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ON RESTRICTED COLOURINGS OF K_n

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Dedicated to Paul Erdős on his seventieth birthday

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Given a sample graph H and two integers, n and r, we colour K_n by r colours and are interested in the following problem.

Which colourings of the subgraphs isomorphic to H in K_n must always occur (and which types of colourings can occur when K_n is coloured in an appropriate way)?

These types of problems include the *Ramsey theory*, where we ask: for which n and r must a monochromatic H occur. They also include the *anti-Ramsey* type problems, where we are trying to ensure a totally multicoloured copy of H, that is, an H each edge of which has different colour.

Notation

We shall consider only simple graphs, that is, graphs without loops and multiple edges. Given a graph G, $V(G)$ resp. $v(G)$ will denote the vertex set and the number of vertices of G, respectively. We shall also use superscripts to indicate the number of vertices: G^n, S^n, ... will always be graphs on n vertices. Given two disjoint sets of vertices, X and Y in $V(G)$, $e(X, Y)$ will denote the number of edges joining them. K_m, P_m and C_m will denote the complete graph, the path and the cycle on m vertices. $N(x)$ and $d(x)$ denote the set of vertices joined to x and the degree of x, respectively. If A is a set of vertices and edges of a graph G, $G-A$ will denote the graph obtained by deleting the edges of A, the vertices of A and the edges incident to vertices in A, from G. If $A=\{a\}$ is just one vertex or one edge, we shall use the notation $G-a$ instead of $G-\{a\}$.

Introduction

In Section 1 we determine the anti-Ramsey numbers foı paths. In the paper Erdős—Simonovits—T. Sós [6] the following type of problems were investigated:

For given n and H determine the maximum integer $r=f(n, H)$ for which there exists a colouring of K_n by r colours without having a *copy of H* in K_n *all the edges of which have different colours*. A copy of a graph H each edge of which is of a different colour will be called *totally multicoloured,* shortly: TMC. While in Ramsey type problems we try to ensure a monochromatic copy of a sample graph H by using

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only a small number of colours, here we use many colours to ensure a TMC copy of H. Therefore these problems will be called *anti-Ramsey* problems.

We have seen in [6] that anti-Ramsey problems are very strongly connected to *Turán type extremal graph problems:*

Given a family $\mathscr{L}$ of forbidden graphs, determine the maximum number of edges a graph G^n can have without containing members of $\mathscr{L}$, as subgraphs. Let us denote this maximum by $\mathrm{ex}(n, \mathscr{L})$. As Erdős and Simonovits [5] proved, if $p+1$ denotes the minimum chromatic number in $\mathscr{L}$, then

$$\mathrm{ex}(n, \mathscr{L}) = \left(1-\frac{1}{p}\right)\binom{n}{2} + o(n^2). \tag{1}$$

(They have also described the asymptotical structure of the extremal graphs, that is, of the graphs S^n attaining the maximum Erdős, [3], Simonovits [8].) We can see from (1) that

$$\mathrm{ex}(n, \mathscr{L}) = o(n^2) \quad \text{iff} \quad \mathscr{L} \text{ contains a bipartite graph } L. \tag{2}$$

The connection between anti-Ramsey problems and Turán type extremal problems is so close that we have

Theorem A (Theorem 2 of [6]). *Let* $\mathscr{H} = \{H-e: e$ *is an edge of* $H\}$. *Then*

$$f(n, H) - \mathrm{ex}(n, \mathscr{H}) = o(n^2), \quad as \quad n \to \infty. \;\blacksquare \tag{3}$$

Theorem A yields an asymptotically sharp estimate on $f(n, H)$ if $H-e$ is always at least 3-chromatic, while e runs through the edges of H. However, the situation gets much more complex when there is an edge e_0 in H for which $H-e_0$ *is bipartite.* The first interesting cases are when *H is a path or a cycle.* In Theorem B $f(n; P_k)$ is determined for $n > n_0(k)$; $f(n; C_k)$ is still unsettled for $k \geqq 5$. (See the conjecture in Section 3.)

Theorem B. *There exists a constant c such that if $t \geqq 5$, $n > ct^2$, then for* $\varepsilon = 0, 1$

$$f(n, P_{2t+3+\varepsilon}) = tn - \binom{t+1}{2} + 1 + \varepsilon. \tag{4}$$

The extremal colouring is the following: Let $a_1, \ldots, a_t, b_1, \ldots, b_{n-t}$ be the vertices of K_n. Colour all the edges represented by the a_i's differently, by $\binom{t}{2} + t(n-t)$ colours, and colour all the pairs (b_i, b_j) by still another colour. In this colouring of K_n we shall not find TMC copies of $P_{2t+3+\varepsilon}$. The longest TMC path will be a P_{2t+2}. If we colour the edges between the b_i's by two colours, the longest TMC path will be a P_{2t+3}.

Remark 1. In fact we can prove the stronger theorem that (4) holds if $n \geqq (5/2)t + c$ with an absolute constant c and for every t. Further, for $t > t_0$,

$$f(n, P_k) = \begin{cases} \binom{k-2}{2} + 1 & \text{if } k \leqq n \leqq \dfrac{5t+3+4\varepsilon}{2}, \\ tn - \binom{t+1}{2} + 1 + \varepsilon & \text{if } n \geqq \dfrac{5t+3+4\varepsilon}{2}. \end{cases} \tag{*}$$

The omitted part of this more complete result can be proven by similar arguments as used in Theorem B but is more involved and rather lengthy. We have conjectured in ESS that (*) holds for *all n and t.*

Remark 2. By the well known theorem of Erdős and Gallai

$$\text{ex}(n, P_k) \leqq \frac{k-2}{2} n. \tag{5}$$

This is sharp: take the union of vertex disjoint K_{k-1}'s if n is divisible by $k-1$. (The results of Faudree and Schelp [7] refine this in the case when $k-1 \nmid n$.) It is easy to see that (5) implies

$$f(n, P_{2t+3}) \leqq \left(t+\frac{1}{2}\right) n \tag{6}$$

and

$$f(n, P_{2t+4}) \leqq (t+1) n. \tag{7}$$

Fix an arbitrary r-colouring of the edges of K_n and select an edge of each colour in an arbitrary way. These edges will form a graph G^n not containing any P_k if the colouring contains no TMC P_k.

It is also worth noticing that $\text{ex}(n, P_{2t+4})-\text{ex}(n, P_{2t+3})=n/2$, while $f(n, P_{2t+4})-f(n, P_{2t+3})=O(1)$.

In Section 2 we consider the following general problem containing Ramsey and anti-Ramsey type problems as well.

The H_0 spectra of colourings. Given a graph $H \subset K_n$ and an r-colouring φ_r: $E(K_n) \to \{1, 2, ..., r\}$ let $c(H; \varphi_r)$ denote the number of colours in H. For a graph H_0 we define the spectrum as

$$S(H_0; n, \varphi_r) := \{i | H \sim H_0, c(H; \varphi) = i\}.$$

The general problem is to characterize those sets $S \subset \{1, ..., r\}$ for which there exists an r-colouring φ_r such that $S(H_0; n, \varphi_r) = S$.

Of course, the description of these sets S in terms of n, r and H_0 is generally very difficult.

Here we shall mostly restrict our considerations to the simplest case, when $H = K_3$. Theorem C shows that even in this simple case quite a lot diffeıent surprising phenomena can be found.

In a recent paper of F. K. Chung and R. L. Graham [2] the following special case of this problem is considered: let $f(p; r)$ be the largest value of n such that it is possible to colour the edges of K_n so that every $K_p \subseteqq K_n$ has exactly $p-1$ different colours. They determine $f(3, r)$ and $f(4, r)$.

1. Proof of Theorem B

We shall use the following results of Erdős and Gallai [4], (see above):

Erdős—Gallai theorem on paths. $\text{ex}(n, P_h) \leqq \frac{h-2}{2} n.$ ∎

Erdős—Gallai theorem on cycles. *If C_r is the longest cycle in G^m, then*

$$e(G^m) \leqq \frac{r(m-1)}{2}. \quad \blacksquare$$

In the proof we shall restrict ourself to the case $t \geqq 5$. The case $t \leqq 4$ can be proved by similar arguments but need to distinguish more cases.

The proof will be in the following setup. First we take a TMC path P_s of maximum length. Then we choose one edge from each remaining colour so that the number of edges joining P_s to the remaining $n-s$ vertices be the maximum possible. Let us denote by G^n the graph spanned by these edges, by G^s the subgraph spanned by P_s and let $G^q = G^n - G^s$. Let us partition $V(G^q)$ into the sets U, V and W as follows:

U is the set of vertices of G^q not joined to P_s at all: neither by edges nor by paths.

V is the set of *isolated* vertices of G^q joined to P_s by edges;

$$W = V(G^q) - U - V.$$

$G(U)$, $G(V)$ and $G(W)$ denote the corresponding induced subgraphs, E_U, E_W the number of edges in $G(U)$ and $G(W)$, F_V and F_W the number of edges joining V and W to P_s. (See Figure 1.) $E_s = e(G^r)$

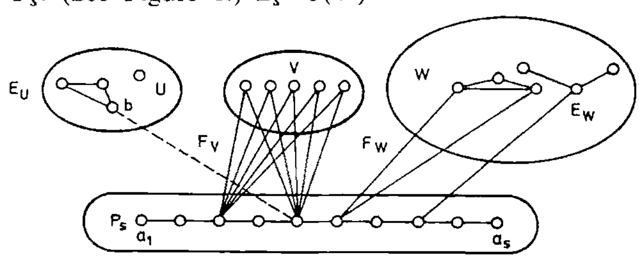

Fig. 1

Let x denote the maximum number of edges joining a vertex of $V \cup W$ to P_s.

Lemma 1. *Using the above notation,*

(a) *$G(U)$ contains no P_l for* $l = \left\lceil \frac{s+1}{2} \right\rceil$.

(b) $E_U \leqq \frac{1}{2} \left\lceil \frac{s-3}{2} \right\rceil |U|$.

(c) $E_W + F_W < \frac{s+1}{3} |W|$.

Proof. Assume that $P_l \subseteq G(U)$ and join its endvertex b_1 to the middle vertex a_i of P_s. Since the number of edges between P_s and G^q is maximum, the colour of (b_1, a_i) cannot occur in P_l. Similarly, it does not occur in one of the 2 segments of P_s. Those yield a $P_{\left\lceil \frac{s+1}{2} \right\rceil + l}$. Since $\left\lceil \frac{s+1}{2} \right\rceil + l > s$, this contradicts the maximality of P_s, proving (a). The Erdős—Gallai theorem implies (b).

To prove (c) take a component H of $G(W)$ and denote by r its longest cycle. If H contains no cycles, write $r=2$. For each vertex u of H a $P_s \subset H$ can be found starting from it. Hence, if $P_s=a_1, ..., a_s$, then $a_1, ..., a_r$ and $a_s, ..., a_{s-r+1}$ cannot be joined to u: otherwise a TMC $P_{s+1} \subseteq G^n$, contradicting the maximality of P_s. Take three consecutive vertices $\{a_i, a_{i+1}, a_{i+2}\}$. Again, by the maximality of P_s, there are no two independent edges between a_i, a_{i+1}, a_{i+2} and H. Hence for $h=v(H)$ there are at most

$$\frac{rh-r}{2}+h\frac{s-2r+2}{3} \leqq \frac{s+1}{3}h$$

edges between H and P_s. Adding this up we get (c). ▮

Lemma 2. $x \leqq \left[\frac{s-3}{2}\right] \leqq t$ *(where x denotes the maximum number of edges joining some vertex of $V \cup W$ to P_s).*

Proof. Let $w \in V \cup W$ be joined to $P_s=(a_1, ..., a_s)$ by x edges. Because of the maximality of s, (w, a_1), (w, a_s) and (a_1, a_s) do not belong to G^n, unless $x=0$. Adding (a_1, a_s) to G^n we must have an edge (a_m, a_{m+1}) of the same colour. Otherwise the cycle $P_s+(a_1, a_s)$ and an edge f joining $V \cup W$ to G^s would form a TMC P_{s+1}. (It is easy to ensure that the colours of f and (a_1, a_s) be different.) Clearly (a_1, a_s) and (a_m, a_{m+1}) are equivalent: w is not joined to a_m and a_{m+1} either.

Since w is not joined to consecutive a_i's and it is not joined to a_1, a_s, a_m, a_{m+1}, therefore $x \leqq \left[\frac{s-3}{2}\right] \leqq t$. (This holds even if some of the 4 vertices a_1, a_s, a_m and a_{m+1} coincide, e.g. $a_1=a_m$.)

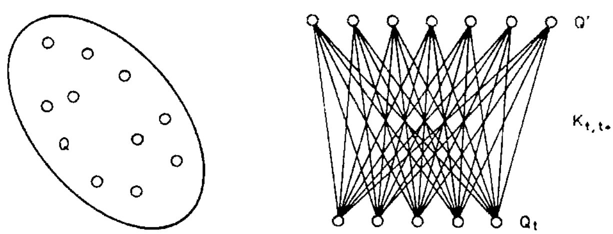

Fig. 2

Lemma 3. *If a colouring of K_n contains a TMC copy of $K_{t,t+2}$ but does not contain a TMC $P_{2t+3+\varepsilon}$ ($\varepsilon=0,1$), then it is the following colouring: denote by Q the vertices of $K_n-K_{t,t+2}$, by Q_t the smaller class of $K_{t,t+2}$ and by Q' the other one. Then there are at most $1+\varepsilon$ colours between the edges of Q'. Let they be red (and blue). Then all the vertices of Q are joined to Q' be red (either by red or by blue).*

We get the most colours if all the edges between Q and Q_t are different, they differ from all the other edges and we use exactly $1+\varepsilon$ colours in Q'. Then the number of colours is

$$m-\binom{t+1}{2}+1+\varepsilon. \quad ▮$$

The proof is trivial.

Proof of Theorem B. We assume that $t \geqq 5$ and $n \geqq n_t$. Clearly,

$$f(n, P_{2t+3+\varepsilon}) \leqq E_S + E_U + F_V + E_W + F_W.$$

Using the estimates of the Lemmas

$$f(n, P_{2t+3+\varepsilon}) \leqq \binom{s}{2} + \frac{t+1}{2}|U| + x \cdot |V| + \frac{2t+1}{3}|W|.$$

Here $\frac{t+1}{2} \leqq t - \frac{1}{2}$ and $\frac{2t+4}{3} \leqq t - \frac{1}{3}$, finally, $x \leqq t$. Let $V^* \subseteqq V$ denote the set of vertices of degree t. Then

$$tn - \binom{t+1}{2} + 1 + \varepsilon \leqq f(n, P_{2t+3+\varepsilon}) \leqq \binom{s}{2} + \left(t - \frac{1}{3}\right)(n - s - |V^*|) + t|V^*|.$$

Hence for $n > c \cdot t^2$ we get at least $t+2$ vertices $b_1, \ldots, b_t \in V^*$ joined to the same t vertices of P^s. By Lemma 3 the proof is completed. ▌

2. On the K_3-spectra of colourings

To describe the H-spectra of the colourings of K_n by r colours seems to be fairly involved. We make a few preliminary remarks on the cases $H = K_p$ and $H = P_h$.

Trivially, if $R(m, H)$ is the m-colour Ramsey number of H, that is, the maximum N such that K_N can be coloured in m colours without having a monochromatic H, then for $n > R(r, H)$ the spectrum of every r-colouring φ must contain 1. Similarly, if H contains v independent edges, then every r-colouring has an H of at least $\min(r, v)$ colours, assumed that $n \geqq v(H)$.

In some of our investigations the following construction plays important role.

The Split-Colouring. Take a K_n on the vertices $a_1, \ldots, a_n$, split the vertex-set into two parts: V_1 and V_2, and use colour "1" for (a_i, a_j) if a_i and a_j belong to different parts. Then split V_1 into two parts: V_{11} and V_{12}, colour the edges between them by "2". Similarly, split V_2 into V_{21} and V_{22} and colour the edges between them by "3". Continue this, always splitting into nonempty parts and stop whenever a set contains only one element. Thus K_n is coloured by $n-1$ colours. An important subcase of this colouring is when (a_i, a_k) is coloured by "$\min(i, k)$".

It is easily seen that each K_p contains exactly $p-1$ colours. It is also interesting to notice that no TMC cycle occurs in this construction. Add now j new vertices to the above construction, $b_1, \ldots, b_j$, and colour all the edges (b_s, a_i) and all the edges (b_s, b_i) by "1". Trivially, the K_p spectrum of this colouring is $\{p-j, \ldots, p-1\}$.

Proposition. *There is a constant R such that if we colour K_n so that each K_p contains at most $\frac{p}{2}$ colours, then we used at most $R + \frac{p}{2}$ colours.*

Proof. In the proof we use that $f(n, P_4) = O(1)$.

We know from Theorem B that for $r > R$ every r-colouring of K_n contains a 3-coloured P_4. Choose $\left[\frac{p-4}{2}\right]$ further colours. The corresponding edges and P_4 define a K_p coloured by at least $\frac{p}{2}+1$ colours. ∎

We intend to return to the investigation of the K_p-spectra and more generally to H-spectra of colourings in another paper. Below we restrict our considerations to the simple case of K_3.

Let $\varrho(n)$ denote the inverse of the Ramsey-function $R(m, K_p)$ i.e. the minimum number m of colours for which K_n can be coloured in m colours without having a monochromatic K_3. We have the following possibilities for the spectrum of a colouring: $S=\{1\}, \{2\}, \{3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}, \{1, 2, 3\}$. We exclude the trivial cases $S=\{1\}$ and $S=\{1, 2, 3\}$.

Theorem C.

Case I. $S=\{2\}$. *If* $\log_2 n \leqq r \leqq n-1$, *one can colour* K_n *by* r *colours so that each* K_3 *contains exactly two colours. If* $r < \varrho(n)$ *or* $r \geqq n$, *no such colouring exists.*

Case II. $S=\{3\}$. *There exists an* r*-colouring of* K_n *in which each* K_3 *is 3-coloured iff* $n^* \leqq r \leqq \binom{n}{2}$, *where* $n^*=n-1$ *if* n *is even,* $n^*=n$ *for* n *odd,*

Case III. $S=\{1, 2\}$. *There exists an* r*-colouring* φ_r *of* K_n *with* $S(K_3, \varphi_r) =\{1, 2\}$ *iff* $2 \leqq r \leqq n-1$.

Case IV. $S=\{1, 3\}$. *If* $2 \leqq r < \sqrt{n}+1$ *and* K_n *is* r*-coloured, there is always a 2-coloured* K_3. *This is sharp: for some constant* c, *if* $\sqrt{n}+o(\sqrt{n}) \leqq r \leqq \binom{n}{2}-c$ *then there exists an* r*-colouring* φ_r *of* K_n *with* $S(K_3, \varphi_r)=\{1, 3\}$.[1]

Case V. $S=\{2, 3\}$. *There exists an* r*-colouring of* K_n *with* $S(K_3, \varphi_r)=\{2, 3\}$ *if* $\varrho(n) \leqq r \leqq \binom{n}{2}-1$, *where* $\varrho(n)$, *as above, denotes the inverse of the Ramsey-function* $R(m, K_3)$.

Proof. *Case of* $\{2\}$. We use a construction obtained from the "split-colouring" when $j=0$ by identifying some colours. The easiest way to formulate our construction is perhaps to represent the n vertices $a_1, \ldots, a_n$ by the endvertices of a rooted binary tree. Each a_i corresponds to a 0—1 sequence (of length $h(a_i)$). Colour (a_i, a_j) by the colour k if the corresponding 0—1 sequences are the same upto the first $k-1$ terms and differ in the k-th one. It is possible to choose the "code-sequences" (the rooted binary tree) so that r colours be used, if $\log_2 n \leqq r \leqq n-1$. One can easily check that each K_3 is 2-coloured. If $r < \varrho(n)$, then we must have a monochromatic K_3, if $r \geqq n$, we have a TMC K_3.

Case of $\{3\}$. Let us partition the edges into n^* edge-disjoint 1-factors and colour each of them by different colours. Clearly, each K_3 is 3-coloured. This remains

[1] As a matter of fact, we know that the spectrum $S=\{1, 3\}$ does not occur for $\binom{n}{2}=c=r$ iff $c=0, 1$, if $n > 4$.

valid if we use for some (or for all) 1-factors more than 1 colour, ensuring, however, that each colour occurs only in one 1-factor. Thus we may colour the edges of K_n in exactly r colours, if $n-1\leqq r\leqq\binom{n}{2}$.

If we use $r<n^*$ colours, then there exist two incident edges (a,b) and (a,c) of the same colour: (a,b,c) is $\leqq$2-coloured.

Case of $\{1,2\}$. The Split Colouring with $n-r-1$ b's has spectrum $\{1,2\}$. Using $n-1$ or more colours we get a TMC K_3.

Case of $\{1,3\}$. *(A)* Take an arbitrary r-colouring of K_n. Fix a vertex x. We may assume that *red* is the colour used the most often to colour the edges (x,y). Assume that no 2-coloured K_3 occurs. Then x and those neighbours y of x for which (x,y) is red form a monochromatic (red) K_m for $m=\left\lceil\frac{n-1}{r}\right\rceil+1$. Take a red K_h of maximum size. Then $h\geqq\left\lceil\frac{n-1}{r}\right\rceil+1$. If w is a vertex not belonging to this K_h, no edge (w,y) $(y\in V(K_h))$ can be coloured in red: otherwise either there were another vertex $y'\in V(K_h)$ for which (x,y') is not red and hence (y,y',w) would be 2-coloured, or else all the edges (w,y') were red, contradicting to the maximality of K_h. Thus the edges (x,y') cannot be red and all they must have different colours: $r-1\geqq\left\lceil\frac{n-1}{r}\right\rceil+1$. This proves that $r\geqq\sqrt{n}+1$.

(B) First let $n=p^2$ and p be a prime-power. Let the vertices of a graph G'' be the points of the *finite affine plane:* the pairs (x,y) mod p. Let the colour of the edge $((x,y),(x',y'))$ be $\frac{y'-y}{x'-x}$ unless $x=x'$, when we colour the edge by a colour "∞". Thus we used $p+1$ colours on n^2 vertices and no 2-coloured K_3 occurs. (The idea was that colouring the edges with the "slope" of the corresponding line we can use that *given a point and a slope, the corresponding line is determined.*)

Observe that according to this construction the lower bound $\sqrt{n}+1$ is the best possible if $n=p^2$ (where p is prime or a prime-power) and the error-term $o(\sqrt{n})$ is needed only since we must approximate $\sqrt{n}$ by prime-powers.

(C) Let n be now arbitrary and choose the least prime power p for which $p^2\geqq n$. Put $q=p^2-n$. We delete from the above construction q vertices from $\left\lceil\frac{q}{p}\right\rceil$ lines coloured by "∞". Thus all the remaining lines have roughly the same number of points, namely $p-\left\lceil\frac{q}{p}\right\rceil=p-o(p)$. Choose the lines $e_1,\ldots,e_m$ so that the number of edges covered by them be at least r, while the number of edges covered by $e_1,\ldots,e_{m-1}$ be less than r. Until now the edges on a line e_i were of the same colour. Now we replace them by different colours. This way we get a new graph each K_3 of which is still monochromatic or 3-chromatic. The number of colours may be larger than r, but then we may use the same colour in e_{m-1} and e_m, the K_3's remain still 1- or 3-chromatic. This way we may achieve that the number of colours be exactly r. If at least one line e_j stays monochromatic, then the sprectrum is really $\{1,3\}$. Clearly, this construction works, if $\sqrt{n}+\varepsilon\sqrt{n}<r<(1-\varepsilon)\binom{n}{2}$ and $n>n_0(\varepsilon)$.

(D) The following trivial construction works between $\binom{n}{2}-c$ and $n+o(n)$: Partition the n-element set S into the subsets $A_1, \ldots, A_k$ and colour each A_i either with colour "i" or with $\binom{|A_i|}{2}$ different colours. Then give to each edge (x, y), $x \in A_i$, $y \in A_j$, $(i \neq j)$ a colour which is used only for this edge. (The colours used for edges in different A_i's may be different and may also be the same.) One can easily see that the spectrum of this colouring is $\{1, 3\}$.

Case of $\{2, 3\}$. Colour K_n by $r-1$ $(\varrho(n) \leqq r-1 \leqq n-2)$ colours without having monochromatic triangles. Clearly, there exists at least one 2-chromatic K_3 on each vertex. This is exactly 2-chromatic. Change one K_3 into 3-chromatic: we get that the spectrum of the r-colouring is $\{2, 3\}$. To cover the case $n-1 \leqq r < \binom{n}{2}$ we may use e.g. Case of $\{3\}$: change (in the colouring given there) the colour of one edge to obtain a 2-chromatic K_3 as well. This will not spoil all the 3-chromatic K_3's. ∎

Remark. In Theorem C all cases but $S=\{2\}$ are sharp.

The result in [2] in this setting means that for $n=5^{k/2}$ (if k is even) $k(=2\log_5 n)$ is the least value of r for which an r-colouring φ_r of K_n exists with $S(K_3, \varphi_r)=\{2\}$. Some further relevant results can also be found in [2].

Some open problems

There are many different results on the spectra of colourings in r colours with respect to an H. Most of them immediately follow from some corresponding extremal graph theorems. The really interesting questions are which show some completely new phenomena, which e.g. do not follow relatively easily from Turán type extremal graph theorems.

Some other problems we started investigating are connected with *uniform colourings*, where one uses each colour approximately for the same number of edges.

Problem 1. *Does there exist a constant c such that one can colour K_n uniformly in $c \cdot k \cdot n$ colours without getting a TMC C_k?*

Problem 2. *What is the maximum number r of colours if we can colour the edges of K_n using each colour at most $(1+o(1))\frac{1}{r}\binom{n}{2}$ times without getting a TMC copy of H?*

More precisely, we assume that for a given function $\varphi(n)$ tending to 0 each colour is used at most $(1+\varphi(n))\frac{1}{r}\binom{n}{2}$ times. Denoting the maximum by $f(n, H, \varphi)$ we are interested in its dependence on n. We may change the uniformity condition above to the weaker one that for a given constant K each colour is used at most $K \cdot \binom{n}{2} \Big/ r$ times.

The motivation of these problems is that in most of our anti-Ramsey theorems most of the colours are used only once, and a few colours very many times. One would like to know what happens if one excludes this unevenness. We do not know the answer even for cycles:

Problem 3. *What is the maximum number $g^*(n, r, H)$ of colours occuring at every r-colouring of K_n in at least one copy of H; what is the minimum number $g_*(n, r, H)$ of colours occuring in every r-colouring of K in at least on H?*

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