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A quasi Poisson-Aradhana distribution

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In this study, a QPAD (quasi Poisson-Aradhana distribution) by compounding a PD (Poisson distribution) with a QAD (quasi Aradhana distribution) is proposed that includes PAD (Poisson-Aradhana distribution) as a particular case. Expressions for its coefficient of variation, coefficient of skewness, coefficient of kurtosis, and index of dispersion are provided and their behaviours are studied for varying values of the parameters. The QPAD is shown to be unimodal and always over-dispersed. The estimation of its parameters using the method of maximum likelihood is discussed. Finally, the goodness of fit of the QPAD is assessed for two real count datasets from ecology and the fit is compared with that of the PD, PLD (Poisson-Lindley distribution), and PAD.

KEYWORDS: compounding, log-concavity, over-dispersion

Shanker [2017] introduced a one-parameter discrete distribution named PAD, defined by the probability mass function:

$$P_1(x;\theta)=\frac{\theta^3}{\theta^2+2\theta+2}\cdot\frac{x^2+(2\theta+5)x+(\theta^2+4\theta+5)}{(\theta+1)^{x+3}};\ x=0,1,2,...,\ \theta>0,\quad /1/$$

where θ is a scale parameter.

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Its statistical properties, parameter estimation using both the method of moments and the method of maximum likelihood, and applications of the PAD in modelling count data have been discussed by *Shanker* [2017]. *Shanker et al.* [2018] have developed the size-biased PAD and discussed its various properties, parameter estimation, and applications. Further, *Shanker* and *Shukla* [2019] developed the zero-truncated PAD and studied its properties, parameter estimation, and applications.

It should be noted that *Shanker* [2017] obtained PAD by compounding a PD with the Aradhana distribution, introduced by *Shanker* [2016], having pdf (probability density function):

$$f_1(x;\theta)=\frac{\theta^3}{\theta^2+2\theta+2}(1+x)^2 e^{-\theta x};\, x>0, \theta>0. \qquad /2/$$

Shanker [2016] has discussed its statistical properties, including *1*. its shape for various values of the parameter; *2*. its coefficient of variation, coefficient of skewness, coefficient of kurtosis, index of dispersion, hazard rate function, mean residual life function, mean deviations, stochastic ordering, Renyi entropy measure, order statistics, Bonferroni and Lorenz indices, and stress-strength reliability; and *3*. its parameter estimation and applications to modelling lifetime data from engineering and biomedical sciences.

The r^{th} factorial moment of the PAD about the origin, $\mu_{(r)}'$, obtained by *Shanker* [2017], is given by

$$\mu_{(r)}'=\frac{r!\left[\theta^2+2(r+1)\theta+(r+1)(r+2)\right]}{\theta^r\left(\theta^2+2\theta+2\right)};\, r=1,2,3,... \qquad /3/$$

Shanker et al. [2018] proposed a two-parameter QAD with pdf:

$$f_2(x;\theta,\alpha)=\frac{\theta}{\alpha^2+2\alpha+2}(\alpha+\theta x)^2 e^{-\theta x};\quad x>0,\quad \theta>0,\quad \alpha^2+2\alpha+2>0, \qquad /4/$$

where α is a shape parameter.

The Aradhana distribution is a particular case of the QAD for $\alpha=\theta$. The QAD's various statistical properties, including moment-based measures, hazard rate function, mean residual life function, stochastic ordering, mean deviations, Bonferroni and Lorenz curves, order statistics, Renyi entropy measure, and stress-strength reliability have been discussed by *Shanker et al.* [2018]. Further, estimation of the parameters using both the method of moments and the method of maximum likelihood, as well as applications of QAD have been studied by *Shanker et al.* [2018].

In this study, a two-parameter QPAD, which includes *Shanker*'s [2017] one-parameter PAD as a particular case, is proposed. Its moments and moment-based measures, including its coefficient of variation, coefficient of skewness, coefficient of kurtosis, and index of dispersion, are studied. Its statistical properties, including over-dispersion, IHR (increasing hazard rate), and unimodality, are also discussed. The method of maximum likelihood estimation is used to estimate the parameters of the distribution. The goodness of fit of the proposed distribution is established through three examples of datasets from the biological sciences.

1. Quasi Poisson-Aradhana distribution

Suppose that the parameter λ of a PD follows a QAD /4/. Then, a Poisson mixture of the QAD can be obtained as follows:

$$P_2\left(x;\theta,\alpha\right)=\int_0^\infty \frac{e^{-\lambda}\lambda^x}{x!}\frac{\theta}{\alpha^2+2\alpha+2}\left(\alpha+\theta\lambda\right)^2 e^{-\theta\lambda}\,d\lambda \qquad /5/$$

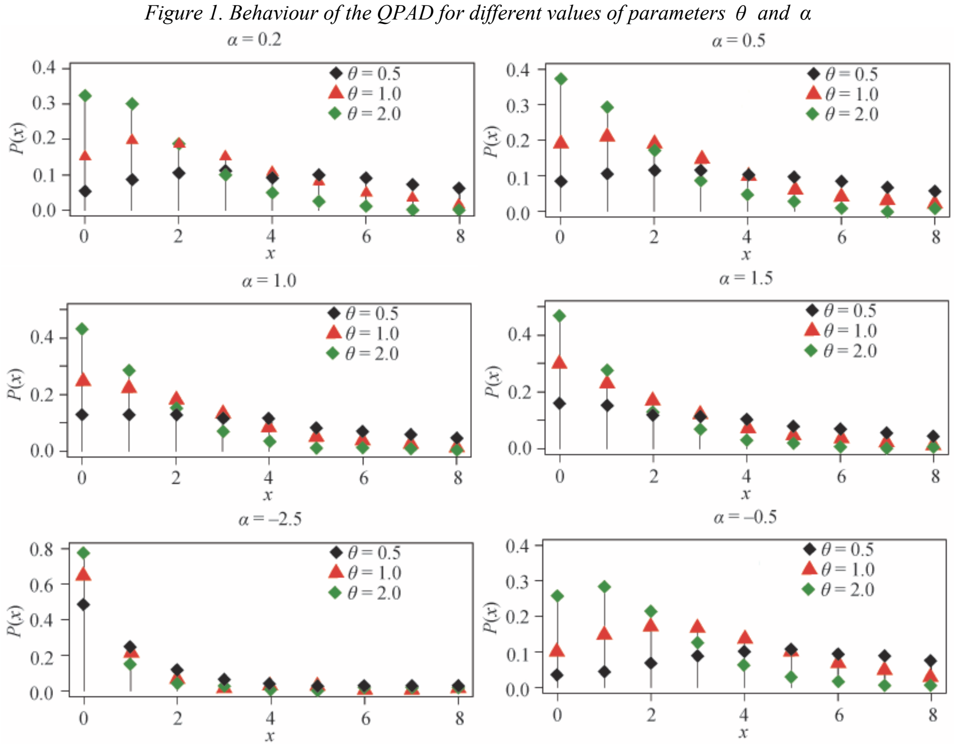
$$=\frac{\theta}{\left(\alpha^2+2\alpha+2\right)x!}\int_0^\infty e^{-(\theta+1)\lambda}\left[\alpha^2\lambda^x+2\alpha\theta\lambda^{x+1}+\theta^2\lambda^{x+2}\right]d\lambda$$

$$=\frac{\theta}{\left(\alpha^2+2\alpha+2\right)x!}\left[\frac{\alpha^2\Gamma\left(x+1\right)}{\left(\theta+1\right)^{x+1}}+\frac{2\alpha\theta\,\Gamma\left(x+2\right)}{\left(\theta+1\right)^{x+2}}+\frac{\theta^2\Gamma\left(x+3\right)}{\left(\theta+1\right)^{x+3}}\right]$$

$$=\frac{\theta}{\left(\alpha^2+2\alpha+2\right)}\frac{\left\{\begin{array}{l}\theta^2x^2+\left(2\alpha\theta^2+2\alpha\theta+3\theta^2\right)x\\+\left(\alpha^2\theta^2+2\alpha^2\theta+2\alpha\theta^2+2\alpha\theta+2\theta^2+\alpha^2\right)\end{array}\right\}}{\left(\theta+1\right)^{x+3}};\ x=0,1,2,..., \qquad /6/$$

where $\Gamma(x)$ is a complete gamma function. We name this probability distribution a QPAD. It can be easily verified that the QPAD /6/ reduces to the PAD /1/ when $\alpha = \theta$. The behaviour of the QPAD for different values of its parameters θ and α is explained through graphs presented in Figure 1.

Figure 1. Behaviour of the QPAD for different values of parameters θ and α



Note. Here and hereinafter, θ is a scale parameter and α is a shape parameter.

2. Statistical constants

The r^{th} factorial moment of QPAD about the origin, $\mu_{(r)}'$ /6/, can be obtained as

$$\mu_{(r)}' = E\left[E\left(X^{(r)} \mid \lambda\right)\right],$$

where $X^{(r)} = X(X-1)(X-2)...(X-r+1)$, and E is an expectation operator.

Using /5/, $\mu_{(r)}'$ can be obtained as

$$\mu_{(r)}' = \frac{\theta}{\alpha^2 + 2\alpha + 2}\int_0^\infty \left[\sum_{x=0}^{\infty} x^{(r)} \frac{e^{-\lambda}\lambda^x}{x!}\right](\alpha + \theta\lambda)^2 e^{-\theta\lambda} d\lambda$$

$$= \frac{\theta}{\alpha^2 + 2\alpha + 2}\int_0^\infty \lambda^r \left[\sum_{x=r}^{\infty} \frac{e^{-\lambda}\lambda^{x-r}}{(x-r)!}\right](\alpha + \theta\lambda)^2 e^{-\theta\lambda} d\lambda .$$

Substituting $x + r$ for x within the brackets, we obtain

$$\mu_{(r)}' = \frac{\theta}{\alpha^2 + 2\alpha + 2}\int_0^\infty \lambda^r \left[\sum_{x=0}^{\infty} \frac{e^{-\lambda}\lambda^{x}}{x!}\right](\alpha + \theta\lambda)^2 e^{-\theta\lambda} d\lambda$$

$$= \frac{\theta}{\alpha^2 + 2\alpha + 2}\int_0^\infty \lambda^r (\alpha + \theta\lambda)^2 e^{-\theta\lambda} d\lambda .$$

After a few algebraic simplifications, we finally obtain a general expression for $\mu_{(r)}'$:

$$\mu_{(r)}' = \frac{r!\left\{\alpha^2 + 2(r+1)\alpha + (r+1)(r+2)\right\}}{\theta^r\left(\alpha^2 + 2\alpha + 2\right)}; \; r = 1, 2, 3, ... \qquad /7/$$

It can be easily verified that when $\alpha = \theta$, expression /7/ reduces to the corresponding expression for the PAD /3/. Substituting $r = 1, 2, 3,$ and 4 in /7/, the first four factorial moments of QPAD about the origin can be obtained and using the relationship between factorial moments about the origin and moments about the origin, the first four moments of QPAD about the origin /6/ are obtained:

$$\mu_1' = \frac{\alpha^2 + 4\alpha + 6}{\theta\left(\alpha^2 + 2\alpha + 2\right)},$$

$$\mu_2' = \frac{(\theta+2)\alpha^2 + (4\theta + 12)\alpha + (6\theta + 24)}{\theta^2\left(\alpha^2 + 2\alpha + 2\right)},$$

$$\mu_3' = \frac{\left(\theta^2 + 6\theta + 6\right)\alpha^2 + \left(4\theta^2 + 36\theta + 48\right)\alpha + \left(6\theta^2 + 72\theta + 120\right)}{\theta^3\left(\alpha^2 + 2\alpha + 2\right)},$$

$$\mu_4' = \frac{\left\{\begin{matrix}\left(\theta^3 + 14\theta^2 + 36\theta + 24\right)\alpha^2 + \left(4\theta^3 + 84\theta^2 + 288\theta + 240\right)\alpha + \\ + \left(6\theta^3 + 168\theta^2 + 720\theta + 720\right)\end{matrix}\right\}}{\theta^4\left(\alpha^2 + 2\alpha + 2\right)}.$$

Using the relationship between moments about the mean and moments about the origin, the QPAD's moments about the mean are:

$$\mu_2 = \frac{\left(\theta + 1\right)\alpha^4 + \left(6\theta + 8\right)\alpha^3 + \left(16\theta + 24\right)\alpha^2 + \left(20\theta + 24\right)\alpha + \left(12\theta + 12\right)}{\theta^2\left(\alpha^2 + 2\alpha + 2\right)^2},$$

$$\mu_3 = \frac{\left\{\begin{matrix}\left(\theta^2 + 3\theta + 2\right)\alpha^6 + \left(8\theta^2 + 30\theta + 24\right)\alpha^5 + \left(30\theta^2 + 126\theta + 108\right)\alpha^4 + \\ + \left(64\theta^2 + 264\theta + 200\right)\alpha^3 + \left(84\theta^2 + 324\theta + 216\right)\alpha^2 + \\ \left(64\theta^2 + 216\theta + 144\right)\alpha + \left(24\theta^2 + 72\theta + 48\right)\end{matrix}\right\}}{\theta^3\left(\alpha^2 + 2\alpha + 2\right)^3},$$

$$\mu_4 = \frac{\left\{\begin{matrix}\left(\theta^3 + 10\theta^2 + 18\theta + 9\right)\alpha^8 + \left(10\theta^3 + 120\theta^2 + 252\theta + 144\right)\alpha^7 + \\ + \left(48\theta^3 + 652\theta^2 + 1{,}488\theta + 912\right)\alpha^6 + \\ + \left(140\theta^3 + 2{,}040\theta^2 + 4{,}680\theta + 2{,}832\right)\alpha^5 + \\ + \left(272\theta^3 + 4{,}136\theta^2 + 9{,}264\theta + 5{,}448\right)\alpha^4 + \\ + \left(360\theta^3 + 5{,}600\theta^2 + 12{,}048\theta + 6{,}912\right)\alpha^3 + \\ + \left(320\theta^3 + 5{,}040\theta^2 + 10{,}368\theta + 5{,}760\right)\alpha^2 + \\ + \left(176\theta^3 + 2{,}784\theta^2 + 5{,}472\theta + 2{,}880\right)\alpha + \\ + \left(48\theta^3 + 768\theta^2 + 1{,}440\theta + 720\right)\end{matrix}\right\}}{\theta^4\left(\alpha^2 + 2\alpha + 2\right)^4}.$$

The coefficient of variation (CV), coefficient of skewness $\left(\sqrt{\beta_1}\right)$, coefficient of kurtosis $\left(\beta_2\right)$, and index of dispersion $\left(\gamma\right)$ of the QPAD are thus given by

$$CV = \frac{\sigma}{\mu_1'} = \frac{\sqrt{(\theta+1)\alpha^4 + (6\theta+8)\alpha^3 + (16\theta+24)\alpha^2 + (20\theta+24)\alpha + (12\theta+12)}}{\alpha^2+4\alpha+6},$$

$$\sqrt{\beta_1} = \frac{\mu_3}{\mu_2^{3/2}} =$$

$$= \frac{\left\{\begin{aligned}&(\theta^2+3\theta+2)\alpha^6 + (8\theta^2+30\theta+24)\alpha^5 + (30\theta^2+126\theta+108)\alpha^4 + \\ &+ (64\theta^2+264\theta+200)\alpha^3 + (84\theta^2+324\theta+216)\alpha^2 + \\ &+ (64\theta^2+216\theta+144)\alpha + (24\theta^2+72\theta+48)\end{aligned}\right\}}{\left\{(\theta+1)\alpha^4 + (6\theta+8)\alpha^3 + (16\theta+24)\alpha^2 + (20\theta+24)\alpha + (12\theta+12)\right\}^{3/2}},$$

$$\beta_2 = \frac{\mu_4}{\mu_2^{2}} =$$

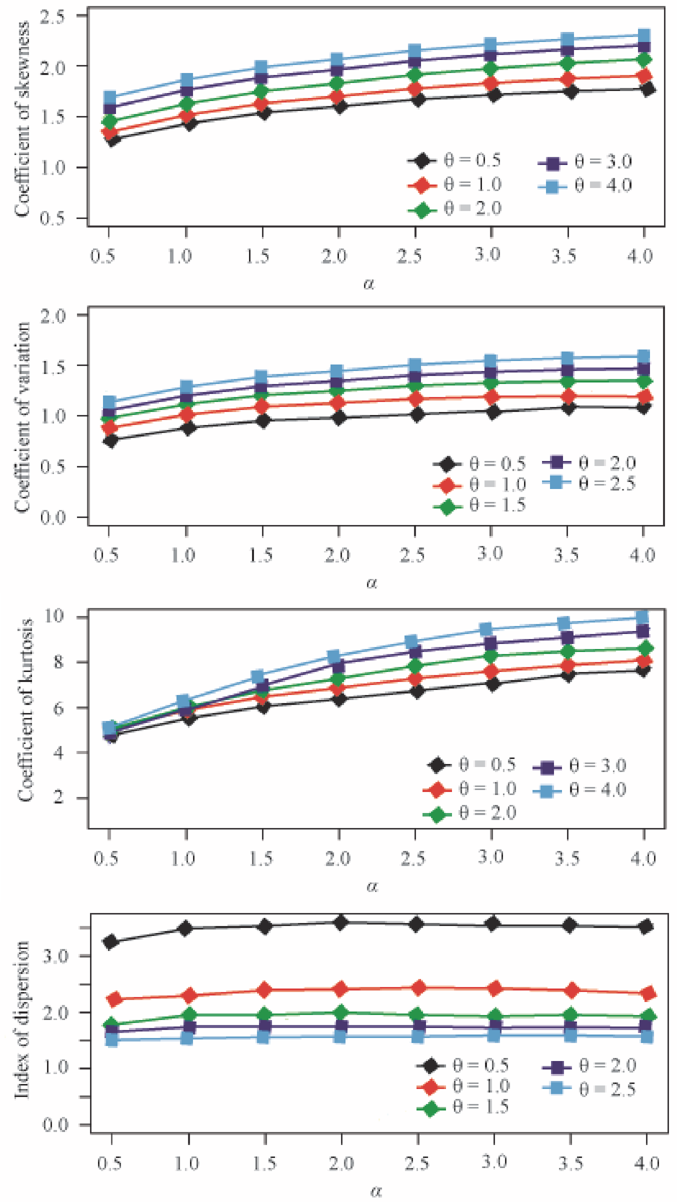
$$= \frac{\left\{\begin{aligned}&(\theta^3+10\theta^2+18\theta+9)\alpha^8 + (10\theta^3+120\theta^2+252\theta+144)\alpha^7 + \\ &+ (48\theta^3+652\theta^2+1{,}488\theta+912)\alpha^6 + \\ &+ (140\theta^3+2{,}040\theta^2+4{,}680\theta+2{,}832)\alpha^5 + \\ &+ (272\theta^3+4{,}136\theta^2+9{,}264\theta+5{,}448)\alpha^4 + \\ &+ (360\theta^3+5{,}600\theta^2+12{,}048\theta+6{,}912)\alpha^3 + \\ &+ (320\theta^3+5{,}040\theta^2+10{,}368\theta+5{,}760)\alpha^2 + \\ &+ (176\theta^3+2{,}784\theta^2+5{,}472\theta+2{,}880)\alpha + \\ &+ (48\theta^3+768\theta^2+1{,}440\theta+720)\end{aligned}\right\}}{\left\{(\theta+1)\alpha^4 + (6\theta+8)\alpha^3 + (16\theta+24)\alpha^2 + (20\theta+24)\alpha + (12\theta+12)\right\}^{2}},$$

$$\gamma = \frac{\sigma^2}{\mu_1'} = \frac{(\theta+1)\alpha^4 + (6\theta+8)\alpha^3 + (16\theta+24)\alpha^2 + (20\theta+24)\alpha + (12\theta+12)}{\theta(\alpha^2+2\alpha+2)(\alpha^2+4\alpha+6)}.$$

It can be easily verified that when $\alpha = \theta$, the expressions of these statistical constants for QPAD reduce to the corresponding expressions for PAD.

The behaviours of QPAD's coefficient of variation, coefficient of skewness, coefficient of kurtosis, and index of dispersion for varying values of parameters θ and α have been explained through graphs and presented in Figure 2.

Figure 2. Behaviours of the QPAD's coefficient of variation, coefficient of skewness, coefficient of kurtosis, and index of dispersion for different values of parameters θ and α



3. Properties

In this section, the properties of the proposed distribution are discussed based on its mean, variance, and unimodality. Distribution whose variance exceeds the mean is regarded over-dispersed $\left(\mu < \sigma^2\right)$, whose variance equals to mean is considered equi-dispersed $\left(\mu = \sigma^2\right)$, and whose variance is less than the mean is known under-dispersed $\left(\mu > \sigma^2\right)$.

3.1. Over-dispersion

The QPAD is always over-dispersed $\left(\sigma^2 > \mu\right)$. We have

$$\sigma^2 = \frac{\left(\theta+1\right)\alpha^4 + \left(6\theta+8\right)\alpha^3 + \left(16\theta+24\right)\alpha^2 + \left(20\theta+24\right)\alpha + \left(12\theta+12\right)}{\theta^2\left(\alpha^2+2\alpha+2\right)^2}$$

$$= \frac{\alpha^2+4\alpha+6}{\theta\left(\alpha^2+2\alpha+2\right)}\left[\frac{\left\{\begin{array}{l}\left(\theta+1\right)\alpha^4 + \left(6\theta+8\right)\alpha^3 + \left(16\theta+24\right)\alpha^2 + \\ + \left(20\theta+24\right)\alpha + \left(12\theta+12\right)\end{array}\right\}}{\theta\left(\alpha^2+2\alpha+2\right)\left(\alpha^2+4\alpha+6\right)}\right]$$

$$= \mu\left[1 + \frac{\alpha^4+8\alpha^3+24\alpha^2+24\alpha+12}{\theta\left(\alpha^2+2\alpha+2\right)\left(\alpha^2+4\alpha+6\right)}\right] > \mu\ .$$

This shows that QPAD is always over-dispersed and, thus, it can be used for modelling over-dispersed data (i.e. data whose variance exceeds the mean).

3.2. IHR and unimodality

The QPAD has an IHR and is unimodal. Since

$$\frac{P_2(x+1;\theta,\alpha)}{P_2(x;\theta,\alpha)}=\frac{1}{\theta+1}\left[1+\frac{2\theta^2 x+\left(2\alpha\theta^2+2\alpha\theta+4\theta^2\right)}{\left\{\begin{array}{l}\theta^2x^2+\left(2\alpha\theta^2+2\alpha\theta+3\theta^2\right)x+\\+\left(\alpha^2\theta^2+2\alpha^2\theta+2\alpha\theta^2+2\alpha\theta+2\theta^2+\alpha^2\right)\end{array}\right\}}\right]$$

is a decreasing function in x, $P_2(x;\theta,\alpha)$ is log-concave. This implies that QPAD has an IHR and is unimodal. The interrelationships among the log-concavity, unimodality, and IHR of discrete distributions are described by *Grandell* [1997].

4. Maximum likelihood estimation of parameters

Let $(x_1,x_2,...,x_n)$ be a random sample of size n from the QPAD and let f_x be the observed frequency in the sample for $X=x\ (x=1,2,3,...,k)$ such that $\sum_{x=1}^{k} f_x=n$, where k is the largest observed value having a non-zero frequency. The likelihood function L of the QPAD is given by

$$L=\left(\frac{\theta}{\alpha^2+2\alpha+2}\right)^n\cdot$$

$$\cdot\frac{1}{(\theta+1)^{\sum_{x=1}^{k}(x+3)f_x}}\prod_{x=1}^{k}\left[\begin{array}{l}\theta^2x^2+\left(2\alpha\theta^2+2\alpha\theta+3\theta^2\right)x+\\+\left(\alpha^2\theta^2+2\alpha^2\theta+2\alpha\theta^2+2\alpha\theta+2\theta^2+\alpha^2\right)\end{array}\right]^{f_x}.$$

Thus, the log-likelihood function is

$$\log L = n\log\left(\frac{\theta}{\alpha^2 + 2\alpha + 2}\right) - \sum_{x=1}^{k}(x+3) f_x \log(\theta+1) + \\ + \sum_{x=1}^{k} f_x \log\left[\begin{array}{l}\theta^2 x^2 + \left(2\alpha\theta^2 + 2\alpha\theta + 3\theta^2\right)x + \\ + \left(\alpha^2\theta^2 + 2\alpha^2\theta + 2\alpha\theta^2 + 2\alpha\theta + 2\theta^2 + \alpha^2\right)\end{array}\right].$$

The MLEs (maximum likelihood estimates; $\hat{\theta}, \hat{\alpha}$) of the QPAD's (θ, α) are the solutions to the following log-likelihood equations:

$$\frac{\partial \log L}{\partial \theta} = \frac{n}{\theta} - \frac{n(\bar{x}+3)}{\theta+1} + \\ + \sum_{x=1}^{k} \frac{\left[\begin{array}{l}2\theta x^2 + \left(4\alpha\theta + 2\alpha + 6\theta\right)x + \\ + \left(2\alpha^2\theta + 2\alpha^2 + 4\alpha\theta + 2\alpha + 4\theta\right)\end{array}\right] f_x}{\left[\begin{array}{l}\theta^2 x^2 + \left(2\alpha\theta^2 + 2\alpha\theta + 3\theta^2\right)x + \\ + \left(\alpha^2\theta^2 + 2\alpha^2\theta + 2\alpha\theta^2 + 2\alpha\theta + 2\theta^2 + \alpha^2\right)\end{array}\right]} = 0,$$

$$\frac{\partial \log L}{\partial \alpha} = -\frac{2n(\alpha+1)}{\alpha^2 + 2\alpha + 2} + \\ + \sum_{x=1}^{k} \frac{\left[\left(2\theta^2 + 2\theta\right)x + \left(2\alpha\theta^2 + 4\alpha\theta + 2\theta^2 + 2\theta + 2\theta^2 + 2\alpha\right)\right] f_x}{\left[\begin{array}{l}\theta^2 x^2 + \left(2\alpha\theta^2 + 2\alpha\theta + 3\theta^2\right)x + \\ + \left(\alpha^2\theta^2 + 2\alpha^2\theta + 2\alpha\theta^2 + 2\alpha\theta + 2\theta^2 + \alpha^2\right)\end{array}\right]} = 0,$$

where $\bar{x}$ is the sample mean.

These log-likelihood equations do not seem to be directly solvable because the solutions cannot be expressed in closed form. The MLEs $(\hat{\theta}, \hat{\alpha})$ of parameters (θ, α) can be computed by solving the log-likelihood equation using the Newton-Raphson iterations available in R until values sufficiently close to $(\hat{\theta}, \hat{\alpha})$ are obtained.

5. Applications

The QPAD is fit to three real count datasets from the biological sciences to test its goodness of fit compared to that of the PD, PLD, and PAD. The maximum likelihood method is used to fit PD, PLD, PAD, and QPAD. The first dataset is Student's historic data on Haemocytometer counts of yeast cells available in *Gosset* [1908], the second dataset is the number of European corn borers from *Mc. Guire–Brindley–Bancroft* [1957], and the third dataset is the number of red mites on apple leaves available in *Fisher–Corpet-Williams* [1943]. The fitted values from these distributions for the datasets, shown in Tables 1, 2, and 3, are plotted in Figure 3. It is clear from the goodness of fit of the QPAD and from the fitted value plots of the distributions that the QPAD provides a much closer fit than those from the PD, PLD, and PAD. Hence, it can be considered as an important distribution for modelling data from the biological sciences.

Table 1

Observed and expected numbers of Haemocytometer yeast cells per square

Number of yeast cells per square	Observed frequency	Expected frequency			
		PD	PLD	PAD	QPAD
0	213	202.1	234.0	231.1	214.4
1	128	138.0	99.4	101.8	122.5
2	37	47.1	40.5	41.5	44.8
3	18	10.7	16.0	16.0	13.4
4	3	1.8	6.2	5.9	3.5
5	1	0.2	2.4	2.1	0.9
6	0	0.1	1.5	1.6	0.5
Total	*400*	*400.0*	*400.0*	*400.0*	*400.0*
MLE		$\hat{\theta}=0.6825$	$\hat{\theta}=1.9502$	$\hat{\theta}=2.4526$	$\hat{\theta}=4.7421$ $\hat{\alpha}=-0.2460$
χ^2		10.08	11.04	9.08	2.29
d*f*		2	2	2	1
p-value		0.0065	0.0040	0.0106	0.1244

Note. Here and hereinafter, PD: Poisson distribution; PLD: Poisson-Lindley distribution; PAD: Poisson-Aradhana distribution; QPAD: quasi Poisson-Aradhana distribution; MLE: maximum likelihood estimate.

Source: *Gosset* [1908].

Table 2

Observed and expected numbers of European corn borers in corn

Number of corn borers per plant	Observed frequency	Expected frequency			
		PD	PLD	PAD	QPAD
0	188	169.4	194.0	192.0	186.5
1	83	109.8	79.5	81.4	88.1
2	36	35.6	31.3	32.0	33.3
3	14	7.8	12.0	11.9	11.1
4	2	1.2	4.5	4.2	3.5
5	1	0.2	2.7	2.5	1.5
Total	*324*	*324.0*	*324.0*	*324.0*	*324.0*
MLE		$\hat{\theta}=0.6481$	$\hat{\theta}=2.0432$	$\hat{\theta}=2.5605$	$\hat{\theta}=3.4932$ $\hat{\alpha}=0.8901$
χ^2		15.19	1.29	0.72	0.58
d*f*		2	2	2	1
p-value		0.0005	0.5247	0.6872	0.4502

Source: *McGuire–Brindley–Bancroft* [1957].

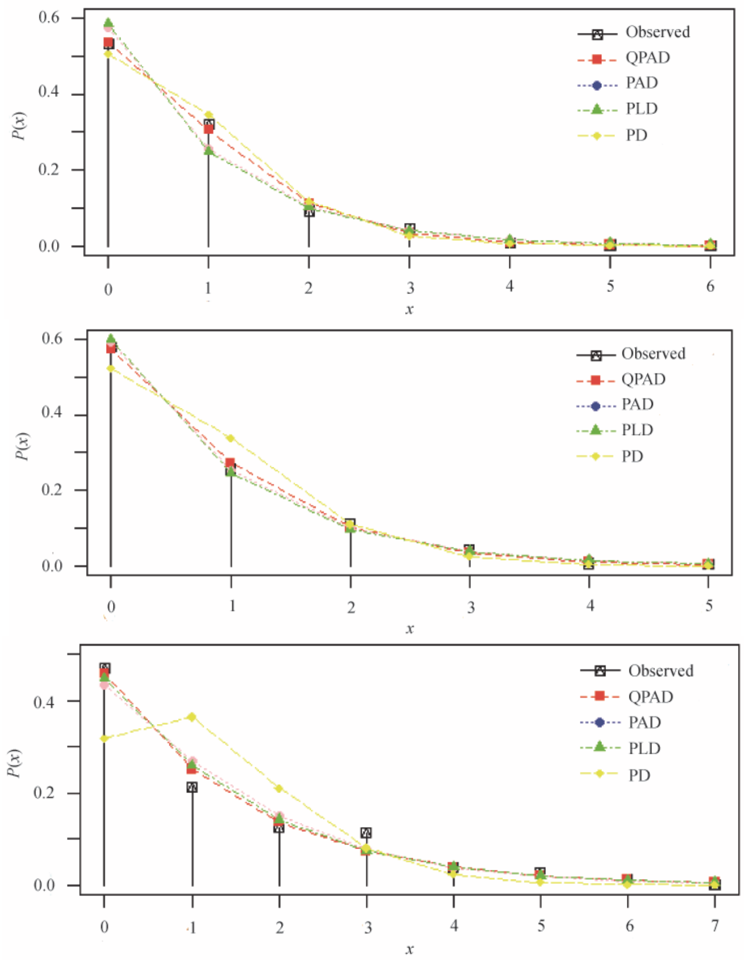
Table 3

Distribution of the number of red mites on apple leaves

Number of red mites per leaf	Observed frequency	Expected frequency			
		PD	PLD	PAD	QPAD
0	38	25.3	35.8	34.7	36.7
1	17	29.1	20.7	21.4	20.1
2	10	16.7	11.4	11.9	10.9
3	9	6.4	6.0	6.1	5.8
4	3	1.8	3.1	3.0	3.1
5	2	0.4	1.6	1.4	1.6
6	1	0.2	0.8	0.7	0.8
7+	0	0.1	0.6	0.8	1.0
Total	*80*	*80.0*	*80.0*	*80.0*	*80.0*
MLE		$\hat{\theta}=1.1500$	$\hat{\theta}=1.2559$	$\hat{\theta}=1.6637$	$\hat{\theta}=0.5806$ $\hat{\alpha}=-5.4363$
χ^2		18.27	2.47	2.90	2.40
d*f*		2	3	3	2
p-value		0.0001	0.4807	0.4073	0.3011

Source: *Fisher–Corpet–Williams* [1943].

Figure 3. Fitted value plots of the distributions for the datasets in Tables 1, 2, and 3



6. Conclusions

In this study, a QPAD by compounding a PD with a QAD has been proposed, and raw and central moments have been derived. The behaviours of the coefficient of variation, skewness, kurtosis, and index of dispersion have been studied for different values of the parameters. The QPAD has been shown to be unimodal and always over-dispersed. The estimation of its parameters using the method of maximum likelihood has been discussed. Finally, the goodness of fit of the QPAD has been discussed with three real count datasets and the fit has been compared with that of the PD, PLD, and PAD. It has been found that the goodness of fit of the QPAD is quite satisfactory compared to that of the other three distributions.

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