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# SOME PROPERTIES OF CERTAIN MEROMORPHIC MULTIVALENT CLOSE-TO-CONVEX FUNCTIONS

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*Abstract.* In this paper, we introduce and investigate a certain subclass of meromorphic multivalent close-to-convex functions. Such results as coefficient inequalities, and radius of meromorphic convexity are derived.

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## 1. INTRODUCTION

Let $\sum_p$ denote the class of functions f of the form

$$f(z) = \frac{1}{z^p} + \sum_{n=0}^{\infty} a_n z^n (p \in \mathbb{N} := \{1,2,3\cdots\}), \tag{1.1}$$

which are analytic in the punctured open unit disk

$$\mathbb{U}^* := \{z : z \in \mathbb{C} \mathrm{and} 0 < |z| < 1\} =: \mathbb{U} \setminus 0.$$

Let $\mathcal{P}$ denote the class of functions p given by

$$p(z) = 1 + \sum_{n=1}^{\infty} b_n z^n (z \in \mathbb{U}), \tag{1.2}$$

which are analytic and convex in $\mathbb{U}$ and satisfy the condition $\mathbb{R}(p(z)) > 0 (z \in \mathbb{U})$.

A function $f \in \sum_p$ is said to be in the class $\mathcal{MS}_p^*(\alpha)$ of meromorphic multivalent starlike functions of order $\alpha$ if it satisfies the inequality

$$\mathbb{R}\left(\frac{zf'(z)}{f(z)}\right) < -\alpha (z \in \mathbb{U}, 0 \le \alpha < p).$$

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Let

$$g(z) = \frac{1}{z^p} + \sum_{n=0}^{\infty} c_n z^n \in \mathcal{MS}_p^*(\alpha), \tag{1.3}$$

a function $f \in \sum_p$ is said to be in the class $\mathcal{MC}_p$ of meromorphic multivalent close-to-convex functions if it satisfies the inequality

$$\mathbb{R}\Big(\frac{zf^{'}(z)}{g(z)}\Big) < 0 (z \in \mathbb{U}, g \in \mathcal{MS}_p^*(0) =: \mathcal{MS}_p^*).$$

In many earlier investigations(for example [2,3,5–7,10–12,15–21,24]), various interesting subclasses of the close-to-convex functions have been studied from a number of different viewpoints. In particular, Gao and Zhou[3](see also [7, 10, 21, 24]) considered a subclass $\mathcal{K}_s$ of close-to-convex functions, which satisfy the condition

$$\mathbb{R}\Big(\frac{z^2 f^{'}(z)}{g(z)g(-z)}\Big) < 0 (z \in \mathbb{U}, g \in \mathcal{S}^*(1/2)),$$

where $f(z) = z + a_2 z^2 + \cdots$, and $\mathcal{S}^*(1/2)$ denotes the usual class of starlike functions of order $1/2$.

Recently, Z.G.Wang et al.[22] introduced the meromophic close-to-convex functions class $\mathcal{MK}$, which satisfy the condition

$$\mathbb{R}\Big(\frac{f^{'}(z)}{g(z)g(-z)}\Big) > 0 (z \in \mathbb{U}, g \in \mathcal{MS}^*(p/2)),$$

where $f(z) = 1/z + a_1 z + a_2 z^2 + \cdots$.

Motivated essentially by the above mentioned works, we introduce a class of meoromorphic multivalent functions related to the meoromorphic multivalent starlike functions, and obtain some interesting results.

**Definition 1.** A function $f \in \sum_p$ is said to be in the $\mathcal{MK}_p$ if it satisfies the inequality

$$\mathbb{R}\Big(\frac{zf^{'}(z)}{z^p g(z)g(-z)}\Big) > 0 (z \in \mathbb{U}), \tag{1.4}$$

where $g \in \mathcal{MS}_p^*(p/2)$.

For some recent investigation of meromorphic multivalent functions, see (for example)the works of [1, 8, 9, 13, 23, 25] and the references cited therein.

In the present paper, we prove that the class $\mathcal{MK}_p$ is a subclass of meromorphic multivalent close-to-convex functions.

**Theorem 1.** *Suppose that* $\mu(z) \in \mathcal{MS}_p^*(\alpha_1)$ *and* $\nu(z) \in \mathcal{MS}_p^*(\alpha_2)$ *with* $0 \le \alpha_1 + \alpha_2 - p < p$. *Then*

$$z^p \mu(z)\nu(z) \in \mathcal{MS}_p^*(\alpha_1 + \alpha_2 - p).$$

*Proof of Theorem 1.* Let $\mu(z) \in \mathcal{MS}_p^*(\alpha_1)$ and $\nu(z) \in \mathcal{MS}_p^*(\alpha_2)$. By definition, we know that

$$\mathbb{R}\left(\frac{z\mu^{'}(z)}{\mu(z)}\right) < -\alpha_1 (z \in \mathbb{U}, 0 \leq \alpha_1 < p),$$

and

$$\mathbb{R}\left(\frac{z\nu^{'}(z)}{\nu(z)}\right) < -\alpha_2 (z \in \mathbb{U}, 0 \leq \alpha_2 < p).$$

Next, we assume that

$$h(z) = z^p \mu(z)\nu(z).$$

Then, we easily get

$$\frac{zh^{'}(z)}{h(z)} = \frac{z\mu^{'}(z)}{\mu(z)} + \frac{z\nu^{'}(z)}{\nu(z)} + p.$$

It follows that

$$\mathbb{R}\left(\frac{zh^{'}(z)}{h(z)}\right) = \mathbb{R}\left(\frac{z\mu^{'}(z)}{\mu(z)}\right) + \mathbb{R}\left(\frac{z\nu^{'}(z)}{\nu(z)}\right) + p < -(\alpha_1 + \alpha_2 - p).$$

Noting that $0 \leq \alpha_1 + \alpha_2 - p < p$, which implies that

$$h(z) \in \mathcal{MS}_p^*(\alpha_1 + \alpha_2 - p).$$

This completes the proof of Theorem 1. $\square$

**Theorem 2.** *Let $g \in \mathcal{MS}_p^*(p/2)$. Then*

$$-z^p g(z)g(-z) \in \mathcal{MS}_p^*(0) =: \mathcal{MS}_p^*.$$

*Proof of Theorem 2.* Similar to the proof of Theorem 1, we can get

$$\mathbb{R}\left(\frac{z(-z^p g(z)g(-z))^{'}}{-z^p g(z)g(-z)}\right) = p + \mathbb{R}\left(\frac{zg^{'}(z)}{g(z)}\right) + \mathbb{R}\left(\frac{-zg^{'}(-z)}{g(-z)}\right) < p - \frac{p}{2} - \frac{p}{2} = 0$$

(noting that $-z \in \mathbb{U}$). This implies the Theorem 2. $\square$

In view of the definitions $\mathcal{MC}_p$ , $\mathcal{MK}_p$ and Theorem 2, we deduce that the class $\mathcal{MK}_p$ is a subclass of the class $\mathcal{MC}_p$ of meromorphic close-to-convex functions.

To derive coefficient inequalities of $f \in \mathcal{MK}_p$, we need consider the parity of $p$. First, we consider the case that $p$ is odd.

## 2. THE CASE P=2K-1

In order to prove our main results, we need the following two lemmas.

**Lemma 1** ([14]). *Suppose that* $h(z) = \frac{1}{z^p} + \sum_{n=0}^{\infty} c_n z^n \in \mathcal{MS}_p^*$. *Then*

$$|c_n| \leq (2p)/(n+p)(n \in \mathbb{N} := \{1,2,3\cdots\}).$$

*Equality holds for the function* $h(z) = z^{-p}(1+z^{n+p})^{2p/(n+p)}$.

**Lemma 2.** *Let* $g(z) = \frac{1}{z^p} + \sum_{n=0}^{\infty} b_n z^n \in \mathcal{MS}_p^*(p/2)$. *Then*

$$|B_{2m-1}| \leq (2p)/(2m-1+p)(m \in \mathbb{N}),$$

*where*

$$B_{2m-1} = \begin{cases} 2b_{2m-1}, & 2m-1 < p, \\ 2b_{2m-1} - 2b_0 b_{2m-1-p} + 2b_1 b_{2m-2-p} - \cdots + & \\ (-1)^{m-\frac{p+1}{2}} 2b_{m-\frac{p+3}{2}} b_{m-\frac{p-1}{2}} + (-1)^{m-\frac{p-1}{2}} b^2_{m-\frac{p+1}{2}}, & 2m-1 \geq p. \end{cases} \tag{2.1}$$

*Equality holds for the function* $g(z) = z^{-p}(1+z^{n+p})^{p/(n+p)}$.

*Proof.* Suppose that

$$G(z) := -z^p g(z) g(-z). \tag{2.2}$$

In view of Theorem 2, we know that $G(z) \in \mathcal{MS}_p^*$. When $p = 2k-1(k = 1,2,\cdots)$, it is easy to verify that

$$G(-z) = -G(z),$$

which implies that $G(z)$ is a meromorphic odd starlike multivalent function. If we set

$$G(z) = \frac{1}{z^p} + \sum_{m=1}^{\infty} B_{2m-1} z^{2m-1}, \tag{2.3}$$

it follows from Lemma 1 that

$$|B_{2m-1}| \leq (2p)/(2m-1+p)(m \in \mathbb{N}). \tag{2.4}$$

By substituting the series expressions of $g(z)$ and $G(z)$ into ( 2.2) and carefully comparing the similar items of two sides of resulting equation, we get the desired expression of $B_{2m-1}$ given by ( 2.1). □

**Theorem 3.** *Suppose that* $f(z) = \frac{1}{z^p} + \sum_{n=0}^{\infty} a_n z^n \in \mathcal{MK}_p$. *Then*

$$|a_{2n}| \leq \begin{cases} p/n, & 2n-3 < p, \\ \frac{p}{n}\left(1 + \frac{2p}{n-1} + \frac{2p}{n-2} + \cdots + \frac{2p}{1+p}\right), & 2n-3 \geq p, \end{cases} (n \in \mathbb{N}) \tag{2.5}$$

*and*

$$(2n-1)|a_{2n-1}| \le \begin{cases} 2p+\frac{2p^2}{(2n-1+p)}, & 2n-3<p, \\ 2p+\frac{2p^2}{n-1}+\frac{2p^2}{n-2}+\cdots+\frac{2p^2}{1+p}+\frac{2p^2}{2n-1+p}, & 2n-3\ge p. \end{cases} \tag{2.6}$$

*Proof of Theorem 3.* Suppose that $f\in\mathcal{MK}_p$. Then, we know that $\mathbb{R}\left(\frac{zf'(z)}{G(z)}\right)<0$, where $G$ is given by (2.2). If we set

$$q(z) := -\frac{zf'(z)}{pG(z)}, \tag{2.7}$$

it follows that

$$q(z) = 1+d_{p+1}z^{p+1}+d_{p+2}z^{p+2}+\cdots\in\mathcal{P}.$$

By substituting the series expressions of $f, G$ and $q$ into (2.7), we get

$$p(1+d_{p+1}z^{p+1}+\cdots+d_{p+n}z^{p+n}+\cdots)\left(\frac{1}{z^p}+B_1z+B_3z^3+\cdots+B_{2n-1}z^{2n-1}+\cdots\right)$$

$$=\frac{p}{z^p}-a_1z-2a_2z^2-\cdots-2na_{2n}z^{2n}-(2n+1)a_{2n+1}z^{2n+1}-\cdots. \tag{2.8}$$

We get from (2.8) that

$$\frac{-2n}{p}a_{2n} = \begin{cases} d_{p+2n}, & 2n-3<p, \\ d_{p+2n}+d_{p+2}B_{2n-2-p}+\cdots+d_{2n-1}B_1, & 2n-3\ge p, \end{cases} \tag{2.9}$$

and

$$\frac{2n-1}{-p}a_{2n-1} = \begin{cases} d_{p+2n-1}+B_{2n-1}, & 2n-3<p, \\ d_{p+2n-1}+B_{2n-1}+d_{p+1}B_{2n-p-2} & \\ +d_{p+2}B_{2n-p-3}+\cdots+d_{2n-2}B_1, & 2n-3\ge p. \end{cases} \tag{2.10}$$

For $q(z)\in\mathcal{P}$, we know that $|d_{n+p}|\le 2$ ([4]). Moreover, combining (2.4), (2.8), (2.9) and (2.10), we get (2.5) and (2.6). $\square$

**Theorem 4.** *Let $g(z)=\frac{1}{z^p}+\sum_{n=0}^{\infty}b_nz^n\in\mathcal{MS}_p^*(p/2)$. If $f\in\sum_p$ satisfies condition*

$$\sum_{n=1}^{\infty}n|a_n|+p\sum_{n=1}^{\infty}|B_{2n-1}|\le p, \tag{2.11}$$

*where $B_{2n-1}$ is given by (2.1), then $f\in\mathcal{MK}_p$.*

*Proof of Theorem 4.* To prove $f\in\mathcal{MK}_p$, it needs to show that

$$\mathbb{R}\left(\frac{f'(z)}{g(z)g(-z)}\right)=\mathbb{R}\left(\frac{zf'(z)}{G(z)}\right)>0,$$

i,e, it suffices to show that

$$\left|\frac{zf'(z)}{G(z)}+p\right| < \left|\frac{zf'(z)}{G(z)}-p\right|,$$

where $G$ is given by ( 2.3). From ( 2.11), it is easy to know that

$$\sum_{n=1}^{\infty} n|a_n| + p\sum_{n=1}^{\infty}|B_{2n-1}| \le 2p - \sum_{n=1}^{\infty} n|a_n| - p\sum_{n=1}^{\infty}|B_{2n-1}|. \tag{2.12}$$

Now, by the maximum principle, we deduce from ( 1.1) and ( 2.12) that

$$\left|\frac{\frac{zf'(z)}{G(z)}+p}{\frac{zf'(z)}{G(z)}-p}\right| = \left|\frac{\sum_{n=1}^{\infty} na_n z^{n+p} + \sum_{n=1}^{\infty} pB_{2n-1}z^{2n+p-1}}{\sum_{n=1}^{\infty} na_n z^{n+p} - \sum_{n=1}^{\infty} pB_{2n-1}z^{2n+p-1} - 2p}\right|$$
$$< \frac{\sum_{n=1}^{\infty} n|a_n| + \sum_{n=1}^{\infty} p|B_{2n-1}|}{2p - \sum_{n=1}^{\infty} n|a_n| - \sum_{n=1}^{\infty} p|B_{2n-1}|} \le 1.$$

This evidently complete proof of Theorem 4. □

Moreover, we consider the case that $p$ is even.

## 3. THE CASE P=2K

By similarly applying the method of proof of Lemma 3, we easily get the following Lemma.

**Lemma 3.** *Let* $p = 2k, k \in \mathbb{N}$ *and*

$$g(z) = \frac{1}{z^p} + \sum_{n=0}^{\infty} b_n z^n \in \mathcal{MS}_p^*(p/2).$$

*Then*

$$|B_{2m}| \le (2p)/(2m+p)(m \in \mathbb{N}), \tag{3.1}$$

*where*

$$B_{2m} = \begin{cases} 2b_{2m}, & 2m < p, \\ 2b_{2m} + 2b_0 b_{2m-p} - 2b_1 b_{2m-1-p} + \cdots + & \\ (-1)^{m-\frac{p+2}{2}} 2b_{m-\frac{p+2}{2}} b_{m-\frac{p-2}{2}} + (-1)^{m-\frac{p}{2}} b_{m-\frac{p}{2}}^2, & 2m \ge p. \end{cases} \tag{3.2}$$

*Equality holds for the function* $h(z) = z^{-p}(1+z^{n+p})^{p/(n+p)}$.

**Theorem 5.** *Suppose that*

$$f(z) = \frac{1}{z^p} + \sum_{n=0}^{\infty} a_n z^n \in \mathcal{MK}_p.$$

*Then*

$$|a_{2n}| \leq \begin{cases} p/n + p^2/(2n^2+np), & 2n < p, \\ \frac{p}{n}\left(1+\frac{2p}{n-1}+\frac{2p}{n-2}+\cdots+\frac{2p}{1+p}+2\right)+\frac{p^2}{2n^2+np}, & 2n \geq p, \end{cases} (n \in \mathbb{N}) \tag{3.3}$$

*and*

$$(2n-1)|a_{2n-1}| \leq \begin{cases} 2p, & 2n-2 < p, \\ 2p+\frac{2p^2}{n-1}+\frac{2p^2}{n-2}+\cdots+\frac{2p^2}{1+p}, & 2n-2 \geq p. \end{cases} \tag{3.4}$$

*Proof of Theorem 5.* Suppose that $f \in \mathcal{MK}_p$. Then, we know that $\mathbb{R}\left(\frac{zf'(z)}{G(z)}\right) < 0$, where $G$ is given by ( 2.2). For $p = 2k, k \in \mathbb{N}$ , it is easy to deduce that $G(z)$ is a meromorphic even starlike multivalent function. If we set

$$G(z) = \frac{1}{z^p} + \sum_{m=0}^{\infty} B_{2m} z^{2m}, \tag{3.5}$$

where $B_{2m}$ is defined by (3.2) and

$$\tau(z) := -\frac{zf'(z)}{pG(z)}, \tag{3.6}$$

it follows that

$$\tau(z) = 1 + d_p z^p + d_{p+1} z^{p+1} + \cdots \in \mathcal{P}.$$

By substituting the series expressions of $f, G$ and $\tau$ into (3.6), we get

$$p(1+d_p z^p + d_{p+1} z^{p+1} + \cdots + d_{p+n} z^{p+n} + \cdots)\left(\frac{1}{z^p} + B_0 + B_2 z^2 + \cdots + B_{2n} z^{2n} + \cdots\right)$$

$$= \frac{p}{z^p} - a_1 z - 2a_2 z^2 - \cdots - 2n a_{2n} z^{2n} - (2n+1) a_{2n+1} z^{2n+1} - \cdots . \tag{3.7}$$

We get from (3.7) that

$$\frac{-2n}{p} a_{2n} = \begin{cases} d_{p+2n} + B_{2n}, & 2n < p, \\ d_{p+2n} + B_{2n} + d_{p+2} B_{2n-2-p} + \cdots + d_{2n} B_0, & 2n \geq p, \end{cases} \tag{3.8}$$

and

$$\frac{2n-1}{-p} a_{2n-1} = \begin{cases} d_{p+2n-1}, & 2n-2 < p, \\ d_{p+2n-1} + d_{p+1} B_{2n-p-2} + \cdots + d_{2n-2} B_1, & 2n-2 \geq p. \end{cases} \tag{3.9}$$

For $\tau(z) \in \mathcal{P}$, we know that $|d_{n+p}| \leq 2$(see [4]). Moreover, combining (3.1),(3.7),(3.8) and (3.9), we get (3.3) and (3.4). □

**Theorem 6.** *If $f \in \sum_p$ satisfies condition*

$$\sum_{n=1}^{\infty} n|a_n| + p \sum_{n=0}^{\infty} |B_{2n}| \leq p, \tag{3.10}$$

*where $B_{2n}$ is given by (3.2), then $f \in \mathcal{MK}_p$.*

*Proof of Theorem 6.* The proof of Theorem 6 is similar to Theorem 4, we here omit the details. □

## 4. On the convexity radius of the functions in $\mathcal{MK}_p$

We say a function $f(z) \in \mathcal{MK}_p$ is meromorphic convex, if $f(z)$ satisfies condition:

$$\mathbb{R}\Big(1+\frac{zf''(z)}{f'(z)}\Big) < 0(z \in \mathbb{U}).$$

When we give the convexity radius of the functions in $\mathcal{MK}_p$ , we need the following lemmas.

**Lemma 4.** *Let $G(z)$ is given by (2.2) and $r < 1$, then*

$$\mathbb{R}\Big(\frac{zG'(z)}{G(z)}\Big) \le -\frac{1-r^2}{1+r^2}p(|z|=r).$$

*Proof.* Suppose that

$$H(z) := -\frac{zG'(z)}{pG(z)}(G(z) \in \mathcal{MS}_p^*), \tag{4.1}$$

where $G(z)$ is given by (2.2), we easily know that $G(z)$ is an odd or even meromorphic starlike function, also $H(z) \in \mathcal{P}$ and is an even function, which imply that

$$H(z) = \frac{1+[w(z)]^2}{1-[w(z)]^2}, \tag{4.2}$$

where $w(z)$ is Schwarz function with $w(0) = 0$ and $|w(z)| < 1$. Thus, we get from (4.2) that

$$[w(z)]^2 = \frac{H(z)-1}{H(z)+1}.$$

So

$$\Big|\frac{H(z)-1}{H(z)+1}\Big| = |w(z)|^2 \le |z|^2,$$

this inequality can be written as

$$|H(z)|^2 - 2Re\{H(z)\}+1 \le |z|^4\{|H(z)|^2+2Re\{H(z)\}+1\}.$$

From above inequality we can get

$$\Big|H(z)-\frac{1+|z|^4}{1-|z|^4}\Big|^2 \le \Big(\frac{1+|z|^4}{1-|z|^4}\Big)^2 - 1 \le \Big(\frac{2|z|^2}{1-|z|^4}\Big)^2,$$

that is

$$\left|H(z)-\frac{1+|z|^4}{1-|z|^4}\right|\leq\frac{2|z|^2}{1-|z|^4}.$$

From this inequality we get

$$\mathbb{R}\{-H(z)\}\leq-\frac{1-|z|^2}{1+|z|^2}=-\frac{1-r^2}{1+r^2},$$

this implies Lamma 4. □

**Lemma 5** (see [3]). *Let $q(z)$ satisfy $q(0)=1,\mathbb{R}\{q(z)\}>0$, then we have*

$$\left|\frac{zq^{'}(z)}{q(z)}\right|\leq\frac{2r}{1-r^2}(|z|=r<1).$$

**Theorem 7.** *Let $f(z)\in\mathcal{MK}_p$, then $f(z)$ is meromorphic convex in*

$$0<|z|<r_p=\frac{1}{2}\sqrt{4+\frac{1}{p^2}}-\frac{\sqrt{\frac{1}{\sqrt{4+\frac{1}{p^2}}p^3}+\frac{1}{p^2}+\frac{4}{\sqrt{4+\frac{1}{p^2}}p}}}{\sqrt{2}}+\frac{1}{2p}. \tag{4.3}$$

*Proof of Theorem 7.* When $f(z)\in\mathcal{MK}_p$, there exists $g(z)\in\mathcal{MS}_p^*(p/2)$ such that ( 1.4) holds, also $G(z)=-z^pg(z)g(-z)$ is an odd or even meromophic starlike multivalent function, so from (2.7) and (3.5) we have

$$zf'(z)=-pG(z)\cdot q(z),$$

where $q(z)$ satisfies the condition of Lemma 5, and

$$1+\frac{zf^{''}(z)}{f'(z)}=\frac{zG^{'}(z)}{G(z)}+\frac{zq^{'}(z)}{q(z)}.$$

So using Lemma 4 and 5 we can get

$$\begin{aligned}\mathbb{R}\left\{1+\frac{zf^{''}(z)}{f'(z)}\right\}=\mathbb{R}\left\{\frac{zG^{'}(z)}{G(z)}\right\}+\mathbb{R}\left\{\frac{zq^{'}(z)}{q(z)}\right\}&\\ \leq-\frac{1-r^2}{1+r^2}p+\left|\frac{zq^{'}(z)}{q(z)}\right|&\\ \leq-\frac{1-r^2}{1+r^2}p+\frac{2r}{1-r^2}=\frac{-pr^4+2r^3+2pr^2+2r-p}{1-r^4}&.\end{aligned}$$

It is easy to know that if $-pr^4+2r^3+2pr^2+2r-p<0$, we have $\mathbb{R}\left(1+\frac{zf^{''}(z)}{f'(z)}\right)<0$. Let

$$T_p(r)=-pr^4+2r^3+2pr^2+2r-p<0,$$

because $T_p(0) = -p < 0, T_p(1) = 4$, and

$$T_p'(r) = -4pr^3 + 6r^2 + 4pr + 2 = 4pr(1-r^2) + 6r^2 + 2 > 0(0 < r < 1).$$

It follows that $T_p(r)$ are strictly monotone increasing functions of $r$, and for very $p$, equation $T_p(r) = 0$ has only a root $r_p$ in interval (0, 1), solve those equations we get the $r_p$ in (4.3). Thus when $0 < |z| < r_p$, $\mathbb{R}\left(1 + \frac{zf''(z)}{f'(z)}\right) < 0$, that is, $f(z)$ is meromophic convex in $0 < |z| < r_p$. □

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