Optimal Sequencing of Items In a Consecutive-2-out-of-*n* System

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Reader Aids—
Purpose: Widen state of the art
Special math required for explanations: Elementary probability
Special math needed to use results: None
Results useful to: Reliability analysis and theoreticians

Abstract—**A *consecutive-2-out-of-n system* is an array of *n* items in a line such that the system fails if and only if two consecutive items both fail. Suppose that the items have different probabilities of failing and that the system can be arranged into any sequence of the *n* items. Which sequence minimizes the probability of a system failure? It has been conjectured that the best sequence is one which essentially interlaces the more reliable items with the less reliable items. This paper partially supports the conjecture by proving it for the case that: a) the *n* probabilities take on only two distinct values, and b) the *n* probabilities take on only three distinct values, including either a zero or a one.**

1. INTRODUCTION

A consecutive-2-out-of-*n* system [1] is an array of *n* items in a line such that the system fails if and only if two consecutive items fail. An example of such a system is a broadcasting network consisting of relay stations. Suppose that the stations are lined up and the signal broadcast from one station is only strong enough to reach the next two stations. Then the relay can be interrupted if and only if two consecutive stations fail simultaneously. Now for a given set of *n* items with known but unequal working probabilities, what is the best way to arrange them into a sequence so that the probability of system failure is minimized? Derman et al. [2] conjectured that the best sequence is one which essentially interlaces the more reliable items with the less reliable items. This paper partially supports the conjucture by proving it for the two special cases:

a. There are two types of working probabilities.
b. There are three types of working probabilities one of which is either zero or one.

The assumption of a "zero" or "one" for working probability may sound unrealistic. But case b is studied for two purposes. Mathematically, it is a nontrivial extention of the 2-class case towards the final goal of *n* distinct probabilities. Practically it is the limiting case for three genuine probabilities with one of them approaching zero or one.

2. NOTATION

I_i	item i
p_i	working probability for item i
S^*	the conjectured optimum sequence $I_{i_1}, I_{i_n}, I_{i_3}, I_{i_{n-2}}, \ldots, I_{i_{n-3}}, I_{i_4}, I_{i_{n-1}}, I_{i_2}$ where $p_{i_1} \leqslant p_{i_2} \leqslant \ldots \leqslant p_{i_n}$
S	the sequence $I_1, I_2, \ldots, I_n$
S_{ij}	the sequence $I_1, I_2, \ldots, I_{i-1}, I_j, I_{j-1}, \ldots, I_i, I_{j+1}, \ldots, I_n$ (defined for $i<j$ and obtained from S by reversing the subsequence $I_i, I_{i+1}, \ldots, I_{j-1}, I_j$)
$P(S')$	the probability that the sequence S' does not contain two consecutive failing items
x_1	$P(I_1, I_2, \ldots, I_{i-1}\|I_{i-1}$ works$)\ p_{i-1}\ (1 - p_{j+1})P(I_{j+1}, I_{j+2}, \ldots, I_n\|I_{j+1}$ fails$)$
x_2	$P(I_1, I_2, \ldots, I_{i-1}\|I_{i-1}$ fails$)\ (1 - p_{i-1})\ p_{j+1}P(I_{j+1}, I_{j+2}, \ldots, I_n\|I_{j+1}$ works$)$
y_1	$p_iP(I_i, I_{i+1}, \ldots, I_j\|I_i$ works, I_j fails$)\ (1 - p_j)$
y_2	$(1 - p_i)P(I_i, I_{i+1}, \ldots, I_j\|I_i$ fails, I_j works$)\ p_j$ (x_1, x_2, y_1 and y_2 are defined for $1 \leqslant i < j \leqslant n$)
$[u]^+$	the largest integer not greater than u
$[u]^-$	the smallest integer not less than u.

3. MAIN RESULTS

Assumptions: The independent case of a linear consecutive-2-out-of-*n* system.

Consider the sequence S, and the sequence S_{ij} which is obtained from S by reversing the subsequence $I_i, I_{i+1}, \ldots, I_j$. We shall derive a sufficient condition for S_{ij} to be better than S. By repeated applications of this kind of reversals, we arrive at the optimum arrangement in the 2-probability case. Proofs of all theorems and lemmas are in the appendix.

Let S and S_{ij} both be augmented by a starting item I_0 and an ending item I_{n+1} where $p_0 = p_{n+1} = 1$ by definition. The augmentation does not affect the comparison of S and S_{ij}.

Lemma 1. $P(S_{ij}) - P(S) = (x_1 - x_2)(y_1 - y_2)$.

Next we give sufficient conditions for determining the signs of $x_1 - x_2$ and $y_1 - y_2$.

Lemma 2. Let $m = [(j - i + 1)/2]^+$, then we have $y_1 \leqslant y_2$ if both:

$$p_{i+k} \leqslant p_{j-k}, \text{ for all even } k,\ 0 \leqslant k \leqslant m, \tag{1a}$$

$$p_{i+k} \geqslant p_{j-k}, \text{ for all odd } k,\ 0 \leqslant k \leqslant m, \tag{1b}$$

(we have $y_1 \geqslant y_2$ if the words "even" and "odd" are interchanged.)

Remark: Assume $0 < p_l < 1$ for all l, $i \leqslant l \leqslant j$. If at least one strict inequality in (1) holds, then $y_1 < y_2$. ($y_1 > y_2$ if the words "even" and "odd" in (1) are interchanged).

Lemma 3. Let $i - 1 \leqslant n - j$. If:

$$p_k \geqslant p_{j+i-k}, \text{ for all even } k, 0 \leqslant k \leqslant i - 1, \tag{2a}$$

$$p_k \leqslant p_{j+i-k}, \text{ for all odd } k, 0 \leqslant k \leqslant i - 1, \tag{2b}$$

then we have

$$x_1 \geqslant x_2, \text{ if } i \text{ odd}, \tag{3a}$$

$$x_1 \leqslant x_2, \text{ if } i \text{ even}. \tag{3b}$$

Remark: An analogous statement holds when $i - 1 \geqslant n - j$. This can be best visualized by looking at S backwards.

Remark: Assume $0 < p_l < 1$ for all l, $1 \leqslant l < i$ or $j < l \leqslant n$. Then the strict inequality in (3) holds if at least one strict inequality in (2) holds. An analogous statement holds when $i - 1 \geqslant n - j$.

Combining lemmas 1 - 3, we obtain a sufficient condition for S_{ij} to be better than S.

Theorem 1. Let $i - 1 \leqslant n - j$ and $m = [(j - i + 1)/2]^+$. For odd i we have $P(S_{ij}) \geqslant P(S)$ if both:

$$p_{i-k} \geqslant p_{j+k}, \text{ for odd } k, 1 \leqslant k \leqslant i, \tag{4a}$$

$$p_{i-k} \leqslant p_{j+k}, \text{ for even } k, 1 \leqslant k \leqslant i, \tag{4b}$$

$$p_{i+k} \geqslant p_{j-k}, \text{ for all even } k, 0 \leqslant k \leqslant m, \tag{5a}$$

$$p_{i+k} \leqslant p_{j-k}, \text{ for all odd } k, 0 \leqslant k \leqslant m. \tag{5b}$$

For even i, we have $P(S_{ij}) \geqslant P(S)$ if the words "even" and "odd" in (4) and (5) are interchanged.

Remark: An analogous statement can be obtained for $i - 1 \geqslant n - j$.

Remark: Assume $0 < p_l < 1$ for all l, $1 \leqslant l \leqslant n$, and $i - 1 \leqslant n - j$. The strict inequality $P(S_{ij}) > P(S)$ holds if (4) and (5) hold, and at least one strict inequality holds in (4) and (5) each.

Assumptions: There are a items, A, of working probability p and b items, B, of working probability q, where $a + b = n$ and $p < q$. The sequence S^* as defined in section 2 takes the form:

$AB \ldots AB$	$Z \ldots Z$	$BA \ldots BA$
$[z/2]^-$ pairs of AB	$z' - z$	$[z/2]^+$ pairs of BA

where $z \equiv \min\{a, b\}$; $z' \equiv \max\{a, b\}$; and $Z = A$ if there are more A's than B's, $Z = B$ otherwise.

Let $S = (I_1, I_2, \ldots, I_n)$ be a sequence different from S^* or its reverse. We shall show that $P(S_{ij}) > P(S)$ for some $i < j$.

Let i be the first position where S and S^* differ, and $n + 1 - i'$ the last position they differ in. Without loss of generality, assume $i \leqslant i'$ (or else we deal with the reverse of S). Let j be the first position after i such that I_i and I_j are of different types.

Checking the conditions in theorem 1 carefully, we obtain:

Theorem 2. Let S be a sequence different from S^* or its reverse, and i, j be defined as above, then $P(S_{ij}) \geqslant P(S)$ for $0 \leqslant p < q \leqslant 1$, and $P(S_{ij}) > P(S)$ for $0 < p < q < 1$.

Since any sequence other than S^* and its reverse is "unstable", we establish S^* and its reverse as the only optimum sequences.

We now give two results which extend the 2-class case to some special 3-class cases.

Assumptions: There are a of type p, b of type q, and c of type 1 with $a + b + c = n$.

Theorem 3. For $0 \leqslant p < q < 1$, $S^*(I_1, I_2, \ldots, I_n)$ is optimal.

Assumptions: There are a of type p, b of type q and c of type 0 with $a + b + c = n$.

Theorem 4. For $0 < p < q \leqslant 1$ $S^*(I_1, I_2, \ldots, I_n)$ is optimal.

APPENDIX: The Proofs

Proof of lemma 1: Since S_{ij} and S will both fail if any of the pairs (I_t, I_{t+1}), $1 \leqslant t < n$, $t \neq i - 1$, $t \neq j$, fails, we assume that this is not the case. Then it suffices to study the following four scenarios.

i. I_{i-1}, I_i, I_{j-1}, work but I_j, I_{j+1} fail. Then S fails and S_{ij} works. The probability of this occurring is $x_1 y_1$.

ii. I_{i+1}, I_j, I_{j+1} work but I_{i-1}, I_i fail. Again, S fails and S_{ij} works. The probability of this occurring is $x_2 y_2$.

iii. I_i, I_{j-1}, I_{j+1} work but I_{i-1}, I_j fail. Now S works and S_{ij} fails. The probability of this occurring is $x_2 y_1$.

iv. I_{i-1}, I_{i+1}, I_j work but I_i, I_{j+1} fail. Again S works and S_{ij} fails. The probability of this occurring is $x_1 y_2$.

Combining the four scenarios, we obtain lemma 1.

Proof of lemma 2: The proof is by induction on m. When $m = 1$, we have either $j = i + 1$ or $i + 2$. In the former, we have:

$$y_1 - y_2 = p_i(1 - p_j) - (1 - p_i)p_j \leqslant 0.$$

In the latter, we have:

$$y_1 - y_2 = p_i p_{i+1}(1 - p_j) - (1 - p_i)p_{i+1}p_j \leqslant 0.$$

Assume lemma 2 true for 1, 2, ..., $m - 1$. Rearranging, we have:

$$y_1 = p_i(1 - p_j)[P(I_{i+1}, \ldots, I_{j-1}|I_{i+1} \text{ works}, I_{j-1} \text{ works})p_{i+1}p_{j-1} + P(I_{i+1}, \ldots, I_{j-1}|I_{i+1} \text{ fails}, I_{j-1} \text{ works})(1 - p_{i+1})p_{j-1}],$$

$$y_2 = (1 - p_i)p_j[P(I_{i+1}, \ldots, I_{j-1}|I_{i+1} \text{ works}, I_{j-1} \text{ works})p_{i+1}p_{j-1} + P(I_{i+1}, \ldots, I_{j-1}|I_{i+1} \text{ works}, I_{j-1} \text{ fails})p_{i+1}(1 - p_{j-1})],$$

$$y_1 - y_2 = [p_i(1 - p_j) - (1 - p_i)p_j]P(I_{i+1}, \ldots, I_{j-1}|I_{i+1} \text{ works}, I_{j-1} \text{ works})p_{i+1}p_{j-1} + p_i(1 - p_j)$$
$$[P(I_{i+1}, \ldots, I_{j-1}|I_{i+1} \text{ fails}, I_{j-1} \text{ works})(1 - p_{i+1})p_{j-1}] - (1 - p_i)p_j[P(I_{i+1}, \ldots, I_{j-1}|I_{i+1} \text{ works}, I_{j-1} \text{ fails})p_{i+1}(1 - p_{j-1})]. \quad (6)$$

Since $p_i \leqslant p_j$, we have:

$$p_i(1 - p_j) - (1 - p_i)p_j \leqslant 0. \quad (7)$$

Letting $i' = i + 1, j' = j - 1, m' = m - 1$, we find:

$$p_{i'+k} \geqslant p_{j'-k}, \quad \text{for all even } k, \quad 0 \leqslant k \leqslant m',$$

$$p_{i'+k} \leqslant p_{j'-k}, \quad \text{for all odd } k, \quad 1 \leqslant k \leqslant m'.$$

The induction hypothesis implies:

$$P(I_{i'}, \ldots, I_{j'}|I_{i'} \text{ fails}, I_{j'} \text{ works})(1 - p_{i'})p_{j'}$$
$$\leqslant P(I_{i'}, \ldots, I_{j'}|I_{i'} \text{ works}, I_{j'} \text{ fails})p_{i'}(1 - p_{j'}). \quad (8)$$

Combining (6)-(8), we obtained the desired result.

Proof of lemma 3: Let T be the following sequence of items:

$$T = T_1T_2 \ldots T_t = I_{j+1}, \ldots, I_n, J_1, J_0, J_1, \ldots, J_0, J_1, I_1, \ldots, I_{i-1}.$$

The item J_1 always works and J_0 never works. A total of $n - j + 1$ J_1's is alternatively interlaced with $n - j$ J_0's. The length of T is $t = 3(n - j) + i$. Note that:

$$x_2 - x_1 = P(T|I_{j+1} \text{ works}, I_{i-1} \text{ fails})p_{j+1}(1 - p_{i-1}) - P(T|I_{j+1} \text{ fails}, I_{i-1} \text{ works})(1 - p_{j+1})p_{i-1}.$$

Let q_l denote the probability that T_l works. For odd i we have:

$$q_l = p_{j+l} \leqslant p_{i-l} = q_{t+1-l}, \text{ for odd } l, 1 \leqslant l \leqslant i - 1,$$

$$q_l = p_{j+l} \geqslant p_{i-l} = q_{t+1-l}, \text{ for even } l, 1 \leqslant l \leqslant i - 1,$$

$$q_l \leqslant 1 = q_{t+1-l}, \text{ for odd } l, i \leqslant l \leqslant [t/2]^+,$$

$$q_l \geqslant 0 = q_{t+1-l}, \text{ for even } l, i \leqslant l \leqslant [t/2]^+.$$

By lemma 2, we have:

$$P(T| T_1 \text{ works}, T_t \text{ fails})q_1(1 - q_t) \leqslant P(T| T_1 \text{ fails}, T_t \text{ works})(1 - q_1)q_t.$$

Therefore, for odd i, we have $x_2 - x_1 \leqslant 0$. Similarly, for even i, we have $x_2 - x_1 \geqslant 0$.

Proof of theorem 2: If i is odd, then $p_i = q > p = p_j$. The fact that $p_{i+1} = p_{i+2} = \ldots = p_{j-1} = q$ makes it clear that the inequalities (5) in theorem 1 are satisfied. Furthermore,

$$p_0 = 1 \geqslant p_{j+1}$$

$$p_{i-k} = q \geqslant p_{j+k} \text{ for } k \text{ odd}, 1 \leqslant k < i - 2,$$

$$p_{i-k} = p \leqslant p_{j+k} \text{ for } k \text{ even}, 2 \leqslant k \leqslant i - 1.$$

So the inequalities (4) are also satisfied. It follows that $P(S_{ij}) \geqslant P(S)$.

An analogous argument takes care of the even i case. Further, we have:

$$p_0 = 1 > p_{j+i} \text{ and } p_i = q > p = p_j.$$

Note that $p_{j+i} < 1$ because $p_{j+i} = 1$ implies $j + i = n + 1$ and S = reverse of S^*, a contradiction. By the remark following theorem 1, we obtain $P(S_{ij}) > P(S)$.

Proof of theorem 3: We prove theorem 3 by induction on n. Theorem 3 is trivially true for $n = 3$. For $n > 3$ we assume that a, b, c are all nonzero, for otherwise either theorem 3 is trivial or theorem 2 applies. Let $S = (I_1, I_2, \ldots, I_n)$ be an optimal sequence and again add two items I_0 and I_{n+1} of type 1 to S. Let $I_{i_0} = I_0, I_{i_1}, I_{i_2}, \ldots, I_{i_c}, I_{i_{c+1}} = I_{n+1}$ denote the $c + 2$ items of type 1. Let S_k denote the sequence of items between $I_{i_{k-1}}$ and I_{i_k} in S. Then—

$$P(S) = \prod_{k=1}^{c+1} P(S_k).$$

Note that we can permute the S_k's without affecting $P(S)$.

We show that if $(p_1, p_2) \neq (p, 1)$ in S, then we can always find a sequence S' such that $(p_1', p_2') = (p, 1)$ and $P(S') \geqslant P(S)$. Note that $S^*(I_1, I_2, \ldots, I_n)$ is (I_{i_1}, I_{i_n}) juxtaposed with $\bar{S}$, where $\bar{S}$ is the reverse of $S^*(I_{i_2}, I_{i_3}, \ldots, I_{i_{n-1}})$. Now $(p_{i_1}, p_{i_n}) = (p, 1)$ and by the induction hypothesis $\bar{S}$ is an optimal sequence for $n - 2$ items consisting of $a - 1$ items of type p, b items of type q, and $c - 1$ items of type 1. Therefore—

$$P(S^*) \geqslant P(S') \geqslant P(S) \geqslant P(S^*),$$

i.e., S^* is an optimal sequence.

We now show the existence of S' with the specified properties.

Since $a \geqslant 1$ one of the S_k must contain an object of type p. Since permuting S_k's does not affect $P(S)$, we may assume that S_1 contains an object of type p. Let $I_{11}, I_{12}, \ldots I_{1m}$ denote the items in S_1. From theorem 2 we may assume that $S_1 = S^*(I_{11}, I_{12}, \ldots, I_{1m})$. Therefore $p_1 = p$. Now if I_{12} does not exist, i.e., $S_1 = (I_{11})$, then $I_2 = 1$ and we are through with $S' = S$. If I_{12} exists, define $I_i = I_{12}$, $I_j = I_{i_1}$ and $S' = S_{ij}$. Then $p'_2 = 1$. We show $P(S') \geqslant P(S)$.

First note that $i_1 < n$ for otherwise we can interchange I_{j-1} with I_j and obtain a strict improvement. Hence $i - 1 \leqslant n - j$. Let $m = [(j - i + 1)/2]^+$. By using the definition of $S^*(I_{11}, I_{12}, \ldots, I_{1m})$, it is straightforward to verify that:

$$p_i < 1 = p_j$$

$$p_{i+k} \geqslant p_{j-k} \text{ for } k \text{ odd, } 0 \leqslant k \leqslant m,$$

$$p_{i+k} \leqslant p_{j-k} \text{ for } k \text{ even, } 0 \leqslant k \leqslant m.$$

Furthermore,

$$p_{i-1} = p_1 = p \leqslant p_{j+1},$$

$$p_{i-2} = p_0 = 1 \geqslant p_{j+2}.$$

It follows from Theorem 1 that $P(S') \geqslant P(S)$.

Proof of theorem 4: We prove theorem 4 again by induction on n. Theorem 4 is trivially true for $n \leqslant 3$. If a, b and c are not all positive, then again theorem 4 is either trivial or follows from theorem 2. Therefore we assume $n > 3$ and a, b, c are all positive.

Let $S = (I_1, I_2, \ldots, I_n)$ be an optimal sequence. We show that we may assume that I_1 is of type 0. If not, let I_j be the first item of type 0. We assume $j \neq n$ for otherwise we will consider the reverse of S. Suppose that I_{j+1} is of type r' where r' is either p or q. Then since I_{j+1} must work,

$$\begin{aligned} P(S) &= P(I_1, I_2, \ldots, I_j)\, r'P(I_{j+2}, I_{j+3}, \ldots, I_n) \\ &\leqslant P(S^*(I_1, I_2, \ldots, I_j))\, r'P(I_{j+2}, I_{j+3}, \ldots, I_n). \end{aligned}$$

Therefore we may assume that $(I_1, I_2, \ldots, I_j) = S^*(I_1, I_2, \ldots, I_j)$ and I_1 is of type 0.

Suppose that I_2 is of type r. Since I_2 must work,

$$\begin{aligned} P(S) &= rP(I_3, I_4, \ldots, I_n) \\ &\leqslant rP(S^*(I_3, I_4, \ldots, I_n)). \end{aligned}$$

Clearly, we may assume that r is either p or q. Suppose $r = p$. Let $i = 2$ and let I_j be the first item of type q. By theorem 1, we can verify that $P(S_{ij}) \geqslant P(S)$ straightforwardly. Therefore, we may assume that I_2 is of type q. Since S works only if I_2 works,

$$\begin{aligned} P(S) &= qP(I_3, I_4, \ldots, I_n) \\ &\leqslant qP(S^*(I_3, I_4, \ldots, I_n)) \\ &= qP(\text{the reverse of } S^*(I_3, I_4, \ldots, I_n)) \\ &= P(S^*(I_1, I_2, \ldots, I_n)). \end{aligned}$$

Therefore $S^*(I_1, I_2, \ldots, I_n)$ is optimal.

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