

**EFFICIENT ESTIMATE OF THE REMAINDER FOR THE
DIRICHLET FUNCTION $\eta(p)$ FOR $p \in \mathbb{R}^+$**

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Abstract. For any $k, n, q \in \mathbb{N}$, where $n \geq 2k + 1 \geq 3$, and $p \in \mathbb{R}^+$ the n th remainder $\rho(n, p)$ of the Dirichlet eta function $\eta(p)$, known also as the alternating Riemann zeta function,

$$\eta(p) := \sum_{i=1}^{\infty} (-1)^{i+1} \frac{1}{i^p},$$

is given in the form

$$\rho(n, p) = \Delta_q(n, p) + \delta_q(k, p),$$

where k and q are parameters controlling the magnitude of the error term $\delta_q(k, p)$. The function $\Delta_q(n, p)$ consists of $\lfloor q/2 \rfloor + 1$ simple summands and $\delta_q(k, n, p)$ is estimated as

$$|\delta_q(k, p)| < \frac{5}{3} \cdot \frac{p^{(q-1)}}{\pi^{q-1}} \cdot \frac{1}{(2k)^{p+q-1}},$$

where $p^{(q-1)}$ is the rising Pochhammer product.

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1. INTRODUCTION

To obtain the numerical values of the sequence $n \mapsto H^*(n, \frac{1}{1000})$,

$$\begin{aligned} H^*(n, \frac{1}{1000}) &:= \sum_{j=1}^n (-1)^{j+1} \frac{1}{j^{1/1000}} \\ &= 1.0000 \dots - 0.9993 \dots + 0.9989 \dots - 0.9986 \dots \\ &\quad + 0.9983 - 0.9982 \dots + \dots + \frac{(-1)^{n+1}}{n^{1/1000}} \quad (n \in \mathbb{N}), \end{aligned}$$

the Dirichlet eta function,¹

$$\eta(p) := \sum_{j=1}^{\infty} (-1)^{j+1} \frac{1}{j^p} \quad (p > 0),$$

is not a very useful tool although $H^*(n, \frac{1}{1000}) \approx \lim_{n \rightarrow \infty} H^*(n, \frac{1}{1000}) = \eta(\frac{1}{1000}) \approx \lim_{p \downarrow 0} \eta(p) = \frac{1}{2}$ [1, Proposition 5.3]. The approximation $H^*(n, \frac{1}{1000}) \approx \eta(\frac{1}{1000})$ is good only for “very large” n due to slow convergence of $H^*(n, \frac{1}{1000})$ illustrated² in Figure 1.

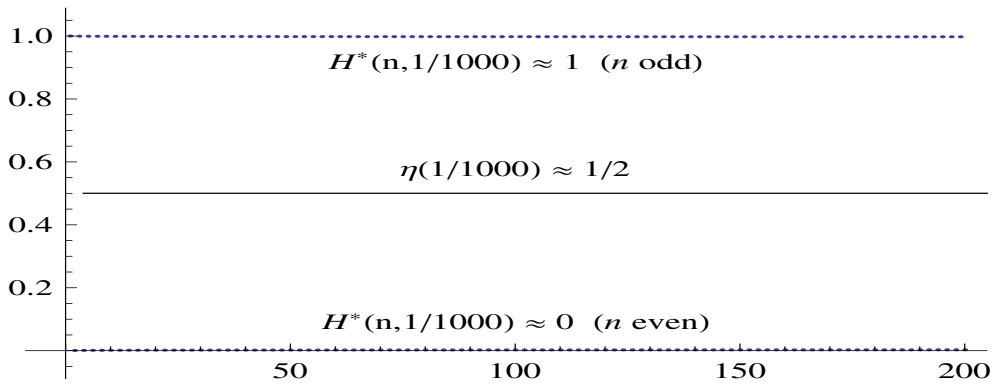


FIGURE 1. The graph of the sequence $n \mapsto H^*(n, \frac{1}{1000})$.

We also remark that for any (fixed) $n \in \mathbb{N}$ we have

$$\lim_{p \downarrow 0} \sum_{j=1}^n \frac{(-1)^{j+1}}{j^p} = \lim_{p \downarrow 0} \begin{cases} 1 - \sum_{i=1}^{(n-1)/2} \left(\frac{1}{(2i)^p} - \frac{1}{(2i+1)^p} \right), & n \text{ odd} \\ \sum_{i=1}^{n/2} \left(\frac{1}{(2i-1)^p} - \frac{1}{(2i)^p} \right), & n \text{ even} \end{cases} = \begin{cases} 1, & n \text{ odd} \\ 0, & n \text{ even.} \end{cases}$$

Motivated by the results obtained by L. Tóth – J. Bukor [5], V. Timofte [4], and the Sîntămărian’s papers [2, 3], we shall accurately estimate the remainder

$$\rho(n, p) := \eta(p) - H^*(n, p) \quad (n \in \mathbb{N}, p \in \mathbb{R}^+), \tag{1.1}$$

where

$$H^*(n, p) := \sum_{j=1}^n (-1)^{j+1} \frac{1}{j^p} \tag{1.2}$$

is called the n th alternating generalized harmonic number of order p .

¹known also as the alternating Riemann zeta function

²In this article all figures are plotted by Stephen Wolfram’s Mathematica Software, Version 7.0.

2. AUXILIARY RESULTS

In the sequel we shall use special sums³

$$\sigma_q^*(x, p) := \sum_{i=1}^{\lfloor q/2 \rfloor} (4^i - 1) \frac{B_{2i} \cdot p^{(2i-1)}}{x^{p+2i-1} \cdot (2i)!} \quad (q \in \mathbb{N}, p, x \in \mathbb{R}), \quad (2.1)$$

where B_k is the k th Bernoulli coefficient (or Bernoulli number)⁴, defined by the identity $\frac{t}{e^t-1} \equiv \sum_{k=0}^{\infty} B_k \frac{t^k}{k!}$ ($|t| < 2\pi$). Here, for $x \in \mathbb{R}$ and $k \in \mathbb{N}_0 := \{0\} \cup \mathbb{N}$ the symbol $x^{(k)}$ means the upper (rising) Pochhammer product defined as

$$x^{(0)} := 1 \quad \text{and} \quad x^{(k)} := \prod_{i=0}^{k-1} (x+i) = x(x+1) \cdot \dots \cdot (x+k-1) \quad (k \in \mathbb{N}). \quad (2.2)$$

According to [1, Theorem 3.2] there holds the following lemma.

Lemma 1. For $p \in \mathbb{R}^+$ and for any $k, n, q \in \mathbb{N}$, where $n \geq 2k + 1 \geq 3$, we have

$$H^*(n, p) = H_q^*(k, n, p) + r_q^*(k, p), \quad (2.3)$$

where

$$H_q^*(k, n, p) := A_q^*(k, p) + B_q^*(n, p), \quad (2.4)$$

$$A_q^*(k, p) := H^*(2k-1, p) - \frac{1}{2(2k)^p} - \sigma_q^*(2k, p), \quad (2.5)$$

$$B_q^*(n, p) := \frac{(-1)^{n+1}}{2(2\lfloor \frac{n+1}{2} \rfloor)^p} + \sigma_q^*(2\lfloor \frac{n+1}{2} \rfloor, p), \quad (2.6)$$

$$|r_q^*(k, p)| \leq + \frac{p^{(q-1)} \cdot \pi^{p+1}}{3(2k\pi)^{p+q-1}} \quad (2.7)$$

$$\cdot \quad (2.8)$$

Referring to [1, Theorem 5.2] we have also the next lemma.

Lemma 2. For $p \in \mathbb{R}^+$ and for any $k, q \in \mathbb{N}$, there holds the following relation (see (2.1)),

$$\eta(p) = H^*(2k-1, p) - \frac{1}{2(2k)^p} - \sigma_q^*(2k, p) + \varepsilon_q(k, p), \quad (2.9)$$

where

$$|\varepsilon_q(k, p)| < \frac{p^{(q-1)}}{3\pi^{q-2}(2k)^{p+q-1}}. \quad (2.10)$$

³By definition, $\sum_{i=m}^n x_i = 0$ if $m > n$.

⁴ $B_1 = -\frac{1}{2}, B_2 = \frac{1}{6}, B_3 = B_5 = B_7 = \dots = 0, B_4 = B_8 = -\frac{1}{30}, B_6 = \frac{1}{42}, \dots$

3. THE MAIN RESULT

From Lemmas 2 and 1 immediately follows the next result.

Theorem 1. For $p \in \mathbb{R}^+$ and every $k, n, q \in \mathbb{N}$, with $n \geq 2k + 1 \geq 3$, the n th remainder $\rho(n, p) := \eta(p) - \sum_{j=1}^n (-1)^{j+1} \frac{1}{j^p}$ is given in the form

$$\rho(n, p) = \Delta_q(n, p) + \delta_q(k, p), \tag{3.1}$$

where

$$\Delta_q(n, p) = \frac{(-1)^n}{2(2\lfloor \frac{n+1}{2} \rfloor)^p} - \sigma_q^*(2\lfloor \frac{n+1}{2} \rfloor, p), \tag{3.2}$$

$$|\delta_q(k, p)| < \frac{5p^{(q-1)}}{3\pi^{q-1}} \cdot \frac{1}{(2k)^{p+q-1}}. \tag{3.3}$$

Proof. We have

$$\begin{aligned} \rho(n, p) &= \eta(p) - H^*(n, p) \\ &= \left(\frac{H^*(2k-1, p)}{2(2k)^p} - \frac{1}{2(2k)^p} - \sigma_q^*(2k, p) + \varepsilon_q(k, p) \right) \\ &\quad - \left(\frac{H^*(2k-1, p)}{2(2k)^p} - \frac{1}{2(2k)^p} - \sigma_q^*(2k, p) + \frac{(-1)^{n+1}}{2(2\lfloor \frac{n+1}{2} \rfloor)^p} \right. \\ &\quad \left. + \sigma_q^*(2\lfloor \frac{n+1}{2} \rfloor, p) + r_q^*(k, p) \right) \\ &= \varepsilon_q(k, p) - \frac{(-1)^{n+1}}{2(2\lfloor \frac{n+1}{2} \rfloor)^p} - \sigma_q^*(2\lfloor \frac{n+1}{2} \rfloor, p) - r_q^*(k, p), \end{aligned}$$

where

$$\begin{aligned} |\delta_q(k, p)| &= |\varepsilon_q(k, p) - r_q^*(k, p)| \\ &< \frac{p^{(q-1)}}{3\pi^{q-2}(2k)^{p+q-1}} + \frac{p^{(q-1)}}{3(2k\pi)^{p+q-1}} \\ &= \frac{p^{(q-1)}}{3\pi^{q-2}(2k)^{p+q-1}} \left(1 + \pi^{-(p+1)} \right) \\ &< \frac{p^{(q-1)}}{3\pi^{q-2}(2k)^{p+q-1}} \cdot \frac{5}{\pi}. \quad \square \end{aligned}$$

In our Theorem k and q are parameters controlling the magnitude of the error term $\delta_q(k, p) = \varepsilon_q(k, p) - r_q^*(k, p)$. We add that, for $0 < p \leq 1$ and $q \in \mathbb{N}$, using the Stirling factorial formula, we estimate

$$p^{(q-1)} = p(p+1) \cdots (p+q-2) \leq (q-1)! = \frac{q!}{q} < \frac{1}{q} \left(\frac{q}{e} \right)^q \sqrt{2\pi q} \cdot e^{\frac{1}{12q}}.$$

Thus, for $n \geq 2k + 1 \geq 3$ and $0 < p \leq 1$, we obtain the estimates

$$\begin{aligned} |\delta_q(k, p)| &< \frac{5p^{(q-1)}}{3\pi^{q-1}} \cdot \frac{1}{(2k)^{p+q-1}} < \frac{5}{3} \sqrt{\frac{2\pi}{q}} \left(\frac{q}{e}\right)^q e^{\frac{1}{12q}} \cdot \frac{1}{(2\pi k)^{q-1}} \\ &= \frac{5}{3e} \sqrt{2\pi q} \left(\frac{q}{2e\pi k}\right)^{q-1} e^{\frac{1}{12q}} < 2\sqrt{q} \left(\frac{q}{2e\pi k}\right)^{q-1}. \end{aligned} \tag{3.4}$$

Figure 2 shows the graph of the sequence $n \mapsto \Delta_5(n, 1/1000)$.

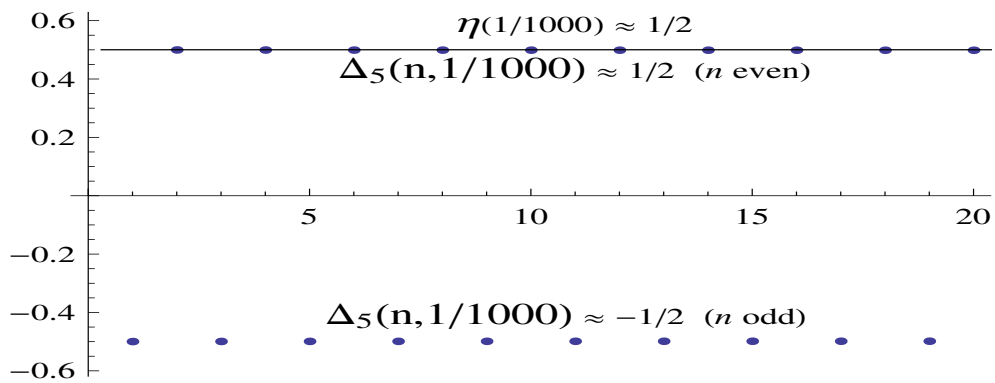


FIGURE 2. The graph of the sequence $n \mapsto \Delta_5\left(n, \frac{1}{1000}\right)$.

Since $\sigma_1(x, p) \equiv 0$, we obtain from Theorem the first corollary.

Corollary 1. For $p \in \mathbb{R}^+$ and any $k, n \in \mathbb{N}$, with $n \geq 2k + 1 \geq 3$, we have

$$\begin{aligned} \sum_{j=1}^n (-1)^{j+1} \frac{1}{j^p} + \frac{(-1)^n}{2(2\lfloor \frac{n+1}{2} \rfloor)^p} - \frac{5}{3(2k)^p} \\ < \eta(p) < \sum_{j=1}^n (-1)^{j+1} \frac{1}{j^p} + \frac{(-1)^n}{2(2\lfloor \frac{n+1}{2} \rfloor)^p} + \frac{5}{3(2k)^p}. \end{aligned}$$

Setting $q = 5$ in Theorem we obtain the following result.

Corollary 2. For $p \in \mathbb{R}^+$ and $k, n \in \mathbb{N}$, such that $n \geq 2k + 1 \geq 3$, the following equation follows

$$\begin{aligned} \eta(p) = \sum_{j=1}^n (-1)^{j+1} \frac{1}{j^p} + \frac{(-1)^n}{2(2\lfloor \frac{n+1}{2} \rfloor)^p} - \frac{p}{4(2\lfloor \frac{n+1}{2} \rfloor)^{p+1}} \\ + \frac{p(p+1)(p+2)}{48(2\lfloor \frac{n+1}{2} \rfloor)^{p+3}} + \delta_5(k, p), \end{aligned}$$

where

$$|\delta_5(k, p)| < \frac{5p(p+1)(p+2)(p+3)}{3\pi^4(2k)^{p+4}}.$$

Example 1. Since $|\delta_5(10, 0.001)| < 10^{-9}$ the approximation

$$\eta(0.001) \approx S_n^* := \sum_{j=1}^n (-1)^{j+1} \frac{1}{j^{0.001}} + \frac{(-1)^n}{2 \left(2^{\lfloor \frac{n+1}{2} \rfloor}\right)^{0.001}} - \frac{0.001}{4 \left(2^{\lfloor \frac{n+1}{2} \rfloor}\right)^{1.001}} + \frac{0.001 \times 1.001 \times 2.001}{48 \left(2^{\lfloor \frac{n+1}{2} \rfloor}\right)^{3.001}}$$

is very accurate for every $n \geq 21$ as is seen from Figure 3 where are plotted the graphs of the sequence $n \mapsto S_n^*$ and the constant function $n \mapsto \eta(0.001) = H^*(\infty, 0.001) = 0.5002257\dots$. The corrected term $\Delta_5(n, 0.001)$ is crucial since, for example, we have

$H^*(100000, 0.001) = 0.005949215\dots$, $H^*(100001, 0.001) = 0.994502300\dots$, $\Delta_5(100000, 0.001) = 0.494276544\dots$ and $\Delta_5(100001, 0.001) = -0.494276539\dots$

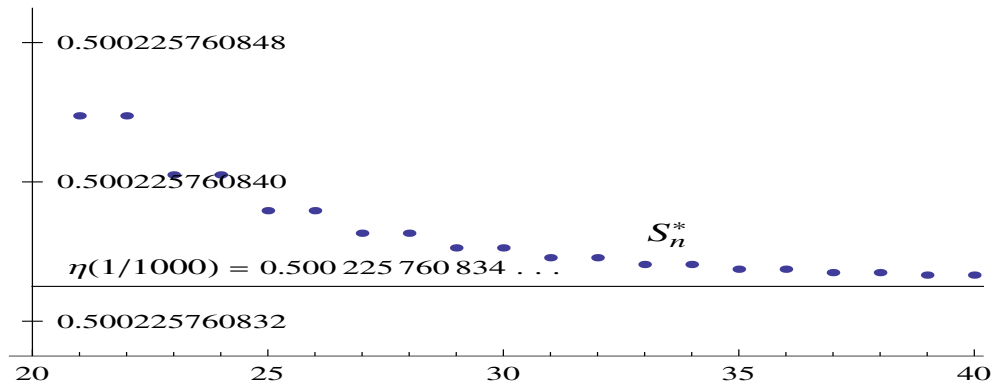


FIGURE 3. The graph of the sequence $n \mapsto S_n^*$.

Obviously, using the approximation $\eta(0.001) \approx S_n^*$ we directly obtain accurate approximation of $H^*\left(n, \frac{1}{1000}\right)$ from Introduction:

$$H^*\left(n, \frac{1}{1000}\right) \approx \eta(0.001) + \frac{(-1)^{n+1}}{2 \left(2^{\lfloor \frac{n+1}{2} \rfloor}\right)^{0.001}} + \frac{0.001}{4 \left(2^{\lfloor \frac{n+1}{2} \rfloor}\right)^{1.001}} - \frac{0.001 \times 1.001 \times 2.001}{48 \left(2^{\lfloor \frac{n+1}{2} \rfloor}\right)^{3.001}} \quad (n \in \mathbb{N}).$$

CONCLUSION

The paper provides an estimate of the remainder of the series representing Dirichlet's function $\eta(p)$ for $p \in \mathbb{R}^+$. The presented estimate is effective also in the case of slowly convergent series occurring for $p \approx 0$.

REFERENCES

- [1] V. Lampret, “Efficient approximations of finite and infinite real alternating p -series.” *J. Nonlinear Sci. Appl.*, vol. 11, no. 11, pp. 1250–1261, 2018, doi: <http://dx.doi.org/10.22436/jnsa.011.11.05>.
- [2] A. Sîntămărian, “A new proof for estimating the remainder of the alternating harmonic series.” *Creat. Math. Inform.*, vol. 21, no. 2, pp. 221–225, 2012.
- [3] A. Sîntămărian, “Sharp estimates regarding the remainder of the alternating harmonic series.” *Math. Inequal. Appl.*, vol. 18, no. 1, pp. 347–352, 2015, doi: [10.7153/mia-18-24](https://doi.org/10.7153/mia-18-24).
- [4] V. Timofte, “On Leibniz series defined by convex functions.” *J. Math. Anal. Appl.*, vol. 300, no. 1, pp. 160–171, 2004, doi: [10.1016/j.jmaa.2004.06.038](https://doi.org/10.1016/j.jmaa.2004.06.038).
- [5] L. Tóth and J. Bukor, “On the alternating series $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \dots$.” *J. Math. Anal. Appl.*, vol. 282, no. 1, pp. 21–25, 2003, doi: [10.1016/S0022-247X\(02\)00344-X](https://doi.org/10.1016/S0022-247X(02)00344-X).

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