# Limit cycles of piecewise polynomial perturbations of higher dimensional linear differential systems

Jaume Llibre, Douglas D. Novaes and Iris O. Zeli

**Abstract.** The averaging theory has been extensively employed for studying periodic solutions of smooth and nonsmooth differential systems. Here, we extend the averaging theory for studying periodic solutions a class of regularly perturbed non–autonomous $n$-dimensional discontinuous piecewise smooth differential system. As a fundamental hypothesis, it is assumed that the unperturbed system has a manifold $\mathcal{Z} \subset \mathbb{R}^n$ of periodic solutions satisfying $\dim(\mathcal{Z}) < n$. Then, we apply this result to study limit cycles bifurcating from periodic solutions of linear differential systems, $x' = Mx$, when they are perturbed inside a class of discontinuous piecewise polynomial differential systems with two zones. More precisely, we study the periodic solutions of the following differential system

$$x' = Mx + \varepsilon F_1^n(x) + \varepsilon^2 F_2^n(x),$$

in $\mathbb{R}^{d+2}$ where $\varepsilon$ is a small parameter, $M$ is a $(d+2) \times (d+2)$ matrix having one pair of pure imaginary conjugate eigenvalues, $m$ zeros eigenvalues, and $d - m$ non–zero real eigenvalues.

## 1. Introduction

The analysis of discontinuous piecewise smooth differential systems has recently had a large and fast growth due to its applications in several areas of the knowledge. Such systems model many phenomena in control systems (see [1]), impact on mechanical systems (see [2]), economy (see [17]), biology (see [18]), nonlinear oscillations (see [27]), neuroscience (see [8, 13, 28]), and other fields of science.

Establishing the existence of limit cycles is one of the major problem in the theory of differential systems. The interest in detecting such objects is due to the

fact that they are non-local invariant sets providing information on the qualitative behavior of the system. The first studies on this subject considered smooth differential systems and, since then, many contributions have been made in this direction (see [15] and the references therein). The study of limit cycles has also been considered for continuous (see, for instance, [4,23,25]) and discontinuous piecewise smooth differential systems (see, for instance, [11,14,19,20,26]). Most of them are concentrated on planar piecewise differential systems.

The *averaging theory* is one of the main tools for studying periodic solutions in regularly perturbed differential systems of the form

$$(1.1) \quad \dot{x} = F_0(t, \mathbf{x}) + \sum_{i=1}^{k} \varepsilon^i F_i(t, \mathbf{x}) + \varepsilon^{k+1} R(t, \mathbf{x}, \varepsilon), \ (t, \mathbf{x}, \varepsilon) \in \mathbb{R} \times D \times (-\varepsilon_0, \varepsilon_0),$$

where $D$ is an open bounded subset of $\mathbb{R}^n$ and the functions $F_i$, $i = 0, 1, \ldots, k$, and $R$ are $T$-periodic in the first variable. Here, $k$ is called *order of perturbation* in $\varepsilon$. As a fundamental hypothesis, it is assumed that the *unperturbed system*,

$$(1.2) \qquad\qquad\qquad\qquad \dot{x} = F_0(t, \mathbf{x}),$$

has a manifold $\mathcal{Z} \subset \mathbb{R}^n$ of periodic solutions. Roughly speaking, this theory provides a sequence of functions, called *averaged functions*, which have their simple zeros associated with limit cycles of system (1.1).

The averaging theory has been extensively employed for studying periodic solutions of smooth and nonsmooth differential systems. First, considering $F_0 = 0$ (consequently, $\dim(\mathcal{Z}) = n$) one can find in [31, 32] results providing sufficient condition on $F_1$ ensuring the existence of periodic solutions of system (1.1) under smoothness and boundedness conditions. Topological methods were used in [4] to generalize these results for Lipschitz continuous differential systems. In [23], assuming the weaker hypothesis $\dim(\mathcal{Z}) = n$, the averaging theory was developed at any order for Lipschitz continuous differential systems. Then, in [20, 24], the averaging theory was extended up to order 2 for detecting periodic orbits of discontinuous piecewise smooth differential systems. Some applications of these results can be found in [26, 29]. Finally, in [16, 22], the averaging theory was developed at any order for a class of discontinuous piecewise smooth systems.

When $\dim(\mathcal{Z}) < n$, the averaging theory has to be combined with other techniques, for instance *Lyapunov-Schmidt reduction method*, to provide sufficient conditions for the existence of periodic solutions. Here, we also obtain a sequence of function, now called *bifurcation functions*, which have their simple zeros associated with limit cycles of system (1.1). In the smooth case, the averaging theory is developed at any order [3, 5, 10]. For the nonsmooth case, the first order averaging theory has been addressed in [30], however it is lacking in a higher order analysis.

In this paper, our first main goal is to develop the averaging theory up to order 2 in $\varepsilon$ for a class of discontinuous piecewise smooth differential systems assuming $\dim(\mathcal{Z}) = d < n$. The study of any finite order in $\varepsilon$ could be performed in a similar way, however the general expression for higher order bifurcation functions would be more complex because it involves higher derivatives of composite functions. As

our second main goal, we apply this result to study the number of limit cycles bifurcating from the periodic orbits of a linear differential system $x' = Mx$, where $M$ is a $(d + 2) \times (d + 2)$ matrix having one pair of pure imaginary conjugate eigenvalues, $m$ zeros eigenvalues, and $d-m$ real eigenvalues. We focus our attention when this system is perturbed up to order 2 in the small parameter $\varepsilon$ inside a class of discontinuous piecewise polynomial functions having two zones.

This paper is organized as follows. In Section 2, we state our main results: Theorem 1, improving the averaging theory for nonsmooth systems; and Theorems 3-5, regarding piecewise polynomial perturbations of higher dimensional linear systems. In Section 3, we provide some preliminary results. The remainder Sections 4-7 are devoted to the proofs of Theorem 1 and Theorems 3-5.

## 2. Statements of the main results

### 2.1. Advances on averaging theory

In this subsection we improve the averaging theory of first and second order to study the limit cycles of a class of discontinuous piecewise smooth differential systems.

Let $D$ be an open bounded subset of $\mathbb{R}^{d+1}$ and for a positive real number $T$ we consider the $\mathcal{C}^3$ differentiable functions $F_i^{\pm} : \mathbb{S}^1 \times D \to \mathbb{R}^{d+1}$ for $i = 0, 1, 2$, and $R^{\pm} : \mathbb{S}^1 \times D \times (-\varepsilon_0, \varepsilon_0) \to \mathbb{R}^{d+1}$ where $\mathbb{S}^1 \equiv \mathbb{R}/(\mathbb{Z}T)$. Thus, we define the following $T$-periodic *discontinuous piecewise smooth differential system*

$$(2.1) \qquad \mathbf{x}' = \begin{cases} F^+(\theta, \mathbf{x}, \varepsilon) & \text{if} \quad 0 \leq \theta \leq \phi, \\ F^-(\theta, \mathbf{x}, \varepsilon) & \text{if} \quad \phi \leq \theta \leq T, \end{cases}$$

where the prime denotes derivative with respect to the variable $\theta \in \mathbb{S}^1$, and

$$F^{\pm}(\theta, \mathbf{x}, \varepsilon) = F_0^{\pm}(\theta, \mathbf{x}) + \varepsilon F_1^{\pm}(\theta, \mathbf{x}) + \varepsilon^2 F_2^{\pm}(\theta, \mathbf{x}) + \varepsilon^3 R^{\pm}(\theta, \mathbf{x}, \varepsilon),$$

with $\mathbf{x} \in D$. The set of discontinuity of system (2.1) is given by $\Sigma = \{\theta = 0\} \cup \{\theta = \phi\}$.

For $\mathbf{z} \in D$, let $\varphi(\theta, \mathbf{z})$ be the solution of the unperturbed system

$$(2.2) \qquad \mathbf{x}' = F_0(\theta, \mathbf{x}),$$

such that $\varphi(0, \mathbf{z}) = \mathbf{z}$, where

$$F_0(\theta, \mathbf{x}) = \begin{cases} F_0^+(\theta, \mathbf{x}) & \text{if} \quad 0 \leq \theta \leq \phi, \\ F_0^-(\theta, \mathbf{x}) & \text{if} \quad \phi \leq \theta \leq T. \end{cases}$$

Clearly,

$$\varphi(\theta, \mathbf{z}) = \begin{cases} \varphi^+(\theta, \mathbf{z}) & \text{if} \quad 0 \leq \theta \leq \phi, \\ \varphi^-(\theta, \mathbf{z}) & \text{if} \quad \phi \leq \theta \leq T, \end{cases}$$

where $\varphi^\pm(\theta, \mathbf{z})$ are the solutions of the systems

$$(2.3) \qquad \mathbf{x}' = F_0^\pm(\theta, \mathbf{x}),$$

such that $\varphi^\pm(0, \mathbf{z}) = \mathbf{z}$.

We assume that there exists a manifold $\mathcal{Z}$ embedded in $D$ such that the solutions starting in $\mathcal{Z}$ are all $T$-periodic. More precisely, for $p = d + 1$, $q \le p$ and $V$ an open bounded subset of $\mathbb{R}^q$, let $\sigma : \overline{V} \to \mathbb{R}^{p-q}$ be a $\mathcal{C}^3$ function and define

$$(2.4) \qquad \mathcal{Z} = \{\mathbf{z}_\nu = (\nu, \sigma(\nu)) : \nu \in \overline{V}\}.$$

We shall assume that

$(H)$  $\mathcal{Z} \subset D$ and for each $\mathbf{z}_\nu$ the unique solution $\varphi(\theta, \mathbf{z}_\nu)$ such that $\varphi(0, \mathbf{z}_\nu) = \mathbf{z}_\nu$ is $T$-periodic.

For $\mathbf{z} \in D$ we consider the first order variational equations of systems (2.3) along the solution $\varphi^\pm(\theta, \mathbf{z})$, that is

$$(2.5) \qquad Y' = D_\mathbf{x} F_0^\pm(\theta, \varphi^\pm(\theta, \mathbf{z})) \, Y.$$

Denote by $Y^\pm(\theta, \mathbf{z})$ a fundamental matrix of the differential system (2.5).

Let $\xi : \mathbb{R}^q \times \mathbb{R}^{p-q} \to \mathbb{R}^q$ and $\xi^\perp : \mathbb{R}^q \times \mathbb{R}^{p-q} \to \mathbb{R}^{p-q}$ be the orthogonal projections onto the first $q$ coordinates and onto the last $p - q$ coordinates, respectively. For a point $\mathbf{z} \in \mathrm{D}$ denote $\mathbf{z} = (u, v) \in \mathbb{R}^q \times \mathbb{R}^{p-q}$. Before defining the bifurcation functions we have to define some auxiliar functions. Let

$$y_0^\pm(\theta, \mathbf{z}) = \varphi^\pm(\theta, \mathbf{z}),$$

$$y_1^\pm(\theta, \mathbf{z}) = Y^\pm(\theta, \mathbf{z}) \int_0^\theta Y^\pm(s, \mathbf{z})^{-1} F_1^\pm(s, \varphi^\pm(s, \mathbf{z})) ds,$$

$$(2.6)$$

$$y_2^\pm(\theta, \mathbf{z}) = Y^\pm(\theta, \mathbf{z}) \int_0^\theta Y^\pm(s, \mathbf{z})^{-1} \bigg( 2 F_2^\pm(s, \varphi^\pm(s, \mathbf{z})) +$$

$$2 \frac{\partial F_1^\pm}{\partial \mathbf{x}}(s, \varphi^\pm(s, \mathbf{z})) y_1^\pm(s, \mathbf{z}) + \frac{\partial^2 F_0^\pm}{\partial \mathbf{x}^2}(s, \varphi^\pm(s, \mathbf{z})) y_1^\pm(s, \mathbf{z})^2 \bigg) ds.$$

In the formula of $y_2^\pm(\theta, \mathbf{z})$, the second derivative $\dfrac{\partial^2 F_0^\pm}{\partial \mathbf{x}^2}(s, \varphi^\pm(s, \mathbf{z}))$ is a bilinear form defined on $\mathbb{R}^p \times \mathbb{R}^p$ which is applied to a "product" of two vectors, in our case $y_1^\pm(s, \mathbf{z})^2$.

Now, consider

$$(2.7) \qquad g_i(\mathbf{z}) = y_i^+(\phi, \mathbf{z}) - y_i^-(\phi - T, \mathbf{z}), \text{ for } i = 0, 1, 2.$$

The functions $g_1$ and $g_2$ are usually called averaged functions of order 1 and 2, respectively. Finally, assuming that the lower right corner $(p - q) \times (p - q)$ matrix of $Y^+(\phi, \nu) - Y^-(\phi - T, \nu)$, denoted by $\Delta_\nu$, is invertible, we define

$$(2.8) \qquad \gamma(\nu) = -\Delta_\nu^{-1} \xi^\perp g_1(\mathbf{z}_\nu).$$

Hence, the bifurcation functions $f_1, f_2 : \overline{V} \to \mathbb{R}^q$ of order 1 and 2 are given, respectively, by

$$
\begin{aligned}
(2.9) \quad & f_1(\nu) = \xi g_1(\mathbf{z}_\nu), \\
& f_2(\nu) = 2\frac{\partial \xi g_1}{\partial v}(\mathbf{z}_\nu)\gamma(\nu) + \frac{\partial^2 \xi g_0}{\partial v^2}(\mathbf{z}_\nu)\gamma(\nu)^2 + 2\xi g_2(\mathbf{z}_\nu).
\end{aligned}
$$

Again, in the formula of $f_2$, the second derivative $\dfrac{\partial^2 \xi g_0}{\partial v^2}(\mathbf{z}_\nu)$ is a bilinear form defined on $\mathbb{R}^{(p-q)} \times \mathbb{R}^{(p-q)}$. Thus, as before, we say that it is applied to a "product" of two vectors, in our case, $\gamma(\nu)^2$.

Our main result on the periodic solutions of system (2.1) is the following.

**Theorem 1.** *In addition to hypothesis $(H)$, we assume that for any $\nu \in \overline{V}$ the matrix $Y^+(\phi, \nu) - Y^-(\phi - T, \nu)$ has in the upper right corner the null $q \times (p-q)$ matrix, and in the lower right corner has the $(p-q) \times (p-q)$ matrix $\Delta_\nu$ with $\det(\Delta_\nu) \neq 0$. Then, the following statements hold.*

(a) *If there exists $\nu^* \in V$ such that $f_1(\nu^*) = 0$ and $\det(f_1'(\nu^*)) \neq 0$, then for $|\varepsilon| > 0$ sufficiently small there exists a $T$–periodic solution $\mathbf{x}(\theta, \varepsilon)$ of system (2.1) such that $\mathbf{x}(0, \varepsilon) \to \mathbf{z}_{\nu^*}$ as $\varepsilon \to 0$.*

(b) *Assume that $f_1 \equiv 0$. If there exists $\nu^* \in V$ such that $f_2(\nu^*) = 0$ and $\det(f_2'(\nu^*)) \neq 0$, then for $|\varepsilon| > 0$ sufficiently small there exists a $T$–periodic solution $\mathbf{x}(\theta, \varepsilon)$ of system (2.1) such that $\mathbf{x}(0, \varepsilon) \to \mathbf{z}_{\nu^*}$ as $\varepsilon \to 0$.*

Theorem 1 is proved in Section 4. The following result is an immediate consequence of Theorem 1.

**Corollary 2.** *Assume the hypothesis $(H)$ and that $q = p$, in this case $\mathcal{Z} = \overline{V} \subset D$ is a compact bounded $p$–dimensional manifold. Then, statements $(a)$ and $(b)$ of Theorem 1 hold by taking $f_1 = g_1$ and $f_2 = 2g_2$.*

### 2.2. Perturbations of higher dimensional linear systems

Consider a $(d+2) \times (d+2)$ matrix $M$ given by

$$
M = \begin{pmatrix} 0 & -1 & 0_{1\times d} \\ 1 & 0 & 0_{1\times d} \\ 0_{d\times 1} & 0_{d\times 1} & \widetilde{M} \end{pmatrix},
$$

where $0_{i\times j}$ denotes a null $i \times j$ matrix. When $0 < m < d$ assume that $\widetilde{M}$ is the diagonal matrix $\mathrm{diag}(\mu_1, \mu_2, \ldots, \mu_d)$ with $\mu_1 = \ldots = \mu_m = 0$ and $\mu_{m+1} \neq 0, \ldots, \mu_d \neq 0$. If $m = 0$, then $\widetilde{M}$ is a diagonal matrix with all entries distinct from zero, and if $m = d$ we assume that $\widetilde{M}$ is the null matrix.

6

Let $L_1 = \{(x, 0, z) : x \geq 0, z \in \mathbb{R}^d\}$ and $L_2 = \{(\lambda \cos\phi, \lambda \sin\phi, z) : \lambda \geq 0, z \in \mathbb{R}^d\}$ be two half–hyperplanes of $\mathbb{R}^{d+2}$ sharing the boundary $\{0, 0, z) : z \in \mathbb{R}^d\}$. The set $\Sigma = L_1 \cup L_2$ splits $D \subset \mathbb{R}^{d+2}$ in 2 disjoint open sectors, namely $C^+$ and $C^-$ (see Figure 1).
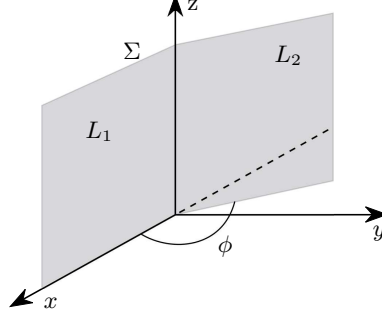


Figure 1: Set of discontinuity $\Sigma$.

We will denote by $X_\lambda$ and $Y_\lambda$ two polynomials of degree $n$ in the variables $x, y \in \mathbb{R}$ and $z = (z_1, \ldots, z_d) \in \mathbb{R}^d$, more precisely

$$X_\lambda(x, y, z) = \sum_{i+j+k_1+\ldots+k_d=0}^{n} \lambda_{ijk_1\ldots k_d} x^i y^j z_1^{k_1} \ldots z_d^{k_d}, \text{ and}$$

$$Y_\lambda(x, y, z) = \sum_{i+j+k_1+\ldots+k_d=0}^{n} \lambda_{ijk_1\ldots k_d} x^i y^j z_1^{k_1} \ldots z_d^{k_d},$$

for $\lambda_{ijk_1\ldots k_d} \in \mathbb{R}$ and $i, j, k_1, \ldots, k_d \in \mathbb{N}$. Then, take

$$(2.10) \quad X^\pm = (X_{a^\pm}, X_{b^\pm}, X_{c_1^\pm}, \ldots, X_{c_d^\pm}), \quad Y^\pm = (Y_{\alpha^\pm}, Y_{\beta^\pm}, Y_{\gamma_1^\pm}, \ldots, Y_{\gamma_d^\pm}),$$

and let $\mathcal{X}(x, y, z)$ and $\mathcal{Y}(x, y, z)$ be polynomial vector fields defined by

$$\mathcal{X}(x, y, z) = X^\pm(x, y, z) \quad \text{if} \quad (x, y, z) \in C^\pm,$$
$$\mathcal{Y}(x, y, z) = Y^\pm(x, y, z) \quad \text{if} \quad (x, y, z) \in C^\pm.$$

Now, consider the discontinuous piecewise polynomial differential systems

$$(2.11) \qquad (\dot{x}, \dot{y}, \dot{z}) = M(x, y, z) + \varepsilon \mathcal{X}(x, y, z) + \varepsilon^2 \mathcal{Y}(x, y, z),$$

where $x, y \in \mathbb{R}$ and $z = (z_1, z_2, \ldots, z_d) \in \mathbb{R}^d$. The dot denotes derivative with respect to the time $t$, and $\Sigma$ denotes the set of discontinuity for system (2.11). Also, $M(x, y, z)$ is an abuse of notation and denotes the matrix $M$ applied to the vector $(x, y, z)$, which is defined as the product between the matrix $M$ with the column matrix associated with the vector $(x, y, z)$. This abuse of notation will be recurrent throughout the paper.

Denote by $N_i(m, n, \phi)$ the maximum number of limit cycles of system (2.11) that can be detected using averaging theory of order $i$ when $|\varepsilon| \neq 0$ is sufficiently small.

**Theorem 3.** *Assume $0 \leq m \leq d$, $n \in \mathbb{N}$, and $\phi \in (0, 2\pi) \setminus \{\pi\}$. Then,*

(a) $N_1(m, n, \phi) = n^{m+1}$ *and*

(b) $2n(2n - 1)^m \leq N_2(m, n, \phi) \leq (2n)^{m+1}$.

Theorem 3 generalizes the particular case $m = d$ of [26]. Comparing itens $(a)$ and $(b)$ of Theorem 3, we can easily check that $N_2(m, n, \phi) > N_1(m, n, \phi)$ for every $0 \leq m \leq d$, $n \in \mathbb{N}$, and $\phi \in (0, 2\pi) \setminus \{\pi\}$.

Notice that, the lower and upper bounds given in statement $(b)$ of Theorem 3 coincide for $m = 0$. In this case, $N_2(0, n, \phi) = 2n$. In general, the lower bound of statement $(b)$ of Theorem 3 is not optimal and can be improved in some cases (see Proposition 5.1).

Theorems 3 is proved in section 5.

If $\phi = \pi$ we note that the maximum number of limit cycles eventually decreases as stated in the following result.

**Theorem 4.** *Assume $0 \leq m \leq d$ and $\phi = \pi$. Then,*

(a) $N_1(m, n, \pi) = n^{m+1}$ *and*

(b) $N \leq N_2(m, n, \pi) \leq (2n)^{m+1}$ *where $N = (2n - 1)^{m+1}$ if $n$ is odd, and*
   $N = (2n - 2)(2n - 1)^m$ *if $n$ is even.*

Theorem 4 is proved in Section 6.

Comparing itens $(a)$ and $(b)$ of Theorem 4, we can check that $N_2(m, n, \pi) \geq N_1(m, n, \pi)$ for every $0 \leq m \leq d$ and $n \in \mathbb{N}$, with strictly inequality for $n \neq 1$.

When $\phi = 2\pi$, system (2.11) is continuous. In this case $\mathcal{X}(x, y, z) = X^+(x, y, z)$ and $\mathcal{Y}(x, y, z) = Y^+(x, y, z)$. So, we get the following result.

**Theorem 5.** *Assume that $0 \leq m \leq d$ and $\phi = 2\pi$. Then,*

(a) $N_1(m, n, 2\pi) = n^m(n - 1)/2$ *for all $m \neq 0$, and*

$$N_1(0, n, 2\pi) = \begin{cases} \dfrac{n - 1}{2} & \text{if $n$ is odd,} \\[2mm] \dfrac{n - 2}{2} & \text{if $n$ is even.} \end{cases}$$

(b) $n \leq N_2(0, n, 2\pi) \leq 2n$.

Theorem 5 generalizes the particular cases $m = d = 0$ and $m = d = 1$ of [7] (see Theorems 2 and 3). Moreover, statement $(a)$ of Theorem 5 also generalizes Theorem 1 of [26] when $m = d$. We prove Theorem 5 in Section 7.

## 3. Preliminary results

In this section we present some preliminaries results that we shall need in Sections 5, 6 and 7. In Section 3.1, we present a change of coordinates so that system (2.11) reads in the standard form (2.1) to apply the averaging method. In Section 3.2, we construct the averaging functions $f_1$ and $f_2$ for system (2.11), defined in (2.9). Finally, in Section 3.3 we present some trigonometric relations that will be used in the calculus of the zeros of the functions $f_1$ and $f_2$.

### 3.1. Standard form

Let $x, y \in \mathbb{R}$ and $z = (z_1, \ldots, z_d) \in \mathbb{R}^d$. Using the change of variables

$$(3.1) \qquad x = r \cos\theta \quad \text{and} \quad y = r \sin\theta,$$

with $r \in \mathbb{R}_+$ and $\theta \in \mathbb{S}^1 \equiv \mathbb{R}/(2\pi\mathbb{Z})$, system (2.11) becomes

$$(3.2) \qquad \left(\dot{\theta}, \dot{r}, \dot{z}\right) = (1, 0, \widetilde{M}z) + \varepsilon A(\theta, r, z) + \varepsilon^2 B(\theta, r, z),$$

where $A, B : \mathbb{S}^1 \times \mathbb{R}_+ \times \mathbb{R}^d \to \mathbb{R}^{d+2}$ are piecewise smooth functions given by

$$A = \begin{cases} A^+ & \text{if} \quad 0 \le \theta \le \phi, \\ A^- & \text{if} \quad \phi \le \theta \le 2\pi, \end{cases} \quad \text{and} \quad B = \begin{cases} B^+ & \text{if} \quad 0 \le \theta \le \phi, \\ B^- & \text{if} \quad \phi \le \theta \le 2\pi, \end{cases}$$

where

$$A^\pm(\theta, r, z) = \quad (A_1^\pm(\theta, r, z), \ldots, A_{d+2}^\pm(\theta, r, z)),$$

$$B^\pm(\theta, r, z) = \quad (B_1^\pm(\theta, r, z), \ldots, B_{d+2}^\pm(\theta, r, z)),$$

with

$$\begin{aligned}
A_1^\pm &= \quad \frac{1}{r}\left(X_{b^\pm}\left(r\cos\theta, r\sin\theta, z\right)\cos\theta - X_{a^\pm}\left(r\cos\theta, r\sin\theta, z\right)\sin\theta\right), \\
B_1^\pm &= \quad \frac{1}{r}\left(Y_{\beta^\pm}\left(r\cos\theta, r\sin\theta, z\right)\cos\theta - Y_{\alpha^\pm}\left(r\cos\theta, r\sin\theta, z\right)\sin\theta\right), \\
A_2^\pm &= \quad X_{a^\pm}\left(r\cos\theta, r\sin\theta, z\right)\cos\theta + X_{b^\pm}\left(r\cos\theta, r\sin\theta, z\right)\sin\theta, \\
B_2^\pm &= \quad Y_{\alpha^\pm}\left(r\cos\theta, r\sin\theta, z\right)\cos\theta + Y_{\beta^\pm}\left(r\cos\theta, r\sin\theta, z\right)\sin\theta, \\
A_{\ell+2}^\pm &= \quad X_{c_\ell^\pm}(r\cos\theta, r\sin\theta, z), \\
B_{\ell+2}^\pm &= \quad Y_{\gamma_\ell^\pm}(r\cos\theta, r\sin\theta, z),
\end{aligned}$$

(3.3)

for $1 \le \ell \le d$. Clearly the discontinuity $\Sigma$ is now given by

$$\Sigma = \{(0, r, z) : r \in \mathbb{R}_+, z \in \mathbb{R}^d\} \cup \{(\phi, r, z) : r \in \mathbb{R}_+, z \in \mathbb{R}^d\}.$$

Taking the angle $\theta$ as the new time, system (3.2) reads

(3.4)
$$r' = \frac{\dot{r}}{\dot{\theta}} = \frac{\varepsilon A_2(\theta, r, \mathrm{z}) + \varepsilon^2 B_2(\theta, r, \mathrm{z})}{1 + \varepsilon A_1(\theta, r, \mathrm{z}) + \varepsilon^2 B_1(\theta, r, \mathrm{z})},$$

$$z'_\ell = \frac{\dot{z}_\ell}{\dot{\theta}} = \frac{\mu_\ell z_\ell + \varepsilon A_{\ell+2}(\theta, r, \mathrm{z}) + \varepsilon^2 B_{\ell+2}(\theta, r, \mathrm{z})}{1 + \varepsilon A_1(\theta, r, \mathrm{z}) + \varepsilon^2 B_1(\theta, r, \mathrm{z})},$$

for $1 \le \ell \le d$. Note that now the prime denotes derivative with respect to the independent variable $\theta$.

Expanding system (3.4) in Taylor series around $\varepsilon = 0$, it can be written as system (2.1) by taking $\mathbf{x} = (r, \mathrm{z}) \in D \subset \mathbb{R}_+ \times \mathbb{R}^d$ and

(3.5) $$F_j^\pm(\theta, r, \mathrm{z}) = (F_{j0}^\pm(\theta, r, \mathrm{z}), \ldots, F_{jd}^\pm(\theta, r, \mathrm{z})), \quad \text{for} \quad j = 0, 1, 2,$$

where

$$F_{0\ell}^\pm(\theta, r, \mathrm{z}) = 0,$$

$$F_{0\omega}^\pm(\theta, r, \mathrm{z}) = \mu_\omega z_\omega,$$

$$F_{1\ell}^\pm(\theta, r, \mathrm{z}) = A_{\ell+2}^\pm(\theta, r, \mathrm{z}),$$

(3.6) $$F_{1\omega}^\pm(\theta, r, \mathrm{z}) = A_{\omega+2}^\pm(\theta, r, \mathrm{z}) - \mu_\omega z_\omega A_1^\pm(\theta, r, \mathrm{z}),$$

$$F_{2\ell}^\pm(\theta, r, \mathrm{z}) = B_{\ell+2}^\pm(\theta, r, \mathrm{z}) - A_1^\pm(\theta, r, \mathrm{z}) A_{\ell+2}^\pm(\theta, r, \mathrm{z}),$$

$$F_{2\omega}^\pm(\theta, r, \mathrm{z}) = B_{\omega+2}^\pm(\theta, r, \mathrm{z}) + \mu_\omega z_\omega \left(A_1^\pm(\theta, r, \mathrm{z})\right)^2$$

$$\qquad\qquad - A_1^\pm(\theta, r, \mathrm{z}) A_{\omega+2}^\pm(\theta, r, \mathrm{z}) - \mu_\omega z_\omega B_1^\pm(\theta, r, \mathrm{z}),$$

for $0 \le \ell \le m$ and $m+1 \le \omega \le d$.

When $m = d$ the functions $F_{j\omega}^\pm$, for $j = 0, 1, 2$, do not be considered.

### 3.2. Construction of the averaging functions

Now, we shall use the notations introduced in subSection 2.1. Since the unperturbed system (2.2) is continuous, we have $\varphi^+(\theta, \mathbf{z}) = \varphi^-(\theta, \mathbf{z})$. Therefore, when $0 \le m < d$ the solution of system (2.2) is given by

$$\varphi(\theta, \mathbf{z}) = (r, z_1, \ldots, z_m, e^{\mu_{m+1}\theta} z_{m+1}, \ldots, e^{\mu_d \theta} z_d),$$

for $\mathbf{z} = (r, \mathrm{z}) = (r, z_1, \ldots, z_d)$. Note that if $\mathbf{z}_\nu = (r, z_1, \ldots, z_m, 0, \ldots, 0)$ then $\varphi(\theta, \mathbf{z}_\nu) = \mathbf{z}_\nu$ for every $\theta \in \mathbb{S}^1$. Then, taking an open bounded subset $V \subset \mathbb{R}^{m+1}$ and the zero function $\sigma : \overline{V} \to \mathbb{R}^{d-m}$, the manifold $\mathcal{Z}$, defined in (2.4), becomes

$$\mathcal{Z} = \{\mathbf{z}_\nu = (\nu, 0) \in \mathbb{R}^{d+1} : \nu = (r, z_1, \ldots, z_m) \in \overline{V}\}.$$

For $\mathbf{z} \in D$ a fundamental matrix of system (2.5) is

$$Y(\theta, \mathbf{z}) = \begin{pmatrix} \mathrm{Id}_{1+m} & 0 \\ 0 & \Delta \end{pmatrix},$$

where $\mathrm{Id}_{1+m}$ is the $(1+m) \times (1+m)$ identity matrix, and $\Delta$ is the diagonal matrix $\mathrm{diag}(e^{\mu_{m+1}\theta}, \ldots, e^{\mu_d \theta})$. Since $Y(\theta, \mathbf{z})$ does not depend of $\mathbf{z}$ we denote $Y(\theta, \mathbf{z}) = Y(\theta)$. Then, we have

$$Y(\phi) - Y(\phi - 2\pi) = \begin{pmatrix} 0 & 0 \\ 0 & \Delta_\nu \end{pmatrix},$$

where

(3.7) $$\Delta_\nu = \mathrm{diag}\big(e^{\mu_{m+1}\phi}(1 - e^{-\mu_{m+1}2\pi}), \ldots, e^{\mu_d\phi}(1 - e^{-\mu_d 2\pi})\big).$$

According to the notation introduced in Theorem 1 we have $p = d + 1$ and $p - q = d - m$, with $q = m + 1$. Since $\mathcal{Z}$ has dimension $m + 1$, we consider the projections $\xi : \mathbb{R}^{m+1} \times \mathbb{R}^{d-m} \to \mathbb{R}^{m+1}$ and $\xi^\perp : \mathbb{R}^{m+1} \times \mathbb{R}^{d-m} \to \mathbb{R}^{d-m}$, with $u = (r, z_1, \ldots, z_m) \in \mathbb{R}^{m+1}$ and $v = (z_{m+1}, \ldots, z_d) \in \mathbb{R}^{d-m}$.

From (2.6) and (3.6) we have $y_1(\theta, \mathbf{z}) = \big(y_{10}(\theta, \mathbf{z}), \ldots, y_{1d}(\theta, \mathbf{z})\big)$ where

(3.8)
$$y_{1\ell}^\pm(\theta, \mathbf{z}) = \int_0^\theta A_{\ell+2}^\pm(s, \varphi(s, \mathbf{z}))ds,$$

$$y_{1\omega}^\pm(\theta, \mathbf{z}) = \int_0^\theta e^{\mu_\omega(\theta - s)}\big(A_{\omega+2}^\pm(s, \varphi(s, \mathbf{z})) - \mu_\omega z_\omega A_1^\pm(s, \varphi(s, \mathbf{z}))\big)ds,$$

for $0 \le \ell \le m$ and $m + 1 \le \omega \le d$.

Moreover, from (2.7) we have $g_1(\mathbf{z}_\nu) = \big(g_{10}(\mathbf{z}_\nu), \ldots, g_{1d}(\mathbf{z}_\nu)\big)$ with

(3.9)
$$g_{1\ell}(\mathbf{z}_\nu) = \int_0^\phi A_{\ell+2}^+(s, \varphi(s, \mathbf{z}_\nu))\,ds + \int_\phi^{2\pi} A_{\ell+2}^-(s, \varphi(s, \mathbf{z}_\nu))\,ds,$$

$$g_{1\omega}(\mathbf{z}_\nu) = \int_0^\phi e^{\mu_\omega(\phi - s)} A_{\omega+2}^+(s, \varphi(s, \mathbf{z}_\nu))\,ds + \int_\phi^{2\pi} e^{\mu_\omega(\phi - 2\pi - s)} A_{\omega+2}^-(s, \varphi(s, \mathbf{z}_\nu))\,ds$$

for $0 \le \ell \le m$ and $m + 1 \le \omega \le d$.

Therefore, the bifurcation function $f_1 : \overline{V} \to \mathbb{R}^{m+1}$, defined in (2.9), is given by

(3.10) $$f_1(\nu) = \xi g_1(\mathbf{z}_\nu) = \big(f_{10}(\nu), \ldots, f_{1m}(\nu)\big),$$

with $f_{1\ell}(\nu) = g_{1\ell}(\mathbf{z}_\nu)$, where $g_{1\ell}$ is given in (3.9) for $0 \le \ell \le m$.

Now, we compute the bifurcation function $f_2$ defined also in (2.9).

Since $g_0$ is linear (see (2.6) and (2.7)) we have $\dfrac{\partial^2 \xi g_0}{\partial v^2}(\mathbf{z}_\nu) = 0$.

Moreover, as $\xi^\perp g_1(\mathbf{z}_\nu) = \big(g_{1\,m+1}(\mathbf{z}_\nu), \dots, g_{1\,d}(\mathbf{z}_\nu)\big)$, it follows from (2.8), (3.7) and (3.9) that
$$\gamma(\nu) = \big(\gamma_{m+1}(\nu), \dots, \gamma_d(\nu)\big),$$

where
(3.11)
$$\gamma_\omega(\nu) = \frac{-1}{1 - e^{-\mu_\omega 2\pi}} \left( \int_0^\phi e^{-\mu_\omega s} A_{\omega+2}^+(s, \mathbf{z}_\nu)\,ds + \int_\phi^{2\pi} e^{-\mu_\omega(2\pi+s)} A_{\omega+2}^-(s, \mathbf{z}_\nu)\,ds \right),$$

for $m+1 \le \omega \le d$. Furthermore, for $v = (z_{m+1}, \dots, z_d)$ we have
$$\frac{\partial \xi g_1}{\partial v}(\mathbf{z}_\nu)\gamma(\nu) = \big(\widetilde{G}_{10}(\nu), \dots, \widetilde{G}_{1m}(\nu)\big),$$

with

(3.12)
$$\widetilde{G}_{1\ell}(\nu) = \sum_{\omega=m+1}^d \frac{\partial g_{1\ell}}{\partial z_\omega}(\mathbf{z}_\nu)\gamma_\omega(\nu),$$

where $g_{1\ell}$ is given in (3.9) for $0 \le \ell \le m$. Additionally from (2.7) and (2.6) we obtain
$$\xi g_2(\mathbf{z}_\nu) = \xi(y_2^+(\phi, \mathbf{z}_\nu)) - \xi(y_2^-(\phi - 2\pi, \mathbf{z}_\nu)),$$

where
$$\xi y_2^\pm(\theta, \mathbf{z}_\nu) = 2\int_0^\theta \xi\big(F_2^\pm(s, \mathbf{z}_\nu)\big) + \xi\left(\frac{\partial F_1^\pm}{\partial \mathbf{x}}(s, \mathbf{z}_\nu)y_1^\pm(s, \mathbf{z}_\nu)\right)ds,$$

because $F_0^\pm$ is linear.

On the other hand
$$\xi F_2^\pm(s, \mathbf{z}_\nu) = \big(F_{20}^\pm(s, \mathbf{z}_\nu), \dots, F_{2m}^\pm(s, \mathbf{z}_\nu)\big), \text{ and}$$

$$\xi\left(\frac{\partial F_1^\pm}{\partial \mathbf{x}}(s, \mathbf{z}_\nu)y_1^\pm(s, \mathbf{z}_\nu)\right) = \big(\widetilde{F}_{10}^\pm(s, \mathbf{z}_\nu), \dots, \widetilde{F}_{1m}^\pm(s, \mathbf{z}_\nu)\big),$$

being

(3.13) $$\widetilde{F}_{1\ell}^\pm(s, \mathbf{z}_\nu) = \frac{\partial F_{1\ell}^\pm}{\partial r}(s, \mathbf{z}_\nu)y_{10}^\pm(s, \mathbf{z}_\nu) + \dots + \frac{\partial F_{1\ell}^\pm}{\partial z_d}(s, \mathbf{z}_\nu)y_{1d}^\pm(s, \mathbf{z}_\nu),$$

for $F_{1\ell}^\pm$ and $F_{2\ell}^\pm$ defined in (3.6) for $0 \le \ell \le m$. Hence

(3.14) $$f_2(\nu) = 2\frac{\partial \xi g_1}{\partial v}(\mathbf{z}_\nu)\gamma(\nu) + 2\,\xi g_2(\mathbf{z}_\nu) = \big(f_{20}(\nu), \dots, f_{2m}(\nu)\big),$$

where

(3.15)
$$\begin{aligned}
f_{2\ell}(\nu) = \ & 2\widetilde{G}_{1\ell}(\nu) + 4\int_0^\phi \big(F_{2\ell}^+(s, \mathbf{z}_\nu) + \widetilde{F}_{1\ell}^+(s, \mathbf{z}_\nu)\big)ds \\
& + 4\int_\phi^{2\pi} \big(F_{2\ell}^-(s, \mathbf{z}_\nu) + \widetilde{F}_{1\ell}^-(s, \mathbf{z}_\nu)\big)ds,
\end{aligned}$$

for $0 \leq \ell \leq m$. See the explicit expression of all functions that appear in (3.15) in the Appendix.

If $m = d$, then the functions $\widetilde{G}_{1\ell}(\nu)$ are not considered because $f_2 = 2g_2$ (see Corollary 2).

### 3.3. Some trigonometric integrals

In order to study the zeros of the averaging functions $f_1$ and $f_2$, we need to know some results about trigonometric integrals. Then, we shall state Lemma 6. The proof of this lemma will be omitted here, but it can easily be proven using some trigonometric relations found in Chapter 2 of [12].

For $p, q \in \mathbb{N}$ and $\phi \in (0, 2\pi]$ consider the functions

$$(3.16) \qquad I_{(p,q,\phi)} = \int_0^\phi \cos^p s \sin^q s \, ds, \quad J_{(p,q,\phi)} = \int_\phi^{2\pi} \cos^p s \sin^q s \, ds.$$

**Lemma 6.** *Let $I_{(p,q,\phi)}$ and $J_{(p,q,\phi)}$ be the functions defined in (3.16) for $\phi \in (0, 2\pi]$. Then, the following statements hold.*

(a) *If $\phi \neq \pi$ and $\phi \neq 2\pi$ then $I_{(p,q,\phi)}$, $J_{(p,q,\phi)}$, $\int_0^\phi \cos^i s \sin^j s \, I_{(p,q,\phi)} \, ds$, and $\int_\phi^{2\pi} \cos^i s \sin^j s \, I_{(p,q,\phi)} \, ds$ are non–zero;*

(b) *If $\phi = \pi$ then $I_{(p,q,\pi)} = 0$ or $J_{(p,q,\pi)} = 0$ if and only if $p$ is odd.*
*Moreover*

$$\int_0^\pi \cos^i s \sin^j s \, I_{(p,q,s)} \, ds = 0 \quad or \quad \int_\pi^{2\pi} \cos^i s \sin^j s \, I_{(p,q,s)} \, ds = 0$$

*if and only if one of the following statements hold:*

  (i) *$i$, $j$, $p$ and $q$ are odd;*
  (ii) *$i$, $p$ and $q$ are odd, and $j$ is even;*
  (iii) *$i$ and $p$ are odd, and $q$ and $j$ are even;*
  (iv) *$i$, $p$ and $j$ are odd, and $q$ is even.*

(c) *If $\phi = 2\pi$ then $I_{(p,q,2\pi)} \neq 0$ if and only if $p$ and $q$ are simultaneously even.*

## 4. Proof of Theorem 1

The proof of Theorem 1 is based on the next lemma which is a particular case of the *Lyapunov-Schmidt reduction* for a finite dimensional function (see for instance [6]).

**Lemma 7.** *Assuming $q \leq p$ are positive integers, let $D$ and $V$ be open bounded subsets of $\mathbb{R}^p$ and $\mathbb{R}^q$, respectively. Let $g : D \times (-\varepsilon_0, \varepsilon_0) \to \mathbb{R}^p$ and $\sigma : \overline{V} \to \mathbb{R}^{p-q}$ be $\mathcal{C}^3$ functions such that $g(\mathbf{z}, \varepsilon) = g_0(\mathbf{z}) + \varepsilon g_1(\mathbf{z}) + \varepsilon^2 g_2(\mathbf{z}) + \mathcal{O}(\varepsilon^3)$ and $\mathcal{Z} = \{\mathbf{z}_\nu = (\nu, \sigma(\nu)) : \nu \in \overline{V}\} \subset D$. We denote by $\Gamma_\nu$ the upper right corner $q \times (p-q)$ matrix of $D g_0(\mathbf{z}_\nu)$, and by $\Delta_\nu$ the lower right corner $(p-q) \times (p-q)$ matrix of $D g_0(\mathbf{z}_\nu)$. Assume that for each $\mathbf{z}_\nu \in \mathcal{Z}$, $\det(\Delta_\nu) \neq 0$ and $g_0(\mathbf{z}_\nu) = 0$. We consider the functions $f_1, f_2 : \overline{V} \to \mathbb{R}^q$ defined in (2.9). Then, the following statements hold.*

(a) *If there exists $\nu^* \in V$ with $f_1(\nu^*) = 0$ and $\det(D f_1(\nu^*)) \neq 0$, then there exists $\nu_\varepsilon$ such that $g(\mathbf{z}_{\nu_\varepsilon}, \varepsilon) = 0$ and $\mathbf{z}_{\nu_\varepsilon} \to \mathbf{z}_{\nu^*}$ when $\varepsilon \to 0$.*

(b) *Assume that $f_1 = 0$. If there exists $\nu^* \in V$ with $f_2(\nu^*) = 0$ and $\det(D f_2(\nu^*)) \neq 0$, then there exists $\nu_\varepsilon$ such that $g(\mathbf{z}_{\nu_\varepsilon}, \varepsilon) = 0$ and $\mathbf{z}_{\nu_\varepsilon} \to \mathbf{z}_{\nu^*}$ when $\varepsilon \to 0$.*

The proof of this lemma can be found in [21].

Note that in Lemma 7 the functions $g_i$ for $i = 0, 1, 2$ which appears in the expression of (2.9) and (2.8) are the ones of the function

$$(4.1) \qquad g(z, \varepsilon) = g_0(z) + \varepsilon g_1(z) + \varepsilon^2 g_2(z) + \mathcal{O}(\varepsilon^3),$$

instead of the functions which appear in (2.7).

*Proof of Theorem 1.* Let $\psi(\theta, \mathbf{z}, \varepsilon)$ be a periodic solution of system (2.1) such that $\psi(0, \mathbf{z}, \varepsilon) = \mathbf{z}$. Similarly let $\psi^\pm(\theta, \mathbf{z}, \varepsilon)$ be the solutions of the systems $\mathbf{x}' = F^\pm(\theta, \mathbf{x}, \varepsilon)$ such that $\psi^\pm(0, \mathbf{z}, \varepsilon) = \mathbf{z}$. So

$$\psi(\theta, \mathbf{z}, \varepsilon) = \begin{cases} \psi^+(\theta, \mathbf{z}, \varepsilon) & \text{if} \quad 0 \leq \theta \leq \phi, \\ \psi^-(\theta, \mathbf{z}, \varepsilon) & \text{if} \quad \phi \leq \theta \leq T. \end{cases}$$

Since the vector field (2.1) is $T$–periodic, it may also read

$$\psi(\theta, \mathbf{z}, \varepsilon) = \begin{cases} \psi^+(\theta, \mathbf{z}, \varepsilon) & \text{if} \quad 0 \leq \theta \leq \phi, \\ \psi^-(\theta, \mathbf{z}, \varepsilon) & \text{if} \quad \phi - T \leq \theta \leq 0. \end{cases}$$

Now, we consider the function $g(\mathbf{z}, \varepsilon) = \psi^+(\phi, \mathbf{z}, \varepsilon) - \psi^-(\phi - T, \mathbf{z}, \varepsilon)$. It is easy to see that the solution $\psi(\theta, \mathbf{z}, \varepsilon)$ is $T$–periodic in $\theta$ if and only if $g(\mathbf{z}, \varepsilon) = 0$. So, from hypothesis $(H)$ we have that $g(\mathbf{z}_{\nu, \varepsilon}) = 0$ for every $\mathbf{z}_{\nu, \varepsilon} \in \mathcal{Z}$.

Using Taylor series to expand the functions $\psi^\pm(\theta, \mathbf{z}, \varepsilon)$ in powers of $\varepsilon$ we obtain

$$(4.2) \qquad \psi^\pm(\theta, \mathbf{z}, \varepsilon) = y_0^\pm(\theta, \mathbf{z}) + \varepsilon y_1^\pm(\theta, \mathbf{z}) + \varepsilon^2 \frac{y_2^\pm(\theta, \mathbf{z})}{2} + \mathcal{O}(\varepsilon^2),$$

where $y_i(\theta, \mathbf{z})$ is given in (2.6). We shall omit the computations for obtaining (4.2), nevertheless they can be found in [23]. Therefore, $g(\mathbf{z}, \varepsilon) = g_0(\mathbf{z}) + \varepsilon g_1(\mathbf{z}) + \varepsilon^2 g_2(\mathbf{z}) + \mathcal{O}(\varepsilon^2)$, where $g_i(\mathbf{z}) = y_i^+(\phi, \mathbf{z}) - y_i^-(\phi - T, \mathbf{z})$ for $i = 0, 1, 2$. Moreover

$$D g_0(\mathbf{z}) = \frac{\partial \varphi^+}{\partial \mathbf{z}}(\phi, \mathbf{z}) - \frac{\partial \varphi^-}{\partial \mathbf{z}}(\phi - T, \mathbf{z}) = Y^+(\phi, \mathbf{z}) - Y^-(\phi - T, \mathbf{z}).$$

So, from hypothesis of Theorem 1 we have that the matrix $Dg_0(\mathbf{z})$ has in the upper right corner the zero $q \times (d - q)$ matrix, and in the lower right corner has the $(p - q) \times (p - q)$ matrix $\Delta_\nu$ with $\det(\Delta_\nu) \neq 0$.

We conclude the proof of this theorem by applying Lemma 7 to the function $g(\mathbf{z}, \varepsilon)$ defined in (4.1). □

## 5. Proof of Theorem 3

In order to prove Theorem 3 we shall study the zeros of the averaging functions $f_1$ and $f_2$, given in (3.10) and (3.14), respectively, when $\phi \in (0, 2\pi) \setminus \{\pi\}$.

**Remark 8.** *For sake of simplicity we shall denote by $\lambda_{ijk_1\ldots k_m 0}$ the coefficient of $x^i y^j z_1^{k_1} \ldots z_m^{k_m}$, and by $\lambda_{ij0}$ the coefficient of $x^i y^j$ of system (2.11), when $\lambda = a^\pm, b^\pm, c_\ell^\pm$ for all $1 \leq \ell \leq m$.*

From statement $(a)$ of Lemma 6 we have $f_1(\nu) = (f_{10}(\nu), \ldots, f_{1m}(\nu))$ where

(5.1)

$$f_{10}(\nu) = \sum_{i+j+k_1+\ldots+k_m=0}^{n} r^{i+j} z_1^{k_1} \ldots z_m^{k_m} \Big( a_{ijk_1\ldots k_m 0}^+ I_{(i+1,j,\phi)}$$

$$+ b_{ijk_1\ldots k_m 0}^+ I_{(i,j+1,\phi)} + a_{ijk_1\ldots k_m 0}^- J_{(i+1,j,\phi)} + b_{ijk_1\ldots k_m 0}^- J_{(i,j+1,\phi)} \Big),$$

$$f_{1\ell}(\nu) = \sum_{i+j+k_1+\ldots+k_m=0}^{n} r^{i+j} z_1^{k_1} \ldots z_m^{k_m} \Big( c_{\ell,ijk_1\ldots k_m 0}^+ I_{(i,j,\phi)} + c_{\ell,ijk_1\ldots k_m 0}^- J_{(i,j,\phi)} \Big),$$

with $\nu = (r, z_1, \ldots, z_m)$ and $1 \leq \ell \leq m$.

**Proposition 9.** *Assume $0 \leq m \leq d$ and $\phi \neq \pi$. Then $f_1$ has at most $n^{m+1}$ simple zeros and this number can be reached.*

*Proof.* For each $0 \leq \ell \leq m$ and $\nu = (r, z_1, \ldots, z_m)$, $f_{1\ell}(\nu)$ is a complete polynomial of degree $n$. Recall that a *complete polynomial of degree $k$* means a polynomial that appears all its monomials. By Bezout Theorem (see [9]), $f_1(\nu)$ can be at most $n^{m+1}$ simple zeros. Since all the coefficients of $f_1(\nu)$ are independent, we can choose them in order that $f_1(\nu)$ has exactly $n^{m+1}$ zeros with $r > 0$, and $\det f_1'(\nu^*) \neq 0$ for each zero $\nu^*$ of $f_1(\nu)$ (that is, $\nu^*$ is a simple zero). □

**Proposition 10.** *Take $0 \leq m \leq d$ and $\phi \neq \pi$. If $f_1 \equiv 0$ then $f_2$ has at most $(2n)^{m+1}$ simple zeros, and a lower bound for the maximum number of simple zeros is $(2n)(2n - 1)^m$.*

*Proof.* Assume that $f_1 \equiv 0$. From (5.1) it follows that

(5.2)
$$\sum_{i+j=s} a_{ijk_1\ldots k_m 0}^+ I_{(i+1,j,\phi)} + b_{ijk_1\ldots k_m 0}^+ I_{(i,j+1,\phi)}$$

$$+ a_{ijk_1\ldots k_m 0}^- J_{(i+1,j,\phi)} + b_{ijk_1\ldots k_m 0}^- J_{(i,j+1,\phi)} = 0,$$

$$\sum_{i+j=s} c_{\ell,ijk_1\ldots k_m 0}^+ I_{(i,j,\phi)} + c_{\ell,ijk_1\ldots k_m 0}^- J_{(i,j,\phi)} = 0,$$

for $1 \leq \ell \leq m$, $0 \leq s \leq n$, $0 \leq k_\ell \leq n$ with $0 \leq k_1 + \ldots + k_m \leq n - s$.

Moreover, $f_2(\nu) = (f_{20}(\nu), \ldots, f_{2m}(\nu))$ with $\nu = (r, z_1, \ldots, z_m)$. In particular, if $m = 0$ then $f_2(\nu) = f_{20}(r)$. Considering the expression for $f_{2\ell}(\nu)$, given in (3.15) for $0 \leq \ell \leq m$, we conclude that $\widetilde{G}_{1\ell}(\nu)$ and $\int_0^\phi \widetilde{F}_{1\ell}^+(s, \mathbf{z}_\nu) \, ds + \int_\phi^{2\pi} \widetilde{F}_{1\ell}^-(s, \mathbf{z}_\nu) \, ds$ are complete polynomials of degree $2n - 1$ in the variables $(r, z_1, \ldots, z_m)$, and

$$\int_0^\phi F_{2\ell}^+(s, \mathbf{z}_\nu) \, ds + \int_\phi^{2\pi} F_{2\ell}^-(s, \mathbf{z}_\nu) \, ds = \frac{1}{r} \sum_{k=0}^{2n} Q_k(z_1, \ldots, z_m) \, r^k,$$

where $\mathbf{z}_\nu = (r, z_1, \ldots, z_m, 0, \ldots, 0) \in \mathbb{R}^{d+1}$, $Q_k(z_1, \ldots, z_m)$ is a complete polynomial of degree $2n - k$ in the variables $(z_1, \ldots, z_m)$ if $m \neq 0$, and $Q_k(z_1, \ldots, z_m)$ is constant if $m = 0$. The above equality is evident if we take into account statement $(a)$ of Lemma 6 and conditions (5.2). Therefore, each $r f_{2\ell}(\nu)$ is a complete polynomial of degree $2n$ in the variables $(r, z_1, \ldots, z_m)$. Since $r > 0$, it is known that $r f_{2\ell}(\nu) = 0$ if and only if $f_{2\ell}(\nu) = 0$ for each $0 \leq \ell \leq m$. Then, by Bezout Theorem, $f_2(\nu)$ has at most $(2n)^{m+1}$ simple zeros for all $0 \leq m \leq d$.

In order to show that the maximum number is greater than or equal to $(2n)(2n-1)^m$ we provide a particular example. So, take $a_{i00}^\pm \neq 0$, $c_{\ell,00\ldots 0k_\ell 0}^\pm \neq 0$, and we take zero all the other coefficients for $1 \leq \ell \leq m$. From (3.15) we obtain $f_{20}(\nu) = f_{20}(r)$ and $f_{2\ell}(\nu) = f_{2\ell}(r, z_\ell)$, where

$$\begin{aligned}
f_{20}(r) = \quad & \frac{4}{r} \sum_{i=0}^n \sum_{p=0}^n r^{i+p} \Big( a_{i00}^+ a_{p00}^+ I_{(i+p+1,1,\phi)} + a_{i00}^- a_{p00}^- J_{(i+p+1,1,\phi)} \\
& + i \, a_{i00}^+ a_{p00}^+ \int_0^\phi \cos^{i+1} s \, I_{(p+1,0,s)} ds + i a_{i00}^- a_{p00}^- \int_\phi^{2\pi} \cos^{i+1} s I_{(p+1,0,s)} ds \Big),
\end{aligned}$$

$$\begin{aligned}
f_{2\ell}(r, z_\ell) = \quad & \frac{4}{r} \sum_{i=0}^n \sum_{k_l=0}^n r^i z_\ell^{k_\ell} \Big( a_{i00}^+ c_{\ell,0\ldots 0k_\ell 0}^+ I_{(i,1,\phi)} + a_{i00}^- c_{\ell,0\ldots 0k_\ell 0}^- J_{(i,1,\phi)} \Big) \\
& + 4 \sum_{k_\ell=1}^n \sum_{L_\ell=0}^n z_\ell^{k_\ell + L_\ell - 1} \Big( \frac{\phi^2}{2} k_\ell \, c_{\ell,0\ldots 0k_\ell 0}^+ \, c_{\ell,0\ldots 0L_\ell 0}^+ \\
& + \frac{(2\pi)^2 - \phi^2}{2} k_\ell \, c_{\ell,0\ldots 0k_\ell 0}^- \, c_{\ell,0\ldots 0L_\ell 0}^- \Big),
\end{aligned}$$

where $a_{i00}^+ I_{(i+1,0,\phi)} = -a_{i00}^- J_{(i+1,0,\phi)}$ and $c_{\ell,00\ldots 0k_\ell 0}^+ I_{(0,0,\phi)} = -c_{\ell,00\ldots 0k_\ell 0}^- J_{(0,0,\phi)}$ for $1 \leq \ell \leq m$ (see (5.2)).

From statement $(a)$ of Lemma 6, $r f_{20}(r)$ is a complete polynomial of degree $2n$ in the variable $r$, whose coefficients are independent. Furthermore, if $f_{20}(r^*) = 0$ with $r^* > 0$, then $f_{2\ell}(r^*, z_\ell)$ is a polynomial of degree $2n - 1$ in the variable $z_\ell$, and all their coefficients are independent for $1 \leq \ell \leq m$. Therefore, By Bezout Theorem, $f_2(\nu)$ has at most $(2n)(2n-1)^m$ simple zeros, and this number can be reached due to the independence of coefficients. $\square$

*Proof of Theorem 3.* We apply Theorem 1 to the function $f_1$ of Proposition 9 and we conclude statement $(a)$. Statement $(b)$ is proved applying Theorem 1 to the functions $f_1$ and $f_2$ given in Proposition 10. $\square$

## 5.1. Improving the lower bound

As mentioned in the introduction, the lower bound of statement $(b)$ of Theorem 3 is not optimal and can be improved. From Theorem 1 we need to solve the equation $f_2(\nu) = 0$, assuming $f_1 \equiv 0$. This can be a hard task due to the complexity of $f_2$. In what follows, we provide a simpler polynomial system for which their simple zeros imply the existence of simple zeros of $f_2$.

From (3.14) we have $f_2(\nu) = (f_{20}(\nu), \ldots, f_{2m}(\nu))$. In (3.15) we can take $\widetilde{G}_{1\ell}(\nu) = 0$ and, since $1/r$ appears as a common factor in the expression of $A_1^{\pm}$ (3.3), we define $\widetilde{A}_1^{\pm} = r A_1^{\pm}$. Finally, for $1 \leq \ell \leq m$, we assume that $A_{\ell+2}^{\pm} = \delta \widetilde{A}_{\ell+2}^{\pm}$ and $B_{\ell+2}^{\pm} = \delta \widetilde{B}_{\ell+2}^{\pm}$ for $\delta > 0$ sufficiently small. Notice that, the assumption is equivalent to ask that the coefficients of the perturbation (2.10) for $1 \leq \ell \leq m$ are of order $\delta$.

Now, for $1 \leq \ell \leq m$, we define

$$
\begin{aligned}
P_\ell(\nu) &= \int_0^\phi \widetilde{B}_{\ell+2}^+(s, \mathbf{z}_\nu)\, ds + \int_\phi^{2\pi} \widetilde{B}_{\ell+2}^-(s, \mathbf{z}_\nu)\, ds, \\
Q_\ell(\nu) &= \int_0^\phi \widetilde{A}_1^+(s, \mathbf{z}_\nu)\widetilde{A}_{\ell+2}^+(s, \mathbf{z}_\nu)\, ds + \int_\phi^{2\pi} \widetilde{A}_1^-(s, \mathbf{z}_\nu)\widetilde{A}_{\ell+2}^-(s, \mathbf{z}_\nu)\, ds.
\end{aligned}
$$

(5.3)

Thus, from (3.3), (3.6), (3.8) and (3.13) we have $\int \widetilde{F}_{1\ell}^{\pm}(s, \mathbf{z}_\nu)ds = \mathcal{O}_2(\delta)$ and, therefore,

$$
\frac{r}{4\delta} f_{2\ell}(\nu) = r P_\ell(\nu) - Q_\ell(\nu) + \mathcal{O}(\delta), \quad \text{for } 1 \leq \ell \leq m.
$$

Hence, taking $\delta > 0$ sufficiently small, we obtain the following proposition.

**Proposition 11.** *If the polynomial system*

(5.4)
$$
f_{20}(\nu) = 0 \quad and \quad r P_\ell(\nu) - Q_\ell(\nu) = 0, \quad for \ 1 \leq \ell \leq m,
$$

*has $N$ isolated solutions, then $N_2(m, n, \phi) \geq N$.*

# 6. Proof of Theorem 4

In this section we study the zeros of the functions $f_1$ and $f_2$, given in (3.10) and (3.14), respectively, when $\phi = \pi$. Then, we conclude Theorem 4 applying Theorem 2.1.

From statement $(b)$ of Lemma 6 we have $f_1(\nu) = (f_{10}(\nu), \ldots, f_{1\ell}(\nu))$ where
(6.1)
$$
\begin{aligned}
f_{10}(\nu) = & \sum_{i\,odd,\,P=0}^{n} r^{i+j} z_1^{k_1} \ldots z_m^{k_m} \left( a_{ijk_1\ldots k_m 0}^{+} I_{(i+1,j,\pi)} + a_{ijk_1\ldots k_m 0}^{-} J_{(i+1,j,\pi)} \right) \\
& + \sum_{i\,even,\,P=0}^{n} r^{i+j} z_1^{k_1} \ldots z_m^{k_m} \left( b_{ijk_1\ldots k_m 0}^{+} I_{(i,j+1,\pi)} + b_{ijk_1\ldots k_m 0}^{-} J_{(i,j+1,\pi)} \right), \\
f_{1\ell}(\nu) = & \sum_{i\,even,\,P=0}^{n} r^{i+j} z_1^{k_1} \ldots z_m^{k_m} \left( c_{\ell,ijk_1\ldots k_m 0}^{+} I_{(i,j,\pi)} + c_{\ell,ijk_1\ldots k_m 0}^{-} J_{(i,j,\pi)} \right),
\end{aligned}
$$

where $\nu = (r, z_1, \ldots, z_m)$, $1 \le \ell \le m$ and $P = i + j + k_1 + \ldots + k_m$.

**Proposition 12.** *Take $0 \le m \le d$ and $\phi = \pi$. Then, $f_1$ has at most $n^{m+1}$ simple zeros and this number can be reached.*

*Proof.* This proof is analogously to the proof of Proposition 9, noticing that for each $0 \le \ell \le m$, $f_{1\ell}(\nu)$ is a complete polynomial of degree $n$ in the variables $(r, z_1, \ldots, z_m)$ and all their coefficients are independent. $\square$

**Proposition 13.** *Assume $0 \le m \le d$ and $\phi = \pi$. If $f_1 \equiv 0$ then $f_2$ has at most $(2n)^{m+1}$ simple zeros, and the lower bound for the number of simple zeros is $(2n-1)^{m+1}$ if $n$ is odd, and $(2n-2)(2n-1)^m$ if $n$ is even.*

*Proof.* Assume that $f_1 \equiv 0$. From (6.1) it follows that

$$
\sum_{i\,odd,\,i+j=s} a_{ijk_1\ldots k_m 0}^{+} I_{(i+1,j,\pi)} + a_{ijk_1\ldots k_m 0}^{-} J_{(i+1,j,\pi)}
$$

(6.2)
$$
+ \sum_{i\,even,\,i+j=s} b_{ijk_1\ldots k_m 0}^{+} I_{(i,j+1,\pi)} + b_{ijk_1\ldots k_m 0}^{-} J_{(i,j+1,\pi)} = 0,
$$

$$
\sum_{i\,even,\,i+j=s} c_{\ell,ijk_1\ldots k_m 0}^{+} I_{(i,j,\pi)} + c_{\ell,ijk_1\ldots k_m 0}^{-} J_{(i,j,\pi)} = 0,
$$

for $1 \le \ell \le m$, $0 \le s \le n$, $0 \le k_\ell \le n$ with $0 \le k_1 + \ldots + k_m \le n - s$.

Moreover, $f_2(\nu) = (f_{20}(\nu), \ldots, f_{2m}(\nu))$ with $\nu = (r, z_1, \ldots, z_m)$. If $m = 0$ then $f_2(\nu) = f_{20}(r)$. Analogously to the proof of Proposition 10 we conclude that $f_2(\nu)$ has at most $(2n)^{m+1}$ simple zeros for all $0 \le m \le d$.

Now, we provide a particular example to exhibit the lower bound for the maximum number of simple zeros. So, take $a_{i00}^{\pm} \ne 0$, $c_{\ell,00\ldots 0k_\ell 0}^{\pm} \ne 0$, and take zero all the other coefficients for $1 \le \ell \le m$. From (3.15) we obtain $f_{20}(\nu) = f_{20}(r)$ and

$f_{2\ell}(\nu) = f_{2\ell}(r, z_\ell)$, where

$$f_{20}(r) = \frac{4}{r}\Bigg( \sum_{i\,even,\,i=0}^{n} \sum_{p\,odd,\,p=0}^{n} r^{i+p}\big(a_{i00}^+ a_{p00}^+ I_{(i+p+1,1,\pi)} + a_{i00}^- a_{p00}^- J_{(i+p+1,1,\pi)}\big)$$

$$+ i\,a_{i00}^+ a_{p00}^+ \int_0^\pi \cos^{i+1} s\, I_{(p+1,0,s)}ds + i\,a_{i00}^- a_{p00}^- \int_\pi^{2\pi} \cos^{i+1} s\, I_{(p+1,0,s)}ds\big)$$

$$+ \sum_{i\,odd,\,i=0}^{n} \sum_{p\,even,\,p=0}^{n} r^{i+p}\big(a_{i00}^+ a_{p00}^+ I_{(i+p+1,1,\pi)} + a_{i00}^- a_{p00}^- J_{(i+p+1,1,\pi)}\big)$$

$$+ i\,a_{i00}^+ a_{p00}^+ \int_0^\pi \cos^{i+1} s\, I_{(p+1,0,s)}ds + i\,a_{i00}^- a_{p00}^- \int_\pi^{2\pi} \cos^{i+1} s\, I_{(p+1,0,s)}ds\big)$$

$$+ \sum_{i\,odd,\,i=0}^{n} \sum_{p\,odd,\,p=0}^{n} r^{i+p}\big(a_{i00}^+ a_{p00}^+ I_{(i+p+1,1,\pi)} + a_{i00}^- a_{p00}^- J_{(i+p+1,1,\pi)}\big)$$

$$+ i\,a_{i00}^+ a_{p00}^+ \int_0^\pi \cos^{i+1} s\, I_{(p+1,0,s)}ds + i a_{i00}^- a_{p00}^- \int_\pi^{2\pi} \cos^{i+1} s I_{(p+1,0,s)}ds\Bigg),$$

and

$$f_{2\ell}(r, z_\ell) = \frac{4}{r}\sum_{i=0}^{n}\sum_{k_l=0}^{n} r^i z_\ell^{k_\ell}\big(a_{i00}^+ c_{\ell,0\ldots0k_\ell0}^+ I_{(i,1,\phi)} + a_{i00}^- c_{\ell,0\ldots0k_\ell0}^- J_{(i,1,\phi)}\big)$$

$$+ \sum_{k_\ell=1}^{n}\sum_{L_\ell=0}^{n} z_\ell^{k_\ell+L_\ell-1} k_\ell\left(\frac{\phi^2}{2} c_{\ell,0\ldots0k_\ell0}^+ c_{\ell,0\ldots0L_\ell0}^+\right.$$

$$\left. + \frac{(2\pi)^2-\phi^2}{2} c_{\ell,0\ldots0k_\ell0}^- c_{\ell,0\ldots0L_\ell0}^-\right),$$

for $1 \le \ell \le m$, where $a_{i00}^+ I_{(i+1,0,\pi)} = -a_{i00}^- J_{(i+1,0,\pi)}$ if $i$ is odd and $c_{\ell,00\ldots0k_\ell0}^+ I_{(0,0,\pi)} = -c_{\ell,00\ldots0k_\ell0}^- J_{(0,0,\pi)}$ (see (6.2)). Therefore, from statement $(b)$ of Lemma 6, $r f_{20}(r)$ is a complete polynomial in the variable $r$ of degree $2n-1$ if $n$ is odd, and $2n-2$ if $n$ is even, and its coefficients are independent. Furthermore, if $f_{20}(r^*) = 0$ with $r^* > 0$, then $f_{2\ell}(r^*, z_\ell)$ is a polynomial of degree $2n-1$ in the variable $z_\ell$ for each $1 \le \ell \le m$. Then, the number of simple zeros with $r > 0$ of $f_2(\nu)$ can be $(2n-1)^{m+1}$ if $n$ is odd, and $(2n-2)(2n-1)^m$ if $n$ is even. By the independence of all coefficients these numbers can be reached. $\square$

*Proof of Theorem 4.* From Theorem 1 and Proposition 12, statement $(a)$ holds, and applying Theorem 1 to the functions $f_1$ and $f_2$ given in Proposition 13 we conclude statement $(b)$. $\square$

## 7. Proof of Theorem 5

When $\phi = 2\pi$ system (2.11) is continuous. Then, considering the cylindrical coordinates given in (3.1), and taking $\theta$ as the new time, system (2.11) can be

written as system (2.1) that is,

$$\mathbf{x}' = F^+(\theta, \mathbf{x}, \varepsilon), \quad \text{for } 0 \le \theta \le 2\pi,$$

where

$$F^+(\theta, \mathbf{x}, \varepsilon) = F_0^+(\theta, \mathbf{x}) + \varepsilon F_1^+(\theta, \mathbf{x}) + \varepsilon^2 F_2^+(\theta, \mathbf{x}) + \varepsilon^3 R^+(\theta, \mathbf{x}, \varepsilon),$$

for $\mathbf{x} = (r, \mathbf{z})$ and $\mathbf{z} = (z_1, \ldots, z_d)$, with $F_j^+$ given in (3.5) and (3.6) for $j = 0, 1, 2$.

From statement $(c)$ of Lemma 6 we have $f_1(\nu) = (f_{10}(\nu), \ldots, f_{1m}(\nu))$ with

$$
\begin{aligned}
f_{10}(\nu) &= \sum_{\substack{i \, odd, \, j \, even, \, P=0}}^{n} r^{i+j} z_1^{k_1} \ldots z_m^{k_m} \, a_{ijk_1\ldots k_m 0}^+ I_{(i+1,j,2\pi)} \\
(7.1) \qquad &+ \sum_{\substack{i \, even, \, j \, odd, \, P=0}}^{n} r^{i+j} z_1^{k_1} \ldots z_m^{k_m} \, b_{ijk_1\ldots k_m 0}^+ I_{(i,j+1,2\pi)}, \\
f_{1\ell}(\nu) &= \sum_{\substack{i, \, j \, even, \, P=0}}^{n} r^{i+j} z_1^{k_1} \ldots z_m^{k_m} \, c_{\ell,ijk_1\ldots k_m 0}^+ I_{(i,j,2\pi)},
\end{aligned}
$$

where $\nu = (r, z_1, \ldots, z_m)$, $1 \le \ell \le m$ and $P = i + j + k_1 + \ldots + k_m$.

**Proposition 14.** *Assume $0 \le m \le d$ and $\phi = 2\pi$. If $m \neq 0$ then $f_1$ has at most $n^m(n-1)/2$ simple zeros and this number can be reached. If $m = 0$ then $f_1$ has at most $(n-1)/2$ simple zeros if $n$ is odd, and $(n-2)/2$ if $n$ is even, and these numbers can be reached.*

*Proof.* We have $f_{10}(\nu) = r\widetilde{f}_{10}(\nu)$ with

$$\widetilde{f}_{10}(\nu) = h_1 + r^2 h_3 + r^4 h_5 + r^6 h_7 + \ldots + \begin{cases} r^{n-1} h_n & \text{if n is odd,} \\ r^{n-2} h_{n-1} & \text{if n is even,} \end{cases}$$

where

$$
\begin{aligned}
h_k = \sum_{k_1+\ldots+k_m=0}^{n-k} z_1^{k_1} \ldots z_m^{k_m} \Bigg( &\sum_{\substack{i \, odd, \, j \, even, \, i+j=k}} a_{ijk_1\ldots k_m 0}^+ I_{(i+1,j,2\pi)} \\
+ &\sum_{\substack{i \, even, \, j \, odd, \, i+j=k}} b_{ijk_1\ldots k_m 0}^+ I_{(i,j+1,2\pi)} \Bigg).
\end{aligned}
$$

If $m \neq 0$ then $\widetilde{f}_{10}(\nu)$ and $f_{1\ell}(\nu)$ are polynomials in the variables $(r, z_1, \ldots, z_m)$ of degree $n - 1$ and $n$, respectively, for $1 \le \ell \le m$. From Bezout Theorem the maximum number of simple zeros of $f_1(\nu)$ is $n^m(n-1)$. Since the exponents of $r$ in the function $\widetilde{f}_{10}(\nu)$ are always even numbers, the maximum number of simple zeros of $f_1(\nu)$ is $n^m(n-1)/2$. In what follows we provide a particular example to prove that this number is reached.

First if $n$ is even we take $a^+_{10k_10} \neq 0$, $b^+_{01k_10} \neq 0$, $c^+_{1,ij0} \neq 0$, $c^+_{\ell,00k_\ell 0} \neq 0$ and take zero all the other coefficients in other that $\widetilde{f}_{10}(\nu) = \widetilde{f}_{10}(z_1)$, $f_{11}(\nu) = f_{11}(r)$, and $f_{1\ell}(\nu) = f_{11}(z_\ell)$, where

$$\widetilde{f}_{10}(z_1) = \sum_{k_1=0}^{n-1} z_1^{k_1} \left( a^+_{10k_10} I_{(2,0,2\pi)} + b^+_{01k_10} I_{(0,2,2\pi)} \right),$$

$$f_{11}(r) = \sum_{i,j \, even, \, i+j=0}^{n} r^{i+j} \, c^+_{1,ij0} \, I_{(i,j,2\pi)},$$

$$f_{1\ell}(z_\ell) = \sum_{k_\ell=0}^{n} z_\ell^{k_\ell} \, c^+_{\ell,00k_\ell 0} \, I_{(0,0,2\pi)},$$

for $2 \leq \ell \leq m$. Thus, $\widetilde{f}_{10}(z_1)$ is a complete polynomial of degree $n-1$ in the variable $z_1$, $f_{11}(r)$ is an even polynomial of degree $n$ in the variable $r$, and $f_{1\ell}(z_\ell)$ is a complete polynomial of degree $n$ in the variable $z_\ell$ for all $2 \leq \ell \leq m$. Since the exponents of $r$ in $f_{11}(r)$ is even, then $f_1(\nu)$ can have $n^m(n-1)/2$ simple zeros with $r > 0$.

On the other hand, if $n$ is odd we take $a^+_{ij0} \neq 0$, $b^+_{ij0} \neq 0$, $c^+_{\ell,00k_\ell 0} \neq 0$ and we take zero all the other coefficients and then we obtain $\widetilde{f}_{10}(\nu) = \widetilde{f}_{10}(r)$ and $f_{1\ell}(\nu) = f_{1\ell}(z_\ell)$, where

$$\widetilde{f}_{10}(r) = \quad h_1 + r h_2 + r^2 h_3 + \ldots + r^{n-1} h_n,$$

$$f_{1\ell}(\nu) = \quad \sum_{k_\ell=0}^{n} z_\ell^{k_\ell} \, c_{\ell,00k_\ell 0} \, I_{(0,0,2\pi)},$$

for $1 \leq \ell \leq m$. Then, $\widetilde{f}_{10}(r)$ is a polynomial of degree $n-1$ in the variable $r$, whose exponents are always even. In a similar way $f_{1\ell}(z_\ell)$ is a polynomial of degree $n$ in the variable $z_\ell$ for $1 \leq \ell \leq m$. Therefore, $f_1(\nu)$ can have $n^m(n-1)/2$ simple zeros with $r > 0$.

If $m = 0$ then $\nu = r$ and $f_1(\nu) = r\widetilde{f}_{10}(r)$. So the number of simple zeros can be $n-1$ if $n$ is odd, and $n-2$ if $n$ is even. Since the exponent of $r$ in $\widetilde{f}_{10}$ is even, the maximum number of simple zeros with $r > 0$ of $f_1(\nu)$ is $(n-1)/2$ if $n$ is odd, and $(n-2)/2$ if $n$ is even.

Now, we exhibit a particular example where the maximum number of simple zeros of $f_1(\nu)$ can be reached. Take $a^+_{ij0} \neq 0$, $b^+_{ij0} \neq 0$ and we take zero all the other coefficients so that $\widetilde{f}_{10}(r)$ is an even polynomial in the variable $r$ of degree $n-1$ if $n$ is odd, and $n-2$ is $n$ if even. So, the number of simple zeros of $f_1(\nu)$ with $r > 0$ can be $(n-1)/2$ if $n$ is odd, and $(n-2)/2$ if $n$ is even.

In both particular cases, $m \neq 0$ and $m = 0$, the coefficients of $f_1(\nu)$ are independent. Therefore, the maximum number of simple zeros with $r > 0$ of $f_1(\nu)$ can be reached. □

Now, we emphasize that the averaging function $f_2$ of the continuous system (2.11), for $\phi = 2\pi$, is given by $f_2(\nu) = \big(f_{20}(\nu), \ldots, f_{2m}(\nu)\big)$ being

$$(7.2) \qquad f_{2\ell}(\nu) = 2\widetilde{G}_{1\ell}(\nu) + 4 \int_0^{2\pi} \big(F_{2\ell}^+(s, \mathbf{z}_\nu) + \widetilde{F}_{1\ell}^+(s, \mathbf{z}_\nu)\big)\, ds,$$

for $0 \leq \ell \leq m$, $F_{2\ell}^+$, $\widetilde{G}_{1\ell}$ and $\widetilde{F}_{1\ell}^+$ given in (3.6), (3.12) and (3.13), respectively.

**Proposition 15.** *Assume $m = 0$ and $\phi = 2\pi$. If $f_1 \equiv 0$ then $f_2$ has at most $2n$ simple zeros. Moreover, the lower bound for the number of simple zeros is $n$.*

*Proof.* If $m = 0$ then $\nu = r$ and $f_1(\nu) = f_{10}(r)$. Assume that $f_1 \equiv 0$. From (7.1) we obtain

$$(7.3) \qquad \sum_{\substack{i\,odd,\,j\,even,\,P=s}}^{n} a_{ij0}^+ I_{(i+1,j,2\pi)} + \sum_{\substack{i\,even,\,j\,odd,\,P=s}}^{n} b_{ij0}^+ I_{(i,j+1,2\pi)} = 0,$$

where $P = i + j$ and $0 \leq s \leq n$.

Furthermore, by (7.2) we have $f_2(\nu) = f_{20}(r)$. Therefore, from statement $(c)$ of Lemma 6 and (7.3), we conclude that $\widetilde{G}_{10}(\nu)$ and $\displaystyle\int_0^{2\pi} \widetilde{F}_{10}(s, \mathbf{z}_\nu)\, ds$ are complete polynomials of degree $2n - 1$ in the variable $r$, and

$$\int_0^{2\pi} F_{20}^+(s, \mathbf{z}_\nu)\, ds = \sum_{s=0}^{N_1} R_s\, r^{2s+1} + \frac{1}{r} \sum_{k=0}^{n} Q_k\, r^{2k},$$

where $\mathbf{z}_\nu = (r, 0, \ldots, 0) \in \mathbb{R}^{d+1}$, $R_s$ and $Q_k$ are constants, $N_1 = \dfrac{n-2}{2}$ if $n$ is even, and $N_1 = \dfrac{n-1}{2}$ if $n$ is odd. Therefore, $r \displaystyle\int_0^{2\pi} F_{20}^+(s, \mathbf{z}_\nu)\, ds$ is an even polynomial in the variable $r$. Since $r > 0$ it follows that $r f_2(\nu) = 0$ if and only if $f_2(\nu) = 0$. By Bezout Theorem the maximum number of simple zeros of $f_2(\nu)$ is $2n$.

In order to exhibit the lower bound for the number of simple zeros of $f_2(\nu)$, we provide a particular example. Then, take $a_{ij0}^\pm \neq 0$, $b_{ij0}^\pm \neq 0$, $\alpha_{ij0}^\pm \neq 0$, and $\beta_{ij0}^\pm \neq 0$ and we take zero all the other coefficients in such a way that $f_{20}(r) = 4 \displaystyle\int_0^{2\pi} F_{20}^+(r, \mathbf{z}_\nu)\, d\theta$. Therefore, $r f_{20}(r)$ is a polynomial in $r$ of degree $2n$. Since $r f_{20}(r)$ is an even polynomial in $r$, then the number of simple zeros of $f_2(\nu)$ with $r > 0$ can be $n$, and this number can be reached due to the independence of all coefficients. $\square$

*Proof of Theorem 5.* Applying Theorem 2.1 to the function $f_1$ given in Proposition 14, statement $(a)$ holds. We apply Theorem 2.1 to the function $f_2$ given in Proposition 15 and we conclude statement $(b)$. $\square$

## Appendix

In this appendix we shall exhibit some general expression of functions that appears in subSection 3.2.

We denote by $\lambda_{ijk_1\ldots k_m 1_\omega}$ the coefficient of $x^i y^j z_1^{k_1} \ldots z_m^{k_m} z_\omega$, and by $\lambda_{ijk_1\ldots k_m 0}$ the coefficient of $x^i y^j z_1^{k_1} \ldots z_m^{k_m}$ in system (2.11) when $\lambda = a^\pm, b^\pm, \alpha^\pm, \beta^\pm, c_\ell^\pm, \gamma_\ell^\pm$ for all $0 \leq \ell \leq m$ and $m + 1 \leq \omega \leq d$. Recall that $\nu = (r, z_1, \ldots, z_m)$ and $\mathbf{z}_\nu = (r, z_1, \ldots, z_m, 0, \ldots, 0) \in \mathbb{R}^{d+1}$.

For the next expressions, take $P = i + j + k_1 + \ldots + k_m$ and $Q = p + q + L_1 + \ldots + L_m$.

From (3.3) and (3.9) we obtain

$$
\begin{aligned}
\frac{\partial g_{10}}{\partial z_\omega}(\mathbf{z}_\nu) = \ &\sum_{P=0}^{n-1} r^{i+j} z_1^{k_1} \ldots z_m^{k_m} \Bigg( a_{ijk_1\ldots k_m 1_\omega}^+ \int_0^\phi e^{\mu_\omega s} \cos^{i+1} s\, \sin^j s\, ds \\
&+ b_{ijk_1\ldots k_m 1_\omega}^+ \int_0^\phi e^{\mu_\omega s} \cos^i s\, \sin^{j+1} s\, ds \\
&+ a_{ijk_1\ldots k_m 1_\omega}^- \int_\phi^{2\pi} e^{\mu_\omega s} \cos^{i+1} s\, \sin^j s\, ds \\
&+ b_{ijk_1\ldots k_m 1_\omega}^- \int_\phi^{2\pi} e^{\mu_\omega s} \cos^i s\, \sin^{j+1} s\, ds \Bigg), \\
\frac{\partial g_{1\ell}}{\partial z_\omega}(\mathbf{z}_\nu) = \ &\sum_{P=0}^{n-1} r^{i+j} z_1^{k_1} \ldots z_m^{k_m} \Bigg( c_{l,ijk_1\ldots k_m 1_\omega}^+ \int_0^\phi e^{\mu_\omega s} \cos^i s\, \sin^j s\, ds \\
&+ c_{\ell,ijk_1\ldots k_m 1_\omega}^- \int_\phi^{2\pi} e^{\mu_\omega s} \cos^i s\, \sin^j s\, ds \Bigg),
\end{aligned}
$$

for $1 \leq \ell \leq m$ and $m + 1 \leq \omega \leq d$.

From (3.11) we get

$$
\begin{aligned}
\gamma_\omega(\nu) = \ &\frac{-1}{1 - e^{-\mu_\omega 2\pi}} \sum_{P=0}^{n} r^{i+j} z_1^{k_1} \ldots z_m^{k_m} \Bigg( c_{\omega,ijk_1\ldots k_m 0}^+ \int_0^\phi e^{-\mu_\omega s} \cos^i s\, \sin^j s\, ds \\
&+ c_{\omega,ijk_1\ldots k_m 0}^- \int_\phi^{2\pi} e^{-\mu_\omega (2\pi + s)} \cos^i s\, \sin^j s\, ds \Bigg),
\end{aligned}
$$

for $m + 1 \leq \omega \leq d$.

From the above equalities and (3.12) we obtain for $1 \leq \ell \leq m$ that

$$\widetilde{G}_{10}(\nu) = \sum_{\omega=m+1}^{d} \left[ \sum_{P=0}^{n-1} r^{i+j} z_1^{k_1} \dots z_m^{k_m} \left( a_{ijk_1\dots k_m 1_\omega}^{+} \int_0^{\phi} e^{\mu_\omega s} \cos^{i+1} s \, \sin^j s \, ds \right. \right.$$

$$+ b_{ijk_1\dots k_m 1_\omega}^{+} \int_0^{\phi} e^{\mu_\omega s} \cos^i s \, \sin^{j+1} s \, ds + a_{ijk_1\dots k_m 1_\omega}^{-} \int_\phi^{2\pi} e^{\mu_\omega s} \cos^{i+1} s \, \sin^j s \, ds$$

$$\left. + b_{ijk_1\dots k_m 1_\omega}^{-} \int_\phi^{2\pi} e^{\mu_\omega s} \cos^i s \, \sin^{j+1} s \, ds \right) \Bigg]$$

$$\left[ \frac{-1}{1-e^{-\mu_\omega 2\pi}} \sum_{Q=0}^{n} r^{i+j} z_1^{k_1} \dots z_m^{k_m} \left( c_{\omega, ijL_1\dots L_m 0}^{+} \int_0^{\phi} e^{-\mu_\omega s} \cos^i s \, \sin^j s \, ds \right. \right.$$

$$\left. \left. + c_{\omega, ijL_1\dots L_m 0}^{-} \int_\phi^{2\pi} e^{-\mu_\omega (2\pi+s)} \cos^i s \, \sin^j s \, ds \right) \right],$$

$$\widetilde{G}_{1\ell}(\nu) = \sum_{\omega=m+1}^{d} \left[ \sum_{P=0}^{n-1} r^{i+j} z_1^{k_1} \dots z_m^{k_m} \left( c_{\ell, ijk_1\dots k_m 1_\omega}^{+} \int_0^{\phi} e^{\mu_\omega s} \cos^i s \, \sin^j s \, ds \right. \right.$$

$$\left. + c_{\ell, ijk_1\dots k_m 1_\omega}^{-} \int_\phi^{2\pi} e^{\mu_\omega s} \cos^i s \, \sin^j s \, ds \right) \Bigg]$$

$$\left[ \frac{-1}{1-e^{-\mu_\omega 2\pi}} \sum_{Q=0}^{n} r^{i+j} z_1^{L_1} \dots z_m^{L_m} \left( c_{\omega, ijL_1\dots L_m 0}^{+} \int_0^{\phi} e^{-\mu_\omega s} \cos^i s \, \sin^j s \, ds \right. \right.$$

$$\left. \left. + c_{\omega, ijL_1\dots L_m 0}^{-} \int_\phi^{2\pi} e^{-\mu_\omega (2\pi+s)} \cos^i s \, \sin^j s \, ds \right) \right].$$

Now, from (3.6) we compute

$$\frac{\partial F_{10}^{\pm}}{\partial r}(s, \varphi(s, \mathbf{z}_\nu)) = \frac{1}{r} \sum_{P=0}^{n} (i+j) \, r^{i+j} z_1^{k_1} \dots z_m^{k_m}$$
$$\left( a_{ijk_1\dots k_m 0}^{\pm} \cos^{i+1} s \sin^j s + b_{ijk_1\dots k_m 0}^{\pm} \cos^i s \sin^{j+1} s \right),$$

$$\frac{\partial F_{10}^{\pm}}{\partial z_\rho}(s, \varphi(s, \mathbf{z}_\nu)) = \sum_{P=0}^{n} k_\rho \, r^{i+j} z_1^{k_1} \dots z_\rho^{k_\rho-1} \dots z_m^{k_m}$$
$$\left( a_{ijk_1\dots k_m 0}^{\pm} \cos^{i+1} s \sin^j s + b_{ijk_1\dots k_m 0}^{\pm} \cos^i s \sin^{j+1} s \right),$$

$$\frac{\partial F_{10}^{\pm}}{\partial z_\omega}(s, \varphi(s, \mathbf{z}_\nu)) = \sum_{P=0}^{n-1} r^{i+j} z_1^{k_1} \dots z_m^{k_m}$$
$$\left( a_{ijk_1\dots k_m 1_\omega}^{\pm} \cos^{i+1} s \sin^j s + b_{ijk_1\dots k_m 1_\omega}^{\pm} \cos^i s \sin^{j+1} s \right),$$

$$\frac{\partial F_{1\ell}^{\pm}}{\partial r}(s,\varphi(s,\mathbf{z}_\nu)) = \frac{1}{r}\sum_{P=0}^{n}(i+j)\,r^{i+j}z_1^{k_1}\ldots z_m^{k_m}\,c_{\ell,ijk_1\ldots k_m0}^{\pm}\cos^i s\sin^j s,$$

$$\frac{\partial F_{1\ell}^{\pm}}{\partial z_\rho}(s,\varphi(s,\mathbf{z}_\nu)) = \sum_{P=0}^{n}k_\rho\,r^{i+j}z_1^{k_1}\ldots z_p^{k_\rho-1}\ldots z_m^{k_m}\,c_{\ell,ijk_1\ldots k_m0}^{\pm}\cos^i s\sin^j s,$$

$$\frac{\partial F_{1\ell}^{\pm}}{\partial z_\omega}(s,\varphi(s,\mathbf{z}_\nu)) = \sum_{P=0}^{n-1}r^{i+j}z_1^{k_1}\ldots z_m^{k_m}\,c_{\ell,ijk_1\ldots k_m1_\omega}^{\pm}\cos^i s\sin^j s,$$

for $1\le\ell\le m$, $1\le\rho\le m$ and $m+1\le\omega\le d$.

Note that when $m=d$ we do not consider the functions $\dfrac{\partial F_{10}^{\pm}}{\partial z_\omega}$ and $\dfrac{\partial F_{1\ell}^{\pm}}{\partial z_\omega}$.

Now, from (3.3) and (3.8) we get

$$
\begin{aligned}
y_{10}^{\pm}(s,\mathbf{z}_\nu) &= \sum_{P=0}^{n}r^{i+j}z_1^{k_1}\ldots z_m^{k_m}\left(a_{ijk_1\ldots k_m0}^{\pm}I_{(i+1,j,s)}+b_{ijk_1\ldots k_m0}^{\pm}I_{(i,j+1,s)}\right),\\
y_{1\rho}^{\pm}(\theta,\mathbf{z}_\nu) &= \sum_{P=0}^{n}r^{i+j}z_1^{k_1}\ldots z_m^{k_m}\,c_{\rho,ijk_1\ldots k_m0}^{\pm}I_{(i,j,s)},\\
y_{1\omega}^{\pm}(\theta,\mathbf{z}_\nu) &= \sum_{P=0}^{n}r^{i+j}z_1^{k_1}\ldots z_m^{k_m}\,c_{\omega,ijk_1\ldots k_m0}^{\pm}\int_0^s e^{\mu_\omega(s-\tau)}\cos^i\tau\sin^j s\,d\tau,
\end{aligned}
$$

for $1\le\rho\le m$ and $m+1\le\omega\le d$. Therefore

$$\int\frac{\partial F_{10}^{\pm}}{\partial r}(s,\varphi(s,\mathbf{z}_\nu))y_{10}^{\pm}(s,\mathbf{z}_\nu)\,ds = \frac{1}{r}\sum_{P=0}^{n}\sum_{Q=0}^{n}(i+j)\,r^{i+j+p+q}\,z_1^{k_1+L_1}\ldots z_m^{k_m+L_m}$$

$$\left(a_{ijk_1\ldots k_m0}^{\pm}a_{pqL_1\ldots L_m0}^{\pm}\int\cos^{i+1}s\sin^j s\,I_{(p+1,q,s)}ds\right.$$

$$+b_{ijk_1\ldots k_m0}^{\pm}a_{pqL_1\ldots L_m0}^{\pm}\int\cos^i s\sin^{j+1}s\,I_{(p+1,q,s)}ds$$

$$+a_{ijk_1\ldots k_m0}^{\pm}b_{pqL_1\ldots L_m0}^{\pm}\int\cos^{i+1}s\sin^j s\,I_{(p,q+1,s)}ds$$

$$\left.+b_{ijk_1\ldots k_m0}^{\pm}b_{pqL_1\ldots L_m0}^{\pm}\int\cos^i s\sin^{j+1}s\,I_{(p,q+1,s)}ds\right),$$

$$\int\frac{\partial F_{10}^{\pm}}{\partial z_\rho}(s,\varphi(s,\mathbf{z}_\nu))y_{1\rho}^{\pm}(s,\mathbf{z}_\nu)ds = \sum_{P=0}^{n}\sum_{Q=0}^{n}k_\rho r^{i+j+p+q}z_1^{k_1+L_1}\ldots z_\rho^{k_\rho+L_\rho-1}\ldots z_m^{k_m+L_m}$$

$$\left(a_{ijk_1\ldots k_m0}^{\pm}c_{\rho,pqL_1\ldots L_m0}^{\pm}\int\cos^{i+1}s\sin^j s\,I_{(p,q,s)}\,ds\right.$$

$$\left.+b_{ijk_1\ldots k_m0}^{\pm}c_{\rho,pqL_1\ldots L_m0}^{\pm}\int\cos^i s\sin^{j+1}s\,I_{(p,q,s)}\,ds\right),$$

$$\int \frac{\partial F_{10}^{\pm}}{\partial z_{\omega}}(s, \varphi(s, \mathbf{z}_{\nu}))y_{1\omega}^{\pm}(s, \mathbf{z}_{\nu})ds = \sum_{P=0}^{n-1} \sum_{Q=0}^{n} r^{i+j+p+q} z_1^{k_1+L_1} z_2^{k_2+L_2} z_3^{k_3+L_3} ... z_m^{k_m+L_m}$$

$$\left( a_{ijk_1...k_m1_{\omega}}^{\pm} c_{\omega,pqL_1...L_m0}^{\pm} \int \cos^{i+1} s \sin^j s \left( \int_0^s e^{\mu\omega(s-\tau)} \cos^p \tau \sin^q \tau d\tau \right) ds \right.$$

$$\left. + b_{ijk_1...k_m1_{\omega}}^{\pm} c_{\omega,pqL_1...L_m0}^{\pm} \int \cos^i s \sin^{j+1} s \left( \int_0^s e^{\mu\omega(s-\tau)} \cos^p \tau \sin^q \tau d\tau \right) ds \right),$$

$$\int \frac{\partial F_{1\ell}^{\pm}}{\partial r}(s, \varphi(s, \mathbf{z}_{\nu}))y_{10}^{\pm}(s, \mathbf{z}_{\nu})ds = \frac{1}{r} \sum_{P=0}^{n} \sum_{Q=0}^{n} (i+j) \, r^{i+j+p+q} z_1^{k_1+L_1} z_2^{k_2+L_2} ... z_m^{k_m+L_m}$$

$$\left( c_{\ell,ijk_1...k_m0}^{\pm} a_{pqL_1...L_m0}^{\pm} \int \cos^i s \sin^j s \, I_{(p+1,q,s)} \, ds \right.$$

$$\left. + c_{\ell,ijk_1...k_m0}^{\pm} b_{pqL_1...L_m0}^{\pm} \int \cos^i s \sin^j s \, I_{(p,q+1,s)} \, ds \right),$$

$$\int \frac{\partial F_{1\ell}^{\pm}}{\partial z_{\rho}}(s, \varphi(s, \mathbf{z}_{\nu}))y_{1\rho}^{\pm}(s, \mathbf{z}_{\nu})ds = \sum_{P=0}^{n} \sum_{Q=0}^{n} k_{\rho} \, r^{i+j+p+q} z_1^{k_1+L_1} ... z_{\rho}^{k_{\rho}+L_{\rho}-1} ... z_m^{k_m+L_m}$$

$$c_{\ell,ijk_1...k_m0}^{\pm} c_{\rho,pqL_1...L_m0}^{\pm} \int \cos^i s \sin^j s \, I_{(p,q,s)} ds,$$

$$\int \frac{\partial F_{1\ell}^{\pm}}{\partial z_{\omega}}(s, \varphi(s, \mathbf{z}_{\nu}))y_{1\omega}^{\pm}(s, \mathbf{z}_{\nu})ds = \sum_{P=0}^{n-1} \sum_{Q=0}^{n} r^{i+j+p+q} z_1^{k_1+L_1} z_2^{k_2+L_2} z_3^{k_3+L_3} ... z_m^{k_m+L_m}$$

$$c_{\ell,ijk_1...k_m1_{\omega}}^{\pm} c_{\omega,pqL_1...L_m0}^{\pm} \int \cos^i s \sin^j s \left( \int_0^s e^{\mu\omega(s-\tau)} \cos^p \tau \sin^q \tau d\tau \right) ds,$$

for $1 \le \ell \le m$, $1 \le \rho \le m$ and $m+1 \le \omega \le d$.

Moreover, from (3.6) we get

$$\int_0^\phi F_{20}^+(s,\mathbf{z}_\nu)\,ds = \sum_{P=0}^n \alpha_{ijk_1\ldots k_m0}^+ \, r^{i+j} z_1^{k_1}\ldots z_m^{k_m}\, I_{(i+1,j,\phi)}$$

$$+\sum_{P=0}^n \beta_{ijk_1\ldots k_m0}^+ \, r^{i+j} z_1^{k_1}\ldots z_m^{k_m}\, I_{(i,j+1,\phi)}$$

$$-\frac{1}{r}\sum_{P=0}^n\sum_{Q=0}^n a_{ijk_1\ldots k_m0}^+ b_{pqL_1\ldots L_m0}^+ \, r^{i+j+p+q}\, z_1^{k_1+L_1}\ldots z_m^{k_m+L_m}\, I_{(i+p+2,j+q,\phi)}$$

$$+\frac{1}{r}\sum_{P=0}^n\sum_{Q=0}^n a_{ijk_1\ldots k_m0}^+ a_{pqL_1\ldots L_m0}^+ \, r^{i+j+p+q}\, z_1^{k_1+L_1}\ldots z_m^{k_m+L_m}\, I_{(i+p+1,j+q+1,\phi)}$$

$$-\frac{1}{r}\sum_{P=0}^n\sum_{Q=0}^n b_{ijk_1\ldots k_m0}^+ b_{pqL_1\ldots L_m0}^+ \, r^{i+j+p+q}\, z_1^{k_1+L_1}\ldots z_m^{k_m+L_m}\, I_{(i+p+1,j+q+1,\phi)}$$

$$+\frac{1}{r}\sum_{P=0}^n\sum_{Q=0}^n a_{ijk_1\ldots k_m0}^+ b_{pqL_1\ldots L_m0}^+ \, r^{i+j+p+q}\, z_1^{k_1+L_1}\ldots z_m^{k_m+L_m}\, I_{(i+p,j+q+2,\phi)},$$

$$\int_\phi^{2\pi} F_{20}^-(s,\mathbf{z}_\nu)\,ds = \sum_{P=0}^n \alpha_{ijk_1\ldots k_m0}^- \, r^{i+j} z_1^{k_1}\ldots z_m^{k_m}\, J_{(i+1,j,\phi)}$$

$$+\sum_{P=0}^n \beta_{ijk_1\ldots k_m0}^- \, r^{i+j} z_1^{k_1}\ldots z_m^{k_m}\, J_{(i,j+1,\phi)}$$

$$-\frac{1}{r}\sum_{P=0}^n\sum_{Q=0}^n a_{ijk_1\ldots k_m0}^- b_{pqL_1\ldots L_m0}^- \, r^{i+j+p+q}\, z_1^{k_1+L_1}\ldots z_m^{k_m+L_m}\, J_{(i+p+2,j+q,\phi)}$$

$$+\frac{1}{r}\sum_{P=0}^n\sum_{Q=0}^n a_{ijk_1\ldots k_m0}^- a_{pqL_1\ldots L_m0}^- \, r^{i+j+p+q}\, z_1^{k_1+L_1}\ldots z_m^{k_m+L_m}\, J_{(i+p+1,j+q+1,\phi)}$$

$$-\frac{1}{r}\sum_{P=0}^n\sum_{Q=0}^n b_{ijk_1\ldots k_m0}^- b_{pqL_1\ldots L_m0}^- \, r^{i+j+p+q}\, z_1^{k_1+L_1}\ldots z_m^{k_m+L_m}\, J_{(i+p+1,j+q+1,\phi)}$$

$$+\frac{1}{r}\sum_{P=0}^n\sum_{Q=0}^n a_{ijk_1\ldots k_m0}^- b_{pqL_1\ldots L_m0}^- \, r^{i+j+p+q}\, z_1^{k_1+L_1}\ldots z_m^{k_m+L_m}\, J_{(i+p,j+q+2,\phi)},$$

$$\int_0^\phi F_{2\ell}^+(s,\mathbf{z}_\nu)\,ds = \sum_{P=0}^n \gamma_{\ell,ijk_1\ldots k_m}^+ \, r^{i+j} z_1^{k_1}\ldots z_m^{k_m}\, I_{(i,j,\phi)}$$

$$-\frac{1}{r}\sum_{P=0}^n\sum_{Q=0}^n b_{ijk_1\ldots k_m0}^+ c_{\ell,pqL_1\ldots L_m0}^+ \, r^{i+j+p+q}\, z_1^{k_1+L_1}\ldots z_m^{k_m+L_m}\, I_{(i+p+1,j+q,\phi)}$$

$$+\frac{1}{r}\sum_{P=0}^n\sum_{Q=0}^n a_{ijk_1\ldots k_m0}^+ c_{\ell,pqL_1\ldots L_m0}^+ \, r^{i+j+p+q}\, z_1^{k_1+L_1}\ldots z_m^{k_m+L_m}\, I_{(i+p,j+q+1,\phi)},$$

$$\int_{\phi}^{2\pi} F_{2\ell}^{-}(s, \mathbf{z}_\nu)\, ds = \sum_{P=0}^{n} \gamma_{\ell,ijk_1\ldots k_m 0}^{-} r^{i+j} z_1^{k_1} \ldots z_m^{k_m} J_{(i,j,\phi)}$$

$$-\frac{1}{r} \sum_{P=0}^{n} \sum_{Q=0}^{n} b_{ijk_1\ldots k_m 0}^{-} c_{\ell,pqL_1\ldots L_m 0}^{-} r^{i+j+p+q} z_1^{k_1+L_1} \ldots z_m^{k_m+L_m} J_{(i+p+1,j+q,\phi)}$$

$$+\frac{1}{r} \sum_{P=0}^{n} \sum_{Q=0}^{n} a_{ijk_1\ldots k_m 0}^{-} c_{\ell,pqL_1\ldots L_m 0}^{-} r^{i+j+p+q} z_1^{k_1+L_1} \ldots z_m^{k_m+L_m} J_{(i+p,j+q+1,\phi)},$$

for $1 \le \ell \le m$.

On the other hand when the perturbation is continuous that is, $\phi = 2\pi$, we have

$$\widetilde{G}_{10}(\nu) = \sum_{\omega=m+1}^{d} \left[ \sum_{P=0}^{n-1} r^{i+j} z_1^{k_1} \ldots z_m^{k_m} \right.$$

$$\left( a_{ijk_1\ldots k_m 1_\omega}^{+} \int_0^{2\pi} e^{\mu_\omega s} \cos^{i+1} s \, \sin^j s \, ds + b_{ijk_1\ldots k_m 1_\omega}^{+} \int_0^{2\pi} e^{\mu_\omega s} \cos^i s \, \sin^{j+1} s \, ds \right) \Big]$$

$$\left[ \frac{-1}{1 - e^{-\mu_\omega 2\pi}} \sum_{Q=0}^{n} r^{i+j} z_1^{L_1} \ldots z_m^{L_m} c_{\omega,ijL_1\ldots L_m 0}^{+} \int_0^{2\pi} e^{-\mu_\omega s} \cos^i s \, \sin^j s \, ds \right],$$

and

$$\int_0^{2\pi} F_{20}^{+}(s, \mathbf{z}_\nu)\, ds = \sum_{P=0,\, i\,odd,\, j\,even}^{n} \alpha_{ijk_1\ldots k_m 0}^{+} r^{i+j} z_1^{k_1} \ldots z_m^{k_m} I_{(i+1,j,2\pi)}$$

$$+ \sum_{P=0,\, i\,even,\, j\,odd}^{n} \beta_{ijk_1\ldots k_m 0}^{+} r^{i+j} z_1^{k_1} \ldots z_m^{k_m} I_{(i,j+1,2\pi)}$$

$$-\frac{1}{r} \sum_{i,j\,even,P=0}^{n} \sum_{p,q\,even,Q=0}^{n} a_{ijk_1\ldots k_m 0}^{+} b_{pqL_1\ldots L_m 0}^{+} r^{i+j+p+q} z_1^{k_1+L_1} \ldots$$

$$z_m^{k_m+L_m} I_{(i+p+2,j+q,2\pi)}$$

$$-\frac{1}{r} \sum_{i,j\,odd,P=0}^{n} \sum_{p,q\,odd,Q=0}^{n} a_{ijk_1\ldots k_m 0}^{+} b_{pqL_1\ldots L_m 0}^{+} r^{i+j+p+q} z_1^{k_1+L_1} \ldots$$

$$z_m^{k_m+L_m} I_{(i+p+2,j+q,2\pi)}$$

$$-\frac{1}{r} \sum_{i\,even,j\,odd,P=0}^{n} \sum_{p\,even,q\,odd,Q=0}^{n} a_{ijk_1\ldots k_m 0}^{+} b_{pqL_1\ldots L_m 0}^{+} r^{i+j+p+q} z_1^{k_1+L_1} \ldots$$

$$z_m^{k_m+L_m} I_{(i+p+2,j+q,2\pi)}$$

$$-\frac{1}{r}\sum_{i\,odd,j\,even,P=0}^{n}\sum_{p\,odd,q\,even,Q=0}^{n}a^+_{ijk_1...k_m0}b^+_{pqL_1...L_m0}r^{i+j+p+q}z_1^{k_1+L_1}...$$

$$z_m^{k_m+L_m}I_{(i+p+2,j+q,2\pi)}$$

$$+\frac{1}{r}\sum_{i,j\,odd,P=0}^{n}\sum_{p,q\,even,Q=0}^{n}a^+_{ijk_1...k_m0}a^+_{pqL_1...L_m0}r^{i+j+p+q}z_1^{k_1+L_1}...$$

$$z_m^{k_m+L_m}I_{(i+p+1,j+q+1,2\pi)}$$

$$+\frac{1}{r}\sum_{i,j\,even,P=0}^{n}\sum_{p,q\,odd,Q=0}^{n}a^+_{ijk_1...k_m0}a^+_{pqL_1...L_m0}r^{i+j+p+q}z_1^{k_1+L_1}...$$

$$z_m^{k_m+L_m}I_{(i+p+1,j+q+1,2\pi)}$$

$$+\frac{1}{r}\sum_{i\,even,j\,odd,P=0}^{n}\sum_{p\,odd,q\,even,Q=0}^{n}a^+_{ijk_1...k_m0}a^+_{pqL_1...L_m0}r^{i+j+p+q}z_1^{k_1+L_1}...$$

$$z_m^{k_m+L_m}I_{(i+p+1,j+q+1,2\pi)}$$

$$+\frac{1}{r}\sum_{i\,odd,j\,even,P=0}^{n}\sum_{p\,even,q\,odd,Q=0}^{n}a^+_{ijk_1...k_m0}a^+_{pqL_1...L_m0}r^{i+j+p+q}z_1^{k_1+L_1}...$$

$$z_m^{k_m+L_m}I_{(i+p+1,j+q+1,2\pi)}$$

$$-\frac{1}{r}\sum_{i,j\,odd,P=0}^{n}\sum_{p,q\,even,Q=0}^{n}b^+_{ijk_1...k_m0}b^+_{pqL_1...L_m0}r^{i+j+p+q}z_1^{k_1+L_1}...$$

$$z_m^{k_m+L_m}I_{(i+p+1,j+q+1,2\pi)}$$

$$-\frac{1}{r}\sum_{i,j\,even,P=0}^{n}\sum_{p,q\,odd,Q=0}^{n}b^+_{ijk_1...k_m0}b^+_{pqL_1...L_m0}r^{i+j+p+q}z_1^{k_1+L_1}...$$

$$z_m^{k_m+L_m}I_{(i+p+1,j+q+1,2\pi)}$$

$$-\frac{1}{r}\sum_{i\,even,j\,odd,P=0}^{n}\sum_{p\,odd,q\,even,Q=0}^{n}b^+_{ijk_1...k_m0}b^+_{pqL_1...L_m0}r^{i+j+p+q}z_1^{k_1+L_1}...$$

$$z_m^{k_m+L_m}I_{(i+p+1,j+q+1,2\pi)}$$

$$-\frac{1}{r}\sum_{i\,odd,j\,even,P=0}^{n}\sum_{p\,even,q\,odd,Q=0}^{n}b^+_{ijk_1...k_m0}b^+_{pqL_1...L_m0}r^{i+j+p+q}z_1^{k_1+L_1}...$$

$$z_m^{k_m+L_m}I_{(i+p+1,j+q+1,2\pi)}$$

$$+\frac{1}{r}\sum_{i,j\,even,P=0}^{n}\sum_{p,q\,even,Q=0}^{n}a^+_{ijk_1...k_m0}b^+_{pqL_1...L_m0}r^{i+j+p+q}z_1^{k_1+L_1}...$$

$$z_m^{k_m+L_m}I_{(i+p,j+q+2,2\pi)}$$

$$+\frac{1}{r}\sum_{i,j\,odd,P=0}^{n}\sum_{p,q\,odd,Q=0}^{n}a^+_{ijk_1...k_m0}b^+_{pqL_1...L_m0}r^{i+j+p+q}z_1^{k_1+L_1}...$$

$$z_m^{k_m+L_m}I_{(i+p,j+q+2,2\pi)}$$

$$+\frac{1}{r}\sum_{i\,even,j\,odd,P=0}^{n}\sum_{p\,even,q\,odd,Q=0}^{n}a_{ijk_1...k_m0}^{+}b_{pqL_1...L_m0}^{+}\,r^{i+j+p+q}z_1^{k_1+L_1}...$$
$$z_m^{k_m+L_m}I_{(i+p,j+q+2,2\pi)}$$
$$+\frac{1}{r}\sum_{i\,odd,j\,even,P=0}^{n}\sum_{p\,odd,q\,even,Q=0}^{n}a_{ijk_1...k_m0}^{+}b_{pqL_1...L_m0}^{+}\,r^{i+j+p+q}z_1^{k_1+L_1}...$$
$$z_m^{k_m+L_m}I_{(i+p,j+q+2,2\pi)}.$$

## Acknowledgements

## References

[1] E. A. Barbashin: Introduction to the theory of stability. *Translated from the Russian by Transcripta Service.*, London. Edited by T. Lukes. Wolters-Noordhoff Publishing, Groningen, 1970.

[2] F. Bizzarri, M. Storace, and A. Colombo: Bifurcation analysis of an impact model for forest fire prediction. *Internat. J. Bifur. Chaos Appl. Sci. Engrg.* **18** (2008), no. 8, 2275–2288.

[3] A. Buică, J. P. Françoise, and J. Llibre: Periodic solutions of nonlinear periodic differential systems with a small parameter. *Commun. Pure Appl. Anal.* **6** (2007), no. 1, 103–111.

[4] A. Buică and J. Llibre: Averaging methods for finding periodic orbits via Brouwer degree. *Bull. Sci. Math.* **128** (2004), no. 1, 7–22.

[5] M. R. Cândido, J. Llibre, and D. D. Novaes: Persistence of periodic solutions for higher order perturbed differential systems via Lyapunov-Schmidt reduction. *Nonlinearity* **30** (2017), no. 9, 3560–3586.

[6] C. Chicone: Lyapunov-Schmidt reduction and Melnikov integrals for bifurcation of periodic solutions in coupled oscillators. *J. Differential Equations* **112** (1994), no. 2, 407–447.

[7] A. Cima, J. Llibre, and M. A. Teixeira: Limit cycles of some polynomial differential systems in dimension 2, 3 and 4, via averaging theory. *Appl. Anal.* **87** (2008), no. 2, 149–164.

[8] S. Coombes: Neuronal networks with gap junctions: a study of piecewise linear planar neuron models. *SIAM J. Appl. Dyn. Syst.* **7** (2008), no. 3, 1101–1129.

[9] W. Fulton: Algebraic curves. *Advanced Book Classics.* Addison-Wesley Publishing Company, Advanced Book Program, Redwood City, CA, 1989. An introduction to algebraic geometry, Notes written with the collaboration of Richard Weiss, Reprint of 1969 original.

[10] J. Giné, J. Llibre, K. Wu, and X. Zhang: Averaging methods of arbitrary order, periodic solutions and integrability. *J. Differential Equations* **260** (2016), no. 5, 4130–4156.

[11] M. R. A. Gouveia, J. Llibre, D. D. Novaes, and C. Pessoa: Piecewise smooth dynamical systems: persistence of periodic solutions and normal forms. *J. Differential Equations* **260** (2016), no. 7, 6108–6129.

[12] I. S. Gradshteyn and I. M. Ryzhik: *Table of integrals, series, and products.* Elsevier/Academic Press, Amsterdam, eighth edition, 2015. Translated from the Russian, Translation edited and with a preface by Daniel Zwillinger and Victor Moll, Revised from the seventh edition [MR2360010].

[13] J. Harris and B. Ermentrout: Bifurcations in the Wilson-Cowan equations with nonsmooth firing rate. *SIAM J. Appl. Dyn. Syst.* **14** (2015), no. 1, 43–72.

[14] S. M. Huan and X. S. Yang: On the number of limit cycles in general planar piecewise linear systems. *Discrete Contin. Dyn. Syst.* **32** (2012), no. 6, 2147–2164.

[15] Y. Ilyashenko: Centennial history of Hilbert's 16th problem. *Bull. Amer. Math. Soc. (N.S.)* **39** (2002), no. 3, 301–354.

[16] J. Itikawa, J. Llibre, and D. D. Novaes: A new result on averaging theory for a class of discontinuous planar differential systems with applications. *Rev. Mat. Iberoam.* **33** (2017), no. 4, 1247–1265.

[17] T. Ito: A Filippov solution of a system of differential equations with discontinuous right-hand sides. *Econom. Lett.* **4** (1979/80) , no. 4, 349–354.

[18] V. Křivan: On the Gause predator-prey model with a refuge: a fresh look at the history. *J. Theoret. Biol.* **274** (2011), 67–73.

[19] J. Llibre and A. C. Mereu: Limit cycles for discontinuous quadratic differential systems with two zones. *J. Math. Anal. Appl.* **413** (2014), no. 2, 763–775.

[20] J. Llibre, A. C. Mereu, and D. D. Novaes: Averaging theory for discontinuous piecewise differential systems. *J. Differential Equations* **258** (2015), no. 11, 4007–4032.

[21] J. Llibre and D. D. Novaes: Improving the averaging theory for computing periodic solutions of the differential equations. *Z. Angew. Math. Phys.* **66** (2015), no. 4, 1401–1412.

[22] J. Llibre, D. D. Novaes, and C. A. B. Rodrigues: Averaging theory at any order for computing limit cycles of discontinuous piecewise differential systems with many zones. *Phys. D* **353/354** (2017), 1–10.

[23] J. Llibre, D. D. Novaes, and M. A. Teixeira: Higher order averaging theory for finding periodic solutions via Brouwer degree. *Nonlinearity* **27** (2014), no. 3, 563–583.

[24] J. Llibre, D. D. Novaes, and M. A. Teixeira: On the birth of limit cycles for non-smooth dynamical systems. *Bull. Sci. Math.* **139** (2015), no. 3, 229–244.

[25] J. Llibre, M. Ordóñez, and E. Ponce: On the existence and uniqueness of limit cycles in planar continuous piecewise linear systems without symmetry. *Nonlinear Anal. Real World Appl.* **14** (2013), no. 5, 2002–2012.

[26] J. Llibre, M. A. Teixeira, and I. O. Zeli: Birth of limit cycles for a class of continuous and discontinuous differential systems in $(d + 2)$-dimension. *Dyn. Syst.* **31** (2016), no. 3, 237–250.

[27] N. Minorsky: *Nonlinear oscillations.* D. Van Nostrand Co., Inc., Princeton, N.J. Toronto-London-New York, 1962.

[28] W. Nicola and S. A. Campbell: Nonsmooth bifurcations of mean field systems of two-dimensional integrate and fire neurons. *SIAM J. Appl. Dyn. Syst.* **15** (2016), no. 1, 391–439.

[29] D. D. Novaes: On nonsmooth perturbations of nondegenerate planar centers. *Publ. Mat.* **58** (2014), 395–420.

[30] D. D. Novaes: *Regularization and minimal sets for non-smooth dynamical systems.* PhD thesis. Universidade Estadual de Campinas, 31, 2015.

[31] J. A. Sanders, F. Verhulst, and J. A. Murdock: *Averaging methods in nonlinear dynamical systems.* **59** Springer, 2007.

[32] F. Verhulst: *Nonlinear differential equations and dynamical systems.* Springer Science & Business Media, 2006.

Jaume Llibre: Departament de Matemàtiques, Universitat Autònoma de Barcelona (UAB), 08193 Bellaterra, Barcelona, Catalonia, Spain
E-mail: jllibre@mat.uab.cat

Douglas Duarte Novaes: Departamento de Matemática, Universidade Estadual de Campinas, Rua Sérgio Baruque de Holanda, 651, Cidade Universitária Zeferino Vaz, 13083–859, Campinas, SP, Brazil.
E-mail: ddnovaes@unicamp.br

Iris de Oliveira Zeli: Departamento de Matemática, Instituto Tecnológico de Aeronáutica (ITA), Praça Marechal Eduardo Gomes, 50, Vila das Acácias, 12228–900, São José dos Campos, SP, Brazil
E-mail: iriszeli@ita.br