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OSCILLATORY BEHAVIOR OF SECOND ORDER NONLINEAR DIFFERENCE EQUATIONS WITH A NONLINEAR NONPOSITIVE NEUTRAL TERM

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Abstract. The authors present some new oscillation criteria for second order nonlinear difference equations with a nonlinear nonpositive neutral term of the form

$$\Delta\left(a(t)\left(\Delta\left(x(t)-p(t)x^{\alpha}(t-k)\right)\right)^{\gamma}\right)+q(t)x^{\beta}(t+1-m)=0,$$

with positive coefficients. Examples are given to illustrate the main results.

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1. Introduction

This paper deals with the oscillatory behavior of solutions of the nonlinear second order difference equations with a nonlinear nonpositive neutral term

$$\Delta\left(a(t)\left(\Delta\left(x(t)-p(t)x^{\alpha}(t-k)\right)\right)^{\gamma}\right)+q(t)x^{\beta}(t+1-m)=0, \quad t\geq t_0, \tag{1.1}$$

where $\Delta x(t)=x(t+1)-x(t)$ and:

(i) α, γ, and β are the ratios of positive odd integers with $\gamma\geq\beta$ and $0<\alpha\leq 1$;
(ii) $\{a(t)\}$, $\{p(t)\}$ and $\{q(t)\}$ are positive real sequences for $t\geq t_0$, and $0<p(t)<p_0<1$;
(iii) k is a positive integer and m is a nonnegative integer;
(iv) $h(t)=t-m+k+1\leq t$, i.e., $m\geq k+1$.

We set

$$A(v,u)=\sum_{s=u}^{v}\frac{1}{a^{1/\gamma}(s)} \quad \text{for} \quad v\geq u\geq t_0,$$

and assume that

$$A(t,t_0)\to\infty \text{ as } t\to\infty. \tag{1.2}$$

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Let $\theta = \max\{k, m-1\}$. By a *solution* of equation (1.1), we mean a real sequence $\{x(t)\}$ defined for all $t \geq t_0 - \theta$ that satisfies equation (1.1) for all $t \geq t_0$. A solution of equation (1.1) is called *oscillatory* if its terms are neither eventually positive nor eventually negative; otherwise, it is called *nonoscillatory*. If all solutions of the equation are oscillatory, then the equation itself is called oscillatory.

In recent years there has been a great deal of research activity on the oscillation and asymptotic behavior of solutions of various classes of difference equations; for example, see the monographs [1, 2, 5, 6], and the papers listed below. There are numerous results for second order neutral functional difference equations due to their increasing use as models in the natural sciences and in theoretical studies. Some such recent results on the oscillatory and asymptotic behavior of second order difference equations can be found in [3, 4, 7–22]. However, there does not appear to be any known results on the oscillation of second order difference equations of the type (1.1). Our aim here is to present some new sufficient conditions that ensure all solutions of (1.1) are oscillatory.

2. Main results

For $t \geq T$ for any $T \geq t_0$, we let

$$\mu(t) = a^{1/\gamma}(t)A(t,T) \quad \text{and} \quad Q(t) = \sum_{s=t}^{\infty} q(s).$$

For any constant $c > 0$, we set

$$g_c(t) = \begin{cases} 1, & \text{if } \beta = \gamma, \\ cA^{(\beta-\gamma)/\gamma}(t), & \text{if } \beta < \gamma. \end{cases} \tag{2.1}$$

We begin with the following new result.

Theorem 1. *Let conditions (i)–(iv) and (1.2) hold. Assume there exists a positive nondecreasing sequence $\{\rho(t)\}$ such that for any constant $c > 0$,*

$$\limsup_{t\to\infty} \left(\rho(t)Q(t) + \sum_{s=t_2}^{t} \left[\rho(s)q(s) - \frac{\gamma^\gamma}{(1+\gamma)^{\gamma+1}} \frac{a(t-m)}{(\beta g_c(s))^\gamma} \left(\frac{(\Delta\rho(s))^{\gamma+1}}{\rho^\gamma(s)} \right) \right] \right) = \infty, \tag{2.2}$$

$$\limsup_{t\to\infty} \sum_{s=h(t)}^{t} A^{\beta/\alpha}(h(t),h(s))q(s) > 1 \ \text{ if } \ \beta = \alpha\gamma, \tag{2.3}$$

and

$$\limsup_{t\to\infty} \sum_{s=h(t)}^{t} A^{\beta/\alpha}(h(t),h(s))q(s) = \infty \ \text{ if } \ \beta < \alpha\gamma. \tag{2.4}$$

Then equation (1.1) is oscillatory.

Proof. Let $x(t)$ be a nonoscillatory solution of equation (1.1), say $x(t) > 0$, $x(t-m+1) > 0$, and $x(t-k) > 0$ for $t \geq t_1$ for some $t_1 \geq t_0$. Then with $y(t) = x(t) - p(t)x^{\alpha}(t-k)$, it follows from (1.1) that

$$\Delta\left(a(t)\left(\Delta y(t)\right)^{\gamma}\right) = -q(t)x^{\beta}(t-m+1) \leq 0. \tag{2.5}$$

Hence, $a(t)\left(\Delta y(t)\right)^{\gamma}$ is nonincreasing and eventually of one sign. That is, there exists $t_2 \geq t_1$ such that $\Delta y(t) > 0$ or $\Delta y(t) < 0$ for $t \geq t_2$. We claim that $\Delta y(t) > 0$ for $t \geq t_2$. To see this, assume that $\Delta y(t) < 0$ for $t \geq t_2$. Then,

$$a(t)\left(\Delta y(t)\right)^{\gamma} \leq -c < 0 \quad \text{for } t \geq t_2,$$

where $c = -a(t_2)\left(\Delta y(t_2)\right)^{\gamma} < 0$, so

$$y(t) \leq y(t_2) - c^{1/\gamma} \sum_{s=t_2}^{t} a^{-1/\gamma}(s).$$

In view of (1.2), $\lim_{t\to\infty} y(t) = -\infty$. Now, we consider the following two cases.

Case 1. If $x(t)$ is unbounded, then there exists an increasing sequence $\{t_n\}$ such that $\lim_{n\to\infty} t_n = \infty$ and $\lim_{n\to\infty} x(t_n) = \infty$ where $x(t_k) = \max\{x(s) : t_0 \leq s \leq t_k\}$. This implies

$$x(t_n - m + 1) \leq \max\{x(s) : t_0 \leq s \leq t_n\} = x(t_n).$$

Therefore, since $\{x(t_n)\} \to \infty$ and (ii) holds for all large n,

$$\begin{aligned} y(t_n) = x(t_n) - p(t_n)x^{\alpha}(t_n - k) &\geq x(t_n) - p(t_n)x^{\alpha}(t_n) \\ &\geq \left(1 - \frac{p(t_n)}{x^{1-\alpha}(t_n)}\right)x(t_n) > 0. \end{aligned}$$

which contradicts the fact that $\lim_{t\to\infty} y(t) = -\infty$.

Case 2. If $x(t)$ is bounded, then $y(t)$ is also bounded, which contradicts $\lim_{t\to\infty} y(t) = -\infty$. This completes the proof of the claim so we conclude that $\Delta y(t) > 0$ for $t \geq t_2$.

Next, we have two possibilities to consider: (I) $y(t) > 0$ or (II) $y(t) < 0$ for $t \geq t_2$. If (I) holds, then in view of (2.5) and the fact that $x(t) \geq y(t)$, we have

$$\Delta\left(a(t)\left(\Delta y(t)\right)^{\gamma}\right) \leq -q(t)y^{\beta}(t-m+1) \leq 0. \tag{2.6}$$

Summing Δy from t_2 to t gives

$$\begin{aligned} y(t) &= y(t_2) + \sum_{s=t_2}^{t} \frac{\left(a(s)\left(\Delta y(s)\right)^{\gamma}\right)^{1/\gamma}}{a^{1/\gamma}(s)} \\ &\geq a^{1/\gamma}(t)\Delta y(t) \sum_{s=t_2}^{t} a^{-1/\gamma}(s) := \mu(t)\Delta y(t). \end{aligned} \tag{2.7}$$

Summing (2.6) from t to u, letting $u \to \infty$, and using the fact that $y(t)$ is increasing, we have

$$\begin{aligned} a(t)(\Delta y(t))^{\gamma} &\geq \sum_{s=t}^{\infty} q(s) y^{\beta}(s-m+1) \\ &\geq y^{\beta}(t-m+1) \sum_{s=t}^{\infty} q(s) := Q(t) y^{\beta}(t-m+1) \\ &\geq Q(t) y^{\beta}(t-m). \end{aligned} \tag{2.8}$$

Define

$$w(t) = \rho(t) \frac{a(t)(\Delta y(t))^{\gamma}}{y^{\beta}(t-m)} > 0 \quad \text{for } t \geq t_2. \tag{2.9}$$

Then, it follows that $w(t) > 0$ and

$$w(t) = \rho(t) \frac{a(t)(\Delta y(t))^{\gamma}}{y^{\beta}(t-m)} \geq \rho(t) \sum_{s=t}^{\infty} q(s). \tag{2.10}$$

Now,

$$\begin{aligned} \Delta w(t) &= \Delta \left(\frac{\rho(t)}{y^{\beta}(t-m)} \right) a(t+1)(\Delta y(t+1))^{\gamma} \\ &\quad + \Delta \left(a(t)(\Delta y(t))^{\gamma} \right) \left(\frac{\rho(t)}{y^{\beta}(t-m)} \right) \\ &\leq -\rho(t) q(t) + \left(\frac{\Delta \rho(t)}{\rho(t+1)} \right) w(t+1) \\ &\quad - \left(\frac{\rho(t)}{\rho(t+1)} \right) \frac{\Delta y^{\beta}(t-m)}{y^{\beta}(t-m)} w(t+1). \end{aligned} \tag{2.11}$$

By the Generalized Mean Value Theorem for Derivatives,

$$\beta y^{\beta-1}(t-m+1) \Delta y(t-m) \geq \Delta y^{\beta}(t-m) \geq \beta y^{\beta-1}(t-m) \Delta y(t-m).$$

Using this in (2.11) gives

$$\begin{aligned} \Delta w(t) &\leq -\rho(t) q(t) + \left(\frac{\Delta \rho(t)}{\rho(t+1)} \right) w(t+1) \\ &\quad - \beta \left(\frac{\rho(t)}{\rho(t+1)} \right) \frac{y^{\beta-1}(t-m) \Delta y(t-m)}{y^{\beta}(t-m)} w(t+1). \end{aligned} \tag{2.12}$$

Since $a(t)(\Delta y(t))^{\gamma}$ is decreasing and $y(t)$ is increasing, we have

$$\frac{\Delta y(t-m)}{\Delta y(t)} \geq \left(\frac{a(t)}{a(t-m)} \right)^{1/\gamma} \quad \text{and} \quad \frac{w(t+1)}{\rho(t+1)} \leq \frac{w(t)}{\rho(t)}. \tag{2.13}$$

Using (2.13) in (2.12), we obtain

$$\Delta w(t) \leq -\rho(t)q(t) + \left(\frac{\Delta\rho(t)}{\rho(t+1)}\right) w(t+1) - \beta\left(\frac{\rho(t)}{\rho(t+1)}\right)\left(\frac{a(t)}{a(t-m)}\right)^{1/\gamma} \frac{\Delta y(t)}{y(t-m)} w(t+1).$$

Now,

$$\frac{\Delta y(t)}{y^{\beta/\gamma}(t-m)} = \rho^{-1/\gamma}(t)a^{-1/\gamma}(t)w^{1/\gamma}(t) \geq \rho^{-1/\gamma}(t)a^{-1/\gamma}(t)\left(\frac{\rho(t)}{\rho(t+1)}\right)^{1/\gamma} w^{1/\gamma}(t+1).$$

Thus,

$$\Delta w(t) \leq -\rho(t)q(t) + \left(\frac{\Delta\rho(t)}{\rho(t+1)}\right) w(t+1) - \frac{\beta}{a^{1/\gamma}(t-m)}\left(\frac{\rho(t)}{\rho^{1+1/\gamma}(t+1)}\right) w^{1+1/\gamma}(t+1)y^{(\beta-\gamma)/\gamma}(t-m),$$

and so,

$$\Delta w(t) \leq -\rho(t)q(t) + \left(\frac{\Delta\rho(t)}{\rho(t+1)}\right) w(t+1) - \frac{\beta\rho(t)}{a^{1/\gamma}(t-m)\rho^{1+1/\gamma}(t+1)} w^{1+1/\gamma}(t+1)y^{(\beta-\gamma)/\gamma}(t-m).$$

For the case $\beta = \gamma$, we see that $y^{(\beta-\gamma)/\gamma}(t) = 1$ while for the case $\beta < \gamma$, since $a(t)\,(\Delta y(t))^{\gamma}$ is decreasing, there exists a constant $c_1 > 0$ such that

$$a(t)\,(\Delta y(t))^{\gamma} \leq c_1 \quad \text{for } t \geq t_2.$$

Summing this inequality from t_2 to t, we have

$$y(t) \leq y(t_2) + A(t,t_2) \leq c_2 A(t,t_2)$$

for $t \geq t_3$ for some $c_2 > 0$ and $t_3 \geq t_2$. Thus,

$$y^{(\beta-\gamma)/\gamma}(t) \geq c_2^{(\beta-\gamma)/\gamma} A^{(\beta-\gamma)/\gamma}(t,t_2) := c^* A^{(\beta-\gamma)/\gamma}(t,t_2),$$

where $c^* = c_2^{(\beta-\gamma)/\gamma}$. Combining the two cases on β and the definition of $g_{c^*}(t)$ gives

$$\Delta w(t) \leq -\rho(t)q(t) + \left(\frac{\Delta\rho(t)}{\rho(t+1)}\right) w(t+1) - \frac{\beta\rho(t)}{a^{1/\gamma}(t-m)\rho^{1+1/\gamma}(t+1)} g_{c^*}(t) w^{1+1/\gamma}(t+1). \tag{2.14}$$

Setting

$$B := \frac{\Delta\rho(t)}{\rho(t+1)} \quad \text{and} \quad C := \frac{\beta\rho(t)g_{c^*}(t)}{a^{1/\gamma}(t-m)\rho^{1+1/\gamma}(t+1)},$$

and using the inequality (see [7])

$$Bu - Cu^{(1+\gamma)/\gamma} \le \frac{\gamma^\gamma}{(1+\gamma)^{\gamma+1}}\left(\frac{B^{\gamma+1}}{C^\gamma}\right),$$

with $u = w(t+1)$, we have

$$\Delta w(t) \le -\rho(t)q(t) + \frac{\gamma^\gamma}{(1+\gamma)^{\gamma+1}}\frac{a(t-m)}{(\beta g_{c^*}(t))^\gamma}\left(\frac{(\Delta\rho(t))^{\gamma+1}}{\rho^\gamma(t)}\right).$$

Summing this inequality from t_2 to t gives

$$w(t) \le w(t_2) - \sum_{s=t_2}^{t}\left[\rho(s)q(s) - \frac{\gamma^\gamma}{(1+\gamma)^{\gamma+1}}\frac{a(t-m)}{(\beta g_{c^*}(s))^\gamma}\left(\frac{(\Delta\rho(s))^{\gamma+1}}{\rho^\gamma(s)}\right)\right].$$

Taking into account (2.8), we see that

$$w(t_2) \ge \rho(t)Q(t) + \sum_{s=t_2}^{t}\left[\rho(s)q(s) - \frac{\gamma^\gamma}{(1+\gamma)^{\gamma+1}}\frac{a(t-m)}{(\beta g_{c^*}(s))^\gamma}\left(\frac{(\Delta\rho(s))^{\gamma+1}}{\rho^\gamma(s)}\right)\right].$$

Taking the lim sup of both sides in the above inequality as $t \to \infty$, we obtain a contradiction to condition (2.2).

Now consider Case (II). If we set $z(t) = -y(t) > 0$ for $t \ge t_2$, then $\Delta z(t) = -\Delta y(t) < 0$, and from equation (1.1),

$$\Delta\left(a(t)(\Delta z(t))^\gamma\right) = q(t)x^\beta(t-m+1) \ge 0. \tag{2.15}$$

Moreover,

$$z(t) = -y(t) = p(t)x^\alpha(t-k) - x(t) \le p(t)x^\alpha(t-k),$$

so

$$x^\alpha(t-k) \ge z(t) \quad \text{or} \quad z^{1/\alpha}(t+k) \le x(t).$$

Using this inequality in (1.1), we have

$$\Delta\left(a(t)(\Delta z(t))^\gamma\right) \ge q(t)z^{\beta/\alpha}(t-m+k+1) := q(t)z^{\beta/\alpha}(h(t)). \tag{2.16}$$

For $t_2 \le u \le v$, we may write

$$\begin{aligned} z(u) - z(v) &= -\sum_{s=u}^{v} a^{-1/\gamma}(s)\left(a(s)(\Delta z(s))^\gamma\right)^{1/\gamma} \\ &\ge A(v,u)\left(-\left(a(v)(\Delta z(v))^\gamma\right)^{1/\gamma}\right) \end{aligned}$$

for $t \geq s \geq t_2$. Setting $u = h(s)$ and $v = h(t)$ in the above inequality gives

$$z(h(s)) \geq A\,(h(t),h(s))\left(-\left(a(h(t))(\Delta z(h(t)))^{\gamma}\right)^{1/\gamma}\right).$$

Summing inequality (2.16) from $h(t) \geq t_2$ to t, we find that

$$\begin{aligned} Z(t) &:= -a(h(t))(\Delta z(h(t)))^{\gamma} \\ &\geq \left(-a(h(t))(\Delta z(h(t)))^{\gamma}\right)^{\beta/\alpha\gamma} \sum_{s=h(t)}^{t} A^{\beta/\alpha}\,(h(t),h(s))\,q(s) \\ &= Z^{\beta/\alpha\gamma}(t) \sum_{s=h(t)}^{t} A^{\beta/\alpha}\,(h(t),h(s))\,q(s), \end{aligned}$$

and hence

$$Z^{1-\beta/\alpha\gamma}(t) \geq \sum_{s=h(t)}^{t} A^{\beta/\alpha}\,(h(t),h(s))\,q(s).$$

Taking the lim sup of both sides of this inequality as $t \to \infty$, we arrive at a contradiction to (2.3) if $\beta = \alpha\gamma$. Since $\Delta Z(t) \leq 0$ by (2.15), $Z(t)$ is bounded, and so we obtain a contradiction to (2.4) if $\beta < \alpha\gamma$. This completes the proof of the theorem. □

Remark 1. We note that Theorem 1 holds if $Q(t) < \infty$ so the presence of the additional term $\rho(t)Q(t)$ in condition (2.2) may improve some of well-known existing results in the literature.

In case $Q(t)$ does not exist as $t \to \infty$, we see that condition (2.2) can be replaced by

$$\limsup_{t\to\infty} \sum_{s=t_2}^{t} \left[\rho(s)q(s) - \frac{\gamma^{\gamma}}{(1+\gamma)^{\gamma+1}} \frac{a(t-m+1)}{(\beta g_c(s))^{\gamma}} \left(\frac{(\Delta\rho(s))^{\gamma+1}}{\rho^{\gamma}(s)}\right)\right] = \infty \qquad (2.17)$$

and the conclusion of Theorem 1 still holds.

For the non-neutral equation, i.e., equation (1.1) with $p(t) \equiv 0$, and $q(t)$ is either nonnegative or nonpositive for all large t, equation (1.1) reduces to

$$\Delta\left(a(t)\,(\Delta\,(x(t)))^{\gamma}\right) + \delta q(t) x^{\beta}(t+1-m) = 0, \qquad (2.18)$$

where $\delta = \pm 1$. From Theorem 1, we extract the following immediate results.

Corollary 1. *Let conditions (i)–(iii) and (1.2) hold. If there exists a positive sequence $\{\rho(t)\}$ with $\Delta\rho(t) \geq 0$ such that condition (2.2) holds, then equation (2.18) with $\delta = \pm 1$ is oscillatory.*

Proof. The proof is contained in the proof of Case (I) in Theorem 1 and hence is omitted. □

We note that Corollary 1 is related to some of the results in [3–5, 12–16, 19] and the references cited therein.

Corollary 2. *Let conditions (i)–(iv) and (1.2) hold. If condition (2.3) or (2.4) holds, then every bounded solution of equation (2.18) with $\delta = \pm 1$ is oscillatory.*

Proof. The proof is contained in the proof of Case (II) of Theorem 1 and hence is omitted. □

The following example illustrates the above theorem.

Example 1. Consider the neutral equation

$$\Delta\left(\Delta\left(x(t) - \frac{1}{2}x^{1/3}(t-3)\right)^3\right) + 8x(t-7) = 0. \tag{2.19}$$

Here, $k = 3$ and $m = 8$, so $h(t) = t - 4$. All conditions of Theorem 1 with $\rho(t) \equiv 1$ and condition (2.2) replaced by (2.17) are satisfied, so equation (2.19) is oscillatory.

Our next result follows directly from Theorem 1.

Theorem 2. *Let the hypotheses of Theorem 1 hold with $\Delta\rho(t) \le 0$ for $t \ge t_0$ and condition (2.2) replaced by*

$$\limsup_{t\to\infty}\left[\rho(t)Q(t) + \sum_{s=t_0}^{t}\rho(s)q(s)\right] = \infty. \tag{2.20}$$

Then equation (1.1) is oscillatory.

In the following theorem we employ a different approach to replacing condition (2.2) in Theorem 1.

Theorem 3. *Let the hypotheses of Theorem 1 hold with condition (2.2) replaced by*

$$\limsup_{t\to\infty}\left[\rho(t)Q(t) + \sum_{s=t_0}^{t}\rho(s)q(s) - \frac{a^{1/\gamma}(s-m)(\Delta\rho(s))^2}{4\beta g_c(s)\rho(s)Q^{(1/\gamma)-1}(s+1)}\right] = \infty. \tag{2.21}$$

Then equation (1.1) is oscillatory.

Proof. Let $x(t)$ be a nonoscillatory solution of equation (1.1), say $x(t) > 0$, $x(t-m+1) > 0$, and $x(t-k) > 0$ for $t \ge t_1$ for some $t_1 \ge t_0$. Proceeding as in the proof of Theorem 1, we conclude that $\Delta y(t) > 0$ for $t \ge t_2$ and $y(t)$ satisfies either Case (I) or Case (II) for $t \ge t_2$. If (I) holds, then as in the proof of Theorem 1, we again obtain (2.12). Since $a(t)(\Delta y(t))^\gamma$ is nonincreasing and $y(t)$ is nondecreasing, we have

$$a^{1/\gamma}(t-m)\Delta y(t-m) \ge a^{1/\gamma}(t+1)\Delta y(t+1) \quad \text{and} \quad 1/y(t-m) \ge 1/y(t-m+1),$$

so

$$
\begin{aligned}
\Delta w(t) \leq & -\rho(t)q(t) + \left(\frac{\Delta\rho(t)}{\rho(t+1)}\right) w(t+1) \\
& - \frac{\beta\rho(t)}{\rho(t+1)} \frac{a^{1/\gamma}(t+1)\Delta y(t+1)}{a^{1/\gamma}(t-m)y(t-m+1)} w(t+1) \\
\leq & -\rho(t)q(t) + \left(\frac{\Delta\rho(t)}{\rho(t+1)}\right) w(t+1) \\
& - \frac{\beta\rho(t)}{\rho^{1+1/\gamma}(t+1)} \frac{y^{\frac{\beta-\gamma}{\gamma}}(t-m+1)}{a^{1/\gamma}(t-m)} w^{1+1/\gamma}(t+1) \\
\leq & -\rho(t)q(t) + \left(\frac{\Delta\rho(t)}{\rho(t+1)}\right) w(t+1) \\
& - \frac{\beta\rho(t)}{\rho^{1+1/\gamma}(t+1)} \frac{g_{c^*}(t)}{a^{1/\gamma}(t-m)} w^{1+1/\gamma}(t+1).
\end{aligned}
$$

From (2.10) we see that $w^{1-1/\gamma}(t+1)/\rho^{1-1/\gamma}(t+1) \geq Q^{1-1/\gamma}(t+1)$, so

$$
\begin{aligned}
\Delta w(t) \leq & -\rho(t)q(t) + \left(\frac{\Delta\rho(t)}{\rho(t+1)}\right) w(t+1) \\
& - \frac{\beta\rho(t)}{a^{1/\gamma}(t-m)\rho^2(t+1)} g_{c^*}(t) Q^{1/\gamma-1}(t+1) w^2(t+1).
\end{aligned}
$$

Completing the square on the second and third terms on the right gives

$$
\Delta w(t) \leq -\rho(t)q(t) + + \frac{a^{1/\gamma}(t-m)(\Delta\rho(t))^2}{4\beta g_{c^*}(t)\rho(t)Q^{(1/\gamma)-1}(t+1)}.
$$

The remainder of the proof is similar to that of Theorem 1 and is omitted. □

Example 2. Consider the neutral equation

$$
\Delta\left(t^3\Delta\left(x(t) - \frac{1}{3}x^{1/3}(t-2)\right)^3\right) + \frac{1}{\ln t}x(t-3) = 0,\ t > 1. \tag{2.22}
$$

Here, $k = 2$, $m = 4$, $\alpha = 1/3$, and $\gamma = 3$. All conditions of Theorem 3 are satisfied with $\rho \equiv 1$ and hence equation (2.22) is oscillatory.

Next, we present some new and easily verifiable oscillation criteria for equation (1.1).

Theorem 4. *Let $\alpha = 1$ and conditions (i)–(iv) and (1.2) hold. Assume that condition (2.3) holds and*

$$
\limsup_{t\to\infty} A^{\beta}(t-m, t_0)Q(t) > 1 \tag{2.23}
$$

if $\beta = \gamma$, and condition (2.4) holds and

$$\limsup_{t\to\infty} A^{\beta}(t-m,t_0)Q(t) = \infty \tag{2.24}$$

if $\beta < \gamma$. Then equation (1.1) is oscillatory.

Proof. Let $x(t)$ be a nonoscillatory solution of equation (1.1), say $x(t) > 0$, $x(t-m+1) > 0$, and $x(t-k) > 0$ for $t \geq t_1$ for some $t_1 \geq t_0$. Proceeding as in the proof of Theorem 1, we conclude that $\Delta y(t) > 0$ for $t \geq t_2$ and $y(t)$ satisfies either (I) or (II) for $t \geq t_2$. If (I) holds, then as in the proof of Theorem 1, we obtain (2.7) and (2.8). Using the fact that $a(t)(\Delta y(t))^{\gamma}$ is decreasing, we have

$$\begin{aligned}
w(t) &:= a(t)(\Delta y(t))^{\gamma} \geq Q(t)\mu^{\beta}(t-m)(\Delta y(t-m))^{\beta} \\
&= Q(t)\mu^{\beta}(t-m)\left(a^{-\beta/\gamma}(t-m)\right)\left(a(t-m)(\Delta y(t-m))^{\gamma}\right)^{\beta/\gamma} \\
&\geq Q(t)\mu^{\beta}(t-m)\left(a^{-\beta/\gamma}(t-m)\right)\left(a(t)(\Delta y(t))^{\gamma}\right)^{\beta/\gamma} \\
&= Q(t)\mu^{\beta}(t-m)\left(a^{-\beta/\gamma}(t-m)\right)w^{\beta/\gamma}(t),
\end{aligned}$$

or

$$\begin{aligned}
w^{1-\beta/\gamma}(t) &\geq Q(t)\mu^{\beta}(t-m)\left(a^{-\beta/\gamma}(t-m)\right) \\
&= Q(t)\left(\sum_{s=t_2}^{t-m} a^{-1/\gamma}(s)\right)^{\beta} = A^{\beta}(t-m,t_2)Q(t).
\end{aligned}$$

Taking lim sup of both sides of this inequality as $t \to \infty$, we arrive at a contradiction to condition (2.23) if $\beta = \gamma$ and to condition (2.24) and the boundedness of $w(t)$ if $\beta < \gamma$. The proof of Case (II) is similar to that in the proof of Theorem 1 and is omitted. □

Remark 2. We may note that corollaries similar to Corollaries 1 and 2 can be also drawn from Theorems 2–4. The details are left to the reader.

In conclusion, we would like to point out that our results in this paper can be extended to higher order equations of the form

$$\Delta\left(a(t)\left(\Delta^{n-1}(x(t)-p(t)x(t-k))\right)^{\gamma}\right) + q(t)x^{\beta}(t+1-m) = 0,$$

where n is a positive integer. Also, it would be of interest to study equation (1.1) in the case where $\beta > \gamma$.

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