SASTRA Ramanujan Prize for this achievement and other results. See [27] for the latest results on the bounded gap problem. In the last few months, Ford-Green-Konyagin-Tao [24] and Maynard [26] have announced a solution to the Erdős $10,000 problem by showing that the constant in Rankin's lower bound can be made arbitrarily large. The methods in [24] and [26] are different.

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László Lovász and Vera T. Sós

Erdős Centennial

It would be impossible to discuss the tremendous work of Paul Erdős especially in such a short article. All we can do is to contribute some impressions and ideas about the nature of his work, flavored by some quotations from letters of Erdős and by a few personal impressions and experiences. Several "mathematical," and not only mathematical, biographies of Erdős were written, among these we mention here two thorough ones written by Babai [2] and Bollobás [6]. His work was treated in depth in a number of volumes containing expert articles [20], [21], [22], [24], and even on the pages of these *Notices* [4].

The idea of the present issue of the *Notices* arose in connection with the Erdős Centennial Conference we organized in summer 2013. To be precise, we organized three Paul Erdős conferences in Budapest: The first took place in 1996, one day after his funeral. At that one-day meeting, our goal was to give an immediate short survey of his oeuvre, a demonstration of his unique role in mathematics in the past seven decades. The second conference took place three years later, in 1999, when our primary aim was to cover as much as possible the full scope and richness of his mathematics and its impact. The third conference was in 2013 to celebrate the hundredth anniversary of his birth. The intention of this third one was to give a panorama of the monumental development originating in his mathematics, of the wide-ranging influence of his work, and to give some indication of possible trends in the future. The success of the conference surpassed all our expectations: more than twice as many mathematicians participated as we first expected, illustrating the tremendous interest in his work and its proceeds.

Courtesy of Vera Sós

Paul Erdős and Paul Turán were great mathematicians, close friends, and partners in several important collaborations.

Trying to collect a few thoughts about the character of Erdős's mathematics, our starting point could be what he wrote in a letter that included a scientific biography written by him in the late 1970s:

> To finish this short outline of my scientific biography, I observe that most of my papers contain some type of combinatorial reasoning and most of them contain unsolved problems.

Indeed, a special trait all across his work was his unparalleled power of formulating and posing problems and conjectures. He had a special sense for asking just the right questions: how else can we explain that many of his innocent-looking problems have opened up new areas, in some cases after several decades? He wrote the first "problem paper" in 1956 [9], which contained six problems. After more than half a century, in spite of the many important results and methods initiated by this paper, none of these six problems is completely solved.

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In one of his letters from 1979 he wrote:

> I am writing a paper with the title: 'Combinatorial problems I would most like to see solved'—the subjective title is better because then I do not have to write about my opinion on the importance of the problems.

While Erdős often just asked a problem in a very compact form without mentioning any reasons, these problems were not as spontaneous as it would seem. An example: In a letter to Paul Turán in 1938 he formulated a conjecture about the maximum number of k-element subsets of an n-element set with the property that any two of them intersect in at least r points. He mentioned that with the help of Ko he could prove the case $r = 1$, and he added the remark: "The theorem could have beautiful applications in number theory." The proof was published only in 1961 in a famous paper by Erdős, Ko, and Rado [15], but possible applications in number theory are not mentioned in the paper.

In its simplest form, the Erdős-Ko-Rado Theorem says that to get the largest number of k-subsets of an n-set ($n \geq 2k$) that mutually intersect, one should take all k-subsets containing a given element. When reading such a statement, one realizes that many similar problems can be raised about subsets of a finite set, and then, depending on one's temperament, one might escape, or one might be challenged by, the fact that such basic questions are unsolved. Luckily, Erdős and several others felt the challenge, and over a relatively short period a wealth of basic questions in extremal set theory were answered. The theorems of Sperner (about sets not containing each other), Erdős-de Bruijn (about sets, any two intersecting in exactly one element), Erdős-Rado (about sets among which no three mutually have the same intersection) and Kruskal-Katona (about k-sets covering the least number of r-sets) are not only standard theorems in combinatorics textbooks, but they have very important applications in geometry, number theory, computer science, and elsewhere. These problems, which arise in a very simple and natural way, are often quite difficult to solve, and in some cases a complete solution is still missing after decades of intensive research.

Another characteristic of his mathematics was that very often his questions and proofs reveal deep relationships between different areas in mathematics. Even though he was never directly involved in computer science, he had an essential influence on it, mostly through extremal set theory and the probabilistic method. These connections could not have been foreseen, except perhaps by Erdős himself. (About this aspect of his work see Babai [3].) Using his own words from the late seventies:

> I am basically a pure mathematician and had little contact with applied mathematics, I expect that my paper with Rényi on the evolution of random graphs will be used in several branches in science—Rényi planned to work in this direction but was prevented by his untimely death. Graham, Szemerédi and I [16] have a paper on problems raised by computer scientist but I am not competent enough to judge their importance for applications.

Erdős started out as a number theorist, and number theory remained present in his mathematics all the time. There are several survey articles dealing with his work on the theory of primes, equidistribution, diophantine approximation, additive and multiplicative number theory, and many more. Discovering the combinatorial nature of some of his early number theory problems led him to general questions in combinatorics and in graph theory. He writes in his above-quoted "scientific biography":

> My main subjects are: number theory (a subject which interested me since early childhood when I learned from my father Euclid's proof that the number of primes is infinite), combinatorial analysis, set theory, probability, geometry and various branches of analysis.

His work in set theory often arose from combinatorics as infinite versions of finite problems. His problems and results in geometry and algebra also have a combinatorial flavor. He was the driving force behind the development of large areas of modern combinatorics, including extremal graph theory and extremal set theory. Since combinatorics is the best-known area of his work (which is due, at least in part, to the fact that this was the focus of his work in his later years), we will not go into the details of these results.

Another area that Erdős introduced into several branches of mathematics is probability. The interaction with probability is a very hot topic in number theory, combinatorics, computer science, and other areas, and the pioneering work of Erdős is present all over this work. We could talk about four different ways in which he contributed to this field.

1. He studied problems in pure probability theory (often with a combinatorial flavor but belonging to mainstream probability), like random walks or the Law of Iterated Logarithm. As to this last work, let us quote Bollobás [6]: "There are very few people who have contributed more to the fundamental theorems in probability theory than Paul Erdős."

2. Starting with problems in number theory, he showed how to exploit the random-like behavior of different structures. Let's quote his own words [13]:

> Heuristic probability arguments can often be used to make plausible but often hopeless conjectures on primes and on other branches of number theory.

The deliberate and systematic application of probability theory to number theory started with the celebrated Erdős-Kac theorem [14]. For a detailed review of the story of this theorem, see the article of Alladi-Krantz in this issue of the *Notices*. Erdős himself wrote about the formation of the Erdős-Kac theorem several times; let's quote from [10]:

> I conjectured that the convergence of the three series is both necessary and sufficient for the existence of the distribution function (of an additive function) but this I could not prove due to my gaps of knowledge in Probability Theory.…After the lecture (of Kac) we got together…and thus with a little impudence we would say, that probabilistic number theory was born.

Elliott writes in his book [8] about this theorem:

> This result, of immediate appeal, was the archetype of many results to follow. It firmly established the application of the theory of probability to the study of fairly wide class of additive and multiplicative functions.

3. Perhaps most important of Erdős's achievements is the "probabilistic method," the use of probability to prove the existence of certain objects without explicitly constructing them (and whose explicit construction is sometimes still open sixty years later). This issue of the *Notices* contains other papers that describe this fundamental method and its applications, and we can also refer to the books of Alon and Spencer [1] and Erdős and Spencer [19].

4. The Erdős-Rényi theory of random graphs is the first major example of the investigation of random structures. To be precise, random sets of integers, random polynomials, random matrices, and other random structures were considered before by several mathematicians (including Erdős and Rényi themselves), but random graphs were the first where a comprehensive theory arose that showed how basic properties of these graphs are different from their deterministic counterparts. Several books have been written about random graphs [5], [23]. The Erdős-Rényi random graphs serve as basic examples in the recent explosion of random graph models for many real-life networks (like the Internet and social networks), where the understanding and explanation of the differences from this basic model is the main goal.

Paul Erdős with his mother, who travelled with him around the world until her death in 1971.

Analysis, in particular approximation, interpolation, polynomials, complex functions, and infinite series, were also in the foreground of his research from the thirties through the sixties. His analytic power can be felt in his papers all along. It is best to quote Paul Turán, who was an early collaborator of Erdős and wrote a detailed survey on Erdős's work on the occasion of his fiftieth birthday [27]. (This became an important source for many later articles on Erdős.) Out of the several topics in analysis which Turán discussed in this paper, let's quote what he wrote about the application of probability in analysis:

"The application of probabilistic methods runs right through the whole oeuvre of Erdős and this holds for his works in analysis as well. In this connection I have in my mind especially three of his papers, the first of which was published in 1956 in the *Proc. London Math. Soc.* with Offord [17], the second in 1959 in the *Michigan Math. J.* with Dvoretzky [7], the third will be published with Rényi in the volume to be issued to celebrate the 75th birthday of György Pólya [18]. In the first they showed that if $\varepsilon_\nu = \pm 1$, then the 2^n equations

$$1 + \varepsilon_1 x + \cdots + \varepsilon_n x^n = 0$$

have, with at most $o(2^n/\sqrt{\log\log n})$ exceptions,

$$\frac{2}{\pi}\log n + o\left(\log^{\frac{2}{3}} n \log\log n\right)$$

real roots each.

"The second gives an *existence proof* of the nice theorem that there exists a power series $\sum_0^\infty \frac{e^{i\alpha_n}}{\sqrt{n}} z^n$ with real α_n that diverges on the *whole* unit circle (that this can be achieved excluding a set

Courtesy of Vera Sós

(From l to r) George Graetzer, Paul Erdős, Paul Turán and Alfred Renyi.

of measure zero was known). In the third they solve an old problem of Zygmund in connection with a theorem of N. Wiener. This theorem of Wiener states (in a weakened form) that if the series

$$\sum_{\nu}(a_\nu \cos l_\nu x + b_\nu \sin l_\nu x),$$

where the l_ν's are positive integers satisfying

$$\lim_{\nu\to\infty}(l_{\nu+1} - l_\nu) = \infty$$

is Abel-summable in an arbitrarily small interval (a, b) to a function $f(x)$ belonging to L_2, when we have $\sum(a_\nu^2 + b_\nu^2) < \infty$, hence the series is the Fourier series of a function belonging to L_2 on, the whole $[0, 2\pi]$, thus $f(x)$ has an extension to $[0, 2\pi]$ that is in L_2 and whose Fourier series is the given series. Ingham, Zygmund and Marcinkiewicz and the author of these lines gave much simpler proofs of this theorem than the original; some twenty years ago Zygmund raised the question whether the theorem can be extended to a class L_q with $q > 2$ in the place of L_2. Now Erdős and Rényi with probabilistic methods showed for every $q > 2$ the *existence* of a trigonometric series satisfying the above lacunarity condition that is summable to a function continuous in (a, b) for every $0 < a < b < 2\pi$ and still the series is not the Fourier series of any function belonging to L_q on $[0, 2\pi]$."

Erdős was always very supportive of young people. In the 1960s, when the Cold War began to melt and he started to spend more time in Budapest, he would often sit in the lobby of his hotel all day, with students and young researchers coming and going, discussing their new results, and learning about new developments and new problems from all over the world. One of us (the first author) was lucky enough, as a high school student, to have the opportunity to stay there and take part in these discussions. The effect of these discussions on how to look at mathematics, research, colleagues, science, and the world has lasted a lifetime.

From this experience, and in general from the attitude of Erdős towards open problems, conjectures, dissemination of ideas and collaboration, his basic (probably unstated) philosophy can be distilled: he believed in total openness in research, where the goal is to advance knowledge, and we all work together to achieve it.

Let me (the second author) also mention my first and last meeting with Paul Erdős—the beginning and the end of almost fifty years of acquaintance and more than three decades of collaboration, partly in several hundreds of letters. I met Erdős the first time in 1948, when he returned to Hungary after a break of ten years. My high school teacher, Tibor Gallai, one of Erdős's best friends, introduced me to him. I cannot recall the particulars of our conversation, but I am sure he asked mathematical questions, as he usually did when meeting young people interested in mathematics. However, I remember that because of a long break his visit had a special significance. Let me say a few words about this.

Erdős and Gallai were members of the now legendary "Anonymus group."[1] The members of this group met regularly during their university years at the *Statue of Anonymus* in City Park in Budapest. Lifelong friendships were formed between them, and their meetings had a deep impact on their professional lives as well.

Arranged by Mordell, Erdős spent the years 1934–38 in Manchester. During this period he returned to Hungary quite regularly three times a year for shorter visits. In 1938 he decided to leave Hungary, with its adverse and deteriorating political situation. He had to leave his family, he had to leave his friends. Then came the war years; Erdős returned to Budapest only ten years later to see his mother and his friends. This was the occasion when, in December 1948, I met Erdős for the first time.

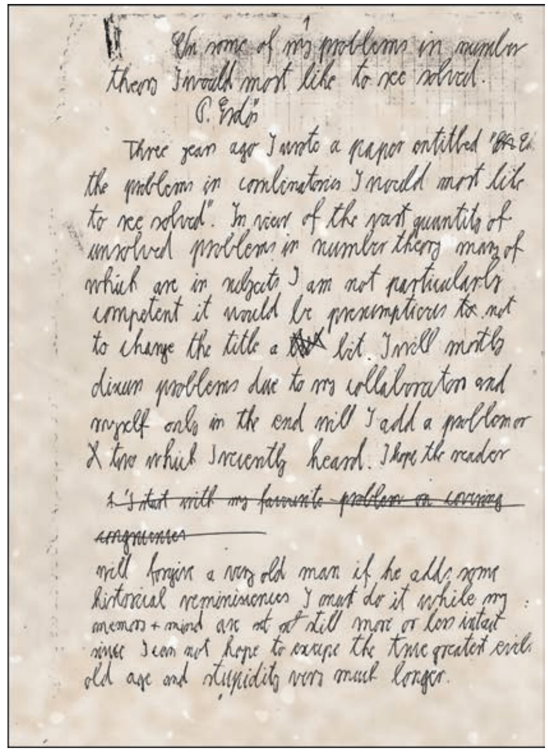
In September 1996 we both attended a graph theory conference in Warsaw. Our plan was to go from Warsaw, together with András Sárközy, to Vilnius to participate in a number theory conference the following week. On the morning of Wednesday, September 18, he gave his very last problem lecture. The last problem he mentioned was a problem of Hajnal (and perhaps himself). He got stuck, started again, and this was repeated two

[1] *László Alpár (1914–1991), Pál Erdős (1913–1996), János Erőds, (1912–1944), Ervin Feldheim (1912–1944), Géza Grünwald (1913–1944), Tibor Grünwald (Gallai) (1912–1992), Eszter Klein (1910–1975), Dezső Lázár (1913–1943), György Szekeres (1911–1975), Pál Turán (1910–1976), Márta Wachsberger (Sved) (1911–2005), Endre Weiszfeld (1913–1976).*

more times. After the third attempt, he put down the chalk and finished the talk. The audience broke out in applause, and he responded, "Thank you. I know this is meant as a consolation!" There was an excursion the same afternoon, which he skipped, partly because of the cold weather. Instead of that, the rest of the day became the last hours we spent together, switching between topics and problems perhaps more often than at other times. Paul Erdős passed away on Friday, September 20 [26].

Erdős's brilliant mathematical thinking, pure character, helpful and sympathizing nature; his quest for truth in science, politics, everyday life—these are what motivated his untiring, relentless activity and creativity until his last days. His personality is perhaps evoked by the simple lines he wrote one morning in 1976:

> It is six in the morning, the house is still asleep, I am listening to lovely music, while writing and conjecturing.[2]


On some of my problems in number theory I would most like to see solved.
P. Erdős
Three years ago I wrote a paper entitled "On the problems in combinatorics I would most like to see solved". In view of the vast quantity of unsolved problems in number theory many of which are in subjects I am not particularly competent it would be presumptious to not to change the title a bit. I will mostly discuss problems due to my collaborators and myself only in the end will I add a problem or two which I recently heard. I hope the reader ~~I start with my favorite problem on covering congruences~~ will forgive a very old man if he adds some historical reminiscences I must do it while my memory + mind are at at still more or less intact since I can not hope to escape the two greatest evils old age and stupidity very much longer.

Manuscript courtesy of Krishnaswami Alladi

Erdős contribution to the Proceedings of the Conference on Number Theory in his honor for his 70th birthday held in Ootacamund, India in January 1984, and referred to in the Lovász-Sós article.

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Ronald L. Graham and Joel Spencer

Ramsey Theory and the Probabilistic Method

Ramsey Theory was a lifelong interest of Paul Erdős. It began [11] in the winter of 1931–32. George Szekeres recalled:

> We had a very close circle of young mathematicians, foremost among them Erdős, Turán and Gallai; friendships were forged which became the most lasting that I have ever known and which outlived the upheavals of the thirties, a vicious world war and our scattering to the four corners of the world. I […] often joined the mathematicians at weekend excursions in the charming hill country around Budapest and (in the summer) at open air meetings on the benches of the city park.

Szekeres, Esther Klein, and Erdős attacked an unusual geometric problem: Is it true that for every k there exists an n so that given any n points in the plane, no three collinear, some k of them form a convex k-gon? Szekeres, in finding a proof of this conjecture, actually proved Ramsey's Theorem, which none of the three knew about at the time.

The mantra for Ramsey Theory is "Complete disorder is impossible." Let s, r, k be positive integers. Then, Ramsey showed, for n sufficiently large (dependent on s, r, k), the following holds: Let Ω have size n. Take *any* partition of the s-element subsets of Ω into r colors. Then there will be a k-element set $S \subset \Omega$ which is monochromatic, in the sense that *all* of its s-element subsets are the same color. In the important special case $s = 2$ one may think of an r-coloring of the edges of the complete graph K_n. While Erdős was not the originator of Ramsey Theory, he was its chief proponent, with conjectures and theorems in myriad directions that truly turned Ramsey's Theorem into Ramsey Theory.

A natural question arose: Just how big does n need to be? We'll restrict here to $s = 2$, though the other cases are also important. The Ramsey function $r(k)$ is the least n such that if the edges of the complete graph K_n are red/blue colored, then there will necessarily be a monochromatic K_k. The proof of Szekeres worked for $n = \binom{2k-2}{k-1}$ so that, thinking asymptotically, $r(k) < (4 + o(1))^k$. In 1947 Erdős published a three-page paper [3] in the *Bulletin of the AMS* that had a profound effect on both the Probabilistic Method and on Ramsey Theory.

Theorem. *Let n, k satisfy*

$$\binom{n}{k} 2^{1-\binom{k}{2}} < 1.$$

Then $r(k) > n$. That is, there exists *a two-coloring of the edges of K_n such that there is no monochromatic K_k.*

Today, for those in the area, the proof is two words: Color Randomly! Consider a random coloring of the edges. For each of the $\binom{n}{k}$ sets S of k vertices there is a probability 2^{1-m}, $m = \binom{k}{2}$, that the m edges are all colored the same. The probability of a disjunction is at most the sum of the probabilities, and so the disjunction has probability strictly less than one. Thus with positive probability the coloring is as desired. But (this part is sometimes called Erdős Magic) if there were no such coloring, then the probability would be zero, so, reversing, the coloring absolutely positively must exist.

Asymptotic analysis (from Erdős's paper) gives $r(k) > (\sqrt{2} + o(1))^k$. There have been some improvements in both the upper and lower bounds, most notably by David Conlon, but only for lower-order terms. The gap between $\sqrt{2}$ and 4 has not moved since 1947 and is a central question in the field.

In 1950 [7], with Richard Rado, Erdős began the area of canonical Ramsey Theory. Let S be an ordered set. They gave four special colorings of the pairs of S: They could all have the same color; they could all have different colors; the color of $\{x, y\}$ with $x < y$ could be different for different x and the same for the same x; the color of $\{x, y\}$ with $x < y$

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