# ON THE THEORY OF DIOPHANTINE APPROXIMATIONS. I[1]
## (ON A PROBLEM OF A. OSTROWSKI)

By
VERA T. SÓS (Budapest)
(*Presented by* A. RÉNYI)

In what follows we denote by $\langle x\rangle$ the fractional part of the positive $x$, and by $\alpha$ any number with $0<\alpha<1$. As well known,

$$\lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N}\langle n\alpha\rangle=\frac{1}{2}.$$

We put for an $\alpha$

$$\sum_{n=1}^{N}\langle n\alpha\rangle-\frac{N}{2}=C_\alpha(N).$$

As A. OSTROWSKI [1][2] has shown, for any irrational $\alpha$ the quantity $C_\alpha(N)$ is unbounded. In the same paper he raised the question whether or not $C_\alpha(N)$ can for an appropriate $\alpha$ be *onesidedly* bounded. In this paper we are going to give to this question an affirmative answer, i. e. to prove the following

THEOREM. *There is an irrational $\alpha$ and a constant $C$ such that for $N=1,2,\dots$ the inequality*

$$C_\alpha(N)>C$$

*holds.*

A slight modification of our construction gives at the same time the existence of a set of such $\alpha$'s having the power of the continuum.

As A. RÉNYI remarked, the constant $C$ of our theorem cannot be $\geqq 0$; a slight modification of our construction would show that for $C$ we could prescribe any negative number. We shall omit this modification.

In the proof we start from a geometrical interpretation of continued fractions which is applicable also to other questions of diophantine approximations. So I intend to return in this sequence of papers to a theorem of A. KHINTCHINE, i. e. to the lower estimation of

$$\sup_{\beta}\ \inf_{\substack{x\geqq 0\\ y,\,x\text{ integers}}} x|\alpha x+\beta-y|,$$

[1] The results of this sequence of papers were contained in my dissertation, defended in June 1957, and I lectured on some parts of it in Lublin and Lodz in September 1956.

[2] The numbers in brackets refer to the Bibliography given at the end of the paper.

investigated previously by A. KHINTCHINE [2], S. FUKASAWA [3], H. DAVENPORT [4] and A. V. PRASAD [5], and to the upper estimation of

$$\inf_{\alpha} \inf_{\substack{x \geqq 0 \\ x,\, y \text{ integers}}} x|\alpha x+\beta-y|,$$

investigated by J. W. S. CASSELS [6]. Further applications I shall publish elsewhere.

In § 1 of this paper we give this geometrical interpretation of continued fractions and announce some lemmas whose proof will be postponed owing to their general character to the Appendix. In § 2 we deduce from it an "exact formula" for $\sum_{n=1}^{N}\langle n\alpha\rangle$ (Main Lemma). In § 3 we prove the announced theorem.

## § 1

Starting from a fixed point $O$ of the periphery of a circle $E$ with unity periphery we put up in positive direction the arc with the length $\alpha$ $(0<\alpha<1)$ once, twice, ..., $n$-times, .... The endpoints of these arcs we shall call the "$n\alpha$-points" $(n=1, 2, \ldots)$. We need the following

DEFINITION. We call the $s\alpha$-point "adjacent to $O$", and the corresponding $s$ an "adjacent multiplum of $O$", if no $n\alpha$-point with $0<n<s$ is contained in one of the two closed arcs determined by $O$ and the $s\alpha$-point.

So we obtained to our fixed $\alpha$ a sequence

$$(1.1) \qquad 0=s_0,\ 1=s_1<s_2<s_3<\cdots$$

of adjacent multipla; we shall denote the "empty" open arc corresponding to the $s_\nu\alpha$-point by $\Delta_\nu$, the length of this arc by $\bar{\delta}_\nu$. We shall use also the directed "empty" arc between $O$ and the $s_\nu\alpha$-point, the sign of its length being positive and negative, respectively, according to the direction in which the arc $\Delta_\nu$ starts from $O$. This length with sign we shall denote by $\delta_\nu$.

Particularly important are those $s_\nu$-multipla from (1.1) for which $\delta_\nu$ and $\delta_{\nu+1}$ have different signs. We shall call these $s_{\nu_1}, s_{\nu_2}, \ldots, s_{\nu_k}, \ldots$, forming a subsequence of the sequence $s_1, s_2, \ldots, s_\nu, \ldots$, the "jumping-multipla" and denote them by

$$(1,2) \qquad q_1, q_2, \ldots, q_k=s_{\nu_k}, \ldots.$$

In the case when $\frac{1}{2}<\alpha<1$, the definition needs an additional remark, it is suitable to define $q_2=q_1=1$. If $k>1$, then $q_{k+1}>q_k$. The corresponding quantities $\bar{\delta}_{\nu_k}$ and $\delta_{\nu_k}$ we shall denote simply by $\bar{d}_k$ and $d_k$, respectively.

We define $\bar{d}_0 = 1$. Next we define

$$a_k = \left[\frac{\bar{d}_{k-1}}{\bar{d}_k}\right] \qquad (k=1,2,\ldots). \tag{1.3}$$

LEMMA I. *If the $s_\nu\alpha$-point is adjacent to $O$ and from the opposite side of $O$ the nearest to $O$ among the $\alpha$-, $2\alpha$-, ..., $s_\nu\alpha$-points is the $s_{\nu-l}\alpha$-point ($l$ positive integer), then we have*

$$s_{\nu+1} = s_\nu + s_{\nu-l}, \tag{1.4}$$

$$\delta_{\nu+1} = \delta_\nu + \delta_{\nu-l}. \tag{1.5}$$

The geometrical meaning of Lemma I is that one obtains the arc $\Delta_{\nu+1}$ from the arcs $\Delta_\nu$ and $\Delta_{\nu-l}$ in the following way: considering the larger of $\Delta_\nu$ and $\Delta_{\nu-l}$, from its endpoint different from the point $O$ we draw back the smaller of the arcs $\Delta_\nu$ and $\Delta_{\nu-l}$. This remark will be often used explicitly or implicitly.

LEMMA II. *We have for the above-defined quantities the recursive formulae for $k=1,2,\ldots$:*

$$q_{k+1} = q_{k-1} + a_k q_k, \tag{1.6}$$

$$d_{k+1} = d_{k-1} + a_k d_k, \qquad \bar{d}_{k+1} = \bar{d}_{k-1} - a_k \bar{d}_k, \tag{1.7}$$

$$\left.\begin{aligned} s_{\nu_k+r} &= q_{k-1} + r q_k, \\ \delta_{\nu_k+r} = d_{k-1} + r d_k, &\qquad \bar{\delta}_{\nu_k+r} = \bar{d}_{k-1} - r\bar{d}_k \end{aligned}\right\} \quad (0<r<a_k), \tag{1.8, 1.9}$$

$$q_{k+1}\bar{d}_k + q_k\bar{d}_{k+1} = 1. \tag{1.10}$$

LEMMA III. *If the positive integer $n$ is not an $s$ from (1.1), then there exists an $s_\nu$ with $s_\nu < n$ and*

$$s_\nu\bar{\delta}_\nu < n t_n \tag{1.11}$$

*where*

$$t_n = \min(\langle n\alpha\rangle, 1-\langle n\alpha\rangle).$$

*Further, if $s_\nu$ is not a $q_k$ from (1.2), then there is a $q_k < s_\nu$ such that*

$$q_k < s_\nu < q_{k+1}$$

*and*

$$q_{k+1}\bar{d}_{k+1} < s_\nu\bar{\delta}_\nu. \tag{1.12}$$

The proof of these lemmas will follow in the Appendix. This shows that the $s_\nu$-multipla in (1.1) are identical with the set of all denominators and by-denominators (Neben-Nenner) of the convergents of the continued fraction of $\alpha$, whereas the $q$-multipla in (1.2) are the denominators of its convergents and the $a_k$'s its digits.[3]

[3] In this paper we use the term "digit" instead of the usual "partial quotient".

Moreover we shall need the following special

LEMMA IV. *An arbitrary positive integer N can be represented in the form*

$$N = s_{\mu_1} + s_{\mu_2} + \cdots + s_{\mu_k} \tag{1.13}$$

*where the $s_{\mu_j}$-numbers are the adjacent multipla of an arbitrarily prescribed irrational $\alpha$ in the sense of* (1.1), *further*

$$\mu_1 > \mu_2 > \cdots > \mu_k$$

*and*[4]

$$n_j \equiv N - \sum_{i=1}^{j} s_{\mu_i} < s_{\mu_j - 1} \qquad (j = 1, 2, \ldots, k). \tag{1.14}$$

PROOF. Is $N$ one of our $s_\nu$-numbers, we have nothing to prove. If not, then there is an index $\mu_1$ with

$$s_{\mu_1} < N < s_{\mu_1+1}.$$

Owing to (1.4) we have

$$n_1 = N - s_{\mu_1} < s_{\mu_1+1} - s_{\mu_1} = s_{\mu_1-1} \leqq s_{\mu_1-1}.$$

Next there is an index $\mu_2$ with $\mu_2 < \mu_1$ and

$$s_{\mu_2} \leqq n_1 < s_{\mu_2+1}.$$

Again we have, using (1.4),

$$n_2 = n_1 - s_{\mu_2} < s_{\mu_2+1} - s_{\mu_2} \leqq s_{\mu_2-1}$$

and this process is obviously finished after a finite number of steps.

## § 2

Let $N$ be a positive integer, $\alpha$ a positive irrational number and we represent $N$ in the form (1.13). Then we assert the following

MAIN LEMMA.[5] *With the notation of § 1 and the representation* (1.13) *the formula*

$$\begin{aligned} C_\alpha(N) \equiv \sum_{n=1}^{N} \langle n\alpha \rangle - \frac{N}{2} = \left( \bar{\delta}_{\mu_k} \frac{s_{\mu_k}+1}{2} - \frac{1}{2} \right) \operatorname{sign} \delta_{\mu_k} + \\ + \sum_{j=1}^{k-1} \left\{ \bar{\delta}_{\mu_j} \left( \frac{s_{\mu_j}+1}{2} + s_{\mu_{j+1}} + \cdots + s_{\mu_k} \right) - \frac{1}{2} \right\} \operatorname{sign} \delta_{\mu_j} \end{aligned} \tag{2.1}$$

*holds.*

[4] In the case when $\mu_k = 1$, the last inequality for $n_k$ must be dropped.

[5] An exact formula occurs also in OSTROWSKI's paper [1]. His formula contains only the denominators of the convergents of the continued fraction of $\alpha$.

For the proof we shall need the following lemmas:

LEMMA V. *The Main Lemma is true in case of* $k=1$, *i. e.*

$$\sum_{n=1}^{s_\mu} \langle n\alpha \rangle = \frac{s_\mu}{2} + \left\{ \bar{\delta}_\mu \frac{s_\mu+1}{2} - \frac{1}{2} \right\} \operatorname{sign} \delta_\mu .$$

PROOF. The $n\alpha$-points $(n=0, 1, 2, \dots, s_\mu)$ divide the periphery of the circle $E$ into $(s_\mu+1)$ disjunct arcs; starting from the point $O$ in positive direction we denote the length of these arcs by $t_0, t_1, \dots, t_{s_\mu}$, respectively. Since the arcs with the length $\langle n\alpha \rangle$ put up on $E$ from $O$ in positive direction $(n=0, 1, \dots, s_\mu)$ cover the arc with the length $t_l$ $(l=0, 1, \dots, s_\mu)$ obviously $(s_\mu - l)$-times, we have on the one hand

$$\sum_{n=1}^{s_\mu} \langle n\alpha \rangle = \sum_{l=0}^{s_\mu} (s_\mu - l) t_l . \tag{2.2}$$

On the other hand, we can determine the sum on the left side putting up the arcs $\alpha, 2\alpha, \dots, s_\mu\alpha$ in the negative direction, starting from the $s_\mu\alpha$-point. These points in their totality coincide obviously with the $n\alpha$-points $(n=0, 1, \dots, s_\mu)$. Thus now the $s_\mu\alpha$-point plays the role of $O$ and we have to sum the distances of our points from the $s_\mu\alpha$-point.

*Case I.* $\delta_\mu > 0$ (i. e. $\delta_\mu = t_0$). Then expressing our sum again by means of the $t_l$'s we obviously get

$$\sum_{n=1}^{s_\mu} \langle n\alpha \rangle = s_\mu t_0 + (s_\mu - 1) t_{s_\mu} + (s_\mu - 2) t_{s_\mu - 1} + \cdots + 2 t_3 + t_2 . \tag{2.3}$$

Adding (2. 2) and (2. 3) we obtain

$$\sum_{n=1}^{s_\mu} \langle n\alpha \rangle = s_\mu t_0 + \frac{s_\mu - 1}{2} (t_1 + t_2 + \cdots + t_{s_\mu}). \tag{2.4}$$

Since

$$t_0 + t_1 + \cdots + t_{s_\mu} = 1,$$

(2. 4) gives

$$\sum_{n=1}^{s_\mu} \langle n\alpha \rangle = \frac{1}{2} (s_\mu - 1 + (s_\mu + 1) t_0) =$$

$$= \frac{s_\mu}{2} + \left\{ \bar{\delta}_\mu \frac{s_\mu+1}{2} - \frac{1}{2} \right\} = \frac{s_\mu}{2} + \left\{ \bar{\delta}_\mu \frac{s_\mu+1}{2} - \frac{1}{2} \right\} \operatorname{sign} \delta_\mu .$$

*Case II.* $\delta_\mu < 0$ (i. e. $\delta_\mu = -t_{s_\mu}$). Then the identity corresponding to (2. 3) is

$$\sum_{n=1}^{s_\mu} \langle n\alpha \rangle = s_\mu t_{s_\mu - 1} + (s_\mu - 1) t_{s_\mu - 2} + \cdots + 2 t_1 + t_0 . \tag{2.5}$$

Adding (2. 2) and (2. 5) we obtain

$$\sum_{n=1}^{s_\mu}\langle n\alpha\rangle=\frac{s_\mu+1}{2}(t_0+\cdots+t_{s_\mu-1})=\frac{s_\mu+1}{2}(1-t_{s_\mu})= \tag{2.6}$$
$$=\frac{s_\mu}{2}+\left\{\delta_\mu\frac{s_\mu+1}{2}+\frac{1}{2}\right\}=\frac{s_\mu}{2}+\left\{\overline{\delta}_\mu\frac{s_\mu+1}{2}-\frac{1}{2}\right\}\operatorname{sign}\delta_\mu.$$

Further we need the simpler

LEMMA VI. *Let $m$, $S$ be positive integers and let us consider the $(m+j)\alpha$-points $(j=1,2,\dots,S)$. If one of the arcs determined by $O$ and the $m\alpha$-point is empty and the directed length of this empty arc is $d(m)$, then we have*

$$\sum_{j=1}^{S}\langle(m+j)\alpha\rangle=\sum_{j=1}^{S}\langle j\alpha\rangle+Sd(m).$$

PROOF. The directed distance from the $j\alpha$-point to the $(m+j)\alpha$-point on the circle is the same as between $O$ and the $m\alpha$-point, i. e.

$$\langle(m+j)\alpha\rangle-\langle j\alpha\rangle=d(m)$$

from which summation for $j=1,2,\dots,S$ already proves the lemma.

Finally we prove the

LEMMA VII. *Using the representation* (1. 13) *it holds for $j=1,2,\dots,k$ that one of the two arcs of the circle $E$ determined by $O$ and the $(s_{\mu_1}+\cdots+s_{\mu_j})\alpha$-point does not contain any $n\alpha$-point whenever*

$$s_{\mu_1}+\cdots+s_{\mu_j}<n\leqq s_{\mu_1}+\cdots+s_{\mu_{j+1}}. \tag{2.7}$$

PROOF. From the point $O$ we can reach the $(s_{\mu_1}+s_{\mu_2}+\cdots+s_{\mu_j})\alpha$-point starting from $O$ going first to the $s_{\mu_1}\alpha$-point along the arc $\Delta_{\mu_1}$, then from the $s_{\mu_1}\alpha$-point to the $(s_{\mu_1}+s_{\mu_2})\alpha$-point along the arc with the directed length $\delta_{\mu_2}$, and so forth, and finally from the $(s_{\mu_1}+\cdots+s_{\mu_{j-1}})\alpha$-point to the $(s_{\mu_1}+\cdots+s_{\mu_j})\alpha$-point along the arc with the directed length $\delta_{\mu_j}$. We shall prove our lemma a fortiori by showing that for the $n$'s in (2. 7) no $n\alpha$-points lie in these arcs with the directed length $\delta_{\mu_1},\delta_{\mu_2},\dots,\delta_{\mu_j}$. First of all from (2. 7) it follows that for $i=1,2,\dots,j$

$$n>s_{\mu_1}+\cdots+s_{\mu_i}. \tag{2.8}$$

If for an $n$ the $n\alpha$-point would lie on the above-mentioned arc with the directed length $\delta_{\mu_{i+1}}$, then the ordering of the points

$$(s_{\mu_1}+\cdots+s_{\mu_i})\alpha,\ n\alpha,\ (s_{\mu_1}+\cdots+s_{\mu_{i+1}})\alpha$$

would be the same as the ordering of the points

$$O,\ (n-s_{\mu_1}-\cdots-s_{\mu_i})\alpha,\ s_{\mu_{i+1}}\alpha.$$

Taking into account (2. 8) and the definition of $s_{\mu_{i+1}}$ it would follow

$$n - s_{\mu_1} - \cdots - s_{\mu_i} \geqq s_{\mu_{i+1}+1},$$

i. e.

$$n \geqq s_{\mu_1} + \cdots + s_{\mu_i} + s_{\mu_{i+1}+1}.$$

But owing to Lemma IV

$$s_{\mu_{i+1}+1} > s_{\mu_{i+1}} + \cdots + s_{\mu_k},$$

i. e. $n > N$ would follow, which is a contradiction.

From the above lemmas the proof of the Main Lemma can be completed as follows. We write

$$\sum_{n=1}^{N} \langle n\alpha \rangle = \sum_{n=1}^{s_{\mu_1}} \langle n\alpha \rangle + \sum_{n=s_{\mu_1}+1}^{s_{\mu_1}+s_{\mu_2}} \langle n\alpha \rangle + \cdots + \sum_{n=s_{\mu_1}+\cdots+s_{\mu_{k-1}}+1}^{s_{\mu_1}+\cdots+s_{\mu_k}} \langle n\alpha \rangle.$$

Owing to Lemma VII, Lemma VI is applicable; using also Lemma V we obtain

$$\begin{aligned} \sum_{n=1}^{N} \langle n\alpha \rangle = {} & \frac{s_{\mu_1}}{2} + \left\{ \bar{\delta}_{\mu_1} \frac{s_{\mu_1}+1}{2} - \frac{1}{2} \right\} \operatorname{sign} \delta_{\mu_1} + \\ & + \frac{s_{\mu_2}}{2} + \left\{ \bar{\delta}_{\mu_2} \frac{s_{\mu_2}+1}{2} - \frac{1}{2} \right\} \operatorname{sign} \delta_{\mu_2} + s_{\mu_2} d(s_{\mu_1}) + \\ & \vdots \\ & + \frac{s_{\mu_k}}{2} + \left\{ \bar{\delta}_{\mu_k} \frac{s_{\mu_k}+1}{2} - \frac{1}{2} \right\} \operatorname{sign} \delta_{\mu_k} + s_{\mu_k} d(s_{\mu_1} + \cdots + s_{\mu_k}). \end{aligned} \tag{2. 9}$$

From what has been said in the proof of Lemma VII it follows

$$d(s_{\mu_1} + \cdots + s_{\mu_j}) = \delta_{\mu_1} + \cdots + \delta_{\mu_j}.$$

Putting it into (2. 9) the proof of the Main Lemma is complete.

## § 3

We shall prove the announced theorem. We use again the representation (1. 13) of $N$ and divide the $s_{\mu_j}$'s into two classes according to the sign of the corresponding $\delta_{\mu_j}$. Let

$$\begin{aligned} s_{\mu_j} &= s'_{\mu_j}, & \delta_{\mu_j} &= \delta'_{\mu_j} \quad \text{for} \quad \delta_{\mu_j} > 0, \\ s_{\mu_j} &= s''_{\mu_j}, & \delta_{\mu_j} &= \delta''_{\mu_j} \quad \text{for} \quad \delta_{\mu_j} < 0. \end{aligned} \tag{3. 1}$$

Introducing this notation in the Main Lemma we get

$$\begin{aligned} C_\alpha(N) = {} & \sum_j \left\{ \delta'_{\mu_j} \left( \frac{s'_{\mu_j}+1}{2} + \sum_{s_{\mu_l} < s'_{\mu_j}} s_{\mu_l} \right) - \frac{1}{2} \right\} + \\ & + \sum_j \left\{ \frac{1}{2} - \delta''_{\mu_j} \left( \frac{s''_{\mu_j}+1}{2} + \sum_{s_{\mu_l} < s''_{\mu_j}} s_{\mu_l} \right) \right\} \equiv \Sigma_1 + \Sigma_2. \end{aligned} \tag{3. 2}$$

Omitting from $\Sigma_1$ the positive terms $\sum_{s_{\mu_l}<s_{\mu_j}} s_{\mu_l}$ and taking into account that from Lemma IV

$$\sum_{s_{\mu_l}<s''_{\mu_j}} s_{\mu_l} < s''_{\mu_j},$$

we obtain from (3. 2)

$$C_\alpha(N) > \frac{1}{2}\sum_j (\bar{\delta}'_{\mu_j} s'_{\mu_j} - 1) - \frac{3}{2}\sum_j \delta''_{\mu_j} s''_{\mu_j} \equiv \Sigma'_1 + \Sigma'_2. \tag{3.3}$$

This suggests as a guide for the choice of $\alpha$ that for the $s'_{\mu_j}$-multipla we should have $s'_{\mu_j}\bar{\delta}'_{\mu_j} \sim 1$ and, on the other hand, the products $s''_{\mu_j}\bar{\delta}''_{\mu_j}$ should be small, i. e. $O$ should be approached "badly" from the positive side and well from the negative one.

The actual construction of such an $\alpha$ can be performed as follows. Denoting the digits of the continued fraction of an $\alpha$ by $a_1, a_2, \ldots$,

$$\alpha = \frac{1}{a_1+}\,\frac{1}{a_2+}\,\frac{1}{a_3+\cdots},$$

we define

$$a_{2k-1} = 1, \tag{3.4}$$

$$a_{2k} = k^3 \qquad (k = 1, 2, \ldots); \tag{3.5}$$

owing to $a_1 = 1$ we have $\frac{1}{2} < \alpha < 1$ and owing to the additional remark on p. 462

$$q_1 = q_2 = 1. \tag{3.6}$$

The formulae (1. 6) and (1. 7) give

$$q_{2k} = q_{2k-1} + q_{2k-2}, \tag{3.7}$$

$$q_{2k+1} = q_{2k-1} + k^3 q_{2k}, \tag{3.8}$$

$$\bar{d}_{2k} = \bar{d}_{2k+1} + \bar{d}_{2k+2}, \tag{3.9}$$

$$\bar{d}_{2k+1} = k^3 \bar{d}_{2k+2} + \bar{d}_{2k+3}. \tag{3.10}$$

From (1. 10) and (3. 8) we obtain

$$q_{2k}\bar{d}_{2k} = \frac{1}{\dfrac{q_{2k+1}}{q_{2k}} + \dfrac{\bar{d}_{2k+1}}{\bar{d}_{2k}}} < \frac{q_{2k}}{q_{2k+1}} < \frac{1}{k^3}. \tag{3.11}$$

Again (1. 10) gives

$$q_{2k+1}\bar{d}_{2k+1} = \frac{1}{\dfrac{q_{2k+2}}{q_{2k+1}} + \dfrac{\bar{d}_{2k+2}}{\bar{d}_{2k+1}}}. \tag{3.12}$$

(3. 10) gives at once

$$\frac{\bar{d}_{2k+2}}{\bar{d}_{2k+1}} < \frac{1}{k^3}$$

and from (3. 7) and (3. 8)

$$\frac{q_{2k+2}}{q_{2k+1}} = \frac{q_{2k+1} + q_{2k}}{q_{2k+1}} < 1 + \frac{1}{k^3};$$

putting this into (3. 12)

$$q_{2k+1}\bar{d}_{2k+1} > \frac{1}{1+\frac{2}{k^3}} > 1 - \frac{2}{k^3}. \tag{3.13}$$

In order to extend the estimations (3. 11) and (3. 13) to all $s_\mu \bar{\delta}_\mu$'s we remark first that owing to (3. 4) all $s''_\mu$'s are at the same time $q$'s, i. e. also with some $k$

$$s''_\mu \bar{\delta}''_\mu = q_{2k}\bar{d}_{2k} < \frac{1}{k^3}. \tag{3.14}$$

As to the $s'_\mu$'s (3. 13) and Lemma III give for all $s'_\mu$'s with

$$q_{2k} < s'_\mu < q_{2k+1} \tag{3.15}$$

the estimation

$$s'_\mu \bar{\delta}'_\mu > q_{2k+1}\bar{d}_{2k+1} > 1 - \frac{2}{k^3},$$

i. e.

$$s'_\mu \bar{\delta}'_\mu - 1 > -\frac{2}{k^3}. \tag{3.16}$$

Now (3. 14) and (3. 3) give at once

$$\Sigma'_2 > -\frac{3}{2}\sum_{k=1}^{\infty}\frac{1}{k^3} > -3. \tag{3.17}$$

To obtain a lower bound for $\Sigma'_1$ by the aid of (3. 16) we have to consider how many terms belong to the same $k$ for any fixed $k$. The number of the $s_\nu$-"Neben-Nenner" satisfying (3. 15) is owing to $a_{2k} = k^3$ obviously $k^3$; we need only an upper bound for the number of those which beside fulfilling (3. 15) also occur in the representation (1. 13) of $N$. Let these $s'_\mu$'s be

$$s_{\mu_1, k}, s_{\mu_2, k}, \ldots, s_{\mu_r, k}$$

where

$$s_{\mu_i, k} = q_{2k-1} + \mu_i q_{2k} \qquad (\mu_1 < \mu_2 < \cdots < \mu_r). \tag{3.18}$$

We have to find an upper bound for $r$. Owing to the representation (1. 13) we have

$$s_{\mu_1, k} + \cdots + s_{\mu_{r-1}, k} < s_{\mu_r, k} \leqq q_{2k+1} = q_{2k-1} + k^3 q_{2k}.$$

Using (3. 18) this gives a fortiori

$$\mu_1+\mu_2+\cdots+\mu_{r-1}<k^3.$$

Since

$$\mu_1+\mu_2+\cdots+\mu_{r-1}\geqq 1+2+\cdots+(r-1)=\frac{r(r-1)}{2},$$

we get

$$k^3>\frac{r(r-1)}{2}, \qquad r<3k^{3/2}.$$

Hence, by (3. 3) and (3. 16), we obtain

$$\Sigma_1'>-\sum_{k=1}^{\infty}3k^{3/2}\frac{1}{k^3}.$$

This and (3. 17) complete the proof.

## Appendix

As told in the introduction, we shall prove here the first three lemmas.

PROOF OF LEMMA I. We denote by $\Delta_\nu$ the arc with the endpoints $O$ and the $s_\nu\alpha$-point; then owing to the definition of $l$, $\Delta_\nu$ and $\Delta_{\nu-l}$ have no common point except $O$. Let the $m\alpha$-point fall into $\Delta_\nu+\Delta_{\nu-l}$, then we have

$$m>s_\nu.$$

Since the length of the arc $\Delta_\nu+\Delta_{\nu-l}$ is $\bar{\delta}_\nu+\bar{\delta}_{\nu-l}$, there are two possibilities.

*Case I.* The length of the arc with the endpoints $s_\nu\alpha$-point and $m\alpha$-point (within $\Delta_\nu+\Delta_{\nu-l}$) is $\leqq\bar{\delta}_{\nu-l}$.

*Case II.* The length of the arc with the endpoints $s_{\nu-l}\alpha$-point and $m\alpha$-point (within $\Delta_\nu+\Delta_{\nu-l}$) is $<\bar{\delta}_\nu$.

In Case I the directed distance from the $s_\nu\alpha$-point to the $m\alpha$-point on the circle is the same as that from $O$ to the $(m-s_\nu)\alpha$-point and this last point lies on $\Delta_{\nu-l}$ or in the $s_{\nu-l}\alpha$-endpoint. Owing to the definition of $s_{\nu-l}$ we have in this case $m-s_\nu\geqq s_{\nu-l}$, i. e.

(4. 1) $$m\geqq s_\nu+s_{\nu-l}.$$

In Case II the analogous reasoning gives

(4. 2) $$m>s_\nu+s_{\nu-l}.$$

The smallest $m$ for which the $m\alpha$-point falls into $\Delta_\nu+\Delta_{\nu-l}$ is, according to the definition, $s_{\nu+1}$; from (4. 1) and (4. 2) it follows that

$$s_{\nu+1}\geqq s_\nu+s_{\nu-l}.$$

On the other hand, the $(s_\nu+s_{\nu-l})\alpha$-point lies on the arc $\Delta_\nu+\Delta_{\nu-l}$, indeed, since the directed arc length on the circle from the $s_\nu\alpha$-point to the $(s_\nu+s_{\nu-1})\alpha$-point is the same as that from $O$ to the $s_{\nu-1}\alpha$-point. This proves (1. 4) and (1. 5) consequently.

PROOF OF LEMMA II. It follows from the definition of the $q_k$'s that the arc of the circle $E$ which is bordered by $O$ and the $q_{k-1}\alpha$-point, contains none of the $\alpha$-, $2\alpha$-, ..., $q_k\alpha$-points. Hence, according to Lemma I,

$$(4.3)\qquad \begin{aligned} s_{\nu_k+1} &= q_{k-1}+q_k,\\ \delta_{\nu_k+1} &= d_{k-1}+d_k. \end{aligned}$$

The remark after Lemma I in § 1 and the definition (1. 3) of the digits $a_k$ give that on the one hand the $s_{\nu_k+1}\alpha$-, ..., $s_{\nu_k+a_k}\alpha$-points lie on the same side of $O$ as the $q_{k-1}\alpha$-point, and on the other hand the $s_{\nu_k+a_k+1}\alpha$-point on the opposite side, i. e.

$$(4.4)\qquad q_{k+1}=s_{\nu_k+a_k}.$$

(4. 3) and the repeated use of Lemma I give already (1. 3) and, as easy to see, also (1. 9). Owing to (4. 4) the special case $r=a_k$ gives already (1. 6) and (1. 7).

Since $q_0=0$, $q_1=1$, $d_0=1$, $d_1=\alpha$ and from (1. 6) and (1. 7) $q_2=a_1$, $\bar{d}_2=1-a_1\alpha$, we have

$$q_1\bar{d}_2+q_2\bar{d}_1=1$$

and (1. 10) follows from (1. 6) and (1. 7) by an easy induction.

PROOF OF LEMMA III. (1. 11) follows clearly from the definition of the $s_\nu$'s, since if the $n\alpha$-point is not adjacent to $O$, this gives the existence of an integer $1\leqq s_\nu<n$ for which the $s_\nu\alpha$-point is nearer to $O$ than the $n\alpha$-point.

(1. 12) follows from (1. 8) and (1. 9) in the following way:

$$\begin{aligned} s_\nu\bar{\delta}_\nu &\equiv s_{\nu_k+r}\bar{\delta}_{\nu_k+r}=(q_{k+1}-(a_k-r)q_k)(\bar{d}_{k+1}+(a_k-r)\bar{d}_k)=\\ &=q_{k+1}\bar{d}_{k+1}\left(1-(a_k-r)\frac{q_k}{q_{k+1}}\right)\left(1+(a_k-r)\frac{\bar{d}_k}{\bar{d}_{k+1}}\right). \end{aligned}$$

On account of $\bar{d}_k>\bar{d}_{k+1}$, $q_k<\dfrac{1}{a_k}q_{k+1}$ and $0<a_k-r<a_k$

$$s_\nu\bar{\delta}_\nu>q_{k+1}\bar{d}_{k+1},$$

indeed.

*(Received 9 September 1957)*

## Bibliography

[1] A. Ostrowski, Bemerkungen zur Theorie der diophantischen Approximationen, *Abhandlungen des Mathematischen Seminars Hamburg*, 1 (1921), pp. 77—98.

[2] A. Khintchine, Über eine Klasse linearer diophantischen Approximationen, *Rendiconti di Palermo*, **50** (1926), pp. 170—195.

[3] S. Fukasawa, Über die Größenordnung des absoluten Betrages von einer linearen inhomogenen Form. I, *Jap. Journal of Math.*, **3** (1926), pp. 1—27.

[4] H. Davenport, On a theorem of Khintchine, *Proc. London Math. Soc.* (2), **52** (1950), pp. 66 - 80.

[5] A. V. Prasad, On a theorem of Khintchine, *Proc. London Math. Soc.* (2), **53** (1951), pp. 310—320.

[6] J. W. S. Cassels, Über $\varliminf_{x \to +\infty} x|\vartheta x + \alpha - y|$, *Math. Annalen*, **127** (1954), pp. 288—304.