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Integrating Anticipative Replenishment-Allocation with Reactive Fulfillment for Online Retailing Using Robust Optimization

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Abstract

Problem definition: In each period of a planning horizon, an online retailer decides on how much to replenish each product and how to allocate its inventory to fulfillment centers (FCs) before demand is known. After the demand in the period is realized, the retailer decides on which FCs to fulfill it. It is crucial to optimize the replenishment, allocation, and fulfillment decisions jointly such that the expected total operating cost is minimized. The problem is challenging because the replenishment-allocation is done in an anticipative manner under a "push" strategy, but the fulfillment is executed in a reactive way under a "pull" strategy. We propose a multi-period stochastic optimization model to delicately integrate the anticipative replenishment-allocation decisions with the reactive fulfillment decisions such that they are determined seamlessly as the demands are realized over time.

Academic/practical relevance: The aggressive expansion in e-commerce sales significantly escalates online retailers' operating costs. Our methodology helps boost their competency in this cut-throat industry.

Methodology: We develop a two-phase approach based on robust optimization to solve the problem. The first phase decides whether the products should be replenished in each period (binary decisions). We fix these binary decisions in the second phase, where we determine the replenishment, allocation, and fulfillment quantities.

Results: Numerical experiments suggest that our approach outperforms existing methods from the literature in solution quality and computational time, and performs within 7% of a benchmark with perfect information. A study using real data from a major fashion online retailer in Asia suggests that the two-phase approach can potentially reduce the retailer's cumulative cost significantly.

Managerial implications: By decoupling the binary decisions from the continuous decisions, our methodology can solve large problem instances (up to 1,200 products). The integration, robustness, and adaptability of the decisions under our approach create significant values.

Keywords: online retailing; inventory management; allocation; order fulfillment; robust optimization

1 Introduction

Due to the rapid development of internet technology and mobile devices, more consumers opt for online shopping. This results in a boom of online retail sales during the past decades. According to eMarketer (2016), global e-commerce sales in 2016 grew at a rate of 23.7% and reached $1.915 trillion, which accounts for 8.7% of the total retail sales. By 2020, e-commerce sales will amount to $4.058 trillion, making up 14.6% of the total retail sales. The aggressive expansion in sales not only makes online retailing a promising industry, but also significantly escalates its operating cost (Kaplan, 2017).

This paper is motivated by a common challenge faced by an online retailer selling multiple products to different demand zones over a multi-period horizon. The retailer replenishes the products from different suppliers and stores the products at multiple fulfillment centers (FCs) to satisfy demand. In each period, the retailer makes three types of decisions: (i) At the start of the period, the retailer determines how much to replenish for each product from each supplier given a lead time and a limited production capacity. (ii) The retailer then decides how to allocate the inventory to the different FCs, given that each FC has a limited storage capacity and different allocation and fulfillment costs. (iii) At the end of the period, the demands are realized and the retailer decides on which FCs to fulfill the demands of each zone. In case a product is out of stock, the retailer requests the product to be *drop-shipped* from suppliers to satisfy the demands (for example, CleoCat Fashion in Singapore offers drop-shipping services for fashion products). The retailer's objective is to minimize the expected total operating cost over the multi-period horizon. We have learned of this problem through the interaction with our industrial partner, which is a major fashion online retailer in Asia. The retailer serves the demands of six countries (zones) through three FCs strategically located at different Asian cities.

In contrast to brick-and-mortar retailing, a distinct characteristic of online retailing is that the retailer has the flexibility to satisfy the demands of a zone from any FC that holds the inventory. This fulfillment flexibility improves service levels, but may also increase the retailer's outbound shipping cost, which is a main operating cost of online retailing (Dinlersoz and Li, 2006). The fulfillment flexibility further complicates the inventory allocation to the FCs and the product replenishments from the suppliers. To address these issues in an effective manner, the retailer needs to optimize the replenishment, allocation, and fulfillment decisions *jointly*.

The problem is especially challenging because replenishment and allocation of inventory are typically done before the demand is known in each period. Thus, the replenishment and allocation decisions are made in an *anticipative* manner. In contrast, the fulfillment decisions are made in a *reactive* manner as order fulfillment for online retailing is usually performed after the actual demand is realized in each period. In other words, an online retailer typically adopts a "push" strategy for inventory replenishment-allocation and a "pull" strategy for order fulfillment in each period. In this paper, we propose a multi-period stochastic optimization model that delicately integrates the anticipative replenishment-allocation decisions with the reactive fulfillment decisions to minimize the retailer's expected total cost. These two kinds of decisions (anticipative versus reactive) can be determined seamlessly by our model as the uncertain demands are revealed over time.

We make contributions in three dimensions: constructing a stochastic optimization model, proposing an efficient and scalable solution approach, and obtaining technical insights that may be useful for practitioners.

1. **Model**. To the best of our understanding, this is the first paper establishing a multi-period stochastic optimization model that integrates the anticipative replenishment-allocation decisions with the reactive fulfillment decisions. Note that this is typical for online retailing where replenishment and allocation in each period are done before the demand in the period is realized, but the fulfillment is performed after we know the actual demand. Our paper fills a major gap in the online retailing literature in which all papers address the replenishment, allocation, and fulfillment problems separately.

2. **Solution approach**. We propose a two-phase approach (TPA) based on robust optimization to solve the multi-period stochastic optimization model with binary and continuous decisions. In Phase 1 of the TPA, we solve a target-oriented robust optimization (TRO) model to determine the binary replenishment decisions with a goal to absorb as much demand uncertainty as possible. We fix these binary decisions in Phase 2, where our objective is to minimize the worst-case expected total cost. We use a linear decision rule (LDR) in Phase 2 to adapt the replenishment, allocation, and fulfillment quantities as the demand uncertainty is revealed. *The way that we decouple the binary decisions from the continuous decisions is novel.* We benchmark the TPA with several approaches from the literature through numerical experiments. The results suggest that the TPA outperforms most of these existing approaches by producing high-quality solutions and, more importantly, it is remarkably more scalable than all of these approaches. A further study based on data from a major fashion online retailer in Asia suggests that the TPA can potentially reduce the total cumulative cost of the company's status quo policy by 30%. Our paper is the first to propose an approximation scheme for solving large-scale multi-stage mixed-integer adaptive robust optimization problems.

3. **Technical insights**. We have the following technical insights on the TPA's performance: (i) Decoupling the binary decisions from the continuous decisions yields high-quality solutions while preserving tractability. This decoupling allows the TPA to use a simple decision rule in each phase: a static rule in Phase 1 and an LDR in Phase 2. This results in tractable formulations. (ii) The integration, robustness, and adaptability of the TPA's decisions create significant values. Compared to a heuristic that decouples the replenishment decisions from the other decisions, the TPA yields up to 27% savings. This shows the benefit of integrating all the decisions. Benchmarking the TPA against a deterministic and static policy based on mean demands shows that the TPA yields up to 17% savings, demonstrating the benefit of the TPA's robust and adaptive decisions. (iii) Solving the TRO model in Phase 1 produces more effective binary decisions. The TRO model accommodates as much demand uncertainty as possible when determining the binary decisions. This yields a significantly lower average cost than an alternative two-phase approach that solves a deterministic model using mean demands in Phase 1.

The paper is organized as follows. Section 2 reviews the related literature. Section 3 formulates the problem with deterministic and stochastic demands. Section 4 describes the TPA based on robust optimization. Section 5 conducts a series of numerical experiments to compare the TPA with the benchmark approaches and discusses some technical insights. Section 6 examines the applicability and the performance of the TPA using the real data from our industrial partner. Section 7 gives some concluding remarks. All proofs can be found in the online supplement.

2 Literature review

The problem studied in this paper is related to (i) inventory rationing and allocation and (ii) e-commerce fulfillment. Our methodology is related to research in multi-stage adaptive robust optimization.

Inventory rationing and allocation

There is a large body of literature that studies inventory rationing and allocation in single-depot, multi-warehouse (or multi-location) systems. The goal is to optimize inventory pooling and then to allocate the central inventory to each warehouse facing random demand. Eppen and Schrage (1981) use the classical multi-echelon, multi-period inventory model to solve this problem. Federgruen and Zipkin (1984) derive a dynamic programming model and systematically approximate it by a single-location inventory problem. Erkip et al. (1990) further generalize this problem to incorporate demand correlations across warehouses and over time. They derive a closed-form expression for the optimal safety stock as a function of the level of correlation through time. Jackson and Muckstadt (2015) propose a robust optimization approach to solve a similar problem and extend the inventory policy to incorporate an adaptive and non-anticipative shipment policy.

Cattani and Souza (2002) study an inventory rationing policy for firms operating in a direct-market channel. They show that rationing inventory can be beneficial for the channel under certain conditions. Zhong et al. (2018) consider a manufacturer with a centralized inventory pool that fulfills multiple e-distributors with service-level requirements. They derive necessary and sufficient conditions for the minimum inventory level of the centralized pool, and develop an anticipative allocation policy to satisfy the e-distributors' service levels. Acimovic and Graves (2017) study a phenomenon called demand spillover: A stockout at a fulfillment center leads to demand spilling over to another fulfillment center. By taking possible demand spillover during the replenishment lead time into consideration, they propose a heuristic to allocate inventory to the fulfillment centers. Their simulation results suggest that the heuristic outperforms an online retailer's status quo policy, and captures over 90% of the possible improvement generated by a pseudo-optimal policy.

In contrast to these papers, we consider a three-echelon online retailing network that consists of multiple suppliers, FCs, and demand zones over multiple periods. We also incorporate transportation costs from the suppliers to the FCs and then to the zones, which makes our problem significantly harder to solve.

E-commerce fulfillment

The rapid growth of e-commerce has attracted substantial attention from researchers. See Simchi-Levi et al. (2004) and Agatz et al. (2008) for a comprehensive review. Netessine and Rudi (2006) study a drop-shipping arrangement for online retailers that is also considered in our paper. They investigate the retailers' optimal stocking decisions, and find that drop-shipping and dual channels can be viable choices. Xu et al. (2009) study an e-commerce fulfillment problem, and develop a heuristic to minimize the number of shipments by periodically reevaluating wait-to-pick orders and making real-time reassignments. Their numerical results show that this heuristic yields a 50% reduction in the number of split orders. Mahar and Wright (2009) also investigate the benefits of assigning orders nonmyopically and postponing the assignment decisions. They develop a quasi-dynamic assignment policy that reduces the fulfillment cost by as much as 23% on average.

Acimovic and Graves (2015) study the problem of minimizing the total shipping cost. They develop a heuristic that assigns orders to FCs based on the dual values of a transportation linear program (LP). Their numerical results suggest that the heuristic yields a 1.07% improvement over the myopic policy based on real data from a global online retailer. Jasin and Sinha (2015) develop two heuristics to solve a similar problem. The first heuristic is based on a deterministic LP's solution and the second heuristic improves the first one through a correlated rounding scheme. They analytically and numerically show that the performance of the second heuristic is very close to optimal.

Lei et al. (2018) study joint dynamic pricing and order fulfillment to maximize an online retailer's expected profit, and propose two heuristics to solve the problem. The first heuristic uses the solution of a deterministic approximation as control parameters, and the second heuristic improves the first one by adaptively adjusting the control parameters. They show analytically and numerically that the second heuristic performs very close to a benchmark. In addition, some papers study pricing policies for online retailers (Ferreira et al., 2016) and their interactions with fulfillment-related decisions (Leng and Becerril-Arreola, 2010; Becerril-Arreola et al., 2013; Gümüs et al., 2013). Our work differs from the above papers by considering not only the fulfillment decisions, but also the inventory replenishment and allocation decisions in a single model.

Adaptive robust optimization

Robust optimization (RO) is an approach for solving optimization problems under uncertainty. It uses only partial information of the uncertainty and yields tractable models (Bertsimas and Sim, 2004; Ben-Tal et al., 2009; Bertsimas et al., 2011a). Moreover, the RO approach provides an opportunity to satisfy some pre-specified goals (Chen and Sim, 2009; Lim and Wang, 2017). Recently, adaptive robust optimization (ARO) has attracted considerable interests. The ARO approach addresses multi-period problems where the recourse decisions are determined after the uncertain parameters are revealed. To create tractable models, the ARO approach usually assumes the decision variables as functions of the uncertain parameters. These functions are also called *decision rules*. Many papers have developed decisions rules for the ARO problems with *continuous recourse decisions*. These include the static rule (Bertsimas and Goyal, 2010; Bertsimas et al., 2015; Lim and Wang, 2017), the linear decision rule (Ben-Tal et al., 2004; Chen et al., 2008; Bertsimas et al., 2010; See and Sim, 2010; Kuhn et al., 2011; Ang et al., 2012), and the nonlinear decision rule (Chen and Zhang, 2009; Goh and Sim, 2010; Bertsimas et al., 2011; Georghiou et al., 2014). Applications of these decision rules include contract design (Ben-Tal et al., 2005), revenue management (Adida and Perakis, 2006), inventory management (Bertsimas and Thiele, 2006; See and Sim, 2010; Mamani et al., 2017; Lim and Wang, 2017), warehouse operations (Ang et al., 2012), and vehicle routing (Gounaris et al., 2013).

Unfortunately, the above decision rules cannot be applied to *binary recourse decisions*, which have received less attention in the literature. There are three streams of methodologies that handle the binary recourse decisions: (i) In a K-adaptability approach, a decision maker assumes K second-stage decisions and implements the best of them after the uncertain parameters are revealed. Hanasusanto et al. (2015) approximate a two-stage robust binary program with a K-adaptability problem, and study the approximation quality and the computational complexity. They point out that the K-adaptability approach does not readily extend to multi-stage ARO problems. (ii) A binary decision rule (BDR) represents a binary decision as a function of translated uncertain parameters. Bertsimas and Caramanis (2007) restrict the binary decisions to linear combinations of ceiling functions of uncertain parameters, and solve the resultant semi-infinite optimization problem through constraint sampling. Bertsimas and Georghiou (2015) restrict the binary recourse decisions to piecewise-constant functions of uncertain parameters, and develop an iterative cutting-plane algorithm to solve the resultant semi-infinite program. Bertsimas and Georghiou (2018) restrict binary recourse decisions to linear combinations of translated Heaviside step functions of uncertain parameters, and construct a higher-dimensional probability space using a lifting technology. They show that the resultant mixed-integer reformulation scales polynomially and provides high-quality solutions. (iii) A finite adaptability (FA) approach partitions an uncertainty set and assigns different recourse decisions to each partition. Based on this idea, Bertsimas and Dunning (2016) and Postek and den Hertog (2016) propose similar iterative partition-and-bound methods to approximate a fully adaptive solution. The former uses a Voronoi diagram, whereas the latter uses a single separating hyperplane in each iteration of partition. A linear decision rule for continuous recourse decisions can be incorporated into their methods. All the three approaches above determine the binary and continuous recourse decisions simultaneously in a single model.

In this paper, we propose a two-phase approach based on RO to solve the joint replenishment, allocation, and fulfillment problem for online retailing. The first phase of our approach uses a static rule to determine the binary decisions, and the second phase uses a linear decision rule to determine the continuous decisions. Our approach's novelty is the way we decouple the binary decisions from the continuous decisions, which exhibits a promising performance in producing high-quality solutions while preserving tractability.

3 Problem formulation

Consider an online retailer selling products $n = 1, \ldots, N$ to customers in demand zones $k = 1, \ldots, K$. The retailer replenishes her inventory from suppliers $i = 1, \ldots, I$ and allocates the inventory to FCs $j = 1, \ldots, J$, where she retrieves the inventory to fulfill the demand of each zone. If the retailer is out of stock for a certain product, the product is drop-shipped directly from the suppliers to the customers. For notational convenience, we denote the drop-shipping channel as FC $J + 1$, which incurs significantly higher production and transportation costs.

We divide the planning horizon into periods $t = 1, \ldots, T$. In each period t, the retailer makes the following three decisions in the specified sequence: (1) At the start of period t, the retailer determines the replenishment quantity for each product from each supplier (called the *replenishment decisions*). (2) The retailer then chooses the FCs to store the product (called the *allocation decisions*). (3) At the end of period t, the demand of each zone for each product is realized, the retailer selects the FCs to retrieve the product to fulfill the demand (called the *fulfillment decisions*). For convenience, define $\mathcal{N} = \{1, \ldots, N\}$, $\mathcal{I} = \{1, \ldots, I\}$, $\mathcal{J} = \{1, \ldots, J\}$, $\mathcal{J}^+ = \{1, \ldots, J+1\}$, $\mathcal{K} = \{1, \ldots, K\}$, $\mathcal{T} = \{1, \ldots, T\}$, and $\mathcal{T}^+ = \{1, \ldots, T+1\}$.

3.1 Deterministic optimization model

We first consider a deterministic model in which all demand information throughout the entire planning horizon is available at the start of period $t = 1$. Let y_j^{nt} denote the on-hand inventory level of product n in FC j at the start of period t, for $n \in \mathcal{N}$, $j \in \mathcal{J}$, $t \in \mathcal{T}$. Based on these inventory levels, the retailer replenishes a quantity x_i^{nt} of product n from supplier i at the start of period t. This incurs a fixed *setup cost* S_i^{nt} and a variable *production cost* $p_i^{nt} x_i^{nt}$, where p_i^{nt} is the corresponding unit production cost. Each supplier i has a production capacity $\bar{x}_i^t$ in period t such that $\sum_{n \in \mathcal{N}} x_i^{nt} \leq \bar{x}_i^t$, for $i \in \mathcal{I}, t \in \mathcal{T}$. We assume a constant lead time l_i^n such that a replenishment order for product n from supplier i placed at the start of period $t - l_i^n$ will be received by the retailer at the start of period t. We assume the replenishment quantities $\left\{x_i^{n,1-l_i^n}, \ldots, x_i^{n0}\right\}$ and the initial inventory levels y_j^{n1}, for $n \in \mathcal{N}$, $i \in \mathcal{I}$, $j \in \mathcal{J}$, are given at the start of period $t = 1$.

Define v_{ij}^{nt} as a decision variable representing the quantity of product n from supplier i allocated to FC j in period t, for $n \in \mathcal{N}$, $i \in \mathcal{I}$, $j \in \mathcal{J}$, $t \in \mathcal{T}$. This incurs an *allocation cost* $a_{ij}^{nt} v_{ij}^{nt}$, where a_{ij}^{nt} is the corresponding unit allocation cost. Since all the received quantity $x_i^{n,t-l_i^n}$ at the start of period t must be allocated to the FCs, we have $\sum_{j \in \mathcal{J}} v_{ij}^{nt} = x_i^{n,t-l_i^n}$, for $n \in \mathcal{N}, i \in \mathcal{I}, t \in \mathcal{T}$. The total inventory of each FC j cannot exceed its storage capacity $\bar{y}_j$ such that $\sum_{n \in \mathcal{N}} \left(y_j^{nt} + \sum_{i \in \mathcal{I}} v_{ij}^{nt}\right) \leq \bar{y}_j$, for $j \in \mathcal{J}$, $t \in \mathcal{T}$. Let d_k^{nt} denote the realized demand of zone k for product n in period t, for $n \in \mathcal{N}$, $k \in \mathcal{K}$, $t \in \mathcal{T}$. Define w_{jk}^{nt} as a decision variable representing the quantity of product n retrieved from FC j to fulfill the demand of zone k in period t, for $n \in \mathcal{N}$, $j \in \mathcal{J}^+$, $k \in \mathcal{K}$, $t \in \mathcal{T}$. This incurs a *fulfillment cost* $f_{jk}^{nt} w_{jk}^{nt}$, where f_{jk}^{nt} is the corresponding unit fulfillment cost. Note that $w_{J+1,k}^{nt}$ is the drop-shipping quantity of product n to fulfill the demand of zone k in period t and $f_{J+1,k}^{nt}$ is the corresponding unit drop-shipping cost. We do not allow backlog or lost-sales of demands such that $\sum_{j \in \mathcal{J}^+} w_{jk}^{nt} = d_k^{nt}$, for $n \in \mathcal{N}, k \in \mathcal{K}, t \in \mathcal{T}$, and $y_j^{nt} \geq 0$, for $n \in \mathcal{N}$, $j \in \mathcal{J}$, $t \in \mathcal{T}^+$.

After the demands are fulfilled, the inventory level of product n in FC j at the start of period $t+1$ is $y_j^{n,t+1} = y_j^{nt} + \sum_{i \in \mathcal{I}} v_{ij}^{nt} - \sum_{k \in \mathcal{K}} w_{jk}^{nt}$. Since the leftover inventory at the end of period t is carried over to period $t+1$, a holding cost $h_j^{nt}y_j^{n,t+1}$ is incurred, where h_j^{nt} is the corresponding unit holding cost. Figure 1 illustrates the integration of anticipative replenishment-allocation with reactive fulfillment for the online retailer. The objective is to minimize the online retailer's total cost over the planning horizon. We formulate the *joint replenishment-allocation-fulfillment* (JRAF) problem as the following optimization model:

$$(\mathrm{P_D}) \quad \min \sum_{t\in\mathcal{T}}\sum_{n\in\mathcal{N}}\left[\sum_{i\in\mathcal{I}}\left(S_i^{nt}\delta_i^{nt}+p_i^{nt}x_i^{nt}\right)+\sum_{j\in\mathcal{J}}h_j^{nt}y_j^{n,t+1}+\sum_{i\in\mathcal{I}}\sum_{j\in\mathcal{J}}a_{ij}^{nt}v_{ij}^{nt}+\sum_{j\in\mathcal{J}^+}\sum_{k\in\mathcal{K}}f_{jk}^{nt}w_{jk}^{nt}\right]$$

$$\text{s.t.} \quad \sum_{n\in\mathcal{N}} x_i^{nt} \le \bar{x}_i^t, \qquad i\in\mathcal{I}, t\in\mathcal{T}; \tag{1.1}$$

$$\sum_{j\in\mathcal{J}} v_{ij}^{nt} = x_i^{n,t-l_i^n}, \qquad n\in\mathcal{N}, i\in\mathcal{I}, t\in\mathcal{T}; \tag{1.2}$$

$$\sum_{n\in\mathcal{N}}\left(y_j^{nt}+\sum_{i\in\mathcal{I}}v_{ij}^{nt}\right) \le \bar{y}_j, \qquad j\in\mathcal{J}, t\in\mathcal{T}; \tag{1.3}$$

$$\sum_{j\in\mathcal{J}^+} w_{jk}^{nt} = d_k^{nt}, \qquad n\in\mathcal{N}, k\in\mathcal{K}, t\in\mathcal{T}; \tag{1.4}$$

$$y_j^{n,t+1} = y_j^{nt} + \sum_{i\in\mathcal{I}} v_{ij}^{nt} - \sum_{k\in\mathcal{K}} w_{jk}^{nt}, \qquad n\in\mathcal{N}, j\in\mathcal{J}, t\in\mathcal{T}; \tag{1.5}$$

$$x_i^{nt} \ge 0, \qquad n\in\mathcal{N}, i\in\mathcal{I}, t\in\mathcal{T}; \tag{1.6}$$

$$v_{ij}^{nt} \ge 0, \qquad n\in\mathcal{N}, i\in\mathcal{I}, j\in\mathcal{J}, t\in\mathcal{T}; \tag{1.7}$$

$$w_{jk}^{nt} \ge 0, \qquad n\in\mathcal{N}, j\in\mathcal{J}^+, k\in\mathcal{K}, t\in\mathcal{T}; \tag{1.8}$$

$$y_j^{nt} \ge 0, \qquad n\in\mathcal{N}, j\in\mathcal{J}, t\in\mathcal{T}^+; \tag{1.9}$$

$$x_i^{nt} \le \bar{x}_i^t\delta_i^{nt}, \delta_i^{nt}\in\{0,1\}, \qquad n\in\mathcal{N}, i\in\mathcal{I}, t\in\mathcal{T}. \tag{1.10}$$

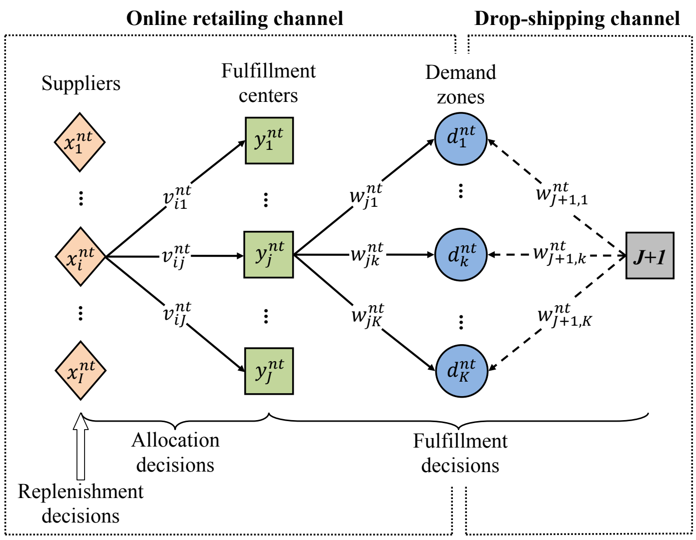


Figure 1: Integrating anticipative replenishment-allocation with reactive fulfillment for an online retailer

The first term of the objective function is the total replenishment cost, the second term is the total holding cost at the FCs, the third term is the total allocation cost to the FCs, and the last term is the total fulfillment cost to the zones. We relax the integrality constraints on the decision variables x_i^{nt}, y_j^{nt}, v_{ij}^{nt}, and w_{jk}^{nt} so that we have a tractable formulation. Problem $\mathrm{P_D}$ is always feasible because the retailer can always request drop-shipping if necessary (for example, one feasible solution is $x_i^{nt} = v_{ij}^{nt} = w_{jk}^{nt} = 0$, for $n \in \mathcal{N}$, $i \in \mathcal{I}$, $j \in \mathcal{J}$, $k \in \mathcal{K}$, $t \in \mathcal{T}$, and $w_{J+1,k}^{nt} = d_k^{nt}$, for $n \in \mathcal{N}$, $k \in \mathcal{K}$, $t \in \mathcal{T}$).

3.2 Stochastic optimization model

We now generalize Problem $\mathrm{P_D}$ to model a more practical situation with uncertain demands. Let $\tilde{d}_k^{nt}$ denote the random demand of zone k for product n in period t with mean $\hat{d}_k^{nt}$, for $n \in \mathcal{N}$, $k \in \mathcal{K}$, $t \in \mathcal{T}$. Let d_k^{nt} be the realization of $\tilde{d}_k^{nt}$ at the end of period t. For convenience, let $\tilde{\mathbf{d}}_k^{nt} = \left(\tilde{d}_k^{n1}, \dots, \tilde{d}_k^{nt}\right)$ denote a collection of demands of zone k for product n from period 1 to period t. Let $\tilde{\mathbf{d}}^t = \left(\tilde{\mathbf{d}}_1^{1t}, \dots, \tilde{\mathbf{d}}_1^{Nt}, \dots, \tilde{\mathbf{d}}_K^{1t}, \dots, \tilde{\mathbf{d}}_K^{Nt}\right)$ denote a collection of all the demands from period 1 to period t, and let $\tilde{\mathbf{d}} = \tilde{\mathbf{d}}^T$. Let $\mathbf{d}^t$ be the realization of $\tilde{\mathbf{d}}^t$ and let $\mathbf{d} = \mathbf{d}^T$.

In practice, we make the replenishment, allocation, and fulfillment decisions after observing some historical demands. We define the following adjustable decision variables: (1) $x_i^{nt}\left(\tilde{\mathbf{d}}^{t-1}\right)$ represents the quantity of product n ordered from supplier i at the start of period t after $\tilde{\mathbf{d}}^{t-1}$ is realized. $\delta_i^{nt}\left(\tilde{\mathbf{d}}^{t-1}\right)$ represents a binary variable that equals 1 if $x_i^{nt}\left(\tilde{\mathbf{d}}^{t-1}\right) > 0$, and equals 0 otherwise. (2) $v_{ij}^{nt}\left(\tilde{\mathbf{d}}^{t-1}\right)$ represents the quantity of product n from supplier i allocated to FC j at the start of period t after $\tilde{\mathbf{d}}^{t-1}$ is realized. (3) $w_{jk}^{nt}\left(\tilde{\mathbf{d}}^{t}\right)$ represents the quantity of product n retrieved from FC j to fulfill the demand of zone k at the end of period t after $\tilde{\mathbf{d}}^t$ is realized. Note that $x_i^{nt}\left(\tilde{\mathbf{d}}^{t-1}\right)$, $\delta_i^{nt}\left(\tilde{\mathbf{d}}^{t-1}\right)$, and $v_{ij}^{nt}\left(\tilde{\mathbf{d}}^{t-1}\right)$ are anticipative decisions because they are determined before we know the demands in period t, and $w_{jk}^{nt}\left(\tilde{\mathbf{d}}^{t}\right)$ is a reactive decision as it is determined after the demands in period t are realized. For notational convenience, let $\mathbf{x}\left(\tilde{\mathbf{d}}\right) = \left(x_i^{nt}\left(\tilde{\mathbf{d}}^{t-1}\right), \forall n \in \mathcal{N}, i \in \mathcal{I}, t \in \mathcal{T}\right)$ denote a collection of the replenishment decisions over the planning horizon. Similarly, let $\boldsymbol{\delta}\left(\tilde{\mathbf{d}}\right) = \left(\delta_i^{nt}\left(\tilde{\mathbf{d}}^{t-1}\right), \forall n \in \mathcal{N}, i \in \mathcal{I}, t \in \mathcal{T}\right)$, $\mathbf{v}\left(\tilde{\mathbf{d}}\right) = \left(v_{ij}^{nt}\left(\tilde{\mathbf{d}}^{t-1}\right), \forall n \in \mathcal{N}, i \in \mathcal{I}, j \in \mathcal{J}, t \in \mathcal{T}\right)$, and $\mathbf{w}\left(\tilde{\mathbf{d}}\right) = \left(w_{jk}^{nt}\left(\tilde{\mathbf{d}}^{t}\right), \forall n \in \mathcal{N}, j \in \mathcal{J}^+, k \in \mathcal{K}, t \in \mathcal{T}\right)$. We summarize the notation of the deterministic and stochastic models in Table 1.

Given $\mathbf{d}$, the retailer's total cost over the planning horizon is given by

$$\Psi\left(\boldsymbol{\delta}\left(\mathbf{d}\right), \mathbf{x}\left(\mathbf{d}\right), \mathbf{v}\left(\mathbf{d}\right), \mathbf{w}\left(\mathbf{d}\right)\right) = \underbrace{\sum_{t\in\mathcal{T}}\sum_{n\in\mathcal{N}}\sum_{i\in\mathcal{I}}\left(S_i^{nt}\delta_i^{nt}\left(\mathbf{d}^{t-1}\right) + p_i^{nt}x_i^{nt}\left(\mathbf{d}^{t-1}\right)\right)}_{\text{total replenishment cost}}$$
$$+\underbrace{\sum_{t\in\mathcal{T}}\sum_{n\in\mathcal{N}}\sum_{j\in\mathcal{J}} h_j^{nt}y_j^{n,t+1}\left(\mathbf{d}^t\right)}_{\text{total holding cost}} + \underbrace{\sum_{t\in\mathcal{T}}\sum_{n\in\mathcal{N}}\sum_{i\in\mathcal{I}}\sum_{j\in\mathcal{J}} a_{ij}^{nt}v_{ij}^{nt}\left(\mathbf{d}^{t-1}\right)}_{\text{total allocation cost}} + \underbrace{\sum_{t\in\mathcal{T}}\sum_{n\in\mathcal{N}}\sum_{j\in\mathcal{J}^+}\sum_{k\in\mathcal{K}} f_{jk}^{nt}w_{jk}^{nt}\left(\mathbf{d}^t\right)}_{\text{total fulfillment cost}}.$$

Table 1: Notation

Sets and parameters	
$\mathcal{N}$:	$\{1,\ldots,N\}$, set of products
$\mathcal{I}$:	$\{1,\ldots,I\}$, set of suppliers
$\mathcal{J}$:	$\{1,\ldots,J\}$, set of FCs
$\mathcal{K}$:	$\{1,\ldots,K\}$, set of demand zones
$\mathcal{T}$:	$\{1,\ldots,T\}$, set of periods
l_i^n:	replenishment lead time of product n from supplier i
$\bar{x}_i^t$:	production capacity of supplier i in period t
$\bar{y}_j$:	storage capacity of FC j
S_i^{nt}:	fixed setup cost for replenishing product n from supplier i in period t
p_i^{nt}:	unit production cost for replenishing product n from supplier i in period t
h_j^{nt}:	unit holding cost for product n in FC j from period t to period $t+1$
a_{ij}^{nt}:	unit allocation cost for product n from supplier i to FC j in period t
f_{jk}^{nt}:	unit fulfillment cost for product n from FC j to zone k in period t
Notation of the determinisitic model	
d_k^{nt}:	demand realization of zone k for product n in period t
$\mathbf{d}_k^{nt}$:	collection of demand realizations of zone k for product n from period 1 to period t, $\mathbf{d}_k^{nt} = \left(d_k^{n1},\ldots,d_k^{nt}\right)$
$\mathbf{d}^t$:	collection of demand realizations for all products from period 1 to period t, $\mathbf{d}^t = \left(\mathbf{d}_1^{1t},\ldots,\mathbf{d}_1^{Nt},\ldots,\mathbf{d}_K^{1t},\ldots,\mathbf{d}_K^{Nt}\right)$
x_i^{nt}:	replenishment quantity for product n from supplier i in period t
δ_i^{nt}:	binary replenishment decision for product n from supplier i in period t, $\delta_i^{nt} \in \{0,1\}$
v_{ij}^{nt}:	allocation quantity for product n from supplier i to FC j in period t
w_{jk}^{nt}:	fulfillment quantity for product n from FC j to zone k in period t
y_j^{nt}:	on-hand inventory level of product n in FC j at the start of period t
Notation of the stochastic model	
$\tilde{d}_k^{nt}$:	demand of zone k for product n in period t
$\tilde{\mathbf{d}}_k^{nt}$:	collection of demands of zone k for product n from period 1 to period t, $\tilde{\mathbf{d}}_k^{nt} = \left(\tilde{d}_k^{n1},\ldots,\tilde{d}_k^{nt}\right)$
$\tilde{\mathbf{d}}^t$:	collection of demands for all products from period 1 to period t, $\tilde{\mathbf{d}}^t = \left(\tilde{\mathbf{d}}_1^{1t},\ldots,\tilde{\mathbf{d}}_1^{Nt},\ldots,\tilde{\mathbf{d}}_K^{1t},\ldots,\tilde{\mathbf{d}}_K^{Nt}\right)$
$x_i^{nt}\left(\tilde{\mathbf{d}}^{t-1}\right)$:	adjustable replenishment quantity for product n from supplier i in period t
$\delta_i^{nt}\left(\tilde{\mathbf{d}}^{t-1}\right)$:	adjustable binary replenishment decision for product n from supplier i in period t
$v_{ij}^{nt}\left(\tilde{\mathbf{d}}^{t-1}\right)$:	adjustable allocation quantity for product n from supplier i to FC j in period t
$w_{jk}^{nt}\left(\tilde{\mathbf{d}}^{t}\right)$:	adjustable fulfillment quantity for product n from FC j to zone k in period t
$y_j^{nt}\left(\tilde{\mathbf{d}}^{t-1}\right)$:	adjustable on-hand inventory level of product n in FC j at the start of period t

We minimize the expected total cost by solving the following stochastic optimization model:

$$(\mathrm{P_S})\ \min\ E_{\tilde{\mathbf{d}}}\ \left[\Psi\left(\boldsymbol{\delta}\left(\tilde{\mathbf{d}}\right),\mathbf{x}\left(\tilde{\mathbf{d}}\right),\mathbf{v}\left(\tilde{\mathbf{d}}\right),\mathbf{w}\left(\tilde{\mathbf{d}}\right)\right)\right]$$

$$\text{s.t.} \sum_{n\in\mathcal{N}} x_i^{nt}\left(\tilde{\mathbf{d}}^{t-1}\right) \le \bar{x}_i^t, \qquad i\in\mathcal{I}, t\in\mathcal{T}; \tag{2.1}$$

$$\sum_{j\in\mathcal{J}} v_{ij}^{nt}\left(\tilde{\mathbf{d}}^{t-1}\right) = x_i^{n,t-l_i^n}\left(\tilde{\mathbf{d}}^{t-l_i^n-1}\right), \qquad n\in\mathcal{N}, i\in\mathcal{I}, t\in\mathcal{T}; \tag{2.2}$$

$$\sum_{n\in\mathcal{N}}\left(y_j^{nt}\left(\tilde{\mathbf{d}}^{t-1}\right)+\sum_{i\in\mathcal{I}} v_{ij}^{nt}\left(\tilde{\mathbf{d}}^{t-1}\right)\right) \le \bar{y}_j, \qquad j\in\mathcal{J}, t\in\mathcal{T}; \tag{2.3}$$

$$\sum_{j\in\mathcal{J}^+} w_{jk}^{nt}\left(\tilde{\mathbf{d}}^{t}\right) = \tilde{d}_k^{nt}, \qquad n\in\mathcal{N}, k\in\mathcal{K}, t\in\mathcal{T}; \tag{2.4}$$

$$y_j^{n,t+1}\left(\tilde{\mathbf{d}}^{t}\right) = y_j^{nt}\left(\tilde{\mathbf{d}}^{t-1}\right)+\sum_{i\in\mathcal{I}} v_{ij}^{nt}\left(\tilde{\mathbf{d}}^{t-1}\right)-\sum_{k\in\mathcal{K}} w_{jk}^{nt}\left(\tilde{\mathbf{d}}^{t}\right), \qquad n\in\mathcal{N}, j\in\mathcal{J}, t\in\mathcal{T}; \tag{2.5}$$

$$x_i^{nt}\left(\tilde{\mathbf{d}}^{t-1}\right) \ge 0, x_i^{nt}\left(\tilde{\mathbf{d}}^{t-1}\right) \in \mathcal{R}^{t-1}, \qquad n\in\mathcal{N}, i\in\mathcal{I}, t\in\mathcal{T}; \tag{2.6}$$

$$v_{ij}^{nt}\left(\tilde{\mathbf{d}}^{t-1}\right) \ge 0, v_{ij}^{nt}\left(\tilde{\mathbf{d}}^{t-1}\right) \in \mathcal{R}^{t-1}, \qquad n\in\mathcal{N}, i\in\mathcal{I}, j\in\mathcal{J}, t\in\mathcal{T}; \tag{2.7}$$

$$w_{jk}^{nt}\left(\tilde{\mathbf{d}}^{t}\right) \ge 0, w_{jk}^{nt}\left(\tilde{\mathbf{d}}^{t}\right) \in \mathcal{R}^{t}, \qquad n\in\mathcal{N}, j\in\mathcal{J}^+, k\in\mathcal{K}, t\in\mathcal{T}; \tag{2.8}$$

$$y_j^{nt}\left(\tilde{\mathbf{d}}^{t-1}\right) \ge 0, y_j^{nt}\left(\tilde{\mathbf{d}}^{t-1}\right) \in \mathcal{R}^{t-1}, \qquad n\in\mathcal{N}, j\in\mathcal{J}, t\in\mathcal{T}^+; \tag{2.9}$$

$$x_i^{nt}\left(\tilde{\mathbf{d}}^{t-1}\right) \le \bar{x}_i^t \delta_i^{nt}\left(\tilde{\mathbf{d}}^{t-1}\right), \delta_i^{nt}\left(\tilde{\mathbf{d}}^{t-1}\right) \in \mathcal{B}^{t-1}, \qquad n\in\mathcal{N}, i\in\mathcal{I}, t\in\mathcal{T}; \tag{2.10}$$

where $\mathcal{R}^\tau$ and $\mathcal{B}^\tau$ are sets of all functions mapping $\mathbb{R}^{N\times K\times\tau}$ to $\mathbb{R}$ and $\{0,1\}$, respectively, for any period τ. The constraints in Problem $\mathrm{P_S}$ must be satisfied essentially for all demand realizations. Note that $y_j^{n1}\left(\tilde{\mathbf{d}}^0\right) = y_j^{n1}$ is given, for $n\in\mathcal{N}$, $j\in\mathcal{J}$. By solving Problem $\mathrm{P_S}$, we obtain replenishment, allocation, and fulfillment decisions that minimize the retailer's expected total cost.

Problem $\mathrm{P_S}$ is a multi-period mixed-integer stochastic optimization problem that is generally intractable in practice. In particular, the adjustable binary decisions $\boldsymbol{\delta}\left(\tilde{\mathbf{d}}\right)$ significantly increase the computational complexity. Unfortunately, the widely used decision rules with continuous recourse decisions in ARO cannot be applied to Problem $\mathrm{P_S}$ directly. For example, the linear rule, which represents the decisions as affine functions of uncertain parameters, cannot be applied to Problem $\mathrm{P_S}$ because the binary variables $\boldsymbol{\delta}\left(\tilde{\mathbf{d}}\right)$ cannot be represented by affine functions of uncertain parameters. The static rule, which fixes the decisions as the demands unfold over time, does not work for Problem $\mathrm{P_S}$ because the fulfillment quantities may not match the random demands (that is, Constraint (2.4) may not be satisfied, for some n, k, and t). Furthermore, we will demonstrate in our numerical experiments that the existing methods (BDR and FA) that handle binary recourse decisions do not scale for large problem instances. This motivates us to develop an efficient computational method to solve Problem $\mathrm{P_S}$.

4 A two-phase approach based on robust optimization

We develop a *two-phase approach* (TPA) based on robust optimization to solve Problem P_S. In Phase 1 of our approach, we solve a *target-oriented robust optimization* (TRO) model to determine the binary replenishment decisions $\boldsymbol{\delta}^*$ with a goal to absorb as much demand uncertainty as possible. We fix these binary decisions in Phase 2, where our objective is to minimize the worst-case expected total cost. We use a linear decision rule (LDR) in Phase 2 to determine the continuous replenishment, allocation, and fulfillment quantities as the uncertainty is revealed. Phase 1 requires solving a mixed-integer linear program, whereas Phase 2 requires solving a linear program. Our idea is to decouple the binary decisions from the continuous decisions. Note that we only require the means and the support sets of the demands in this approach, which allows us to handle demand-distribution ambiguity. As will be evidenced by our numerical experiments, the TPA produces high-quality solutions with acceptable computational time for realistic problem instances.

We assume that the demand $\tilde{d}_k^{nt}$ of zone k for product n in period t falls in a support set $[\underline{d}_k^{nt}, \bar{d}_k^{nt}]$ with mean $\hat{d}_k^{nt}$. We define the uncertainty set for each $\tilde{d}_k^{nt}$ as $D_k^{nt} = \{d_k^{nt} | \underline{d}_k^{nt} \leq d_k^{nt} \leq \bar{d}_k^{nt}\}$, $n \in \mathcal{N}, k \in \mathcal{K}, t \in \mathcal{T}$. Define $\mathbf{D}^t = (D_k^{n\tau}, n \in \mathcal{N}, k \in \mathcal{K}, \tau = 1, \ldots, t)$ and $\mathbf{D} = \mathbf{D}^T$.

4.1 Phase 1: Determining binary replenishment decisions

The goal of Phase 1 is to determine the binary decisions $\boldsymbol{\delta}^*$ by solving Problem P_S using the TRO approach. For each uncertain demand $\tilde{d}_k^{nt}$, we define an *adjustable uncertainty set* as $D_k^{nt}(\gamma) := \left\{ d_k^{nt} \middle| \hat{d}_k^{nt} - \gamma \underline{z}_k^{nt} \leq d_k^{nt} \leq \hat{d}_k^{nt} + \gamma \bar{z}_k^{nt} \right\}$, where $\underline{z}_k^{nt} = \hat{d}_k^{nt} - \underline{d}_k^{nt}$, $\bar{z}_k^{nt} = \bar{d}_k^{nt} - \hat{d}_k^{nt}$, and $\gamma \in [0, 1]$ is called the *uncertainty set parameter*. For convenience, define $\mathbf{D}_k^{nt}(\gamma) = \left(D_k^{n1}(\gamma), \ldots, D_k^{nt}(\gamma)\right)$, $\mathbf{D}^t(\gamma) = \left(\mathbf{D}_1^{1t}(\gamma), \ldots, \mathbf{D}_1^{Nt}(\gamma), \ldots, \mathbf{D}_K^{1t}(\gamma), \ldots, \mathbf{D}_K^{Nt}(\gamma)\right)$, and $\mathbf{D}(\gamma) = \mathbf{D}^T(\gamma)$.

We introduce a cost target ϕ, which is a pre-specified budget that the retailer can spend on her replenishment, allocation, and fulfillment operations for the entire planning horizon. Our goal is to find a solution for the JRAF problem that maximizes the sizes of all the adjustable uncertainty sets, subject to a constraint that all demand realizations from these sets will yield a total cost no more than ϕ. We can achieve this by solving the following optimization problem:

$$
\begin{aligned}
&(\mathrm{P_{TRO}}) \quad \gamma^* = \max\ \gamma \\
\text{s.t.}\quad & \Psi\left(\boldsymbol{\delta}(\mathbf{d}), \mathbf{x}(\mathbf{d}), \mathbf{v}(\mathbf{d}), \mathbf{w}(\mathbf{d})\right) \leq \phi, && \forall \mathbf{d} \in \mathbf{D}(\gamma); \\
& \text{Constraints (2.1)–(2.10)}, && \forall \mathbf{d}^t \in \mathbf{D}^t(\gamma); \\
& 0 \leq \gamma \leq 1.
\end{aligned}
$$

The above is called the *TRO model* of the JRAF problem. The first constraint represents the cost-target constraint. The remaining constraints correspond to the constraints in Problem P_S. The TRO model finds a solution that maximizes the sizes of all the adjustable uncertainty sets such that all demand realizations from these sets will result in a total cost no larger than the pre-specified cost target (budget) ϕ. Instead of minimizing the total cost, the TRO model absorbs as much demand uncertainty as possible so long as the cost target is met.

Theorem 1 of Lim and Wang (2017) shows that a static rule can be optimal if we can identify a *worst-case scenario of uncertainty*. Unfortunately, the theorem does not apply to Problem $\mathrm{P_{TRO}}$. A static rule may not be feasible for Problem $\mathrm{P_{TRO}}$ because of the equality constraint (2.4): $\sum_{j\in\mathcal{J}^+} w_{jk}^{nt}\left(\tilde{\mathbf{d}}\right) = \tilde{d}_k^{nt}$. Note that the left-hand side of the constraint under a static rule will be a constant, which generally does not equal a random demand on the right-hand side. We can overcome this issue by solving the following relaxed problem:

$$\begin{aligned} \gamma' = \max\ & \gamma && (3)\\ \text{s.t.}\quad & \Psi\left(\boldsymbol{\delta}\left(\mathbf{d}\right), \mathbf{x}\left(\mathbf{d}\right), \mathbf{v}\left(\mathbf{d}\right), \mathbf{w}\left(\mathbf{d}\right)\right) \le \phi, && \forall \mathbf{d} \in \mathbf{D}(\gamma);\\ & \sum_{j\in\mathcal{J}^+} w_{jk}^{nt}\left(\mathbf{d}^t\right) \ge d_k^{nt}, n \in \mathcal{N}, k \in \mathcal{K}, t \in \mathcal{T}, && \forall \mathbf{d}^t \in \mathbf{D}^t(\gamma);\\ & \text{Constraints (2.1)–(2.3), (2.5)–(2.10)}, && \forall \mathbf{d}^t \in \mathbf{D}^t(\gamma);\\ & 0 \le \gamma \le 1; \end{aligned}$$

where the second constraint relaxes Constraint (2.4) to an inequality, which leads to $\gamma' \ge \gamma^*$.

We will show that a static rule $\left(\boldsymbol{\delta}\left(\mathbf{d}\right), \mathbf{x}\left(\mathbf{d}\right), \mathbf{v}\left(\mathbf{d}\right), \mathbf{w}\left(\mathbf{d}\right)\right) = \left(\boldsymbol{\delta}, \mathbf{x}, \mathbf{v}, \mathbf{w}\right)$ is optimal for Problem (3), where $\boldsymbol{\delta} = \left(\delta_i^{nt}, \forall n \in \mathcal{N}, i \in \mathcal{I}, t \in \mathcal{T}\right)$, $\mathbf{x} = \left(x_i^{nt}, \forall n \in \mathcal{N}, i \in \mathcal{I}, t \in \mathcal{T}\right)$, $\mathbf{v} = \left(v_{ij}^{nt}, \forall n \in \mathcal{N}, i \in \mathcal{I}, j \in \mathcal{J}, t \in \mathcal{T}\right)$, and $\mathbf{w} = \left(w_{jk}^{nt}, \forall n \in \mathcal{N}, j \in \mathcal{J}^+, k \in \mathcal{K}, t \in \mathcal{T}\right)$. Thus, we do not need to consider other complicated decision rules to solve Problem (3). To show this, we use a vector $\boldsymbol{\pi}\left(\mathbf{d}\right)$ to represent all the decision variables in Problem (3) and let Π denote the feasible ranges of these variables. Recall that $\mathbf{d} = \left\{d_k^{nt}, \forall n \in \mathcal{N}, k \in \mathcal{K}, t \in \mathcal{T}\right\}$ represents a collection of uncertain variables and $\mathbf{D}(\gamma)$ represents the adjustable uncertainty sets of $\mathbf{d}$ given γ. We rewrite Problem (3) in the following general form:

$$\begin{aligned} \gamma' = \max\ & \gamma && (4)\\ \text{s.t.}\quad & \mathbf{A}\left(\mathbf{d}\right)\boldsymbol{\pi}\left(\mathbf{d}\right) \le \mathbf{b}\left(\mathbf{d}\right), && \forall \mathbf{d} \in \mathbf{D}(\gamma);\\ & \mathbf{F}\left(\mathbf{d}\right)\boldsymbol{\pi}\left(\mathbf{d}\right) = \mathbf{g}\left(\mathbf{d}\right), && \forall \mathbf{d} \in \mathbf{D}(\gamma);\\ & \boldsymbol{\pi}\left(\mathbf{d}\right) \in \Pi, && \forall \mathbf{d} \in \mathbf{D}(\gamma); \end{aligned}$$

where $\mathbf{A}\left(\mathbf{d}\right)$, $\mathbf{b}\left(\mathbf{d}\right)$, $\mathbf{F}\left(\mathbf{d}\right)$, and $\mathbf{g}\left(\mathbf{d}\right)$ contain both deterministic and uncertain coefficients. For simplicity, we assume the cost target ϕ and the capacities $\bar{x}_i^t$ and $\bar{y}_j$ are sufficiently large that Problem (3) and its generalized version, Problem (4) are all feasible. The equality constraints $\mathbf{F}\left(\mathbf{d}\right)\boldsymbol{\pi}\left(\mathbf{d}\right) = \mathbf{g}\left(\mathbf{d}\right), \forall \mathbf{d} \in \mathbf{D}(\gamma)$ in the above TRO model require a different way of handling compared to the inequality constraints $\mathbf{A}\left(\mathbf{d}\right)\boldsymbol{\pi}\left(\mathbf{d}\right) \le \mathbf{b}\left(\mathbf{d}\right), \forall \mathbf{d} \in \mathbf{D}(\gamma)$ when we construct a robust counterpart optimization model. This makes the problem more complicated to handle.

We consider a static rule $\boldsymbol{\pi}\left(\mathbf{d}\right) = \boldsymbol{\pi}$ that can be obtained by solving the following problem:

$$\gamma^s = \max \gamma \qquad (5)$$

$$
\begin{aligned}
\text{s.t.}\quad & \mathbf{A}\left(\mathbf{d}\right)\boldsymbol{\pi} \le \mathbf{b}\left(\mathbf{d}\right), \quad \forall \mathbf{d} \in \mathbf{D}(\gamma); \\
& \mathbf{F}\left(\mathbf{d}\right)\boldsymbol{\pi} = \mathbf{g}\left(\mathbf{d}\right), \quad \forall \mathbf{d} \in \mathbf{D}(\gamma); \\
& \boldsymbol{\pi} \in \Pi.
\end{aligned}
$$

We emphasize that Problem (5) may not be feasible because the static rule $\boldsymbol{\pi}$ may not satisfy the constraints for *all* $\mathbf{d} \in \mathbf{D}(\gamma)$. However, we can ensure the feasibility of Problem (5) if we can identify a worst-case scenario of uncertainty defined as follows.

Definition 1. (Worst-case scenario of uncertainty) *Given γ and the coefficients $\mathbf{A}(\mathbf{d})$, $\mathbf{b}(\mathbf{d})$, $\mathbf{F}(\mathbf{d})$, and $\mathbf{g}(\mathbf{d})$, an element $\breve{\mathbf{d}}(\gamma) \in \mathbf{D}(\gamma)$ is called a worst-case scenario of uncertainty if for each $\boldsymbol{\pi} \in \Pi$ that satisfies $\mathbf{A}\left(\breve{\mathbf{d}}(\gamma)\right)\boldsymbol{\pi} \le \mathbf{b}\left(\breve{\mathbf{d}}(\gamma)\right)$ and $\mathbf{F}\left(\breve{\mathbf{d}}(\gamma)\right)\boldsymbol{\pi} = \mathbf{g}\left(\breve{\mathbf{d}}(\gamma)\right)$, it also satisfies $\mathbf{A}(\mathbf{d})\boldsymbol{\pi} \le \mathbf{b}(\mathbf{d})$ and $\mathbf{F}(\mathbf{d})\boldsymbol{\pi} = \mathbf{g}(\mathbf{d})$, for any $\mathbf{d} \in \mathbf{D}(\gamma)$.*

To identify a worst-case scenario of uncertainty for Problem (3), for any γ, we consider the upper bound $\hat{d}_k^{nt} + \gamma \bar{z}_k^{nt}$ of the adjustable uncertainty set $D_k^{nt}(\gamma)$. Let $\breve{\mathbf{d}}(\gamma) = (\hat{d}_k^{nt} + \gamma \bar{z}_k^{nt}$, $\forall n \in \mathcal{N}, k \in \mathcal{K}, t \in \mathcal{T})$. Suppose the static rule $\boldsymbol{\pi} = (\boldsymbol{\delta}, \mathbf{x}, \mathbf{v}, \mathbf{w})$ satisfies the constraints of Problem (3) under the scenario $\breve{\mathbf{d}}(\gamma)$. Specifically, we have $\sum_{j \in \mathcal{J}^+} w_{jk}^{nt} \ge \hat{d}_k^{nt} + \gamma \bar{z}_k^{nt}$, for $n \in \mathcal{N}$, $k \in \mathcal{K}$, $t \in \mathcal{T}$. Now, we show that the static rule $\boldsymbol{\pi}$ is also feasible for Problem (3) under any scenario $\mathbf{d} \in \mathbf{D}(\gamma)$. Since $\sum_{j \in \mathcal{J}^+} w_{jk}^{nt} \ge \hat{d}_k^{nt} + \gamma \bar{z}_k^{nt} \ge d_k^{nt}$, for $d_k^{nt} \in D_k^{nt}(\gamma)$, $n \in \mathcal{N}, k \in \mathcal{K}, t \in \mathcal{T}$, the static rule $\boldsymbol{\pi}$ satisfies the second constraint of Problem (3). The static rule $\boldsymbol{\pi}$ also satisfies the other constraints of Problem (3) because all their coefficients are independent of the demands. According to Definition 1, given any γ, $\breve{\mathbf{d}}(\gamma) = \left(\hat{d}_k^{nt} + \gamma \bar{z}_k^{nt}, \forall n \in \mathcal{N}, k \in \mathcal{K}, t \in \mathcal{T}\right)$ is a worst-case scenario of uncertainty for Problem (3).

Definition 1 implies that for a given γ, a static rule $\boldsymbol{\pi}$ that is feasible for Problem (5) under a worst-case scenario of uncertainty is also feasible for any other scenario $\mathbf{d} \in \mathbf{D}(\gamma)$. This leads to the following lemma.

Lemma 1. *If there exists a worst-case scenario of uncertainty $\breve{\mathbf{d}}(\gamma) \in \mathbf{D}(\gamma)$ for Problem (5) for some $\gamma \in [0, 1]$, then Problem (5) is feasible.*

The following theorem shows that an optimal solution of Problem (5) is also optimal for Problem (4). That is, a static rule is optimal for Problem (4).

Theorem 1. (Optimality of a static rule) *If there exists a worst-case scenario of uncertainty $\breve{\mathbf{d}}(\gamma) \in \mathbf{D}(\gamma)$ for Problem (5) for any $\gamma \in [0, 1]$, then a static rule $\boldsymbol{\pi}^\dagger$ is optimal for Problem (4), where $\boldsymbol{\pi}^\dagger$ represents an optimal solution of the following deterministic optimization problem:*

$$
\begin{aligned}
\gamma^\dagger = & \max \gamma \qquad (6) \\
\text{s.t.}\quad & \mathbf{A}\left(\breve{\mathbf{d}}(\gamma)\right)\boldsymbol{\pi} \le \mathbf{b}\left(\breve{\mathbf{d}}(\gamma)\right); \\
& \mathbf{F}\left(\breve{\mathbf{d}}(\gamma)\right)\boldsymbol{\pi} = \mathbf{g}\left(\breve{\mathbf{d}}(\gamma)\right); \\
& \boldsymbol{\pi} \in \Pi;
\end{aligned}
$$

and $\gamma' = \gamma^s = \gamma^\dagger$.

Theorem 1 is important because for any given $\gamma \in [0,1]$, if Problem (4) has a worst-case scenario of uncertainty, then the problem can be solved by just considering a static rule and the worst-case scenario of uncertainty for the given γ.

We now discuss how to solve Problem (3), which is a relaxed TRO model of the JRAF problem. Recall that we assume Problem (3) is feasible. According to Theorem 1, to solve Problem (3), we just need to consider a static rule $\boldsymbol{\pi}$ and the worst-case scenario of uncertainty $\breve{\mathbf{d}}(\gamma)$ for any γ. Thus, an optimal solution of Problem (3) can be obtained by solving the following mixed-integer program:

$$(\mathrm{P}_{\text{static}}) \quad \gamma^{\dagger} = \max \ \gamma$$

$$\begin{aligned}
\text{s.t.} \quad & \sum_{t \in \mathcal{T}} \sum_{n \in \mathcal{N}} \left[\sum_{i \in \mathcal{I}} \left(S_i^{nt} \delta_i^{nt} + p_i^{nt} x_i^{nt} \right) + \sum_{i \in \mathcal{I}} \sum_{j \in \mathcal{J}} a_{ij}^{nt} v_{ij}^{nt} + \sum_{j \in \mathcal{J}^+} \sum_{k \in \mathcal{K}} f_{jk}^{nt} w_{jk}^{nt} + \sum_{j \in \mathcal{J}} h_j^{nt} y_j^{n,t+1} \right] \le \phi; \\
& \sum_{j \in \mathcal{J}^+} w_{jk}^{nt} \ge \hat{d}_k^{nt} + \gamma \bar{z}_k^{nt}, n \in \mathcal{N}, k \in \mathcal{K}, t \in \mathcal{T}; \\
& \text{Constraints (1.1)–(1.3), (1.5)–(1.10);} \\
& 0 \le \gamma \le 1.
\end{aligned}$$

Define a *target coefficient* $\alpha = \frac{\rho(1) - \phi}{\rho(1) - \rho(0)}$, where $\rho(\gamma)$, $\gamma = 0, 1$, is determined as follows:

$$\begin{aligned}
\rho(\gamma) = \min \quad & \sum_{t \in \mathcal{T}} \sum_{n \in \mathcal{N}} \left[\sum_{i \in \mathcal{I}} \left(S_i^{nt} \delta_i^{nt} + p_i^{nt} x_i^{nt} \right) + \sum_{i \in \mathcal{I}} \sum_{j \in \mathcal{J}} a_{ij}^{nt} v_{ij}^{nt} + \sum_{j \in \mathcal{J}^+} \sum_{k \in \mathcal{K}} f_{jk}^{nt} w_{jk}^{nt} + \sum_{j \in \mathcal{J}} h_j^{nt} y_j^{n,t+1} \right] \qquad (7) \\
\text{s.t.} \quad & \sum_{j \in \mathcal{J}^+} w_{jk}^{nt} \ge \hat{d}_k^{nt} + \gamma \bar{z}_k^{nt}, \ n \in \mathcal{N}, k \in \mathcal{K}, t \in \mathcal{T}; \\
& \text{Constraints (1.1)–(1.3), (1.5)–(1.10).}
\end{aligned}$$

We first choose $\alpha \in [0,1]$ and find the cost target $\phi = \alpha\rho(0) + (1-\alpha)\rho(1)$. By solving Problem $\mathrm{P}_{\text{static}}$, we obtain the binary decisions $\boldsymbol{\delta}^*$ that are optimal for Problem (3). We then pass $\boldsymbol{\delta}^*$ to the second phase to find the replenishment, allocation, and fulfillment quantities.

4.2 Phase 2: Determining replenishment, allocation, and fulfillment quantities

Our goal in the second phase is to determine the replenishment, allocation, and fulfillment quantities. We define a *mean support set* for each $\tilde{d}_k^{nt}$ as $\hat{D}_k^{nt} = \left\{ \hat{d}_k^{nt} | \underline{d}_k^{nt} \le \hat{d}_k^{nt} \le \bar{d}_k^{nt} \right\}$, $n \in \mathcal{N}, k \in \mathcal{K}, t \in \mathcal{T}$. For convenience, define $\hat{\mathbf{D}} = \left(\hat{D}_k^{n\tau}, n \in \mathcal{N}, k \in \mathcal{K}, \tau \in \mathcal{T} \right)$. We define $\mathcal{F}$ as a set of distributions of $\tilde{\mathbf{d}}$ such that, for any distribution $\mathcal{P} \in \mathcal{F}$, $E_{\mathcal{P}}\left[\tilde{\mathbf{d}}\right] \in \hat{\mathbf{D}}$. Since the distributions of the uncertain demands are generally unknown, we formulate Problem $\mathrm{P_S}$ as a robust optimization problem that minimizes the worst-case expected total cost over a family of distributions (Gilboa and Schmeidler, 1989). Fixing the binary decisions $\boldsymbol{\delta}$, we find the replenishment, allocation, and fulfillment quantities by solving the following robust optimization problem in Phase 2:

$$(\mathrm{P_{robust}})\ \min\ \max_{\mathcal{P}\in\mathcal{F}} E_{\mathcal{P}}\left[\sum_{t\in\mathcal{T}}\sum_{n\in\mathcal{N}}\left(\sum_{i\in\mathcal{I}} p_i^{nt} x_i^{nt}\left(\tilde{\mathbf{d}}^{t-1}\right)+\sum_{j\in\mathcal{J}} h_j^{nt} y_j^{n,t+1}\left(\tilde{\mathbf{d}}^{t}\right)+\right.\right.$$
$$\left.\left.\sum_{i\in\mathcal{I}}\sum_{j\in\mathcal{J}} a_{ij}^{nt} v_{ij}^{nt}\left(\tilde{\mathbf{d}}^{t-1}\right)+\sum_{j\in\mathcal{J}^+}\sum_{k\in\mathcal{K}} f_{jk}^{nt} w_{jk}^{nt}\left(\tilde{\mathbf{d}}^{t}\right)\right)\right]$$

$$\begin{aligned} \text{s.t.}\quad & \text{Constraints (2.1)–(2.9)}, && \forall \mathbf{d}^t\in\mathbf{D}^t;\\ & x_i^{nt}\left(\mathbf{d}^{t-1}\right)\le \bar{x}_i^t\delta_i^{nt},\ n\in\mathcal{N}, i\in\mathcal{I}, t\in\mathcal{T}, && \forall \mathbf{d}^{t-1}\in\mathbf{D}^{t-1}.\end{aligned}$$

Note that the fixed binary variables $\boldsymbol{\delta}$ in the last constraint are given by Phase 1.

Given the fixed $\boldsymbol{\delta}$, we solve Problem $\mathrm{P_{robust}}$ using a linear decision rule. Specifically, we restrict the feasible space of the adjustable continuous variables to admit the following linear decision rule:

$$\begin{aligned} x_i^{nt}\left(\tilde{\mathbf{d}}^{t-1}\right) &= x_i^{nt,0}+\sum_{k'\in\mathcal{K}}\sum_{\tau=1}^{t-1} x_i^{nt,k'\tau}\tilde{d}_{k'}^{n\tau}, && n\in\mathcal{N}, i\in\mathcal{I}, t\in\mathcal{T},\\ v_{ij}^{nt}\left(\tilde{\mathbf{d}}^{t-1}\right) &= v_{ij}^{nt,0}+\sum_{k'\in\mathcal{K}}\sum_{\tau=1}^{t-1} v_{ij}^{nt,k'\tau}\tilde{d}_{k'}^{n\tau}, && n\in\mathcal{N}, i\in\mathcal{I}, j\in\mathcal{J}, t\in\mathcal{T},\\ w_{jk}^{nt}\left(\tilde{\mathbf{d}}^{t}\right) &= w_{jk}^{nt,0}+\sum_{k'\in\mathcal{K}}\sum_{\tau=1}^{t} w_{jk}^{nt,k'\tau}\tilde{d}_{k'}^{n\tau}, && n\in\mathcal{N}, j\in\mathcal{J}^+, k\in\mathcal{K}, t\in\mathcal{T}.\end{aligned} \tag{8}$$

Given the coefficients $x_i^{nt,0}, v_{ij}^{nt,0}, w_{jk}^{nt,0}, x_i^{nt,k'\tau}, v_{ij}^{nt,k'\tau}$, and $w_{jk}^{nt,k'\tau}$, the decisions $x_i^{nt}\left(\tilde{\mathbf{d}}^{t-1}\right)$, $v_{ij}^{nt}\left(\tilde{\mathbf{d}}^{t-1}\right)$, and $w_{jk}^{nt}\left(\tilde{\mathbf{d}}^{t}\right)$ can be determined as $\tilde{\mathbf{d}}^t$ unfolds. Note that these decisions only respond to the demands associated with all the zones for product n, and are independent of the other products. Furthermore, we restrict the feasible space of $y_j^{nt}\left(\tilde{\mathbf{d}}^{t-1}\right)$ to admit the following linear decision rule: $y_j^{nt}\left(\tilde{\mathbf{d}}^{t-1}\right)=y_j^{nt,0}+\sum_{k'\in\mathcal{K}}\sum_{\tau=1}^{t-1} y_j^{nt,k'\tau}\tilde{d}_{k'}^{n\tau}$, $n\in\mathcal{N}, j\in\mathcal{J}, t\in\mathcal{T}^+$.

Theorem 2. *Given $\boldsymbol{\delta}$, the optimal coefficients of the linear decision rule (8) can be obtained by solving the following robust optimization problem:*

$$(P_{LDR})\ \min \sum_{t\in\mathcal{T}}\sum_{n\in\mathcal{N}}\left[\sum_{i\in\mathcal{I}} p_i^{nt}\left(x_i^{nt,0}+\sum_{k'\in\mathcal{K}}\sum_{\tau=1}^{t-1} x_i^{nt,k'\tau}\hat{d}_{k'}^{n\tau}\right)+\sum_{i\in\mathcal{I}}\sum_{j\in\mathcal{J}} a_{ij}^{nt}\left(v_{ij}^{nt,0}+\sum_{k'\in\mathcal{K}}\sum_{\tau=1}^{t-1} v_{ij}^{nt,k'\tau}\hat{d}_{k'}^{n\tau}\right)\right.$$
$$\left.+\sum_{j\in\mathcal{J}^+}\sum_{k\in\mathcal{K}} f_{jk}^{nt}\left(w_{jk}^{nt,0}+\sum_{k'\in\mathcal{K}}\sum_{\tau=1}^{t} w_{jk}^{nt,k'\tau}\hat{d}_{k'}^{n\tau}\right)+\sum_{j\in\mathcal{J}} h_j^{nt}\left(y_j^{n,t+1,0}+\sum_{k'\in\mathcal{K}}\sum_{\tau=1}^{t} y_j^{n,t+1,k'\tau}\hat{d}_{k'}^{n\tau}\right)\right]$$

$$\begin{aligned} s.t.\ & \sum_{n\in\mathcal{N}}\left(x_i^{nt,0}+\sum_{k'\in\mathcal{K}}\sum_{\tau=1}^{t-1} x_i^{nt,k'\tau} d_{k'}^{n\tau}\right)\le \bar{x}_i^t, && i\in\mathcal{I}, t\in\mathcal{T}, \forall \mathbf{d}^{t-1}\in\mathbf{D}^{t-1};\\ & \sum_{j\in\mathcal{J}} v_{ij}^{nt,0} = x_i^{n,t-l_i^n,0}, && n\in\mathcal{N}, i\in\mathcal{I}, t\in\mathcal{T};\\ & \sum_{j\in\mathcal{J}} v_{ij}^{nt,k'\tau} = \begin{cases} x_i^{n,t-l_i^n,k'\tau}, & \text{if } \tau=1,\dots,t-l_i^n-1,\\ 0, & \text{if } \tau=t-l_i^n,\dots,t-1,\end{cases} && n\in\mathcal{N}, i\in\mathcal{I}, t\in\mathcal{T}, k'\in\mathcal{K};\end{aligned}$$

$$\sum_{n\in\mathcal{N}}\left[y_j^{nt,0}+\sum_{i\in\mathcal{I}}v_{ij}^{nt,0}+\sum_{k'\in\mathcal{K}}\sum_{\tau=1}^{t-1}\left(y_j^{nt,k'\tau}+\sum_{i\in\mathcal{I}}v_{ij}^{nt,k'\tau}\right)d_{k'}^{n\tau}\right]\le\bar{y}_j,\qquad j\in\mathcal{J},t\in\mathcal{T},\forall\mathbf{d}^{t-1}\in\mathbf{D}^{t-1};$$

$$\sum_{j\in\mathcal{J}^+}w_{jk}^{nt,0}=0,\qquad n\in\mathcal{N},k\in\mathcal{K},t\in\mathcal{T};$$

$$\sum_{j\in\mathcal{J}^+}w_{jk}^{nt,k'\tau}=\begin{cases}1, & \textit{if } k'=k \textit{ and } \tau=t\\ 0, & \textit{otherwise}\end{cases},\qquad n\in\mathcal{N},k\in\mathcal{K},t\in\mathcal{T},k'\in\mathcal{K},\tau=1,\ldots,t;$$

$$y_j^{n,t+1,0}=y_j^{nt,0}+\sum_{i\in\mathcal{I}}v_{ij}^{nt,0}-\sum_{k\in\mathcal{K}}w_{jk}^{nt,0},\qquad n\in\mathcal{N},j\in\mathcal{J},t\in\mathcal{T};$$

$$y_j^{n,t+1,k'\tau}=y_j^{nt,k'\tau}+\sum_{i\in\mathcal{I}}v_{ij}^{nt,k'\tau}-\sum_{k\in\mathcal{K}}w_{jk}^{nt,k'\tau},\qquad n\in\mathcal{N},j\in\mathcal{J},t\in\mathcal{T},k'\in\mathcal{K},\tau=1,\ldots,t-1;$$

$$y_j^{n,t+1,k't}+\sum_{k\in\mathcal{K}}w_{jk}^{nt,k't}=0,\qquad n\in\mathcal{N},j\in\mathcal{J},t\in\mathcal{T},k'\in\mathcal{K};$$

$$x_i^{nt,0}+\sum_{k'\in\mathcal{K}}\sum_{\tau=1}^{t-1}x_i^{nt,k'\tau}d_{k'}^{n\tau}\ge0,\qquad n\in\mathcal{N},i\in\mathcal{I},t\in\mathcal{T},\forall\mathbf{d}^{t-1}\in\mathbf{D}^{t-1};$$

$$v_{ij}^{nt,0}+\sum_{k'\in\mathcal{K}}\sum_{\tau=1}^{t-1}v_{ij}^{nt,k'\tau}d_{k'}^{n\tau}\ge0,\qquad n\in\mathcal{N},i\in\mathcal{I},j\in\mathcal{J},t\in\mathcal{T},\forall\mathbf{d}^{t-1}\in\mathbf{D}^{t-1};$$

$$w_{jk}^{nt,0}+\sum_{k'\in\mathcal{K}}\sum_{\tau=1}^{t}w_{jk}^{nt,k'\tau}d_{k'}^{n\tau}\ge0,\qquad n\in\mathcal{N},j\in\mathcal{J}^+,k\in\mathcal{K},t\in\mathcal{T},\forall\mathbf{d}^{t}\in\mathbf{D}^{t};$$

$$y_j^{nt,0}+\sum_{k'\in\mathcal{K}}\sum_{\tau=1}^{t-1}y_j^{nt,k'\tau}d_{k'}^{n\tau}\ge0,\qquad n\in\mathcal{N},j\in\mathcal{J},t\in\mathcal{T}^+,\forall\mathbf{d}^{t-1}\in\mathbf{D}^{t-1};$$

$$x_i^{nt,0}+\sum_{k'\in\mathcal{K}}\sum_{\tau=1}^{t-1}x_i^{nt,k'\tau}d_{k'}^{n\tau}\le\bar{x}_i^t\delta_i^{nt},\qquad n\in\mathcal{N},i\in\mathcal{I},t\in\mathcal{T},\forall\mathbf{d}^{t-1}\in\mathbf{D}^{t-1};$$

$$x_i^{nt,0},x_i^{nt,k'\tau}\in\mathbb{R},\qquad n\in\mathcal{N},i\in\mathcal{I},t\in\mathcal{T},k'\in\mathcal{K},\tau=1,\ldots,t-1;$$

$$v_{ij}^{nt,0},v_{ij}^{nt,k'\tau}\in\mathbb{R},\qquad n\in\mathcal{N},i\in\mathcal{I},j\in\mathcal{J},t\in\mathcal{T},k'\in\mathcal{K},\tau=1,\ldots,t-1;$$

$$w_{jk}^{nt,0},w_{jk}^{nt,k'\tau}\in\mathbb{R},\qquad n\in\mathcal{N},j\in\mathcal{J}^+,k\in\mathcal{K},t\in\mathcal{T},k'\in\mathcal{K},\tau=1,\ldots,t;$$

$$y_j^{nt,0},y_j^{nt,k'\tau}\in\mathbb{R},\qquad n\in\mathcal{N},j\in\mathcal{J},t\in\mathcal{T}^+,k'\in\mathcal{K},\tau=1,\ldots,t-1.$$

Note that we express the robust optimization formulation $\mathrm{P_{LDR}}$ in terms of demand realizations $d_{k'}^{n\tau}$. Some constraints of Problem $\mathrm{P_{LDR}}$ must hold for *all* $\mathbf{d}^t\in\mathbf{D}^t$. We can transform these constraints into deterministic linear constraints using the duality theory (Ben-Tal et al., 2009). Then, Problem $\mathrm{P_{LDR}}$ is reduced to a linear program.

Figure 2 summarizes the framework of the TPA, which combines the advantages of the static rule and the linear decision rule. The two phases complement each other to overcome their limitations. The static rule in Phase 1 allows us to determine the binary decisions (which are often challenging for the linear decision rule). The linear decision rule in Phase 2 can handle the equality constraints with uncertain parameters (which the static rule cannot handle).

Phase 1

1. Convert Problem P_S to Problem P_TRO. Relax Constraint (2.4) of Problem P_TRO to inequality and obtain Problem (3).
2. Consider the static rule and the worst-case scenario of uncertainty, reduce Problem (3) to Problem P_static.
3. Choose a target coefficient α and obtain a cost target ϕ, solve Problem P_static and obtain the binary decisions $\boldsymbol{\delta}^*$.

$\boldsymbol{\delta}^*$ $\Rightarrow$

Phase 2

1. Given $\boldsymbol{\delta}^*$, convert Problem P_S to Problem P_robust. Consider the LDR, convert Problem P_robust to Problem P_LDR.
2. Solve Problem P_LDR and obtain the coefficients of the LDR in Equations (8).
3. Determine the replenishment, allocation, and fulfillment quantities as the uncertain demands unfold according to Equations (8).

Figure 2: Framework of the two-phase approach (TPA)

5 Numerical experiments and technical insights

In this section, we conduct a series of numerical experiments to compare the TPA with other approaches. Section 5.1 benchmarks the TPA against the existing methods in the literature that handle binary decisions. Section 5.2 tests the TPA's performance on larger problem instances. Section 5.3 summarizes some technical insights from the numerical experiments.

For each policy in the experiments, we first solve the corresponding model to get its decisions, and then conduct simulations to compute its average cost. We round any non-integral x_i^{nt}, v_{ij}^{nt}, and w_{jk}^{nt} to the nearest integers. If the rounded solution does not match the number of replenished (demanded) units, then we allocate (fulfill) any additional units to (from) the nearest available FC. For example, if the realized demand of a zone for a product is 55 but the rounded solution is to fulfill 54 units from FC 1 and 0 from the other FCs, then we fulfill 1 more unit from the nearest FC that has stock. We benchmark all the policies against the *expected value given perfect information*$(EV|PI)$. We first solve Problem P_D using each sample's realized demands, and then compute $EV|PI$ as the average cost over all the samples. Define the *efficiency gap* of a policy with average cost P as $(P - EV|PI)\,/EV|PI \times 100\%$. Unless stated otherwise, we test each policy's performance under three demand distributions: $Beta(\eta,\eta), \eta = 0.3, 1, 4$ in the simulations. We set CPLEX's integrality gap tolerance to 1%.

5.1 Comparing with existing approaches

Since the TPA decouples the binary decisions from the continuous decisions, we first examine the benefits of this decoupling in terms of solution quality and computational tractability. Specifically, we benchmark the TPA against four approaches from the literature that determine the binary and continuous decisions simultaneously in a single model: (i) The BDR by Bertsimas and Georghiou (2018) that handles binary recourse decisions. (ii) The FA approach by Bertsimas and Dunning (2016) that partitions the uncertainty set and assigns different (binary and continuous) recourse decisions to each partition. (iii) An FAB approach that applies the FA approach only on the binary decisions, and solves for the continuous decisions as linear decision rules for the entire uncertainty set. The FAB approach reduces the model size as it treats *only the binary variables* as finitely adaptable. (iv) The nonadaptive binary decision (NBD) approach by See and Sim (2010) that restricts the binary decisions to static binary variables.

Our numerical experiments on small problem instances suggest that the TPA outperforms the BDR, FA, and FAB approaches in terms of average cost and computational time. The BDR, FA, and FAB approaches cannot even solve a small instance with $N = 3$, $I = J = K = 2$, and $T = 10$ within 4 hours. In contrast, the TPA obtains a solution with a lower average cost within about 1 second for all the problem instances examined by us. This suggests a promising advantage of the TPA in both solution quality and computational tractability. Our results also suggest that the TPA's average cost is very close to that of the NBD approach, which determines both the binary and continuous decisions simultaneously. Online Supplement B shows that the NBD approach is also not scalable. Furthermore, the TPA yields a smaller standard deviation of costs compared to the NBD approach.

5.2 Performance of the TPA on larger problem instances

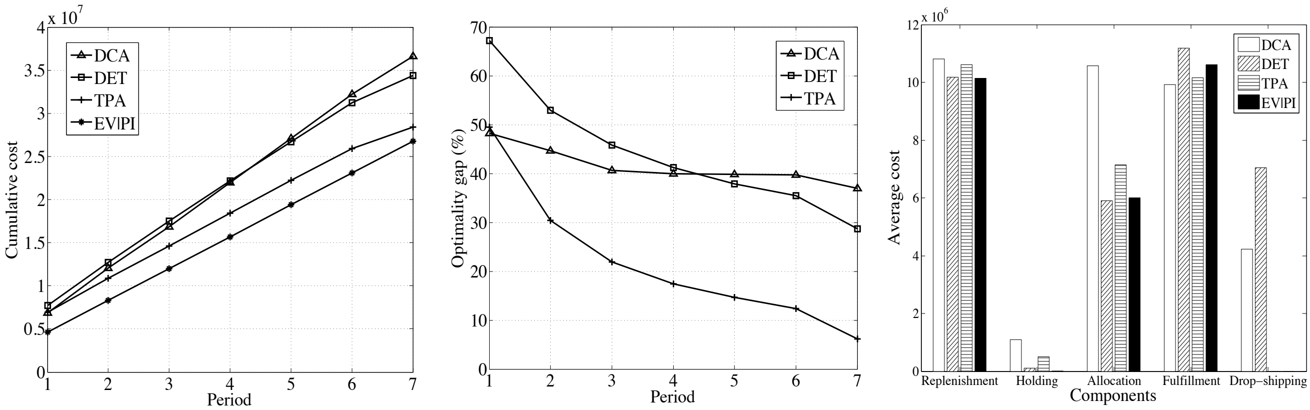
We further evaluate the TPA on larger instances. Given that the BDR, FA, and NBD approaches become intractable as the problem size increases, we compare the TPA with two other heuristics: (i) A decomposition approach (DCA) that decouples the replenishment decisions from the other decisions. Online Supplement F describes the details of the DCA. (ii) A determinisitic (DET) approach that solves Problem $\mathrm{P_D}$ using the mean demands and implements the resultant static policy. We consider five different supply chains shown in column 2 of Table 2. To choose the best target coefficient for the TPA, we first conduct a sensitivity analysis on α in Online Supplement B. As suggested by this analysis, we set $\alpha = 0.7$ for the TPA. Based on a similar analysis, we choose $\alpha = 0.1$ for the DCA. We set the CPLEX's time limit for solving a mixed-integer program to 4 hours (14,400 seconds).

Columns 4–5, 6–7, and 8–9 of Table 2 show the average cost and efficiency gap of the TPA, DCA, and DET policies respectively. The last column shows the $EV|PI$. The results suggest that the TPA significantly outperforms the other policies, with an efficiency gap less than 7%. The subscript of each average cost in Table 2 represents the standard deviation of costs over the 500 samples. The TPA consistently gives a smaller standard deviation than the DCA and

Table 2: Performance of each policy for larger-size instances

$Beta(\eta,\eta)$	$(N\|I,J,K)$	Time (s)	TPA Cost ($\times 10^6$)	TPA Gap (%)	DCA Cost ($\times 10^6$)	DCA Gap (%)	DET Cost ($\times 10^6$)	DET Gap (%)	$EV\|PI$ ($\times 10^6$)
$\eta = 0.3$	$(1,200\|2,5,5)$	19,573	$28.41_{(0.017)}$	6.20	$36.64_{(0.048)}$	36.98	$34.41_{(0.199)}$	28.64	$26.75_{(0.022)}$
	$(900\|3,6,8)$	13,270	$30.27_{(0.018)}$	4.59	$40.36_{(0.079)}$	39.42	$36.05_{(0.223)}$	24.56	$28.95_{(0.021)}$
	$(500\|4,7,10)$	17,546	$19.29_{(0.015)}$	3.11	$25.87_{(0.060)}$	38.22	$23.05_{(0.181)}$	23.19	$18.71_{(0.016)}$
	$(200\|5,10,15)$	11,984	$10.29_{(0.009)}$	4.24	$14.03_{(0.048)}$	42.13	$12.05_{(0.142)}$	22.04	$9.872_{(0.011)}$
	$(100\|6,15,20)$	22,928	$6.270_{(0.006)}$	4.88	$8.656_{(0.044)}$	44.79	$7.248_{(0.115)}$	21.23	$5.978_{(0.008)}$
$\eta = 1$	$(1,200\|2,5,5)$	19,573	$28.41_{(0.013)}$	6.20	$36.39_{(0.039)}$	36.02	$32.45_{(0.152)}$	21.30	$26.75_{(0.017)}$
	$(900\|3,6,8)$	13,270	$30.27_{(0.013)}$	4.59	$40.27_{(0.061)}$	39.11	$34.21_{(0.162)}$	18.20	$28.95_{(0.016)}$
	$(500\|4,7,10)$	17,546	$19.29_{(0.010)}$	3.11	$25.79_{(0.047)}$	37.82	$21.93_{(0.136)}$	17.17	$18.71_{(0.012)}$
	$(200\|5,10,15)$	11,984	$10.29_{(0.007)}$	4.25	$14.01_{(0.040)}$	41.89	$11.48_{(0.107)}$	16.34	$9.872_{(0.008)}$
	$(100\|6,15,20)$	22,928	$6.270_{(0.005)}$	4.89	$8.647_{(0.034)}$	44.64	$6.929_{(0.092)}$	15.91	$5.978_{(0.007)}$
$\eta = 4$	$(1,200\|2,5,5)$	19,573	$28.41_{(0.008)}$	6.20	$36.29_{(0.023)}$	35.67	$30.05_{(0.088)}$	12.33	$26.75_{(0.009)}$
	$(900\|3,6,8)$	13,270	$30.27_{(0.008)}$	4.59	$40.23_{(0.034)}$	38.98	$31.99_{(0.092)}$	10.51	$28.95_{(0.009)}$
	$(500\|4,7,10)$	17,546	$19.29_{(0.006)}$	3.11	$25.77_{(0.025)}$	37.71	$20.56_{(0.078)}$	9.89	$18.71_{(0.007)}$
	$(200\|5,10,15)$	11,984	$10.29_{(0.004)}$	4.25	$14.00_{(0.024)}$	41.81	$10.80_{(0.061)}$	9.45	$9.871_{(0.005)}$
	$(100\|6,15,20)$	22,928	$6.270_{(0.003)}$	4.85	$8.644_{(0.020)}$	44.55	$6.527_{(0.052)}$	9.16	$5.978_{(0.005)}$

DET policies, which suggests that the TPA's cost is less variable under demand ambiguity. One advantage of the TPA is that it adjusts the replenishment quantities as the demands unfold. In contrast, the other two policies fix the replenishment quantities at the start of the planning horizon. Column 3 of Table 2 shows that the TPA's computation time has an average value of about 4 hours and a highest value of about 6 hours on a desktop computer. This is acceptable for making a weekly plan, which demonstrates the applicability of the TPA in practice.



(a) Cumulative costs (b) Efficiency gaps (c) Cost components

Figure 3: Cumulative costs, efficiency gaps, and cost components

Figure 3 compares the policies for $N = 1,200$, $I = 2$, and $J = K = 5$ under the distribution $Beta(0.3, 0.3)$. Figure 3(a) shows that the TPA's cumulative cost is consistently lower than that of the other policies from period 2 onwards (with the cost savings increases with time) and maintains very close to $EV|PI$. Figure 3(b) shows that the TPA's efficiency gap decreases over time to about 7%, and is significantly lower than that of the other policies. Figure 3(c) examines the components of each policy's average cost. The TPA *does not* require any drop-shipping, whereas the other two policies result in large drop-shipping costs. This is the main driver for the TPA to outperform the other two policies because they all yield comparable results for the other components. Note that the TPA's performance is close to the $EV|PI$ in every component. Experiments based on other instances with various unit drop-shipping costs $f^{nt}_{J+1,k}$ in Online Supplement C show similar results.

5.3 Key technical insights from the numerical experiments

We have obtained the following technical insights that may be useful for practitioners from the numerical experiments.

(i) **Decoupling the binary decisions from the continuous decisions yields high-quality solutions while preserving tractability.** Given any set of binary replenishment decisions, there exists a set of continuous decisions so that the problem is feasible. This special problem structure prompts us to design the TPA to first determine the binary variables in Phase 1 and then solve for the continuous variables in Phase 2 with the binary variables fixed. This decoupling allows the TPA to use *a simple decision rule in each phase*: a static rule in Phase 1 and an LDR in Phase 2. This yields tractable formulations and makes the TPA significantly more scalable than the existing approaches (see Sections 5.1–5.2 and Online Supplement B). Compared with the BDR, FA, and FAB approaches, the TPA produces a lower average cost.

(ii) **The integration, robustness, and adaptability of the decisions create significant values.** Compared to the DCA, which decouples the replenishment decisions from the other decisions, the TPA yields a significantly lower average cost (up to 27% savings) and a smaller standard deviation of costs (see Table 2). This shows the benefits of integrating all the decisions. Benchmarking the TPA against the DET policy also shows that the TPA results in a significantly lower average cost (up to 17% savings) and a smaller standard deviation of costs (see Table 2). Online Supplement C reaffirms that the TPA consistently outperforms the DET policy under various drop-shipping costs. Furthermore, as the drop-shipping costs increase, the superiority of the TPA over the DET policy becomes more significant. The TPA's average cost is very robust and remains within 8% of the $EV|PI$ for various drop-shipping costs. This demonstrates the values of the robustness and adaptability of the decisions under the TPA.

(iii) **Solving the TRO model in Phase 1 produces more effective binary decisions.** The TRO model in Phase 1 of our approach absorbs as much demand uncertainty as possible so long as a pre-specified cost target is met. Online Supplement D demonstrates that the TPA yields significant cost savings over an alternative two-phase approach that solves the deterministic Problem P_D using the mean demands in Phase 1. This shows the value of solving the TRO model in Phase 1, which produces more effective binary decisions. We find that the cost savings is especially large when the binary replenishment decisions are crucial. This happens when the demands are so small that frequent replenishments are not necessary and the decisions on which periods to replenish become more important. Online Supplement E further shows that the TPA's performance is especially sensitive to the target coefficient α in this situation. Online Supplement H provides some guidance on the choice of α. We acknowledge that more research is needed to develop a deeper understanding of why the TPA performs well.

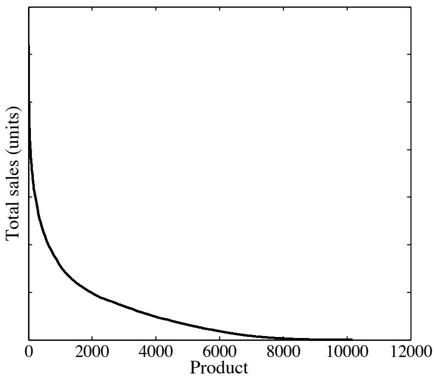
6 Numerical study using real data from an online retailer

To demonstrate the applicability of the TPA, we perform a numerical study using data from a major fashion online retailer in Asia. The retailer sells her products to six countries (or regions): Hong Kong (HK), Indonesia (ID), Malaysia (MY), the Philippines (PH), Singapore (SG), and Taiwan (TW). Each country (or region) corresponds to a zone in our model. The retailer operates three FCs, one each in Jakarta (ID), Kuala Lumpur (MY), and Manila (PH). Thus, we have $J = 3$ and $K = 6$. Currently, the retailer adopts a dedicated strategy with the Jakarta FC serves the ID zone, the Kuala Lumpur FC serves the HK, MY, SG, and TW zones, and the Manila FC serves the PH zone. When a designated FC is out of stock, the demand of a zone is fulfilled by drop-shipping.

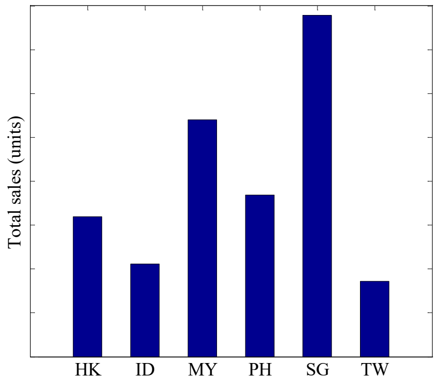
The retailer sells approximately 10,168 different products. We have collected a set of daily sales data for six months, ranging from January 1 to June 30, 2017 (total 181 days). The data set contains the actual demand of each zone for each product in each day. The retailer replenishes these products from a single supplier with multiple production facilities located in Guangzhou, China. Since the supplier can seek help from neighboring production facilities, we assume the production capacity is unlimited.

Figure 4(a) sorts the 10,168 products according to their total sales (demands) of all the zones and days. We remove the scale of the total sales on the Y-axis for confidentiality reasons. The first 1,000 and 3,000 products account for about 40% and 80%, respectively, of the total sales of all the products. Figure 4(b) shows the total sales of each zone over the six months. For computational tractability, we focus on the first 1,000 products for this study.

The values of S_i^{nt}, p_i^{nt}, and h_j^{nt} fall in the ranges $[18.04, 209.92]$, $[1.65, 20.44]$, and $[0.13, 1.86]$



(a) Total sales of each product



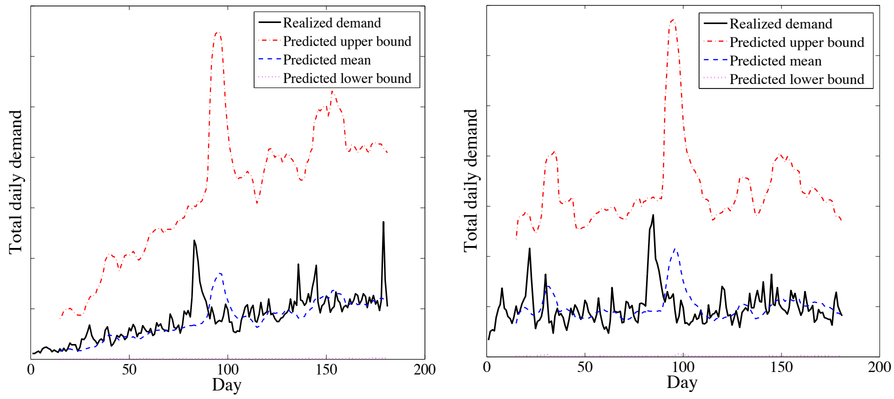
(b) Total sales of each zone

Figure 4: Total sales (demands) of the products and the zones

respectively. Online Supplement G.1 lists the unit shipping costs used in this study. The unit drop-shipping cost, which falls in the range [40.68, 442.11], consists of two components: the unit drop-shipping production cost and the unit shipping cost from Guangzhou to the corresponding zone. We set each period as a day in our model. The replenishment lead time for each product varies from 3 to 5 days. We set the length of the planning horizon $T = 7$.

Since the data does not show a strong demand correlation across the products, we assume that the demand for each product is independent of the other products. However, the autocorrelation of the daily demands for each product suggests a strong correlation across the periods. Thus, we estimate the mean demand $\hat{d}_k^{nt}$ using a moving average method by setting $\hat{d}_k^{nt}$ as the average of the actual demands of zone k for product n from period $t-14$ to period $t-8$. To find the demands' support sets, let σ_k^{nt} denote the standard deviation of the actual demands of zone k for product n from period $t-14$ to period $t-8$. We assume d_k^{nt} falls in $\left[\max\left\{0, \hat{d}_k^{nt} - 3\sigma_k^{nt}\right\}, \hat{d}_k^{nt} + 3\sigma_k^{nt}\right]$. Figures 5(a) and (b) show the forecast means and the upper bounds of the total demands of zones HK and SG, respectively, for the 1,000 products (the lower bounds are almost 0). The scale of the total daily demand on the Y-axis is removed for confidentiality reasons.

We compare the TPA, DCA, and DET policies based on the retailer's data. We use the data of weeks 1–2 to forecast the demands' means and bounds for week 3. We compute the decisions under the three policies and then evaluate them using the actual demands in week 3. We compute the policies' cumulative costs using a rolling-horizon principle: Given the updated FCs' inventory levels at the end of week 3, we use the data of weeks 2–3 to forecast the demands' means and bounds for week 4. We compute the policies and evaluate them for week 4. This procedure repeats for every subsequent week.



(a) Zone HK

(b) Zone SG

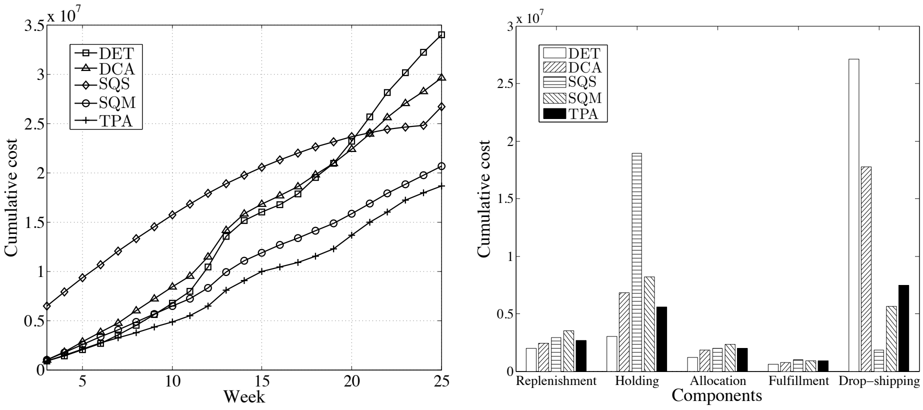
Figure 5: Demand forecast for the 1,000 products over days

We also study the retailer's *status quo single-replenishment* (SQS) policy. Under this policy, the retailer places only a single replenishment and allocates the inventory to the FCs at the start of week 3. The retailer only performs fulfillment for the remaining 23 weeks based on their current dedicated-fulfillment network structure. To optimize the SQS policy, we determine the replenishment and allocation decisions by solving Problem $\mathrm{P_D}$ for a single period using the forecast aggregate demand of each zone for each product over the 23 weeks. Given each FC's allocated inventory at the start of week 3, we then perform fulfillment. We also consider a *status quo multi-replenishment* (SQM) policy, which is the TPA applied to the retailer's current dedicated-fulfillment network structure. The SQS and SQM policies are computed and evaluated based on the same rolling-horizon principle.

6.1 Comparing various policies

We choose $\alpha = 0.1$ and 0.6 for the TPA and the DCA, respectively, based on a preliminary study in Online Supplement G.2. We compare the performance of the TPA, DCA, DET, SQS, and SQM policies for $N = 100$ to $1,000$ using the retailer's data in Online Supplement G.3. The results suggest that the TPA consistently outperforms all the other policies.

We analyze the results for $N = 1,000$ as an illustration. Figure 6(a) shows that the cumulative cost of the TPA is consistently lower than that of all the other policies over the 23 weeks. The performance gaps between the TPA and the other policies generally become larger over time, suggesting that the TPA yields substantial cost savings in the long run. In contrast, the DET policy has the worst cumulative performance. Figure 6(b) compares the cost components of the different policies over the 23 weeks. Since the actual demands can be larger than the forecast demand upper bounds, the TPA requires drop-shipping in some periods.



(a) Cumulative costs (b) Cost components

Figure 6: Performance of the different policies based on the retailer's data with $N = 1,000$

Table 3 compares the daily costs of all the policies (except the SQS policy). The second column suggests that the TPA yields a significantly lower average daily cost than the other policies. Columns 3–5 show that the TPA results in the lowest 75%, 80%, and 85% quantiles of the daily cost among all the policies. The reduction by the TPA in the average daily cost yields substantial cost savings in the long run as shown in Figure 6. We further compare the TPA with the DET policy for $N = 50, 100, 150$, and 200 in two rolling manners: rolling every week and rolling every day. Table 4 shows each policy's cumulative cost and total computational time over 23 weeks. The TPA consistently outperforms the DET policy in terms of the cumulative cost. Note that the TPA rolled every week (column 3) even outperforms the DET policy rolled every day (column 6) with a comparable computational time. Although the DET policy performs substantially better as we change from rolling weekly to rolling daily, the TPA is still preferable to simply updating the DET policy very frequently because of certain limitations in practice (for example, in many cases, the replenishment decisions cannot be updated too frequently).

Table 3: Daily costs of different policies ($\times 10^4$)

	Avg.	**75% quantile**	**80% quantile**	**85% quantile**
DET	21.12	27.32	33.77	38.65
DCA	18.40	21.11	22.58	23.47
SQM	12.82	14.85	16.63	18.27
TPA	11.57	13.41	14.71	16.85

Table 4: Comparing the TPA with the DET policy

		Rolling weekly		**Rolling daily**	
N	**Performance**	**TPA**	**DET**	**TPA**	**DET**
50	Cost($\times 10^4$)	184.61	372.66	164.85	218.40
	Total time(s)	1,360	18	8,068	1,730
100	Cost($\times 10^4$)	333.90	678.57	294.14	386.94
	Total time(s)	4,295	49	416,723	3,257
150	Cost($\times 10^4$)	445.49	922.98	385.36	537.44
	Total time(s)	8,964	68	480,518	8,808
200	Cost($\times 10^4$)	531.53	1,126.7	522.70	660.13
	Total time(s)	14,650	282	513,807	10,959

6.2 The values of multiple replenishments and flexible fulfillment

We next investigate the values of multiple replenishments and flexible fulfillment. The retailer's current practice is to replenish only twice a year (once for six months), and each FC serves only a subset of the zones. By comparing the SQS and SQM policies in Figure 6(a), we see a significant value of multiple replenishments through a flexible production schedule. The total cost savings by the SQM policy over the SQS policy for the 23 weeks is \$6,051,310 (22.67% savings). The value of flexible fulfillment can be assessed by comparing the TPA with the SQM policy. The cost savings of the TPA over the SQM policy reaches \$2,012,640 (9.75% savings). Furthermore, the cost savings of the TPA over the SQS policy (a proxy of the current practice) is \$8,063,950 (30.21% savings). The online retailer is gradually making their replenishment schedule and fulfillment structure more flexible. The above findings represent the potential values of multiple replenishments and flexible fulfillment. Comparing the cost components of the TPA and the SQS policy in Figure 6(b), we find that the SQS policy results in a significantly higher holding cost but a lower drop-shipping cost because of its single replenishment at the start of week 3.

7 Conclusion

We consider an online retailer selling multiple products to different geographical zones over a multi-period horizon. The retailer replenishes inventory from multiple suppliers and fulfills demand through different FCs. The retailer makes three types of decisions in each period: (i) At the start of the period, the retailer decides on the replenishment quantities. (ii) The retailer then decides how to allocate the inventory to the different FCs. (iii) After the demands are realized at the end of the period, the retailer decides on which FCs to fulfill the demands. If a product is out of stock, the retailer requests drop-shipping to satisfy the demands. The objective is to minimize the expected total cost over the selling horizon. To improve the service levels, the retailer can flexibly satisfy the demands from any FCs with the inventory. This fulfillment flexibility may increase the retailer's outbound shipping cost, and complicates the replenishments and allocation of inventory to the FCs. Thus, it is crucial to optimize the three types of decisions jointly. The problem is especially challenging because the replenishment-allocation is done in an anticipative manner under a "push" strategy, but the fulfillment is executed in a reactive way under a "pull" strategy. We propose a multi-period stochastic optimization model that delicately integrates the anticipative replenishment-allocation decisions with the reactive fulfillment decisions such that they can be determined seamlessly as the uncertain demands are revealed over time.

We propose a two-phase approach (TPA) based on robust optimization to solve the multi-period stochastic optimization model with binary and continuous decisions. The novelty of our approach is the way we decouple the binary decisions from the continuous decisions. In Phase

1, we determine the binary replenishment decisions with a goal to absorb as much demand uncertainty as possible so long as a pre-specified cost target is met. We fix these binary decisions in Phase 2, where our objective is to minimize the worst-case expected total cost. We use an LDR in Phase 2 to adapt the replenishment, allocation, and fulfillment quantities as the demands are realized over time. Our numerical experiments suggest that the TPA outperforms the BDR, FA, and FAB approaches, which are specifically designed to handle binary recourse decisions, in solution quality. The TPA also scales remarkably better than the existing approaches (the BDR, FA, FAB, and NBD approaches) in the literature. Despite using very limited demand information, the TPA's average cost is very close to a benchmark with perfect information ($<$ 7% gap). A study based on data from a major fashion online retailer in Asia suggests that the TPA significantly outperforms the DCA and the DET policies, and can potentially reduce the total cumulative cost of the retailer's status quo policy by 30%.

We have obtained the following technical insights that may be useful for practitioners:

(i) Given any set of binary replenishment decisions, there exists a set of continuous decisions so that the problem is feasible. This special problem structure prompts us to design the TPA to first determine the binary variables in Phase 1 and then solve for the continuous variables in Phase 2 with the binary variables fixed. Decoupling the binary decisions from the continuous decisions allows the TPA to use a simple decision rule in each phase: a static rule in Phase 1 and an LDR in Phase 2. This results in tractable formulations and high-quality solutions.

(ii) The integration, robustness, and adaptability of the TPA's decisions create significant values. The TPA yields up to 27% savings over the DCA policy, demonstrating the benefit of integrating all the decisions. The TPA also generates up to 17% savings compared to the DET policy. The TPA consistently outperforms the DET policy under various drop-shipping costs. Furthermore, as the drop-shipping costs increase, the superiority of the TPA over the DET policy becomes more significant. The TPA's average cost is very robust and remains within 8% of the $EV|PI$ for various drop-shipping costs. This demonstrates the values of the robustness and adaptability of the decisions under the TPA.

(iii) Solving the TRO model in Phase 1 of our approach produces more effective binary decisions. The TRO model accommodates as much demand uncertainty as possible when determining the binary decisions. This yields a significantly lower average cost than an alternative two-phase approach that solves a deterministic model using the mean demands in Phase 1. The cost savings is especially large when the binary replenishment decisions are crucial. This happens when the demands are so small that frequent replenishments are not necessary and the decisions on which periods to replenish become more important. We also find that the TPA's performance is especially sensitive to the target coefficient α in this situation. We emphasize that the selection of α requires experimentation and it must be done judiciously to avoid the TPA from being too conservative. We acknowledge that more research is needed to develop a deeper understanding of why the TPA performs well.

The TPA assumes that the demand uncertainty sets are represented as a hypercube. Using demands generated from ellipsoid uncertainty sets, our numerical simulations suggest that the TPA is very robust and consistently produces low average costs. We have also developed the TPA by assuming that the demand uncertainty sets are represented as an ellipsoid. The resultant approach requires a longer computational time on average, and produces average costs comparable to that of the TPA based on hypercube uncertainty sets. Generalizing the TPA to other types of uncertainty sets may extend its applicability to other problems. However, this likely leads to nonlinear constraints, which generally make the resultant model more difficult to solve. We leave this for future research.

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Online Supplement

A Proofs

A.1 Proof of Theorem 1

Since there exists a worst-case scenario of uncertainty $\breve{\mathbf{d}}(\gamma) \in \mathbf{D}(\gamma)$ for Problem (5) for *any* $\gamma \in [0,1]$, Lemma 1 implies that Problem (5) is feasible. The rest of the proof is similar to that of Lim and Wang (2017). First, we have $\gamma^s \le \gamma'$ because γ^s is obtained under a static rule. Then, we have

$$\begin{aligned}
\gamma^s &= \max\left\{\gamma \,|\mathbf{A}(\mathbf{d})\boldsymbol{\pi} \le \mathbf{b}(\mathbf{d}), \mathbf{F}(\mathbf{d})\boldsymbol{\pi} = \mathbf{g}(\mathbf{d}), \forall \mathbf{d} \in \mathbf{D}(\gamma); \boldsymbol{\pi} \in \Pi\right\} \\
&\ge \max\left\{\gamma \,\middle|\mathbf{A}\left(\breve{\mathbf{d}}(\gamma)\right)\boldsymbol{\pi} \le \mathbf{b}\left(\breve{\mathbf{d}}(\gamma)\right); \mathbf{F}\left(\breve{\mathbf{d}}(\gamma)\right)\boldsymbol{\pi} = \mathbf{g}\left(\breve{\mathbf{d}}(\gamma)\right); \boldsymbol{\pi} \in \Pi\right\} \\
&= \gamma^{\dagger} \\
&= \max\left\{\gamma \,\middle|\mathbf{A}\left(\breve{\mathbf{d}}(\gamma)\right)\boldsymbol{\pi}\left(\breve{\mathbf{d}}(\gamma)\right) \le \mathbf{b}\left(\breve{\mathbf{d}}(\gamma)\right); \mathbf{F}\left(\breve{\mathbf{d}}(\gamma)\right)\boldsymbol{\pi}\left(\breve{\mathbf{d}}(\gamma)\right) = \mathbf{g}\left(\breve{\mathbf{d}}(\gamma)\right); \boldsymbol{\pi}\left(\breve{\mathbf{d}}(\gamma)\right) \in \Pi\right\} \\
&\ge \max\left\{\gamma \,|\mathbf{A}(\mathbf{d})\boldsymbol{\pi}(\mathbf{d}) \le \mathbf{b}(\mathbf{d}), \mathbf{F}(\mathbf{d})\boldsymbol{\pi}(\mathbf{d}) = \mathbf{g}(\mathbf{d}), \boldsymbol{\pi}(\mathbf{d}) \in \Pi, \forall \mathbf{d} \in \mathbf{D}(\gamma)\right\} \\
&= \gamma'.
\end{aligned}$$

The first inequality above is due to the definition of the worst-case scenario of uncertainty, and the last inequality is due to $\breve{\mathbf{d}}(\gamma) \in \mathbf{D}(\gamma)$. Since the static rule $\boldsymbol{\pi}^{\dagger}$ is optimal for Problem (6) under the worst-case scenario of uncertainty, it is also feasible for Problem (4). Moreover, $\boldsymbol{\pi}^{\dagger}$ yields the optimal objective γ', thus the static rule is optimal for Problem (4). □

A.2 Proof of Theorem 2

Applying the LDR in (8) to the objective function of Problem $\mathrm{P_S}$, we have

$$\begin{aligned}
&\max_{\mathcal{P}\in\mathcal{F}} E_{\mathcal{P}}\left[\sum_{t\in\mathcal{T}}\sum_{n\in\mathcal{N}}\left(\sum_{i\in\mathcal{I}} p_i^{nt} x_i^{nt}\left(\tilde{\mathbf{d}}^{t-1}\right) + \sum_{j\in\mathcal{J}} h_j^{nt} y_j^{n,t+1}\left(\tilde{\mathbf{d}}^{t}\right) + \sum_{i\in\mathcal{I}}\sum_{j\in\mathcal{J}} a_{ij}^{nt} v_{ij}^{nt}\left(\tilde{\mathbf{d}}^{t-1}\right) + \sum_{j\in\mathcal{J}^+}\sum_{k\in\mathcal{K}} f_{jk}^{nt} w_{jk}^{nt}\left(\tilde{\mathbf{d}}^{t}\right)\right)\right] \\
= &\max_{\mathcal{P}\in\mathcal{F}}\left[\sum_{t\in\mathcal{T}}\sum_{n\in\mathcal{N}}\left(\sum_{i\in\mathcal{I}} p_i^{nt} x_i^{nt,0} + \sum_{j\in\mathcal{J}} h_j^{nt} y_j^{n,t+1,0} + \sum_{i\in\mathcal{I}}\sum_{j\in\mathcal{J}} a_{ij}^{nt} v_{ij}^{nt,0} + \sum_{j\in\mathcal{J}^+}\sum_{k\in\mathcal{K}} f_{jk}^{nt} w_{jk}^{nt,0}\right.\right. \\
&\quad + \sum_{k'\in\mathcal{K}}\sum_{\tau=1}^{t-1}\left(\sum_{i\in\mathcal{I}} p_i^{nt} x_i^{nt,k'\tau} + \sum_{j\in\mathcal{J}} h_j^{nt} y_j^{n,t+1,k'\tau} + \sum_{i\in\mathcal{I}}\sum_{j\in\mathcal{J}} a_{ij}^{nt} v_{ij}^{nt,k'\tau} + \sum_{j\in\mathcal{J}^+}\sum_{k\in\mathcal{K}} f_{jk}^{nt} w_{jk}^{nt,k'\tau}\right) E_{\mathcal{P}}\left[\tilde{d}_{k'}^{n\tau}\right] \\
&\quad \left.\left. + \sum_{k'\in\mathcal{K}}\left(\sum_{j\in\mathcal{J}} h_j^{nt} y_j^{n,t+1,k't} + \sum_{j\in\mathcal{J}^+}\sum_{k\in\mathcal{K}} f_{jk}^{nt} w_{jk}^{nt,k't}\right) E_{\mathcal{P}}\left[\tilde{d}_{k'}^{nt}\right]\right)\right] \\
= &\sum_{t\in\mathcal{T}}\sum_{n\in\mathcal{N}}\left[\sum_{i\in\mathcal{I}} p_i^{nt} x_i^{nt,0} + \sum_{j\in\mathcal{J}} h_j^{nt} y_j^{n,t+1,0} + \sum_{i\in\mathcal{I}}\sum_{j\in\mathcal{J}} a_{ij}^{nt} v_{ij}^{nt,0} + \sum_{j\in\mathcal{J}^+}\sum_{k\in\mathcal{K}} f_{jk}^{nt} w_{jk}^{nt,0}\right. \\
&\quad + \sum_{k'\in\mathcal{K}}\sum_{\tau=1}^{t-1}\left(\sum_{i\in\mathcal{I}} p_i^{nt} x_i^{nt,k'\tau} + \sum_{j\in\mathcal{J}} h_j^{nt} y_j^{n,t+1,k'\tau} + \sum_{i\in\mathcal{I}}\sum_{j\in\mathcal{J}} a_{ij}^{nt} v_{ij}^{nt,k'\tau} + \sum_{j\in\mathcal{J}^+}\sum_{k\in\mathcal{K}} f_{jk}^{nt} w_{jk}^{nt,k'\tau}\right)\hat{d}_{k'}^{n\tau} \\
&\quad \left. + \sum_{k'\in\mathcal{K}}\left(\sum_{j\in\mathcal{J}} h_j^{nt} y_j^{n,t+1,k't} + \sum_{j\in\mathcal{J}^+}\sum_{k\in\mathcal{K}} f_{jk}^{nt} w_{jk}^{nt,k't}\right)\hat{d}_{k'}^{nt}\right].
\end{aligned}$$

Applying the LDR in (8) to the constraint $\sum_{j\in\mathcal{J}} v_{ij}^{nt}\left(\mathbf{d}^{t-1}\right) = x_i^{n,t-l_i^n}\left(\mathbf{d}^{t-l_i^n-1}\right)$, we have

$$\begin{aligned}
&\sum_{j\in\mathcal{J}}\left(v_{ij}^{nt,0} + \sum_{k'\in\mathcal{K}}\sum_{\tau=1}^{t-1} v_{ij}^{nt,k'\tau} d_{k'}^{n\tau}\right) = x_i^{n,t-l_i^n,0} + \sum_{k'\in\mathcal{K}}\sum_{\tau=1}^{t-l_i^n-1} x_i^{n,t-l_i^n,k'\tau} d_{k'}^{n\tau},\ n\in\mathcal{N}, i\in\mathcal{I}, t\in\mathcal{T}, \forall \mathbf{d}^{t-1}\in\mathbf{D}^{t-1} \\
\Leftrightarrow\ &\sum_{j\in\mathcal{J}} v_{ij}^{nt,0} = x_i^{n,t-l_i^n,0} \text{ and } \sum_{j\in\mathcal{J}} v_{ij}^{nt,k'\tau} = \begin{cases} x_i^{n,t-l_i^n,k'\tau}, & \text{if } \tau = 1,\dots,t-l_i^n-1, \\ 0, & \text{if } \tau = t-l_i^n,\dots,t-1, \end{cases}\ n\in\mathcal{N}, i\in\mathcal{I}, t\in\mathcal{T}, k'\in\mathcal{K}.
\end{aligned}$$

The equivalence holds because the set $\mathbf{D}^{t-1}$ is full dimensional. Following a similar procedure, we apply the LDR in (8) to the other constraints and obtain Problem $\mathrm{P_{LDR}}$. □

B Comparing the TPA with the NBD approach

We consider five different supply chains shown in column 1 of Table A1 and set $T = 7$. The values of S_i^{nt}, p_i^{nt}, h_j^{nt}, a_{ij}^{nt}, and f_{jk}^{nt} are generated randomly from the ranges $[60, 80]$, $[2, 8]$, $[0.4, 1]$, $[1, 10]$, and $[2, 15]$, respectively. The unit drop-shipping cost is $f_{J+1,k}^{nt} = 200$, for $n \in \mathcal{N}, k \in \mathcal{K}, t \in \mathcal{T}$. We assume each demand $\tilde{d}_k^{nt}$ falls in $[40, 60]$ with mean 50 and follows a $Beta(1, 1)$ distribution in the simulations. We try $\alpha = 0.1, \ldots, 0.9$ for the TPA and set the time limit of CPLEX to 4 hours (14,400 seconds). Table A1 shows the average cost, the standard deviation of costs, the efficiency gap, and the computational time of each policy. The lowest cost of the TPA for each instance is marked with an asterisk.

Table A1: Comparing the TPA with the NBD approach

(N, I, J, K)	NBD	TPA								
		$\alpha = 0.1$	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
(100, 2, 5, 5)										
Cost($\times 10^4$)	228.97	229.56	229.60	229.48	229.41	229.32	229.47	229.28*	229.60	229.44
Std($\times 10^4$)	(0.405)	(0.401)	(0.401)	(0.402)	(0.401)	(0.402)	(0.404)	(0.403)	(0.403)	(0.403)
Gap(%)	3.121	3.387	3.405	3.349	3.319	3.277	3.347	3.260	3.406	3.333
Time(s)	14,400	170	263	191	641	442	484	352	542	563
(60, 3, 6, 8)										
Cost($\times 10^4$)	216.32	217.39	217.41	217.40	217.25*	217.30	217.32	217.29	217.36	217.46
Std($\times 10^4$)	(0.406)	(0.376)	(0.377)	(0.377)	(0.377)	(0.375)	(0.375)	(0.376)	(0.374)	(0.373)
Gap(%)	3.582	4.096	4.104	4.101	4.028	4.051	4.063	4.050	4.082	4.127
Time(s)	14,400	209	240	233	235	486	628	707	528	524
(40, 4, 7, 10)										
Cost($\times 10^4$)	171.99	171.83	171.81	171.83	171.82	171.80	171.79	171.76*	171.79	171.79
Std($\times 10^4$)	(0.333)	(0.318)	(0.318)	(0.317)	(0.317)	(0.317)	(0.318)	(0.317)	(0.317)	(0.317)
Gap(%)	3.911	3.813	3.803	3.812	3.806	3.793	3.789	3.773	3.791	3.788
Time(s)	14,400	264	274	269	277	349	350	425	419	372
(20, 5, 10, 15)										
Cost($\times 10^4$)	103.80	104.34	104.21	104.28	104.22	104.12	104.32	104.06*	104.08	104.11
Std($\times 10^4$)	(0.230)	(0.211)	(0.211)	(0.211)	(0.212)	(0.212)	(0.210)	(0.213)	(0.213)	(0.212)
Gap(%)	4.566	5.117	4.987	5.050	4.994	4.893	5.092	4.832	4.853	4.885
Time(s)	14,400	402	434	409	415	645	615	477	621	630
(10, 6, 15, 20)										
Cost($\times 10^4$)	1,400	64.72	64.71	64.87	64.90	64.72	64.71	64.62*	64.69	64.69
Std($\times 10^4$)	(4.736)	(0.181)	(0.181)	(0.178)	(0.177)	(0.181)	(0.181)	(0.183)	(0.181)	(0.181)
Gap(%)	2.156	4.265	4.247	4.511	4.556	4.263	4.254	4.111	4.227	4.216
Time(s)	14,400	780	671	726	946	750	690	1,163	987	881

The results suggest that the TPA's average cost is very close to (or, in some cases, even lower than) that of the NBD approach. The standard deviation of costs under the TPA is consistently lower than that of the NBD approach. This suggests that the TPA produces more stabilized costs without significantly sacrificing the solution quality. More importantly, the NBD approach cannot find an optimal solution within 4 hours for all the instances, whereas the TPA obtains a solution within 20 minutes.

To improve the NBD approach's tractability, we have tried a sparser linear decision rule and warm-starting with the binary decisions from the TRO solution. Both methods do not improve its tractability.

C Impact of the drop-shipping costs

In this section, we study the performance of the TPA by varying the unit drop-shipping costs. For each product n, we first set a minimum drop-shipping cost $\underline{f}^n$, which is independent of the zones and the periods. Specifically, we set $\underline{f}^n$ as the sum of the minimum setup cost, the minimum unit production cost, the minimum unit allocation cost, and the minimum unit fulfillment cost for product n. This ensures that drop-shipping is at least as costly as fulfilling the demand through the online retailer's own network. Note that if the drop-shipping costs for product n fall below $\underline{f}^n$, for all n, then the problem may become trivial as the retailer may be better off by simply requesting drop-shipping without replenishing any inventory.

To assess the impact of the drop-shipping costs on the TPA's performance, for each product n, we set the drop-shipping cost $f_{J+1,k}^{nt} = \underline{f}^n + \Delta$, where $\Delta = 0, 50, 100, 200, 300$, for all t, k. The values of S_i^{nt}, p_i^{nt}, h_j^{nt}, a_{ij}^{nt}, and f_{jk}^{nt} fall in the ranges $[60, 80], [4, 10], [0.04, 0.1], [2, 5]$, and $[2.4, 6]$ respectively. We assume each demand $\tilde{d}_k^{nt}$ falls in $[40, 60]$ with mean 50 and follows a $Beta(1, 1)$ distribution in the simulations. We consider $\alpha = 0.1, \dots, 0.9$, and choose the best α for the TPA. We use the DET policy as a benchmark. Table A2 shows the various costs (in 10^3) and the efficiency gaps of the two policies for different values of Δ.

Table A2: Comparing the TPA and DET policies for various magnitudes of drop-shipping costs

(N,I,J,K)	Δ	Replenishment		Holding		Drop-shipping		Total Cost($\times 10^3$)		Gap(%)	
		TPA	DET	TPA	DET	TPA	DET	TPA	DET	TPA	DET
$(100, 2, 5, 5)$	0	825.41	795.47	34.12	8.21	0	206.35	1,387.79	1,512.61	7.15	16.79
	50	826.03	793.49	33.26	8.84	0	350.82	1,386.55	1,656.68	7.06	27.92
	100	826.01	791.07	34.20	8.48	0	498.74	1,387.47	1,804.02	7.13	39.29
	200	824.74	793.64	33.87	8.92	0	788.31	1,387.35	2094.19	7.12	61.70
	300	825.41	793.64	33.46	8.92	0	1,079.69	1,386.43	2,385.58	7.05	84.20
$(60, 3, 6, 8)$	0	696.10	663.84	29.14	5.45	0	74.19	1,288.88	1,303.08	5.79	6.96
	50	697.09	677.73	29.61	5.14	0	262.84	1,290.81	1,497.42	5.83	22.76
	100	696.45	677.73	29.60	5.14	0	373.53	1,290.29	1,608.11	5.78	31.84
	200	697.47	677.73	29.40	5.14	0	594.90	1,291.06	1,829.48	5.85	49.99
	300	696.85	677.73	29.28	5.14	0	816.28	1,290.02	2,050.86	5.76	68.14
$(40, 4, 7, 10)$	0	518.56	505.56	20.11	4.52	0	110.11	1,003.33	1,079.05	4.39	12.27
	50	517.42	505.92	20.39	4.46	0	190.75	1,002.71	1,159.81	4.33	20.67
	100	518.77	505.92	20.36	4.46	0	271.79	1,003.23	1,240.86	4.38	29.11
	200	518.70	505.92	20.10	4.46	0	433.89	1,002.97	1,402.95	4.36	45.97
	300	517.88	505.92	20.19	4.46	0	595.98	1,002.93	1,565.04	4.35	62.84

It is clear that as the drop-shipping costs increase (Δ increases), the DET policy's performance drops (its efficiency gap increases). In contrast, the TPA's performance is *very robust* with its efficiency gap remaining within 8% as Δ increases. For example, the gap for the TPA is about 4.4% for all Δ in the third problem instance, but the gap for the DET policy increases from 12.27% to 62.84% as Δ increases from 0 to 300. The TPA is superior because in Phase 1 it calibrates the replenishment decisions more carefully according to the drop-shipping costs. Thereafter, it fine-tunes the allocation and fulfillment decisions in Phase 2. *Table A2 shows that the TPA incurs larger replenishment and holding costs, but a smaller drop-shipping cost than the DET policy.* This is because the TPA replenishes more products by considering the worst-case demand scenario. As the drop-shipping costs increase, the superiority of the TPA over the DET policy becomes more significant.

D Benefits of solving the TRO model in Phase 1

We examine the benefits of solving the TRO model in Phase 1 for the TPA. We compare the TPA with an alternative two-phase approach (called DTPA) that solves a deterministic model with the mean demands in Phase 1. We consider an instance with $N = 50$, $I = 3$, $J = 8$, $K = 10$, $T = 7$, and three support sets of demands: $[0, 20]$ with mean 10, $[0, 60]$ with mean 30, and $[0, 100]$ with mean 50. We compare the TPA with the DTPA under two settings: (i) A single horizon, where we randomly generate 500 demand samples and compute each policy's average cost. (ii) A rolling horizon over 20 weeks, where we randomly generate a single demand sample and compute each policy's cumulative cost. The ranges of the cost parameters are identical to that of Section 5.1. Table A3 lists the results for the distribution $Beta(4, 4)$ (the results for the other demand distributions are similar). For the single horizon, we report each policy's average cost and its computational time. For the rolling horizon, we report each policy's cumulative cost and its average computational time per week. We try $\alpha = 0.1, \dots, 0.9$ for the TPA. The lowest cost of the TPA for each setting is marked with an asterisk. The second-to-last column of Table A3 shows the DTPA's results. The last column represents the percentage of cost savings by the TPA (using the best α) over the DTPA.

Table A3 shows that for the single horizon, the smallest TPA's average cost (marked with an asterisk) is consistently lower than that of the DTPA. The cost savings by solving the TRO model in Phase 1 becomes even more significant in the rolling horizon setting, where the TPA yields a substantially lower cumulative cost (up to 38% savings) for all the cases. This suggests that the TPA is able to sustain significant cost savings over the DTPA in the long run. Furthermore, the cost savings is especially large when the binary replenishment decisions are crucial (this happens when the demands are so small, with the support set $[0, 20]$, that frequent replenishments are not necessary). The TPA generally requires a longer computational time, which is acceptable given its significant cost advantage.

We also compare the TPA with the DTPA using the online retailer's data for $N = 100, \dots, 1,000$. Table A4 shows each policy's cumulative cost and its average computational time per week. The last row of Table A4 shows the percentage of cost savings by the TPA (using $\alpha = 0.1$) over the DTPA. The

Table A3: Comparing the TPA with the DTPA

Horizon	Setting	TPA									DTPA	Savings
		$\alpha = 0.1$	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9		(%)
Single	$[0, 20]$											
	Cost($\times 10^4$)	41.71	41.02*	41.02	41.02	41.11	41.11	42.64	42.94	43.19	44.08	6.94
	Time(s)	1,270	1,002	1,140	1,317	1,211	1,207	1,655	1,773	7,701	447	
	$[0, 60]$											
	Cost($\times 10^4$)	60.45*	61.29	61.23	61.39	61.13	61.31	61.62	61.51	61.58	62.98	4.01
	Time(s)	702	966	927	1,045	1,059	1,363	2,529	7,758	2,320	493	
	$[0, 100]$											
	Cost($\times 10^4$)	208.3	201.1*	201.3	205.6	208.3	203.6	208.3	208.3	203.7	202.3	0.60
	Time(s)	1,033	902	743	725	887	736	704	744	674	679	
Rolling	$[0, 20]$											
	Cost($\times 10^5$)	70.01*	70.27	70.81	72.44	75.44	79.88	84.08	90.08	98.01	114.02	38.60
	Avg. Time(s)	1,892	1,440	1,362	1,667	1,938	4,398	7,488	7,974	7,838	622	
	$[0, 60]$											
	Cost($\times 10^5$)	108.9*	109.3	110.2	111.9	114.0	116.4	120.3	126.1	131.5	138.3	21.23
	Avg. Time(s)	1,121	1,202	1,357	1,244	1,640	1,614	3,231	6,444	7,136	392	
	$[0, 100]$											
	Cost($\times 10^5$)	345.2*	347.1	349.8	352.7	365.6	371.5	381.3	397.6	426.7	467.7	26.19
	Avg. Time(s)	1,193	1,946	846	1,026	1,651	1,119	1,714	1,069	1,124	884	

results suggest that the TPA consistently yields significant cost savings (up to 24%) over the DTPA for each value of N, which shows an obvious cost advantage of solving the TRO model in Phase 1.

Table A4: Comparing the TPA with the DTPA using the online retailer's data

Policy	Performance	Number of products (N)							
		100	200	300	400	500	600	800	1000
TPA	Cost($\times 10^6$)	3.304	5.502	7.343	9.102	10.39	11.63	15.14	18.63
	Avg. time(s)	187	762	1,897	3,848	7,116	11,758	16,097	18,132
DTPA	Cost($\times 10^6$)	4.198	7.069	9.718	11.93	13.71	15.25	18.39	20.88
	Avg. time(s)	114	296	418	704	791	1,321	1,600	2,975
	Savings(%)	21.29	22.16	24.44	23.71	24.18	23.69	17.73	10.77

E Sensitivity of the TPA's performance on α

In this section, we investigate the impact of α on the TPA's performance. Specifically, we set the demand support set as [0, 20], [40,60], [20, 80], and [0, 100]. The values of S_i^{nt}, p_i^{nt}, h_j^{nt}, a_{ij}^{nt}, f_{jk}^{nt}, and $f_{J+1,k}^{nt}$ fall in the ranges $[60, 80]$, $[4, 10]$, $[0.04, 0.1]$, $[2, 5]$, $[2.4, 6]$, and $[166, 180]$ respectively. Table A5 shows the results. For each problem instance, the lowest cost is marked with an asterisk (*), and we denote the best value of α as α^*. The last column shows the maximum percent deviation from the lowest cost (corresponding to α^*).

The results suggest the following:

(i) The TPA is sensitive to α when the demands are small (that is, [0, 20]). For example, for the problem instance $(100, 2, 5, 5)$ with the demand support set $[0, 20]$ in Table A5, $\alpha^* = 0.1$ gives a cost of 4.17×10^5, while $\alpha = 0.9$ gives a cost of 8.69×10^5, which is more than double. This is because when the demands are sufficiently small, *frequent replenishments* are not necessary. In this situation, optimizing the binary replenishment decisions is more crucial. Thus, the TPA's performance (especially Phase 1) becomes more sensitive to the target coefficient α.

(ii) The TPA is also sensitive to α when the demands have a large support set (that is, [0, 100]). For example, for the problem instance $(60, 3, 6, 8)$ with the demand support set $[0, 100]$ in Table A5, $\alpha^* = 0.7$ gives a cost of 16.13×10^5, and $\alpha = 0.9$ gives a cost that is 6% larger.

F The decomposition approach (DCA)

The DCA decouples the replenishment decisions from the allocation and fulfillment decisions. Specifically, we first determine the replenishment quantities for each period, by solving a backlogged inventory management model using the TRO approach. We fix the replenishment quantities and then determine the allocation and fulfillment quantities using the LDR.

We consider the aggregate demand for each product across all the zones. Let $\tilde{e}^{nt}$ denote the aggregate demand for product n in period t with mean $\hat{e}^{nt}$, for $n \in \mathcal{N}$, $t \in \mathcal{T}$. For convenience, let $\tilde{\mathbf{e}}^{nt} =$

Table A5: The sensitivity of the TPA's performance on α

(N, I, J, K)	**Support Set**	Cost($\times 10^5$) $\alpha =$ **0.1**	**0.2**	**0.3**	**0.4**	**0.5**	**0.6**	**0.7**	**0.8**	**0.9**	**Max Dev**(%)
(100, 2, 5, 5)	[0, 20]	4.17*	4.20	4.25	5.59	5.19	4.38	6.06	8.25	8.69	108.4
	[40, 60]	13.90	13.87*	13.89	13.89	13.91	13.90	13.90	13.90	13.91	0.3
	[20, 80]	16.01	16.10	15.86	15.89	15.92	15.85	15.84	15.93	15.77*	2.1
	[0, 100]	18.86	18.58	18.22*	18.25	18.25	18.78	24.32	28.98	19.00	59.0
(60, 3, 6, 8)	[0, 20]	3.71*	3.77	3.75	3.78	3.83	4.67	4.40	5.60	5.79	56.3
	[40, 60]	12.96	12.92	12.92	12.92	12.94	12.93	12.92	12.90*	12.92	0.4
	[20, 80]	14.64	14.42	14.42	14.45	14.44	14.38	14.38	14.37*	14.47	1.9
	[0, 100]	16.44	17.14	16.44	16.44	16.74	16.13	16.13*	16.42	17.16	6.4
(40, 4, 7, 10)	[0, 20]	2.74	2.75	2.74	2.73*	2.78	2.80	2.80	2.83	2.82	4.0
	[40, 60]	10.05	10.05	10.05	10.04	10.03	10.05	10.03	10.03*	10.03	0.2
	[20, 80]	11.06	11.08	11.05	11.03*	11.05	11.07	11.11	11.11	11.07	0.7
	[0, 100]	12.57	12.57	12.53	12.38	12.88	12.12*	12.16	12.21	12.26	6.3

$\left(\tilde{e}^{n1}, \ldots, \tilde{e}^{nt}\right)$ denote a collection of aggregate demands for product n from period 1 to period t. Let $\tilde{\mathbf{e}}^t = \left(\tilde{\mathbf{e}}^{1t}, \ldots, \tilde{\mathbf{e}}^{Nt}\right)$ denote a collection of aggregate demands for all the products from period 1 to period t. Let e^{nt} be the realization of $\tilde{e}^{nt}$ at the end of period t and let $\mathbf{e}^t$ be the realization of $\tilde{\mathbf{e}}^t$. Let $x_i^{nt}\left(\tilde{\mathbf{e}}^{t-1}\right)$ denote the replenishment quantity of product n from supplier i at the start of period t after $\tilde{\mathbf{e}}^{t-1}$ is realized. Let $\delta_i^{nt}\left(\tilde{\mathbf{e}}^{t-1}\right)$ denote a binary decision that equals 1 if $x_i^{nt}\left(\tilde{\mathbf{e}}^{t-1}\right) > 0$ and equals 0 otherwise. Let $Y^{n,t+1}\left(\tilde{\mathbf{e}}^t\right)$ denote the *system-wide* inventory of product n at the start of period $t+1$, which can be obtained by $Y^{n,t+1}\left(\tilde{\mathbf{e}}^t\right) = Y^{n1} + \sum_{i\in\mathcal{I}}\sum_{\tau=1}^{t} x_i^{n,\tau-l_i^n}\left(\tilde{\mathbf{e}}^{\tau-l_i^n-1}\right) - \sum_{\tau=1}^{t}\tilde{e}^{n\tau}$. Furthermore, define $\hat{h}^{nt} = \frac{\sum_{j\in\mathcal{J}} h_j^{nt}}{J}$ as the *average* unit holding cost for product n in period t, and $\hat{f}^{nt} = \frac{\sum_{k\in\mathcal{K}} f_{J+1,k}^{nt}}{K}$ as the *average* unit drop-shipping cost for product n in period t. We determine the replenishment quantities by solving the following backlogged inventory management model:

$$\min \mathrm{E}_{\tilde{\mathbf{e}}^T}\left[\sum_{t\in\mathcal{T}}\sum_{n\in\mathcal{N}}\sum_{i\in\mathcal{I}}\left(S_i^{nt}\delta_i^{nt}\left(\tilde{\mathbf{e}}^{t-1}\right) + p_i^{nt}x_i^{nt}\left(\tilde{\mathbf{e}}^{t-1}\right)\right) + \hat{h}^{nt}\left(Y^{n,t+1}\left(\tilde{\mathbf{e}}^t\right)\right)^+ + \hat{f}^{nt}\left(Y^{n,t+1}\left(\tilde{\mathbf{e}}^t\right)\right)^-\right] \quad \text{(A1)}$$

$$\begin{aligned}
\text{s.t.} \; & \sum_{n\in\mathcal{N}} x_i^{nt}\left(\tilde{\mathbf{e}}^{t-1}\right) \leq \bar{x}_i^t, && i\in\mathcal{I}, t\in\mathcal{T};\\
& x_i^{nt}\left(\tilde{\mathbf{e}}^{t-1}\right) \geq 0, x_i^{nt}\left(\tilde{\mathbf{e}}^{t-1}\right) \in \mathcal{R}^{t-1}, && n\in\mathcal{N}, i\in\mathcal{I}, t\in\mathcal{T};\\
& x_i^{nt}\left(\tilde{\mathbf{e}}^{t-1}\right) \leq \bar{x}_i^t\delta_i^{nt}\left(\tilde{\mathbf{e}}^{t-1}\right), \delta_i^{nt}\left(\tilde{\mathbf{e}}^{t-1}\right) \in \mathcal{B}^{t-1}, && n\in\mathcal{N}, i\in\mathcal{I}, t\in\mathcal{T}.
\end{aligned}$$

We solve Problem (A1) using the TRO approach. For each $\tilde{e}^{nt}$, we define an adjustable uncertainty set as $E^{nt}(\gamma) = \left\{e^{nt} | \hat{e}^{nt} - \gamma\underline{\xi}^{nt} \leq e^{nt} \leq \hat{e}^{nt} + \gamma\bar{\xi}^{nt}\right\}$, where $\underline{\xi}^{nt} = \hat{e}^{nt} - \underline{e}^{nt}$ and $\bar{\xi}^{nt} = \bar{e}^{nt} - \hat{e}^{nt}$. Furthermore, define $\mathbf{E}^{nt}(\gamma) = \left(E^{n1}(\gamma), \ldots, E^{nt}(\gamma)\right)$ and $\mathbf{E}^t(\gamma) = \left(\mathbf{E}^{1t}(\gamma), \ldots, \mathbf{E}^{Nt}(\gamma)\right)$. Given a cost target ϕ, we reformulate Problem (A1) to the following TRO model:

$$\gamma^* = \max \; \gamma \quad \text{(A2)}$$

$$\begin{aligned}
\text{s.t.} \; & \sum_{t\in\mathcal{T}}\sum_{n\in\mathcal{N}}\left[\sum_{i\in\mathcal{I}}\left(S_i^{nt}\delta_i^{nt}\left(\mathbf{e}^{t-1}\right) + p_i^{nt}x_i^{nt}\left(\mathbf{e}^{t-1}\right)\right) + \hat{h}^{nt}\left(Y^{n,t+1}\left(\mathbf{e}^t\right)\right)^+ + \hat{f}^{nt}\left(Y^{n,t+1}\left(\mathbf{e}^t\right)\right)^-\right] \leq \phi, \; \forall \mathbf{e}^T \in \mathbf{E}^T(\gamma);\\
& \sum_{n\in\mathcal{N}} x_i^{nt}\left(\mathbf{e}^{t-1}\right) \leq \bar{x}_i^t, \quad i\in\mathcal{I}, t\in\mathcal{T}, \forall \mathbf{e}^{t-1}\in\mathbf{E}^{t-1}(\gamma);\\
& x_i^{nt}\left(\mathbf{e}^{t-1}\right) \geq 0, x_i^{nt}\left(\mathbf{e}^{t-1}\right) \in \mathcal{R}^{t-1}, \quad n\in\mathcal{N}, i\in\mathcal{I}, t\in\mathcal{T}, \forall \mathbf{e}^{t-1}\in\mathbf{E}^{t-1}(\gamma);\\
& x_i^{nt}\left(\mathbf{e}^{t-1}\right) \leq \bar{x}_i^{nt}\delta_i^{nt}\left(\mathbf{e}^{t-1}\right), \delta_i^{nt}\left(\mathbf{e}^{t-1}\right) \in \mathcal{B}^{t-1}, \quad n\in\mathcal{N}, i\in\mathcal{I}, t\in\mathcal{T}, \forall \mathbf{e}^{t-1}\in\mathbf{E}^{t-1}(\gamma);\\
& 0 \leq \gamma \leq 1.
\end{aligned}$$

Since the first constraint of Problem (A2) contains nonlinear $(\cdot)^+$ and $(\cdot)^-$ terms, we further approximate

Problem (A2) by the following problem:

$$\gamma'' = \max \quad \gamma$$

$$\begin{aligned}
\text{s.t.} \quad & \sum_{t\in\mathcal{T}}\sum_{n\in\mathcal{N}}\left[\sum_{i\in\mathcal{I}}\left(S_i^{nt}\delta_i^{nt}\left(\mathbf{e}^{t-1}\right)+p_i^{nt}x_i^{nt}\left(\mathbf{e}^{t-1}\right)\right)+\theta^{nt}\right]\le\phi, \quad \forall \mathbf{e}^T\in\mathbf{E}^T(\gamma);\\
& \sum_{n\in\mathcal{N}} x_i^{nt}\left(\mathbf{e}^{t-1}\right)\le \bar{x}_i^t, \qquad i\in\mathcal{I}, t\in\mathcal{T}, \forall\mathbf{e}^{t-1}\in\mathbf{E}^{t-1}(\gamma);\\
& \theta^{nt}\ge \hat{h}^{nt}\left[Y^{n1}+\sum_{\tau=1}^{t}\left(\sum_{i\in\mathcal{I}}x_i^{n,\tau-l_i^n}\left(\mathbf{e}^{\tau-l_i^n-1}\right)-e^{n\tau}\right)\right], \qquad n\in\mathcal{N}, t\in\mathcal{T}, \forall\mathbf{e}^t\in\mathbf{E}^t(\gamma); \qquad \text{(A3.1)}\\
& \theta^{nt}\ge -\hat{f}^{nt}\left[Y^{n1}+\sum_{\tau=1}^{t}\left(\sum_{i\in\mathcal{I}}x_i^{n,\tau-l_i^n}\left(\mathbf{e}^{\tau-l_i^n-1}\right)-e^{n\tau}\right)\right], \qquad n\in\mathcal{N}, t\in\mathcal{T}, \forall\mathbf{e}^t\in\mathbf{E}^t(\gamma);\\
& \qquad \text{(A3.2)}\\
& x_i^{nt}\left(\mathbf{e}^{t-1}\right)\ge 0, x_i^{nt}\left(\mathbf{e}^{t-1}\right)\in\mathcal{R}^{t-1}, \qquad n\in\mathcal{N}, i\in\mathcal{I}, t\in\mathcal{T}, \forall\mathbf{e}^{t-1}\in\mathbf{E}^{t-1}(\gamma);\\
& x_i^{nt}\left(\mathbf{e}^{t-1}\right)\le \bar{x}_i^{nt}\delta_i^{nt}\left(\mathbf{e}^{t-1}\right), \delta_i^{nt}\left(\mathbf{e}^{t-1}\right)\in\mathcal{B}^{t-1}, \qquad n\in\mathcal{N}, i\in\mathcal{I}, t\in\mathcal{T}, \forall\mathbf{e}^{t-1}\in\mathbf{E}^{t-1}(\gamma);\\
& 0\le\gamma\le 1.
\end{aligned}$$

We further tighten Constraints (A3.1) by replacing $e^{n\tau}$ with $\hat{e}^{n\tau}-b^{n\tau}$, and tighten Constraints (A3.2) by replacing $e^{n\tau}$ with $\hat{e}^{n\tau}+c^{n\tau}$, where $b^{n\tau}$ and $c^{n\tau}$ are uncertain variables falling in $B^{n\tau}=\left\{b^{n\tau}\middle|0\le b^{n\tau}\le\gamma\underline{\xi}^{n\tau}\right\}$ and $C^{n\tau}=\left\{c^{n\tau}\middle|0\le c^{n\tau}\le\gamma\bar{\xi}^{n\tau}\right\}$ respectively. For convenience, define $\mathbf{b}^{nt}(\gamma)=\left(b^{n1}(\gamma),\ldots,b^{nt}(\gamma)\right)$, $\mathbf{b}^{t}(\gamma)=\left(\mathbf{b}^{1t}(\gamma),\ldots,\mathbf{b}^{Nt}(\gamma)\right)$, $\mathbf{c}^{nt}(\gamma)=\left(c^{n1}(\gamma),\ldots,c^{nt}(\gamma)\right)$, and $\mathbf{c}^{t}(\gamma)=\left(\mathbf{c}^{1t}(\gamma),\ldots,\mathbf{c}^{Nt}(\gamma)\right)$. Furthermore, define $\mathbf{B}^{nt}(\gamma)=\left(B^{n1}(\gamma),\ldots,B^{nt}(\gamma)\right)$ and $\mathbf{B}^{t}(\gamma)=\left(\mathbf{B}^{1t}(\gamma),\ldots,\mathbf{B}^{Nt}(\gamma)\right)$. Similarly, define $\mathbf{C}^{nt}(\gamma)=\left(C^{n1}(\gamma),\ldots,C^{nt}(\gamma)\right)$ and $\mathbf{C}^{t}(\gamma)=\left(\mathbf{C}^{1t}(\gamma),\ldots,\mathbf{C}^{Nt}(\gamma)\right)$. We further approximate Problem (A3) by the following problem:

$$\gamma' = \max \quad \gamma \qquad \text{(A4)}$$

$$\begin{aligned}
\text{s.t.} \quad & \sum_{n\in\mathcal{N}}\sum_{t\in\mathcal{T}}\left[\sum_{i\in\mathcal{I}}\left(S_i^{nt}\delta_i^{nt}\left(\mathbf{e}^{t-1}\right)+p_i^{nt}x_i^{nt}\left(\mathbf{e}^{t-1}\right)\right)+\theta^{nt}\right]\le\phi, \qquad \forall\mathbf{b}^T\in\mathbf{B}^T(\gamma), \forall\mathbf{c}^T\in\mathbf{C}^T(\gamma);\\
& \sum_{n\in\mathcal{N}} x_i^{nt}\left(\mathbf{e}^{t-1}\right)\le\bar{x}_i^t, i\in\mathcal{I}, t\in\mathcal{T}, \forall\mathbf{b}^{t-1}\in\mathbf{B}^{t-1}(\gamma), \qquad \forall\mathbf{c}^{t-1}\in\mathbf{C}^{t-1}(\gamma);\\
& \theta^{nt}\ge\hat{h}^{nt}\left[Y^{n1}+\sum_{\tau=1}^{t}\left(\sum_{i\in\mathcal{I}}x_i^{n,\tau-l_i^n}\left(\mathbf{e}^{\tau-l_i^n-1}\right)-\hat{e}^{n\tau}+b^{n\tau}\right)\right], \qquad n\in\mathcal{N}, t\in\mathcal{T}, \forall\mathbf{b}^t\in\mathbf{B}^t(\gamma), \forall\mathbf{c}^t\in\mathbf{C}^t(\gamma);\\
& \theta^{nt}\ge -\hat{f}^{nt}\left[Y^{n1}+\sum_{\tau=1}^{t}\left(\sum_{i\in\mathcal{I}}x_i^{n,\tau-l_i^n}\left(\mathbf{e}^{\tau-l_i^n-1}\right)-\hat{e}^{n\tau}-c^{n\tau}\right)\right], \qquad n\in\mathcal{N}, t\in\mathcal{T}, \forall\mathbf{b}^t\in\mathbf{B}^t(\gamma), \forall\mathbf{c}^t\in\mathbf{C}^t(\gamma);\\
& x_i^{nt}\left(\mathbf{e}^{t-1}\right)\ge 0, x_i^{nt}\left(\mathbf{e}^{t-1}\right)\in\mathcal{R}^{t-1}, \qquad n\in\mathcal{N}, i\in\mathcal{I}, t\in\mathcal{T}, \forall\mathbf{b}^{t-1}\in\mathbf{B}^{t-1}(\gamma), \forall\mathbf{c}^{t-1}\in\mathbf{C}^{t-1}(\gamma);\\
& x_i^{nt}\left(\mathbf{e}^{t-1}\right)\le\bar{x}_i^{nt}\delta_i^{nt}\left(\tilde{\mathbf{e}}^{t-1}\right), \delta_i^{nt}\left(\mathbf{e}^{t-1}\right)\in\mathcal{B}^{t-1}, \qquad n\in\mathcal{N}, i\in\mathcal{I}, t\in\mathcal{T}, \forall\mathbf{b}^{t-1}\in\mathbf{B}^{t-1}(\gamma), \forall\mathbf{c}^{t-1}\in\mathbf{C}^{t-1}(\gamma);\\
& 0\le\gamma\le 1.
\end{aligned}$$

According to Theorem 1, the static rule is optimal for Problem (A4). Thus, the solution to Problem (A4) can be obtained by solving the following mixed-integer program:

$$\gamma^{\dagger} = \max \quad \gamma \qquad \text{(A5)}$$

$$\begin{aligned}
\text{s.t.} \quad & \sum_{n\in\mathcal{N}}\sum_{t\in\mathcal{T}}\left[\sum_{i\in\mathcal{I}}\left(S_i^{nt}\delta_i^{nt}+p_i^{nt}x_i^{nt}\right)+\theta^{nt}\right]\le\phi;\\
& \sum_{n\in\mathcal{N}} x_i^{nt}\le\bar{x}_i^t, \qquad i\in\mathcal{I}, t\in\mathcal{T};\\
& \theta^{nt}\ge\hat{h}^{nt}\left[Y^{n1}+\sum_{\tau=1}^{t}\left(\sum_{i\in\mathcal{I}}x_i^{n,\tau-l_i^n}+\gamma\underline{\xi}^{n\tau}\right)-\sum_{\tau=1}^{t}\hat{e}^{n\tau}\right], \qquad n\in\mathcal{N}, t\in\mathcal{T};\\
& \theta^{nt}\ge -\hat{f}^{nt}\left[Y^{n1}+\sum_{\tau=1}^{t}\left(\sum_{i\in\mathcal{I}}x_i^{n,\tau-l_i^n}-\gamma\bar{\xi}^{n\tau}\right)-\sum_{\tau=1}^{t}\hat{e}^{n\tau}\right], \qquad n\in\mathcal{N}, t\in\mathcal{T};
\end{aligned}$$

$$x_i^{nt} \geq 0, \qquad n \in \mathcal{N}, i \in \mathcal{I}, t \in \mathcal{T};$$
$$x_i^{nt} \leq \bar{x}_i^t \delta_i^{nt}, \delta_i^{nt} \in \{0,1\}, \qquad n \in \mathcal{N}, i \in \mathcal{I}, t \in \mathcal{T};$$
$$0 \leq \gamma \leq 1.$$

By solving Problem (A5), we obtain the replenishment quantities $\mathbf{x}^*$. We fix the replenishment quantities and then determine the allocation and fulfillment quantities using the LDR in (8).

G Supplemental material for the numerical study using the online retailer's data

G.1 Unit shipping costs

To determine the unit allocation cost from the supplier to each FC, we choose the cheapest rate for shipping a 1-kg parcel from Guangzhou to each FC from the DHL, UPS, and Fedex websites[1]. Likewise, we use the cheapest rate for shipping a 1-kg parcel from each FC to each zone from the above websites to set the corresponding unit fulfillment cost. We scale the shipping costs such that they are comparable to the unit production costs. Table A6 lists the unit shipping costs used in this study.

Table A6: Unit shipping cost (USD)

	Destinations					
Origins	**HK**	**ID**	**MY**	**PH**	**SG**	**TW**
Guangzhou	2.942	4.361	4.361	4.361	4.361	4.276
Jakarta	4.986	0.335	4.986	4.986	3.757	5.829
Kuala Lumpur	3.527	3.527	1.625	3.527	2.193	3.527
Manila	1.893	2.453	2.453	2.505	1.893	1.893

G.2 A preliminary study

We conduct a preliminary study to choose the best target coefficients for the TPA and the DCA. We try $\alpha = 0.1, \ldots, 0.9$ and vary $N = 50, 100, 150, 200$. Table A7 shows the cumulative cost of each policy. The lowest cost of each policy for each N is marked with an asterisk. We observe that the lowest cost of the TPA policy occurs when $\alpha \in [0.1, 0.3]$. Thus, we choose $\alpha = 0.1$ for the TPA. Similarly, the lowest cost of the DCA policy occurs when $\alpha \in [0.6, 0.7]$ and we choose $\alpha = 0.6$ for the DCA.

Table A7: Choosing the best α

N	Policy	Cumulative cost ($\times 10^4$)								
		$\alpha = 0.1$	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
50	TPA	184.61*	184.95	184.76	185.42	189.81	192.36	192.46	196.15	206.47
	DCA	309.88	315.51	302.72	302.52	300.65	295.28*	297.35	301.76	330.15
100	TPA	333.90	332.01*	333.80	340.93	340.00	342.27	356.82	355.61	374.36
	DCA	604.27	585.42	579.34	576.05	567.79	568.92	564.61*	579.02	635.81
150	TPA	445.49*	446.03	448.72	450.59	455.86	459.45	481.55	491.37	548.57
	DCA	705.87	683.41	682.94	685.24	686.25	681.87*	683.74	715.74	796.34
200	TPA	531.53	534.80	529.48*	560.68	541.42	547.60	562.90	593.17	616.38
	DCA	974.79	975.08	960.62	949.59	947.53	937.71*	942.45	990.84	1,075.4

G.3 Comparing the policies for different numbers of products

We study the performance of each policy for $N = 100$ to 1,000. We set the time limit of CPLEX for solving a single mixed-integer model to 4 hours (14,400 seconds). Table A8 lists each policy's cumulative cost and its average computational time per week. We also report the percentages of cost savings by the TPA over the other policies. The results suggest that the TPA yields significantly lower cumulative costs (up to 54% savings) for all the cases. Although the TPA requires a longer computational time than the other heuristics, we can compute it for 1,000 products within about 5 hours on average. This is acceptable for making a weekly plan.

[1] Sources: http://dct.dhl.com/input.jsp?langId=en#shipping_options, https://www.ups.com/mobile/ratetnthome?loc=en_us, https://www.fedex.com/ratefinder/home?cc=US&language=en&locId=express

Table A8: Comparing the policies for various numbers of products

Policy	Performance	Number of products (N)							
		100	**200**	**300**	**400**	**500**	**600**	**800**	**1,000**
TPA	Cost($\times 10^6$)	3.339	5.315	7.343	9.102	10.39	11.63	15.14	18.63
	Avg. time(s)	187	640	1,897	3,848	7,116	11,758	16,097	18,132
DET	Cost($\times 10^6$)	6.786	11.27	15.74	19.39	22.60	24.94	29.88	34.01
	TPA savings(%)	50.79	52.82	53.35	53.05	54.02	53.36	49.36	45.22
	Avg. time(s)	2	10	10	13	16	33	109	28
DCA	Cost($\times 10^6$)	5.689	9.377	13.16	16.22	18.91	20.97	25.76	29.70
	TPA savings(%)	41.31	43.32	44.19	43.89	45.05	44.51	41.27	37.28
	Avg. time(s)	74	207	233	323	375	831	829	861
SQS	Cost($\times 10^6$)	4.530	7.721	11.37	14.26	16.59	18.71	23.14	26.69
	TPA savings(%)	26.29	31.16	35.40	36.18	37.36	37.82	34.61	30.21
	Avg. time(s)	< 1	< 1	< 1	< 1	< 1	< 1	< 1	< 1
SQM	Cost($\times 10^6$)	4.119	6.743	9.323	11.65	13.49	15.05	18.45	20.64
	TPA savings(%)	18.93	21.17	21.23	21.86	22.95	22.70	18.00	9.75
	Avg. time(s)	75	205	427	839	837	2,186	4,317	3,800

H Choosing the value of α

The TPA relies on the choice of the target coefficient α. To understand how the value of α should be chosen, we conduct experiments to investigate the impact of drop-shipping costs, demand support sets, and setup costs on the choice of α. Table A9 shows the best α for various magnitudes of drop-shipping costs. We consider three supply-chain structures and two demand support sets as shown in columns 1 and 2, respectively, of Table A9. We assume that all demands follow a $Beta(1, 1)$ distribution in the simulations. The values of S_i^{nt}, p_i^{nt}, h_j^{nt}, a_{ij}^{nt}, and f_{jk}^{nt} fall in the ranges $[60, 80]$, $[4, 10]$, $[0.04, 0.1]$, $[2, 5]$, and $[2.4, 6]$ respectively. We set the drop-shipping cost $f_{J+1,k}^{nt} = \underline{f}^n + \Delta$, for $n \in \mathcal{N}$, $k \in \mathcal{K}$, $t \in \mathcal{T}$, where $\underline{f}^n$ represents a minimum drop-shipping cost and $\Delta = 0, 100, 200, 300$, and 400. We try $\alpha = 0.1, \ldots, 0.9$ and report the best α that yields the lowest average cost in the simulations. Table A10 shows the best α for various magnitudes of setup costs. The demand support set is [40,60]. We fix the drop-shipping costs by setting $\Delta = 200$. We set the setup cost $S_i^{nt} = \underline{S}_i^n + \Delta_s$, for $n \in \mathcal{N}$, $i \in \mathcal{I}$, $t \in \mathcal{T}$, where $\underline{S}_i^n$ represents a base setup cost randomly generated from [60, 80] and $\Delta_s = 0, 50, 100, 150$, and 200. The values of p_i^{nt}, h_j^{nt}, a_{ij}^{nt}, and f_{jk}^{nt} fall in the ranges $[4, 10]$, $[0.04, 0.1]$, $[2, 5]$, and $[2.4, 6]$ respectively.

Table A9: The best α for various magnitudes of drop-shipping costs

(N, I, J, K)	Support Set	Δ				
		0	100	200	300	400
$(100, 2, 5, 5)$	[0,20]	0.2	0.2	0.2	0.1	0.1
	[40,60]	0.7	0.4	0.2	0.2	0.1
$(100, 3, 6, 8)$	[0,20]	0.1	0.1	0.1	0.1	0.1
	[40,60]	0.7	0.7	0.6	0.1	0.2
$(100, 4, 7, 10)$	[0,20]	0.1	0.2	0.2	0.1	0.2
	[40,60]	0.7	0.7	0.2	0.1	0.1

Table A10: The best α for various magnitudes of setup costs

(N, I, J, K)	Δ_s				
	0	50	100	150	200
$(100, 2, 5, 5)$	0.4	0.1	0.1	0.2	0.1
$(60, 3, 8, 10)$	0.7	0.4	0.3	0.3	0.3
$(40, 4, 7, 10)$	0.5	0.3	0.3	0.1	0.2

We observe the following trends about the choice of α from Tables A9 and A10:

(i) As the drop-shipping costs increase (Δ increases), we should choose a smaller α (that is, $\alpha = 0.1$ or 0.2). If drop-shipping is costly, we should be more conservative and take larger demands into account when making the binary replenishment decisions. A smaller α corresponds to a larger cost target ϕ, which produces more conservative binary replenishment decisions.

(ii) As the demands become smaller (when the demands fall in $[0, 20]$ in Table A9), we should choose a smaller α. If the demands are sufficiently small that frequent replenishments are not necessary, the binary replenishment decisions become especially important. In this situation, we want to absorb as much uncertainty as possible. We can achieve this by choosing a smaller α.

(iii) As the setup costs become larger (Δ_s increases), a smaller α is generally preferred. When the setup costs become larger, we should be more conservative in making the binary replenishment decisions by absorbing more uncertainty. This can be achieved by setting a smaller α.

The observed trends above provide some guidance for practitioners to choose the value of α.