

Firma Invitada:**Multiple representations and GeoGebra-based learning environments****Markus Hohenwarter**

Resumen	<p>El software de matemáticas Dinámica GeoGebra ofrece la posibilidad de generar applets interactivos para su uso en entornos de aprendizaje. Sus gráficos, álgebra, vistas de álgebra y planilla de cálculo combinan múltiples representaciones matemáticas de uno con el otro de una manera interactiva y conectada. Por un lado, el software facilita la visualización de los datos y conceptos matemáticos. Por otro lado, GeoGebra es compatible con la interacción de diferentes formas de representación de objetos matemáticos. En esta presentación, el uso de applets de GeoGebra está dirigido particularmente para tablets, que se están volviendo cada vez más importantes y se distinguen por su operación intuitiva con los dedos. Matemáticas abstractas están ahora disponibles al alcance de los estudiantes permitiendo una forma muy directa de interacción con objetos y conceptos matemáticos. Una forma de implementar tales ambientes de aprendizaje son colecciones de applets interactivos en el llamado "GeoGebraBooks" que se presenta en este artículo.</p>
Abstract	<p>The dynamic mathematics software GeoGebra offers the possibility of generating interactive applets for use in learning environments. Its graphics, algebra, computer algebra and spreadsheet views combine multiple mathematical representations with each other in an interactive and connected way. On the one hand, the software facilitates the visualization of mathematical concepts and facts. On the other hand, GeoGebra supports the interaction of different forms of representation of mathematical objects. In this paper, the use of GeoGebra applets is particularly addressed concerning tablet computers, which are becoming increasingly important and are distinguished by their intuitive operation with fingers. Abstract mathematics now becomes available at students' fingertips allowing a very direct form of interaction with mathematical concepts and objects. One way to implement such learning environments are collections of interactive applets in so called "GeoGebraBooks" which are presented in this paper.</p>
Resumo	<p>O software de matemática dinâmico GeoGebra oferece a possibilidade de gerar interativo applets para uso em aprender meios Seus gráficos, álgebra, álgebra de computador e spreadsheet as vistas combinam representações matemáticas múltiplas com a cada outro em uma maneira interactiva e conectada. Por um lado, o software facilita a visualização de fatos e conceitos matemáticos. Por outro lado, GeoGebra apoia a interação de formas diferentes de representação de objetos matemáticos. Neste papel, o uso de GeoGebra applets é particularmente dirigido preocupar-se computadores de pastilla, os quais estão devindo a cada vez mais importantes e está distinguido por sua operação intuitiva com dedos. A matemática abstrata agora devém disponível em estudantes' fingertips deixando uma forma muito direta de interação com objetos e conceitos matemáticos. Uma maneira para implementar tais meios de aprendizagem são coleções de interactivos applets em tão chamados "GeoGebraBooks" quais estão apresentados neste papel</p>

Images in mathematics education - iconic representations and visualization

How can images support mathematical understanding and in what way can the use of technology be useful in this context? To answer this question, let us first consider some theoretical aspects concerning the educational use of mathematical images. In particular, let us have a closer look at the terms "iconic" and "visualization". Both are often used in mathematics education in the context of visual representations and the principle of interaction of multiple representations (Wittmann 1981, p.91) which states that mathematical knowledge is retained more easily if it was acquired using multiple representation. The word "iconic" is especially important in Bruner's (1971) theory where he distinguishes between the following three forms of representations allowing to represent and apply knowledge:

- enactive (action-related),
- iconic (visual),
- symbolic.

Let us illustrate these three types of representations with a simple example. The task " $2 + 3 = ?$ " can not only be represented and solved in a symbolic way by writing down numbers, but it also done enactively by using manipulatives: first a child could count two and then three beads in her hand and place them on a table where she can then count the total number of beads by physically touching and counting them. In addition, the task can also be supported by an iconic (visual) representation such as in figure 1.

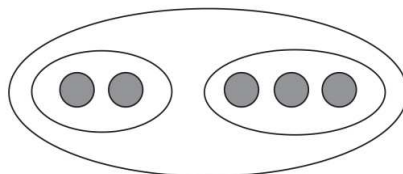


Figure 1: Iconic (visual) representation of $2 + 3$

"Visualization" can be seen as a way to discover and understand mathematics through images (Peter 1999). It should be viewed as a tool to facilitate the gaining of mathematical insights. We can distinguish external and internal visual representations. External iconic representations occur in a variety of ways in the classroom: in school textbooks, drawn in notebooks, sketches on boards at a wall or created on a computer screen. Internal iconic representations, i.e. mental images, also play an important role in the process of understanding mathematical concepts. Thus, we are using the following definition for visualization in this paper which includes both external and internal visual representations:

"Mathematical visualization is the process of forming images (mentally, or with pencil and paper, or with the aid of technology) and using such images effectively for mathematical discovery and understanding."
(Zimmermann und Cunningham 1991, p.3)

Here, understanding means to generate internal representations based on external representations and also includes the ability to externalize internal representations again. In this case, internal and external representations influence each other which does not always need to be a one-to-one mapping (Schnotz 2002).

The definition above also mentions explicitly the role of technology for visualizations. With a laptop or tablet computer we can create so called *dynamic visualizations*, for example in the graphics view of the software GeoGebra, where iconic representations can be changed interactively by moving points or sliders with a mouse or finger. These dynamic representations are fundamentally different from static images that can only represent a fixed situation (Danckwerts & Vogel 2003, p.21). This way, characteristic features of a mathematical object can be investigated and more easily distinguished from those features that result from the random snapshot of a certain situation in a static image (Elschenbroich 2005, p.77). In this way, technology allows a variety of possibilities to manipulate and transform iconic representations, so that students can investigate which properties remain invariant in all positions and therefore are characteristic of a mathematical concept (Dörfler 1991).

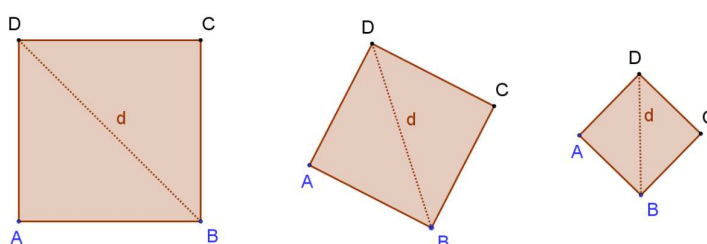


Figure 2: Changing the iconic representation of a square by dynamically moving A and B

By dynamically changing the prototype image of a square like in Figure 2, students can find invariant properties, i.e. that the four sides of the quadrilateral always have the same length and that its interior angles always remain 90 degrees. Such visual manipulations and transformations with dynamic mathematics software like GeoGebra can support the development of student's own internal images. With the help of technology, learners can build such internal images better and more concrete now because of their own actions on the screen (Elschenbroich 2005, p.77).

Interaction of forms of representation with GeoGebra

As mentioned above, enactive, iconic and symbolic representations should be connected according to the principle of interaction of multiple representations which states that knowledge is retained more easily if it was acquired in different representation forms (Wittmann 1981, p.91). The underlying assumption is that student's success in problem solving can be increased when they can switch flexibly between these different forms of representations.

Let us now consider how this interaction of different forms of representations could be achieved using dynamic mathematics software like GeoGebra. We define a dynamic mathematics software (DMS) here as a computer tool that combines features of a spreadsheet software (SSS), a dynamic geometry system (DGS) and a computer algebra system (CAS) in one program.

The dynamic mathematics software GeoGebra is a cognitive tool (Jonassen 1992) allowing to create mathematical models from scratch. Cognitive tools are characterized in that the learners have a lot of freedom in the control of the system, where they are actively working with the system itself and can be creative. Because of their many uses, they are well suited for mathematical modeling and to support discovery learning.

Duval (1999) assumes that mathematical objects are not directly accessible, but only through semiotic representations:

“There is no true understanding in mathematics for students who do not incorporate into their cognitive architecture the various registers of semiotic representations used to do mathematics.” (Duval, 1999)

In this sense, GeoGebra offers two different registers of representations with its graphics and algebra views on the same abstract mathematical object. As a dynamic mathematics software, GeoGebra provides the symbolic and iconic representations of mathematical objects in two connected views side by side in order to support visualization and the principle of interaction of representations. Let's consider the possibilities of the combination of iconic and symbolic representations with the example of a circle's equation. GeoGebra offers the following options:

- A circle can be constructed geometrically like in a dynamic geometry system (iconic) and then show its equation (symbolic).
- A circle can be entered like in a computer algebra system using its equation (symbolic) and then be shown as an image in the graphics view (iconic).
- In addition, the circle entered through its equation (symbolic) can also be dynamically moved and changed with the mouse or finger (interactive/enactive).

The last point distinguishes dynamic mathematics software from traditional computer algebra systems where it allows direct interaction with the iconic representation of a mathematical object entered in symbolic form. Thus, a dynamic mathematics software like GeoGebra allows the bidirectional change between the iconic and symbolic representation of a circle (Hohenwarter 2006b, p.25)

GeoGebraBooks on Tablets

Until recently, the software GeoGebra could only be used on traditional laptop or desktop computers. Since the 2013, it is now possible to also use GeoGebra on tablet computers (Windows 8+, Android and iPad). With a new user interface allowing students to create and manipulate mathematical models not only with their mouse but also with their fingers (Hohenwarter & Kimeswenger 2013, p.355). Such touch interfaces allow for a very direct way of interaction with mathematical objects and are an interesting way how technological tools can also help to support the enactive representation form. Some schools decide now to use tablets in mathematics education (see Figure 3), since they are easy to use, require little space and have long battery life.



Figure 3: Students using GeoGebra on tablets

The GeoGebra application itself is a cognitive tool in the sense that it can only be used effectively in the classroom when learners and teachers have at least some knowledge of the handling of the software such as the use of the geometrical tools available. By using prepared materials such as interactive applets in online environments, much less knowledge about the operation of the software itself is needed in contrast (Preiner 2008).

In 2011, the material sharing platform GeoGebraTube (www.geogebra.org) has been launched which now offers more than 100,000 freely available interactive worksheets created and shared by teachers and students on various topics from mathematics and the natural sciences. From the beginning, there had been the possibility to combine several worksheets into private collections, allowing the creation of very simple learning units. Recently, the desire arose to also allow the possibility of more flexible and better structured units.

This idea has been implemented and is available since January 2014 as a new feature on GeoGebraTube in so-called "GeoGebraBooks" which consist of interactive GeoGebra worksheets organized in chapters. Each chapter can be selected from a menu at the left showing an overview of its dynamic worksheets (see figure 4). Registered users can create their own GeoGebraBooks using all materials of the sharing platform. Both their own worksheets as well as those created by other users can be easily organized into chapters. The resulting GeoGebraBooks can either be shared via unique web addresses with others or downloaded as offline packages. They work on traditional computers as well as tablets and smartphones, the only requirement being an HTML5 ready web browser

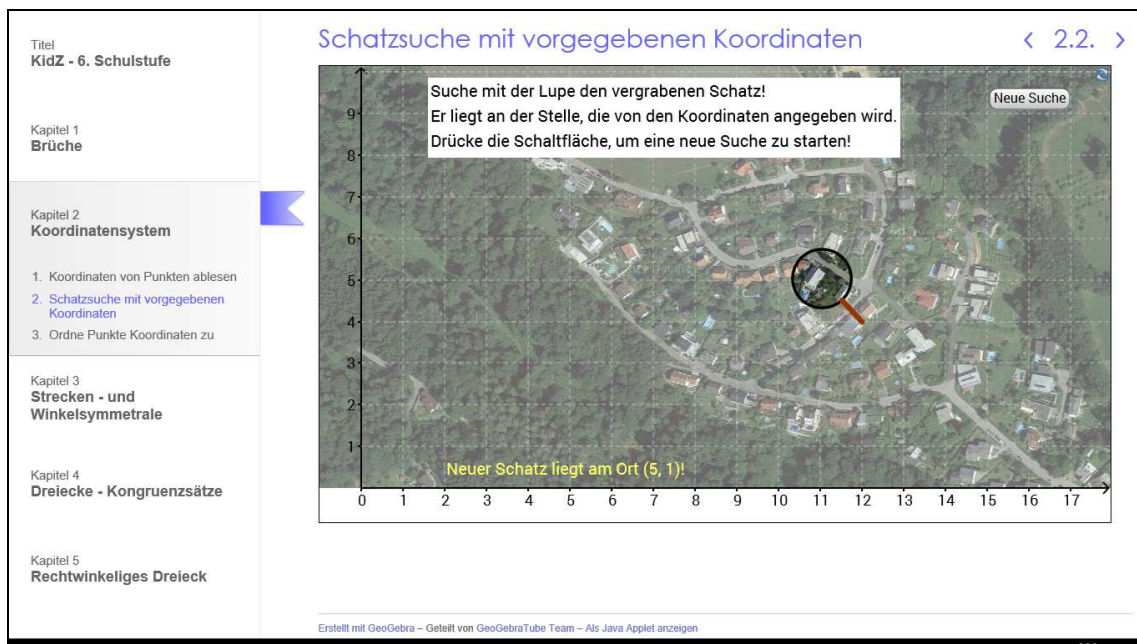


Figure 4: Example of a GeoGebraBook (see <http://www.geogebra.org/book/at/st6/>)

In a project at JKU we have recently investigated the question how such GeoGebraBooks and their interactive worksheets could be optimized for use on tablets. For example, the screen size and way to operate these devices differs significantly from working with a mouse on a large computer screen. The use of tablets does not only bring technical challenges for software development, but also requires changes in the design of interactive worksheets. For instance, control

elements like sliders or input fields should be positioned at the bottom of a worksheet to avoid that your hand covers the rest of the worksheet when you move a slider with your finger (Hohenwarter & Kimeswenger 2013, pp.354).

Based on these considerations and first experiences with tablets in the classroom two prototypical GeoGebraBooks were created with materials for the 11 to 12 year old students. Their design was adapted to the use on tablets concerning screen size and to make it easy to operate all elements with fingers including the size and positioning of the mathematical elements. For example, the worksheet about addition of fractions shown in figure 5 allows students to get immediate feedback on how well they have worked a specific task (Hohenwarter 2006a, p.5). To allow easier handling on tablet computers with onscreen keyboards flying in resp. out from the bottom of the screen, it was important to place the input boxes for the fractions at the bottom of the worksheet.

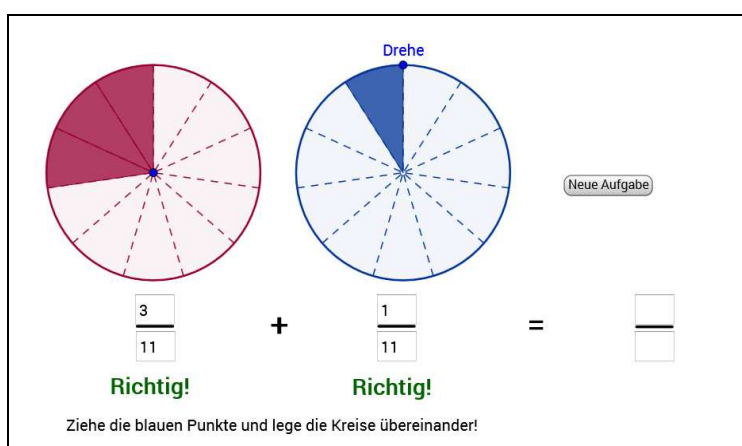


Figure 5: Interactive dynamic exercise for addition of fractions (Meier 2014)

The juxtaposition of fractions in their symbolic form, i.e. the numerator and denominator, and an adequate visual form, i.e. circular sectors, is an example for a concrete implementation of the principle of interaction of representations (Meier 2009, pp.96). In addition, the circles can be moved and rotated to get the visual result of fraction addition by putting them on top of each other. Thus, this exercise shows how the three forms of representation can be related:

- Enactive: control with the finger, move and rotate the iconic representations of the fractions
- Iconic: representation of fractions in circular sectors
- Symbolic: numerators and denominators of the fractions

Such interactive worksheets can enrich learning in the classroom and at home, however it should be noted that they should not replace “real” enactive materials such as representing fractions by folding a sheet of paper. As a teacher, it is important to always reflect which combination of media (paper, real models, tablet computers, etc.) should be used and when as all of them have their specific advantages.

Future of GeoGebraBooks

Based on the feedback from users of GeoGebraBooks it is planned to add new features over the coming months. As a first step, the content of GeoGebraBooks

should become more flexible by allowing to include texts, interactive applets, images and videos flexibly on each worksheet (see figure 6).

Figure 6: Flexible worksheet in GeoGebraBook combining applets, video and text on one page

In addition, it is planned to allow the creation of groups (e.g. for a class in a school), so that a teacher can share a GeoGebraBook only with her own class. This should simplify organizing, structuring and providing relevant teaching materials to students in the direction of mathematical portfolios, i.e. GeoGebraBooks whose contents can be changed or extended by students themselves. On the one hand, a teacher could prepare a GeoGebraBook with exercises where students can create constructions within the book and save their work. Thus, each student has her own copy of this GeoGebraBooks just like a mathematical portfolio where texts, images, videos and applets can be included. On the other hand, it should be possible for multiple students or even a whole class to create a common GeoGebraBook in which everyone can work together on their materials.

Just like in the development of the GeoGebra software, the future directions of GeoGebraBooks will be heavily influenced by the needs and feedback of users. However, the main goal of development should stay in line with those of cognitive tools: learners and teachers should have a lot of freedom in the design of GeoGebraBooks, where they are actively working with the system to support their mathematical creativity.

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