j^k}\|_{p_l'(\cdot)}^{q(Q_j^k)-p(x)} = \|w_l^{-\frac{1}{2}}\chi_{Q_j^k}\|_{2p_l'(\cdot)}^{2q(Q_j^k)-2p(x)} \lesssim 1.$$

This completes the estimate of I_1.

The estimate for I_2. We first form the bilinear Calderón–Zygmund cubes associated with $\mathcal{M}^d(h_1\sigma_1, h_4\sigma_2)$ and we use the same notation as we did in the estimate for I_1. To estimate this term I_2 we need to divide the cubes Q_j^k into three sets: small cubes close to the origin, large cubes close to the origin, and cubes (of all sizes) far from the origin. To make this precise, let $\{P_i\}_{i=1}^{2^n}$ be the 2^n dyadic cubes adjacent to the origin, $|P_i| \geq 1$, that are so large that if Q is any dyadic cube equal to or adjacent to P_i in the same quadrant, and $|P_i| = |Q|$, then $u(Q) \geq 1$ and $\sigma_l(Q) \geq 1$, $l = 1, 2$. The existence of such cubes follows from Lemma 3.1

and Corollary 4.9. Let $P = \bigcup_i P_i$. We can then partition the cubes$\{Q_j^k\}$ into three disjoint sets:

$$\mathscr{F} = \{(k,j) : Q_j^k \subset P_i \text{ for some } i\},$$
$$\mathscr{G} = \{(k,j) : P_i \subset Q_j^k \text{ for some } i\},$$
$$\mathscr{H} = \{(k,j) : Q_j^k \cap P_i = \emptyset \text{ for all } i\}.$$

We now estimate I_2, arguing as we did at the beginning of the estimate for I_1:

$$\begin{aligned}\int_{\mathbb{R}^n} \mathcal{M}^d(h_1\sigma_1, h_4\sigma_2)(x)^{p(x)} u(x)\,dx &\lesssim \sum_{k,j} \int_{E_j^k} \prod_{l=1,4} \langle h_l \sigma_{\rho(l)}\rangle_{Q_j^k}^{p(x)} u(x)\,dx \\ &= \sum_{(k,j)\in\mathscr{F}} + \sum_{(k,j)\in\mathscr{G}} + \sum_{(k,j)\in\mathscr{H}} \\ &= J_1 + J_2 + J_3.\end{aligned}$$

We will estimate each sum in turn.

Remark 6.2. Throughout the rest of this proof, we will allow the implicit constants to depend on $\sigma_l(P)$ or $u(P)$. The choice of the P_i is the one place where the proof depends on the fact that we are working with the dyadic grid $\mathcal{D}_0$. For the grids $\mathcal{D}_t$ we will replace the origin by its translate $\pm t$, where the sign will depend on the scale at which we choose the P_i. See Remark 6.3 below for where the dyadic grid and the choice of the P_i affects the proof.

The estimate for J_1. Since $h_4 \leq 1$ and $p_+ < \infty$, we have that

$$\begin{aligned}J_1 &= \sum_{(k,j)\in\mathscr{F}} \int_{E_j^k} \prod_{l=1,4} \langle h_l \sigma_{\rho(l)}\rangle_{Q_j^k}^{p(x)} u(x)\,dx \\ &\leq \sum_{(k,j)\in\mathscr{F}} \int_{E_j^k} \langle h_1\sigma_1\rangle_{Q_j^k}^{p(x)} \langle \sigma_2\rangle_{Q_j^k}^{p(x)} u(x)\,dx \\ &= \sum_{(k,j)\in\mathscr{F}} \int_{E_j^k} \left(\int_{Q_j^k} h_1\sigma_1\,dy\right)^{p(x)} \\ &\qquad\qquad \times \sigma_2(Q_j^k)^{p(x)-q(Q_j^k)} \sigma_2(Q_j^k)^{q(Q_j^k)} |Q_j^k|^{-2p(x)} u(x)\,dx;\end{aligned}$$

by inequalities (6.2) and (6.3),

$$\leq \sum_{(k,j)\in\mathscr{F}} \int_{E_j^k} \left(\int_{Q_j^k} h_1\sigma_1\,dy\right)^{q(Q_j^k)} \times \sigma_2(Q_j^k)^{p(x)-q(Q_j^k)}\sigma_2(Q_j^k)^{q(Q_j^k)}|Q_j^k|^{-2p(x)}u(x)\,dx$$

$$= \sum_{(k,j)\in\mathscr{F}} \int_{E_j^k} \langle h_1\rangle_{\sigma_1,Q_j^k}^{q(Q_j^k)}\sigma_2(Q_j^k)^{p(x)-q(Q_j^k)} \times \sigma_1(Q_j^k)^{q(Q_j^k)}\sigma_2(Q_j^k)^{q(Q_j^k)}|Q_j^k|^{-2p(x)}u(x)\,dx$$

$$\leq \sum_{(k,j)\in\mathscr{F}} (\sigma_2(Q_j^k)+1)^{p_+(Q_j^k)-q(Q_j^k)}\langle h_1^{\frac{p_1(\cdot)}{(p_1)_-}}\rangle_{\sigma_1,Q_j^k}^{(p_1)_-\frac{q(Q_j^k)}{(p_1)_-(Q_j^k)}} \times \int_{E_j^k}\sigma_1(Q_j^k)^{q(Q_j^k)}\sigma_2(Q_j^k)^{q(Q_j^k)}|Q_j^k|^{-2p(x)}u(x)\,dx.$$

Let q_- be defined as in (4.3). By (6.4) we can estimate the integral:

$$\lesssim (\sigma_2(P)+1)^{p_+-q_-}\sum_{(k,j)\in\mathscr{F}} \langle h_1^{\frac{p_1(\cdot)}{(p_1)_-}}\rangle_{\sigma_1,Q_j^k}^{(p_1)_-\frac{q(Q_j^k)}{(p_1)_-(Q_j^k)}} \times \sigma_1(Q_j^k)^{\frac{q(Q_j^k)}{(p_1)_-(Q_j^k)}}\sigma_2(Q_j^k)^{\frac{q(Q_j^k)}{(p_2)_-(Q_j^k)}}.$$

Therefore, by Young's inequality and by Lemma 3.2,

$$\leq (\sigma_2(P)+1)^{p_+-q_-}\sum_{(k,j)\in\mathscr{F}} [\langle h_1^{\frac{p_1(\cdot)}{(p_1)_-}}\rangle_{\sigma_1,Q_j^k}^{(p_1)_-}\sigma_1(Q_j^k)+\sigma_2(Q_j^k)]$$

$$\lesssim (\sigma_2(P)+1)^{p_+-q_-}\sum_{(k,j)\in\mathscr{F}} [\langle h_1^{\frac{p_1(\cdot)}{(p_1)_-}}\rangle_{\sigma_1,Q_j^k}^{(p_1)_-}\sigma_1(E_j^k)+\sigma_2(E_j^k)]$$

$$\lesssim \sum_{(k,j)\in\mathscr{F}}\int_{E_j^k}\mathcal{M}_{\sigma_1}^d(f_1^{\frac{p_1(\cdot)}{(p_1)_-}})(x)^{(p_1)_-}\sigma_1(x)\,dx + \sum_{(k,j)\in\mathscr{F}}\sigma_2(E_j^k)$$

$$\lesssim \int_{\mathbb{R}^n} f_1(x)^{p_1(x)}\sigma_1(x)\,dx + \sigma_2(P)$$

$$\lesssim 1.$$

The estimate for J_2. Given $(k,j)\in\mathscr{G}$, since $P_i\subset Q_j^k$ we have that $1\le\sigma_2(P_i)\le\sigma_2(Q_j^k)$. Therefore, by Lemma 4.3 applied twice to $w_2^{-\frac{1}{2}}\in\mathcal{A}_{2p_2'(\cdot)}$, we get

$$\frac{1}{|Q_j^k|}\le\frac{|P_i|}{|Q_j^k|}\lesssim\left(\frac{\sigma_2(P_i)}{\sigma_2(Q_j^k)}\right)^{\frac{1}{2(p_2')_\infty}}\lesssim\sigma_2(Q_j^k)^{-\frac{1}{2(p_2')_\infty}}$$
$$\lesssim\|w_2^{-\frac{1}{2}}\chi_{Q_j^k}\|_{2p_2'(\cdot)}^{-1}\lesssim\|w_2^{-1}\chi_{Q_j^k}\|_{p_2'(\cdot)}^{-\frac{1}{2}}.$$

Hence, by Lemma 3.6,

$$\begin{aligned}\frac{1}{|Q_j^k|^2}\int_{Q_j^k}h_4(y)\sigma_2(y)\,dy&\lesssim\|w_2^{-1}\chi_{Q_j^k}\|_{p_2'(\cdot)}^{-1}\int_{Q_j^k}h_4(y)\sigma_2(y)^{\frac{1}{p_2(y)}}\sigma_2(y)^{\frac{1}{p_2'(y)}}\,dy\\&\lesssim\|w_2^{-1}\chi_{Q_j^k}\|_{p_2'(\cdot)}^{-1}\|h_4\|_{L^{p_2(\cdot)}_{\sigma_2}}\|\chi_{Q_j^k}\|_{L^{p_2'(\cdot)}_{\sigma_2}}\\&\le\|w_2^{-1}\chi_{Q_j^k}\|_{p_2'(\cdot)}^{-1}\|f_2\|_{L^{p_2(\cdot)}_{\sigma_2}}\|w_2^{-1}\chi_{Q_j^k}\|_{p_2'(\cdot)}\\&\le c_0.\end{aligned}$$

We can now estimate J_2. By inequality (6.2) and Lemmas 3.13 and 4.6, there exists $t>1$ such that

$$\begin{aligned}J_2&=\sum_{(k,j)\in\mathscr{G}}\int_{E_j^k}c_0^{p(x)}\Big(\int_{Q_j^k}h_1\sigma_1\,dy\Big)^{p(x)}\Big(\frac{c_0^{-1}}{|Q_j^k|^2}\int_{Q_j^k}h_4\sigma_2\,dy\Big)^{p(x)}u(x)\,dx\\&\lesssim\sum_{(k,j)\in\mathscr{G}}c_0^{p_+}\int_{E_j^k}\Big(\int_{Q_j^k}h_1\sigma_1\,dy\Big)^{p_\infty}\Big(\frac{c_0^{-1}}{|Q_j^k|^2}\int_{Q_j^k}h_4\sigma_2\,dy\Big)^{p_\infty}u(x)\,dx\\&\quad+\sum_{(k,j)\in\mathscr{G}}\int_{E_j^k}\frac{u(x)}{(e+|x|)^{ntp_-}}\,dx\\&\lesssim\sum_{(k,j)\in\mathscr{G}}\prod_{l=1,4}\langle h_l\rangle^{p_\infty}_{\sigma_{\rho(l)},Q_j^k}\sigma_1(Q_j^k)^{p_\infty}\sigma_2(Q_j^k)^{p_\infty}|Q_j^k|^{-2p_\infty}u(E_j^k)+1.\end{aligned}$$

We estimate each term in the product separately. First, we claim

$$\sigma_1(Q_j^k)^{p_\infty}\sigma_2(Q_j^k)^{p_\infty}|Q_j^k|^{-2p_\infty}u(E_j^k)\lesssim\sigma_1(Q_j^k)^{\frac{p_\infty}{(p_1)_\infty}}\sigma_2(Q_j^k)^{\frac{p_\infty}{(p_2)_\infty}}.\tag{6.10}$$

Since $\sigma_l(Q_j^k), u(Q_j^k) \geq 1$, by Lemma 4.3 (applied several times) and the definition of $\mathcal{A}_{\vec{p}(\cdot)}$, we have

$$\begin{aligned}
\Big[\sigma_1(Q_j^k)\sigma_2(Q_j^k)\Big]^{p_\infty} &\lesssim \Big(\|w_1^{-\frac{1}{2}}\chi_{Q_j^k}\|_{2p_1'(\cdot)}^{2(p_1')_\infty}\|w_2^{-\frac{1}{2}}\chi_{Q_j^k}\|_{2p_2'(\cdot)}^{2(p_2')_\infty}\Big)^{p_\infty} \\
&= \Big(\|w_1^{-1}\chi_{Q_j^k}\|_{p_1'(\cdot)}^{(p_1')_\infty-1}\|w_2^{-1}\chi_{Q_j^k}\|_{p_2'(\cdot)}^{(p_2')_\infty-1}\Big)^{p_\infty}\Big(\prod_{l=1}^{2}\|w_l^{-1}\chi_{Q_j^k}\|_{p_l'(\cdot)}\Big)^{p_\infty} \\
&\lesssim \Big(\|w_1^{-1}\chi_{Q_j^k}\|_{p_1'(\cdot)}^{(p_1')_\infty-1}\|w_2^{-1}\chi_{Q_j^k}\|_{p_2'(\cdot)}^{(p_2')_\infty-1}\Big)^{p_\infty}\Big(\frac{|Q_j^k|^2}{\|w\chi_{Q_j^k}\|_{p(\cdot)}}\Big)^{p_\infty} \\
&\lesssim \Big(\sigma_1(Q_j^k)^{\frac{(p_1')_\infty-1}{(p_1')_\infty}}\sigma_2(Q_j^k)^{\frac{(p_2')_\infty-1}{(p_2')_\infty}}\Big)^{p_\infty}\frac{|Q_j^k|^{2p_\infty}}{u(Q_j^k)} \\
&\leq \sigma_1(Q_j^k)^{\frac{p_\infty}{(p_1)_\infty}}\sigma_2(Q_j^k)^{\frac{p_\infty}{(p_2)_\infty}}\frac{|Q_j^k|^{2p_\infty}}{u(E_j^k)}.
\end{aligned}$$

This proves (6.10).

Second, by Lemma 3.6 and again by Lemma 4.3 we have

$$\begin{aligned}
\frac{1}{\sigma_1(Q_j^k)}\int_{Q_j^k} h_1(y)\sigma_1(y)\,dy &\lesssim \sigma_1(Q_j^k)^{-1}\|h_1\|_{L_{\sigma_1}^{p_1(\cdot)}}\|\chi_{Q_j^k}\|_{L_{\sigma_1}^{p_1'(\cdot)}} \\
&\leq \sigma_1(Q_j^k)^{-1}\|f\|_{L_{\sigma_1}^{p_1(\cdot)}}\|w_1^{-1}\chi_{Q_j^k}\|_{p_1'(\cdot)} \\
&\lesssim \sigma_1(Q_j^k)^{\frac{1}{(p_1')_\infty}-1} \\
&\leq \sigma_1(P_i)^{-\frac{1}{(p_1)_\infty}} \lesssim 1.
\end{aligned} \tag{6.11}$$

We can now continue the estimate of J_2. Since

$$1 = \frac{p_\infty}{(p_1)_\infty} + \frac{p_\infty}{(p_2)_\infty},$$

by (6.10) and Young's inequality,

$$\begin{aligned}
J_2 &\lesssim \sum_{(k,j)\in\mathscr{G}}\prod_{l=1,4}\langle h_l\rangle_{\sigma_{\rho(l)},Q_j^k}^{p_\infty}\sigma_{\rho(l)}(Q_j^k)^{\frac{p_\infty}{(p_l)_\infty}} + 1 \\
&\lesssim \sum_{(k,j)\in\mathscr{G}}\langle h_1\rangle_{\sigma_1,Q_j^k}^{(p_1)_\infty}\sigma_1(Q_j^k) + \sum_{(k,j)\in\mathscr{G}}\langle h_4\rangle_{\sigma_2,Q_j^k}^{(p_2)_\infty}\sigma_2(Q_j^k) + 1 \\
&\lesssim \sum_{(k,j)\in\mathscr{G}}\langle c_0^{-1}h_1\rangle_{\sigma_1,Q_j^k}^{(p_1)_\infty}\sigma_1(E_j^k) + \sum_{(k,j)\in\mathscr{G}}\langle h_4\rangle_{\sigma_2,Q_j^k}^{(p_2)_\infty}\sigma_2(E_j^k) + 1;
\end{aligned} \tag{6.12}$$

by Lemmas 3.13 and 4.6 there exists $t > 1$ such that,

$$\begin{aligned}
&\lesssim \sum_{(k,j)\in\mathscr{G}} \int_{E_j^k} \langle c_0^{-1} h_1 \rangle_{\sigma_1, Q_j^k}^{p_1(x)} \sigma_1(x)\, dx \\
&\quad + \sum_{(k,j)\in\mathscr{G}} \int_{E_j^k} \frac{\sigma_1(x)}{(e+|x|)^{tn(p_1)_-}}\, dx \\
&\quad + \sum_{(k,j)\in\mathscr{G}} \int_{E_j^k} M_{\sigma_2}^d h_4(x)^{(p_2)_\infty} \sigma_2(x)\, dx + 1 \\
&\lesssim \sum_{(k,j)\in\mathscr{G}} \int_{E_j^k} \langle c_0^{-1} h_1^{\frac{p_1(\cdot)}{(p_1)_-}} \rangle_{\sigma_1, Q_j^k}^{(p_1)_-} \sigma_1(x)\, dx \\
&\quad + \int_{\mathbb{R}^n} M_{\sigma_2}^d h_4(x)^{(p_2)_\infty} \sigma_2(x)\, dx + 1;
\end{aligned}$$

by Lemma 3.2 applied twice,

$$\begin{aligned}
&\lesssim \int_{\mathbb{R}^n} M_{\sigma_1}^d (h_1^{\frac{p_1(\cdot)}{(p_1)_-}})(x)^{(p_1)_-} \sigma_1(x)\, dx \\
&\quad + \int_{\mathbb{R}^n} h_4(x)^{(p_2)_\infty} \sigma_2(x)\, dx + 1 \\
&\lesssim \int_{\mathbb{R}^n} h_1(x)^{p_1(x)} \sigma_1(x)\, dx + \int_{\mathbb{R}^n} h_4(x)^{(p_2)_\infty} \sigma_2(x)\, dx + 1 \\
&\lesssim \int_{\mathbb{R}^n} h_4(x)^{(p_2)_\infty} \sigma_2(x)\, dx + 1.
\end{aligned}$$

Finally, we again apply Lemmas 3.13 and 4.6 to get

$$\begin{aligned}
&\lesssim \int_{\mathbb{R}^n} h_4(x)^{p_2(x)} \sigma_2(x)\, dx + \int_{\mathbb{R}^n} \frac{\sigma_2(x)}{(e+|x|)^{tn(p_2)_-}}\, dx + 1 \\
&\lesssim 1,
\end{aligned}$$

which completes the estimate for J_2.

The estimate for J_3. If Q_j^k is such that $(k,j) \in \mathscr{H}$, then Q_j^k does not contain the origin. Since it is a dyadic cube, we have that $\operatorname{dist}(Q_j^k, 0) \geq$

$\ell(Q_j^k)$. Therefore, there exists a constant $R > 1$ depending only on n such that

$$\sup_{x\in Q_j^k} |x| \leq R \inf_{x\in Q_j^k} |x|. \tag{6.13}$$

Remark 6.3. The estimate $\operatorname{dist}(Q_j^k, 0) \geq \ell(Q_j^k)$ holds because we are working with the grid $\mathcal{D}_0$. For an arbitrary grid $\mathcal{D}_t$, since the origin will be contained in one of the cubes P_i, we will have that for some $c > 0$, $\operatorname{dist}(Q_j^k, 0) \geq c\ell(Q_j^k)$, and so (6.13) will hold with a possibly larger constant R.

By the continuity of $p(\cdot)$, there exists x_+ in the closure of Q_j^k such that $p_+(Q_j^k) = p(x_+)$. Hence, since $p(\cdot) \in LH$, for all $x \in Q_j^k$, by (6.13),

$$\begin{aligned} 0 \leq p_+(Q_j^k) - p(x) &\leq |p(x_+) - p(x)| + |p(x) - p_\infty| \\ &\leq \frac{C_\infty}{\log(e+|x_+|)} + \frac{C_\infty}{\log(e+|x|)} \lesssim \frac{1}{\log(e+|x|)}. \end{aligned} \tag{6.14}$$

In the same way, for $l = 1, 2$ we have that $p_l(\cdot)$ satisfies

$$|(p_l)_-(Q_j^k) - p_l(x)| \lesssim \frac{1}{\log(e+|x|)}. \tag{6.15}$$

To estimate J_3 we need to divide $\mathscr{H}$ into two subsets depending on the size of the cubes Q_j^k with respect to σ_2:

$$\mathscr{H}_1 = \{(k,j) \in \mathscr{H} : \sigma_2(Q_j^k) \leq 1\}, \quad \mathscr{H}_2 = \{(k,j) \in \mathscr{H} : \sigma_2(Q_j^k) > 1\}.$$

We first estimate the sum over $\mathscr{H}_1$. By (6.14) and Lemmas 3.13 and 4.6,

$$\begin{aligned} &\sum_{(k,j)\in\mathscr{H}_1} \int_{E_j^k} \prod_{l=1,4} \langle h_l \sigma_{\rho(l)} \rangle_{Q_j^k}^{p(x)} u(x)\,dx \\ &\quad\lesssim \sum_{(k,j)\in\mathscr{H}_1} \int_{E_j^k} \prod_{l=1,4} \langle h_l \sigma_{\rho(l)} \rangle_{Q_j^k}^{p_+(Q_j^k)} u(x)\,dx + \sum_{(k,j)\in\mathscr{H}_1} \int_{E_j^k} \frac{u(x)}{(e+|x|)^{tnp_-}}\,dx \\ &\quad\leq \sum_{(k,j)\in\mathscr{H}_1} \int_{E_j^k} \prod_{l=1,4} \langle h_l \sigma_{\rho(l)} \rangle_{Q_j^k}^{p_+(Q_j^k)} u(x)\,dx + 1. \end{aligned}$$

By Lemma 3.11, (6.2), and (6.4), and since $h_1 \geq 1$, $h_4 \leq 1$, and $\sigma_2(Q_j^k) \leq 1$,

$$
\begin{aligned}
&= \sum_{(k,j)\in\mathscr{H}_1} \int_{E_j^k} \Big(\int_{Q_j^k} h_1\sigma_1\,dy\Big)^{p_+(Q_j^k)} \Big(\frac{1}{\sigma_2(Q_j^k)}\int_{Q_j^k} h_4\sigma_2\,dy\Big)^{p_+(Q_j^k)} \\
&\qquad\qquad \times |Q_j^k|^{-2p_+(Q_j^k)}\sigma_2(Q_j^k)^{p_+(Q_j^k)}u(x)\,dx + 1 \\
&\lesssim \sum_{(k,j)\in\mathscr{H}_1} \langle h_1\rangle_{\sigma_1,Q_j^k}^{q(Q_j^k)} \langle h_4\rangle_{\sigma_2,Q_j^k}^{q(Q_j^k)} \\
&\qquad\qquad \times \int_{E_j^k} |Q_j^k|^{-2p(x)}\sigma_1(Q_j^k)^{q(Q_j^k)}\sigma_2(Q_j^k)^{q(Q_j^k)}u(x)\,dx + 1 \\
&\lesssim \sum_{(k,j)\in\mathscr{H}_1} \prod_{l=1,4} \langle h_l\rangle_{\sigma_{\rho(l)},Q_j^k}^{q(Q_j^k)} \sigma_1(Q_j^k)^{\frac{q(Q_j^k)}{(p_1)_-(Q_j^k)}} \sigma_2(Q_j^k)^{\frac{q(Q_j^k)}{(p_2)_-(Q_j^k)}} + 1 \\
&\leq \sum_{(k,j)\in\mathscr{H}_1} \langle h_1^{\frac{p_1(\cdot)}{(p_1)_-(Q_j^k)}}\rangle_{\sigma_1,Q_j^k}^{q(Q_j^k)} \sigma_1(Q_j^k)^{\frac{q(Q_j^k)}{(p_1)_-(Q_j^k)}} \langle h_4\rangle_{\sigma_2,Q_j^k}^{q(Q_j^k)} \sigma_2(Q_j^k)^{\frac{q(Q_j^k)}{(p_2)_-(Q_j^k)}} + 1;
\end{aligned}
$$

by Hölder's inequality and Young's inequality,

$$
\begin{aligned}
&\leq \sum_{(k,j)\in\mathscr{H}_1} \langle h_1^{\frac{p_1(\cdot)}{(p_1)_-}}\rangle_{\sigma_1,Q_j^k}^{(p_1)_-\frac{q(Q_j^k)}{(p_1)_-(Q_j^k)}} \\
&\qquad\qquad \times \sigma_1(Q_j^k)^{\frac{q(Q_j^k)}{(p_1)_-(Q_j^k)}} \langle h_4\rangle_{\sigma_2,Q_j^k}^{q(Q_j^k)} \sigma_2(Q_j^k)^{\frac{q(Q_j^k)}{(p_2)_-(Q_j^k)}} + 1 \\
&\lesssim \sum_{(k,j)\in\mathscr{H}_1} \langle h_1^{\frac{p_1(\cdot)}{(p_1)_-}}\rangle_{\sigma_1,Q_j^k}^{(p_1)_-} \sigma_1(Q_j^k) + \sum_{(k,j)\in\mathscr{H}_1} \langle h_4\rangle_{\sigma_2,Q_j^k}^{(p_2)_-(Q_j^k)} \sigma_2(Q_j^k) + 1 \\
&= K_1 + K_2 + 1.
\end{aligned}
$$

The proof that K_1 is bounded is exactly the same as the final estimate for I_1, beginning at (6.5). Therefore, to complete the estimate for the sum over $\mathscr{H}_1$, we need to bound K_2. By Lemmas 3.13 and 4.6 (applied twice) and by Lemma 3.2,

$$
\begin{aligned}
K_2 &\lesssim \sum_{(k,j)\in\mathscr{H}_1} \int_{E_j^k} \langle h_4\rangle_{\sigma_2,Q_j^k}^{(p_2)_-(Q_j^k)} \sigma_2(x)\,dx \\
&\lesssim \sum_{(k,j)\in\mathscr{H}_1} \int_{E_j^k} \langle h_4\rangle_{\sigma_2,Q_j^k}^{(p_2)_\infty} \sigma_2(x)\,dx + \sum_{(k,j)\in\mathscr{H}_1} \int_{E_j^k} \frac{\sigma_2(x)}{(e+|x|)^{nt(p_2)_-}}\,dx \\
&\leq \int_{\mathbb{R}^n} M_{\sigma_2}^d h_4(x)^{(p_2)_\infty}\sigma_2(x)\,dx + \int_{\mathbb{R}^n} \frac{\sigma_2(x)}{(e+|x|)^{nt(p_2)_-}}\,dx
\end{aligned}
$$

$$
\begin{aligned}
&\lesssim \int_{\mathbb{R}^n} h_4(x)^{(p_2)_\infty} \sigma_2(x)\,dx + 1 \\
&\lesssim \int_{\mathbb{R}^n} h_4(x)^{p_2(x)} \sigma_2(x)\,dx + \int_{\mathbb{R}^n} \frac{\sigma_2(x)}{(e+|x|)^{nt(p_2)_-}}\,dx + 1 \\
&\lesssim 1.
\end{aligned}
$$

To estimate the sum over $\mathscr{H}_2$, first note that by Lemma 3.6 we have

$$
\begin{aligned}
\int_{Q_j^k} h_1\sigma_1\,dy &\lesssim \|h_1\|_{L^{p_1(\cdot)}_{\sigma_1}} \|\chi_{Q_j^k}\|_{L^{p_1'(\cdot)}_{\sigma_1}} \\
&\lesssim \|f_1\|_{L^{p_1(\cdot)}_{\sigma_1}} \|w_1^{-1}\chi_{Q_j^k}\|_{p_1'(\cdot)} \leq c_0 \|w_1^{-1}\chi_{Q_j^k}\|_{p_1'(\cdot)};
\end{aligned}
\tag{6.16}
$$

similarly, we have

$$
\int_{Q_j^k} h_4\sigma_2\,dy \leq c_0 \|w_2^{-1}\chi_{Q_j^k}\|_{p_2'(\cdot)}.
\tag{6.17}
$$

We now divide the cubes in $\mathscr{H}_2$ into two subsets depending on the size of $\sigma_1(Q_j^k)$:

$$
\mathscr{H}_{2a} = \{(k,j) \in \mathscr{H}_2 : \sigma_1(Q_j^k) \geq 1\}, \quad \mathscr{H}_{2b} = \{(k,j) \in \mathscr{H}_2 : \sigma_1(Q_j^k) < 1\}.
$$

We first estimate the sum over $\mathscr{H}_{2a}$. Given (6.16) and (6.17), by Lemma 3.13,

$$
\begin{aligned}
&\sum_{(k,j)\in\mathscr{H}_{2a}} \int_{E_j^k} \prod_{l=1,4} \langle h_l \sigma_{\rho(l)}\rangle_{Q_j^k}^{p(x)} u(x)\,dx \\
&\quad\leq c_0^{2p_+} \sum_{(k,j)\in\mathscr{H}_{2a}} \int_{E_j^k} \prod_{l=1,4} \bigg(c_0^{-1} \|w_{\rho(l)}^{-1}\chi_{Q_j^k}\|_{p'_{\rho(l)}(\cdot)}^{-1} \int_{Q_j^k} h_l\sigma_{\rho(l)}\,dy \bigg)^{p(x)} \\
&\qquad\qquad \times \prod_{l=1}^{2} \bigg(\frac{\|w_l^{-1}\chi_{Q_j^k}\|_{p_l'(\cdot)}}{|Q_j^k|} \bigg)^{p(x)} u(x)\,dx \\
&\quad\lesssim \sum_{(k,j)\in\mathscr{H}_{2a}} \int_{E_j^k} \prod_{l=1,4} \bigg(c_0^{-1} \|w_{\rho(l)}^{-1}\chi_{Q_j^k}\|_{p'_{\rho(l)}(\cdot)}^{-1} \int_{Q_j^k} h_l\sigma_{\rho(l)}\,dy \bigg)^{p_\infty} \\
&\qquad\qquad \times \prod_{l=1}^{2} \bigg(\frac{\|w_l^{-1}\chi_{Q_j^k}\|_{p_l'(\cdot)}}{|Q_j^k|} \bigg)^{p(x)} u(x)\,dx \\
&\qquad + \sum_{(k,j)\in\mathscr{H}_{2a}} \int_{E_j^k} \prod_{l=1}^{2} \bigg(\frac{\|w_l^{-1}\chi_{Q_j^k}\|_{p_l'(\cdot)}}{|Q_j^k|} \bigg)^{p(x)} \frac{u(x)}{(e+|x|)^{tnp_-}}\,dx \\
&\quad= L_1 + L_2.
\end{aligned}
$$

We first estimate L_2. Since $\sigma_2(E_j^k) \gtrsim \sigma_2(Q_j^k) \geqslant 1$, by (6.13), (6.6), and Lemma 4.6,

$$\begin{aligned}
L_2 &\le \sum_{(k,j)\in\mathscr{H}_{2a}} \sup_{x\in Q_j^k} (e+|x|)^{-ntp_-} \int_{Q_j^k} \prod_{l=1}^{2} \|w_l^{-1}\chi_{Q_j^k}\|_{p_l'(\cdot)}^{p(x)} |Q_j^k|^{-2p(x)} u(x)\,dx \\
&\lesssim \sum_{(k,j)\in\mathscr{H}_{2a}} \inf_{x\in Q_j^k} (e+|x|)^{-ntp_-} \sigma_2(E_j^k) \\
&\lesssim \int_{\mathbb{R}^n} \frac{\sigma_2(x)}{(e+|x|)^{ntp_-}}\,dx \\
&\lesssim 1.
\end{aligned}$$

In order to estimate L_1 we first note that for $l = 1, 2$, since $\sigma_l(Q_j^k) \ge 1$, by Lemma 4.3,

$$\left(\frac{\sigma_l(Q_j^k)}{\|w_l^{-1}\chi_{Q_j^k}\|_{p_l'(\cdot)}}\right)^{p_\infty} \lesssim \left(\frac{\sigma_l(Q_j^k)}{\sigma_l(Q_j^k)^{\frac{1}{(p_l')_\infty}}}\right)^{p_\infty} = \sigma_l(Q_j^k)^{\frac{p_\infty}{(p_l)_\infty}}. \tag{6.18}$$

Given this estimate, by (6.6) and Young's inequality we have that

$$\begin{aligned}
L_1 &\lesssim \sum_{(k,j)\in\mathscr{H}_{2a}} \int_{E_j^k} \prod_{l=1,4} \langle h_l\rangle_{\sigma_{\rho(l)},Q_j^k}^{p_\infty} \sigma_1(Q_j^k)^{\frac{p_\infty}{(p_1)_\infty}} \sigma_2(Q_j^k)^{\frac{p_\infty}{(p_2)_\infty}} \\
&\qquad\qquad \times \prod_{l=1}^{2}\left(\frac{\|w_l^{-1}\chi_{Q_j^k}\|_{p_l'(\cdot)}}{|Q_j^k|}\right)^{p(x)} u(x)\,dx \\
&\le \sum_{(k,j)\in\mathscr{H}_{2a}} \prod_{l=1,4} \langle h_l\rangle_{\sigma_{\rho(l)},Q_j^k}^{p_\infty} \sigma_1(Q_j^k)^{\frac{p_\infty}{(p_1)_\infty}} \sigma_2(Q_j^k)^{\frac{p_\infty}{(p_2)_\infty}} \\
&\qquad\qquad \times \int_{Q_j^k} \prod_{l=1}^{2} \|w_l^{-1}\chi_{Q_j^k}\|_{p_l'(\cdot)}^{p(x)} |Q_j^k|^{-2p(x)} u(x)\,dx \\
&\lesssim \sum_{(k,j)\in\mathscr{H}_{2a}} \langle h_1\rangle_{\sigma_1,Q_j^k}^{(p_1)_\infty} \sigma_1(Q_j^k) + \sum_{(k,j)\in\mathscr{H}_{2a}} \langle h_4\rangle_{\sigma_2,Q_j^k}^{(p_2)_\infty} \sigma_2(Q_j^k).
\end{aligned}$$

The estimate of the last term is identical to the estimate for J_2 above, beginning at inequality (6.12); here we use the fact that $\sigma_1(Q_j^k) \ge 1$ to get (6.11).

The estimate over $\mathscr{H}_{2b}$ is similar, but we must replace the exponent p_∞ with $r(Q_j^k)$, which is defined by

$$\frac{1}{r(Q_j^k)} = \frac{1}{(p_1)_-(Q_j^k)} + \frac{1}{(p_2)_\infty}.$$

Then by (6.15), for $x \in Q_j^k$,

$$\left|\frac{1}{p(x)} - \frac{1}{r(Q_j^k)}\right| \leq \left|\frac{1}{p_1(x)} - \frac{1}{(p_1)_-(Q_j^k)}\right| + \left|\frac{1}{p_2(x)} - \frac{1}{(p_2)_\infty}\right| \lesssim \frac{1}{\log(e+|x|)}.$$

We can then argue as we did for the sum over $\mathscr{H}_{2a}$ above to get

$$\begin{aligned}
&\sum_{(k,j)\in\mathscr{H}_{2b}} \int_{E_j^k} \prod_{l=1,4} \langle h_l \sigma_{\rho(l)}\rangle_{Q_j^k}^{p(x)} u(x)\,dx \\
&\quad\lesssim \sum_{(k,j)\in\mathscr{H}_{2b}} \int_{E_j^k} \prod_{l=1,4} \left(c_0^{-1} \|w_{\rho(l)}^{-1}\chi_{Q_j^k}\|_{p'_{\rho(l)}(\cdot)}^{-1} \int_{Q_j^k} h_l \sigma_{\rho(l)}\,dy\right)^{r(Q_j^k)} \\
&\qquad\qquad \times \prod_{l=1}^{2} \left(\frac{\|w_l^{-1}\chi_{Q_j^k}\|_{p'_l(\cdot)}}{|Q_j^k|}\right)^{p(x)} u(x)\,dx \\
&\qquad + \sum_{(k,j)\in\mathscr{H}_{2b}} \int_{E_j^k} \prod_{l=1}^{2} \left(\frac{\|w_l^{-1}\chi_{Q_j^k}\|_{p'_l(\cdot)}}{|Q_j^k|}\right)^{p(x)} \frac{u(x)}{(e+|x|)^{tnp_-}}\,dx \\
&\quad= M_1 + M_2.
\end{aligned}$$

The estimate for M_2 is identical to the estimate for L_2. To estimate M_1, we again use (6.18) for σ_2, replacing p_∞ with $r(Q_j^k)$. Because $\sigma_1(Q_j^k) < 1$ we need to replace (6.18) with a different estimate. Since $(p_1^{'})_\pm(Q_j^k) = (p_1)_\mp(Q_j^k)^{'}$, by the estimate (6.9), replacing $q(Q_j^k)$ with $r(Q_j^k)$, we get

$$\left(\frac{\sigma_1(Q_j^k)}{\|w_1^{-1}\chi_{Q_j^k}\|_{p'_1(\cdot)}}\right)^{r(Q_j^k)} \lesssim \sigma_1(Q_j^k)^{\frac{r(Q_j^k)}{(p_1)_-(Q_j^k)}}.$$

We can now modify the estimate for L_1 to estimate M_1:

$$\begin{aligned}
M_1 &\lesssim \sum_{(k,j)\in\mathscr{H}_{2b}} \int_{E_j^k} \prod_{l=1,4} \langle h_l\rangle_{\sigma_{\rho(l)},Q_j^k}^{r(Q_j^k)} \sigma_1(Q_j^k)^{\frac{r(Q_j^k)}{(p_1)_-(Q_j^k)}} \sigma_2(Q_j^k)^{\frac{r(Q_j^k)}{(p_2)_\infty}} \\
&\qquad\qquad \times \prod_{l=1}^{2} \left(\frac{\|w_l^{-1}\chi_{Q_j^k}\|_{p'_l(\cdot)}}{|Q_j^k|}\right)^{p(x)} u(x)\,dx \\
&\leq \sum_{(k,j)\in\mathscr{H}_{2b}} \prod_{l=1,4} \langle h_l\rangle_{\sigma_{\rho(l)},Q_j^k}^{r(Q_j^k)} \sigma_1(Q_j^k)^{\frac{r(Q_j^k)}{(p_1)_-(Q_j^k)}} \sigma_2(Q_j^k)^{\frac{r(Q_j^k)}{(p_2)_\infty}} \\
&\qquad\qquad \times \int_{Q_j^k} \prod_{l=1}^{2} \|w_l^{-1}\chi_{Q_j^k}\|_{p'_l(\cdot)}^{p(x)} |Q_j^k|^{-2p(x)} u(x)\,dx
\end{aligned}$$

$$\lesssim \sum_{(k,j)\in\mathscr{H}_{2b}} \langle h_1\rangle_{\sigma_1,Q_j^k}^{(p_1)_-(Q_j^k)}\sigma_1(Q_j^k) + \sum_{(k,j)\in\mathscr{H}_{2b}} \langle h_4\rangle_{\sigma_2,Q_j^k}^{(p_2)_\infty}\sigma_2(Q_j^k).$$

The estimate for the second term in the last line is the same as the final estimate for J_2; we use the same argument above to estimate L_1. The estimate for the first term is the same as the estimate for K_1 above, noting that since $h_1 \geq 1$ and by Hölder's inequality,

$$\langle h_1\rangle_{\sigma_1,Q_j^k}^{(p_1)_-(Q_j^k)} \leq \langle h_1^{\frac{p_1(\cdot)}{(p_1)_-(Q_j^k)}}\rangle_{\sigma_1,Q_j^k}^{(p_1)_-(Q_j^k)} \leq \langle h_1^{\frac{p_1(\cdot)}{(p_1)_-}}\rangle_{\sigma_1,Q_j^k}^{(p_1)_-}.$$

This completes the estimate of M_1 and so of I_2.

Remark 6.4. As noted above, the argument for I_3 is the same as that for I_2, replacing $h_1\sigma_1$ with $h_3\sigma_2$ and $h_4\sigma_2$ with $h_2\sigma_1$.

The estimate for I_4. The estimate for I_4 parallels that for I_2. In particular, we will decompose I_4 into essentially the same parts as we did above. For some parts the estimate is very similar to the corresponding part I_2, and so we give the key inequalities but will omit some of the details. For other parts we will need to modify the argument and we will present these in more detail.

Begin by forming the bilinear Calderón–Zygmund cubes associated with $\mathcal{M}^d(h_2\sigma_1, h_4\sigma_2)$. We then decompose the collection of these cubes into the sets $\mathscr{F}$, $\mathscr{G}$, and $\mathscr{H}$, defined as above. Denote the sums over these sets by N_1, N_2, and N_3.

The estimate for N_1. The estimate for N_1 is very similar to that for J_1 above. We replace the arguments used for the h_1 term and estimate the h_2 term and the h_4 term in the same way, using the fact that $h_2, h_4 \leq 1$:

$$\begin{aligned}
N_1 &= \sum_{(k,j)\in\mathscr{F}} \int_{E_j^k} \prod_{l=2,4} \langle h_l\sigma_{\rho(l)}\rangle_{Q_j^k}^{p(x)} u(x)\,dx \\
&\leq \sum_{(k,j)\in\mathscr{F}} \int_{E_j^k} \prod_{l=1}^{2} \langle \sigma_l\rangle_{Q_j^k}^{p(x)} u(x)\,dx \\
&= \sum_{(k,j)\in\mathscr{F}} \int_{E_j^k} \prod_{l=1}^{2} \sigma_l(Q_j^k)^{p(x)-q(Q_j^k)} \\
&\qquad\qquad \times \sigma_1(Q_j^k)^{q(Q_j^k)}\sigma_2(Q_j^k)^{q(Q_j^k)}|Q_j^k|^{-2p(x)}u(x)\,dx \\
&\leq \sum_{(k,j)\in\mathscr{F}} \prod_{l=1}^{2} (1+\sigma_l(Q_j^k))^{p_+(Q_j^k)-q(Q_j^k)} \\
&\qquad\qquad \times \int_{E_j^k} \sigma_1(Q_j^k)^{q(Q_j^k)}\sigma_2(Q_j^k)^{q(Q_j^k)}|Q_j^k|^{-2p(x)}u(x)\,dx
\end{aligned}$$

$$
\begin{aligned}
&\lesssim \prod_{l=1}^{2}(1+\sigma_l(P))^{p_+-q_-} \sum_{(k,j)\in\mathscr{F}} \sigma_1(Q_j^k)^{\frac{q(Q_j^k)}{(p_1)_-(Q_j^k)}} \sigma_2(Q_j^k)^{\frac{q(Q_j^k)}{(p_2)_-(Q_j^k)}} \\
&\lesssim \sum_{(k,j)\in\mathscr{F}} \sigma_1(Q_j^k) + \sum_{(k,j)\in\mathscr{F}} \sigma_2(Q_j^k) \\
&\lesssim \sum_{(k,j)\in\mathscr{F}} \sigma_1(E_j^k) + \sum_{(k,j)\in\mathscr{F}} \sigma_2(E_j^k) \\
&\leq \sigma_1(P)+\sigma_2(P) \\
&\lesssim 1.
\end{aligned}
$$

The estimate for N_2. To estimate N_2 we modify the argument for J_2. By the definition of $\mathcal{A}_{\vec{p}(\cdot)}$ and by Lemma 3.6 we have

$$
\begin{aligned}
\frac{1}{|Q_j^k|^2}\int_{Q_j^k} h_2\sigma_1\,dy \int_{Q_j^k} h_4\sigma_2\,dy
&\lesssim \|w\chi_{Q_j^k}\|_{p(\cdot)}^{-1}\prod_{l=1}^{2}\|w_l^{-1}\chi_{Q_j^k}\|_{p_l'(\cdot)}^{-1}\|h_2\|_{L^{p_1(\cdot)}_{\sigma_1}} \\
&\qquad \times \|h_4\|_{L^{p_2(\cdot)}_{\sigma_2}}\|\chi_{Q_j^k}\|_{L^{p_1'(\cdot)}_{\sigma_1}}\|\chi_{Q_j^k}\|_{L^{p_2'(\cdot)}_{\sigma_2}} \\
&= \|w\chi_{Q_j^k}\|_{p(\cdot)}^{-1}\prod_{l=1}^{2}\|w_l^{-1}\chi_{Q_j^k}\|_{p_l'(\cdot)}^{-1}\|h_2\|_{L^{p_1(\cdot)}_{\sigma_1}} \\
&\qquad \times \|h_4\|_{L^{p_2(\cdot)}_{\sigma_2}}\|w_1^{-1}\chi_{Q_j^k}\|_{p_1'(\cdot)}\|w_2^{-1}\chi_{Q_j^k}\|_{p_2'(\cdot)};
\end{aligned}
$$

since $u(Q)\geq u(P_i)\geq 1$, by Lemma 3.5,

$$
\begin{aligned}
&\lesssim \|w\chi_{Q_j^k}\|_{p(\cdot)}^{-1} \\
&\leq c_0.
\end{aligned}
$$

Therefore, by Lemma 3.13,

$$
\begin{aligned}
N_2 &= \sum_{(k,j)\in\mathscr{G}}\int_{E_j^k}\prod_{l=2,4}\langle h_l\sigma_{\rho(l)}\rangle_{Q_j^k}^{p(x)}u(x)\,dx \\
&\lesssim \sum_{(k,j)\in\mathscr{G}}\int_{E_j^k}\left(\frac{c_0^{-1}}{|Q_j^k|^2}\int_{Q_j^k} h_2\sigma_1\,dy\int_{Q_j^k} h_4\sigma_2\,dy\right)^{p(x)}u(x)\,dx \\
&\lesssim \sum_{(k,j)\in\mathscr{G}}\int_{E_j^k}\prod_{l=2,4}\langle h_l\sigma_{\rho(l)}\rangle_{Q_j^k}^{p_\infty}u(x)\,dx + \sum_{(k,j)\in\mathscr{G}}\int_{E_j^k}\frac{u(x)}{(e+|x|)^{tnp_-}}\,dx.
\end{aligned}
$$

By Lemma 4.6, the second term on the last line is bounded by a constant 1. We estimate the first term using (6.10):

$$\sum_{(k,j)\in\mathscr{G}} \int_{E_j^k} \prod_{l=2,4} \langle h_l \sigma_{\rho(l)} \rangle_{Q_j^k}^{p_\infty} u(x)\, dx$$

$$= \sum_{(k,j)\in\mathscr{G}} \int_{E_j^k} \prod_{l=2,4} \langle h_l \rangle_{\sigma_{\rho(l)},Q_j^k}^{p_\infty} \sigma_1(Q_j^k)^{p_\infty} \sigma_2(Q_j^k)^{p_\infty} |Q_j^k|^{-2p_\infty} u(x)\, dx$$

$$\lesssim \sum_{(k,j)\in\mathscr{G}} \langle h_2 \rangle_{\sigma_1,Q_j^k}^{p_\infty} \sigma_1(Q_j^k)^{\frac{p_\infty}{(p_1)_\infty}} \langle h_4 \rangle_{\sigma_2,Q_j^k}^{p_\infty} \sigma_2(Q_j^k)^{\frac{p_\infty}{(p_2)_\infty}};$$

by Young's inequality and Lemmas 3.2, 3.13, and 4.6,

$$\lesssim \sum_{(k,j)\in\mathscr{G}} \langle h_2 \rangle_{\sigma_1,Q_j^k}^{(p_1)_\infty} \sigma_1(Q_j^k) + \sum_{(k,j)\in\mathscr{G}} \langle h_4 \rangle_{\sigma_2,Q_j^k}^{(p_2)_\infty} \sigma_2(Q_j^k)$$

$$\lesssim \sum_{(k,j)\in\mathscr{G}} \langle h_2 \rangle_{\sigma_1,Q_j^k}^{(p_1)_\infty} \sigma_1(E_j^k) + \sum_{(k,j)\in\mathscr{G}} \langle h_4 \rangle_{\sigma_2,Q_j^k}^{(p_2)_\infty} \sigma_2(E_j^k)$$

$$\le \int_{\mathbb{R}^n} M_{\sigma_1}^d h_2(x)^{(p_1)_\infty} \sigma_1(x)\, dx + \int_{\mathbb{R}^n} M_{\sigma_2}^d h_4(x)^{(p_2)_\infty} \sigma_2(x)\, dx$$

$$\lesssim \int_{\mathbb{R}^n} h_2(x)^{(p_1)_\infty} \sigma_1(x)\, dx + \int_{\mathbb{R}^n} h_4(x)^{(p_2)_\infty} \sigma_2(x)\, dx$$

$$\lesssim \int_{\mathbb{R}^n} h_2(x)^{p_1(x)} \sigma_1(x)\, dx + \int_{\mathbb{R}^n} h_4(x)^{p_2(x)} \sigma_2(x)\, dx$$

$$+ \int_{\mathbb{R}^n} \frac{\sigma_1(x)}{(e+|x|)^{tn(p_1)_-}}\, dx + \int_{\mathbb{R}^n} \frac{\sigma_2(x)}{(e+|x|)^{tn(p_2)_-}}\, dx$$

$$\lesssim 1.$$

The estimate for N_3. The estimate for N_3 is broadly similar to the estimate for J_3 above, but it differs considerably in the details. We first begin by dividing the cubes in $\mathscr{H}$ into the sets $\mathscr{H}_1$ and $\mathscr{H}_2$ as before. However, we now have to subdivide both of these sets and not just $\mathscr{H}_2$. Define

$$\mathscr{H}_{1a} = \{(k,j) \in \mathscr{H}_1 : \sigma_1(Q_j^k) \le 1,\ \sigma_2(Q_j^k) \le 1\}$$

and

$$\mathscr{H}_{1b} = \{(k,j) \in \mathscr{H}_1 : \sigma_1(Q_j^k) > 1,\ \sigma_2(Q_j^k) \le 1\}.$$

The estimate for the sum over $\mathscr{H}_{1a}$ is similar to the estimate over $\mathscr{H}_1$ above for J_3, but we use the fact that both $h_2, h_4 \le 1$. By Lemmas 3.13 and 4.6,

$$\sum_{(k,j)\in\mathscr{H}_{1a}} \int_{E_j^k} \prod_{l=2,4} \langle h_l \sigma_{\rho(l)} \rangle_{Q_j^k}^{p(x)} u(x)\,dx$$

$$\lesssim \sum_{(k,j)\in\mathscr{H}_{1a}} \int_{E_j^k} \prod_{l=2,4} \langle h_l \sigma_{\rho(l)} \rangle_{Q_j^k}^{p_+(Q_j^k)} u(x)\,dx$$

$$+ \sum_{(k,j)\in\mathscr{H}_{1a}} \int_{E_j^k} \frac{u(x)}{(e+|x|)^{tnp_-}}\,dx$$

$$\le \sum_{(k,j)\in\mathscr{H}_{1a}} \prod_{l=2,4} \langle h_l \rangle_{\sigma_{\rho(l)},Q_j^k}^{p_+(Q_j^k)}$$

$$\times \int_{E_j^k} |Q_j^k|^{-2p_+(Q_j^k)} \prod_{l=2,4} \sigma_{\rho(l)}(Q_j^k)^{p_+(Q_j^k)} u(x)\,dx + 1.$$

Since $h_2, h_4 \le 1$ and $\sigma_l(Q_j^k) \le 1$, $l = 1, 2$, by (6.4), replacing $|Q_j^k|^{-2p(x)}$ with $|Q_j^k|^{-2p_+(Q_j^k)}$ (which we can do by Lemma 3.11),

$$\le \sum_{(k,j)\in\mathscr{H}_{1a}} \prod_{l=2,4} \langle h_l \rangle_{\sigma_{\rho(l)},Q_j^k}^{q(Q_j^k)}$$

$$\times \int_{E_j^k} |Q_j^k|^{-2p_+(Q_j^k)} \prod_{l=2,4} \sigma_{\rho(l)}(Q_j^k)^{q(Q_j^k)} u(x)\,dx + 1$$

$$\lesssim \sum_{(k,j)\in\mathscr{H}_{1a}} \prod_{l=2,4} \langle h_l \rangle_{\sigma_{\rho(l)},Q_j^k}^{q(Q_j^k)} \sigma_1(Q_j^k)^{\frac{q(Q_j^k)}{(p_1)_-(Q_j^k)}} \sigma_2(Q_j^k)^{\frac{q(Q_j^k)}{(p_2)_-(Q_j^k)}} + 1;$$

by Young's inequality,

$$\lesssim \sum_{(k,j)\in\mathscr{H}_{1a}} \langle h_2 \rangle_{\sigma_1,Q_j^k}^{(p_1)_-} \sigma_1(Q_j^k) + \sum_{(k,j)\in\mathscr{H}_{1a}} \langle h_4 \rangle_{\sigma_2,Q_j^k}^{(p_2)_-(Q_j^k)} \sigma_2(Q_j^k) + 1.$$

Both of the final terms are estimated as K_2 above.

To estimate the sum over $\mathscr{H}_{1b}$, we first define the exponent $s(Q_j^k)$ by

$$\frac{1}{s(Q_j^k)} = \frac{1}{(p_1)_\infty} + \frac{1}{(p_2)_+(Q_j^k)}.$$

Then, arguing as we did for (6.14), we get that for $x \in Q_j^k$,

$$\left|\frac{1}{p(x)} - \frac{1}{s(Q_j^k)}\right| \leq \left|\frac{1}{p_1(x)} - \frac{1}{(p_1)_\infty}\right| + \left|\frac{1}{p_2(x)} - \frac{1}{(p_2)_+(Q_j^k)}\right| \lesssim \frac{1}{\log(e+|x|)}.$$

Given this, by (6.16) (for h_2 instead of h_1), (6.17), and Lemma 3.13,

$$\begin{aligned}
&\sum_{(k,j)\in\mathscr{H}_{1b}} \int_{E_j^k} \prod_{l=2,4} \langle h_l\sigma_{\rho(l)}\rangle_{Q_j^k}^{p(x)} u(x)\,dx \\
&\quad\lesssim \sum_{(k,j)\in\mathscr{H}_{1b}} \int_{E_j^k} \prod_{l=2,4} \left(c_0^{-1}\|w_{\rho(l)}^{-1}\chi_{Q_j^k}\|_{p'_{\rho(l)}(\cdot)}^{-1} \int_{Q_j^k} h_l\sigma_{\rho(l)}\,dy\right)^{s(Q_j^k)} \\
&\qquad\qquad \times \prod_{l=1}^{2}\left(\frac{\|w_l^{-1}\chi_{Q_j^k}\|_{p'_l(\cdot)}}{|Q_j^k|}\right)^{p(x)} u(x)\,dx \\
&\qquad + \sum_{(k,j)\in\mathscr{H}_{1b}} \int_{E_j^k} \prod_{l=1}^{2}\left(\frac{\|w_l^{-1}\chi_{Q_j^k}\|_{p'_l(\cdot)}}{|Q_j^k|}\right)^{p(x)} \frac{u(x)}{(e+|x|)^{tnp_-}}\,dx \\
&\quad= R_1 + R_2.
\end{aligned}$$

The estimate for R_2 is identical to the estimate for L_2. To estimate R_1, we again use (6.18) for σ_1, replacing p_∞ with $s(Q_j^k)$. Because $\sigma_2(Q_j^k) < 1$ we use a different estimate. Since $(p_2')_-(Q_j^k) = (p_2)_+(Q_j^k)'$, by Lemma 3.5,

$$\left(\frac{\sigma_2(Q_j^k)}{\|w_2^{-1}\chi_{Q_j^k}\|_{p_2'(\cdot)}}\right)^{s(Q_j^k)} \leq \left(\sigma_2(Q_j^k)^{1-\frac{1}{(p_2)_+(Q_j^k)'}}\right)^{s(Q_j^k)} = \sigma_2(Q_j^k)^{\frac{s(Q_j^k)}{(p_2)_+(Q_j^k)}}.$$

We can now argue as in the estimate of L_1 to get

$$\begin{aligned}
R_1 \lesssim \sum_{(k,j)\in\mathscr{H}_{1b}} \int_{E_j^k} \prod_{l=2,4} \langle h_l\rangle_{\sigma_{\rho(l)},Q_j^k}^{s(Q_j^k)} \sigma_1(Q_j^k)^{\frac{s(Q_j^k)}{(p_1)_\infty}} \sigma_2(Q_j^k)^{\frac{s(Q_j^k)}{(p_2)_+(Q_j^k)}} \\
\times \prod_{J=1}^{2}\left(\frac{\|w_J^{-1}\chi_{Q_j^k}\|_{p'_J(\cdot)}}{|Q_j^k|}\right)^{p(x)} u(x)\,dx
\end{aligned}$$

$$\leq \sum_{(k,j)\in\mathscr{H}_{1b}} \prod_{l=2,4} \langle h_l\rangle_{\sigma_{\rho(l)},Q_j^k}^{s(Q_j^k)} \sigma_1(Q_j^k)^{\frac{s(Q_j^k)}{(p_1)_\infty}} \sigma_2(Q_j^k)^{\frac{s(Q_j^k)}{(p_2)_+(Q_j^k)}}$$
$$\times \int_{Q_j^k} \prod_{J=1}^{2} \|w_J^{-1}\chi_{Q_j^k}\|_{p'_J(\cdot)}^{p(x)} |Q_j^k|^{-2p(x)} u(x)\,dx$$
$$\lesssim \sum_{(k,j)\in\mathscr{H}_{1b}} \langle h_2\rangle_{\sigma_1,Q_j^k}^{(p_1)_\infty} \sigma_1(Q_j^k) + \sum_{(k,j)\in\mathscr{H}_{1b}} \langle h_4\rangle_{\sigma_2,Q_j^k}^{(p_2)_+(Q_j^k)} \sigma_2(E_j^k).$$

The estimate for the first term in the last line is the same as the estimate for the h_4 term in J_2. Arguing as we did for (6.14) and (6.15), we get

$$|(p_2)_+(Q_j^k) - p_\infty| \lesssim \frac{1}{\log(e+|x|)}.$$

Then, since $\langle h_4\rangle_{\sigma_1,Q_j^k} \leq 1$, the estimate for the second term follows by (6.15), and by Lemmas 3.13, 3.2, and 4.6:

$$\sum_{(k,j)\in\mathscr{H}_{1b}} \langle h_4\rangle_{\sigma_2,Q_j^k}^{(p_2)_+(Q_j^k)} \sigma_2(E_j^k)$$
$$\lesssim \sum_{(k,j)\in\mathscr{H}_{1b}} \int_{E_j^k} \langle h_4\rangle_{\sigma_2,Q_j^k}^{(p_2)_\infty} \sigma_2(x)\,dx + \int_{\mathbb{R}^n} \frac{\sigma_2(x)}{(e+|x|)^{tn(p_2)_-}}\,dx + 1$$
$$\lesssim \sum_{(k,j)\in\mathscr{H}_{1b}} \int_{E_j^k} \langle h_4\rangle_{\sigma_2,Q_j^k}^{(p_2)_\infty} \sigma_2(x)\,dx + 1.$$

Again, we estimate this last sum as in the final estimate for J_2.

To estimate the sum over $\mathscr{H}_2$ we argue as we did before for J_3, dividing it into sums over $\mathscr{H}_{2a}$ and $\mathscr{H}_{2b}$. The estimate over $\mathscr{H}_{2a}$ is identical to the estimate over this set as before, replacing h_1 by h_2. This yields terms just like L_1 and L_2 above. The estimate for the L_2 term is the same as is the estimate for the L_1 term, except that in the final line the h_2 term is estimated like the h_4 term since both $h_2, h_4 \leq 1$.

To estimate the sum over $\mathscr{H}_{2b}$, we can argue as before, getting terms like M_1 and M_2, replacing h_1 by h_2. The estimate of the M_2 term is again the same. To estimate the M_1 term we argue as before except that we replace the exponent $r(Q_j^k)$ by p_∞. But then the final line of the estimate becomes

$$\sum_{(k,j)\in\mathscr{H}_{2b}} \langle h_1\rangle_{\sigma_1,Q_j^k}^{(p_1)_\infty} \sigma_1(Q_j^k) + \sum_{(k,j)\in\mathscr{H}_{2b}} \langle h_4\rangle_{\sigma_2,Q_j^k}^{(p_2)_\infty} \sigma_2(Q_j^k),$$

and both of these sums are estimated like the final estimate for J_2. This completes the estimate for N_3 and so of I_4. This completes the proof of Theorem 2.4.

7. Proof of Theorem 2.8

Theorem 2.8 follows almost directly from Theorem 2.4. To prove it, we will need two estimates for the Fefferman–Stein sharp maximal operator and an extrapolation theorem in the scale of weighted variable Lebesgue spaces. We first recall the definition of the sharp maximal operator. Given $f \in L^1_{\text{loc}}$, let

$$M^{\#}f(x) = \sup_{Q} \fint_{Q} |f(y) - \langle f\rangle_Q|\, dy\, \chi_Q(x),$$

where the supremum is taken over all cubes Q. For $\delta > 0$, define $M^{\#}_{\delta}f(x) = M^{\#}(|f|^{\delta})(x)^{\frac{1}{\delta}}$. The first estimate relates the norm of f and $M^{\#}$. For a proof, see Journé [**20**] or [**8**].

Proposition 7.1. *Given $w \in A_\infty$, $0 < p < \infty$, and $0 < \delta < 1$,*

$$\|f\|_{L^p_w} \lesssim \|M^{\#}_{\delta}f\|_{L^p_w}.$$

The implicit constant depends on p, n, δ, and w.

The second estimate is a pointwise inequality proved in [**22**].

Proposition 7.2. *Given $0 < \delta < 1/2$ and a bilinear Calderón–Zygmund singular integral T, for all $f_1, f_2 \in L^\infty_c$,*

$$M^{\#}_{\delta}(T(f_1, f_2))(x) \lesssim \mathcal{M}(f_1, f_2)(x).$$

The implicit constant depends only on T, δ, and n.

To apply these results we need to extend Proposition 7.1 to the scale of variable Lebesgue spaces. The following result was proved in [**13**, Theorem 2.25]. The hypotheses are somewhat technical, but they are the right generalization to prove A_∞ extrapolation in this setting ([**8**]). The result is stated in the abstract language of extrapolation pairs; for more on this approach to Rubio de Francia extrapolation, see [**9**].

Proposition 7.3. *Suppose for some $0 < p < \infty$ and every $w_0 \in A_\infty$,*

$$\|f\|_{L^p_{w_0}} \lesssim \|g\|_{L^p_{w_0}} \tag{7.1}$$

for every pair of functions (f, g) in a family $\mathcal{F}$ such that $\|f\|_{L^p(w)} < \infty$. Given $p(\cdot) \in \mathcal{P}_0$, suppose there exists $s \leq p_-$ such that $w^s \in \mathcal{A}_{p(\cdot)/s}$ and the maximal operator is bounded on $L^{(p(\cdot)/s)'}(w^{-s})$. Then for $(f, g) \in \mathcal{F}$ such that $\|f\|_{L^{p(\cdot)}(w)} < \infty$,

$$\|f\|_{L^{p(\cdot)}(w)} \lesssim \|g\|_{L^{p(\cdot)}(w)}.$$

Proof of Theorem 2.8: Fix $\vec{p}(\cdot)$ as in the hypotheses and $\vec{w} \in \mathcal{A}_{\vec{p}(\cdot)}$. Since $(p_j)_- > 1$, $p_- > 1/2$. Let $s = 1/2$. Then by Proposition 4.7 $w^s \in \mathcal{A}_{p(\cdot)/s}$, so $w^{-s} \in \mathcal{A}_{(p(\cdot)/s)'}$. Since $p(\cdot) \in LH$, so is $(p(\cdot)/s)'$. Thus, by the weighted bounds for the maximal operator on variable Lebesgue spaces (see [**7**]), M is bounded on $L^{(p(\cdot)/s)'}(w^{-s})$. Therefore, the main hypothesis of Proposition 7.3 holds.

Fix $0 < \delta < 1/2$ and define the family of extrapolation pairs

$$\mathcal{F} = \{(\min(|T(f_1, f_2)|, N)\chi_{B(0,N)}, M_\delta^\#(T(f_1, f_2))) : f_1, f_2 \in L_c^\infty,\ N > 1\}.$$

Since

$$\min(|T(f_1, f_2)|, N)\chi_{B(0,N)} \in L_c^\infty \subset L_{w_0}^p$$

for any $p > 0$ and $w_0 \in A_\infty$, it follows from Proposition 7.1 that (7.1) holds for every pair in $\mathcal{F}$. Similarly, we have

$$\min(|T(f_1, f_2)|, N)\chi_{B(0,N)} \in L^{p(\cdot)}(w),$$

and so by Propositions 7.3 and 7.2,

$$\begin{aligned}\|\min(|T(f_1, f_2)|, N)\chi_{B(0,N)}\|_{L^{p(\cdot)}(w)} &\lesssim \|M_\delta^\#(T(f_1, f_2))\|_{L^{p(\cdot)}(w)} \\ &\lesssim \|\mathcal{M}(f_1, f_2)\|_{L^{p(\cdot)}(w)}.\end{aligned}$$

If we take the limit as $N \to \infty$, then by Fatou's Lemma (Lemma 3.4) and Theorem 2.4,

$$\|T(f_1, f_2)\|_{L^{p(\cdot)}(w)} \lesssim \|f_1\|_{L_1^p(\cdot)(w_1)} \|f_2\|_{L_2^p(\cdot)(w_2)}.$$

The desired conclusion now follows by a standard approximation argument since L_c^∞ is dense in $L^{p_j(\cdot)}(w_j)$, $j = 1, 2$ [**13**, Lemma 3.1]. □

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