4. On a Conjecture of Roth and Some Related Problems I

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1. Introduction

Let $\mathcal{N}$ denote the set of positive integers and put $[1, N] = \{1, \ldots, N\}$. We use $|S|$ to denote the cardinality of the finite set S. If S is a given set and $\mathcal{A}_1, \ldots, \mathcal{A}_k$ are subsets of S with

$$S = \cup_{i=1}^{k} \mathcal{A}_i, \quad \mathcal{A}_i \cap \mathcal{A}_j = \emptyset \quad \text{for} \quad i \neq j,$$

then $\{\mathcal{A}_1, \ldots, \mathcal{A}_k\}$ will be called a k-partition (or k-colouring) of S, and the subsets $\mathcal{A}_1, \ldots, \mathcal{A}_k$ will be referred to as classes. Let $f : \mathcal{N}^t \to \mathcal{N}$ be a given function. If

$$n = f(a_1, \ldots, a_t) \tag{1}$$

with $a_1, \ldots, a_t$ belonging to the same class, then this will be called a *monochromatic* representation of n in the form (1)

For a fixed k-partition and f we consider the set of integers, which have a monochromatic representation and investigate

a) how dense this set must be?
b) for which $S \subseteq \mathcal{N}$ it must contain an element in S?
c) what sort of structural properties this set has?

We consider first the case $f(x_1, x_2) = x_1 + x_2$.

Let C resp. C^2 denote the set of integers resp. the set of even integers which have a monochromatic representation in the form

$$n = a_1 + a_2 \quad \text{with} \quad a_1 \neq a_2 \tag{2}$$

Put $C_M = C \cap [1, M]$ and $C_M^2 = C^2 \cap [1, M]$.

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K.F. Roth conjectured (see [4] and [9], p.112) that there is an absolute constant $c > 0$ such that for an arbitrary k-partition

$$| C_M | > cM. \tag{3}$$

(Note that if also $a_1 = a_2$ is allowed, then this is trivial.)

We prove this conjecture in a sharper and more general form. We study some related problems too.

The Case $f(x_1, x_2) = x_1 - x_2$

Theorem 1.

(i) *To every $k \geq 2$ there exists an $M_0(k)$ such that for an arbitrary k-partition of $\mathcal{N}$*

$$| C^2_M | > \frac{M}{2} - 3M^{1-2^{-k-1}} \quad \text{if} \quad M > M_0(k). \tag{4}$$

Moreover

(ii) *For every 2-partition*

$$| C^2_M | > \frac{M}{2} - \left(\log\left(\frac{1+\sqrt{5}}{2}\right)\right)^{-1} \log M \tag{5}$$

(iii) *There is a 2-partition so that*

$$2^n \notin C^2 \quad \text{for} \quad n \in \mathcal{N} \tag{6}$$

Proof.

(i) The proof will be based on the following

Lemma 1. *If $d \in \mathcal{N}$, $M > M_0(d)$, $\mathcal{B} \subseteq [1, M]$ and*

$$| \mathcal{B} | > 3M^{1-2^{-d}} \tag{7}$$

then there exist positive integers u, $v_1, \ldots, v_d$ such that $v_i \neq v_j$ for $i \neq j$ and all the 2^d sums

$$u + \sum_{i=1}^{d} \varepsilon_i v_i, \quad \varepsilon_i \in \{0, 1\} \tag{8}$$

belong to $\mathcal{B}$.

This is a density version of Hilbert's lemma [10] (which is considered as the first Ramsey-type result). See also [8]. It can be proved similarly to Lemma 7 in [14] (see also [3] and [20]). However for the sake of completeness, we give the proof here.

Proof of Lemma 1. It suffices to show the existence of sets $\mathcal{B}_0, \mathcal{B}_1, \ldots, \mathcal{B}_d$ and distinct positive integers $v_1, v_2, \ldots, v_d$ such that

$$\mathcal{B}_0 = \mathcal{B}, \tag{9}$$

$$\mathcal{B}_j \cup \{b + v_j : b \in \mathcal{B}_j\} \subset \mathcal{B}_{j-1} \quad \text{for} \quad j = 1, 2, \ldots, d \tag{10}$$

and

$$|\mathcal{B}_j| \geq |\mathcal{B}|^{2^j} (3M)^{-(2^j-1)} \quad \text{for} \quad j = 0, 1, 2, \ldots, d. \tag{11}$$

In fact, if $\mathcal{B}_0, \mathcal{B}_1, \ldots, \mathcal{B}_d$, $v_1, \ldots, v_d$ satisfy these conditions and $u \in \mathcal{B}_d$, then by (9) and (10), $u + \sum_{i=1}^{d} \varepsilon_i v_i \in \mathcal{B}$ for $\varepsilon_i = 0$ or 1, while (7) and (11) imply that $\mathcal{B}_d$ is not empty. This then will complete the proof of Lemma 1.

We are going to construct $\mathcal{B}_0, \mathcal{B}_1 \ldots, \mathcal{B}_d$, $v_1, \ldots, v_d$ recursively. Let $\mathcal{B}_0 = \mathcal{B}$. Assume now that $0 \leq j \leq d-1$ and, in the case $j > 0$, $v_1, \ldots, v_j$ have already been defined. For $1 \leq h \leq M-1$, let $f(\mathcal{B}_j, h)$ denote the number of solutions of

$$b - b' = h, \quad \text{where} \quad b, b' \in \mathcal{B}_j.$$

Then in order to define $\mathcal{B}_{j+1}$ and v_{j+1}, we need an estimate for

$$L = \max f(\mathcal{B}_j, h)$$

where the maximum is over all h with $h \in [1, M]$, $h \notin \{v_1, v_2, \ldots, v_j\}$.

Clearly, for all h we have $f(\mathcal{B}_j, h) \leq |\mathcal{B}_j|$. Also

$$\sum_{h=1}^{M-1} f(\mathcal{B}_j, h) = \binom{|\mathcal{B}_j|}{2} \tag{12}$$

since $b - b' \in [1, M]$ for any pair $b, b' \in \mathcal{B}_j$ with $b > b'$. If we majorize $f(\mathcal{B}_j, h)$ by $|\mathcal{B}_j|$ for $h \in \{v_1, v_2, \ldots, v_j\}$ and by L otherwise, (12) implies

$$\binom{|\mathcal{B}_j|}{2} \leq j\,|\mathcal{B}_j| + (M-1-j)L \leq j\,|\mathcal{B}_j| + LM,$$

so that

$$L > \frac{1}{2M}(|\mathcal{B}_j|^2 - |\mathcal{B}_j| - 2j\,|\mathcal{B}_j|) = \frac{|\mathcal{B}_j|}{3M}\left(\frac{3}{2}\,|\mathcal{B}_j| - \frac{3}{2} - 3j\right). \tag{13}$$

From (7) and (11), we have (for M larger than some absolute and computable constant)

$$|\mathcal{B}_j| \geq |\mathcal{B}|^{2^j} (3M)^{-(2^j-1)} > \left(3M^{1-2^{-d}}\right)^{2^j} (3M)^{-(2^j-1)} =$$

$$= 3M^{1-2^{j-d}} \geq 3M^{1-2^{-1}} > 3 + 6d > 3 + 6j,$$

so that (11) and (13) imply

$$L > \frac{|\mathcal{B}_j|}{3M} \cdot |\mathcal{B}_j| \geq \frac{1}{3M} \left(|\mathcal{B}|^{2^j} (3M)^{(2^j-1)}\right)^2 =$$

(14)

$$=|\mathcal{B}|^{2^{j+1}} (3M)^{-(+2^{j+1}-1)}.$$

Let $v_{j+1} \in [1, M] \setminus \{v_1, v_2, \dots, v_j\}$ denote an integer for which the maximum in the definition of L is attained, i.e., $L = f(\mathcal{B}_j, v_{j+1})$ with $v_{j+1} \notin \{v_1, v_2, \dots, v_j\}$, and let

$$\mathcal{B}_{j+1} = \{b : b \in \mathcal{B}_j,\ b + v_{j+1} \in \mathcal{B}_j\}.$$

Thus (10) holds for $j+1$ in place of j and since $|\mathcal{B}_{j+1}| = L$, (14) implies that (11) holds also for $j+1$ in place of j. This completes the proof of the existence of $\mathcal{B}_0, \mathcal{B}_1, \dots, \mathcal{B}_d,\ v_1, \dots, v_d$ with the desired properties, so that Lemma 1 is proved.

To prove the first statement in Theorem 1, we assume that there are more than $3M^{1-2^{-k-1}}$ even integers not exceeding M which do not have a monochromatic representation in the form (2); let us denote the set of these integers by $\mathcal{B}$. Then (3) holds with $k+1$ in place of d, thus if M is sufficiently large, then by Lemma 1 there exist positive integers $u, v_1, v_2, \dots, v_{k+1}$ such that all the sums

$$u + \sum_{i=1}^{k+1} \varepsilon_i v_i \quad \text{where} \quad \varepsilon_i = 0 \quad \text{or} \quad 1$$

belong to $\mathcal{B}$. Then

$$u = u + \sum_{i=1}^{k+1} 0 \cdot v_i \in \mathcal{B}$$

and since $\mathcal{B}$ consists of even numbers, thus also $u = 2z$ is even. The integers $z+v_1, z+v_2, \dots, z+v_{k+1}$ are distinct, thus by the pigeon hole principle, there exist $1 \leq i < j \leq k+1$ such that $a_1 = z + v_i$ and $a_2 = z + v_j$ belong to the same class. Then $a_1 + a_2$ is a monochromatic sum with $a_1 \neq a_2$, and

$$a_1 + a_2 = (z + v_i) + (z + v_j) = 2z + v_i + v_j = u + v_i + v_j$$

But this contradicts the definition of $\mathcal{B}$, and the proof of the first half of Theorem 1 is completed.

(ii) Let $\mathcal{B} = \{b_1, b_2, \ldots, b_t\}$ (where $b_1 < b_2 < \ldots < b_t$) denote the set of those even integers not exceeding $2M$ which do not have a monochromatic representation in the form (2).
Suppose

$$b_{j+2} < b_j + b_{j+1} \tag{15}$$

for some j. Then there are positive integers x, y, z for which

$$x + y = b_j$$
$$x + z = b_{j+1}$$
$$y + z = b_{j+2}$$

At least two of these numbers belong to the same class. This contradicts to the definition of $\mathcal{B}$. Hence for every j

$$b_{j+2} \geq b_j + b_{j+1} \tag{16}$$

which proves (ii)

To prove (iii) we define the set $\mathcal{A}_1$ recursively. Let $1 \in \mathcal{A}_1$. If $\mathcal{A} \cap [1, 2^{k-1}]$ has been defined, then let $2^k \in \mathcal{A}_1$ and for $2^{k-1} < n < 2^k$, $n \in \mathcal{A}_1$ iff $2^k - n \notin \mathcal{A}_1 \cap [1, 2^{k-1}]$. Furthermore let $\mathcal{A}_2 = \mathcal{N} \setminus \mathcal{A}_1$. Then obviously $2^n \notin \mathrm{C}$ for $n = 1, 2, \ldots$.

Observe that $| \mathrm{C}_M |$ need not be much greater then $| \mathrm{C}_M^2 |$ as the following example shows: $\mathcal{A}_1 = \{2j - 1 : j \in \mathcal{N}\}$, $\mathcal{A}_2 = \{2j : j \in \mathcal{N}\}$. However the situation is different for $k \leq 3$ and for $k \geq 4$.

Theorem 2.

(i) *There is an absolute constant C so that if $k \leq 3$ then at any k-partition*

$$| \mathrm{C}_M | \geq \left[\frac{M}{2} \right] - 1 \quad \text{if} \quad M > C. \tag{17}$$

(ii) *If $k \geq 4$, there exists a k-partition such that*

$$| \mathrm{C}_M | < \frac{M}{2} - ck \log M, \tag{18}$$

where c is an absolute constant.

Proof of (i). *Case $k = 2$.*

Without loss of generality we can assume that $x \in \mathcal{A}_1$ for $1 \leq x \leq a$ and $a + 1 \in \mathcal{A}_2$.

Then $y \in \mathrm{C}$ for $3 \leq y \leq 2a - 1$. On the other hand for every $y > 0$ either $y + a \in \mathrm{C}$ or $y + a + 1 \in \mathrm{C}$.

Case $k = 3$

Suppose $2x - 1 \in A_1$ if $1 \leq x \leq a$ and $2a + 1 \in A_2$. Then

(19) $$2y \in C \quad \text{if} \quad 2 \leq y \leq 2a.$$

We may assume that there is an $n > 2a$ such that

(20) $$2n \notin C \quad \text{and} \quad 2n - 1 \notin C$$

and

(21) $$| C_{2n} | < \left[\frac{M}{2}\right]$$

Case 1 $2n \leq 6a$. Put $2n = 4a + 2t$, $(t \leq a)$. First we prove

$$2a + 2 \in A_2.$$

Namely if $2a + 2 \in A_1$, then $2x - 1 + 2a + 2 \in C$ for $1 \leq x \leq a$. Hence

$$| C_{2n} | > 2a + a$$

which contradicts (21).

Now suppose $2a + 2 \in A_3$. Then $2n - (2a + 2) = 2a + 2t - 2 \in A_1 \cup A_2$. In case $2a + 2t - 2 \in A_1$

$$2x - 1 + 2a + 2t - 2 \in C \quad \text{for} \quad 1 \leq x \leq a.$$

This implies

$$C_{2n} \geq 2a + a$$

which contradicts again to (21).

In case $2a + 2t - 2 \in A_2$

$$2n - 1 = (2a + 1) + (2a + 2t - 2) \in C$$

would follow, which contradicts to (20).

Thus $2a + 2 \in A_2$.

Consider now the integers in $[2a + 2,\ 2a + 2t]$. For every y, $0 \leq y \leq 2t$

$$2a + y \in A_3 \quad \text{implies} \quad 2a + 2t - y \in A_3.$$

Therefore at least t integers in $[2a + 2,\ 2a + 2t]$ belong to $A_1 \cup A_2$.

If there is an even $x \in A_1 \cup [2a + 2,\ 2a + 2t]$, then

$$4a < x + 2v - 1 < 4a + 2t = n \quad \text{for} \quad 1 \leq v \leq a.$$

Hence

$$| C_{2n} | > 2a + a$$

which contradicts (21).

If all the t even integers in $[2a + 2,\ 2a + 2t]$ belong to A_2, then for $1 \leq u \leq t - 1$

$$(2a + 2 + 2u) + 2a + 1 \in C$$

and

$$(2a+2+2u)+2a+2 \in \mathrm{C}$$

This would imply

$$\mid \mathrm{C}_{2n} \mid > 3a.$$

This finishes the case when $2n \le 6a$.

Case 2 $2n > 6a$.

Since $2n \notin \mathrm{C}$, at least $\frac{n-2}{2}$ even numbers below $2n$ are in $\mathcal{A}_1 \cup \mathcal{A}_2$. Thus at least $\frac{n-2}{2} - a$ even numbers below $2n - 2a$ are in $\mathcal{A}_1 \cup \mathcal{A}_2$. Therefore at least $\frac{n-2}{4} - \frac{a}{2} > \frac{n-2}{12}$ are in $\mathcal{A}_1$ resp. in $\mathcal{A}_2$. Adding to these numbers $2a-1$ or $2a+1$ we gain $\frac{n-2}{12}$ odd numbers in C. Hence by Theorem 1

$$\mathrm{C}_{2n} > n - 6n^{\frac{15}{16}} + \frac{n-2}{12} > n \quad \text{if}$$

n is large enough.

Proof of (ii). We may suppose that $k = 4\ell$ where ℓ is odd. Define t_0 by

$$2^{t_0 - 1} \le 2\ell < 2^{t_0}$$

For $i = 1, 2, \ldots, \ell$ we are going to define subsets $\mathcal{A}_{4i-j}$, $j = 0,1,2,3$ recursively. Let for $j = 1, 3$

$$\mathcal{A}_{4i-j} \cap [1, 2^{t_0}] = \{n : n \equiv i \ (\mathrm{mod}\ \ell), n \equiv \left[\frac{j}{2}\right] \ (\mathrm{mod}\ 2)\} \cap [1, 2^{t_0}]$$

and

$$\mathcal{A}_{4i-j} \cap [1, 2^{t_0}] = \emptyset \quad \text{if} \quad j = 0, 2.$$

Assume now that $\mathcal{A}_{4i-j} \cap [1, 2^t]$ have been defined for $j = 0,1,2,3$, $i = 1, \ldots, 2\ell+1$. Let $r_i(t)$ defined by

$$2i \equiv 2^{t+1} + r_i(t) \ (\mathrm{mod}\ 2\ell), \quad 0 \le r_i(t) < 2\ell.$$

Now we define $\mathcal{A}_{4i-j} \cap \left[2^t + 1, 2^{t+1}\right]$ in the following way: let $2^t < n \le 2^{t+1}$. For $2^t < n < 2^{t+1}$ $n \in \mathcal{A}_{4i-3}$ iff n is even and

$$n \equiv i \ (\mathrm{mod}\ \ell), \quad 2^{t+1} + r_i(t) - n \notin \mathcal{A}_{4i-3} \cap [1, 2^t], 2 \mid n$$

$n \in \mathcal{A}_{4i-2}$ iff n is even and

$$n \equiv i \ (\mathrm{mod}\ \ell), \quad n \notin \mathcal{A}_{4i-3},$$

$n \in \mathcal{A}_{4i-1}$ iff n is odd and

$$n \equiv i \ (\mathrm{mod}\ \ell),\ 2^{t+1} + r_i(t) - n \notin \mathcal{A}_{4i-1} \cap [1, 2^t], 2 \mid n$$

$n \in \mathcal{A}_{4i}$ iff n is odd and

$$n \equiv i \ (\mathrm{mod}\ \ell),\ n \notin \mathcal{A}_{4i-1},$$

Then clearly the sets $\mathcal{A}_{4i-j}$, $1 \le i \le \ell$, $0 \le j \le 3$ give a 4ℓ-partition of $\mathcal{N}$. Furthermore it can be seen easily that all the monochromatic sums

$a_1 + a_2$, $a_1 \neq a_2$ are even and none of these sums is equal to a number of the form $2^t + 2^j$ where $t > t_0$ and $0 \leq j \leq \ell - 1$. This completes the proof of Theorem 2.

By Theorem 1, there are more than $\frac{M}{2} - c_1 M^{1-2^{k-1}}$ integers in $[1, M]$ which have a monochromatic representation in the form (2), and by Theorem 2, the number of these integers can be less than $\frac{M}{2} - c_2 k \log M$. It follows from a result of Erdős and Sárközy (Theorem 8 in [5]) that if $k \in \mathcal{N}$, $M \in \mathcal{N}$, $M > M_0(k)$, $t \in \mathcal{N}$ and $M^{2/3}(\log M)^2 < t \leq M$, then almost all the sets $\mathcal{B}$ with $\mathcal{B} \subset [1, M]$, $|\mathcal{B}| = t$ are such that for every k-partition of $[1, M]$ there is (at least one) element in $\mathcal{B}$ which has a monochromatic representation in the form (2). (In fact, the following sharper statement is true: almost all of these sets $\mathcal{B}$ are such that for every $\mathcal{A}$ with $\mathcal{A} \subset \left[1, \frac{M}{2}\right]$ and $|\mathcal{A}| > \frac{1}{k}[M/2]$, there is an element in $\mathcal{B}$ which can be represented in the form (2) with $a \in \mathcal{A}$, $a' \in \mathcal{A}$.) Ruzsa [16] proved that if $f(x) \to +\infty$, then there exists an infinite sequence $\mathcal{D}$ of positive integers such that $D(x) = \sum_{\substack{d \leq x \\ d \in \mathcal{D}}} | = 0(f(x)(\log x)^2)$, and if $\mathcal{A}$ is a sequence of positive integers with positive upper asymptotic density, then $\mathcal{D}$ intersects the set of the integers of the form $a + a'$ where $a \in \mathcal{A}$, $a' \in \mathcal{A}$. These results suggest that the upper bound $\frac{M}{2} - ck \log M$ is closer to the truth than the lower bound.

Recently Balog, Fürstenberg, Sárközy, Stewart, Lagarias, Odlyzko, Schearer [1], [7], [13], [14], [17], [18], [19] and others have studied the solvability of the equations

$$\begin{aligned} a - a' &= x^2 \\ a - a' &= p - 1 \\ a + a' &= x^2 \\ a + a' &= px,\ x \text{ "small" } (= 0(1)) \end{aligned}$$

with $a, a' \in \mathcal{A}$ where $\mathcal{A}$ is a "dense" sequence of positive integers. These results and Hindman's theorem [2], [11] led us to consider the corresponding "monochromatic" questions.

Theorem 1 implies that e.g. the equations

$$\begin{aligned} a_1 + a_2 &= 2p \\ a_1 + a_2 &= p - 1 \end{aligned}$$

have monochromatic solutions with $a_1 \neq a_2$.

Our result is not strong enough to obtain for arbitrary k that

$$a_1 + a_2 = x^2$$

has a monochromatic solution with $a_1 \neq a_2$. However a simple argument leads to

Theorem 3. *If $k \leq 3$, then for any k-partition of $\mathcal{N}$ there are infinitely many squares in* C.

Proof. We use the following simple (and well known)

Lemma 2. *For every $\varepsilon > 0$ there are infinitely many integers n so that*

$$n = x^2 + y^2$$

has at least three (in fact arbitrary many) integer solutions where

$$x^2, y^2 \in \left[\frac{n}{2}(1-\varepsilon),\ \frac{n}{2}(1+\varepsilon)\right].$$

Now let

$$x_1^2 + x_6^2 = x_2^2 + x_5^2 = x_3^2 + x_4^2$$

with $x_i \in \left[\frac{n}{2}(1-\varepsilon), \frac{n}{2}(1+\varepsilon)\right]$, $1 \leq i \leq 6$.

Then an easy calculation shows, that the system

$$\begin{aligned} u_1 + u_2 &= x_1^2 \\ u_3 + u_4 &= x_6^2 \\ u_2 + u_3 &= x_2^2 \\ u_1 + u_4 &= x_5^2 \\ u_1 + u_3 &= x_3^2 \\ u_2 + u_4 &= x_4^2 \end{aligned}$$

in $u_i (1 \leq i \leq 4)$ has a solution in distinct positive numbers. Since at least two of the u_i's belong to the same class, one of the x_i^2 $(1 \leq i \leq 6)$ squares must have a monochromatic representation.

If we have some information on the structure of the classes A_i in the given partition then the lower bound given for the integers that have a monochromatic representation in form (2) can be sharpened. In fact we have

Theorem 4.

(i) *For every $\varepsilon > 0$ and k there exists an $M_0(\varepsilon, k)$ such that if we have a k-partition of $\mathcal{N}$ where every class contains both even and odd integers then.*

$$| \mathrm{C}_M | > \left(\frac{1}{2} + \frac{1}{2k} - \varepsilon\right) M \text{ if } M > M_0(\varepsilon, k).$$

(ii) *For every $k \in \mathcal{N}$ there is a k-partition of $\mathcal{N}$ so that every class contains both even and odd integers and*

$$| \mathrm{C}_M | < \left(\frac{1}{2} + \frac{1}{k}\right) M + 1.$$

Proof.

(i) can be proved by the method used in the proof of Theorem 2,

(ii) follows from the following construction: for $i = 1, 2, \ldots, k$ let

$$\mathcal{A}_i = \{n : n \equiv 2i \ (\text{mod} 2k)\} \cup \{n : n \equiv 1 - 2i \ (\text{mod} 2k)\}.$$

It is easy to see that this k-partition of $\mathcal{N}$ has the desired properties.

The Case $f(x_1, x_2) = | rx_1 + sx_2 |$.

Let r, s be integers. As before, let C denote the set of integers which have a monochromatic representation in the form

$$n = | ra_1 + sa_2 | \quad \text{with} \quad a_1 \neq a_2. \tag{22}$$

Let $C_M =: C \cap [1, M]$. The following result is merely a simple modification of Theorem 1.

Theorem 5. *Let $r \neq 0$, $s \neq 0$, $r + s \neq 0$. Put $| r + s | = m$. For every $\varepsilon > 0$, k, r, s and for every k-partition*

$$| C_M | \geq (1 - \varepsilon) \frac{M}{m}.$$

This can not be essentially improved, since choosing

$$k = m \text{ and } \mathcal{A}_i = \{n : n \equiv i \ (\text{mod} m)\}, \ 1 \leq i \leq m \tag{23}$$

only the multiples of m have a monochromatic representation in the form (22)

Note furthermore that Theorem 5 does not cover the case of the *differences* $a_1 - a_2$. Namely, in this case the density of the integers having a monochromatic representation in the form (22) need not be greater than a positive absolute constant. To see this, let us consider a large integer m and define the partition as in (23). Then only the multiples of m have a monochromatic representation in the form (22) so that their density is $\frac{1}{m}$ which $\to 0$ if $m \to \infty$.

Proof. Assume that there are more than $\varepsilon \frac{M}{m}$ positive multiples of m in $[1, M]$ which do not have a monochromatic representation in the form [22]. Then by Szemerédi's theorem [20], for $M > M_0(k, \varepsilon, r, s)$ their set must contain an arithmetic progression of $2(| r | + | s |)k + 1$ terms; let us write this arithmetic progression (all whose terms are multiples of m) in the form

$$um - (| r | + | s |)kv, \ um - ((| r | + | s |)k - 1)v, \ldots, \ um + (| r | + | s |)kv. \tag{24}$$

Let us consider the integers $u, u+v, \ldots, u+kv$. By the pigeon hole principle, two of them, say $a_1 = u+iv$ and $a_2 = u+jv$ (where $i \neq j$) belong to the same class. Then

$$ra_1 + sa_2 = r(u+iv) + s(u+jv) = (r+s)u + (ri+sj)v.$$

Here we have

$$| ri + sj | \leq | r | k + | s | k = (| r | + | s |)k.$$

Since $| r+s | = m$ and all the numbers in (24) are positive, $| ra_1 + sa_2 |$ is equal to one of the numbers in (24). But this contradicts the fact that none of these numbers has a monochromatic representation in the form (22), and the proof is completed.

2. Some Unsolved Problems

Problem 1. Do there exist α and β which depend only on k, so that for an arbitrary k-partition

$$| \mathrm{C}_M | > \frac{M}{2} - (\log M)^{\alpha(k)}$$

or even more $| C_M^2 | > \frac{M}{2} - (\log M)^{\beta(k)}$.

Problem 2. Let $f(x)$ be a polynomial of integer coefficients such that 2 is a prime divisor of it. Is it true that for any k-partition for some x (or for infinitely many x)

$$a_1 + a_2 = f(x),$$

have a monochromatic solution with $a_1 \neq a_2$?

Problem 3. Is it true that for every k-partition of $[1, M]$ almost all the even integers $2n$ in $[1, M]$ have more than $c(k)n$ monochromatic representations in form (2)? (Perhaps this holds with $c(k) = \frac{c_1}{k}$.)

Problem 4 a) For a given k-partition let $n_1 < n_1 < \ldots$ be the sequence of those integers which have a monochromatic representation in form (2). ($\mathrm{C} = \{n_i\}$). What can be said about the structure of the sequence $\{n_i\}$? (For example it is easy to see that $| n_{i+1} - n_i | < 2k$.)

b) The complementary problem is to study the structure of the set $\mathcal{B} = \mathcal{N} - \mathrm{C}$ (the set of those integers which do not have a monochromatic representation in form (2)).

Let $\mathcal{G}(\mathcal{N}; E)$ be the graph with edgeset $\{(x,y) \mid x+y \in \mathcal{B},\ x,y \in \mathcal{N}\}$. Obviously at any k-partition the chromatic number of $\mathcal{G}(\mathcal{N}; E)$ is $\leq k$. Basically this was used in the proofs above.

Problem 5. So far we have studied monochromatic representations in form (1) in the special case when $f(x_1, \dots, x_t)$ is a linear polynomial and $t = 2$. In the paper Erdős–Sárközy [6] the case $f(x_1, x_2) = x_1 x_2$ is considered.

What can one say on general polynomials $f(x_1, \dots, x_t)$ (whose coefficients are integers)? What can be said in the most important special case when $f(x_1, x_2, \dots, x_t)$ is of the form $g(x_1) + \dots + g(x_t)$?

As Ruzsa [15] observed, if

$$f(x_1, x_2, x_3, x_4) = x_1^2 + x_2^2 + x_3^2 + x_4^2$$

then for every k-partition

$$| C_M | > c(k) \cdot M$$

and $| C_M | > cM$ cannot hold with an absolute constant c.

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