Partitioning transitive tournaments into isomorphic digraphs

Attila Sali[a,b,1]   **Gábor Simonyi**[a,b,2]   **Gábor Tardos**[a,3]

[a]Alfréd Rényi Institute of Mathematics, Hungarian Academy of Sciences

[b]Department of Computer Science and Information Theory,
Budapest University of Technology and Economics

sali@renyi.hu   simonyi@renyi.hu   tardos@renyi.hu

arXiv:1806.00729v1 [math.CO] 3 Jun 2018

Abstract

In an earlier paper (see [14]) the first two authors have shown that self-complementary graphs can always be oriented in such a way that the union of the oriented version and its isomorphically oriented complement gives a transitive tournament. We investigate the possibilities of generalizing this theorem to decompositions of the complete graph into three or more isomorphic graphs. We find that a complete characterization of when an orientation with similar properties is possible seems elusive. Nevertheless, we give sufficient conditions that generalize the earlier theorem and also imply that decompositions of odd vertex complete graphs to Hamiltonian cycles admit such an orientation. These conditions are further generalized and some necessary conditions are given as well.

1   Introduction

Investigating the relationship between the Shannon capacity of graphs and the Sperner capacity of their oriented versions the first two authors proved the following theorem.

Theorem 1. ([14], cf. also [8]) *Let G be a graph isomorphic to its complement $F = \overline{G}$. Then G and F can be oriented so that they remain isomorphic as digraphs while the tournament formed by their union is the transitive tournament.*

   Moreover, the above can be done for any fixed isomorphism between G and F. That is, for any such isomorphism f one can find oriented versions $\vec{G}$ and $\vec{F}$ of G and F, respectively, such that f provides and isomorphism between $\vec{G}$ and $\vec{F}$ and the union of $\vec{G}$ and $\vec{F}$ is a transitive tournament.

   The goal of the present paper is to investigate the possibilities of generalizing the above theorem to three or more graphs, that is, to the situation when (the edge set of) the complete graph is partitioned into three or more isomorphic graphs. As already observed by Görlich, Kalinowski, Meszka, Pilśniak, and Woźniak [4] in this case it will not be true that the three graphs can always be oriented in an isomorphic manner so that their union forms a transitive tournament. Moreover, a complete characterization of when this is possible seems to be elusive. In [4, 5] the authors determine all digraphs with at most four edges that can decompose a transitive tournament. (For related results, see also [7] and [6].)

   In our approach we fix the number of isomorphic graphs in the decompositions considered. We start with the case when this number is 3. We will give some sufficient conditions when the isomorphic parts of a decomposition of K_n can be isomorphically oriented to get a decomposition of the transitive tournament. This result gives a generalization of Theorem 1.

   It is well-known that complete graphs on an odd number of vertices decompose into Hamiltonian cycles. One can directly show how to obtain a decomposition of the transitive tournament of odd order into isomorphically oriented Hamiltonian cycles, but this also follows from our sufficient condition mentioned above.

   First we extend our sufficient condition to a more general one, then we show with an example (found by computer) that this more general condition is still not necessary. A complete characterization seems out of reach, but we are able to give some non-trivial necessary conditions.

   As usual K_n denotes the complete graph on n vertices, while we denote the transitive tournament on n vertices by T_n. The vertex set of K_n and T_n is assumed to be $[n] = \{0, 1 \ldots, n-1\}$ and we consider these vertices as residue classes modulo n, that is equality between vertices will be understood modulo n. We denote the cyclic permutation of $[n]$ bringing i to $i+1$ by σ_n.

2   Small examples and problem formulation

First we recall an example from [4] of three isomorphic graphs partitioning the complete graph that cannot be isomorphically oriented so that their union is a transitive tournament even if the functions giving the isomorphism among them are not fixed.

Let $n = |V(G)| = 4$ and the three isomorphic graphs be paths on 3 vertices. One easily partitions K_4 into three such graphs. It is also easy to see that whatever way we orient these paths in an isomorphic manner, we cannot put them together to obtain a transitive tournament on 4 vertices. This is simply because from no orientation can we produce simultaneously a vertex of outdegree 0 and a vertex of outdegree 3. Note that this example is just a very special case of Theorem 5 from [4].

Let us assume that the edge set of the complete graph K_n is partitioned into three isomorphic graphs F, G and H. We can ask whether there are isomorphic orientations $\vec{F}$, $\vec{G}$ and $\vec{H}$ of the graphs F, G and H, respectively, such that their union gives a transitive tournament. But we can be more specific and fix an isomorphism σ from F to G, an isomorphism ρ from G to H and ask whether there are orientations $\vec{F}$, $\vec{G}$ and $\vec{H}$ of the graphs F, G and H whose union is a transitive tournament and such that σ is an isomorphism between $\vec{F}$ and $\vec{G}$ and ρ is an isomorphism between $\vec{G}$ and $\vec{H}$.

To illustrate the difference between these two questions let us consider the smallest possible example. The graph K_3 can be partitioned into three (isomorphic) single edge graphs: F, G and H. Clearly, the three oriented edges of T_3 also form isomorphic graphs. This answers the first question above for this specific partition affirmatively. If, however, we fix a cyclic permutation $\sigma = \rho$ that brings F to G and G to H, then the answer to the second question is negative. Indeed, if $\vec{F}$ is either orientation of F, $\vec{G} = \sigma(\vec{F})$ and $\vec{H} = \sigma^2(\vec{F})$, then the union of these three directed graphs is a directed cycle and thus not transitive.

In this paper we will concentrate on the question with fixed permutations σ and ρ. We will only consider the special case $\sigma = \rho$. Although this assumption is restrictive, it is in complete analogy with the case of two self-complementary graphs, and we believe that understanding this special case would largely improve our knowledge about the situation.

Definition 1. *Let σ be a permutation of the vertex set of K_n. We call the partition of the edge set of K_n into three graphs F, G and H such that the permutation σ brings F to G and G to H a σ-partition. In this case σ brings H to F and σ^3 is an automorphism of all three of these graphs. We call a transitive orientation T of K_n a transitive σ-orientation of this σ-partition if the subgraphs $\vec{F}$, $\vec{G}$ and $\vec{H}$ of T that are the orientations of the graphs F, G and H, respectively, satisfy $\sigma(\vec{F}) = \vec{G}$ and $\sigma(\vec{G}) = \vec{H}$. We say that σ reverses the orientation of an edge e in T if $\sigma(e)$ is oriented in the other direction, that is, if e goes from a to b, and the edge of T between $\sigma(a)$ and $\sigma(b)$ is oriented toward $\sigma(a)$. Observe that a transitive orientation T of K_n is a transitive σ-orientation of the σ-partition of K_n to F, G and H if and only if σ reverses no edges of T that belong to F or G.*

Just as it was the case with self-complementary graphs, we may assume that the

permutation σ is cyclic as the case of general σ reduces to the cyclic case. Indeed, let the cycle decomposition of the permutation of σ be $\rho_1 \rho_2 \ldots \rho_k$. Let F, G and H form a σ-partition P. The subgraphs F_i, G_i, H_i induced by the domain of the cycle ρ_i form a ρ_i-partition P_i for all i. Clearly, if P has a transitive σ-orientation, then it restricts to transitive ρ_i-orientations of P_i. On the other hand, if P_i has a transitive ρ_i-orientation for all i, then P has a transitive σ-orientation. To see this last statement simply keep the orientations of the edges in the transitive ρ_i-orientations and orient edges connecting vertices from distinct cycles ρ_i and ρ_j toward the higher indexed cycle. None of these latter type of edges is reversed by σ.

It is easy to see that a σ-partition exists if and only if σ has at most one fixed point and the length of all non-trivial cycles of the cycle decomposition of σ is divisible by 3.

From now on we do make the assumption that σ consists of a single cycle on $n > 1$ vertices, namely $\sigma = \sigma_n$, where σ_n stands for the permutation on the set $[n] = \{0, 1, 2, \ldots, n-1\}$ bringing i to $i + 1 \bmod n$. The vertices of our graphs will therefore be the elements of $[n]$ and we consider them as the residue classes modulo n, that is, equalities about them are always understood modulo n. We assume n is divisible by 3 as otherwise there is no σ_n-partition.

We denote the graphs of the σ_n-partition by F_0, $F_1 = \sigma_n(F_0)$ and $F_2 = \sigma_n^2(F_0)$. The *label* $\ell(a, b)$ of an edge $\{a, b\}$ of K_n is the index of the subgraph the edge belongs to, so the label of the edges of F_i are i. As σ_n brings F_0 to F_1 to F_2 and back to F_0 we must have $\ell(a+1, b+1) \equiv \ell(a, b) + 1$ for all a and b, where the congruence is modulo 3 (and, as noted above, the vertices are understood modulo n). With the same convention we have the more general congruence for any edge $\{a, b\}$ and integer i:

$$\ell(a + i, b + i) \equiv \ell(a, b) + i \pmod 3. \tag{1}$$

Definition 2. *The* defining sequence *of the σ_n-partition $\{F_0, F_1, F_2\}$ is $a_1, a_2, \ldots, a_m$, where $m = \lfloor n/2 \rfloor$ and $a_j = \ell(0, j)$. By the congruence above, this sequence determines all other labels and thus the entire σ_n-partition. On the other hand, it is easy to see that (as n is divisible by 3) every sequence of length $\lfloor n/2 \rfloor$ over the alphabet $\{0, 1, 2\}$ is a defining sequence of a σ_n-partition. This is analogous to the case of self-complementary graphs, cf. [3, 12, 13]. By symmetry, we may and will often assume that $\ell(0, 1) = 0$, that is, the defining sequence starts with $a_1 = 0$. This can be achieved by shifting the σ_n-partition by σ_n or σ_n^2.*

In the smallest $n = 3$ case, there is just one σ_3-partition and we have already seen that it has no transitive σ_3-orientation. Let us look at the next case $n = 6$ a bit closer. By the foregoing, there are $3^2 = 9$ σ_6-partitions to consider according to the labeling of the edges $\{0, 2\}$ and $\{0, 3\}$.

The corresponding graphs F_0 are depicted in Figure 1. It turns out that transitive σ_6-orientations exist in exactly four of the nine cases. (We have indicated such an orientation in Figure 1 whenever it exists.) Notice that the F_0 is simply a path on the six vertices in four cases but a transitive σ_6-orientation exists for only two of them. (The truth of

this statement will follow from the results of the next section.) Thus, in spite of the isomorphism of these four graphs they behave differently according to the different effect of permutation σ_6 on them.

3   The standard orientation

We want to decide whether a given σ_n-partition P has a transitive σ_n-orientation. For our notation including the labeling of edges and the definition of the defining sequence see the previous section. We will describe a transitive σ_n-orientation with an ordering $\tau(1), \tau(2), \ldots, \tau(n)$ on the vertices. We say that an orientation is *consistent* with τ if all edges point towards the vertex that come later in the order. Clearly, the transitive orientation of K_n and the ordering it is consistent with mutually determine each other. Recall that a transitive orientation T is a transitive σ_n-orientation of our σ_n-partition P if and only if the orientation of no label-0 or label-1 edge is reversed by σ_n.

Theorem 2. *If $n = 2m$ and the defining sequence $a_1 \ldots a_m \in \{0, 1, 2\}^m$ of a σ_n-partition satisfies that for every $j \in \{1, \ldots, m-1\}$ either $a_{j+1} = a_j$ or $a_{j+1} \equiv a_j + 1 \pmod 3$, then there exists a transitive σ_n-orientation for this σ_n-partition.*

Proof. The proof is an extension of the argument given by Gyárfás [8] for our Theorem 1. We give a linear order τ of the vertices $0, 1, \ldots, n-1$ and show that orienting the edges consistently with this order gives a transitive σ_n-orientation.

Let us first recall our assumption that $a_1 = 0$. This can be achieved by appropriately relabeling the vertices. The relabeling changes the defining sequence but does not affect the condition in the theorem. Now we define τ. We set $\tau(1) = 0$ and declare that τ will have the property, that for any i, the set of vertices $A_i := \{\tau(1), \tau(2), \ldots, \tau(i)\}$ forms a consecutive arc of the cycle formed by the vertices $0, 1, \ldots, n-1$, i.e., it is equal to $\{j_i + 1, \ldots, j_i + i\}$ for some $j_i \in \{n - i, n - i + 1, \ldots, n - 1\}$. Recall that the names of the vertices are understood modulo n. Now $\tau(i+1)$, that is the unique element of $A_{i+1} \setminus A_i$ is either j_i or $j_i + i + 1$ for every i. Thus τ is determined if we give a rule for deciding which of the two elements j_i and $j_i + i + 1$ should be taken as $\tau(i + 1)$ if $i < n - 1$. (No rule is needed for $i = n - 1$ as then $j_i = j_i + i + 1$ is the only vertex outside A_i.) This choice for $\tau(i + 1)$ depends on the label of the edge $\{j_i, j_i + i + 1\}$. If $\ell(j_i, j_i + i + 1) = 0$, then we set $\tau(i + 1) = j_i$ making $j_{i+1} = j_i - 1$. If $\ell(j_i, j_i + i + 1) = 2$, then we set $\tau(i + 1) = j_i + i + 1$ making $j_{i+1} = j_i$. We claim that the third possibility, namely $\ell(j_i, j_i + i + 1) = 1$ will not happen for any $1 \le i < n - 1$.

First we show this last statement by induction. Note that all congruences are modulo 3. The base case is all right as $\tau(1) = 0$ and $\ell(n - 1, 1) \equiv a_2 - 1$, see Equation (1). By the assumption on the defining sequence this is either $a_1 - 1 \equiv 2$ or $(a_1 + 1) - 1 = 0$. Now assume the statement to be true for the edge $\{j_{i-1}, j_{i-1} + i\}$, and we show that it is also true for $\{j_i, j_i + i + 1\}$. By Equation(1), we have $\ell(j_{i-1}, j_{i-1} + i) \equiv \ell(0, i) + j_{i-1}$ and $\ell(j_i, j_i + i + 1) \equiv \ell(0, i + 1) + j_i$. We have $j_i = j_{i-1} - 1$ if $\tau(i) = j_{i-1}$, i.e., if
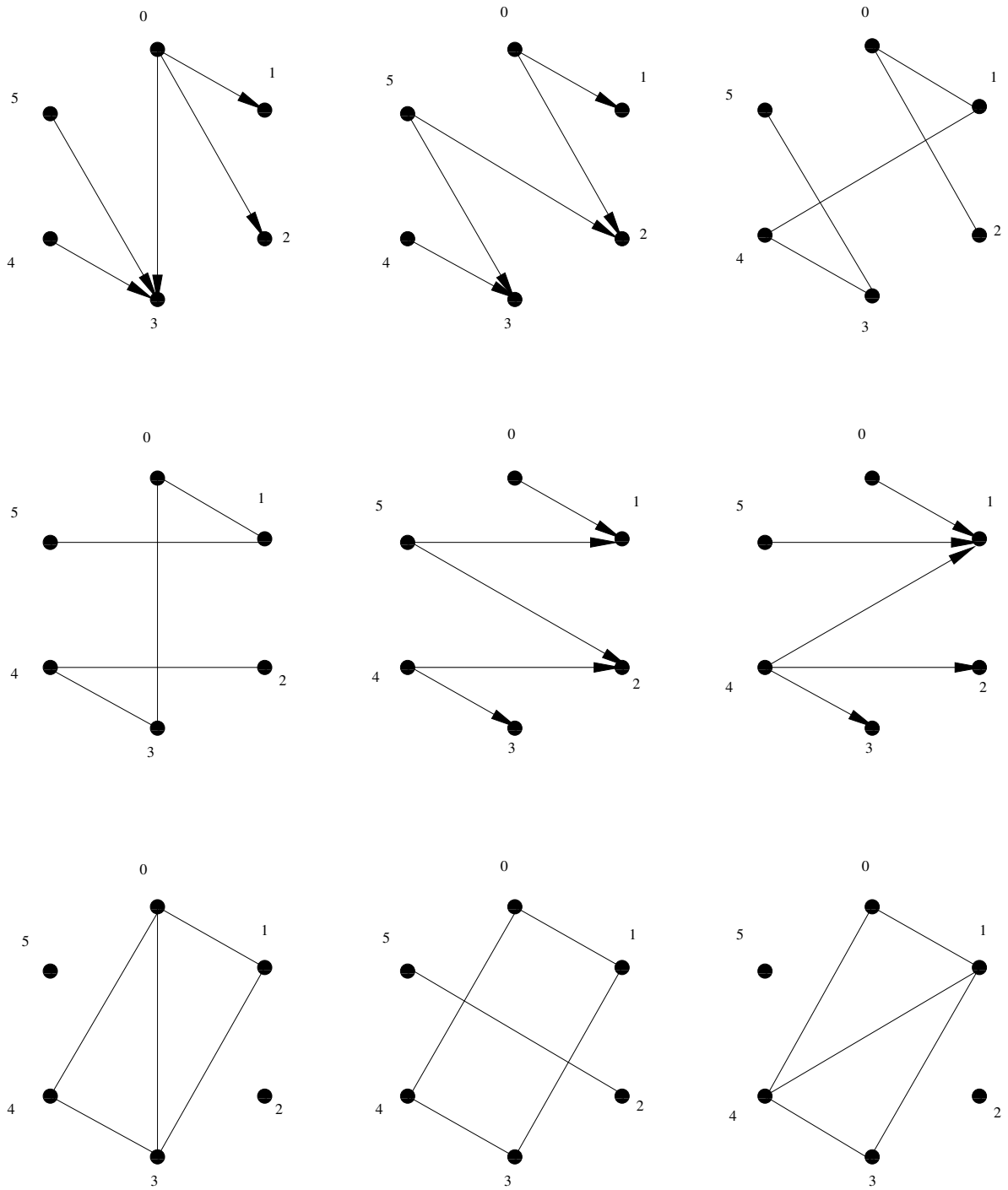
4

Figure 1: Four of the nine possible 3-partitions of K_6 can be oriented as required. In the remaining five cases such orientations do not exist.

$\ell(j_{i-1}, j_{i-1} + i) = 0$, while otherwise this label is 2, so we have $j_i = j_{i-1}$. Therefore, we can formulate the congruence:

$$j_i \equiv j_{i-1} - \ell(j_{i-1}, j_{i-1} + i) - 1 \equiv j_{i-1} - (\ell(0, i) + j_{i-1}) - 1 = -\ell(0, i) - 1.$$

We also have:

$$\ell(j_i, j_i + i + 1) \equiv \ell(0, i + 1) + j_i \equiv \ell(0, i + 1) - \ell(0, i) - 1. \tag{2}$$

For $1 \le i \le n/2 - 1$ we simply have $\ell(0, i + 1) = a_{i+1}$ and $\ell(0, i) = a_i$, so

$$\ell(j_i, j_i + i + 1) \equiv a_{i+1} - a_i - 1, \tag{3}$$

so by assumption it cannot be 1. If $n/2 \le i < n$ we use Equation (1) again to see that $\ell(0, i) = \ell(i, 0) \equiv \ell(0, n - i) + i = a_{n-i} + i$ and similarly, $\ell(0, i + 1) \equiv a_{n-i-1} + i + 1$. Therefore

$$\ell(j_i, j_i + i + 1) \equiv a_{n-i-1} - a_{n-i}, \tag{4}$$

which cannot be 1 either by the same assumption. Note that we used the fact that n is even. For n odd and $i = (n-1)/2$ we would have $\ell(j_i, j_i + i + 1) \equiv \ell(0, i + 1) - \ell(0, i) - 1 \equiv (a_i - i) - a_i - 1 = -i - 1 \equiv 1$.

We need to show that the orientation consistent with the order τ is a transitive σ_n-orientation. As noted above, for this we have to show that σ_n reverses the orientation only of edges of label 2. Equivalently, if an edge $\{u, v\}$ is oriented from u to v and it has label 1 or 2 then the edge $\{u - 1, v - 1\}$ is oriented from $u - 1$ to $v - 1$. Assume $\{u, v\}$ is oriented from u to v, that is, $\tau^{-1}(u) < \tau^{-1}(v)$. We distinguish cases according to the order of u and v. Note that while in most formulas we consider the vertices as residue classes modulo n (and thus equality really means congruence modulo n) in inequalities the vertices are treated as integers between 0 and $n - 1$.

In the simplest case we have $0 < u < v$. In this case $u - 1 \in A_{\tau^{-1}(u)}$ while $v \notin A_{\tau^{-1}(u)}$ and either $(v - 1) \notin A_{\tau^{-1}(u)}$ or $v - 1 = u$. In both cases we have $\tau^{-1}(u - 1) < \tau^{-1}(v - 1)$ implying that the edge $\{u - 1, v - 1\}$ is oriented from $u - 1$ to $v - 1$ as we need.

If $u = 0$ and $\{u - 1, v - 1\}$ is oriented toward $u - 1$, then $\tau^{-1}(u - 1) > \tau^{-1}(v - 1)$ and therefore $\ell(u - 1, v - 1) = 2$ implying $\ell(u, v) = 0$ and therefore it does not matter that $\{u - 1, v - 1\}$ is not oriented from $u - 1$ to $v - 1$.

Finally assume $v < u$. Note that $v > 0$ as otherwise the edge $\{u, v\}$ could not be directed toward v. If $\{u - 1, v - 1\}$ is not oriented from $u - 1$ to $v - 1$, then the arc $\{u, u+1, \ldots, v - 1\}$ is either $A_{\tau^{-1}(u)}$ or $A_{\tau^{-1}(v-1)}$. In the former case our rule implies that $\ell(u, v) = 0$, in the latter case it implies $\ell(u - 1, v - 1) = 2$ and thus again $\ell(u, v) = 0$ in which case we have no problem.

This proves that our rule gives a transitive σ_n-orientation and completes the proof of the theorem. $\qquad \square$

Note that the linear order obtained on the vertex set of K_n by the orientation in the proof above has some special properties. To formulate them we introduce the following notions.

Definition 3. *Let* $\tau(1)\dots\tau(n)$ *be an ordering of the numbers* $0,1,\dots,n-1$. *We say that* $j \in \{0,1,\dots,n-1\}$ *is a* local minimum *in this order if* j *precedes both* $j-1$ *and* $j+1$ *(addition is meant modulo* n*), that is* $\tau^{-1}(j) < \tau^{-1}(j-1)$ *and* $\tau^{-1}(j) < \tau^{-1}(j+1)$. *Similarly,* $j \in \{0,1,\dots,n-1\}$ *is a* local maximum *if* j *is preceded by both* $j-1$ *and* $j+1$, *that is* $\tau^{-1}(j) > \tau^{-1}(j-1)$ *and* $\tau^{-1}(j) > \tau^{-1}(j+1)$.

We call τ bitonic *if it has a unique local minimum and a unique local maximum. We call a transitive* σ_n*-orientation of a* σ_n*-partition* standard *if it is consistent with a bitonic ordering of the vertices.*

Proposition 1. *The transitive* σ_n*-orientations given in the proof of Theorem 2 are standard. The unique local minimum of the corresponding ordering is at* 0, *while the unique local maximum is at* $n/2$.

Proof. The fact that τ given in the proof is bitonic and thus the transitive σ_n-orientations are standard follows immediately from the construction. It is also clear that $\tau(1) = 0$ is the local minimum and we need only prove that the local maximum (that is $\tau(n)$) is $n/2$.

Recall from the proof of Theorem 2 that $A_i = \{\tau(1),\dots,\tau(i)\}$ is a consecutive interval in the cycle formed by the i vertices from $j_i + 1$ to $j_i + i$ for all $1 \le i \le n$. The sequence $(A_i)_{i=1}^{n}$ starts at $A_1 = \{0\}$ and we obtain A_{i+1} from A_i by extending A_i with $\tau(i+1)$ at one end of this interval. The label $\ell(j_i, j_i + i + 1)$ (which is never 1) determines which end we place $\tau(i+1)$, namely if the label is 0 we chose one end, while if it is 2 we chose the other end. For $1 \le i \le n/2 - 1$ we have $\ell(j_i, j_i + i + 1) \equiv a_{i+1} - a_i - 1$ by Equation (3) and $\ell(j_{n-i-1}, j_{n-i-1} + n - i) \equiv a_i - a_{i+1}$ by Equation (4). This means that $\ell(j_i, j_i + i + 1) + \ell(j_{n-i-1}, j_{n-i-1} + n - i) \equiv 2$, so one of these labels must be 0 and the other 2 and thus we extend the interval A_i on one end to get A_{i+1} while we extend A_{n-i-1} on the other end to obtain A_{n-i}. This partitions the $n-2$ extension steps bringing A_1 to A_{n-1} into $n/2 - 1$ pairs and shows that we use $n/2 - 1$ extensions of the interval at either side. Thus $n/2$ (the vertex in distance $n/2$ from $A_1 = \{0\}$ in either direction) must be the only vertex outside A_{n-1} and therefore we have $\tau(n) = n/2$ as stated. $\square$

The following proposition is a sort of converse of the previous one.

Proposition 2. *If there exists a standard transitive* σ_n*-orientation for a* σ_n*-partition, then the conditions of Theorem 2 are satisfied, namely the number* n *of vertices is even and for the defining sequence* $a_1 \dots a_{n/2}$ *of the* σ_n*-partition either* $a_{j+1} = a_j$ *or* $a_{j+1} \equiv a_j + 1$ (mod 3) *holds for each* $1 \le j \le n/2 - 1$. *If there exists a standard transitive* σ_n*-orientation for a* σ_n*-partition, then it is unique up to shifts of the automorphism* σ_n^3.

Proof. Consider a standard transitive σ_n-orientation T for a σ_n-partition. Without loss of generality we may assume that the defining sequence of the σ_n-partition starts with $a_1 = 0$. Consider the bitonic ordering τ of the vertices consistent with T. The first element $a = \tau(1)$ is clearly the local minimum. All edges are directed away from a, so σ reverses the orientation of the edge $\{a - 1, a\}$, as it brings it to $\{a, a + 1\}$. Therefore we

have $\ell(a, a+1) = 0$ and thus $a \equiv 0 \pmod 3$. We may and will assume $a = 0$ as this can be achieved with a shift of a suitable power of the automorphism σ_n^3.

As τ is bitonic, the set $A_i = \{\tau(1), \ldots, \tau(i)\}$ must be an interval $\{j_i + 1, \ldots, j_i + i\}$ along the cycle formed by the vertices. Clearly, $\tau(i+1)$ is either j_i or $j_i + i + 1$. We show the uniqueness of τ by observing that the value of $\tau(i+1)$ depends on the label of the edge $e = \{j_i, j_i + i + 1\}$ exactly as in the construction in the proof of Theorem 2, namely $\tau(i+1) = j_i$ if this label is 0 and $\tau(i+1) = j_i + i + 1$ if the label is 2 and the label of e cannot be 1. Indeed, if $\tau(i+1) = j_i + i + 1$, then σ_n reverses the orientation of the edge e, so its label is 2, but if $\tau(i+1) = j_i$, then σ_n^{-1} reverses the orientation of e, so its label must be 0.

From the rule established above we can derive Equation (2) just as in the proof of Theorem 2. In case n is odd we can apply this formula to $i = (n-1)/2$ and using also Equation (1) we obtain $\ell(j_i, j_i+i+1) \equiv \ell(0, i+1) - \ell(0, i) - 1 = (\ell(0, i) + i + 1) - \ell(0, i) - 1 \equiv i \equiv 1$ contradicting that we saw that the label of the edge $\{j_i, j_i + i + 1\}$ cannot be 1. This proves that n is even.

From Equation (2) we can deduce Equation (3) for $1 \leq i \leq n/2 - 1$. Using that the label of the edge $\{j_i, j_i + i + 1\}$ cannot be 1 this implies that $a_{i+1} = a_i$ or $a_{i+1} \equiv a_i + 1$ finishing the proof of the proposition. $\qquad\square$

3.1 Several parts

In this subsection we extend the earlier results to partitions to more than three parts. The extensions are straightforward, the proofs carry over almost verbatim. We just list the results here and show how they apply to decompositions into Hamiltonian paths and cycles. Note that in the discussion below we also allow the case when the number of parts is $k = 2$.

Definition 4. *Let k and n be integers larger than 1 and (as before) let σ_n be the cyclic permutation on the set $[n] = \{0, 1, \ldots, n-1\}$ bringing i to $i+1$. Recall that elements of $[n]$ are understood modulo n. Let us call a partition of the edge set of the complete graph on the vertex set $[n]$ into the subgraphs $F_0, \ldots, F_{k-1}$ a σ_n-k-partition if $\sigma_n(F_d) = F_{d+1}$ for $0 \leq d < k - 1$. This implies that n is divisible by k, further if k is even, then n is divisible by $2k$. It also implies that $\sigma_n(F_{k-1}) = F_0$ and that σ_n^k is an automorphism of all the graphs F_d. We say that the* label *of an edge e of F_d is $\ell(e) = d$. The defining sequence of this partition is the sequence $a_1, \ldots, a_{\lfloor n/2 \rfloor}$, where $a_i = \ell(\{0, i\})$. The defining sequence uniquely determines the σ_n-k-partition, namely F_d consists of the edges $\{b, b+i\}$ for which $b \equiv d - a_i \pmod k$. This is indeed a σ_n-k-partition for any sequence $a_1, \ldots, a_{\lfloor n/2 \rfloor}$ over the letters $0, 1, \ldots, k-1$ if the divisibility conditions are satisfied by n and k.*

We call a transitive orientation T of K_n a transitive σ_n-orientation *of a σ_n-k-partition if $\sigma_n(\vec{F}_d) = \vec{F}_{d+1}$ for $0 \leq d < k - 1$, where $\vec{F}_d$ is the orientation of F_d obtained as a subgraph of T. A transitive orientation T satisfies this condition if and only if σ_n reverses the orientation of no edges of F_d with $d \neq k - 1$. We call a transitive σ_n-orientation*

standard *if it is consistent with a bitonic ordering of the vertices.*

The common generalization of Theorems 1 and 2 is as follows.

Theorem 3. *There exists a standard orientation for a σ_n-k-partition if and only if n is divisible by $2k$ and the defining sequence $a_1 a_2 \ldots a_{n/2}$ of the partition satisfies that for every $1 \leq j \leq n/2-1$ either $a_{j+1} = a_j$ or $a_{j+1} \equiv a_j+1 \pmod{k}$. If a standard orientation exists, then it is unique up to shifts of σ^k and the corresponding bitonic ordering of the vertices satisfies that the first and last elements in the ordering (the unique local minimum and maximum) differ by $n/2$.*

Remark 1. *It is straightforward to see that Theorem 3 generalizes Theorem 2. Note that Theorem 1 is also implied by Theorem 3 since in case of two self-complementary graphs, that is, when $k = 2$, the condition $a_{j+1} \equiv a_j$ or $a_j +1 \pmod{k}$ is automatically satisfied as a_{j+1} cannot take any value other than 0 or 1.*

We formulate the following simple generalization of the trivial observation that the (only) σ_3-partition has no transitive σ_3-orientation.

Proposition 3. *If there exists a transitive σ_n-orientation for a σ_n-k-partition, then $n \geq 2k$ and $n \neq 3k$. If $n = 2k$, then any transitive σ_n-orientation of a σ_n-k-partition is standard.*

Proof. From the existence of this partition we know that k divides n. Note that σ_n reverses the orientation of an even number of the n edges $\{i, i + 1\}$ in any orientation T of K_n. But if $k = n$, then only one of these edges has label $k - 1$, so if T is a transitive σ_n-orientation, then the orientation of none of these edges are reversed. But then they form a directed cycle, so T is not transitive.

If $n = 2k$ or $3k$, then two or three of these n edges have label $k - 1$, so exactly two of them have to be reversed by σ_n to avoid the directed cycle. This makes the transitive σ_n-orientation standard. But we know from Theorem 3 that if a σ_n-k-partition admits a standard orientation, then n is divisible by $2k$, so $n = 3k$ is not an option. $\square$

Complete graphs of odd order can be decomposed into Hamiltonian cycles. This is a classical result as mentioned in Adrian Bondy's chapter [1] of the *Handbook of Combinatorics.* (It is added there that "one such construction, due to a Monsieur Walecki, is described in the book by Lucas (1891, pp. 161–164)", cf. [11].)

This result extends to the decomposition of odd order transitive tournaments to identically oriented Hamiltonian cycles. This decomposition can be found directly, but it can also be arrived at by applying our result to a certain decomposition of the non-oriented complete graph as shown below.

Corollary 1. *If n is even, then we can decompose T_n into $\frac{n}{2}$ alternatingly oriented Hamiltonian paths.*

Proof. Let F_0 be the path defined as $0, 1, n-1, 2, n-2, \ldots, i, n-i, (i+1), \ldots, (\frac{n}{2}+1), \frac{n}{2}$. It is easy to see that the graphs $F_d := \sigma_n^d(F_0)$ for $d = 0, 1, \ldots, n/2 - 1$ partition the edge set of K_n into $\frac{n}{2}$ Hamiltonian paths. This is a σ_n-$n/2$-partition.

One can also readily check that $\{0, j\} \in E(F_{\lfloor j/2 \rfloor})$ holds for every $j \in \{1, \ldots, n-1\}$. This means that the defining sequence of this σ-$n/2$-partition is $a_1, \ldots, a_{n/2}$ with $a_i = \lfloor i/2 \rfloor$. By Theorem 3 there is a standard orientation for this partition that is unique up to a shift with $\sigma_n^{n/2}$. It is not hard to check that the standard orientation orients the Hamiltonian paths in this partition alternatingly, that is, each vertex will be a source or a sink of each Hamiltionian path. This construction is illustrated for $n = 6$ by the oriented path in the very central picture of Figure 1. $\qquad\square$

Corollary 2. *If n is odd, then we can decompose T_n into $\frac{n-1}{2}$ isomorphically oriented Hamiltonian cycles.*

Proof. Consider the decomposition of T_{n-1} given in Corollary 1 on vertices labeled by $[n-1]$. Add the extra vertex v and connect it to the two endpoints of each of the Hamiltonian paths thus extending them to Hamiltonian cycles. Orient all edges incident to v away from v. These isomorphically oriented Hamiltonian cycles decompose the transitive tournament on n vertices. $\qquad\square$.

Remark 2. *The orientation of the Hamiltonian cycles in our decomposition in Corollary 2 is such that all but one of the vertices is either a source or a sink. This kind of orientation of odd cycles is called alternating in [9], where it is shown that these oriented versions of odd cycles have maximal Sperner capacity. In the special case of $n = 5$ this orientation already appeared in [2], where it was observed that its Sperner capacity is $\sqrt{5}$, that is, it achieves the Shannon capacity of the underlying undirected graph which is C_5, whose capacity was determined in the celebrated paper by Lovász [10]. This observation was the starting point of our investigations in [14].*

In Section 2 we explained how to find transitive σ-orientations for σ-partitions if the permutation σ is not cyclic. First we solve the restriction of the problem for the domain of each cycle in the cycle decomposition of σ, then extend the obtained orientations by orienting the edges between distinct cycles consistently with a linear ordering of these cycles. We used the same strategy here for the two cycles of the permutation σ that has v as a fixed point and acts on $[n-1]$ as σ_{n-1}. The decomposition has v as the domain of a trivial cycle and $[n-1]$ as the domain of σ_{n-1}. Had we defined these notions for arbitrary permutations (not just for cyclic ones), we could call the decomposition of the complete graph in Corollary 2 a σ-$(n-1)/2$-partition of K_n and the orientations of the Hamiltonian cycles would form a transitive σ-orientation of this partition.

4 Non-standard orientations

The results in the previous section may make one hope that the conditions of Theorem 2 are not just sufficient but also necessary for a σ_n-partition to have transitive σ_n-

orientations. This is not the case. In this section we give some sufficient conditions that go beyond the ones in Theorems 2 and 3 .

Our construction takes a transitive σ_m-orientation for a σ_m-k-partition and uses that to get transitive σ_n-orientations of related σ_n-k-partitions. Here n is a multiple of m. Even for some standard σ_m-k-orientations, the resulting transitive σ_n-orientations are not always standard and in some cases the σ_n-k-partitions do not have standard orientations at all.

Definition 5. *Let $k > 1$ and m be such that σ_m-k-partitions exist (that is, m is a multiple of k and if k is even m is also a multiple of $2k$). Let n be a multiple of m. We call the σ_n-k-partition P with defining sequence $a_1, \ldots, a_{\lfloor n/2 \rfloor}$ a blow-up of the σ_m-k-partition Q with defining sequence $b_1, \ldots, b_{\lfloor m/2 \rfloor}$ if $a_i = b_{(i \bmod m)}$ whenever $1 \le i \le \lfloor n/2 \rfloor$ and i is not divisible by m. For this to make sense even if $i \bmod m > \lfloor m/2 \rfloor$ we extend the sequence b_i by setting $b_i = (b_{m-i} + i) \bmod k$ for $\lfloor m/2 \rfloor < i < m$. This makes b_i the label of the edge $\{0, i\}$ in the σ_m-k-partition Q for any i. Note that we have no requirement for the value of a_i if i is divisible by m, so P is not determined by Q and n.*

Notice that if the σ_m-k-partition Q has a standard orientation, then its defining sequence satisfies the requirements of Theorem 3 and therefore the defining sequence $a_1, \ldots, a_{n/2}$ of its blow-up P also satisfies $a_{i+1} = a_i$ or $a_{i+1} \equiv a_i + 1 \pmod{k}$ whenever neither i nor $i+1$ is divisible by m. However, by the free choice of the value a_i whenever i is divisible by m, this property need not hold for the other indices, thus P may violate the conditions of Theorem 3. Nevertheless, as we will show, P admits a transitive σ_n-orientation in this case. We call the transitive orientation constructed in the next theorem the *blow-up* of the transitive σ_m-orientation of Q.

Theorem 4. *If the σ_n-k-partition P is a blow-up of the σ_m-k-partition Q and Q admits a transitive σ_m-orientation, then P admits a transitive σ_n-orientation.*

Proof. Let T be the transitive σ_m-orientation of Q we assumed to exist and let τ be the ordering of the vertex set $[m]$ that T is consistent with. The theorem claims that P has transitive σ_n-orientation T'. We construct T' by finding the ordering τ' on the vertex set $[n]$ that T' is consistent with. The only requirement we have to satisfy is that all edges of T' whose orientation σ_n reverses should have label $k - 1$.

Let us set $d = n/m$ and for $i \in [m]$, let $H_i = \{jm + i \mid j \in [d]\}$. Each of these m sets has d elements and together they partition $[n]$. The ordering τ' starts with the elements of $H_{\tau(1)}$ followed by the elements of $H_{\tau(2)}, \ldots, H_{\tau(m-1)}$. The order within the sets H_i will be specified later. We call an edge of K_n an *outer edge* if it connects vertices from distinct sets H_i and H_j, otherwise it is an *inner edge*. The orientation of the outer edges in T' are not influenced by the order within the sets H_i. namely for $a \in H_i$ and $b \in H_j$ $(i \ne j)$ the orientation of the edge $\{a, b\}$ in T' is determined by the orientation of the of edge $\{i, j\}$ in T. This means that σ_n reverses the orientation of the edge $\{a, b\}$ in T' if and only if σ_m reverses the orientation of the edge $\{i, j\}$ in T. Our definition of a blow-up ensures

11

that the label of the edge $\{a, b\}$ for the partition P is the same as the label of the edge $\{i, j\}$ for the partition Q. Therefore σ_n reverses the orientation of outer edges in T' only if their label is $k - 1$.

To finish the proof we have to specify the ordering τ' within the sets H_i in such a way that the same can be said about inner edges of T': σ_n reserves the orientation of them only if their label is $k - 1$.

For $i \in [m]$ let $\tau_i(1), \tau_i(2), \ldots, \tau_i(d)$ be an ordering of $[d]$ to be specified later and let us say that τ' orders the elements of H_i in the following order: $\tau_i(1)m + i, \tau_i(2)m + i, \ldots, \tau_i(d)m + i$.

Consider an inner edge $e = \{a, b\}$, with $a, b \in H_i$, $i \in [m]$. We have $a = Am + i$ and $b = Bm + i$ for some $A, B \in [d]$ and the orientation of e in T' is determined by the order of A and B in τ_i. In case $i \neq m - 1$ we have $\sigma_n(e) = \{a+1, b+1\}$, $a+1, b+1 \in H_{i+1}$ and the orientation of $\sigma_n(e)$ is determined by the order of A and B in τ_{i+1}. Thus, σ_n reverses the orientation of e in T' if and only if A and B are in different order in τ_i and in τ_{i+1}. We must make sure that this only happens if the label of e is $k - 1$.

The situation is a bit different for inner edges $e = \{a, b\}$ with $a, b \in H_{m-1}$. We still have $\sigma_n(e) = \{a + 1, b + 1\}$ (recall that these vertices are understood modulo n). But now $a + 1, b + 1 \in H_0$ and $a + 1 = (A + 1)m$, $b + 1 = (B + 1)m$. Thus, σ_n reverses the orientation of e if and only if A and B are in different order in τ_{m-1}, then $A+1$ and $B+1$ are in τ_0. Here both $A + 1$ and $B + 1$ are understand modulo d. We will make sure that this only happens if the label of e is $k - 1$.

We will now specify the orderings τ_i. As explained above, this finishes the proof if we can show the following two properties:

(a) If for some $0 \leq i < m - 1$ and $A \neq B \in [d]$ the order of A and B is different in τ_i and τ_{i+1}, then we have $\ell(Am + i, Bm + i) = k - 1$.

(b) If A precedes B in τ_{m-1} but $(B + 1) \bmod d$ precedes $(A + 1) \bmod d$ in τ_0, then we have $\ell(Am + m - 1, Bm + m - 1) = k - 1$.

We set τ_0 to be the order: $0, 1, d-1, 2, d-2, \ldots, \lceil d/2 \rceil$. Note that the pair of elements j and $d - j$ are consecutive for all $1 \leq j < d/2$. For $0 \leq i < k$ we obtain τ_{i+1} from τ_i by swapping the order of j and $d - j$ for each j satisfying $\ell(jm + i, (d - j)m + i) = k - 1$. In this way we maintain that the j and $d - j$ are consecutive in τ_i for all $0 \leq i \leq k$ and $1 \leq j < d/2$. Also, with this rule condition (a) is satisfied for $0 \leq i < k$. For any i and j as above we have $\ell(jm+i, (d-j)m+i) \equiv \ell(jm, (d-j)m)+i \pmod{k}$. Therefore, for any such j, the label will be $k-1$ for exactly one of the indices $0 \leq i < k$ and thus τ_k will have all the pairs $(j, d-j)$ swapped. Namely, τ_k is the order $0, d-1, 1, d-2, 2, d-3, \ldots, \lfloor d/2 \rfloor$.

Proposition 3 tells us that $m \geq 2k$. Let us assume for now that $m > 2k$. We will come back to the case $m = 2k$ later. Observe that j and $d - j - 1$ are consecutive in τ_k for any $0 \leq j \leq d/2 - 1$. For $k \leq i < 2k$ we obtain τ_{i+1} from τ_i by swapping the order of j and $d - j - 1$ for each j satisfying $\ell(jm + i, (d - j - 1)m + i) = k - 1$. In this way we maintain that the vertices j and $d - j - 1$ are consecutive in τ_i for all $k \leq i \leq 2k$

and $0 \le j \le d/2 - 1$. Also, this rule makes condition (a) satisfied for $k \le i < 2k$. Just as before, for any j as above the label condition is satisfied for exactly one index $k \le i < 2k$ and thus τ_{2k} will have all the pairs $(j, d - j - 1)$ swapped. Namely, τ_{2k} is the order $d - 1, 0, d - 2, 1, d - 3, 2, \ldots \lceil d/2 \rceil - 1$.

We set $\tau_i = \tau_{2k}$ for $2k < i < m$. This makes condition (a) hold vacuously for $2k \le i \le m - 1$ as $\tau_i = \tau_{i+1}$. Condition (b) is also satisfied vacuously, since A precedes B in $\tau_{m-1} = \tau_{2k}$ if and only if $(A + 1) \bmod d$ precedes $(B + 1) \bmod d$ in τ_0. This is so because τ_0 can be obtained from τ_{2k} by replacing each element j by $(j + 1) \bmod d$. This finishes the proof of the theorem in the case $m > 2k$.

It remains to consider the case $m = 2k$. We define the orders τ_i for $k < i \le 2k$ the same way as above. We do not use τ_{2k} in the definition of τ', but we will use it in our argument below.

Condition (a) is satisfied as before. But now condition (b) is not vacuous. We still have that A precedes B in τ_{2k} if and only if $(A + 1) \bmod d$ precedes $(B + 1) \bmod d$ in τ_0, so if A precedes B in τ_{2k-1} but $(B + 1) \bmod d$ precedes $(A + 1) \bmod d$ in τ_0 as called for in condition (b), then A and B appear in different order in τ_{2k-1} and τ_{2k} and therefore $\ell(Am + m - 1, Bm + m - 1) = k - 1$. This makes condition (b) satisfied and finishes the proof of the theorem. □

To illustrate Theorem 4 we show a σ_n-partition that admits a transitive σ_n-orientation but does not admit a standard one. We take $n = 12$ and consider the σ-partition P with defining sequence 000121. Recall that this is a partition of the edge set of K_{12} to three isomorphic subgraphs and it has no standard orientation by Proposition 2. But it is a blow-up of the σ_6-partition Q with defining sequence 000. Q has a transitive σ_6-orientation, even a standard one by Theorem 2. The first part of Q and its orientation is depicted in the first illustration on Figure 1. By Theorem 4 P has a transitive σ_{12}-orientation. For example orienting the edges consistent with the following ordering of the vertices gives a transitive σ_{12}-orientation: $0, 6, 1, 7, 2, 8, 11, 5, 4, 10, 3, 9$.

5 Complications

After seeing Theorems 2 and 4 one might be curious to know whether they describe all transitive σ_n-orientations of a σ_n-partition, namely if all such orientations are blow-ups of standard orientations. This is, however, not the case, moreover, there exist σ_n-partitions that neither admit a standard orientation nor are blow-ups of σ_m-partitions for some $m < n$, yet they do admit a transitive σ_n-orientation. We found such examples by computer and do not see a general pattern that would still suggest a complete characterization. (We note that our examples in this section all concern the simplest possible case $k = 3$ again.)

The σ_{24}-partition with the defining sequence

$$0, 0, 0, 1, 2, 0, 0, 0, 1, 1, 2, 1$$

is such an example. The orientation consistent with the following order of the vertices is a transitive σ_{24}-orientation for this partition:

13

$$0, 1, 2, 23, 22, 21, 3, 9, 4, 10, 20, 5, 11, 8, 7, 19, 6, 18, 12, 13, 14, 17, 16, 15.$$

(To check that this defines a transitive σ_{24}-orientation one has only to verify that the edges whose orientation σ_{24} reverses have label 2. For example, the edge $e = \{4, 8\}$ is oriented towards 8 as 8 appears later in this sequence than 4. But $\sigma_{24}(e) = \{5, 9\}$ is oriented towards 5 as 5 appears later than 9. So σ_{24} reverses the orientation of e. Now $\ell(4, 8) \equiv \ell(0, 4) + 4 \pmod{3}$ and $\ell(0, 4)$ is the fourth number of the defining sequence, namely 1, so $\ell(4, 8) = 2$ as required.)

6   Necessary conditions

So far we have seen sufficient conditions for σ_n-partitions (or σ_n-k-partitions) to posses a transitive σ_n-orientation. While a complete characterization seems elusive it makes sense to look also for non-trivial necessary conditions. Here we give a simple such condition.

Theorem 5. *Let P be a σ_n-k-partition with defining sequence $a_1, \ldots, a_{\lfloor n/2 \rfloor}$. Assume $a_1 = 0$ and let i be an index with $1 \leq i < \lfloor n/2 \rfloor$ such that $a_j \neq k - 1$ for $1 \leq j \leq i$. If P has a transitive σ_n-orientation, then $a_{i+1} \leq a_i + 1$.*

Proof. Fix a transitive σ_n-orientation T of P. When referring to the orientation of edges we speak about the orientation in T.

We call a vertex $m \in [n]$ a *leader* if it is divisible by k. Notice that if m is a leader, then the label $\ell(m, m + j)$ is a_j for all $1 \leq j \leq \lfloor n/2 \rfloor$. (Recall that vertices are always understood modulo n.) We call a leader m an *in-leader* if the edge $\{m, m + 1\}$ is oriented towards m, otherwise it is an *out-leader*.

We claim that if m is an in-leader, then all the edges $\{m, m + j\}$ for $1 \leq j \leq i + 1$ are oriented toward m, while if m is an out-leader, then all these edges are oriented away from m. By symmetry, it is enough to prove one of the statements. We prove the latter one by induction on j. The statement of the claim is assumed for $j = 1$. So let $1 \leq j \leq i$ and assume $\{m, m + j\}$ is oriented away from m. The label of this edge is $a_j \neq k - 1$, so σ_n does not reverse its orientation. Therefore the edge $\{m + 1, m + j + 1\}$ is oriented towards $m + j + 1$. As m is an out-vertex the edge $\{m, m + 1\}$ is oriented toward $m + 1$. To get a transitive orientation $\{m, m + j + 1\}$ must therefore be oriented away from m finishing the inductive proof.

Consider the cycle formed by the edges $\{m, m + 1\}$ for all $m \in [n]$. If all leader vertices were in-leaders or all of them were out-leaders, then this cycle would be a directed cycle contradicting the transitivity of T. So we must have at least one in-leader and also at least one out-leader. We can therefore fix an out-leader m such that the very next leader vertex, namely $m + k$ is an in-leader.

By the claim above (and since m is an out-leader) the edge $\{m, m + i\}$ is oriented towards $m + i$. The label of this edge is a_i, so one can apply the permutation σ_n to this edge $k - 1 - a_i$ times without it reversing the orientation. Thus, the edge $\{m + k - 1 -$

14

$a_i, m+i+k-1-a_i\}$ is oriented toward $m+i+k-1-a_i$. Recall that $a_i \neq k-1$, so (as m is an out-leader) the edge $\{m+k-2-a_i, m+k-1-a_i\}$ is oriented toward $m+k-1-a_i$. The transitivity of T therefore implies that the edge $e = \{m+k-2-a_i, m+i+k-1-a_i\}$ is oriented toward $m+i+k-1-a_i$. We have $\sigma_n^{a_i+2}(e) = \{m+k, m+k+i+1\}$ and this edge is oriented toward the in-leader $m+k$ by the claim above. We see that $\sigma_n^{a_i+2}$ reverses the orientation of e, therefore the label a_{i+1} of $\sigma_n^{a_i+2}(e)$ must be less than $a_i + 2$. $\square$

Definition 6. *Let $a_1, \ldots, a_{\lfloor n/2 \rfloor}$ be the defining sequence of a σ_n-k-partition P. We say that the sequence* halts *at the index i ($1 \leq i < \lfloor n/2 \rfloor$) if $a_{i+1} = a_i$. We say that it* steps *at the index i if $a_{i+1} \equiv a_i + 1 \pmod{k}$. We say that it* jumps *at the index i if it neither halts nor steps there.*

We call the σ_n-k-partition with defining sequence $b_1, \ldots, b_{\lfloor n/2 \rfloor}$ the dual *of P if $b_i \equiv i - 1 - a_i \pmod{k}$ for all i.*

Note that if a the defining sequence of a σ_n-k-partition halts at an index i, then the defining sequence of its dual steps there and vice versa. The defining sequences of a σ_n-k-partition and its dual jump at the same indices.

We can rephrase Theorem 3 as follows: A σ_n-k-partition has a standard orientation if and only if its defining sequence does not jump at all.

Theorem 6. *If a σ_n-k-partition admits a transitive σ_n-orientation, then so does its dual.*

If a σ_n-k-partition P admits a transitive σ_n-orientation and the defining sequence of P jumps at an index i, then there is an index $j < i$ where it halts and at least $k-1$ distinct indices $j' < i$ where it steps, or the other way around: there is an index $j < i$ where it steps and at least $k-1$ distinct indices $j' < i$ where it halts. In particular, there is no jump at indices $i \leq k$.

Proof. Let P be the σ_n-k-partition, with parts $F_0, \ldots, F_{k-1}$. The dual Q of P can be obtained by first relabeling the vertices, namely switching the label v and $n-v$ for $1 \leq v < k$, and then considering the parts in reverse order, namely as $F_{k-1}, F_{k-2}, \ldots, F_0$. If a transitive orientation T of K_n is a transitive σ_n-orientation of P, then T (after the relabeling) is also a transitive σ_n-orientation of Q. This proves the first claim in the theorem.

To verify the second claim we assume without loss of generality that the defining sequence (a_j) of P starts with $a_1 = 0$. Note that this makes the defining sequence (b_j) of its dual Q start with $b_1 = 0$. Assume that the first jump in (a_j) is at the index i, so at indices $1 \leq j < i$ the sequence halts or steps. By Theorem 5 we have $a_{i+1} \leq a_i + 1$ unless the sequence steps for at least $k-1$ such indices. Further, it has to step for at least one such index, as otherwise we would have $a_j = 0$ for all $j \leq i$ and $a_{i+1} \leq a_i + 1 = 1$ contradicting our assumption that the sequence jumps at i. Note that i is also the first index where the sequence (b_j) jumps, so we similarly have that this sequence steps for at least one index $j < i$ and we have $b_{i+1} \leq b_i + 1$ unless it steps at $k-1$ or more such indices. To finish the proof it is enough to note that (a_j) steps where (b_j) halts and vice versa and

15

if both step at fewer than $k-1$ indices $j < i$, then $a_i \leq a_{i+1} \leq a_i + 1$ contradicting our assumption that the sequence (a_j) jumps at i. $\square$

We conclude the paper with two conjectures. All σ_n-partitions for which we found a transitive σ_n-orientation were for even values of n. We tried to find such examples with odd n but failed even with a computer. This suggests the following.

Conjecture 1. *No σ_n-partition with n odd has a transitive σ_n-orientation.*

Although we did not do any computer search for partitions with more than three parts, we still venture the following stronger conjecture:

Conjecture 2. *No σ_n-k-partition with $k > 1$ and n odd has a transitive σ_n-orientation.*

References

[1] J. A. Bondy, Basic graph theory: paths and circuits, in: *Handbook of combinatorics, Vol. 1*, 3–110, Elsevier Sci. B. V., Amsterdam, 1995.

[2] A. Galluccio, L. Gargano, J. Körner, G. Simonyi, Different capacities of a digraph, *Graphs Combin.*, **10** (1994), 105–121.

[3] R. A. Gibbs, Self-complementary graphs, *J. Combin. Theory ser. B*, **16** (1974), 106–123.

[4] A. Görlich, R. Kalinowski, M. Meszka, M. Pilsniak, M. Woźniak, A note on decompositions of transitive tournaments. *Discrete Math.* 307 (2007), no. 7-8, 896904.

[5] A. Görlich, R. Kalinowski, M. Meszka, M. Pilsniak, M. Woźniak, A note on decompositions of transitive tournaments. II. *Australas. J. Combin.* 37 (2007), 5766.

[6] A. Görlich, M. Pilśniak, A note on an embedding problem in transitive tournaments, *Discrete Math.* 310 (2010), no. 4, 681686.

[7] A. Görlich, M. Pilsniak, M. Woźniak, A note on a packing problem in transitive tournaments, *Graphs Combin.* 22 (2006), no. 2, 233239.

[8] A. Gyárfás, Transitive tournaments and self-complementary graphs. *J. Graph Theory*, **38** (2001), no. 2, 111–112.

[9] J. Körner, C. Pilotto, G. Simonyi, Local chromatic number and Sperner capacity, *J. Combin. Theory Ser. B*, **95** (2005), 101–117.

[10] L. Lovász, On the Shannon capacity of a graph, *IEEE Trans. Inform. Theory*, **25** (1979), 1–7.

[11] É. Lucas, *Récréations Mathematiques*, Vol. 2, Gauthiers-Villars, Paris, 1891.

[12] G. Ringel, Selbstkomplementäre Graphen, *Arch. Math.*, **14** (1963), 354-358.

[13] H. Sachs, Über selbstkomplementäre Graphen, *Publ. Math. Debrecen*, **9** (1962), 270–288.

[14] A. Sali, G. Simonyi, Orientations of self-complementary graphs and the relation of Sperner and Shannon capacities, *Europ. J. Combin.*, **20** (1999), 93–99.