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A NOTE ON PARADOXICAL METRIC SPACES (ANNOTATED VERSION, 2004)

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[1]

1. Introduction

Given a metric space, $(\mathcal{M}, d)$, we shall call a mapping $\varphi : \mathcal{M} \to \mathcal{M}$ **wobbling** if $d(x, \varphi(x))$ is bounded for $x \in \mathcal{M}$.

Such mappings were investigated by Laczkovich in his fundamental study on squaring the disk [6]. He considered sets which may be mapped into the regular grid by wobbling mappings.

The simple idea behind this concept is related also to physics and crystallography. Consider, for example, an amount of iron filings distributed in the plane to which an electrical field of finite energy is applied. The filings will move into an arranged position along the lines of the field. As long as the electrical field has small energy it is expected that no element is moved too far. Similarly, a faulty crystal can be imagined to be obtained from a regular crystal by moving certain elements by some small distance. Such mappings occur in many applications and may be treated in several ways [11].

In this note we outline some aspects of wobbling mappings in arbitrary metric spaces. The Banach-Tarski's theorem states that the unit ball in $\mathbb{R}^3$ may be decomposed into two parts which are piecewise congruent to the unit ball. We shall consider analogues of the Banach-Tarski's phenomenon in arbitrary metric spaces. We will characterize those metric spaces which may be decomposed into two parts, where both parts are equivalent to the whole metric space by a wobbling bijection.

[1]**Added in 2004:** This is a new, extended version of our earlier paper with the above title. This version is created solely for posting it on our homepage, to correct some of the minor misprints and add just a few remarks. The *NEW* notes, extensions, remarks are indicated either by being boxed or by putting them into footnotes. The first author, Walter Deuber died on ...., the last author gave a lecture in his memory and the G. Elek and the last author wrote a longer survey in his memory [17]. While writing that survey, we realized that posting such an annotated version of our paper on our homepage could be useful.

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2. Wobbling Equivalences

Definition 1. Let $(\mathcal{M}, d)$ be a metric space and $X, Y \subseteq M$. An injective mapping $\varphi : X \to Y$ is called k–**wobbling** if

$$\sup_{x \in X} d(\varphi(x), x) < k.$$

We call X and Y **wobbling equivalent** if for some k there is a k–wobbling bijection $\varphi : X \to Y$.

Copying the proof of the Cantor-Bernstein theorem one gets

Lemma 2.1. *Let X_1 and X_2 be subsets of a metric space $(\mathcal{M}, d)$ and φ_1 be a wobbling mapping of X_1 onto a subset $Y_2 \subseteq X_2$ and φ_2 be a wobbling mapping of X_2 onto a subset $Y_1 \subseteq X_1$. Then X_1 and X_2 are wobbling equivalent.*

Definition 2. For two sets $X, Y \subseteq \mathcal{M}$ the bipartite k–distance graph $G_k(X, Y)$ is the bipartite graph with colour classes X and Y, where $x \in X$ is joined to $y \in Y$ iff $d(x, y) < k$.

Clearly, X has a wobbling injection into Y if there exists a constant k such that $G_k(X, Y)$ contains a matching covering X. A metric space $(\mathcal{M}, d)$ is **discrete** if every bounded subset of $\mathcal{M}$ is finite. For discrete metric spaces the k–distance graphs $G_k(X, Y)$ are locally finite.

We denote by $N_k(Z)$ the k–neighborhood of a set Z in M.

Applying the Rado-Hall theorem [10] for matchings in countable locally finite bipartite graphs to the k–distance graphs above gives the following.

Claim 2.2. *Let $(\mathcal{M}, d)$ be a discrete countable metric space. Two subsets X, Y of $\mathcal{M}$ are wobbling–equivalent iff there exists a constant $k > 0$ such that*

(i) For every finite subset X' of X $|N_k(X') \cap Y| \geq |X'|$.

(ii) For every finite subset Y' of Y $|N_k(Y') \cap X| \geq |Y'|$.

Sets which are equivalent to $\mathbb{Z}^d$ are called **uniformly spread** [7]. In the geometric setup one can make the transition from "counting" as in Claim 2.4 to "measuring volumes": Let $X \subseteq \mathbb{R}$. To each $x \in X$ associate the unit cube with lower left corner in x:

$$C^d(x) = x + [0, 1)^d.$$

Heuristically one would say that if a set X is uniformly spread, then for every finite set $X' \subseteq X$ the cardinality $|X'|$ may be approximated by the volume $\lambda_d(\bigcup_{x \in X} C^d(x))$ where λ_d is the d–dimensional volume.

For a subset $X \subseteq \mathbb{R}^d$ of a $C \subseteq \mathbb{R}^d$, the quantity $||X \cap C| - \lambda_d(C)|$ is called the **discrepancy** of X relative to C and denoted by $\Delta(X, C)$.

To prove his famous result on "squaring the disk", Laczkovich proved the following [7].

THEOREM 2.3 (Laczkovich). *A subset X of $\mathbb{R}^d$ is equivalent to $\mathbb{Z}^d$ if there exists a constant L such that for every measurable set $C \subset \mathbb{R}^d$ the following holds:*

$$\Delta(X, Y) \leq L\lambda_d(N_1(\partial C)),$$

where ∂C denotes the boundary of C.

For $d = 2$ there is a variant of this theorem [7].

THEOREM 2.4 (Laczkovich). *A set $X \subset \mathbb{R}^2$ is equivalent to $\mathbb{Z}^2$ if there exists a constant L such that for every Jordan domain C of diameter at least 1*

$$\Delta(X, C) \leq L\lambda_1(\partial C)$$

holds.

As a corollary of the above theorem, one can easily show

COROLLARY 2.5. *(See also P. Pleasants* [9] *) Every Penrose tiling is equivalent to $\tau\mathbb{Z}^2$ for some $\tau \in \mathbb{R}$.*

Proof. By De Brujin's theorem [2] every Penrose tiling P is obtained as follows: There exists a 2 dimensional plane $E \subset \mathbb{R}^5$ and a constant ℓ such that the orthogonal projection Π from $\mathbb{R}^5$ to E satisfies

$$\prod(N_\ell(E) \cap \mathbb{Z}^5) = P.$$

It is easy to verify that $N_\ell(E) \cap \mathbb{Z}^5$ satisfies the discrepancy condition of the theorem of Laczkovich. Then the projection - which is injective in this case - is a wobbling mapping, since ℓ is fixed. ∎

In a metric space much less is known in general about wobbling equivalence.[2] Of course, there are general theorems guaranteeing the existence of injections such as the extensions of Hall's theorem by Michael Holz; Klaus Peter Podewski; Karsten Steffens [4]. It could well be that an application of these theorems gives new insight in the context of wobbling equivalences.

PROBLEM 1. *Characterize the sets which are wobbling equivalent to $\mathbb{Z}^2$.*

The same problem could be of interest for $X \subset \mathbb{Q}^2$.

3. PARADOXICAL SETS

DEFINITION 3. Two sets A, B in $\mathbb{R}^3$ are called **piecewise congruent** if there exist decompositions $A = A_1 \dot\cup \ldots \dot\cup A_n$, and $B = B_1 \dot\cup \ldots \dot\cup B_n$ such that each A_i is congruent to B_i.

In their classical paper Banach and Tarski [1], (see also Wagon [13]) proved that the unit ball B in $\mathbb{R}^3$ is paradoxical in the following sense: B can be decomposed into two disjoint sets B_1, B_2 so that B_1, B_2 and B are pairwise piecewise congruent. Whenever one has an equivalence relation on

[2]**Added in 2004:** see the survey of Elek and Sós [17] and also [15] for developments after we have published this paper.

the power-set of some set, one can define paradoxical sets. Here we define paradoxical sets only for the wobbling equivalence.

Definition 4. Let $(\mathcal{M}, d)$ be a metric space. $(\mathcal{M}, d)$ is **paradoxical** if there exists a decomposition $\mathcal{M} = M_1 \dot{\cup} M_2$ such that M_1, M_2 and $\mathcal{M}$ are pairwise wobbling equivalent.

Example 1. $\mathbb{R}^2$ is paradoxical. Take a checkerboard tiling of the plane. A translation moves the black tiles into the white ones. Moreover any single square is equivalent to a domino. This shows that $\mathbb{R}^2$ and the set of black tiles are equivalent.

[3]

Example 2. Let $\mathcal{M} = \{\log n \mid n \in \mathbb{N}\}$. Then $M_1 = \{\log(2n+1) \mid n \in \mathbb{N}\}$ and $M_2 = \{\log 2n \mid n \in \mathbb{N}\}$ show that $\mathcal{M}$ is paradoxical.

In order to characterize paradoxical sets (for wobbling equivalence) we introduce the following

Definition 5. Let $(\mathcal{M}, d)$ be a discrete metric space. $\mathcal{M}$ has **exponential growth rate** if

$(*)$ there exists a k (the doubling radius) such that for every finite set M' the k–neighborhood $N_k(M')$ contains at least $2 \cdot |M'|$ elements.

[4]

Remark. Obviously, the condition $(*)$ above is equivalent to that *for some fixed $q > 1$ there exists a k such that for every finite set M' the k–neighborhood $N_k(M')$ has at least qM' elements.*

Theorem 3.1. *Let $(\mathcal{M}, d)$ be a discrete countable metric space. Then the following are equivalent.*

(i) M is paradoxical.

(ii) M has exponential growth rate.

One should be aware that this theorem is not just a rewriting of definitions. To check exponential growth rate one has local tests: For every finite set one establishes the doubling radius. $\mathcal{M}$ is paradoxical if all these local doubling radii remain bounded. On the other hand, paradoxicity is a global property.

For the proof we need a variant of Hall's theorem.

[3]**Added in 2004:** $\mathbb{R}^d$ is not paradoxical for the isometries. This shows that being paradoxical depends very much on the equivalence relation (or, in some specific cases, on the group/family of mappings we consider).

[4]**Added in 2004:** Here we kept our original definition, though later we realized that this was slightly unfortunate since the expression **"exponential growth rate"** had been used slightly differently in the group theory literature, see [16], [17].

Definition 6. Let $G = (A, B)$ be a bipartite graph. A set E of edges is an (ℓ_1, ℓ_2)**-matching** if every vertex of A is contained in exactly ℓ_1 edges of E and every vertex of B is contained in exactly ℓ_2 edges of E.

We need the following.

Generalized Hall-Rado theorem. *Let $G = (A, B)$ be a countable locally finite bipartite graph. G contains an (ℓ_1, ℓ_2)-matching if the following two conditions are satisfied.*

(i) For every finite subset A' of A there are at least $\ell_1 \cdot |A'|$ neighbours in B.

(ii) For every finite subset B' of B there are at least $\ell_2 \cdot |B'|$ neighbours in A.

Originally, a misprint was here: we wrote "iff".

Proof of Theorem 3.1. Let $\mathcal{M}$ be paradoxical. Then there exists a $k \in \mathbb{R}$ such that for every finite subset M' of $\mathcal{M}$ the k–neighborhood $N_k(M')$ contains two disjoint sets of cardinality $|M'|$.

Indeed, let $\mathcal{M} = M_1 \dot{\cup} M_2$ be paradoxical decomposition with wobbling distance k. Then both $M_1 \cap N_k(M')$ and $M_2 \cap N_k(M')$ have at least $|M'|$ elements. Hence $\mathcal{M}$ has exponential growth rate.

To see the converse statement, observe that M is paradoxical iff for some k the k–distance graph $G_k(\mathcal{M}, \mathcal{M})$ contains a (2,1) matching. To ensure a (2,1)-matching, we use the condition of the generalized Hall-Rado theorem for $G_k(\mathcal{M}, \mathcal{M})$ with $(\ell_1, \ell_2) = (2, 1)$. The exponential growth rate implies the Hall condition for $G_k(\mathcal{M}, \mathcal{M})$ with $(2, 2)$, and therefore with $(2, 1)$ as well. ∎

4. Paradoxical Graphs

Any graph G can be regarded as a metric space, where the distance $d(x, y)$ is the length of the shortest path between x and y in G.

Problem 2. *When is an infinite graph G paradoxical?*

For trees this question can be answered easily. Let us call a path $P_k \subseteq G$ a **hanging chain** if all its inner vertices have degree 2 in G.

Theorem 4.1. *A locally finite infinite tree T without endvertices is paradoxical iff the lengths of hanging chains in T is bounded.*

Here it should be remarked that when a tree T is decomposed into 2 subsets wobbling equivalent with each other and with the whole tree, these subsets are not trees.

Corollary 4.2. *An infinite tree is paradoxical if its minimum degree is at least 3.*

Proof. Assume that T contains no hanging chain of K inner vertices (i.e. $K + 1$ edges). For any $S \subseteq V(T)$ we define ∂S as the set of vertices joined to S but not in S. We may apply our characterisation, Theorem 3.4 to T: the only thing to be proven is that if $S \subseteq V(T)$, then $|\partial S| \geq c_K|S|$. Indeed, let F_n be the forest induced by the set $S \cup \partial S$. We shall count the vertices of degree 1 in F_n since all they belong to ∂S. Let n_i be the set of vertices of degree i. The number of vertices of degree at least 3 is $n_{\geq 3} = n_3 + n_4 + \ldots$. Then (for any tree or forest) $n_1 \geq n_{\geq 3} + 2$. Further, $n - n_2 > n/K$. Indeed, fix a vertex w of degree 1 and map each x of degree 2 to the y for which x is on a hanging chain yy^* and y is farther from w than y^*. We get each y at most K times. Thus $n_1 = n - n_2 - n_{\geq 3} > n - n/K - n_1$ implying $|\partial S| > n/2K$. Hence T has exponential growth rate. ∎

One feels that in case of trees a directly constructed partition should also exist. One can easily provide the partition $V(T) = V_1 \dot{\cup} V_2$, e.g., if T is a tree of minimum degree 3.

Problem 3. *When is an infinite graph paradoxical? Is it true that if an infinite graph G is paradoxical, then there is an infinite spanning tree $T \subseteq G$ which is paradoxical?* [5]

5. Recursive Sets

Often one would like to ensure some extra properties of the (wobbling) mapping or of the parts in a paradoxical partition under the condition that the original sets have additional properties. From the point of view of mathematical logic, those things are interesting for us which can be generated by a Turing Machine. This motivates the problems below.

Problem 4. *Let $X \subseteq \mathbb{Z}^2$ be recursive and equivalent to $\mathbb{Z}^2$. Is there a recursive wobbling bijection $X \to \mathbb{Z}^2$ which is recursive?*

Problem 5. *Are there recursive paradoxical sets $\mathcal{M}$ in $\mathbb{Q}^2$ for which there is no recursive paradoxical decomposition $\mathcal{M} = M_1 \dot{\cup} M_2$?*

Remark. We do not think that there is a trivial positive answer. There are analogous situations with negative answers.

(a) There exists a recursive countable locally finite tree (i.e. the characteristic function of the edge set is recursive) which has no recursive infinite path [12].

(b) There exists a recursive k– regular bipartite graph $G(A, B)$ which has a 1-factor but has no recursive 1- factor. [8].

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[5] **Added in 2004:** The answer YES can be found in [14]

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