Asymptotics, Equidistribution and Inequalities for Partition Functions

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Emil-Alexandru Ciolan

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Berichterstatter/in:	Prof. Dr. Kathrin Bringmann Prof. Dr. Sander Zwegers
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On the cover: In red, yellow and blue are represented the plots of the functions $p_2(0,3,n)$, $p_2(1,3,n)$ *and* $p_2(2,3,n)$ *for* $52 \leq n \leq 123$, *obtained using* Mathematica 8.0.

Abstract

This thesis consists of three research projects on asymptotics, equidistribution properties and inequalities for partition and overpartition functions. We start by proving that the number of partitions into squares with an even number of parts is asymptotically equal to that of partitions into squares with an odd number of parts. We further show that, for n large enough, the two quantities are different, and that which of the two is bigger depends on the parity of n. By doing so, we answer a conjecture formulated by Bringmann and Mahlburg (2012). We continue by placing this problem in a broader context and by proving that the same results are true for partitions into any powers. For this, we invoke an estimate on Gauss sums found by Banks and Shparlinski (2015) using the effective lower bounds on center density from the sphere packing problem established by Cohn and Elkies (2003). Finally, we compute asymptotics for the coefficients of an infinite class of overpartition rank generating functions, and we show that $\overline{N}(a, c, n)$, the number of overpartitions of n with rank congruent to a modulo c, is equidistributed with respect to $0 \leq a < c$, as $n \to \infty$, for any $c \geq 2$. In addition, we prove some inequalities between ranks of overpartitions recently conjectured by Ji, Zhang and Zhao (2018), and Wei and Zhang (2018).

Acknowledgments

Before going deeper into the results of this thesis, I would like to express my gratitude to all those whose help was, in many and diverse ways, essential in this endeavor.

To begin with, I need to make a jump in time up to the end of the previous millenium. Then, on a rainy September day, I got to meet Marius Pălărie, who, much more than just my primary school instructor, proved to be a *mentor* and a *teacher* in the deepest meaning of these words. Without the passion for study that I was so lucky to develop under his guidance, my academic path would have certainly not been the same.

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Contents

Chapter 1

Introduction and scope of the thesis

This thesis is centered on answering some recent open questions on asymptotics, equidistribution and inequalities for certain partition functions, and it essentially consists of the research articles [Cio20], [Cio] and [Cio19] written by the author. The first two projects revolve around partitions into powers and a rather interesting relation between those partitions with an even and those with an odd number of parts, whereas the third project focuses on the study of overpartition rank generating functions.

In what follows, we give the reader an overview on how the thesis should best be read. We continue by summarizing the basic definitions and results existing in the literature that inspired the study of our problems. In Chapter 1.2 we give the reader a quick introduction to partitions and we state the main results from Chapters 2 and 3. In Chapter 1.3 we do the same for overpartitions and we give a brief overview of the results from Chapter 4.

What our findings had in common was, to some extent surprising, a certain equidistribution pattern that was a priori not expected. Partly for this reason, we decided to unify the three projects under a common theme. We will elaborate on this in Chapter 1.5, in which we summarize the main ideas of this introductory part and in which we explain our motivation for putting together the topics of this thesis. Another reason for doing so and, at the same time, another point that our projects had in common, is given by the techniques used in the proofs, some of which are outlined in Chapter 1.4.

1.1 How to read the thesis

The actual content of this thesis is represented by Chapters 2, 3 and 4, in which we present our results. In Chapter 5 we give a short reminder of our findings, and we suggest some open problems and possible directions for future research.

The chapters are self-contained and can be read in any preferred order. We recommend, however, that Chapters 2 and 3 be read together, as they deal with the same problem of studying partitions into squares, respectively into rth powers, and the relation between the number of partitions with an odd and that of partitions with an even number of parts. Chapter 2 answers this question, formulated in Conjecture 1.1, in the case $r = 2$, whereas Chapter 3 shows how the same result generalizes to partitions into higher powers.

While one could have gone another way, suppressing some material from Chapter 2 (or the entire chapter, for that matter) and showing only the argument for general $r \geq 2$, we find it in the benefit of the reader to work out in detail both cases and to show how one project originated in the other. In this way, the reader can easily go back and forth between the two arguments and see what is the main difference between the two cases. The proof in the case $r = 2$ is in itself very instructive, showing ideas that can also be employed in other problems.

We aimed to keep the presentation of Chapters 2–4 as close as possible to the content of the articles on which they are based. For this reason, there are certain similarities between Chapters 2 and 3 and we hope the reader will not find this upsetting. Also, since we wanted to give the reader an overview of our results, there is some unavoidable repetition of material in the introductory sections.

Nevertheless, in order to avoid too much repetition, we shortened the introductory sections here and there and made some slight reformulations. The rest of the material is, up to minor layout changes, identical to the one from the original articles. We also corrected a few typos that had missed the eyes of both the author and the referee in the published versions of the articles, the most significant of which we point out in the form of footnotes throughout the chapters.

As a token of appreciation, we kept the individual Acknowledgments for each chapter in the same form as they were originally written in the corresponding articles, with the hope that the author expressed his gratitude to all those that contributed to the completion of the projects resulting in this thesis.

Each of the Chapters 2, 3 and 4 contains the same references as the respective article on which it is based. Inside these chapters, apart from the introductory paragraphs and some explanatory footnotes referring to supplementary literature, each reference is cited only from and within the chapter. For consistency, the citation style throughout uses authors initials and years. All individual references, together with those mentioned in Chapters 1 and 5, are collected in the main Bibliography. As the sources are cited unitarily, the reader has the liberty to consult, according to his or her own preference, either the references lists at the end of each chapter or the main bibliography.

The notation we follow in this thesis is the one traditionally used in the literature. The most frequent notation is explained in an appendix following the Bibliography. Well-established symbols such as $\ll$ or O are used without further ado. However, they are all to be found in the notation index. Every new definition – particularly the various partition functions $p(n)$, $p_r(n)$, $N(m,n)$, etc. – is explained within each chapter. In this sense, every chapter is local and self-contained.

1.2 Partitions into powers

1.2.1 Preliminaries

A *partition* of a positive integer n is a non-increasing sequence of positive integers, called *parts*, usually written as a sum, adding up to n. The number of partitions of n is denoted by $p(n)$. For example, $p(4) = 5$ as the partitions of 4 are 4, $3+1$, $2+2$, $2+1+1$ and $1+1+1+1$. We set, by convention, $p(0) = 1$. This is the case of the so-called *unrestricted* partitions, i.e., partitions for which the parts can be any positive integers and repetitions are allowed. One can consider, however, partitions with various other properties, such as partitions into odd parts, partitions into distinct parts, etc. Indeed, it is often the case that we focus on partitions with parts belonging to some particular subset of $\mathbb{N}$. As presented in detail in Chapters 2 and 3 and summarized here, we will first be dealing with partitions into squares and, more generally, into powers.

1.2.2 Partitions and generating functions

While the concept of partitions is easy to explain, some of their properties have eluded proof for decades. At the same time, trying to prove simple congruence statements about partition functions gave birth to deep techniques, such as the work of Serre [Ser76] and Swinnerton-Dyer [Swi73, Swi87] on ℓ-adic representations and modular forms.

There are, however, a few facts that can be established without difficulty and, at the same time, a few elementary methods that can help a great deal in partition problems. One such example, an object otherwise intimately related to the partition function, is its generating function, defined for $|x| < 1$ by

$$\prod_{n=1}^{\infty} \frac{1}{1-x^n} = \sum_{n=0}^{\infty} p(n)x^n. \tag{1.2.1}$$

If we ignore for the moment any possible issues about convergence and regard this result strictly as an identity of formal power series, in order to prove (1.2.1) we might first want to express the infinite product as a product of geometric series:

$$\prod_{n=1}^{\infty} \frac{1}{1-x^n} = (1+x+x^2+\cdots)(1+x^2+x^4+\cdots)\cdots. \tag{1.2.2}$$

We can next simply multiply the infinite sums on the right of (1.2.2), collecting powers of x, in order to obtain a power series of the form

$$\sum_{n=0}^{\infty} a(n)x^n.$$

We would like to prove that $a(n) = p(n)$. The first step is easy, as we have $a(0) = 1$ and, by convention, this equals $p(0)$. Now, assume we pick the term x^{s_1} from the first sum, the term x^{2s_2} from the second, the term x^{3s_3} from the third, and so on until we pick the (final) term from the kth sum to be x^{ks_k}, with $s_i \geq 0$ for $i = 1, \dots, k$. The product of these powers of x will be another power of x, let us call it x^n, and so

$$x^{s_1}x^{2s_2}x^{3s_3}\cdots x^{ks_k} = x^n,$$

with

$$n = s_1 + 2s_2 + 3s_3 + \cdots + ks_k,$$

which can also be written as

$$n = (1+\cdots+1)+(2+\cdots+2)+(3+\cdots+3)+\cdots+(k+\cdots+k), \quad (1.2.3)$$

with each paranthesis of the form $(i + \cdots + i)$ containing s_i terms. However, one notices that (1.2.3) is nothing else than a partition of n. Thus, each partition of n produces a term of the form x^n and, conversely, each monomial x^n comes from a unique partition of n, from where it follows indeed that $a(n) = p(n)$. One only needs to make things rigorous and understand why the condition $|x| < 1$ has to be imposed to ensure convergence.

Following the notation of Andrews [And98, pp. 2–3], for $S \subseteq \mathbb{N}$ a set of positive integers, we denote by $p(\text{“}S\text{”}, n)$ the number of partitions of n that have all their parts in S, and by $p(\text{“}S\text{”}(\leq d), n)$ the number of such partitions that have all their parts at most equal to d. For a subset $\mathcal{S}$ of the set of all partitions, we denote by $p(\mathcal{S}, n)$ the set of those partitions of n belonging to $\mathcal{S}$. For instance, we let $\mathcal{O}$ be the set of all partitions into odd parts and $\mathcal{D}$ that of partitions into distinct parts.

A similar reasoning as above can be employed to prove a variety of other identities for partition generating functions (see [And98, Ch. 1]).

Theorem 1.1 ([And98, Th. 1.1]). *If $|q| < 1$, then*

$$\sum_{n=0}^{\infty} p(\text{“}S\text{”}, n)q^n = \prod_{n\in S} \frac{1}{1-q^n}$$

and

$$\sum_{n=0}^{\infty} p(\text{“}S\text{”}(\leq d), n)q^n = \prod_{n\in S}(1 + q^n + \cdots + q^{dn}) = \prod_{n\in S} \frac{1-q^{(d+1)n}}{1-q^n}.$$

Two consequences of Theorem 1.1 follow easily.

Corollary 1.1 (Euler, 1748). *If $n \geq 1$, we have $p(\mathcal{O}, n) = p(\mathcal{D}, n)$.*

Corollary 1.2 (Glaisher [Gla83]). *If N_d denotes the set of positive integers not divisible by d, then for all $n \geq 1$ we have*

$$p(\text{“}N_{d+1}\text{”}, n) = p(\text{“}N\text{”}(\leq d), n).$$

The list of such examples can continue, but we would like to introduce now another simple, more visual, and often very useful method to study partitions.

1.2.3 Partitions and graphical representations

To each partition λ of n we associate its *graphical representation* (also called *Ferrers graph*) $\mathcal{G}_\lambda$, which is the set of points with integral coordinates (i, j) such that if $n = \lambda_1 + \cdots + \lambda_k$ with $\lambda_1 \geq \ldots \geq \lambda_k$, then $(i, j) \in \mathcal{G}_\lambda$ if and only if $0 \geq i \geq -k+1$ and $0 \leq j \leq \lambda_{|i|+1} - 1$. As this formal definition might not reveal much at first, let us exemplify it in a concrete case. The graphical representation of the partition $20 = 7 + 5 + 5 + 2 + 1$, which we can also write as $\lambda = (7, 5, 5, 2, 1)$, is given by

$$\begin{array}{ccccccc}
. & . & . & . & . & . & . \\
. & . & . & . & . & & \\
. & . & . & . & . & & \\
. & . & & & & & \\
. & & & & & &
\end{array}$$

What is important to remember, is that the ith row from the top contains λ_i points (or dots, or nodes). There are several other (equivalent) ways of drawing this representation and some authors use square units instead of dots, which proves particularly useful when one considers applications to *plane partitions* or *Young tableaux* (see [And98, Ch. 11]). As we do not intend to spend much time on this matter, we refer the reader interested to familiarize him or herself with the topic to the beautiful book of Andrews [And98]. For the rest of this section, we only want to illustrate how certain facts about partitions can sometimes follow simply by having a close look at their graphical representation.

For this, let us first define another useful object. If $\lambda = (\lambda_1, \ldots, \lambda_k)$ is a partition, we define a new partition, called the *conjugate* of λ, as $\lambda' = (\lambda'_1, \ldots, \lambda'_\ell)$ by choosing λ'_i to be the number of parts of λ that are greater than or equal to i. To better visualize this by an example, the conjugate of the partition $\lambda = (7, 5, 5, 2, 1)$ is given by

$$\begin{array}{ccccc}
. & . & . & . & . \\
. & . & . & . & \\
. & . & . & & \\
. & . & . & & \\
. & . & . & & \\
. & & & & \\
. & & & &
\end{array}$$

The way to remember this is that the conjugate partition is obtained by counting (and placing) the dots on the rows in successive columns; equivalently, the graphical representation of the conjugate partition is obtained by reflecting the graph of the original partition along the main diagonal. An interesting result which is an immediate consequence of the graphical representation is the following.

Theorem 1.2 ([And98, Th. 1.4]). *The number of partitions of n with at most m parts equals the number of partitions of n in which no part exceeds m.*

The proof follows simply by inspecting the graphs of the partition and its conjugate. One can easily identify a bijection between the two classes of partitions considered by mapping each partition onto its conjugate. From the graphical representation one sees that the condition "at most m parts" is transformed into "no part exceeding m" and the other way around.

1.2.4 Motivation and previous results

An example of result that can be proven on combining identities of generating functions with combinatorial arguments deduced from graphical representations is Franklin's proof of Euler's pentagonal number theorem, a remarkable achievement of the 19th century mathematics. For details, see [And98, Th. 1.6].

Theorem 1.3 (Franklin [Fra81]). *Let $p_{\mathrm{e}}(\mathcal{D}, n)$ and $p_{\mathrm{o}}(\mathcal{D}, n)$ denote the number of partitions of n into an even, respectively an odd number of parts. Then*

$$p_{\mathrm{e}}(\mathcal{D}, n) - p_{\mathrm{o}}(\mathcal{D}, n) = \begin{cases} (-1)^m & \text{if } n = \frac{1}{2}m(3m \pm 1), \\ 0 & \text{otherwise.} \end{cases}$$

An immediate consequence is Euler's famous result.

Corollary 1.3 (Euler's pentagonal number theorem). *If $|q| < 1$, then*

$$\begin{aligned} \prod_{n=1}^{\infty}(1 - q^n) &= 1 + \sum_{m=1}^{\infty}(-1)^m q^{\frac{1}{2}m(3m-1)}(1 + q^m) \\ &= \sum_{m=-\infty}^{\infty}(-1)^m q^{\frac{1}{2}m(3m-1)}. \end{aligned}$$

Another result that goes in the spirit of Theorem 1.3 is the identity

$$p_1(0,2,n) - p_1(1,2,n) = (-1)^n p_{\mathrm{odd}}(n),$$

established using generating functions by Glaisher [Gla76] in 1876. Here, for $r \in \mathbb{N}$ we let $p_r(a,m,n)$ be the number of partitions of n into rth powers with a number of parts that is congruent to a modulo m, and $p_{\mathrm{odd}}(n)$ the number of partitions of n into odd parts without repeated parts.

Glaisher's result tells us that an even number n has more partitions into an even number of parts than into an odd number of parts, and conversely if n is odd. It is natural then to investigate if the same result holds, say, for partitions into rth powers with $r \geq 2$. In this regard, Bringmann and Mahlburg [BM] noticed, based on computer experiments, that the answer seems to be "yes" in case $r = 2$.

Conjecture 1.1 (Bringmann–Mahlburg, 2012)**.**

(i) *As $n \to \infty$, we have*

$$p_2(0,2,n) \sim p_2(1,2,n).$$

(ii) *We have*

$$\begin{cases} p_2(0,2,n) > p_2(1,2,n) & \text{if } n \text{ is even,} \\ p_2(0,2,n) < p_2(1,2,n) & \text{if } n \text{ is odd.} \end{cases}$$

1.2.5 Epilogue: Waring's problem

The problem of finding representations of a positive integer n as a sum of squares, or powers in general, is by all means not new. In fact, a discussion about partitions, even if brief, would not be complete without mentioning Waring's problem, one of the central questions in additive number theory, named so after the English mathematician E. Waring, who stated in 1770, without proof and with limited numerical evidence, that every positive integer is the sum of 4 squares, of 9 cubes, of 19 fourth powers, etc.

Waring's problem. Determine whether, for a given positive integer k, there exists an integer s (depending only on k) such that the equation

$$n = x_1^k + \cdots + x_s^k \tag{1.2.4}$$

has solutions for every $k \geq 1$.

Over the centuries, many results linked to famous mathematicians were found in this direction. That the answer to Waring's problem is positive, is known due to Hilbert [Hil09], who proved this in 1909. If $r_k(n)$ denotes the number of solutions of (1.2.4), where the x_i may be positive, negative or zero, and the order of the summands is taken into account, Jacobi [Jac29] expressed $r_k(n)$ in terms of divisor functions for $k = 2, 4, 6$ and 8. For instance, he proved that

$$r_2(n) = 4(d_1(n) - d_3(n)),$$

where $d_1(n)$ and $d_3(n)$ are the number of divisors of n congruent to 1, respectively 3 modulo 4. Exact formulas for $r_k(n)$ have also been found for $k = 3, 5$ and 7. For larger values of k, the study of $r_k(n)$ is significantly more difficult. Hardy and Littlewood gave an asymptotic formula for the number of solutions of (1.2.4) using a novel method designed by Hardy and Ramanujan. We will come back to this in Chapter 1.4.1. For a historical account on Waring's problem, see the survey of Vaughan and Wooley [VW02].

1.2.6 Description of results

In Chapter 2 we prove that Conjecture 1.1 is true for n sufficiently large. For this we partly follow an approach taken by Meinardus [Mei54] combined with an application of the circle method and the saddle-point method. One step of the argument requires running a rather technical computer check. In Chapter 3 we show how to extend this result to the general case of partitions into powers on avoiding any numerical check.

In addition to the notation introduced at the end of Chapter 1.2.4, let us denote by $p_r(n)$ the number of partitions of n into rth powers. Our main result from Chapter 3, proven with a different approach in Chapter 2 for the case $r = 2$, states the following.

Theorem 1.4. *For any $r \geq 2$ and n sufficiently large, we have*

$$p_r(0,2,n) \sim p_r(1,2,n) \sim \frac{p_r(n)}{2}$$

and

$$\begin{cases} p_r(0,2,n) > p_r(1,2,n) & \text{if } n \text{ is even,} \\ p_r(0,2,n) < p_r(1,2,n) & \text{if } n \text{ is odd.} \end{cases}$$

One part of the proof is a straightforward generalization of the argument from the case $r = 2$. However, in order to extend the result to any $r \geq 2$, we have to substantially modify the rest of the proof and find a way to avoid doing a computer check as in the case $r = 2$. This is possible on invoking a result of Banks and Shparlinski [BS15] which, somewhat surprisingly, is related to the work of Cohn and Elkies [CE03] on effective lower bounds for center density in the sphere packing problem. This result was unbeknownst to the author at the time of completing the project [Cio20] on which Chapter 2 is built.

1.3 Overpartition ranks

1.3.1 Preliminaries

An *overpartition* of n is a partition in which the first occurrence of a part may (or may not) be overlined. We denote by $\overline{p}(n)$ the number of overpartitions of n. For a comparison with the usual partitions, we have seen that $p(4) = 5$, while $\overline{p}(4) = 14$, as the overpartitions of 4 are $4,\ \overline{4},\ 3+1,\ \overline{3}+1,\ 3+\overline{1},\ \overline{3}+\overline{1},\ 2+2,\ \overline{2}+2,\ 2+1+1,\ \overline{2}+1+1,\ 2+\overline{1}+1,\ \overline{2}+\overline{1}+1,\ 1+1+1+1$ and $\overline{1}+1+1+1$.

Overpartitions are natural combinatorial objects appearing, most notably, in q-series and combinatorics (see, e.g., [CP04, CH04, CL02, CL04, Lov04b, Yee04]), but also in areas such as mathematical physics [FJM05a, FJM05b], the theory of symmetric functions [Bre93, DLM03], representation theory [KK04] and algebraic number theory [Lov04a, LM08]. In [JS87] they led to an algorithmic approach to the combinatorics of basic hypergeometric series, while in [Cor03] and [CL04] they played a central role in the bijective proofs of Ramanujan's ${}_1\psi_1$ summation and the q-Gauss summation. For a survey of results, the reader is referred to [Pak06] and the references therein.

1.3.2 From Ramanujan to Dyson: ranks and cranks

An easy consequence of the result stated in Corollary 1.3 is a recursion for the partition function obtained by Euler, which says that if $n > 0$, then

$$p(n) - p(n-1) - p(n-2) + p(n-5) + p(n-7) + \cdots$$

$$+(-1)^m p\left(n-\frac{1}{2}m(3m-1)\right)+(-1)^m p\left(n-\frac{1}{2}m(3m+1)\right)+\cdots=0,$$

where we set $p(n)=0$ for any $n<0$. Using this recursion, MacMahon was able to compute $p(n)$ up to the value $n=200$ and made tables with these values. Studying these tables, Ramanujan discovered some remarkable congruences that seemed to hold for the partition function. These congruences stated that, for $n\geq 0$,

$$\begin{aligned} p(5n+4) &\equiv 0 \pmod 5,\\ p(7n+5) &\equiv 0 \pmod 7,\\ p(11n+6) &\equiv 0 \pmod{11}. \end{aligned}$$

Here and in what follows we write, for $a,b\in\mathbb{C}$ and $n\in\mathbb{N}\cup\{\infty\}$,

$$(a)_n:=\prod_{r=0}^{n-1}(1-aq^r) \quad \text{and} \quad (a;b)_n:=\prod_{r=0}^{n-1}(1-aq^r)(1-bq^r).$$

Using q-series identities such as

$$\sum_{n=0}^{\infty}p(5n+4)q^n=5\frac{(q^5)_\infty^5}{(q)_\infty^6} \quad \text{or} \quad \sum_{n=0}^{\infty}p(7n+5)q^n=7\frac{(q^7)_\infty^3}{(q)_\infty^4}+49q\frac{(q^7)_\infty^7}{(q)_\infty^8},$$

Ramanujan [Ram21] proved the first two congruences, while claiming that “it appears there are no equally simple properties for any moduli involving primes other than these.” Nevertheless, his proof gives very little combinatorial insight as to why the above congruences hold.

In order to give a combinatorial answer to this question, Dyson [Dys44] introduced the *rank* of a partition, often known also as *Dyson's rank*, which is defined to be the largest part of the partition minus the number of its parts. Dyson conjectured that the partitions of $5n+4$ form 5 groups of equal size when sorted by their ranks modulo 5, and that the same is true for the partitions of $7n+5$ when sorted modulo 7, conjecture which was proven by Atkin and Swinnerton-Dyer [AS54]. Dyson also conjectured the existence of a *crank* function for partitions that would provide a combinatorial proof of Ramanujan's congruences modulo 11. Some forty years later, in a celebrated paper, Andrews and Garvan [AG88] found this function and managed to show how the crank simultaneously explains the three Ramanujan congruences modulo 5, 7 and 11.

1.3.3 Motivation and previous results

There are several ways to define the *rank* of an overpartition. One way is to simply extend the definition we have seen for partitions, by letting the rank be the largest part of the overpartition minus the number of its parts. This is also known as the *D-rank*, in light of its direct relation with Dyson's rank for partitions. Another rank studied in the literature is defined to be one less than the largest part ℓ of the overpartition minus the number of overlined parts less than ℓ. In this thesis, however, we will only deal with the D-rank, which, for simplicity, will just be called "rank."

We denote by $N(m,n)$ the number of partitions of n with rank m and by $N(a,m,n)$ the number of partitions of n with rank congruent to a modulo m, whilst by $\overline{N}(m,n)$ and $\overline{N}(a,m,n)$ we denote the same quantities for overpartitions. It is well-known (see, e.g., [AG88]) that

$$R(w;q) := \sum_{m=-\infty}^{\infty} \sum_{n=0}^{\infty} N(m,n) w^m q^n = 1 + \sum_{n=1}^{\infty} \frac{q^{n^2}}{(wq;q)_n (w^{-1}q;q)_n}.$$

On denoting

$$R(\zeta_c^a;q) =: 1 + \sum_{n=1}^{\infty} A\left(\frac{a}{c};n\right) q^n,$$

Bringmann and Ono [BO10] proved that $R(\zeta_c^a;q)$ is the holomorphic part of a harmonic Maass form of weight $1/2$, and showed that the rank partition function satisfies some other Ramanujan type congruences.

Theorem 1.5 ([BO10, Th. 1.5]). *Let t be a positive odd integer, and let $Q \nmid 6t$ be prime. If j is a positive integer, then there are infinitely many non-nested arithmetic progressions $An + B$ such that for every $0 \le r < t$ we have*

$$N(r, t, An + B) \equiv 0 \pmod{Q^j}.$$

In the same spirit, Bringmann and Lovejoy [BL07] showed that the overpartition rank generating function is the holomorphic part of a harmonic Maass form of weight $1/2$ and proved similar congruences for overpartitions.

Theorem 1.6 ([BL07, Th. 1.2]). *Let t be a positive odd integer, and let $\ell \nmid 6t$ be a prime. If j is a positive integer, then there are infinitely* *many non-nested arithmetic progressions* $An+B$ *such that for every* $0 \le r < t$ *we have*

$$\overline{N}(r,t,An+B) \equiv 0 \pmod{\ell^j}.$$

Remark 1.1. By *non-nested* the authors mean that there are infinitely many arithmetic progressions $An+B$, with $0 \le B < A$, such that no progression contains another.

In [Bri09] Bringmann obtained asymptotic formulas for the coefficients $A\left(\frac{a}{c};n\right)$, which in turn were used to compute asymptotics for the rank partition function and to answer a conjecture of Andrews and Lewis.

Theorem 1.7 ([Bri09, Th. 1.1])**.** *If* $0 < a < c$ *are coprime integers and* c *is odd, then for positive integers* n *we have that*

$$\begin{aligned} A\left(\frac{a}{c};n\right) = {} & \frac{4\sqrt{3}i}{\sqrt{24n-1}} \sum_{\substack{1 \le k \le \sqrt{n} \\ c \mid k}} \frac{B_{a,c,k}(-n,0)}{\sqrt{k}} \sinh\left(\frac{\pi\sqrt{24n-1}}{6k}\right) \\ & + \frac{8\sqrt{3}\sin\left(\frac{\pi a}{c}\right)}{\sqrt{24n-1}} \sum_{\substack{1 \le k \le \sqrt{n} \\ c \nmid k \\ r \ge 0,\ \delta_{c,k,r} > 0}} \frac{D_{a,c,k}(-n,m_{a,c,k,r})}{\sqrt{k}} \\ & \times \sinh\left(\frac{\pi\sqrt{2\delta_{c,k,r}(24n-1)}}{\sqrt{3}k}\right) \\ & + O_c(n^{\varepsilon}). \end{aligned}$$

Here the parameters $m_{a,c,k,r}$ and $\delta_{c,k,r}$ depend only on the values of a,c,k,r and $B_{a,c,k}, D_{a,c,k}$ are Kloosterman sums depending on a,c,k that can be computed without much difficulty for small values of c. As a consequence of Theorem 1.7, asymptotics for $N(a,c,n)$ follow. In [AL00] and [Lew97] Andrews and Lewis showed that

$$\begin{aligned} N(0,2,2n) &< N(1,2,2n) && \text{if } n \ge 1, \\ N(0,4,n) &> N(2,4,n) && \text{if } n > 26 \text{ and } n \equiv 0,1 \pmod 4, \\ N(0,4,n) &< N(2,4,n) && \text{if } n > 26 \text{ and } n \equiv 2,3 \pmod 4, \end{aligned}$$

and formulated an open problem.

Conjecture 1.2 (Andrews–Lewis, 2000)**.** *For all $n > 0$, we have*

$$N(0,3,n) < N(1,3,n) \quad \text{if } n \equiv 0,2 \pmod 3,$$
$$N(0,3,n) > N(1,3,n) \quad \text{if } n \equiv 1 \pmod 3.$$

Using the asymptotics obtained for $N(a,c,n)$, Bringmann [Bri09] proved that Conjecture 1.2 is true for $n \notin \{3,9,21\}$, in which cases we have equality.

1.3.4 Description of results

There has been a vivid study of similar questions for overpartition ranks. Lovejoy and Osburn [LO08] computed the rank differences $\overline{N}(s,\ell,n) - \overline{N}(t,\ell,n)$ for $\ell = 3$ and $\ell = 5$, while Jennings-Shaffer [Jen16] did so for $\ell = 7$. Ji, Zhang and Zhao [JZZ18] proved some identities and inequalities between overpartition ranks for moduli 6 and 10, conjecturing some others. Further inequalities, stated below, were conjectured modulo 6 by Wei and Zhang [WZ20].

Conjecture 1.3 (Ji–Zhang–Zhao, 2018)**.**

(i) *For $n \geq 0$ and $1 \leq i \leq 4$, we have*

$$\overline{N}(0,10,5n+i)+\overline{N}(1,10,5n+i) \geq \overline{N}(4,10,5n+i)+\overline{N}(5,10,5n+i).$$

(ii) *For $n \geq 0$, we have*

$$\overline{N}(1,10,n) + \overline{N}(2,10,n) \geq \overline{N}(3,10,n) + \overline{N}(4,10,n).$$

Conjecture 1.4 (Wei–Zhang, 2018)**.** *For $n \geq 11$, we have*

$$\overline{N}(0,6,3n) \geq \overline{N}(1,6,3n) = \overline{N}(3,6,3n) \geq \overline{N}(2,6,3n),$$

$$\overline{N}(0,6,3n+1) \geq \overline{N}(1,6,3n+1) = \overline{N}(3,6,3n+1) \geq \overline{N}(2,6,3n+1),$$
$$\overline{N}(1,6,3n+2) \geq \overline{N}(2,6,3n+2) \geq \overline{N}(0,6,3n+2) \geq \overline{N}(3,6,3n+2).$$

In Chapter 4 we compute asymptotics for the overpartition rank generating function and, as a corollary, we prove that Conjectures 1.3 and 1.4 are true. We prove, in addition, that the rank $\overline{N}(a,c,n)$ is equidistributed as $n \to \infty$ with respect to $0 \leq a \leq c-1$. While the main ideas are essentially those used by Bringmann [Bri09] in computing asymptotics for partition ranks, complications arise and certain modifications need to be made.

To make this more precise, similarly to the case of partitions, there is an overpartition rank generating function (see, e.g, [Lov05]) of the form

$$\mathcal{O}(u;q) := \sum_{m=-\infty}^{\infty}\sum_{n=0}^{\infty}\overline{N}(m,n)u^m q^n = \sum_{n=0}^{\infty}\frac{(-1)_n q^{\frac{1}{2}n(n+1)}}{(uq;q/u)_n}$$
$$= \frac{(-q)_\infty}{(q)_\infty}\left(1+2\sum_{n\geq 1}\frac{(1-u)(1-u^{-1})(-1)^n q^{n^2+n}}{(1-uq^n)(1-u^{-1}q^n)}\right).$$

On letting

$$\mathcal{O}\left(\frac{a}{c};q\right) := \mathcal{O}\left(\zeta_c^a;q\right) = 1+\sum_{n=1}^{\infty}A\left(\frac{a}{c};n\right)q^n,$$

we prove the following.

Theorem 1.8. *If $0<a<c$ are coprime positive integers with $c>2$, and $\varepsilon>0$ is arbitrary, then*

$$A\left(\frac{a}{c};n\right) = i\sqrt{\frac{2}{n}}\sum_{\substack{1\leq k\leq\sqrt{n}\\ c|k,\ 2\nmid k}}\frac{B_{a,c,k}(-n,0)}{\sqrt{k}}\sinh\left(\frac{\pi\sqrt{n}}{k}\right)$$
$$+2\sqrt{\frac{2}{n}}\sum_{\substack{1\leq k\leq\sqrt{n}\\ c\nmid k,\ 2\nmid k,\ c_1\neq 4\\ r\geq 0,\ \delta_{c,k,r}>0}}\frac{D_{a,c,k}(-n,m_{a,c,k,r})}{\sqrt{k}}\sinh\left(\frac{4\pi\sqrt{\delta_{c,k,r}n}}{k}\right)$$
$$+O_c(n^\varepsilon).$$

Here we set $c_1=\frac{c}{(c,k)}$. The quantities $m_{a,c,k,r},\delta_{c,k,r}$ and the Kloosterman sums $B_{a,c,k},D_{a,c,k}$ depend only on a,c,k,r and can be easily computed for small values of c. We will explain this in detail in Chapter 4, where we use Theorem 1.8 to prove Conjectures 1.3 and 1.4. In addition, we prove some other similar inequalities and we obtain an interesting consequence.

Corollary 1.4. *If $c\geq 2$, then for any $0\leq a\leq c-1$ we have, as $n\to\infty$,*

$$\overline{N}(a,c,n)\sim\frac{\overline{p}(n)}{c}\sim\frac{1}{c}\cdot\frac{e^{\pi\sqrt{n}}}{8n}.$$

1.4 Methods used in the proofs

Other than the common flavor of the results of this thesis, all dealing with asymptotics, inequalities and equidistribution properties for partition functions, our projects also share a few techniques. One of them, and perhaps the technique that cannot be separated from the asymptotic study of partitions, is the *circle method*. Another tool pertains to the modular transformations for the partition functions involved that are provided to us by the work of Wright [Wri34], and Bringmann and Lovejoy [BL07].

1.4.1 Circle method

The circle method is perhaps the crowning achievement of the joint efforts of Hardy, Littlewood and Ramanujan. Originally used to find asymptotics for the number of partitions of n and in the study of Waring's problem, the method was later perfected or slightly modified by Rademacher, who came up with a particular application for computing coefficients of modular forms of negative weight, Davenport, Vinogradov, and many others. As such, there are several versions in which the method can be applied, all more or less equivalent. The classical approach, which we outline below and which we use in Chapter 4, relies on the so-called *Farey arcs*. Another variant, used in Chapters 2 and 3, deals with *major* and *minor arcs*. We briefly explain in what follows the main idea of the method as it was essentially thought of by Hardy and Ramanujan [HR18].

We recall that, for $0 < |z| < 1$, the partition function is generated by the product

$$F(z) := \prod_{n=1}^{\infty} \frac{1}{1-z^n} = \sum_{n=0}^{\infty} p(n) z^n, \tag{1.4.1}$$

from where it follows that

$$\frac{F(z)}{z^{n+1}} = \sum_{k=0}^{\infty} \frac{p(k) z^k}{z^{n+1}} \tag{1.4.2}$$

for each $n \geq 0$. As the series on the right-hand side of (1.4.2) is the Laurent expansion of $F(z)/z^{n+1}$ in the punctured disk $0 < |z| < 1$ and this function has a pole at $z = 0$ with residue $p(n)$, by Cauchy's residue theorem we obtain

$$p(n) = \frac{1}{2\pi i}\int_{\mathcal{C}} \frac{F(z)}{z^{n+1}}dz,$$

where $\mathcal{C}$ is any positively oriented simple closed contour that encircles the origin and lies inside the unit circle. The factors appearing in the product from (1.4.1) vanish at each root of unity, i.e., whenever $z = 1$, $z^2 = 1$, $z^3 = 1$, etc.

The main idea of the circle method, as groundbreaking as it is simple, is to choose $\mathcal{C}$ a circle of radius very close (but not equal) to 1 and to divide it into arcs $\mathcal{C}_{h,k}$ lying near the roots of unity $e^{\frac{2\pi ih}{k}}$, with $0 \le h < k$, $(h,k) = 1$ and $k = 1, 2, \dots, N$, for some value N suitably chosen in terms of n. The integral along $\mathcal{C}$ can then be splitted into a finite sum of integrals along these arcs as

$$\int_{\mathcal{C}} \frac{F(z)}{z^{n+1}}dz = \sum_{k=1}^{N} \sum_{\substack{h=0 \\ (h,k)=1}}^{k-1} \int_{\mathcal{C}_{h,k}} \frac{F(z)}{z^{n+1}}dz,$$

and on each arc $\mathcal{C}_{h,k}$ the function $F(z)$ appearing in the integrand is to be replaced by some elementary function $\psi_{h,k}(z)$ which has essentially the same behavior as $F(z)$, but gives more information near the singularities $e^{\frac{2\pi ih}{k}}$. Replacing F by $\psi_{h,k}$ introduces an error that needs to be estimated as $N \to \infty$. It is often this error term that is the most difficult to compute, and this will indeed be the case in our problems. The arcs $\mathcal{C}_{h,k}$ are called *Farey arcs* and a beautiful exposition of the circle method and of the Farey dissection is given by Apostol [Apo90, Ch. 5].

1.4.2 Modular transformations

After applying the circle method, we need to use some modular transformations that capture the behavior of our generating functions in a better way.

Through a skillful application of the method developed by Hardy, Littlewood and Ramanujan, Wright [Wri34] found modular transformations for the generating function

$$H_r(q) := \sum_{n=0}^{\infty} p_r(n)q^n$$

of partitions into rth powers. As too much notation would need to be introduced in order to formulate his result in full, we keep things brief for the moment, while coming back to it in Chapters 2 and 3. If $0 \leq a < b$ are coprime integers and $\mathrm{Re}(y) > 0$, Wright's transformation formula [Wri34, Th. 4] says that

$$H_r\left(e^{\frac{2\pi i a}{b} - y}\right) = C_{a,b} y^{\frac{1}{2}} e^{jy} \exp\left(\frac{\Lambda_{a,b}}{\sqrt[r]{y}}\right) P_{a,b},$$

where $C_{a,b}$ is a constant depending on a and b,

$$j = j(r) = (-1)^{\frac{1}{2}(r+1)} \frac{\Gamma(r+1)\zeta(r+1)}{(2\pi)^{r+1}}$$

if r is odd and $j = 0$ otherwise. The expression $\Lambda_{a,b}$, defined in terms of the rth *Gauss sums*

$$S_r(a,b) := \sum_{n=1}^{b} \exp\left(\frac{2\pi i a n^r}{b}\right)$$

as

$$\Lambda_{a,b} := \frac{\Gamma\left(1+\frac{1}{r}\right)}{b} \sum_{m=1}^{\infty} \frac{S_r(ma,b)}{m^{1+\frac{1}{r}}},$$

will play a major role in our analysis. The factors $P_{a,b}$ only depend on y and, as we shall see, $\log|P_{a,b}(y)| = O(b)$. In Chapter 2, with the help of a numerical check, some explicit bounds for the sum $S_r(a,b)$ will be given in case $r = 2$. In Chapter 3 we use an estimate of Banks and Shparlinski [BS15] on the sums $S_r(a,b)$ and we show how to change the proof from the case $r = 2$ in order to extend it to any $r \geq 2$ and avoid running the computer checks which would now be of no help. This bound was found using the work of Cochrane and Pinner [CP11] on Gauss sums with prime moduli and that of Cohn and Elkies [CE03] on center density bounds in the sphere packing problem.

As for the third project, we move our attention to overpartitions and their rank generating functions. Using Poisson summation and following ideas from [And66] and [Bri09], Bringmann and Lovejoy [BL07] found transformation laws for the overpartition generating functions. Based on certain divisibility conditions, these fall into six classes. We exemplify one particular instance of the transformation behavior.

Let $0 \le h < k$ be coprime integers, $q = e^{\frac{2\pi i}{k}(h+iz)}$ and $q_1 = e^{\frac{2\pi i}{k}\left(h'+\frac{i}{z}\right)}$, with $z \in \mathbb{C}$ and $\mathrm{Re}(z) > 0$. Let $h' \in \mathbb{Z}$ be defined by $hh' \equiv -1 \pmod{k}$ and set $k_1 = \frac{k}{(c,k)}$. If $0 < a < c$ are coprime integers with $c \mid k$ and $2 \mid k$, then the transformation law found by Bringmann and Lovejoy [BL07] tells us that

$$\begin{aligned}\mathcal{O}\left(\frac{a}{c};q\right) = (-1)^{k_1+1} i e^{-\frac{2\pi a^2 h' k_1}{c}} \tan\left(\frac{\pi a}{c}\right)\cot\left(\frac{\pi a h'}{c}\right)\\ \times \frac{\omega_{h,k}^2}{\omega_{h,k/2}} z^{-\frac{1}{2}} \mathcal{O}\left(\frac{ah'}{c};q_1\right)\\ + \frac{4\sin^2\left(\frac{\pi a}{c}\right)\cdot \omega_{h,k}^2}{\omega_{h,k/2}\cdot k} z^{-\frac{1}{2}} \sum_{\nu=0}^{k-1} (-1)^\nu e^{-\frac{2\pi i h' \nu^2}{k}} I_{a,c,k,\nu}(z),\end{aligned}$$

where $\omega_{h,k}$ is a certain root of unity and $I_{a,c,k,\nu}$, defined for $\nu \in \mathbb{Z}$, is a Mordell-type integral. We will evaluate and make all these quantities precise in Chapter 4.

1.5 General motivation

Having had a first glimpse at the results which we are going to discuss in the following chapters and at what motivated them, we would like to conclude this preliminary part by explaining the reasoning behind the content of this thesis, which consists of the three articles [Cio20], [Cio19] and [Cio] written by the author.

Chapter 2 of the thesis coincides with the content of [Cio20], paper in which we prove that $p_r(a,2,n)$, the number of partitions of n into rth powers with a number of parts congruent to a modulo 2, is equidistributed with respect to $a \in \{0,1\}$ as $n \to \infty$, and that which quantity is bigger between $p_r(0,2,n)$ and $p_r(1,2,n)$ alternates with the parity of n for $r = 2$. In particular, this solves a conjecture formulated by Bringmann and Mahlburg (2012).

In Chapter 3 we continue by presenting the results of [Cio]. In this paper, with partly different methods than those used in [Cio20], we show how to generalize the previous results to all values $r \ge 2$. The methods used in [Cio20] would alone not suffice for a generalization. As the author was not aware of the techniques used in [Cio] at the time of completing [Cio20], we chose to present in detail both projects, highlighting the similarities and the differences between them.

Chapter 4 is, up to minor reformulations, a reproduction of the material from [Cio19], paper in which we study overpartitions, a very natural and important generalization of the usual partitions. More precisely, we compute asymptotics for an infinite class of overpartition rank generating functions and we show that $\overline{N}(a, c, n)$, the number of overpartitions of n having a rank congruent to a modulo c, is equidistributed with respect to $0 \leq a < c$, while proving, at the same time, several inequalities between overpartition ranks conjectured by Ji, Zhang and Zhao (2018), and Wei and Zhang (2018).

The projects mentioned above found inspiration in a few recent conjectures and focused on the study of partitions and overpartitions. Trying to answer these questions did not only reveal that the conjectures hold true, but also led to new results on the asymptotic behavior of these partition functions. This, in turn, allowed us to prove several inequalities for our partition objects, as well as a certain asymptotic uniformity of the quantities which we investigated.

While the problems that motivated our work were of course different in nature, there are several proof techniques that they had in common, such as the circle method, which was used in combination with modular transformations to obtain asymptotics for the quantities we were interested in. At the same time, all our results eventually dealt with the same topics of asymptotics, inequalities, and uniform distribution.

For these reasons, we believe that the aforementioned articles fit naturally together and, consequently, we decided to unify them under the common theme of this thesis.

Chapter 2

Partitions into squares

This chapter is based on the paper [Cio20] published in the *Journal of International Number Theory.*

2.1 Introduction

A *partition* of a positive integer n is a non-increasing sequence of positive integers (called its *parts*), usually written as a sum, which add up to n. The number of partitions of n is denoted by $p(n)$. For example, $p(5) = 7$ as the partitions of 5 are 5, $4+1$, $3+2$, $3+1+1$, $2+2+1$, $2+1+1+1$ and $1+1+1+1+1$. By convention, $p(0) = 1$. This is the case of the so-called *unrestricted* partitions, but one can consider partitions with various other properties, such as partitions into odd parts, partitions into distinct parts, etc.

Studying congruence properties of partition functions fascinated many people and we limit ourselves to mentioning the famous congruences of Ramanujan [Ram21], who proved that if $n \geq 0$, then

$$\begin{aligned} p(5n+4) &\equiv 0 \pmod{5}, \\ p(7n+5) &\equiv 0 \pmod{7}, \\ p(11n+6) &\equiv 0 \pmod{11}. \end{aligned}$$

In this chapter we study partitions based on their number of parts being in certain congruence classes. For $r \in \mathbb{N}$ let $p_r(a, m, n)$ be the number of partitions of n into rth powers with a number of parts that is congruent to a modulo m. Glaisher [Gla76] proved (with different notation) that

$$p_1(0, 2, n) - p_1(1, 2, n) = (-1)^n p_{\text{odd}}(n),$$

where $p_{\text{odd}}(n)$ denotes the number of partitions of n into odd parts without repeated parts.

It is as such of interest to ask what happens for partitions into rth powers with $r \geq 2$, and a natural point to start is by investigating partitions into squares. Based on computer experiments, Bringmann and Mahlburg [BM] observed an interesting pattern and conjectured the following.

Conjecture 2.1 (Bringmann–Mahlburg, 2012)**.**

(i) *As $n \to \infty$, we have*

$$p_2(0, 2, n) \sim p_2(1, 2, n).$$

(ii) *We have*

$$\begin{cases} p_2(0,2,n) > p_2(1,2,n) & \textit{if } n \textit{ is even,} \\ p_2(0,2,n) < p_2(1,2,n) & \textit{if } n \textit{ is odd.} \end{cases}$$

We build on the initial work done by Bringmann and Mahlburg [BM] towards solving Conjecture 2.1, the goal of this chapter being to prove that the inequalities stated in part (ii) hold true asymptotically. In turn, this will show that part (i) of Conjecture 2.1 holds true as well. More precisely, we prove the following.

Theorem 2.1.

(i) *As $n \to \infty$, we have*

$$p_2(0, 2, n) \sim p_2(1, 2, n).$$

(ii) *Furthermore, for n sufficiently large, we have*

$$\begin{cases} p_2(0,2,n) > p_2(1,2,n) & \textit{if } n \textit{ is even,} \\ p_2(0,2,n) < p_2(1,2,n) & \textit{if } n \textit{ is odd.} \end{cases}$$

In other words, we prove that the number of partitions into squares with an even number of parts is asymptotically equal to that of partitions into squares with an odd number of parts. However, for n large enough, the two quantities are always different, which of the two is bigger depending on the parity of n. Given that asymptotics for partitions into rth powers (in particular, for partitions into squares) are known due to Wright [Wri34], we can make the asymptotic value in part (i) of Theorem 2.1 precise. We will come back to this after we give the proof of Theorem 2.1.

As for the structure of this chapter, in Chapters 2.2 and 2.3 we introduce the notation needed in the sequel and do some preliminary work required for the proof of Theorem 2.1, which we give in detail in Chapter 2.4.

2.2 Preliminaries

2.2.1 Notation

Before going into details, we recall some notation and well-known facts that will be used throughout. By $\Gamma(s)$ and $\zeta(s)$ we denote the usual Gamma and Riemann zeta functions, while by

$$\zeta(s,q)=\sum_{n=0}^{\infty}\frac{1}{(q+n)^s}\quad (\text{for } \mathrm{Re}(s)>1 \text{ and } \mathrm{Re}(q)>0)$$

we denote the Hurwitz zeta function. For reasons of space, we will sometimes use $\exp(z)$ for e^z. Whenever we take logarithms of complex numbers, we use the principal branch and denote it by Log. By $\zeta_n = e^{\frac{2\pi i}{n}}$ we denote the standard primitive nth root of unity.

2.2.2 A key identity

If by $p_r(n)$ we denote the number of partitions of n into rth powers, then it is well-known (see, for example, [And98, Ch. 1]) that

$$\prod_{n=1}^{\infty}(1-q^{n^r})^{-1}=\sum_{n=0}^{\infty}p_r(n)q^n,$$

where, as usual, $q=e^{2\pi i\tau}$ and $\tau\in\mathbb{H}$ (the upper half-plane). Let

$$H_r(w;q):=\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}p_r(m,n)w^mq^n,$$

with $p_r(m,n)$ being the number of partitions of n into rth powers with m parts, and

$$H_{r,a,m}(q):=\sum_{n=0}^{\infty}p_r(a,m,n)q^n,$$

where $p_r(a,m,n)$ stands, as defined in Chapter 2.1, for the number of partitions of n into rth powers with a number of parts that is congruent to a modulo m.

By using the orthogonality of roots of unity, we obtain

$$H_{r,a,m}(q) = \frac{1}{m}H_r(q) + \frac{1}{m}\sum_{j=1}^{m-1} \zeta_m^{-aj} H_r(\zeta_m^j; q), \tag{2.2.1}$$

where we denote

$$H_r(q) := \prod_{n=1}^{\infty} (1-q^{n^r})^{-1}.$$

2.2.3 A reformulation of our result

For the rest of this chapter we only deal with the case $r=2$, which corresponds to partitions into squares. To prove Theorem 2.1, part (ii), it is enough to show that the series

$$H_{2,0,2}(-q) - H_{2,1,2}(-q) = \sum_{n=0}^{\infty} a_2(n) q^n$$

has positive coefficients for sufficiently large n, since

$$a_2(n) = \begin{cases} p_2(0,2,n) - p_2(1,2,n) & \text{if } n \text{ is even,} \\ p_2(1,2,n) - p_2(0,2,n) & \text{if } n \text{ is odd.} \end{cases}$$

Using, in turn, (2.2.1) and eq. (2.1.1) of Andrews [And98, p. 16], we obtain

$$H_{2,0,2}(q) - H_{2,1,2}(q) = H_2(-1;q) = \prod_{n=1}^{\infty} \frac{1}{1+q^{n^2}}.$$

Changing $q \mapsto -q$ gives

$$\begin{aligned} H_2(-1;-q) = \prod_{n=1}^{\infty} \frac{1}{1+(-q)^{n^2}} &= \prod_{n=1}^{\infty} \frac{1}{\left(1+q^{4n^2}\right)\left(1-q^{(2n+1)^2}\right)} \\ &= \prod_{n=1}^{\infty} \frac{\left(1-q^{4n^2}\right)^2}{\left(1-q^{8n^2}\right)\left(1-q^{n^2}\right)}. \end{aligned}$$

Therefore, by setting

$$G(q) := H_{2,0,2}(-q) - H_{2,1,2}(-q),$$

we obtain

$$G(q) = \prod_{n=1}^{\infty} \frac{\left(1 - q^{4n^2}\right)^2}{\left(1 - q^{8n^2}\right)\left(1 - q^{n^2}\right)} = \sum_{n=0}^{\infty} a_2(n) q^n$$

and we want to prove that the coefficients $a_2(n)$ are positive as $n \to \infty$. We will come back to this in the next section.

2.3 Preparations for the proof

2.3.1 Meinardus' asymptotics

Our approach is to some extent similar to that taken by Meinardus [Mei54] in proving his famous theorem on asymptotics of certain infinite product generating functions and described by Andrews in more detail in [And98, Ch. 6]. Our case is however slightly different and, whilst we can follow some of the steps, we cannot apply his result directly and we need to make certain modifications. One of them pertains to an application of the *circle method.*

Under certain conditions on which we do not insist for the moment, as we shall formulate similar assumptions in the course of our proof, Meinardus gives an asymptotic formula for the coefficients $r(n)$ of the infinite product

$$f(\tau) = \prod_{n=1}^{\infty} (1 - q^n)^{-a_n} = \sum_{r=0}^{\infty} r(n) q^n, \tag{2.3.1}$$

where $a_n \geq 0$ and $q = e^{-\tau}$ with $\mathrm{Re}(\tau) > 0$.

Theorem 2.2 ([And98, Th. 6.2], cf. Meinardus [Mei54, Satz 1])**.** *As $n \to \infty$, we have*

$$r(n) = Cn^{\kappa} \exp\left(n^{\frac{\alpha}{\alpha+1}} \left(1 + \frac{1}{\alpha}\right) \left(A\Gamma(\alpha+1)\zeta(\alpha+1)\right)^{\frac{1}{\alpha+1}}\right) (1+O(n^{-\kappa_1})),$$

where

$$C = e^{D'(0)} \left(2\pi(1+\alpha)\right)^{-\frac{1}{2}} \left(A\Gamma(\alpha+1)\zeta(\alpha+1)\right)^{\frac{1-2D(0)}{2+2\alpha}},$$

$$\kappa = \frac{D(0) - 1 - \frac{1}{2}\alpha}{1+\alpha},$$
$$\kappa_1 = \frac{\alpha}{\alpha+1} \min\left\{\frac{C_0}{\alpha} - \frac{\delta}{4}, \frac{1}{2} - \delta\right\},$$

with $\delta > 0$ *arbitrary.*

Here the Dirichlet series

$$D(s) = \sum_{n=1}^{\infty} \frac{a_n}{n^s} \quad (s = \sigma + it)$$

is assumed to converge for $\sigma > \alpha > 0$ and to possess a meromorphic continuation to the region $\sigma > -c_0$ $(0 < c_0 < 1)$. In this region $D(s)$ is further assumed to be holomorphic except for a simple pole at $s = \alpha$ with residue A.

2.3.2 Circle method

We now turn attention to our problem. Let $\tau = y - 2\pi i x$ and $q = e^{-\tau}$, with $y > 0$ (so that $\mathrm{Re}(\tau) > 0$ and $|q| < 1$). Recall that, as defined in Chapter 2.2.3,

$$G(q) = \sum_{n=0}^{\infty} a_2(n) q^n = \prod_{n=1}^{\infty} \frac{\left(1 - q^{4n^2}\right)^2}{\left(1 - q^{n^2}\right)\left(1 - q^{8n^2}\right)}. \tag{2.3.2}$$

As one can easily see, unlike the product in (2.3.1), where all factors appear to non-positive powers, the factors $\left(1 - q^{4n^2}\right)$ have positive exponents in the product from the right-hand side of (2.3.2). Therefore we cannot directly apply Theorem 2.2 to obtain asymptotics for the coefficients $a_2(n)$. We will, nevertheless, follow certain steps from the proof of Meinardus [Mei54].

Let $s = \sigma + it$ and

$$D(s) = \sum_{n=1}^{\infty} \frac{1}{n^{2s}} + \sum_{n=1}^{\infty} \frac{1}{(8n^2)^s} - 2\sum_{n=1}^{\infty} \frac{1}{(4n^2)^s} = (1 + 8^{-s} - 2^{1-2s})\zeta(2s),$$

which is convergent for $\sigma > \frac{1}{2} = \alpha$, has a meromorphic continuation to $\mathbb{C}$ (thus we may choose $0 < c_0 < 1$ arbitrarily) and a simple pole at $s = \frac{1}{2}$ with residue $A = \frac{1}{4\sqrt{2}}$. We have

$$D(0) = 0,$$

$$D'(0) = \zeta(0)(-3\log 2 + 4\log 2) = -\frac{\log 2}{2}.$$

By Cauchy's Theorem we have, for $n > 0$,

$$a_2(n) = \frac{1}{2\pi i}\int_{\mathcal{C}} \frac{G(q)}{q^{n+1}}dq = e^{ny}\int_{-\frac{1}{2}}^{\frac{1}{2}} G(e^{-y+2\pi ix})e^{-2\pi inx}dx,$$

where $\mathcal{C}$ is taken to be the positively oriented circle of radius e^{-y} around the origin.

We choose

$$y = n^{-\frac{2}{3}}\left(\frac{\sqrt{\pi}}{8\sqrt{2}}\zeta\left(\frac{3}{2}\right)\right)^{\frac{2}{3}} > 0 \tag{2.3.3}$$

and set

$$m = n^{\frac{1}{3}}\left(\frac{\sqrt{\pi}}{8\sqrt{2}}\zeta\left(\frac{3}{2}\right)\right)^{\frac{2}{3}},$$

so that $ny = m$. The reason for this choice of y is motivated by the *saddle-point method*, which was also employed by Meinardus [Mei54], and will become apparent later in the proof.

Moreover, let

$$\beta = 1 + \frac{\alpha}{2}\left(1 - \frac{\delta}{2}\right), \quad \text{with } 0 < \delta < \frac{2}{3}, \tag{2.3.4}$$

so that

$$\frac{7}{6} < \beta < \frac{5}{4}. \tag{2.3.5}$$

We can then rewrite

$$a_2(n) = e^{ny}\int_{-y^\beta}^{y^\beta} G(e^{-y+2\pi ix})e^{-2\pi inx}dx + R(n), \tag{2.3.6}$$

where

$$R(n) := e^{ny}\int_{y^\beta \le |x| \le \frac{1}{2}} G(e^{-y+2\pi ix})e^{-2\pi inx}dx.$$

The idea is that the main contribution for $a_2(n)$ will be given by the integral from (2.3.6), while $R(n)$ will go into an error term. We first prove the following estimate.

Lemma 2.1. *If* $|x| \le \frac{1}{2}$ *and* $|\operatorname{Arg}(\tau)| \le \frac{\pi}{4}$*, then*

$$G(e^{-\tau}) = \frac{1}{\sqrt{2}} \exp\left(\frac{\sqrt{\pi}\zeta\left(\frac{3}{2}\right)}{4\sqrt{2}\sqrt{\tau}} + O(y^{c_0}) \right)$$

holds uniformly in x *as* $y \to 0$*, with* $0 < c_0 < 1$.

Proof. We have

$$\operatorname{Log} G(e^{-\tau}) = \sum_{k=1}^{\infty} \frac{1}{k} \sum_{n=1}^{\infty} \left(e^{-kn^2\tau} + e^{-8kn^2\tau} - 2e^{-4kn^2\tau} \right).$$

Using the Mellin inversion formula (see, e.g., [Apo90, p. 54]) we get

$$e^{-\tau} = \frac{1}{2\pi i} \int_{\sigma_0 - i\infty}^{\sigma_0 + i\infty} \tau^{-s} \Gamma(s) ds$$

for $\operatorname{Re}(\tau) > 0$ and $\sigma_0 > 0$, thus $2\pi i \operatorname{Log} G(e^{-\tau})$ equals

$$\int_{1+\alpha-i\infty}^{1+\alpha+i\infty} \Gamma(s) \sum_{k=1}^{\infty} \frac{1}{k} \sum_{n=1}^{\infty} \left(\frac{1}{(kn^2\tau)^s} + \frac{1}{(8kn^2\tau)^s} - \frac{2}{(4kn^2\tau)^s} \right) ds$$

$$= \int_{\frac{3}{2}-i\infty}^{\frac{3}{2}+i\infty} \Gamma(s) D(s) \zeta(s+1) \tau^{-s} ds. \tag{2.3.7}$$

By assumption,

$$|\tau^{-s}| = |\tau|^{-\sigma} e^{t \cdot \operatorname{Arg}(\tau)} \le |\tau|^{-\sigma} e^{\frac{\pi}{4}|t|}.$$

Classical results (see, e.g., [AAR99, Ch. 1] and [Tit86, Ch. 5]) tell us that the bounds

$$D(s) = O(|t|^{c_1}),$$
$$\zeta(s+1) = O(|t|^{c_2}),$$
$$\Gamma(s) = O\big(e^{-\frac{\pi|t|}{2}} |t|^{c_3}\big)$$

hold uniformly in $-c_0 \le \sigma \le \frac{3}{2} = 1 + \alpha$ as $|t| \to \infty$, for some c_1, c_2 and $c_3 > 0$.

Thus we may shift the path of integration to $\sigma = -c_0$. The integrand in (2.3.7) has poles at $s = \frac{1}{2}$ and $s = 0$, with residues

$$\operatorname{Res}_{s=\frac{1}{2}} (\Gamma(s) D(s) \zeta(s+1) \tau^{-s}) = A\Gamma\left(\frac{1}{2}\right) \zeta\left(\frac{3}{2}\right) \tau^{-\frac{1}{2}},$$

and

$$\begin{aligned}
&\operatorname{Res}_{s=0}(\Gamma(s)D(s)\zeta(s+1)\tau^{-s}) \\
&= \operatorname{Res}_{s=0}\left(\left(\frac{1}{s}+O(1)\right)(D'(0)s+O(s^2))\left(\frac{1}{s}+O(1)\right)(1+O(s))\right) \\
&= D'(0) = -\frac{\log 2}{2}.
\end{aligned}$$

The remaining integral equals

$$\begin{aligned}
\frac{1}{2\pi i}\int_{-c_0-i\infty}^{-c_0+i\infty} \tau^{-s}\Gamma(s)D(s)\zeta(s+1)ds &\ll |\tau|^{c_0}\int_0^\infty t^{c_1+c_2+c_3}e^{-\frac{\pi t}{4}}dt \\
&\ll |\tau|^{c_0} = |y-2\pi i x|^{c_0} \\
&\leq (\sqrt{2}y)^{c_0}
\end{aligned}$$

since, again by the assumption,

$$\frac{2\pi|x|}{y} = \tan(|\operatorname{Arg}(\tau)|) \leq \tan\left(\frac{\pi}{4}\right) = 1.$$

We therefore obtain

$$\operatorname{Log} G\left(e^{-\tau}\right) = \left(\frac{\zeta\left(\frac{3}{2}\right)\sqrt{\pi}}{4\sqrt{2}\sqrt{\tau}} - \frac{\log 2}{2}\right) + O(y^{c_0}),$$

which completes the proof. □

The proof of the upcoming Lemma 2.2 is similar in spirit with that of part (b) of the "Hilfssatz" (Lemma) in [Mei54, p. 390] or, what is equivalent, the second part of Lemma 6.1 in [And98, Ch. 6]. Our case is however more subtle, in that it involves some extra factors $P_{a,b}$ (which will be explained in what follows) and requires certain modifications.

For this we need a setup in which to apply the circle method as described by Wright [Wri34, p. 172]. For a nice introduction to the circle method and the theory of Farey fractions, the reader is referred to [Apo90, Ch. 5.4].

We consider the Farey dissection of order $\lfloor y^{-\frac{2}{3}} \rfloor$ of $\mathcal{C}$ and we distinguish two kinds of arcs:

(i) major arcs, denoted $\mathfrak{M}_{a,b}$, such that $b \leq y^{-\frac{1}{3}}$;

(ii) minor arcs, denoted $\mathfrak{m}_{a,b}$, such that $y^{-\frac{1}{3}} < b \leq y^{-\frac{2}{3}}$.

We write any $\tau \in \mathfrak{M}_{a,b} \cup \mathfrak{m}_{a,b}$ as

$$\tau = y - 2\pi i x = \tau' - 2\pi i \frac{a}{b}, \tag{2.3.8}$$

with $\tau' = y - 2\pi i x'$. From basics of Farey theory (see also [Wri34, p. 172]) it follows that[1]

$$|x'| \leq \frac{y^{\frac{2}{3}}}{b}. \tag{2.3.9}$$

2.3.3 Wright's modular transformations

Our next step requires us to apply the modular transformations found by Wright [Wri34] for the generating functions of partitions into rth powers. In what follows, we choose the principal branch of the square root. In the notation introduced in the previous subsection, the transformation law obtained by Wright [Wri34, Th. 4] rewrites as

$$H_2(q) = H_2\left(e^{\frac{2\pi i a}{b} - \tau'}\right) = C_b\sqrt{\tau'}\exp\left(\frac{\Lambda_{a,b}}{\sqrt{\tau'}}\right)P_{a,b}(\tau'), \tag{2.3.10}$$

where

$$\Lambda_{a,b} = \frac{\Gamma\left(\frac{3}{2}\right)}{b}\sum_{m=1}^{\infty}\frac{S_2(ma,b)}{m^{\frac{3}{2}}}, \tag{2.3.11}$$

$$S_2(a,b) = \sum_{n=1}^{b}\exp\left(\frac{2\pi i a n^2}{b}\right), \tag{2.3.12}$$

and

$$C_b = \frac{b_1}{2\pi},$$

with $0 \leq a < b$ coprime integers and b_1 the least positive integer such that $b \mid b_1^2$ and $b = b_1 b_2$,

$$P_{a,b}(\tau') = \prod_{h=1}^{b}\prod_{s=1}^{2}\prod_{\ell=0}^{\infty}\left(1 - g(h,\ell,s)\right)^{-1},$$

[1] In the paper [Cio20] on which this chapter is based, the equivalent version of ineq. (2.3.9) contains also a lower estimate for $|x'|$, which is a misprint and is incorrect; see (3.9) from [Cio20, p. 128]. However, that lower bound is not used anywhere in the proof.

with

$$g(h,\ell,s) = \exp\left(\frac{(2\pi)^{\frac{3}{2}}(\ell+\mu_{h,s})^{\frac{1}{2}}e^{\frac{\pi i}{4}(2s+1)}}{b\sqrt{\tau'}} - \frac{2\pi i h}{b}\right),$$

where $0 \le d_h < b$ is defined by the congruence

$$ah^2 \equiv d_h \pmod{b}$$

and

$$\mu_{h,s} = \begin{cases} \frac{d_h}{b} & \text{if } s=1, \\ \frac{b-d_h}{b} & \text{if } s=2, \end{cases}$$

for $d_h \neq 0$. If $d_h = 0$, we let $\mu_{h,s} = 1$.

Our goal is to establish the following result, the proof of which we give at the end of the section.

Lemma 2.2. *There exists $\varepsilon > 0$ such that, as $y \to 0$,*

$$G(e^{-\tau}) = O\left(e^{\frac{\Lambda_{0,1}}{2\sqrt{2y}} - cy^{-\varepsilon}}\right)$$

holds uniformly in x with $y^\beta \le |x| \le \frac{1}{2}$, for some $c > 0$.

Recall that $q = e^{-\tau}$, with $y > 0$ (so that $\mathrm{Re}(\tau) > 0$ and $|q| < 1$). From (2.3.2), (2.3.8) and (2.3.10) we have, for some positive constant C that can be made explicit,

$$G(q) = \frac{H(q)H(q^8)}{H(q^4)^2} = C\exp\left(\frac{\lambda_{a,b}}{\sqrt{\tau'}}\right)\frac{P_{a,b}(\tau')P'_{a,b}(8\tau')}{P''_{a,b}(4\tau')^2}, \tag{2.3.13}$$

where

$$P'_{a,b} = P_{\frac{8a}{(b,8)},\frac{b}{(b,8)}}, \quad P''_{a,b} = P_{\frac{4a}{(b,4)},\frac{b}{(b,4)}}$$

and

$$\lambda_{a,b} = \Lambda_{a,b} + \frac{1}{2\sqrt{2}}\Lambda_{\frac{8a}{(b,8)},\frac{b}{(b,8)}} - \Lambda_{\frac{4a}{(b,4)},\frac{b}{(b,4)}}. \tag{2.3.14}$$

Additionally, set

$$\Lambda^*_{a,b} = \frac{\Lambda_{a,b}}{\Gamma\left(\frac{3}{2}\right)} \quad \text{and} \quad \lambda^*_{a,b} = \frac{\lambda_{a,b}}{\Gamma\left(\frac{3}{2}\right)}. \tag{2.3.15}$$

We want to study the behavior of $P_{a,b}(\tau')$.

Lemma 2.3. *If* $\tau \in \mathfrak{M}_{a,b} \cup \mathfrak{m}_{a,b}$, *then*

$$\log |P_{a,b}(\tau')| \ll b \quad \textit{as } y \to 0.$$

Proof. Using (2.3.9) and letting $y \to 0$, we have

$$|\tau'|^{\frac{3}{2}} = (y^2 + 4\pi^2 x'^2)^{\frac{3}{4}} \leq \left(y^2 + \frac{4\pi^2 y^{\frac{4}{3}}}{b^2} \right)^{\frac{3}{4}} \leq \frac{c_4 y}{b^{\frac{3}{2}}} = \frac{c_4 \mathrm{Re}\,(\tau')}{b^{\frac{3}{2}}},$$

for some $c_4 > 0$. Thus, [Wri34, Lem. 4] gives

$$|g(h, \ell, s)| \leq e^{-c_5 (\ell+1)^{\frac{1}{2}}},$$

with $c_5 = \frac{2\sqrt{2\pi}}{c_4}$, which in turn leads to

$$\begin{aligned} |\log |P_{a,b}(\tau')|| &\leq \sum_{h=1}^{b} \sum_{s=1}^{2} \sum_{\ell=1}^{\infty} |\log(1 - g(h, \ell, s))| \\ &\leq 2b \sum_{\ell=1}^{\infty} \left| \log \left(1 - e^{-c_5 (\ell+1)^{\frac{1}{2}}} \right) \right| \\ &\ll b, \end{aligned}$$

concluding the proof. $\square$

2.3.4 Final lemmas

We first want to bound $G(q)$ on the minor arcs.

Lemma 2.4. *If* $\varepsilon > 0$ *and* $\tau \in \mathfrak{m}_{a,b}$, *then*

$$|\operatorname{Log} G(q)| \ll_{\varepsilon} y^{\frac{1}{6} - \varepsilon}.$$

Proof. In the proof and notation of [Wri34, Lem. 17], replace $a = \frac{1}{2}$, $b = \frac{1}{3}$, $c = 2$, $\gamma = \varepsilon$ and $N = y^{-1}$. $\square$

Before delving into the proof of Lemma 2.2 we need two final, though tedious, steps.

Lemma 2.5. *If* $0 \leq a < b$ *are coprime integers with* $b \geq 2$, *we have*

$$\max \left\{ |\mathrm{Re}\,(\lambda_{a,b})|, |\mathrm{Im}\,(\lambda_{a,b})| \right\} < \frac{\zeta\left(\frac{3}{2}\right) \Gamma\left(\frac{3}{2}\right)}{1.14 \cdot 2\sqrt{2}}.$$

Proof. A well-known result due to Gauss (for a proof see, e.g., [BEW98, Ch. 1]) says that, for $(a,b)=1$, the sum $S_2(a,b)$, defined in (2.3.12), can be computed by the formula

$$S_2(a,b)=\begin{cases}0 & \text{if } b\equiv 2 \pmod 4,\\ \varepsilon_b\sqrt{b}\left(\frac{a}{b}\right) & \text{if } 2\nmid b,\\ (1+i)\varepsilon_a^{-1}\sqrt{b}\left(\frac{b}{a}\right) & \text{if } 4\mid b,\end{cases}$$

where $\left(\frac{a}{b}\right)$ is the usual Jacobi symbol and

$$\varepsilon_b=\begin{cases}1 & \text{if } b\equiv 1 \pmod 4,\\ i & \text{if } b\equiv 3 \pmod 4.\end{cases}$$

On recalling (2.3.11), (2.3.14) and (2.3.15), it is enough to prove that

$$\max\{|\mathrm{Re}(\lambda^*_{a,b})|,|\mathrm{Im}(\lambda^*_{a,b})|\}<\frac{\zeta\left(\frac{3}{2}\right)}{1.14\cdot 2\sqrt{2}}.$$

We explicitly evaluate $\Lambda^*_{a,b}$. Let us, for simplicity, remove the subscript dependence and write $S(a,b)=S_2(a,b)$. We have, on using the fact that $S(ma,b)=dS\left(\frac{ma}{d},\frac{b}{d}\right)$ to prove the second equality below, and on replacing $m\mapsto md$ and $d\mapsto\frac{b}{d}$ to prove the third and fourth respectively,

$$\begin{aligned}\Lambda^*_{a,b}&=\frac{1}{b}\sum_{m=1}^{\infty}\frac{S(ma,b)}{m^{\frac{3}{2}}}=\frac{1}{b}\sum_{d\mid b}\sum_{\substack{m\geq 1\\(m,b)=d}}\frac{dS\left(\frac{ma}{d},\frac{b}{d}\right)}{m^{\frac{3}{2}}}\\&=\frac{1}{b}\sum_{d\mid b}d\sum_{\substack{m\geq 1\\(m,b/d)=1}}\frac{S\left(ma,\frac{b}{d}\right)}{(md)^{\frac{3}{2}}}=\frac{1}{b}\sum_{d\mid b}d^{-\frac{1}{2}}\sum_{\substack{m\geq 1\\(m,b/d)=1}}\frac{S\left(ma,\frac{b}{d}\right)}{m^{\frac{3}{2}}}\\&=\frac{1}{b}\sum_{d\mid b}\left(\frac{b}{d}\right)^{-\frac{1}{2}}\sum_{\substack{m\geq 1\\(m,d)=1}}\frac{S(ma,d)}{m^{\frac{3}{2}}}=\frac{1}{b^{\frac{3}{2}}}\sum_{d\mid b}d^{\frac{1}{2}}\sum_{\substack{m\geq 1\\(m,d)=1}}\frac{S(ma,d)}{m^{\frac{3}{2}}}.\end{aligned}$$

We distinguish several cases, in all of which we shall apply the following bound for divisor sums. If $\beta,L,\ell\in\mathbb{N}$ and γ is the *Euler-Mascheroni* constant, then

$$\sum_{\substack{d\mid\beta\\ d\equiv\ell \pmod L}}\frac{1}{d}\leq\sum_{\substack{1\leq Ld+\ell\leq\beta\\ 0\leq d\leq\frac{\beta-\ell}{L}}}\frac{1}{Ld+\ell}\leq\frac{1}{\ell}+\frac{1}{L}\sum_{1\leq d\leq\frac{\beta}{L}}\frac{1}{d}$$

$$\leq \frac{1}{\ell} + \frac{1}{L}\left(\log\left(\frac{\beta}{L}\right) + \gamma + \frac{1}{\frac{2\beta}{L} + \frac{1}{3}}\right). \tag{2.3.16}$$

Remark 2.1. The first inequality in (2.3.16) can be easily deduced, while the second one was posed as a problem in the *American Mathematical Monthly* by Tóth [Tót91, Problem E3432] and can be solved by usual techniques like summation by parts and integral estimates.

Case 1: $2 \nmid b$. We have

$$\begin{aligned}
\lambda^*_{a,b} &= \Lambda^*_{a,b} + \frac{1}{2\sqrt{2}}\Lambda^*_{8a,b} - \Lambda^*_{4a,b} \\
&= \frac{1}{b^{\frac{3}{2}}} \sum_{d|b} d^{\frac{1}{2}} \sum_{\substack{m\geq 1\\(m,d)=1}} \frac{1}{m^{\frac{3}{2}}}\left(S(ma,d) + \frac{S(8ma,d)}{2\sqrt{2}} - S(4ma,d)\right) \\
&= \frac{1}{2\sqrt{2}b^{\frac{3}{2}}} \sum_{d|b} d\varepsilon_d\left(\frac{2a}{d}\right) \sum_{\substack{m\geq 1\\(m,d)=1}} \frac{\left(\frac{m}{d}\right)}{m^{\frac{3}{2}}}.
\end{aligned}$$

In case $b \equiv 1 \pmod 4$ we bound both the real and imaginary parts of $\lambda^*_{a,b}$ (for $j = 1, 3$ respectively) by

$$\begin{aligned}
\frac{1}{2\sqrt{2}b^{\frac{3}{2}}} \sum_{\substack{d|b\\ d\equiv j \pmod 4}} d\zeta\left(\frac{3}{2}\right) &= \frac{1}{2\sqrt{2}b^{\frac{3}{2}}} \sum_{\substack{d|b\\ d\equiv j \pmod 4}} \frac{b}{d}\zeta\left(\frac{3}{2}\right) \\
&= \frac{1}{2\sqrt{2}b^{\frac{1}{2}}} \sum_{\substack{d|b\\ d\equiv j \pmod 4}} \frac{1}{d}\zeta\left(\frac{3}{2}\right),
\end{aligned}$$

whilst for $b \equiv 3 \pmod 4$ we can bound the two quantities by

$$\begin{aligned}
\frac{1}{2\sqrt{2}b^{\frac{3}{2}}} \sum_{\substack{d|b\\ d\equiv j \pmod 4}} d\zeta\left(\frac{3}{2}\right) &= \frac{1}{2\sqrt{2}b^{\frac{3}{2}}} \sum_{\substack{d|b\\ d\equiv j+2 \pmod 4}} \frac{b}{d}\zeta\left(\frac{3}{2}\right) \\
&= \frac{1}{2\sqrt{2}b^{\frac{1}{2}}} \sum_{\substack{d|b\\ d\equiv j+2 \pmod 4}} \frac{1}{d}\zeta\left(\frac{3}{2}\right).
\end{aligned}$$

Using the bound (2.3.16) in the worst possible case (that is, $d \equiv 1 \pmod 4$) gives

$$\sum_{\substack{d|b \\ d\equiv 1 \ (\mathrm{mod}\ 4)}} \frac{1}{d} \leq 1 + \frac{1}{4}\left(\log\left(\frac{b}{4}\right) + \gamma + \frac{1}{\frac{b}{2}+\frac{1}{3}}\right).$$

We checked in MAPLE that, for $b > 1$,

$$\frac{\zeta\left(\frac{3}{2}\right)}{2\sqrt{2}b^{\frac{1}{2}}}\left(1 + \frac{1}{4}\left(\log\left(\frac{b}{4}\right) + \gamma + \frac{1}{\frac{b}{2}+\frac{1}{3}}\right)\right) < \frac{\zeta\left(\frac{3}{2}\right)}{1.14\cdot 2\sqrt{2}}. \tag{2.3.17}$$

Since the left-hand side of (2.3.17) is a decreasing function of b, we are done in this case.

Case 2: $2 \parallel b$. As $S(a,b) = 0$ for $b \equiv 2 \pmod 4$, we have

$$\begin{aligned}
\lambda^*_{a,b} &= \Lambda^*_{a,b} + \frac{1}{2\sqrt{2}}\Lambda^*_{4a,\frac{b}{2}} - \Lambda^*_{2a,\frac{b}{2}} \\
&= \frac{1}{b^{\frac{3}{2}}}\sum_{d|\frac{b}{2}} d^{\frac{1}{2}} \sum_{\substack{m\geq 1 \\ (m,d)=1}} \frac{1}{m^{\frac{3}{2}}}\left(S(ma,d) + S(4ma,d) - 2\sqrt{2}S(2ma,d)\right) \\
&= \frac{2}{b^{\frac{3}{2}}}\sum_{d|\frac{b}{2}} d^{\frac{1}{2}} \sum_{\substack{m\geq 1 \\ (m,d)=1}} \frac{\varepsilon_d\left(\frac{ma}{d}\right)\left(1-\sqrt{2}\left(\frac{2}{d}\right)\right)\sqrt{d}}{m^{\frac{3}{2}}} \\
&= \frac{2}{b^{\frac{3}{2}}}\sum_{d|\frac{b}{2}} d\left(\frac{a}{d}\right)\varepsilon_d\left(1-\sqrt{2}\left(\frac{2}{d}\right)\right)\sum_{\substack{m\geq 1 \\ (m,d)=1}} \frac{\left(\frac{m}{d}\right)}{m^{\frac{3}{2}}}.
\end{aligned}$$

Taking real and imaginary parts gives (for $j = 1, 3$ respectively, and some $\ell = 1, 3$ depending on the congruence class of $\frac{b}{2} \pmod 8$)

$$\begin{aligned}
&\frac{2}{b^{\frac{3}{2}}}\sum_{\substack{d|\frac{b}{2} \\ d\equiv j \ (\mathrm{mod}\ 4)}} d\left(\frac{a}{d}\right)\left(1-\sqrt{2}\left(\frac{2}{d}\right)\right)\sum_{\substack{m\geq 1 \\ (m,d)=1}} \frac{\left(\frac{m}{d}\right)}{m^{\frac{3}{2}}} \\
&\leq \frac{\zeta\left(\frac{3}{2}\right)}{b^{\frac{1}{2}}}\sum_{\substack{d|\frac{b}{2} \\ d\equiv \ell \ (\mathrm{mod}\ 8)}} \frac{1}{d}(\sqrt{2}-1) + \frac{\zeta\left(\frac{3}{2}\right)}{b^{\frac{1}{2}}}\sum_{\substack{d|\frac{b}{2} \\ d\equiv \ell+4 \ (\mathrm{mod}\ 8)}} \frac{1}{d}(\sqrt{2}+1).
\end{aligned} \tag{2.3.18}$$

We now use (2.3.16) in the worst possible case (that is, $\ell+4 \equiv 1 \pmod 8$) to bound the expression in (2.3.18) by

$$\frac{\zeta\left(\frac{3}{2}\right)}{b^{\frac{1}{2}}}(\sqrt{2}-1)\left(\frac{1}{5}+\frac{1}{8}\left(\log\left(\frac{b}{16}\right)+\gamma+\frac{1}{\frac{b}{8}+\frac{1}{3}}\right)\right)$$
$$+\frac{\zeta\left(\frac{3}{2}\right)}{b^{\frac{1}{2}}}(\sqrt{2}+1)\left(1+\frac{1}{8}\left(\log\left(\frac{b}{16}\right)+\gamma+\frac{1}{\frac{b}{8}+\frac{1}{3}}\right)\right).$$

This is a decreasing function of b and a computer check in MAPLE shows that it is bounded above by $\frac{\zeta\left(\frac{3}{2}\right)}{1.14\cdot 2\sqrt{2}}$ for $b \geq 124$. For the remaining cases we use the well-known relation between a Dirichlet L-series and the Hurwitz zeta function (see, e.g., [Apo76, Ch. 12]) to write

$$\lambda^*_{a,b} = \frac{2}{b^{\frac{3}{2}}}\sum_{d\mid\frac{b}{2}} d^{-\frac{1}{2}}\varepsilon_d\left(1-\sqrt{2}\left(\frac{2}{d}\right)\right)\sum_{\ell=1}^{d}\left(\frac{\ell a}{d}\right)\zeta\left(\frac{3}{2},\frac{\ell}{d}\right).$$

For $b \leq 124$ we checked in MAPLE that

$$\max\{|\mathrm{Re}(\lambda^*_{a,b})|, |\mathrm{Im}(\lambda^*_{a,b})|\} < \frac{\zeta\left(\frac{3}{2}\right)}{1.14\cdot 2\sqrt{2}}.$$

Case 3: $4 \parallel b$. We have

$$\begin{aligned}
\lambda^*_{a,b} &= \Lambda^*_{a,b} + \frac{1}{2\sqrt{2}}\Lambda^*_{2a,\frac{b}{4}} - \Lambda^*_{a,\frac{b}{4}}\\
&= \frac{1}{b^{\frac{3}{2}}}\sum_{d\mid b} d^{\frac{1}{2}}\sum_{\substack{m\geq 1\\(m,d)=1}}\frac{S(ma,d)}{m^{\frac{3}{2}}}\\
&\quad + \frac{8}{b^{\frac{3}{2}}}\sum_{d\mid\frac{b}{4}} d^{\frac{1}{2}}\sum_{\substack{m\geq 1\\(m,d)=1}}\frac{1}{m^{\frac{3}{2}}}\left(\frac{S(2ma,d)}{2\sqrt{2}}-S(ma,d)\right)\\
&= \frac{1}{b^{\frac{3}{2}}}\sum_{d\mid\frac{b}{4}} d^{\frac{1}{2}}\sum_{\substack{m\geq 1\\(m,d)=1}}\frac{1}{m^{\frac{3}{2}}}(S(ma,d)+2\sqrt{2}S(2ma,d)-8S(ma,d))\\
&\quad + \frac{1}{b^{\frac{3}{2}}}\sum_{d\mid\frac{b}{4}}(4d)^{\frac{1}{2}}\sum_{\substack{m\geq 1\\(m,2d)=1}}\frac{S(ma,4d)}{m^{\frac{3}{2}}}
\end{aligned}$$

$$= \frac{1}{b^{\frac{3}{2}}} \sum_{d | \frac{b}{4}} d^{\frac{1}{2}} \sum_{\substack{m \geq 1 \\ (m,d)=1}} \frac{\varepsilon_d \sqrt{d} \left(\frac{ma}{d}\right) \left(-7 + 2\sqrt{2} \left(\frac{2}{d}\right)\right)}{m^{\frac{3}{2}}}$$
$$+ \frac{1}{b^{\frac{3}{2}}} \sum_{d | \frac{b}{4}} (4d)^{\frac{1}{2}} \sum_{\substack{m \geq 1 \\ (m,2d)=1}} \frac{(1+i)\varepsilon_{ma}^{-1} 2\sqrt{d} \left(\frac{4d}{ma}\right)}{m^{\frac{3}{2}}}.$$

In the same way as before, the real and imaginary parts of $\lambda^*_{a,b}$ can be bounded (for some $j = 1, 3$ depending on the congruence class of $\frac{b}{4} \pmod 4$) by

$$\frac{\zeta\left(\frac{3}{2}\right)}{b^{\frac{3}{2}}} \sum_{\substack{d | \frac{b}{4} \\ d \equiv j \pmod 8}} d(7 + 2\sqrt{2}) + \frac{\zeta\left(\frac{3}{2}\right)}{b^{\frac{3}{2}}} \sum_{\substack{d | \frac{b}{4} \\ d \equiv j+4 \pmod 8}} d(7 - 2\sqrt{2})$$
$$+ 4\big(1 - 2^{-\frac{3}{2}}\big) \frac{\zeta\left(\frac{3}{2}\right)}{b^{\frac{3}{2}}} \sum_{d | \frac{b}{4}} d$$
$$= \frac{\zeta\left(\frac{3}{2}\right)}{b^{\frac{3}{2}}} \sum_{\substack{d | \frac{b}{4} \\ d \equiv j \pmod 8}} \frac{b}{4d}(7 + 2\sqrt{2}) + \frac{\zeta\left(\frac{3}{2}\right)}{b^{\frac{3}{2}}} \sum_{\substack{d | \frac{b}{4} \\ d \equiv j+4 \pmod 8}} \frac{b}{4d}(7 - 2\sqrt{2})$$
$$+ 4\big(1 - 2^{-\frac{3}{2}}\big) \frac{\zeta\left(\frac{3}{2}\right)}{b^{\frac{3}{2}}} \sum_{d | \frac{b}{4}} \frac{b}{4d},$$

quantity which, by using (2.3.16) in the worst possible case (that is, $j + 4 \equiv 5 \pmod 8$), is seen to be less than

$$(7 + 2\sqrt{2}) \frac{\zeta\left(\frac{3}{2}\right)}{4b^{\frac{1}{2}}} \left(1 + \frac{1}{8}\left(\log\left(\frac{b}{32}\right) + \gamma + \frac{1}{\frac{b}{16} + \frac{1}{3}}\right)\right)$$
$$+ (7 - 2\sqrt{2}) \frac{\zeta\left(\frac{3}{2}\right)}{4b^{\frac{1}{2}}} \left(\frac{1}{5} + \frac{1}{8}\left(\log\left(\frac{b}{32}\right) + \gamma + \frac{1}{\frac{b}{16} + \frac{1}{3}}\right)\right)$$
$$+ 4\big(1 - 2^{-\frac{3}{2}}\big) \frac{\zeta\left(\frac{3}{2}\right)}{4b^{\frac{1}{2}}} \left(1 + \frac{1}{2}\left(\log\left(\frac{b}{8}\right) + \gamma + \frac{1}{\frac{b}{4} + \frac{1}{3}}\right)\right).$$

In turn, a computer check in MAPLE shows that this decreasing function in b is bounded above by $\frac{\zeta\left(\frac{3}{2}\right)}{1.14 \cdot 2\sqrt{2}}$ for $b \geq 390$. For the remaining cases we rewrite

$$\lambda^*_{a,b} = \frac{1}{b^{\frac{3}{2}}} \sum_{d \mid \frac{b}{4}} d^{-\frac{1}{2}} \varepsilon_d \left(-7 + 2\sqrt{2}\left(\frac{2}{d}\right)\right) \sum_{\ell=1}^{d} \left(\frac{\ell a}{d}\right) \zeta\left(\frac{3}{2}, \frac{\ell}{d}\right)$$
$$+ \frac{1}{b^{\frac{3}{2}}} \sum_{d \mid \frac{b}{4}} (4d)^{-\frac{1}{2}} (1+i) \sum_{\ell=1}^{4d} \varepsilon_{\ell a}^{-1} \left(\frac{4d}{\ell a}\right) \zeta\left(\frac{3}{2}, \frac{\ell}{4d}\right).$$

A MAPLE check shows that, for $b \leq 390$, we have

$$\max\{|\mathrm{Re}(\lambda^*_{a,b})|, |\mathrm{Im}(\lambda^*_{a,b})|\} < \frac{\zeta\left(\frac{3}{2}\right)}{1.14 \cdot 2\sqrt{2}}.$$

Case 4: $8 \mid b$. We write $b = 2^\nu b'$, with b' odd. If we define $\delta_{d,4} = 0$ for $4 \nmid d$ and $\delta_{d,4} = 1$ for $4 \mid d$, we have

$$\begin{aligned}
\lambda^*_{a,b} &= \Lambda^*_{a,b} + \frac{1}{2\sqrt{2}} \Lambda^*_{a,\frac{b}{8}} - \Lambda^*_{a,\frac{b}{4}} \\
&= \frac{1}{b^{\frac{3}{2}}} \sum_{d \mid b} d^{\frac{1}{2}} \sum_{\substack{m \geq 1 \\ (m,d)=1}} \frac{1}{m^{\frac{3}{2}}} \left(\varepsilon_d \left(\frac{4ma}{d}\right)\sqrt{d} + \delta_{d,4} \varepsilon_{ma}^{-1} (1+i)\sqrt{d}\left(\frac{d}{ma}\right)\right) \\
&\quad + \frac{1}{\left(\frac{b}{8}\right)^{\frac{3}{2}} 2\sqrt{2}} \sum_{d \mid \frac{b}{8}} d^{\frac{1}{2}} \sum_{\substack{m \geq 1 \\ (m,d)=1}} \frac{1}{m^{\frac{3}{2}}} \left(\varepsilon_d \left(\frac{4ma}{d}\right)\sqrt{d}\right. \\
&\quad \left. + \delta_{d,4} \varepsilon_{ma}^{-1} (1+i)\sqrt{d}\left(\frac{d}{ma}\right)\right) \\
&\quad - \frac{1}{\left(\frac{b}{4}\right)^{\frac{3}{2}}} \sum_{d \mid \frac{b}{4}} d^{\frac{1}{2}} \sum_{\substack{m \geq 1 \\ (m,d)=1}} \frac{1}{m^{\frac{3}{2}}} \left(\varepsilon_d \left(\frac{4ma}{d}\right)\sqrt{d}\right. \\
&\quad \left. + \delta_{d,4} \varepsilon_{ma}^{-1} (1+i)\sqrt{d}\left(\frac{d}{ma}\right)\right) \\
&= \frac{1}{b^{\frac{3}{2}}} \sum_{d \mid b'} d \sum_{\substack{m \geq 1 \\ (m,d)=1}} \frac{\varepsilon_d \left(\frac{ma}{d}\right)}{m^{\frac{3}{2}}} + \frac{1+i}{b^{\frac{3}{2}}} \sum_{\substack{d \mid b' \\ 2 \leq j \leq \nu-3}} 2^j d \sum_{\substack{m \geq 1 \\ (m,2d)=1}} \frac{\varepsilon_{ma}^{-1} \left(\frac{2^j d}{ma}\right)}{m^{\frac{3}{2}}} \\
&\quad - \frac{7(i+1)}{b^{\frac{3}{2}}} \sum_{d \mid b'} 2^{\nu-2} d \sum_{\substack{m \geq 1 \\ (m,2d)=1}} \frac{\varepsilon_{ma}^{-1} \left(\frac{2^{\nu-2} d}{ma}\right)}{m^{\frac{3}{2}}}
\end{aligned}$$

$$+\frac{1+i}{b^{\frac{3}{2}}}\sum_{\substack{d\mid b'\\ \nu-1\le j\le\nu}} 2^j d\sum_{\substack{m\ge 1\\ (m,2d)=1}}\frac{\varepsilon_{ma}^{-1}\left(\frac{2^j d}{ma}\right)}{m^{\frac{3}{2}}}.$$

Taking real and imaginary parts gives, for $\ell, k\in\{1,3\}$ depending on the congruence class of b' (mod 4),

$$\frac{\zeta\left(\frac{3}{2}\right)}{b^{\frac{3}{2}}}\sum_{\substack{d\mid b'\\ d\equiv\ell\ (\mathrm{mod}\ 4)}} d+\frac{\zeta\left(\frac{3}{2}\right)\left(1-2^{-\frac{3}{2}}\right)}{b^{\frac{3}{2}}}\sum_{d\mid b'} d\left(3\cdot 2^{\nu-1}+\sum_{2\le j\le\nu}2^j\right)$$

$$=\frac{\zeta\left(\frac{3}{2}\right)}{b^{\frac{3}{2}}}\sum_{\substack{d\mid b'\\ d\equiv k\ (\mathrm{mod}\ 4)}}\frac{b}{2^\nu d}+\frac{\zeta\left(\frac{3}{2}\right)\left(1-2^{-\frac{3}{2}}\right)}{b^{\frac{3}{2}}}\sum_{d\mid b'}\frac{b}{2^\nu d}\left(3\cdot 2^{\nu-1}+\sum_{2\le j\le\nu}2^j\right)$$

as bound for $\max\{|\mathrm{Re}(\lambda^*_{a,b})|, |\mathrm{Im}(\lambda^*_{a,b})|\}$. The expression inside the brackets from the inner sum equals $7\cdot 2^{\nu-1}-4<7\cdot 2^{\nu-1}$, and thus we obtain as overall bound, in the worst possible case (that is, $d\equiv 1\ (\mathrm{mod}\ 4)$),

$$\zeta\left(\frac{3}{2}\right)\left(\frac{1}{b^{\frac{1}{2}}2^\nu}\sum_{\substack{d\mid b'\\ d\equiv 1\ (\mathrm{mod}\ 4)}}\frac{1}{d}+\frac{7\left(1-2^{-\frac{3}{2}}\right)}{2b^{\frac{1}{2}}}\sum_{\substack{d\mid b'\\ d\equiv 1\ (\mathrm{mod}\ 2)}}\frac{1}{d}\right)$$

$$\le\frac{\zeta\left(\frac{3}{2}\right)}{8b^{\frac{1}{2}}}\left(1+\frac{1}{4}\left(\log\left(\frac{b'}{4}\right)+\gamma+\frac{1}{\frac{b'}{2}+\frac{1}{3}}\right)\right)$$

$$+7\left(1-2^{-\frac{3}{2}}\right)\frac{\zeta\left(\frac{3}{2}\right)}{2b^{\frac{1}{2}}}\left(1+\frac{1}{2}\left(\log\left(\frac{b'}{2}\right)+\gamma+\frac{1}{b'+\frac{1}{3}}\right)\right)$$

$$\le\frac{\zeta\left(\frac{3}{2}\right)}{8b^{\frac{1}{2}}}\left(1+\frac{1}{4}\left(\log\left(\frac{b}{32}\right)+\gamma+\frac{1}{\frac{b}{16}+\frac{1}{3}}\right)\right)$$

$$+7\left(1-2^{-\frac{3}{2}}\right)\frac{\zeta\left(\frac{3}{2}\right)}{2b^{\frac{1}{2}}}\left(1+\frac{1}{2}\left(\log\left(\frac{b}{16}\right)+\gamma+\frac{1}{\frac{b}{8}+\frac{1}{3}}\right)\right).$$

A computer check in MAPLE shows that this last expression, which is a decreasing function in b, is bounded above by $\frac{\zeta\left(\frac{3}{2}\right)}{1.14\cdot 2\sqrt{2}}$ for $b\ge 527$.

For the remaining cases we rewrite

$$
\begin{aligned}
\lambda_{a,b}^{*} &= \frac{1}{b^{\frac{3}{2}}}\sum_{d|b} d\varepsilon_d \sum_{m\geq 1}\frac{\left(\frac{4ma}{d}\right)}{m^{\frac{3}{2}}} + \frac{1}{b^{\frac{3}{2}}}\sum_{d|\frac{b}{4}} 4d(1+i)\sum_{m\geq 1}\frac{\varepsilon_{ma}^{-1}\left(\frac{4d}{ma}\right)}{m^{\frac{3}{2}}} \\
&\quad + \frac{8}{b^{\frac{3}{2}}}\sum_{d|\frac{b}{32}}\sum_{m\geq 1} 4d(1+i)\frac{\varepsilon_{ma}^{-1}\left(\frac{4d}{ma}\right)}{m^{\frac{3}{2}}} \\
&\quad - \frac{8}{b^{\frac{3}{2}}}\sum_{d|\frac{b}{16}} 4d(1+i)\sum_{m\geq 1}\frac{\varepsilon_{ma}^{-1}\left(\frac{4d}{ma}\right)}{m^{\frac{3}{2}}} \\
&= \frac{1}{b^{\frac{3}{2}}}\sum_{d|b} d^{-\frac{1}{2}}\varepsilon_d\sum_{\ell=1}^{d}\left(\frac{4\ell a}{d}\right)\zeta\left(\frac{3}{2},\frac{\ell}{d}\right) \\
&\quad + \frac{1+i}{b^{\frac{3}{2}}}\sum_{d|\frac{b}{4}}(4d)^{-\frac{1}{2}}\sum_{\ell=1}^{4d}\varepsilon_{\ell a}^{-1}\left(\frac{4d}{\ell a}\right)\zeta\left(\frac{3}{2},\frac{\ell}{4d}\right) \\
&\quad + \frac{8(i+1)}{b^{\frac{3}{2}}}\sum_{d|\frac{b}{32}}(4d)^{-\frac{1}{2}}\sum_{\ell=1}^{4d}\varepsilon_{\ell a}^{-1}\left(\frac{4d}{\ell a}\right)\zeta\left(\frac{3}{2},\frac{\ell}{4d}\right) \\
&\quad - \frac{8(i+1)}{b^{\frac{3}{2}}}\sum_{d|\frac{b}{16}}(4d)^{-\frac{1}{2}}\varepsilon_{\ell a}^{-1}\left(\frac{4d}{\ell a}\right)\zeta\left(\frac{3}{2},\frac{\ell}{4d}\right)
\end{aligned}
$$

and check that

$$\max\{|\mathrm{Re}(\lambda_{a,b}^{*})|, |\mathrm{Im}(\lambda_{a,b}^{*})|\} < \frac{\zeta\left(\frac{3}{2}\right)}{1.14\cdot 2\sqrt{2}}.$$

This finishes the proof of the lemma. $\square$

Lemma 2.6. *If $0 \leq a < b$ are coprime integers with $b \geq 2$, for some $c > 0$ we have*

$$\frac{\lambda_{0,1}}{\sqrt{y}} - \mathrm{Re}\left(\frac{\lambda_{a,b}}{\sqrt{\tau'}}\right) \geq \frac{c}{\sqrt{y}}.$$

Proof. We write $\tau' = y + ity$ for some $t \in \mathbb{R}$. We have

$$\mathrm{Re}\left(\frac{\lambda_{a,b}}{\sqrt{\tau'}}\right) = \frac{1}{\sqrt{y}}\mathrm{Re}\left(\frac{\lambda_{a,b}}{\sqrt{1+it}}\right) = \frac{1}{\sqrt{y}}\mathrm{Re}\left(\frac{\lambda_{a,b}}{(1+t^2)^{\frac{1}{4}}e^{\frac{i}{2}\arctan t}}\right)$$

$$= \frac{1}{\sqrt{y}(1+t^2)^{\frac{1}{4}}}\left(\cos\left(\frac{\arctan t}{2}\right)\mathrm{Re}\left(\lambda_{a,b}\right)+\sin\left(\frac{\arctan t}{2}\right)\mathrm{Im}\left(\lambda_{a,b}\right)\right).$$

We aim to find the maximal absolute value of

$$f(t) := \frac{1}{(1+t^2)^{\frac{1}{4}}}\left(\left|\cos\left(\frac{\arctan t}{2}\right)\right|+\left|\sin\left(\frac{\arctan t}{2}\right)\right|\right).$$

Using the trigonometric identities

$$\cos\frac{\Theta}{2}=\sqrt{\frac{1+\cos\Theta}{2}},\ \sin\frac{\Theta}{2}=\sqrt{\frac{1-\cos\Theta}{2}},\ \cos(\arctan t)=\frac{1}{\sqrt{1+t^2}},$$

as well as the fact that $|\arctan t|<\frac{\pi}{2}$, we obtain

$$f(t)=\frac{1}{\sqrt{2}}\left(\sqrt{\frac{1}{\sqrt{1+t^2}}+\frac{1}{1+t^2}}+\sqrt{\frac{1}{\sqrt{1+t^2}}-\frac{1}{1+t^2}}\right),$$

and an easy calculus exercise shows that the maximum value of f occurs for $t=\pm\frac{1}{\sqrt{3}}$ and equals

$$f\left(\pm\frac{1}{\sqrt{3}}\right)=\frac{3^{\frac{3}{4}}}{2}=1.13975\ldots<1.14.$$

On noting that $\lambda_{0,1}=\frac{\Lambda_{0,1}}{2\sqrt{2}}=\frac{\Gamma(\frac{3}{2})\zeta(\frac{3}{2})}{2\sqrt{2}}$ and that by Lemma 2.5 there exists a small enough $c>0$ such that

$$\mathrm{Re}\left(\frac{\lambda_{a,b}}{\sqrt{\tau'}}\right)\le\frac{\lambda_{0,1}-c}{\sqrt{y}},$$

we conclude the proof. $\square$

Proof of Lemma 2.2. If we are on a minor arc, then it suffices to apply Lemma 2.4 (because, as $y\to 0$, a negative power of y will dominate any positive power of y), so let us assume that we are on a major arc. We first consider the behavior near 0, which corresponds to $a=0$, $b=1$, $\tau=\tau'=y-2\pi ix$. Writing $y^\beta=y^{\frac{5}{4}-\varepsilon}$ with $\varepsilon>0$ (here we use the second inequality from (2.3.5)), we have, on setting $b=1$ in (2.3.9),

$$y^{\frac{5}{4}-\varepsilon}\le|x|=|x'|\le y^{\frac{2}{3}}. \tag{2.3.19}$$

By (2.3.13) we get

$$G(q) = Ce^{\frac{\Lambda_{0,1}}{2\sqrt{2}\sqrt{\tau}}}\frac{P_{0,1}(\tau)P_{0,1}(8\tau)}{P_{0,1}(4\tau)^2}$$

for some $C > 0$ and thus, by Lemma 2.3,

$$\log|G(q)| = \frac{\Lambda_{0,1}}{2\sqrt{2}\sqrt{|\tau|}} + O(1).$$

On using (2.3.19) to prove the first inequality below and expanding into Taylor series to prove the second one, we obtain, by letting $y \to 0$,

$$\begin{aligned}\frac{1}{\sqrt{|\tau|}} = \frac{1}{\sqrt{y}}\frac{1}{\left(1+\frac{4\pi^2x^2}{y^2}\right)^{\frac{1}{4}}} &\leq \frac{1}{\sqrt{y}}\frac{1}{\left(1+4\pi^2y^{\frac{1}{2}-2\varepsilon}\right)^{\frac{1}{4}}}\\ &\leq \frac{1}{\sqrt{y}}\left(1-c_6y^{\frac{1}{2}-2\varepsilon}\right)\end{aligned}$$

for some $c_6 > 0$, and this concludes the proof in this case.

To finish the claim we assume $2 \leq b \leq y^{-\frac{1}{3}}$. If $\tau \in \mathfrak{M}_{a,b}$, then by (2.3.13) and Lemma 2.3 we obtain

$$\log|G(q)| = \operatorname{Re}\left(\frac{\lambda_{a,b}}{\sqrt{\tau'}}\right) + O\big(y^{-\frac{1}{3}}\big)$$

as $y \to 0$. Since by Lemma 2.6 there exists $c_7 > 0$ such that

$$\operatorname{Re}\left(\frac{\lambda_{a,b}}{\sqrt{\tau'}}\right) \leq \frac{\lambda_{0,1}}{\sqrt{y}} - \frac{c_7}{\sqrt{y}}, \tag{2.3.20}$$

we infer from (2.3.20) that, as $y \to 0$, we have

$$\log|G(q)| \leq \frac{\lambda_{0,1}}{\sqrt{y}} - \frac{c_8}{\sqrt{y}}$$

for some $c_8 > 0$ and the proof is complete. $\square$

2.4 Proof of the main theorem

We have now all necessary ingredients for the proof of our main result.

Proof of Theorem 2.1. We begin by proving part (ii). By Lemma 2.2 and the fact that $\Lambda_{0,1} = \Gamma\left(\frac{3}{2}\right)\zeta\left(\frac{3}{2}\right)$, we have

$$\begin{aligned}
R(n) &= e^{ny}\int_{y^\beta\le|x|\le\frac{1}{2}} G\left(e^{-y+2\pi ix}\right)e^{-2\pi inx}dx\\
&\ll e^{ny}\int_{y^\beta\le|x|\le\frac{1}{2}} e^{\frac{1}{2\sqrt{2}}\Gamma\left(\frac{3}{2}\right)\zeta\left(\frac{3}{2}\right)\frac{1}{\sqrt{y}}-cy^{-\varepsilon}}dx\\
&\le e^{ny+\frac{1}{2\sqrt{2}}\Gamma\left(\frac{3}{2}\right)\zeta\left(\frac{3}{2}\right)\frac{1}{\sqrt{y}}-cy^{-\varepsilon}}\\
&= e^{3n^{\frac{1}{3}}\left(\frac{1}{4\sqrt{2}}\Gamma\left(\frac{3}{2}\right)\zeta\left(\frac{3}{2}\right)\right)^{\frac{2}{3}}-Cn^{\varepsilon_1}}, \qquad (2.4.1)
\end{aligned}$$

with $\varepsilon_1 = \frac{2\varepsilon}{3} > 0$ and some $C > 0$.

We next turn to the asymptotic main term integral. Let $n \ge n_1$ be large enough so that $y^{\beta-1} \le \frac{1}{2\pi}$. This choice allows us to apply Lemma 2.1, as it ensures $|x| \le \frac{1}{2}$ and $|\operatorname{Arg}(\tau)| \le \frac{\pi}{4}$. Denoting, for simplicity,

$$I(n) := e^{ny}\int_{-y^\beta}^{y^\beta} G(e^{-y+2\pi ix})e^{-2\pi inx}dx,$$

and recalling that $\Gamma\left(\frac{3}{2}\right) = \frac{\sqrt{\pi}}{2}$, we obtain

$$I(n) = \frac{e^{ny}}{\sqrt{2}}\int_{-y^\beta}^{y^\beta} e^{\frac{1}{4\sqrt{2}}\Gamma\left(\frac{1}{2}\right)\zeta\left(\frac{3}{2}\right)\frac{1}{\sqrt{\tau}}+O(y^{c_0})-2\pi inx}dx. \qquad (2.4.2)$$

Splitting

$$\frac{1}{\sqrt{\tau}} = \frac{1}{\sqrt{y}} + \left(\frac{1}{\sqrt{\tau}} - \frac{1}{\sqrt{y}}\right),$$

we can rewrite the expression for $I(n)$ from (2.4.2) as

$$\begin{aligned}
&\frac{e^{ny}}{\sqrt{2}}\int_{-y^\beta}^{y^\beta} e^{\frac{1}{4\sqrt{2}}\Gamma\left(\frac{1}{2}\right)\zeta\left(\frac{3}{2}\right)\frac{1}{\sqrt{y}}}e^{\frac{1}{4\sqrt{2}}\Gamma\left(\frac{1}{2}\right)\zeta\left(\frac{3}{2}\right)\left(\frac{1}{\sqrt{\tau}}-\frac{1}{\sqrt{y}}\right)}e^{-2\pi inx+O(y^{c_0})}dx\\
&= \frac{1}{\sqrt{2}}\int_{-y^\beta}^{y^\beta}\left(e^{ny+\frac{1}{4\sqrt{2}}\Gamma\left(\frac{1}{2}\right)\zeta\left(\frac{3}{2}\right)\frac{1}{\sqrt{y}}}\right)e^{\frac{1}{4\sqrt{2}}\Gamma\left(\frac{1}{2}\right)\zeta\left(\frac{3}{2}\right)\left(\frac{1}{\sqrt{\tau}}-\frac{1}{\sqrt{y}}\right)}e^{-2\pi inx+O(y^{c_0})}dx\\
&= C(n)\int_{-y^\beta}^{y^\beta} e^{\frac{1}{2\sqrt{2}}\Gamma\left(\frac{3}{2}\right)\zeta\left(\frac{3}{2}\right)\frac{1}{\sqrt{y}}\left(\frac{1}{\sqrt{1-\frac{2\pi ix}{y}}}-1\right)}e^{-2\pi inx+O(y^{c_0})}dx,
\end{aligned}$$

where

$$C(n) := \frac{e^{3n^{\frac{1}{3}}\left(\frac{1}{4\sqrt{2}}\Gamma\left(\frac{3}{2}\right)\zeta\left(\frac{3}{2}\right)\right)^{\frac{2}{3}}}}{\sqrt{2}}.$$

Putting $u = -\frac{2\pi x}{y}$, we get that $I(n)$ equals

$$\frac{yC(n)}{2\pi}\int_{-2\pi y^{\beta-1}}^{2\pi y^{\beta-1}} e^{\frac{1}{2\sqrt{2}}\Gamma\left(\frac{3}{2}\right)\zeta\left(\frac{3}{2}\right)\frac{1}{\sqrt{y}}\left(\frac{1}{\sqrt{1+iu}}-1\right)+inuy+O(y^{c_0})}du. \tag{2.4.3}$$

Set $B = \frac{1}{2\sqrt{2}}\Gamma\left(\frac{3}{2}\right)\zeta\left(\frac{3}{2}\right)$. We have the Taylor series expansion

$$\frac{1}{\sqrt{1+iu}} = 1 - \frac{iu}{2} - \frac{3u^2}{8} + \frac{5iu^3}{16} + \cdots = 1 - \frac{iu}{2} - \frac{3u^2}{8} + O(|u|^3),$$

thus

$$B\frac{1}{\sqrt{y}}\left(\frac{1}{\sqrt{1+iu}} - 1\right) + inuy = -\frac{Biu}{2\sqrt{y}} + inuy - \frac{3Bu^2}{8\sqrt{y}} + O\left(\frac{|u|^3}{\sqrt{y}}\right).$$

However, an easy computation shows that for y chosen as in (2.3.3) we have $B = 2ny^{\frac{3}{2}}$, hence

$$-\frac{Biu}{2\sqrt{y}} + inuy = 0,$$

and, using (2.3.3) and the fact that $|u| \leq 2\pi y^{\beta-1}$,

$$\begin{aligned} B\frac{1}{\sqrt{y}}\left(\frac{1}{\sqrt{1+iu}} - 1\right) + inuy &= -\frac{3Bu^2}{8\sqrt{y}} + O\left(\frac{|u|^3}{\sqrt{y}}\right) \\ &= -\frac{3Bu^2}{8\sqrt{y}} + O\left(n^{\frac{1}{3}\left(1+\frac{3(1-\beta)}{\alpha}\right)}\right). \end{aligned}$$

Thus, if $C_1 = 2\pi\left(\frac{B}{2n}\right)^{\frac{2}{3}(\beta-1)}$, we may change the integral from the right-hand side of (2.4.3) into

$$\begin{aligned} &\int_{|u|\leq 2\pi y^{\beta-1}} e^{B\frac{1}{\sqrt{y}}\left(\frac{1}{\sqrt{1+iu}}-1\right)+inuy+O(y^{c_0})}du \\ =&\int_{|u|\leq C_1} e^{-\frac{3Bu^2}{8\sqrt{y}}}\, e^{O\left(y^{c_0}+\frac{u^3}{\sqrt{y}}\right)}du \end{aligned}$$

$$= \int_{|u|\le C_1} e^{-\frac{3Bu^2}{8\sqrt{y}}} e^{O\left(n^{-\frac{2c_0}{3}}+n^{\frac{1}{3}+2(1-\beta)}\right)} du$$
$$= \int_{|u|\le C_1} e^{-\frac{3\sqrt[3]{2n}\sqrt[3]{B^2}u^2}{8}} \left(1+\left(e^{O\left(n^{-\frac{2c_0}{3}}+n^{\frac{1}{3}+2(1-\beta)}\right)}-1\right)\right)du.$$

From (2.3.4), or equivalently, from the first inequality in (2.3.5), we now infer that
$$\frac{1}{3}+2(1-\beta)=-\frac{1}{6}+\frac{\delta}{4}<0$$
and thus
$$e^{O\left(n^{-\frac{2c_0}{3}}+n^{\frac{1}{3}+2(1-\beta)}\right)}-1=e^{O\left(n^{-\frac{2c_0}{3}}+n^{-\frac{1}{6}+\frac{\delta}{4}}\right)}-1=O(n^{-\kappa}),$$
where $\kappa=\min\left\{\frac{2c_0}{3},\frac{1}{6}-\frac{\delta}{4}\right\}$. We further get
$$J(n)=\int_{|u|\le C_1} e^{-\frac{3\sqrt[3]{2n}\sqrt[3]{B^2}u^2}{8}}(1+O(n^{-\kappa})du,$$
where we denote
$$J(n):=\int_{|u|\le 2\pi y^{\beta-1}} e^{B\frac{1}{\sqrt{y}}\left(\frac{1}{\sqrt{1+iu}}-1\right)+inuy+O(y^{c_0})}du,$$

and, on using (2.3.3) again and setting $v=\frac{\sqrt{3}\sqrt[6]{2n}\sqrt[3]{B}u}{2\sqrt{2}}$ and $C_2=2^{\frac{1}{3}-\frac{2}{3}\beta}\sqrt{3}\pi B^{\frac{2}{3}\beta-\frac{1}{3}}>0$, we obtain
$$\begin{aligned} J(n)&=\int_{|u|\le C_1} e^{-\frac{3\sqrt[3]{2n}\sqrt[3]{B^2}u^2}{8}}(1+O(n^{-\kappa}))du \\ &=\frac{2\sqrt{2}}{\sqrt{3}\sqrt[6]{2nB^2}}\int_{|v|\le C_2 n^{\frac{\delta}{12}}} e^{-v^2}(1+O(n^{-\kappa}))dv. \end{aligned} \tag{2.4.4}$$

By letting $n\to\infty$, we turn the integral from (2.4.4) into a Gauss integral. This introduces an exponentially small error and yields
$$J(n)=\frac{2\sqrt{2}}{\sqrt{3}\sqrt[6]{2n}\sqrt[3]{B}}\cdot\sqrt{\pi}(1+O(n^{-\kappa_1})),$$
where $\kappa_1=\min\left\{\frac{2c_0}{3}-\frac{\delta}{12},\frac{1}{6}-\frac{\delta}{3}\right\}$. Putting together (2.3.6), (2.4.1) and (2.4.3), we obtain that, as $n\to\infty$, the main asymptotic contribution for our coefficients $a_2(n)$ is given by
$$a_2(n)\sim\frac{y}{2\sqrt{2}\pi}\cdot e^{3n^{\frac{1}{3}}\left(\frac{1}{4\sqrt{2}}\Gamma\left(\frac{3}{2}\right)\zeta\left(\frac{3}{2}\right)\right)^{\frac{2}{3}}}\cdot\frac{2\sqrt{2}}{\sqrt{3}\sqrt[6]{2n}\sqrt[3]{B}}\int_{-\infty}^{\infty}e^{-v^2}dv$$

$$
\begin{aligned}
&= \frac{y}{2\sqrt{2}\pi} \cdot e^{3n^{\frac{1}{3}}\left(\frac{1}{4\sqrt{2}}\Gamma\left(\frac{3}{2}\right)\zeta\left(\frac{3}{2}\right)\right)^{\frac{2}{3}}} \cdot \frac{2\sqrt{2}\cdot\sqrt{\pi}}{\sqrt{3}\sqrt[6]{2n}\sqrt[3]{B}} \\
&= \frac{\sqrt[3]{B}}{\sqrt{3\pi}\cdot(2n)^{\frac{5}{6}}} e^{3n^{\frac{1}{3}}\left(\frac{1}{4\sqrt{2}}\Gamma\left(\frac{3}{2}\right)\zeta\left(\frac{3}{2}\right)\right)^{\frac{2}{3}}}. \qquad (2.4.5)
\end{aligned}
$$

This shows that $a_2(n) > 0$ as $n \to \infty$, hence part (ii) of Theorem 2.1 is proven.

We now turn to part (i). Clearly, $p_2(n) = p_2(0,2,n) + p_2(1,2,n)$. By applying either Meinardus' Theorem (Theorem 2.2) or Wright's Theorem ([Wri34, Th. 2]) we have, on keeping the notation from [Wri34, pp. 144–145],

$$p_2(n) \sim B_0 n^{-\frac{7}{6}} e^{\Lambda n^{\frac{1}{3}}},$$

where

$$B_0 = \frac{\Lambda}{2\cdot(3\pi)^{\frac{3}{2}}}$$

and

$$\Lambda = 3\left(\frac{\Gamma\left(\frac{3}{2}\right)\zeta\left(\frac{3}{2}\right)}{2}\right)^{\frac{2}{3}} = 6\left(\frac{1}{4\sqrt{2}}\Gamma\left(\frac{3}{2}\right)\zeta\left(\frac{3}{2}\right)\right)^{\frac{2}{3}}.$$

We thus obtain

$$p_2(n) \sim B_0 n^{-\frac{7}{6}} e^{6n^{\frac{1}{3}}\left(\frac{1}{4\sqrt{2}}\Gamma\left(\frac{3}{2}\right)\zeta\left(\frac{3}{2}\right)\right)^{\frac{2}{3}}}. \qquad (2.4.6)$$

On adding (2.4.5) and (2.4.6) and recalling that

$$a_2(n) = \begin{cases} p_2(0,2,n) - p_2(1,2,n) & \text{if } n \text{ is even,} \\ p_2(1,2,n) - p_2(0,2,n) & \text{if } n \text{ is odd,} \end{cases}$$

we have

$$p_2(0,2,n) \sim p_2(1,2,n) \sim \frac{B_0}{2} n^{-\frac{7}{6}} e^{6n^{\frac{1}{3}}\left(\frac{1}{4\sqrt{2}}\Gamma\left(\frac{3}{2}\right)\zeta\left(\frac{3}{2}\right)\right)^{\frac{2}{3}}}$$

as $n \to \infty$, and the proof is complete. □

Remark 2.2. As promised in the beginning and already revealed by our proof, by plugging in the values of B_0 and Λ we obtain that, as $n \to \infty$, the asymptotic value of $p_2(0,2,n)$ and $p_2(1,2,n)$ is given by

$$\frac{1}{2\pi\sqrt{3\pi}}\left(\frac{1}{4\sqrt{2}}\Gamma\left(\frac{3}{2}\right)\zeta\left(\frac{3}{2}\right)\right)^{\frac{2}{3}} n^{-\frac{7}{6}} e^{6n^{\frac{1}{3}}\left(\frac{1}{4\sqrt{2}}\Gamma\left(\frac{3}{2}\right)\zeta\left(\frac{3}{2}\right)\right)^{\frac{2}{3}}}.$$

Remark 2.3. Note that, although we could not apply Meinardus' Theorem to our product from (2.3.2), the asymptotic value we obtained for $a_2(n)$ in (2.4.5) agrees, surprisingly or not, precisely with that given for $r(n)$ in Theorem 2.2. This indicates that, even if it may not directly apply to certain generating products, Meinardus' Theorem is a powerful enough tool to provide correct heuristics.

Remark 2.4. We notice that, in its original formulation, part (ii) of Conjecture 2.1 is not entirely true since there are cases when $p_2(0,2,n) = p_2(1,2,n)$, as it happens, e.g., for $n \in \{4,5,6,7,13,14,15,16,22,23,24,31,39,47,48,56,64\}$. No other values of n past 64 revealed such a behavior and, based on the pattern we observed, we strongly believe that the inequalities hold true for $n \geq 65$. In particular, we checked this is the case up to $n = 50,000$.

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References

[And98] G. E. Andrews, *The theory of partitions*, Reprint of the 1976 original, Cambridge Mathematical Library. Cambridge University Press, Cambridge, 1998.

[AAR99] G. E. Andrews, R. Askey and R. Roy, *Special functions*, Encyclopedia of Mathematics and its Applications 71. Cambridge University Press, Cambridge, 1999.

[Apo76] T. M. Apostol, *Introduction to Analytic Number Theory*, Undergraduate Texts in Mathematics, Springer-Verlag, New York-Heidelberg, 1976.

[Apo90] T. M. Apostol, *Modular Functions and Dirichlet Series in Number Theory*, Second edition, Graduate Texts in Mathematics 41. Springer-Verlag, New York, 1990.

[BEW98] B. C. Berndt, R. J. Evans and K. S. Williams, *Gauss and Jacobi sums*, Canadian Mathematical Society Series of Monographs and Advanced Texts. A Wiley-Interscience Publication. John Wiley & Sons, Inc., New York, 1998.

[BM] K. Bringmann and K. Mahlburg, Transformation laws and asymptotics for nonmodular products, unpublished preprint.

[Gla76] J. W. L. Glaisher, On formulae of verification in the partition of numbers, *Proc. Royal Soc. London* **24** (1876), 250–259.

[Mei54] G. Meinardus, Asymptotische aussagen über Partitionen, *Math. Z.* **59** (1954), 388–398.

[Ram21] S. Ramanujan, Congruence properties of partitions, *Math. Z.* **9** (1921), 147–153.

[Tit86] E. C. Titchmarsh, *The theory of the Riemann zeta-function*, Second edition, The Clarendon Press, Oxford University Press, New York, 1986.

[Tót91] L. Tóth, Elementary Problems: E 3432, *Amer. Math. Monthly* **98** (1991), no. 3, 263–264.

[Wri34] E. M. Wright, Asymptotic partition formulae. III. Partitions into k-th powers, *Acta Math.* **63** (1934), no. 1, 143–191.

Chapter 3

Partitions into powers

This chapter is based on the manuscript [Cio] submitted for publication in the *Journal of Combinatorial Theory, Series A* and extends the results from the previous chapter to partitions into rth powers, for any $r \geq 2$. Consequently, it presents a few unavoidable similarities with it, Chapters 3.2 and 3.3 being a straightforward generalization of their counterparts from Chapters 2.2 and 2.3.

While we could have either skipped some proofs from this chapter by saying that they are a generalization of the results from the case $r = 2$, or omitted parts from Chapter 2 by pointing out that they are a particular case of the results from this chapter, we believe it is in the benefit of the reader to present most of the arguments again. In this way, the reader can easily compare this chapter with the previous one, and understand the similarities as well as the differences between them.

In particular, we decided to keep the computer check presented in Lemma 2.5 from Chapter 2 because we think that the various techniques used there are interesting and instructive, but also since, at the time of completing the project [Cio20], the author was not aware of the bound given in Theorem 3.4 from the present chapter. It is this bound that makes possible the generalization to all $r \geq 2$.

3.1 Introduction

3.1.1 Motivation

We recall that a *partition* of a positive integer n is a non-increasing sequence (often written as a sum) of positive integers, called *parts*, adding up to n. By $p(n)$ we denote the number of partitions of n, and by convention we set $p(0) = 1$. For example, $p(4) = 5$ as the partitions of 4 are 4, $3 + 1$, $2 + 2$, $2 + 1 + 1$ and $1 + 1 + 1 + 1$, this being the case of *unrestricted* partitions. One can consider, however, partitions with various conditions imposed on their parts, such as partitions with all their parts being in a set $\mathcal{S}$ satisfying certain properties.

For $r \in \mathbb{N}$ we let $p_r(n)$ denote the number of partitions of n into rth powers, $p_r(m,n)$ that of partitions of n into rth powers with exactly m parts, and $p_r(a,m,n)$ that of partitions of n into rth powers with a number of parts that is congruent to a modulo m. Motivated by an interesting pattern noticed by Bringmann and Mahlburg and by their initial work [BM] on the problem, the author [Cio20] proved the following.

Theorem 3.1 ([Cio20, Th. 1])**.** *For n sufficiently large, we have*

$$p_2(0,2,n) \sim p_2(1,2,n) \sim \frac{p_2(n)}{2}$$

and

$$\begin{cases} p_2(0,2,n) > p_2(1,2,n) & \text{if } n \text{ is even,} \\ p_2(0,2,n) < p_2(1,2,n) & \text{if } n \text{ is odd.} \end{cases}$$

The only analogous results of which the author is aware are due to Glaisher [Gla76], who proved that if $p_{\mathrm{odd}}(n)$ denotes the number of partitions of n into odd parts without repeated parts, then

$$p_1(0,2,n) - p_1(1,2,n) = (-1)^n p_{\mathrm{odd}}(n), \tag{3.1.1}$$

and to Zhou [Zho], who has recently proven the equidistribution of partitions into parts that are certain polynomial functions. The identity established in (3.1.1) tells us that an even number n has more partitions into an even number of parts than into an odd number of parts, and the other way around if n is odd.

3.1.2 Statement of results

The goal of this chapter is to prove that Theorem 3.1 extends to partitions into any higher powers. We manage to do so by using a bound on Gauss sums established by Banks and Shparlinski [BS15] with the perhaps unexpected help of the effective lower estimates on center density found by Cohn and Elkies [CE03] for the sphere packing problem. More precisely, we prove the following.

Theorem 3.2. *For any $r \geq 2$ and n sufficiently large, we have*

$$p_r(0,2,n) \sim p_r(1,2,n) \sim \frac{p_r(n)}{2}$$

and

$$\begin{cases} p_r(0,2,n) > p_r(1,2,n) & \text{if } n \text{ is even,} \\ p_r(0,2,n) < p_r(1,2,n) & \text{if } n \text{ is odd.} \end{cases} \tag{3.1.2}$$

In other words, we prove that the number of partitions of n into rth powers with an even number of parts is greater than that with an odd number of parts if n is even, and conversely if n is odd, and that the two quantities are asymptotically equal as $n \to \infty$. The last claim will follow easily from our proof, but it is also a straightforward consequence of the aforementioned work of Zhou [Zho], who proved that

$$p_f(a,m,n) \sim \frac{p_f(n)}{m}$$

holds uniformly, as $n \to \infty$, for all $a, m, n \in \mathbb{N}$ with $m^{2+2\deg(f)} \ll n$ satisfying a certain congruence condition (see [Zho, (1.4)]), where $f \in \mathbb{Q}[x]$ is any non-constant polynomial such that the values $f(n)$ are coprime positive integers (that is, for every prime p there exists n such that $p \nmid f(n)$), $p_f(n)$ denotes the number of partitions of n with parts from the set $\mathcal{S} = \{f(n) : n \in \mathbb{N}\}$ and $p_f(a,m,n)$ that of partitions of n with parts from $\mathcal{S}$ having a number of parts congruent to a modulo m. The asymptotic equidistribution stated in Theorem 3.2 is a consequence of this result for $f(x) = x^r$. (As $f \in \mathbb{Q}[x]$ is a non-constant polynomial, we believe there is no reason for confusion between this notation and $p_r(n)$, used to denote partitions into rth powers.)

3.1.3 Notation

Before concluding this section, let us introduce some notation used in the sequel. By $\zeta_n = e^{\frac{2\pi i}{n}}$ we will denote the standard primitive nth root of unity. For reasons of space, and with the hope that the reader will not regard this as an inconsistency, we will sometimes use $\exp(z)$ for e^z. Whenever required to take logarithms or to extract roots of complex numbers, we will use the principal branch; in the case of complex logarithms, this will be denoted by Log. The symbols o, O and $\ll$ are used throughout with their standard meaning.

3.1.4 Outline

This chapter is structured as follows. In Chapter 3.2 we explain the strategy of the proof, and the similarities and differences with the proof of the same result from [Cio20] in the case $r = 2$. This will also be done throughout in the form of commentaries at the end of each section. We consider this to be for the benefit of the reader interested in comparing the present and the previous chapters. In Chapters 3.3 and 3.4 we prove two estimates which, combined, will provide the proof of Theorem 3.2, given in Chapter 3.5.

3.2 Philosophy of the proof

In view of what has already been mentioned, it is only of interest to us to prove the asymptotic inequalities from Theorem 3.2. In doing so, we will first reformulate the claim of our problem so that it becomes equivalent with proving that the coefficients of a certain product generating function are positive.

3.2.1 A reformulation

It is well-known (see, for example, [And98, Ch. 1]) that

$$\prod_{n=1}^{\infty}(1-q^{n^r})^{-1} = \sum_{n=0}^{\infty} p_r(n)q^n,$$

where, as usual, for $\tau \in \mathbb{H}$ (the upper half-plane) we set $q = e^{2\pi i\tau}$. Letting

$$\begin{aligned}
H_r(q) &:= \sum_{n=0}^{\infty} p_r(n)q^n,\\
H_r(w;q) &:= \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} p_r(m,n)w^m q^n,\\
H_{r,a,m}(q) &:= \sum_{n=0}^{\infty} p_r(a,m,n)q^n,
\end{aligned}$$

it is not difficult to see, by the orthogonality relations for roots of unity, that

$$H_{r,a,m}(q) = \frac{1}{m} H_r(q) + \frac{1}{m} \sum_{j=1}^{m-1} \zeta_m^{-aj} H_r(\zeta_m^j; q). \tag{3.2.1}$$

On noting now that

$$H_{r,0,2}(-q) - H_{r,1,2}(-q) = \sum_{n=0}^{\infty} a_r(n) q^n,$$

where

$$a_r(n) = \begin{cases} p_r(0,2,n) - p_r(1,2,n) & \text{if } n \text{ is even,} \\ p_r(1,2,n) - p_r(0,2,n) & \text{if } n \text{ is odd,} \end{cases}$$

proving the asymptotic inequalities from Theorem 3.2 is equivalent to showing that $a_r(n) > 0$ as $n \to \infty$. Using, in turn, (3.2.1) and eq. (2.1.1) from [And98, p. 16], we obtain

$$H_{r,0,2}(q) - H_{r,1,2}(q) = H_r(-1; q) = \prod_{n=1}^{\infty} \frac{1}{1 + q^{n^r}}.$$

Changing $q \mapsto -q$ gives

$$\begin{aligned} H_r(-1; -q) = \prod_{n=1}^{\infty} \frac{1}{1 + (-q)^{n^2}} &= \prod_{n=1}^{\infty} \frac{1}{(1 + q^{2^r n^r})(1 - q^{(2n+1)^r})} \\ &= \prod_{n=1}^{\infty} \frac{(1 - q^{2^r n^r})^2}{(1 - q^{2^{r+1} n^r})(1 - q^{n^r})}, \end{aligned}$$

from where, by setting

$$G_r(q) := H_{r,0,2}(-q) - H_{r,1,2}(-q),$$

we get

$$G_r(q) = \prod_{n=1}^{\infty} \frac{(1 - q^{2^r n^r})^2}{(1 - q^{2^{r+1} n^r})(1 - q^{n^r})} = \sum_{n=0}^{\infty} a_r(n) q^n. \tag{3.2.2}$$

In conclusion, what we need to prove now is that the coefficients $a_r(n)$ are positive as $n \to \infty$ and a natural way to verify this would be to compute asymptotics for them, which is precisely what we are going to do.

3.2.2 A result by Meinardus

The reader familiar with asymptotics for infinite product generating functions might recognize at this point the similarity between the infinite product from (3.2.2) and that studied by Meinardus [Mei54]. Writing $q = e^{-\tau}$ with $\mathrm{Re}(\tau) > 0$, the product in question is of the form

$$F(q) = \prod_{n=1}^{\infty} (1 - q^n)^{-a_n} = \sum_{n=0}^{\infty} r(n) q^n,$$

with $a_n \geq 0$ and, under certain assumptions on which we do not elaborate now, Meinardus found asymptotic formulas for the coefficients $r(n)$. More precisely, if the Dirichlet series

$$D(s) = \sum_{n=1}^{\infty} \frac{a_n}{n^s} \quad (s = \sigma + it)$$

converges for $\sigma > \alpha > 0$ and admits a meromorphic continuation to the region $\sigma > -c_0$ $(0 < c_0 < 1)$, region in which $D(s)$ is holomorphic everywhere except for a simple pole at $s = \alpha$ with residue A, then the following holds.

Theorem 3.3 ([And98, Th. 6.2], cf. Meinardus [Mei54, Satz 1])**.** *As $n \to \infty$, we have*

$$r(n) = cn^{\kappa} \exp\left(n^{\frac{\alpha}{\alpha+1}} \left(1 + \frac{1}{\alpha}\right) (A\Gamma(\alpha+1)\zeta(\alpha+1))^{\frac{1}{\alpha+1}} \right) (1 + O(n^{-\kappa_1})),$$

where

$$\begin{aligned} c &= e^{D'(0)} \left(2\pi(\alpha+1)\right)^{-\frac{1}{2}} \left(A\Gamma(\alpha+1)\zeta(\alpha+1)\right)^{\frac{1-2D(0)}{2+2\alpha}}, \\ \kappa &= \frac{2D(0) - 2 - \alpha}{2(\alpha+1)}, \\ \kappa_1 &= \frac{\alpha}{\alpha+1} \min\left\{ \frac{c_0}{\alpha} - \frac{\delta}{4}, \frac{1}{2} - \delta \right\}, \end{aligned}$$

with $\delta > 0$ arbitrary.

Writing $\tau = y - 2\pi i x$, an application of Cauchy's Theorem gives

$$r(n) = \frac{1}{2\pi i} \int_{\mathcal{C}} \frac{F(q)}{q^{n+1}} dq = e^{ny} \int_{-\frac{1}{2}}^{\frac{1}{2}} F(e^{-y+2\pi i x}) e^{-2\pi i n x} dx, \qquad (3.2.3)$$

where $\mathcal{C}$ is the (positively oriented) circle of radius e^{-y} around the origin. Meinardus found the estimate stated in Theorem 3.3 by splitting the integral from (3.2.3) into two integrals evaluated over $|x| \le y^\beta$ and over $y^\beta \le |x| \le \frac{1}{2}$, for a certain choice of β in terms of α, and by showing that the former integral gives the main contribution for the coefficients $r(n)$, while the latter is only an error term.

The positivity condition $a_n \ge 0$ is, however, essential in Meinardus' proof and, as one can readily note, this is not satisfied by all the factors from the product in (3.2.2). For this reason, we need to come up with a certain modification using the *circle method* and Wright's modular transformations [Wri34] for the function $G_r(q)$. This will be used to show that the integral over $y^\beta \le |x| \le \frac{1}{2}$ does not contribute.

On comparing with what was done for the case $r = 2$, the reader might notice that, up to this point, the strategy described here is analogous to that from [Cio20], or equivalently, from Chapter 2 of this thesis. The essential difference is that, in the case $r = 2$, a numerical check ([Cio20, Lem. 5] or, what is the same, Lemma 2.5 from Chapter 2) had to be carried out in order to prove a certain estimate ([Cio20, Lem. 6], or alternatively, Lemma 2.6 from Chapter 2). This numerical check is rather technical and certainly cannot be run for all $r \ge 2$. In the present chapter, we show how to avoid it by using a bound on Gauss sums due to Banks and Shparlinski [BS15] and by modifying an argument from [Cio20]. It is precisely this step that allows for a significantly simpler proof and, at the same time, for a generalization of our results to any $r \ge 2$.

3.2.3 Two estimates

We keep the notation introduced in the previous subsection and write $q = e^{-\tau}$, with $\tau = y - 2\pi i x$ and $y > 0$. Recall that

$$G_r(q) = \prod_{n=1}^{\infty} \frac{(1 - q^{2^r n^r})^2}{(1 - q^{2^{r+1} n^r})(1 - q^{n^r})}. \tag{3.2.4}$$

Let $s = \sigma + it$ and

$$\begin{aligned} D_r(s) &= \sum_{n=1}^{\infty} \frac{1}{n^{rs}} + \sum_{n=1}^{\infty} \frac{1}{(2^{r+1} n^r)^s} - 2 \sum_{n=1}^{\infty} \frac{1}{(2^r n^r)^s} \\ &= (1 + 2^{-s(r+1)} - 2^{1-sr})\zeta(rs), \end{aligned}$$

which is convergent for $\sigma > \frac{1}{r} = \alpha$, has a meromorphic continuation to $\mathbb{C}$ (thus we may choose $0 < c_0 < 1$ arbitrarily) and a simple pole at $s = \frac{1}{r}$ with residue $A = \frac{1}{r} \cdot 2^{-\frac{r+1}{r}}$.

If $\mathcal{C}$ is the (positively oriented) circle of radius e^{-y} around the origin, Cauchy's Theorem tells us that

$$a_r(n) = \frac{1}{2\pi i} \int_{\mathcal{C}} \frac{G_r(q)}{q^{n+1}} dq = e^{ny} \int_{-\frac{1}{2}}^{\frac{1}{2}} G_r(e^{-y+2\pi i x}) e^{-2\pi i n x} dx, \quad (3.2.5)$$

for $n > 0$. Set

$$\beta = 1 + \frac{\alpha}{2}\left(1 - \frac{\delta}{2}\right), \quad \text{with } 0 < \delta < \frac{2}{3}, \quad (3.2.6)$$

so that

$$\frac{3r+1}{3r} < \beta < \frac{2r+1}{2r}, \quad (3.2.7)$$

and rewrite

$$a_r(n) = I_r(n) + J_r(n),$$

where

$$I_r(n) := e^{ny} \int_{-y^\beta}^{y^\beta} G_r(e^{-y+2\pi i x}) e^{-2\pi i n x} dx$$

and

$$J_r(n) := e^{ny} \int_{y^\beta \le |x| \le \frac{1}{2}} G_r(e^{-y+2\pi i x}) e^{-2\pi i n x} dx.$$

As already mentioned, the idea is that the main contribution for $a_r(n)$ is given by $I_r(n)$, and we will be able to prove this using standard integration techniques. To show, however, that $J_r(n)$ is an error term will prove to be much more tricky.

3.3 The main term $I_r(n)$

In this section, we prove the following estimate.

Lemma 3.1. *If $|x| \le \frac{1}{2}$ and $|\operatorname{Arg}(\tau)| \le \frac{\pi}{4}$, then*

$$G_r(e^{-\tau}) = 2^{-\frac{r-1}{2}} \exp\left(A\Gamma\left(\frac{1}{r}\right)\zeta\left(1 + \frac{1}{r}\right)\tau^{-\frac{1}{r}} + O(y^{c_0})\right)$$

holds uniformly in x as $y \to 0$, with $0 < c_0 < 1$.

Proof. By taking logarithms in (3.2.4), we obtain

$$\operatorname{Log} G_r(e^{-\tau}) = \sum_{k=1}^{\infty} \frac{1}{k} \sum_{n=1}^{\infty} \left(e^{-kn^r\tau} + e^{-2^{r+1}kn^r\tau} - 2e^{-2^r kn^r\tau} \right).$$

Using the Mellin inversion formula (see, e.g., [Apo90, p. 54]) we get

$$e^{-\tau} = \frac{1}{2\pi i} \int_{\sigma_0 - i\infty}^{\sigma_0 + i\infty} \tau^{-s}\Gamma(s)ds$$

for $\operatorname{Re}(\tau) > 0$ and $\sigma_0 > 0$, thus $2\pi i \operatorname{Log} G_r(e^{-\tau})$ equals

$$\int_{\alpha+1-i\infty}^{\alpha+1+i\infty} \Gamma(s) \sum_{k=1}^{\infty} \frac{1}{k} \sum_{n=1}^{\infty} \left(\frac{1}{(kn^r\tau)^s} + \frac{1}{(2^{r+1}kn^r\tau)^s} - \frac{2}{(2^r kn^r\tau)^s} \right) ds$$

$$= \int_{\frac{r+1}{r}-i\infty}^{\frac{r+1}{r}+i\infty} \Gamma(s) D_r(s) \zeta(s+1) \tau^{-s} ds. \tag{3.3.1}$$

By assumption, we have

$$|\tau^{-s}| = |\tau|^{-\sigma} e^{t\cdot \operatorname{Arg}(\tau)} \leq |\tau|^{-\sigma} e^{\frac{\pi}{4}|t|}.$$

Well-known results (see, e.g., [AAR99, Ch. 1] and [Tit86, Ch. 5]) state that the bounds

$$\begin{aligned} D_r(s) &= O(|t|^{c_1}), \\ \zeta(s+1) &= O(|t|^{c_2}), \\ \Gamma(s) &= O\big(e^{-\frac{\pi|t|}{2}}|t|^{c_3}\big) \end{aligned}$$

hold uniformly in $-c_0 \leq \sigma \leq \frac{r+1}{r} = \alpha + 1$ as $|t| \to \infty$, for some c_1, c_2 and $c_3 > 0$. We may thus shift the path of integration from $\sigma = \alpha + 1$ to $\sigma = -c_0$. A quick computation gives $D_r(0) = 0$ and $D_r'(0) = -(r-1)\frac{\log 2}{2}$. The integrand in (3.3.1) has poles at $s = \frac{1}{r}$ and $s = 0$, with residues

$$\operatorname{Res}_{s=\frac{1}{2}} A(\Gamma(s)D(s)\zeta(s+1)\tau^{-s}) = \Gamma\left(\frac{1}{r}\right)\zeta\left(1+\frac{1}{r}\right)\tau^{-\frac{1}{r}},$$

and

$$\operatorname{Res}_{s=0} A(\Gamma(s)D(s)\zeta(s+1)\tau^{-s})$$

$$= \operatorname{Res}_{s=0}\left(\left(\frac{1}{s}+O(1)\right)(D'(0)s+O(s^2))\left(\frac{1}{s}+O(1)\right)(1+O(s))\right)$$
$$= -\frac{(r-1)\log 2}{2},$$

whereas the remaining integral equals

$$\begin{aligned}\frac{1}{2\pi i}\int_{-c_0-i\infty}^{-c_0+i\infty}\tau^{-s}\Gamma(s)D(s)\zeta(s+1)ds &\ll |\tau|^{c_0}\int_0^\infty t^{c_1+c_2+c_3}e^{-\frac{\pi t}{4}}dt\\ &\ll |\tau|^{c_0} = |y-2\pi i x|^{c_0}\\ &\le (\sqrt{2}y)^{c_0}\end{aligned}$$

since, again by the assumption, we know that

$$\frac{2\pi|x|}{y} = \tan(|\operatorname{Arg}(\tau)|) \le \tan\left(\frac{\pi}{4}\right) = 1.$$

In conclusion, integration along the shifted contour gives

$$\operatorname{Log} G_r(e^{-\tau}) = \left(A\Gamma\left(\frac{1}{r}\right)\zeta\left(1+\frac{1}{r}\right)\tau^{-\frac{1}{r}} - \frac{(r-1)\log 2}{2}\right) + O(y^{c_0}).$$

□

Commentary. This part is a straightforward generalization of [Cio20, Lem. 1] (or, what is the same, Lemma 2.1 from Chapter 2). We thought it best for the reader to keep the reasoning here as close as possible to that presented in [Cio20, §3.2]. On replacing $r = 2$, the proof of [Cio20, Lem. 1] can be easily traced back.

3.4 The error term $J_r(n)$

This section is dedicated to proving that $J_r(n)$ does not contribute to the coefficients $a_r(n)$. More precisely, we prove the following estimate.

Lemma 3.2. *There exists $\varepsilon > 0$ such that, as $y \to 0$,*

$$G_r(e^{-\tau}) = O\left(\exp\left(A\Gamma\left(\frac{1}{r}\right)\zeta\left(1+\frac{1}{r}\right)y^{-\frac{1}{r}} - cy^{-\varepsilon}\right)\right) \tag{3.4.1}$$

holds uniformly in x with $y^\beta \le |x| \le \frac{1}{2}$, for some $c > 0$.

The proof is slightly more involved and will come in several steps. We start by describing the setup needed to apply the circle method.

3.4.1 Circle method

Inspired by Wright [Wri34], we consider the Farey dissection of order $\lfloor y^{-\frac{r}{r+1}} \rfloor$ of the circle $\mathcal{C}$ over which we integrate in (3.2.5). We further distinguish two kinds of arcs:

(i) major arcs, denoted $\mathfrak{M}_{a,b}$, such that $b \leq y^{-\frac{1}{r+1}}$;

(ii) minor arcs, denoted $\mathfrak{m}_{a,b}$, such that $y^{-\frac{1}{r+1}} < b \leq y^{-\frac{r}{r+1}}$.

In what follows, we express any $\tau \in \mathfrak{M}_{a,b} \cup \mathfrak{m}_{a,b}$ in the form

$$\tau = y - 2\pi i x = \tau' - 2\pi i \frac{a}{b}, \tag{3.4.2}$$

with $\tau' = y - 2\pi i x'$. From basics of Farey theory it follows that

$$|x'| \leq \frac{y^{\frac{r}{r+1}}}{b}. \tag{3.4.3}$$

For a neat introduction to Farey fractions and the circle method, the reader is referred to [Apo90, Ch. 5.4].

3.4.2 Modular transformations

Recalling the definition of $H_r(q)$, we can rewrite (3.2.4) as

$$G_r(q) = \frac{H_r(q) H_r(q^{2^{r+1}})}{H_r(q^{2^r})^2}. \tag{3.4.4}$$

In order to obtain more information about $G_r(q)$, we would next like to use Wright's transformation law [Wri34, Th. 4] for the generating function $H_r(q)$ of partitions into rth powers.

Before doing so, we need to introduce a bit of notation. In what follows, $0 \leq a < b$ are assumed to be coprime positive integers, with b_1 the least positive integer such that $b \mid b_1^2$ and $b = b_1 b_2$. First, set

$$j = j(r) = 0, \qquad \omega_{a,b} = 1$$

if r is even, and

$$j = j(r) = \frac{(-1)^{\frac{1}{2}(r+1)}}{(2\pi)^{r+1}} \Gamma(r+1)\zeta(r+1),$$

and

$$\omega_{a,b} = \exp\left(\pi\left(\frac{1}{b^2}\sum_{h=1}^{b} hd_h - \frac{1}{4}(b-b_2)\right)\right)$$

if r is odd, where $0 \le d_h < b$ is defined by the congruence

$$ah^2 \equiv d_h \pmod{b}$$

and

$$\mu_{h,s} = \begin{cases} \frac{d_h}{b} & \text{if } s \text{ is odd,} \\ \frac{b-d_h}{b} & \text{if } s \text{ is even,} \end{cases}$$

for $d_h \neq 0$. If $d_h = 0$, we set $\mu_{h,s} = 1$. Further, let

$$S_r(a,b) = \sum_{n=1}^{b} \exp\left(\frac{2\pi i a n^r}{b}\right) \tag{3.4.5}$$

be the so-called *Gauss sums* (of order r), and

$$\Lambda_{a,b} = \frac{\Gamma\left(1+\frac{1}{r}\right)}{b}\sum_{m=1}^{\infty}\frac{S_r(ma,b)}{m^{1+\frac{1}{r}}}. \tag{3.4.6}$$

Finally, put

$$C_{a,b} = \left(\frac{b_1}{2\pi}\right)^{\frac{r}{2}}\omega_{a,b},$$

and

$$P_{a,b}(\tau') = \prod_{h=1}^{b}\prod_{s=1}^{r}\prod_{\ell=0}^{\infty}\left(1-g(h,\ell,s)\right)^{-1},$$

with

$$g(h,\ell,s) = \exp\left(\frac{(2\pi)^{\frac{r+1}{r}}(\ell+\mu_{h,s})^{\frac{1}{r}}e^{\frac{\pi i}{2r}(2s+r+1)}}{b\sqrt[r]{\tau'}} - \frac{2\pi i h}{b}\right).$$

Having introduced all the required definitions, we can now state Wright's modular transformation [Wri34, Th. 4], which says, in our notation, that

$$H_r(q) = H_r\left(e^{\frac{2\pi i a}{b}-\tau'}\right) = C_{a,b}\sqrt{\tau'}e^{j\tau'}\exp\left(\frac{\Lambda_{a,b}}{\sqrt[r]{\tau'}}\right)P_{a,b}(\tau'). \tag{3.4.7}$$

On combining (3.4.4) and (3.4.7) we obtain, for some positive constant C that can be made explicit if necessary,

$$G_r(q) = Ce^{j\tau'} \exp\left(\frac{\lambda_{a,b}}{\sqrt[r]{\tau'}}\right) \frac{P_{a,b}(\tau')P'_{a,b}(2^{r+1}\tau')}{P''_{a,b}(2^r\tau')^2}, \tag{3.4.8}$$

where

$$P'_{a,b} = P_{\frac{2^{r+1}a}{(b,2^{r+1})}, \frac{b}{(b,2^{r+1})}}, \quad P''_{a,b} = P_{\frac{2^r a}{(b,2^r)}, \frac{b}{(b,2^r)}}$$

and

$$\lambda_{a,b} = \Lambda_{a,b} + 2^{-\frac{r+1}{r}} \Lambda_{\frac{2^{r+1}a}{(2^{r+1},b)}, \frac{b}{(2^{r+1},b)}} - \Lambda_{\frac{2^r a}{(2^r,b)}, \frac{b}{(2^r,b)}}. \tag{3.4.9}$$

3.4.3 A bound on Gauss sums

As we shall soon see, a crucial step in our proof is finding an upper bound for $\mathrm{Re}(\lambda_{a,b})$ or, what is equivalent, a bound for $|\lambda_{a,b}|$. This is given by the following sharp estimate found by Banks and Shparlinski [BS15] for the Gauss sums defined in (3.4.5).

Theorem 3.4 ([BS15, Th. 1]). *For any coprime positive integers a, b with $b \geq 2$ and any $r \geq 2$, we have*

$$|S_r(a,b)| \leq \mathcal{A} b^{1-\frac{1}{r}}, \tag{3.4.10}$$

where $\mathcal{A} = 4.709236\ldots$.

The constant $\mathcal{A}$ is known as *Stechkin's constant.* Stechkin [Ste75] conjectured in 1975 that the quantity

$$\mathcal{A} = \sup_{b,n\geq 2} \max_{(a,b)=1} \frac{|S_r(a,b)|}{b^{1-\frac{1}{r}}}$$

is finite, this being proven in 1991 by Shparlinski [Shp91]. In the absence of any effective bounds on the sums $S_r(a,b)$, the precise value of $\mathcal{A}$ remained a mystery until 2015 when, using the work of Cochrane and Pinner [CP11] on Gauss sums with prime moduli and that of Cohn and Elkies [CE03] on lower bounds for the center density in the sphere packing problem, Banks and Shparlinski [BS15] were finally able to determine it.

Coming back to our problem, we can now prove the following estimate.

Lemma 3.3. *If* $0 \le a < b$ *are coprime integers with* $b \ge 2$, *we have*

$$|\lambda_{a,b}| < 3\mathcal{A} \cdot \Gamma\left(1+\frac{1}{r}\right)\zeta\left(1+\frac{1}{r}\right) b^{-\frac{1}{r}} \sum_{d|b} d^{-\frac{1}{r}},$$

where $\mathcal{A}$ *is Stechkin's constant.*

Proof. Let us first give a bound for $|\Lambda_{a,b}|$. If we recall (3.4.6) and write $\Lambda_{a,b} = \Gamma\left(1+\frac{1}{r}\right)\Lambda^*_{a,b}$, we have, on using the fact that $S_r(ma,b) = dS_r\left(\frac{ma}{d}, \frac{b}{d}\right)$ to prove the second equality below, and on replacing $m \mapsto md$ and $d \mapsto \frac{b}{d}$ to prove the third and fourth respectively,

$$\begin{aligned}
\Lambda^*_{a,b} &= \frac{1}{b}\sum_{m=1}^{\infty} \frac{S_r(ma,b)}{m^{1+\frac{1}{r}}} = \frac{1}{b}\sum_{d|b} \sum_{\substack{m\ge 1\\(m,b)=d}} \frac{dS_r\left(\frac{ma}{d}, \frac{b}{d}\right)}{m^{1+\frac{1}{r}}} \\
&= \frac{1}{b}\sum_{d|b} d \sum_{\substack{m\ge 1\\(m,b/d)=1}} \frac{S_r(ma, \frac{b}{d})}{(md)^{1+\frac{1}{r}}} \\
&= \frac{1}{b}\sum_{d|b} d^{-\frac{1}{r}} \sum_{\substack{m\ge 1\\(m,b/d)=1}} \frac{S_r(ma, \frac{b}{d})}{m^{1+\frac{1}{r}}} \\
&= \frac{1}{b}\sum_{d|b} \left(\frac{b}{d}\right)^{-\frac{1}{r}} \sum_{\substack{m\ge 1\\(m,d)=1}} \frac{S_r(ma, d)}{m^{1+\frac{1}{r}}} \\
&= \frac{1}{b^{1+\frac{1}{r}}}\sum_{d|b} d^{\frac{1}{r}} \sum_{\substack{m\ge 1\\(m,d)=1}} \frac{S_r(ma, d)}{m^{1+\frac{1}{r}}}.
\end{aligned}$$

Invoking (3.4.10), we obtain

$$\begin{aligned}
|\Lambda_{a,b}| &\le \frac{\Gamma\left(1+\frac{1}{r}\right)}{b^{1+\frac{1}{r}}}\sum_{d|b} d^{\frac{1}{r}} \sum_{\substack{m\ge 1\\(m,d)=1}} \frac{|S_r(ma,d)|}{m^{1+\frac{1}{r}}} \\
&\le \frac{\mathcal{A}\Gamma\left(1+\frac{1}{r}\right)\zeta\left(1+\frac{1}{r}\right)}{b^{\frac{1}{r}}} \sum_{d|b} \frac{1}{d^r}.
\end{aligned}$$

The claim follows easily on applying this bound to the expression for $\lambda_{a,b}$ from (3.4.9). $\square$

3.4.4 Final estimates

We are now getting closer to our purpose and we only need a few last steps before giving the proof of Lemma 3.2. Let us begin by estimating the factors of the form $P_{a,b}$ appearing in (3.4.8).

Lemma 3.4. *If* $\tau \in \mathfrak{M}_{a,b} \cup \mathfrak{m}_{a,b}$, *then*

$$\log|P_{a,b}(\tau')| \ll b \quad \textit{as } y \to 0.$$

Proof. Using (3.4.3) and letting $y \to 0$, we have

$$|\tau'|^{1+\frac{1}{r}} = (y^2+4\pi^2 x'^2)^{\frac{r+1}{2r}} \leq \left(y^2 + \frac{4\pi^2 y^{\frac{2r}{r+1}}}{b^2}\right)^{\frac{r+1}{2r}} \leq \frac{c_4 y}{b^{\frac{r+1}{r}}} = \frac{c_4 \mathrm{Re}\,(\tau')}{b^{\frac{r+1}{r}}},$$

for some $c_4 > 0$. Thus, [Wri34, Lem. 4] gives

$$|g(h,\ell,s)| \leq e^{-c_5(\ell+1)^{\frac{1}{r}}},$$

with $c_5 = \frac{4\sqrt[r]{2\pi}}{rc_4}$, which in turn leads to

$$\begin{aligned}|\log|P_{a,b}(\tau')|| &\leq \sum_{h=1}^{b}\sum_{s=1}^{r}\sum_{\ell=1}^{\infty}|\log(1-g(h,\ell,s))| \\ &\leq rb\sum_{\ell=1}^{\infty}\left|\log\left(1-e^{-c_5(\ell+1)^{\frac{1}{r}}}\right)\right| \ll b,\end{aligned}$$

concluding the proof. □

The next result gives a bound for $G_r(q)$ on the minor arcs. As it is an immediate consequence of replacing $a = \frac{1}{r}$, $b = \frac{1}{r+1}$, $c = 2^{r-1}$, $\gamma = \varepsilon$ and $N = y^{-1}$ in [Wri34, Lem. 17], we omit its proof.

Lemma 3.5. *If* $\varepsilon > 0$ *and* $\tau \in \mathfrak{m}_{a,b}$, *then*

$$|\operatorname{Log} G(q)| \ll_\varepsilon y^{\frac{2^{r-1}r-r-1}{r(r+1)}-\varepsilon}.$$

Remark 3.1. Note that $2^{r-1}r > r+1$ for any $r \geq 2$, therefore the exponent of y in Lemma 3.5 is positive for a small enough choice of $\varepsilon > 0$.

At last, we need the following estimate, a modified version of [Cio20, Lem. 6].

Lemma 3.6. *If $0 \leq a < b$ are coprime integers with $b \geq 2$ and $x \notin \mathbb{Q}$, we have as $y \to 0$, for some $c > 0$,*

$$\operatorname{Re}\left(\frac{\lambda_{a,b}}{\sqrt[r]{\tau'}}\right) \leq \frac{\lambda_{0,1} - c}{\sqrt[r]{y}}. \tag{3.4.11}$$

Proof. Note that

$$\lambda_{0,1} = 2^{-\frac{r+1}{r}}\Lambda_{0,1} = \frac{1}{2^{1+\frac{1}{r}}}\Gamma\left(1+\frac{1}{r}\right)\zeta\left(1+\frac{1}{r}\right) = A\Gamma\left(\frac{1}{r}\right)\zeta\left(1+\frac{1}{r}\right).$$

Writing $\tau' = y + ity$ for some $t \in \mathbb{R}$, we have

$$\begin{aligned}\operatorname{Re}\left(\frac{\lambda_{a,b}}{\sqrt[r]{\tau'}}\right) &= \frac{1}{\sqrt[r]{y}}\operatorname{Re}\left(\frac{\lambda_{a,b}}{\sqrt[r]{1+it}}\right) = \frac{1}{\sqrt[r]{y}}\operatorname{Re}\left(\frac{\lambda_{a,b}}{\sqrt[2r]{1+t^2}e^{\frac{i}{r}\arctan t}}\right)\\ &= \frac{1}{\sqrt[r]{y}\,\sqrt[2r]{1+t^2}}\left(\cos\left(\frac{\arctan t}{r}\right)\operatorname{Re}\left(\lambda_{a,b}\right)\right.\\ &\quad\left.+\sin\left(\frac{\arctan t}{r}\right)\operatorname{Im}\left(\lambda_{a,b}\right)\right).\end{aligned}$$

If we denote by $f_r(t)$ the function given by

$$\frac{1}{\sqrt[2r]{1+t^2}}\left(\cos\left(\frac{\arctan t}{r}\right)\operatorname{Re}\left(\lambda_{a,b}\right)+\sin\left(\frac{\arctan t}{r}\right)\operatorname{Im}\left(\lambda_{a,b}\right)\right),$$

we clearly have $f_r(t) \to 0$ as $|t| \to \infty$. Note now that the choice of x is independent from that of y, and recall from (3.4.2) that $\tau' = y - 2\pi i x'$, with $x' = x - \frac{a}{b}$, hence $t = -\frac{x'}{2\pi y}$. The assumption $x \notin \mathbb{Q}$ implies $x' \neq 0$, and so $|t| \to \infty$ as $y \to 0$. Consequently, we have $f_r(t) \to 0$ as $y \to 0$. In combination with Lemma 3.3 and the well-known fact that $\sigma_0(n) = o(n^\epsilon)$ for any $\epsilon > 0$ (for a proof see, e.g., [Apo76, p. 296]), where $\sigma_0(n)$ denotes the number of divisors of n, this completes the proof. $\square$

3.4.5 Proof of the second estimate

We are now equipped with all the machinery needed for Lemma 3.2.

Proof of Lemma 3.2. If $\tau \in \mathfrak{m}_{a,b}$, then it suffices to apply Lemma 3.5 (because, as $y \to 0$, a negative power of y will dominate any positive power of y; in particular, also the term jy coming from the factor $e^{j\tau'}$ in the case when r is odd), so let us assume that $\tau \in \mathfrak{M}_{a,b}$.

We start by considering the behavior near 0, which corresponds to $a = 0$, $b = 1$, $\tau = \tau' = y - 2\pi i x$. We have, by writing $y^\beta = y^{\frac{2r+1}{2r} - \varepsilon}$ with $\varepsilon > 0$ (here we use the second inequality from (3.2.7)) and setting $b = 1$ in (3.4.3),

$$y^{\frac{2r+1}{2r} - \varepsilon} \le |x| = |x'| \le y^{\frac{r}{r+1}}. \tag{3.4.12}$$

By (3.4.8) we get

$$G_r(q) = Ce^{j\tau} \exp\left(\frac{\lambda_{0,1}}{\sqrt[r]{\tau}}\right) \frac{P_{0,1}(\tau) P_{0,1}(2^{r+1}\tau)}{P_{0,1}(2^r \tau)^2}$$

for some $C > 0$. Thus, by Lemma 3.4 we obtain

$$\log |G_r(q)| = \frac{\lambda_{0,1}}{\sqrt[r]{|\tau|}} + jy + O(1).$$

Using (3.4.12) to prove the first inequality below and expanding into Taylor series to prove the second, we have, on letting $y \to 0$,

$$\begin{aligned} \frac{1}{\sqrt[r]{|\tau|}} = \frac{1}{\sqrt[r]{y}} \frac{1}{\left(1 + \frac{4\pi^2 x^2}{y^2}\right)^{\frac{1}{2r}}} &\le \frac{1}{\sqrt[r]{y}} \frac{1}{\left(1 + 4\pi^2 y^{\frac{1}{r} - 2\varepsilon}\right)^{\frac{1}{2r}}} \\ &\le \frac{1}{\sqrt[r]{y}} \left(1 - c_6 y^{\frac{1}{r} - 2\varepsilon}\right) \end{aligned}$$

for some $c_6 > 0$, which concludes this step of the proof.

To complete the proof, let $\tau \in \mathfrak{M}_{a,b}$, with $2 \le b \le y^{-\frac{1}{r+1}}$. We distinguish two cases. First, let us deal with the case when $x \notin \mathbb{Q}$. By (3.4.8) and Lemma 3.4 we obtain

$$\log |G_r(q)| = \operatorname{Re}\left(\frac{\lambda_{a,b}}{\sqrt[r]{\tau'}}\right) + jy + O\left(y^{-\frac{1}{r+1}}\right) = \operatorname{Re}\left(\frac{\lambda_{a,b}}{\sqrt[r]{\tau'}}\right) + O\left(y^{-\frac{1}{r+1}}\right) \tag{3.4.13}$$

as $y \to 0$. Since by Lemma 3.6 there exists $c_7 > 0$ such that

$$\operatorname{Re}\left(\frac{\lambda_{a,b}}{\sqrt[r]{\tau'}}\right) \le \frac{\lambda_{0,1} - c_7}{\sqrt[r]{y}}, \tag{3.4.14}$$

we infer from (3.4.14) that, as $y \to 0$, we have

$$\log |G_r(q)| \leq \frac{\lambda_{0,1} - c_8}{\sqrt[r]{y}}$$

for some $c_8 > 0$ and the proof is concluded under the assumption that $x \notin \mathbb{Q}$.

Finally, assume that $x = \frac{a}{b}$, that is, $x' = 0$ and $\tau = y - 2\pi i \frac{a}{b}$. We claim that the estimate (3.4.1) is satisfied with the same implied constant, call it C_1. Suppose, by sake of contradiction, that this is not the case. Then there exist infinitely small values of $y > 0$ for which

$$|G_r(e^{-\tau})| \geq C_2 \exp\left(\frac{\lambda_{0,1}}{\sqrt[r]{y}} - cy^{-\varepsilon}\right),$$

with $C_2 > C_1$. However, we can pick now $x' \notin \mathbb{Q}$ infinitely small and set $\tau_1 = y - 2\pi i \left(x' + \frac{a}{b}\right)$. For a fixed choice of y, we have $t \to 0$ as $x' \to 0$; thus, by the same calculation done in the proof of Lemma 3.6, we obtain

$$\operatorname{Re}\left(\frac{\lambda_{a,b}}{\sqrt[r]{\tau_1'}}\right) \to \operatorname{Re}\left(\frac{\lambda_{a,b}}{\sqrt[r]{y}}\right) = \operatorname{Re}\left(\frac{\lambda_{a,b}}{\sqrt[r]{\tau'}}\right), \tag{3.4.15}$$

since $f_r(t) \to 1$. On noting that $\operatorname{Re}(\tau_1') = \operatorname{Re}(\tau) = y$, while clearly all factors of the form $|P_{a,b}(k\tau_1')|$ tend to $|P_{a,b}(k\tau')|$ as $x' \to 0$, we obtain a contradiction in the sense that, on one hand, (3.4.13) and (3.4.15) yield

$$|G_r(e^{-\tau_1})| \to |G_r(e^{-\tau})|$$

as $x' \to 0$, whereas on the other, for a sufficiently small choice of $y > 0$, we have

$$|G_r(e^{-\tau})| - |G_r(e^{-\tau_1})| \geq (C_2 - C_1) \exp\left(\frac{\lambda_{0,1}}{\sqrt[r]{y}} - cy^{-\varepsilon}\right),$$

quantity which gets arbitrarily large for infinitely small choices of $y > 0$. □

Commentary. It is in this part where our proof differs substantially from that given in [Cio20] in the case $r = 2$. More precisely, [Cio20, Lem. 5] (Lemma 2.5 from Chapter 2) was needed to prove the inequality (3.4.11) for all values of y, inequality which was then used in the estimates made in the proof of [Cio20, Lem. 2] (or alternatively, Lemma 2.2 from Chapter 2), the equivalent of the current Lemma 3.2. However, we are only interested in establishing the estimates from Lemma 3.2 on letting $y \to 0$, which is why we only need the bound (3.4.11) to hold as $y \to 0$. The argument presented in Lemma 3.6 further tells us that, in order for this to happen, the estimate (3.4.10), obtained using the bound on Gauss sums found by Banks and Shparlinski [BS15], is enough. As a consequence, we can avoid the rather involved numerical check done in [Cio20, Lem. 5], a check which we would, in fact, not even be able to implement for *all* values $r \geq 2$. In particular, the present argument gives a simplified proof of the results from [Cio20].

3.5 Proof of the main theorem

In this section we give the proof of Theorem 3.2. Having already proven the two estimates from Lemma 3.1 and Lemma 3.2, the rest is only a matter of careful computations. The reader is reminded that, because of the reformulation given in Chapter 3.2.1, what we are interested in is computing asymptotics for the coefficients

$$a_r(n) = e^{ny} \int_{-\frac{1}{2}}^{\frac{1}{2}} G_r(e^{-y+2\pi i x}) e^{-2\pi i n x} dx. \tag{3.5.1}$$

3.5.1 Saddle-point method

Recall that, as defined in Chapter 3.2.3, we denote $\alpha = \frac{1}{r}$ and $A = \frac{1}{r} \cdot 2^{-\frac{r+1}{r}}$, notation which we keep, for simplicity, in what follows. Before proceeding any further, we make a particular choice for y as a function of n. More precisely, let

$$\begin{aligned} y &= n^{-\frac{1}{\alpha+1}} \left(A\Gamma(\alpha+1)\zeta(\alpha+1)\right)^{\frac{1}{\alpha+1}} \\ &= n^{-\frac{r}{r+1}} \left(A\Gamma\left(\frac{1}{r}\right)\zeta\left(1+\frac{1}{r}\right)\right)^{\frac{r}{r+1}}, \end{aligned} \tag{3.5.2}$$

and write $m = ny$.

The reason for this choice of y is motivated by the *saddle-point* method and becomes clear once the reader recognizes in (3.5.2) the quantity appearing in Lemmas 3.1 and 3.2. As the maximum absolute value of the integrand from (3.5.1) occurs for $x = 0$, around which point Lemma 3.1 tells us that the integrand is well approximated by

$$\exp(A\Gamma(\alpha)\zeta(\alpha+1)y^{-\alpha} + ny),$$

the saddle-point method suggests maximizing this expression, that is, finding the value of y for which

$$\frac{d}{dy}(\exp(A\Gamma(\alpha)\zeta(\alpha+1)y^{-\alpha} + ny)) = 0.$$

3.5.2 Proof of the main result

We have now all ingredients necessary to conclude the proof of Theorem 3.2. The proof merely consists of a skillful computation, which can be carried out in two ways. Since Lemma 3.1 and Lemma 3.2 are completely analogous to the two estimates found by Meinardus (combined in the "Hilfssatz" from [Mei54, p. 390]), one way is to follow his approach and carry out the same computations done in [Mei54, pp. 392–394]. The second way is slightly more explicit and is based entirely on the computation done in the proof of the case $r = 2$ from [Cio20, pp. 139–141]. For sake of completeness and for comparison with that computation, we sketch in what follows the main steps of the argument, leaving some details and technicalities as an exercise for the interested reader.

Proof of Theorem 3.2. As already explained, the interesting part is to prove the inequalities from (3.1.2), so let us begin by doing so. By Lemma 3.2 and (3.5.2) we have, as $n \to 0$,

$$\begin{aligned}
J_r(n) &= e^{ny} \int_{y^\beta \le |x| \le \frac{1}{2}} G(e^{-y+2\pi i x}) e^{-2\pi i n x} dx \\
&= e^{ny} \int_{y^\beta \le |x| \le \frac{1}{2}} O\Big(e^{y^{-\alpha} A\Gamma(\alpha)\zeta(\alpha+1) - cy^{-\varepsilon}}\Big) dx \\
&= e^{ny} \cdot O\left(e^{y^{-\alpha} A\Gamma(\alpha)\zeta(\alpha+1) - cy^{-\varepsilon}}\right) \\
&= O\Big(e^{n^{\frac{\alpha}{\alpha+1}}\left(1+\frac{1}{\alpha}\right)(A\Gamma(\alpha+1)\zeta(\alpha+1))^{\frac{1}{\alpha+1}} - C_1 n^{\varepsilon_1}}\Big),
\end{aligned}$$

with $\varepsilon_1 = \frac{r\varepsilon}{r+1} > 0$ and some $C_1 > 0$.

We now compute the main asymptotic contribution, which will be given by $I_r(n)$. Let $n \geq n_1$ be large enough so that $y^{\beta-1} \leq \frac{1}{2\pi}$. This choice allows us to apply Lemma 3.1, as it ensures $|x| \leq \frac{1}{2}$ and $|\operatorname{Arg}(\tau)| \leq \frac{\pi}{4}$. From Lemma 3.1 we obtain

$$I_r(n) = \frac{e^{ny}}{2^{\frac{r-1}{2}}} \int_{-y^\beta}^{y^\beta} e^{A\Gamma(\alpha)\zeta(\alpha+1)\tau^{-\alpha}+O(y^{c_0})-2\pi inx} dx. \tag{3.5.3}$$

Writing

$$\tau^{-\alpha} = \frac{1}{\sqrt[r]{\tau}} = \frac{1}{\sqrt[r]{y}} + \left(\frac{1}{\sqrt[r]{\tau}} - \frac{1}{\sqrt[r]{y}}\right),$$

we can further express the value of $I_r(n)$ from (3.5.3) as

$$\begin{aligned} &\frac{e^{ny}}{2^{\frac{r-1}{2}}} \int_{-y^\beta}^{y^\beta} e^{A\Gamma(\alpha)\zeta(\alpha+1)\frac{1}{\sqrt[r]{y}}} e^{A\Gamma(\alpha)\zeta(\alpha+1)\left(\frac{1}{\sqrt[r]{\tau}}-\frac{1}{\sqrt[r]{y}}\right)} e^{-2\pi inx+O(y^{c_0})} dx \\ &= C_2 \int_{-y^\beta}^{y^\beta} e^{\frac{A\Gamma(\alpha)\zeta(\alpha+1)}{\sqrt[r]{y}}\left(\frac{1}{\sqrt[r]{1-\frac{2\pi ix}{y}}}-1\right)} e^{-2\pi inx+O(y^{c_0})} dx, \end{aligned}$$

where

$$C_2 = 2^{\frac{1-r}{2}} e^{\left(1+\frac{1}{\alpha}\right)n^{\frac{\alpha}{\alpha+1}}(A\Gamma(\alpha+1)\zeta(\alpha+1))^{\frac{1}{\alpha+1}}}.$$

With $u = -\frac{2\pi x}{y}$, we obtain

$$I_r(n) = C_3 \int_{-2\pi y^{\beta-1}}^{2\pi y^{\beta-1}} e^{\frac{A\Gamma(\alpha)\zeta(\alpha+1)}{\sqrt[r]{y}}\left(\frac{1}{\sqrt[r]{1+iu}}-1\right)+inuy+O(y^{c_0})} dx, \tag{3.5.4}$$

where

$$C_3 = 2^{\frac{1-r}{2}} \frac{ye^{\left(1+\frac{1}{\alpha}\right)n^{\frac{\alpha}{\alpha+1}}(A\Gamma(\alpha+1)\zeta(\alpha+1))^{\frac{1}{\alpha+1}}}}{2\pi}.$$

Set, for simplicity, $B = A\Gamma(\alpha)\zeta(\alpha+1)$. We have the Taylor series expansion

$$\frac{1}{\sqrt[r]{1+iu}} = 1 - \frac{iu}{r} - \frac{(r+1)u^2}{2r^2} + O(|u|^3),$$

from where, on recalling that $|u| \leq 2\pi y^{\beta-1}$ and using (3.5.2) to compute $B = rny^{1+\frac{1}{r}}$, it follows that

$$B\frac{1}{\sqrt[r]{y}}\left(\frac{1}{\sqrt[r]{1+iu}} - 1\right) + inuy = -\frac{Biu}{r\sqrt[r]{y}} + inuy - \frac{(r+1)Bu^2}{2r^2\sqrt[r]{y}} + O\left(\frac{|u|^3}{\sqrt[r]{y}}\right)$$

$$= \frac{(r+1)Bu^2}{2r^2\sqrt[r]{y}} + O\left(n^{\frac{1}{r+1}\left(1+\frac{3(1-\beta)}{\alpha}\right)}\right).$$

For an appropriate constant C_4, we may then change the integral from the right-hand side of (3.5.4) into

$$\begin{aligned}
&\int_{|u|\le 2\pi y^{\beta-1}} e^{B\frac{1}{\sqrt[r]{y}}\left(\frac{1}{\sqrt[r]{1+iu}}-1\right)+inuy+O(y^{c_0})}du \\
&= \int_{|u|\le C_4} e^{-\frac{(r+1)Bu^2}{2r^2\sqrt[r]{y}}} e^{O\left(y^{c_0}+\frac{|u|^3}{\sqrt[r]{y}}\right)}du \\
&= \int_{|u|\le C_4} e^{-\frac{(r+1)Bu^2}{2r^2\sqrt[r]{y}}} e^{O\left(n^{-\frac{rc_0}{r+1}}+n^{\frac{1+3r(1-\beta)}{r+1}}\right)}du \\
&= \int_{|u|\le C_4} e^{-\frac{(r+1)Bu^2}{2r^2\sqrt[r]{y}}} \left(1+\left(e^{O\left(n^{-\frac{rc_0}{r+1}}+n^{\frac{1+3r(1-\beta)}{r+1}}\right)}-1\right)\right)du.
\end{aligned}$$

From the first inequality in (3.2.7), we see that $1+3r(1-\beta)<0$, and thus

$$e^{O\left(n^{-\frac{rc_0}{r+1}}+n^{\frac{1+3r(1-\beta)}{r+1}}\right)}-1 = e^{O\left(n^{-\frac{rc_0}{r+1}}+n^{-\frac{1}{6}+\frac{\delta}{4}}\right)}-1 = O(n^{-\kappa}),$$

where $\kappa = \frac{1}{r+1}\min\left\{rc_0, \frac{1}{2}-\frac{3\delta}{4}\right\}$. We further get, on using (3.2.6) when changing the limits of integration,

$$\begin{aligned}
&\int_{|u|\le 2\pi y^{\beta-1}} e^{B\frac{1}{\sqrt{y}}\left(\frac{1}{\sqrt{1+iu}}-1\right)+inuy+O(y^{c_0})}du \\
&= \int_{|u|\le C_4} e^{-\frac{(r+1)Bu^2}{2r^2\sqrt[r]{y}}}(1+O(n^{-\kappa}))du \\
&= c(n)\int_{|v|\le C_5 n^{\frac{\delta}{4(r+1)}}} e^{-v^2}(1+O(n^{-\kappa}))dv, \qquad (3.5.5)
\end{aligned}$$

where $c(n) = \sqrt{\frac{2r}{r+1}}(\alpha Bn^{\alpha})^{-\frac{1}{2(\alpha+1)}}$ and $C_5>0$ is a constant. By letting $n\to\infty$, and turning the integral from (2.4.4) into a Gauss integral, we obtain

$$\int_{|u|\le 2\pi y^{\beta-1}} e^{B\frac{1}{\sqrt[r]{y}}\left(\frac{1}{\sqrt[r]{1+iu}}-1\right)+inuy+O(y^{c_0})}du = c(n)\sqrt{\pi}(1+O(n^{-\kappa_1})), \qquad (3.5.6)$$

where $\kappa_1 = \frac{1}{r+1} \min\left\{rc_0 - \frac{\delta}{4}, \frac{1}{2} - \delta\right\}$. Putting together (3.5.4), (3.5.5) and (3.5.6) we see that, as predicted by Meinardus (see Theorem 3.3), the main asymptotic contribution for our coefficients is given by

$$a_r(n) \sim Cn^{-\frac{\alpha+2}{2(\alpha+1)}} e^{n^{\frac{\alpha}{\alpha+1}}\left(1+\frac{1}{\alpha}\right)(A\Gamma(\alpha+1)\zeta(\alpha+1))^{\frac{1}{\alpha+1}}}, \tag{3.5.7}$$

where

$$C = \frac{1}{\sqrt{2^r(\alpha+1)\pi}} (A\Gamma(\alpha+1)\zeta(\alpha+1))^{\frac{1}{2(\alpha+1)}}.$$

This shows that the inequalities in (3.1.2) are true for $n \to \infty$. The proof can be completed either by adding the estimate (3.5.7) for $a_r(n) = (-1)^n(p_r(0,2,n) - p_r(1,2,n))$ and that obtained by Wright [Wri34, Th. 2] for $p_r(n) = p_r(0,2,n) + p_r(1,2,n)$, or by invoking the recent result of Zhou [Zho, Cor. 1.2]. □

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References

[And98] G. E. Andrews, *The theory of partitions*, Reprint of the 1976 original, Cambridge Mathematical Library. Cambridge University Press, Cambridge, 1998.

[AAR99] G. E. Andrews, R. Askey and R. Roy, *Special functions*, Encyclopedia of Mathematics and its Applications 71. Cambridge University Press, Cambridge, 1999.

[Apo76] T. M. Apostol, *Introduction to Analytic Number Theory*, Undergraduate Texts in Mathematics, Springer-Verlag, New York-Heidelberg, 1976.

[Apo90] T. M. Apostol, *Modular Functions and Dirichlet Series in Number Theory*, Second edition, Graduate Texts in Mathematics 41. Springer-Verlag, New York, 1990.

[BS15] W. Banks and I. E. Shparlinski, On Gauss sums and the evaluation of Stechkin's constant, *Math. Comp.* **85** (2015), no. 301, 2569–2581.

[BM] K. Bringmann and K. Mahlburg, Transformation laws and asymptotics for nonmodular products, unpublished preprint.

[Cio20] A. Ciolan, Asymptotics and inequalities for partitions into squares, *Int. J. Number Theory* **16** (2020), no. 1, 121–143.

[CP11] T. Cochrane and C. Pinner, Explicit bounds on monomial and binomial exponential sums, *Q. J. Math.* **62** (2011), no. 2, 323–349.

[CE03] H. Cohn and N. Elkies, New upper bounds on sphere packings. I, *Ann. of Math. (2)* **157** (2003), no. 2, 689–714.

[Gla76] J. W. L. Glaisher, On formulae of verification in the partition of numbers, *Proc. Royal Soc. London* **24** (1876), 250–259.

[Mei54] G. Meinardus, Asymptotische aussagen über Partitionen, *Math. Z.* **59** (1954), 388–398.

[Shp91] I. E. Shparlinskiĭ, Estimates for Gauss sums (Russian), *Mat. Zametki* **50** (1991), no. 1, 122–130; English transl., *Math. Notes* **50** (1991), no. 1–2, 740–746 (1992).

[Ste75] S. B. Stečkin, An estimate for Gaussian sums (Russian), *Mat. Zametki* **17** (1975), no. 4, 579–588.

[Tit86] E. C. Titchmarsh, *The theory of the Riemann zeta-function*, Second edition, The Clarendon Press, Oxford University Press, New York, 1986.

[Wri34] E. M. Wright, Asymptotic partition formulae. III. Partitions into k-th powers, *Acta Math.* **63** (1934), no. 1, 143–191.

[Zho] N. H. Zhou, Note on partitions into polynomials with number of parts in an arithmetic progression, available as preprint at `https://arxiv.org/abs/1909.13549`.

Chapter 4

Overpartition ranks

This chapter is based on the paper [Cio19] published in the *Journal of Mathematical Analysis and Applications*. To avoid too much repetition with the previous chapters, Chapter 4.1.1 is slightly shortened and rearranged. Remark 4.3 is new and was omitted from [Cio19].

4.1 Introduction

4.1.1 First definitions

The reader should be by now well-acquainted with the concept of partitions of a positive integer n, the number of which we denote by $p(n)$. An *overpartition* of n is a partition in which the first occurrence of a part may be overlined. We denote by $\overline{p}(n)$ the number of overpartitions of n. For example, $\overline{p}(4) = 14$, as the overpartitions of 4 are 4, $\overline{4}$, $3+1$, $\overline{3}+1$, $3+\overline{1}$, $\overline{3}+\overline{1}$, $2+2$, $\overline{2}+2$, $2+1+1$, $\overline{2}+1+1$, $2+\overline{1}+1$, $\overline{2}+\overline{1}+1$, $1+1+1+1$, $\overline{1}+1+1+1$. Overpartitions are natural combinatorial structures associated with the q-binomial theorem, Heine's transformation or Lebesgue's identity. For an overview and further motivation, the reader is referred to [CL04] and [Pak06].

As we have already seen in Chapters 1.3 and 2.1, Ramanujan discovered some remarkable congruences for the partition function $p(n)$, saying that, for $n \geq 0$,

$$\begin{aligned} p(5n+4) &\equiv 0 \pmod{5}, \\ p(7n+5) &\equiv 0 \pmod{7}, \\ p(11n+6) &\equiv 0 \pmod{11}. \end{aligned}$$

In order to give a combinatorial proof of these congruences, Dyson [Dys44] introduced the *rank* of a partition, often known also as *Dyson's rank*. This is defined to be the largest part of the partition minus the number of its parts. Dyson conjectured that the partitions of $5n+4$ form 5 groups of equal size when sorted by their ranks modulo 5, and that the same is true for the partitions of $7n+5$ when working modulo 7, conjecture which was proven by Atkin and Swinnerton-Dyer [AS54].

The *rank* of an overpartition, also called *D-rank* (in order to suggest the connection with the Dyson-rank of a partition), is a straightforward extension of the rank of a partition, being defined as the largest part of the overpartition minus the number of its parts. There is another *rank* of an overpartition, which is defined to be one less than the largest part ℓ of the overpartition minus the number of overlined parts less than ℓ. However, we deal here only with the first definition, to which we refer simply as "rank".

Both the partition and overpartition ranks have been extensively studied. By proving that some generating functions associated to the rank are holomorphic parts of harmonic Maass forms, Bringmann and Ono [BO10] showed that the rank partition function satisfies some other congruences of Ramanujan type. In the same spirit, Bringmann and Lovejoy [BL07] proved that the overpartition rank generating function is the holomorphic part of a harmonic Maass form of weight 1/2, while Dewar [Dew10] made certain refinements.

It is customary to denote by $N(m,n)$ the number of partitions of n with rank m and by $N(a,m,n)$ the number of partitions of n with rank congruent to a modulo m. The same quantities for overpartitions, $\overline{N}(m,n)$ and $\overline{N}(a,m,n)$, are denoted by an overline.

4.1.2 Motivation

By means of generalized Lambert series, Lovejoy and Osburn [LO08] gave formulas for the rank differences $\overline{N}(s,\ell,n)-\overline{N}(t,\ell,n)$ for $\ell=3$ and $\ell=5$, while rank differences for $\ell=7$ were determined by Jennings-Shaffer [Jen16]. Recently, by using q-series manipulations and the 3 and 5-dissection of the overpartition rank generation function, Ji, Zhang and Zhao [JZZ18] proved some identities and inequalities for the rank difference generating functions of overpartitions modulo 6 and 10, and conjectured a few others. Some further inequalities were conjectured by Wei and Zhang [WZ20].

It is one goal of this chapter to prove these conjectures. The other, more general goal is to compute asymptotics for the ranks of overpartitions and this is what we will start with, the inequalities mentioned above, as well as the asymptotic equidistribution of $N(a, c, n)$, following then as a consequence. In doing so, we rely on the Hardy-Ramanujan circle method and the modular transformations for overpartitions established by Bringmann and Lovejoy [BL07]. While the main ideas are essentially those used by Bringmann [Bri09] in computing asymptotics for partition ranks, several complications arise and some modifications need to be carried out.

This chapter is structured as follows. The rest of this section is dedicated to introducing some notation that is needed in the sequel and formulating our main results. An outline of the proof of Theorem 4.1 is given in Chapter 4.2, and its proof, along with that of the equidistribution of $N(a, c, n)$, is given in detail in Chapter 4.3. In the final chapter we show how to use Theorem 4.1 in order to prove the inequalities conjectured by Ji, Zhang and Zhao [JZZ18], and Wei and Zhang [WZ20], which are stated in Theorems 4.2–4.4 together with some other inequalities.

4.1.3 Notation and preliminaries

The overpartition generating function (see, e.g., [CL04]) is given by

$$\overline{P}(q) := \sum_{n\geq 0} \overline{p}(n)q^n = \frac{\eta(2z)}{\eta^2(z)} = 1 + 2q + 4q^2 + 8q^3 + 14q^4 + \cdots, \quad (4.1.1)$$

where

$$\eta(z) := q^{\frac{1}{24}} \prod_{n=1}^{\infty} (1 - q^n)$$

denotes, as usual, Dedekind's eta function and $q = e^{2\pi i z}$, with $z \in \mathbb{C}$ and $\mathrm{Im}(z) > 0$. If we use the standard q-series notation

$$(a)_n := \prod_{r=0}^{n-1} (1 - aq^r),$$
$$(a; b)_n := \prod_{r=0}^{n-1} (1 - aq^r)(1 - bq^r),$$

for $a, b \in \mathbb{C}$ and $n \in \mathbb{N} \cup \{\infty\}$, then we know from [Lov05] that

$$\mathcal{O}(u; q) := \sum_{m=-\infty}^{\infty} \sum_{n=0}^{\infty} \overline{N}(m, n)u^m q^n = \sum_{n=0}^{\infty} \frac{(-1)_n q^{\frac{1}{2}n(n+1)}}{(uq; q/u)_n}$$

$$= \frac{(-q)_\infty}{(q)_\infty} \left(1 + 2 \sum_{n \geq 1} \frac{(1-u)(1-u^{-1})(-1)^n q^{n^2+n}}{(1-uq^n)(1-u^{-1}q^n)} \right). \tag{4.1.2}$$

If $0 < a < c$ are coprime positive integers, and if by $\zeta_n = e^{\frac{2\pi i}{n}}$ we denote the primitive nth root of unity, we set

$$\mathcal{O}\left(\frac{a}{c}; q\right) := \mathcal{O}\left(\zeta_c^a; q\right) = 1 + \sum_{n=1}^{\infty} A\left(\frac{a}{c}; n\right) q^n. \tag{4.1.3}$$

Let k be a positive integer. Set $\widetilde{k} = 0$ if k is even, and $\widetilde{k} = 1$ if k is odd. Moreover, put $k_1 = \frac{k}{(c,k)}$, $c_1 = \frac{c}{(c,k)}$, and let the integer $0 \leq \ell < c_1$ be defined by the congruence $\ell \equiv ak_1 \pmod{c_1}$. If $\frac{b}{c} \in (0,1)$, let

$$s(b,c) := \begin{cases} 0 & \text{if } 0 < \frac{b}{c} \leq \frac{1}{4}, \\ 1 & \text{if } \frac{1}{4} < \frac{b}{c} \leq \frac{3}{4}, \\ 2 & \text{if } \frac{3}{4} < \frac{b}{c} < 1, \end{cases} \quad \text{and} \quad t(b,c) := \begin{cases} 1 & \text{if } 0 < \frac{b}{c} < \frac{1}{2}, \\ 3 & \text{if } \frac{1}{2} < \frac{b}{c} < 1. \end{cases}$$

Throughout we will use, for reasons of space, the shorthand notation $s = s(b,c)$ and $t = t(b,c)$. In what follows, $0 \leq h < k$ are coprime integers (in case $k = 1$, we set $h = 0$ and this is the only case when $h = 0$ is allowed), and $h' \in \mathbb{Z}$ is defined by the congruence $hh' \equiv -1 \pmod{k}$. Further, let

$$\omega_{h,k} := \exp\left(\pi i \sum_{\mu=0}^{k-1} \left(\left(\frac{\mu}{k}\right)\right) \left(\left(\frac{h\mu}{k}\right)\right) \right)$$

be the multiplier occurring in the transformation law of the partition function, where

$$((x)) := \begin{cases} x - \lfloor x \rfloor - \frac{1}{2} & \text{if } x \in \mathbb{R} \setminus \mathbb{Z}, \\ 0 & \text{if } x \in \mathbb{Z}. \end{cases}$$

Remark 4.1. The sums

$$S_{h,k} := \sum_{\mu=0}^{k-1} \left(\left(\frac{\mu}{k}\right)\right) \left(\left(\frac{h\mu}{k}\right)\right)$$

are known in the literature as *Dedekind sums.* For a nice discussion of their properties and how to compute them for small values of h, the reader is referred to [Apo90, p. 62].

We next define several Kloosterman sums. Here and throughout we write $\sum_h'$ to denote summation over the integers $0 \leq h < k$ that are coprime to k.

If $c \mid k$, let

$$A_{a,c,k}(n,m) := (-1)^{k_1+1} \tan\left(\frac{\pi a}{c}\right) \sum_h{}' \frac{\omega_{h,k}^2}{\omega_{h,k/2}} \times \cot\left(\frac{\pi a h'}{c}\right) e^{-\frac{2\pi i h' a^2 k_1}{c}} e^{\frac{2\pi i}{k}(nh+mh')},$$

and

$$B_{a,c,k}(n,m) := -\frac{1}{\sqrt{2}} \tan\left(\frac{\pi a}{c}\right) \sum_h{}' \frac{\omega_{h,k}^2}{\omega_{2h,k}} \times \frac{1}{\sin\left(\frac{\pi a h'}{c}\right)} e^{-\frac{2\pi i h' a^2 k_1}{c}} e^{\frac{2\pi i}{k}(nh+mh')}.$$

If $c \nmid k$ and $0 < \frac{\ell}{c_1} \leq \frac{1}{4}$, let

$$D_{a,c,k}(n,m) := \frac{1}{\sqrt{2}} \tan\left(\frac{\pi a}{c}\right) \sum_h{}' \frac{\omega_{h,k}^2}{\omega_{2h,k}} e^{\frac{2\pi i}{k}(nh+mh')},$$

and if $c \nmid k$ and $\frac{3}{4} < \frac{\ell}{c_1} < 1$, let

$$D_{a,c,k}(n,m) := -\frac{1}{\sqrt{2}} \tan\left(\frac{\pi a}{c}\right) \sum_h{}' \frac{\omega_{h,k}^2}{\omega_{2h,k}} e^{\frac{2\pi i}{k}(nh+mh')}.$$

To state our results, we need at last the following quantities. The motivation behind their expressions becomes clear if one writes down explicitly the computations done in Chapter 4.3. If $c \nmid k$, let

$$\delta_{c,k,r} := \begin{cases} \frac{1}{16} - \frac{\ell}{2c_1} + \frac{\ell^2}{c_1^2} - r\frac{\ell}{c_1} & \text{if } 0 < \frac{\ell}{c_1} \leq \frac{1}{4}, \\ 0 & \text{if } \frac{1}{4} < \frac{\ell}{c_1} \leq \frac{3}{4}, \\ \frac{1}{16} - \frac{3\ell}{2c_1} + \frac{\ell^2}{c_1^2} + \frac{1}{2} - r\left(1 - \frac{\ell}{c_1}\right) & \text{if } \frac{3}{4} < \frac{\ell}{c_1} < 1, \end{cases} \tag{4.1.4}$$

and

$$m_{a,c,k,r} := \begin{cases} -\frac{2(ak_1-\ell)^2+c_1(2r+1)(ak_1-\ell)}{2c_1^2} & \text{if } 0 < \frac{\ell}{c_1} \le \frac{1}{4}, \\ 0 & \text{if } \frac{1}{4} < \frac{\ell}{c_1} \le \frac{3}{4}, \\ -\frac{2(ak_1-\ell)^2-c_1(2r-3)(ak_1-\ell)-c_1^2(2r-1)}{2c_1^2} & \text{if } \frac{3}{4} < \frac{\ell}{c_1} < 1. \end{cases} \tag{4.1.5}$$

4.1.4 Statement of results

We are now in shape to state our main results.

Theorem 4.1. *If* $0 < a < c$ *are coprime positive integers with* $c > 2$, *and* $\varepsilon > 0$ *is arbitrary, then*

$$\begin{aligned} A\left(\frac{a}{c}; n\right) &= i\sqrt{\frac{2}{n}} \sum_{\substack{1 \le k \le \sqrt{n} \\ c \mid k,\ 2 \nmid k}} \frac{B_{a,c,k}(-n,0)}{\sqrt{k}} \sinh\left(\frac{\pi\sqrt{n}}{k}\right) \\ &\quad + 2\sqrt{\frac{2}{n}} \sum_{\substack{1 \le k \le \sqrt{n} \\ c \nmid k,\ 2 \nmid k,\ c_1 \ne 4 \\ r \ge 0,\ \delta_{c,k,r} > 0}} \frac{D_{a,c,k}(-n, m_{a,c,k,r})}{\sqrt{k}} \sinh\left(\frac{4\pi\sqrt{\delta_{c,k,r} n}}{k}\right) \\ &\quad + O_c(n^{\varepsilon}). \end{aligned} \tag{4.1.6}$$

Remark 4.2. In computing the sums $B_{a,c,k}$ and $D_{a,c,k}$ from Theorem 4.1, the integer h' is assumed to be even, cf. [BL07, pp. 14–15].

Remark 4.3. The attentive reader might notice that there is one case (and one only) in which formula (4.1.6) gives no main term, as none of the two sums involving $B_{a,c,k}$ and $D_{a,c,k}$ contributes. This happens for $c = 4$ and $a = 1$ (or, what is the same, $a = 3$). In this case we obtain a very interesting result, namely

$$A\left(\frac{a}{c}; n\right) = \begin{cases} 2 & \text{if } n \text{ is a square}, \\ 0 & \text{otherwise}, \end{cases}$$

or[1] equivalently,

$$\mathcal{O}(i; q) = 1 + 2\sum_{n=1}^{\infty} q^{n^2}. \tag{4.1.7}$$

[1] The function on the right-hand side of (4.1.7) is the classical Jacobi theta function $\theta(z)$, a modular form of weight $1/2$; the coefficient of q^n in the kth power of $\theta(z)$ equals the number $r_k(n)$ introduced in Chapter 1.2.5.

Of course, this trivially satisfies the statement of Theorem 4.1. Identity (4.1.7) is an easy exercise for the reader familiar with standard tools[2] from q-series, such as Watson's transformation or Jacobi's triple product (see, e.g., [Lov05, Prop. 2.1 and Th. 5.6]). The result is implicit in the proof of Theorem 5.6 from [Lov05, p. 330].

While the sums involved in the asymptotic formula of $A\left(\frac{a}{c};n\right)$ might look a bit cumbersome at first, for small values of c they can be computed without much effort. We exemplify below the particular instances when $c=3$ and $c=10$; we will come back to Example 4.2, in more detail, in Chapter 4.4.

Example 4.1. If $a=1$ and $c=3$, the second sum in (4.1.6) does not contribute (as $\delta_{3,k,r}=0$), while the main asymptotic contribution from the first sum is given by the term corresponding to $k=3$. If $h=1$, we have $h'=2$ and $\omega_{1,3}=e^{\frac{\pi i}{6}}$, and if $h=3$, we have $h'=-2$ and $\omega_{2,3}=e^{-\frac{\pi i}{6}}$. Without difficulty, we see that $B_{1,3,3}(-n,0)=-2i\sqrt{2}$ if $n\equiv 1 \pmod 3$, and $B_{1,3,3}(-n,0)=i\sqrt{2}$ if $n\equiv 0,2 \pmod 3$, from where

$$A\left(\frac{1}{3};n\right)\sim\begin{cases}\dfrac{4}{\sqrt{3n}}\tan\left(\dfrac{\pi}{3}\right)\sinh\left(\dfrac{\pi\sqrt{n}}{3}\right) & \text{if } n\equiv 1 \pmod 3,\\ -\dfrac{2}{\sqrt{3n}}\tan\left(\dfrac{\pi}{3}\right)\sinh\left(\dfrac{\pi\sqrt{n}}{3}\right) & \text{if } n\equiv 0,2 \pmod 3.\end{cases}$$

Example 4.2. If $a=1$ and $c=10$, the first sum in (4.1.6) does not contribute, while the main asymptotic contribution from the second sum is given by the term corresponding to $k=1$. In this case, $k_1=\ell=1$, $c_1=10$ and the only positive value of $\delta_{10,1,r}$ is attained for $r=0$. As such, we have $\delta_{10,1,0}=\frac{9}{400}$, $m_{1,10,1,0}=0$ and $D_{1,10,1}(-n,0)=\frac{1}{\sqrt{2}}\tan\left(\frac{\pi}{10}\right)$, hence

$$A\left(\frac{1}{10};n\right)\sim\frac{2}{\sqrt{n}}\tan\left(\frac{\pi}{10}\right)\sinh\left(\frac{3\pi\sqrt{n}}{5}\right).$$

On using Theorem 4.1 together with the identity

$$\sum_{n=0}^{\infty}\overline{N}(a,c,n)q^n=\frac{1}{c}\sum_{n=0}^{\infty}\overline{p}(n)q^n+\frac{1}{c}\sum_{j=1}^{c-1}\zeta_c^{-aj}\cdot\mathcal{O}(\zeta_c^j;q), \tag{4.1.8}$$

[2] The reader unfamiliar with these tools can consult the book of Andrews [And98].

which follows by the orthogonality of roots of unity, and the well-known fact (see, e.g., [HR18]) that

$$\overline{p}(n) \sim \frac{1}{8n} e^{\pi\sqrt{n}}$$

as $n \to \infty$, we obtain the following consequence.

Corollary 4.1. *If $c \geq 2$, then for any $0 \leq a \leq c-1$ we have, as $n \to \infty$,*

$$\overline{N}(a, c, n) \sim \frac{\overline{p}(n)}{c} \sim \frac{1}{c} \cdot \frac{e^{\pi\sqrt{n}}}{8n}.$$

Remark 4.4. A similar result for partition ranks was obtained recently by Males [Mal20].

Remark 4.5. A Rademacher-type convergent series expansion for $\overline{p}(n)$ was found by Zuckerman [Zuc39, p. 321, eq. (8.53)], and is given by

$$\overline{p}(n) = \frac{1}{2\pi} \sum_{2\nmid k} \sqrt{k} {\sum_h}' \frac{\omega_{h,k}^2}{\omega_{2h,k}} \cdot e^{-\frac{2\pi i n h}{k}} \cdot \frac{d}{dn}\left(\frac{1}{\sqrt{n}} \sinh\left(\frac{\pi\sqrt{n}}{k}\right)\right).$$

The following inequalities were conjectured by Ji, Zhang and Zhao [JZZ18, Conj. 1.6 and Conj. 1.7], and Wei and Zhang [WZ20, Conj. 5.10].

Conjecture 4.1 (Ji–Zhang–Zhao, 2018)**.**

(i) *For $n \geq 0$ and $1 \leq i \leq 4$, we have*

$$\overline{N}(0,10,5n+i)+\overline{N}(1,10,5n+i) \geq \overline{N}(4,10,5n+i)+\overline{N}(5,10,5n+i).$$

(ii) *For $n \geq 0$, we have*

$$\overline{N}(1,10,n) + \overline{N}(2,10,n) \geq \overline{N}(3,10,n) + \overline{N}(4,10,n).$$

Conjecture 4.2 (Wei–Zhang, 2018)**.** *For $n \geq 11$, we have*

$$\overline{N}(0,6,3n) \geq \overline{N}(1,6,3n) = \overline{N}(3,6,3n) \geq \overline{N}(2,6,3n), \tag{4.1.9}$$

$$\overline{N}(0,6,3n+1) \geq \overline{N}(1,6,3n+1) = \overline{N}(3,6,3n+1) \geq \overline{N}(2,6,3n+1), \tag{4.1.10}$$

$$\overline{N}(1,6,3n+2) \geq \overline{N}(2,6,3n+2) \geq \overline{N}(0,6,3n+2) \geq \overline{N}(3,6,3n+2). \tag{4.1.11}$$

As an application of Theorem 4.1, we prove these conjectures and, in fact, a bit more.

Theorem 4.2. *For $n \geq 0$, we have*

$$\overline{N}(1,10,n) + \overline{N}(2,10,n) \geq \overline{N}(3,10,n) + \overline{N}(4,10,n), \tag{4.1.12}$$

$$\overline{N}(0,10,n) + \overline{N}(3,10,n) \geq \overline{N}(2,10,n) + \overline{N}(5,10,n), \tag{4.1.13}$$

$$\overline{N}(0,10,n) + \overline{N}(1,10,n) \geq \overline{N}(4,10,n) + \overline{N}(5,10,n). \tag{4.1.14}$$

Theorem 4.3. *For $n \geq 0$, we have*

$$\overline{N}(0,6,n) + \overline{N}(1,6,n) \geq \overline{N}(2,6,n) + \overline{N}(3,6,n), \tag{4.1.15}$$

$$\overline{N}(0,6,3n) + \overline{N}(3,6,3n) \geq \overline{N}(1,6,3n) + \overline{N}(2,6,3n), \tag{4.1.16}$$

$$\overline{N}(0,6,3n+1) + \overline{N}(3,6,3n+1) \geq \overline{N}(1,6,3n+1) + \overline{N}(2,6,3n+1), \tag{4.1.17}$$

$$\overline{N}(0,6,3n+2) + \overline{N}(3,6,3n+2) \leq \overline{N}(1,6,3n+2) + \overline{N}(2,6,3n+2), \tag{4.1.18}$$

$$\overline{N}(0,3,3n) \geq \overline{N}(1,3,3n) = \overline{N}(2,3,3n), \tag{4.1.19}$$

$$\overline{N}(0,3,3n+1) \geq \overline{N}(1,3,3n+1) = \overline{N}(2,3,3n+1), \tag{4.1.20}$$

$$\overline{N}(0,3,3n+2) \leq \overline{N}(1,3,3n+2) = \overline{N}(2,3,3n+2). \tag{4.1.21}$$

Theorem 4.4. *For $n \geq 11$, we have*

$$\overline{N}(0,6,3n) \geq \overline{N}(1,6,3n) \geq \overline{N}(2,6,3n), \tag{4.1.22}$$

$$\overline{N}(0,6,3n+1) \geq \overline{N}(1,6,3n+1) \geq \overline{N}(2,6,3n+1), \tag{4.1.23}$$

$$\overline{N}(1,6,3n+2) \geq \overline{N}(2,6,3n+2) \geq \overline{N}(0,6,3n+2) \geq \overline{N}(3,6,3n+2). \tag{4.1.24}$$

Remark 4.6. Similar identities and inequalities were studied, for instance, by Alwaise, Iannuzzi and Swisher [AIS17], Bringmann [Bri09], and Mao [Mao13] for ranks of partitions, and by Jennings-Shaffer and Reihill [JR], and Mao [Mao15] for M_2-ranks of partitions without repeated odd parts. By establishing identities for the overpartition rank generating functions evaluated at roots of unity analogous to those found in [JR, pp. 38–39] for the M_2-rank, the reader can come up with many other such inequalities.

Remark 4.7. Ji, Zhang and Zhao [JZZ18] proved (4.1.14) for $n \equiv 0 \pmod{5}$, whereas the inequality (4.1.15) is new.

Remark 4.8. The identities from (4.1.9) and (4.1.10) were proven by Ji, Zhang and Zhao [JZZ18], who further proved that $\overline{N}(0,6,3n) > \overline{N}(2,6,3n)$ for $n \geq 1$, and $\overline{N}(0,6,3n+1) > \overline{N}(2,6,3n+1)$ for $n \geq 0$. While (4.1.15) follows easily now for $n \equiv 0, 1 \pmod{3}$, the inequality is not at all clear for $n \equiv 2 \pmod{3}$, as the same authors also showed that $\overline{N}(0,6,3n+2) < \overline{N}(2,6,3n+2)$ for $n \geq 1$ and $\overline{N}(1,6,3n+2) > \overline{N}(3,6,3n+2)$ for $n \geq 0$. For a list of the identities and inequalities already proven, see [JZZ18, Th. 1.4].

Remark 4.9. The identity and inequalities from (4.1.9) were also proven by Wei and Zhang [WZ20, p. 25].

4.2 Strategy of the proof

For the reader's benefit, we outline the main steps in proving Theorem 4.1, along with several other estimates that will be used in what follows.

4.2.1 Circle method

The main idea of our approach is the Hardy-Ramanujan circle method. By Cauchy's Theorem we have, for $n > 0$,

$$A\left(\frac{a}{c}; n\right) = \frac{1}{2\pi i} \int_{\mathcal{C}} \frac{\mathcal{O}\left(\frac{a}{c}; q\right)}{q^{n+1}} dq,$$

where $\mathcal{C}$ may be taken to be the circle of radius $e^{-\frac{2\pi}{n}}$ parametrized by $q = e^{-\frac{2\pi}{n} + 2\pi i t}$ with $t \in [0,1]$, in which case we obtain

$$A\left(\frac{a}{c}; n\right) = \int_0^1 \mathcal{O}\left(\frac{a}{c}; e^{-\frac{2\pi}{n} + 2\pi i t}\right) \cdot e^{2\pi - 2\pi i n t} dt.$$

If $\frac{h_1}{k_1} < \frac{h}{k} < \frac{h_2}{k_2}$ are adjacent Farey fractions in the Farey sequence of order $N := \lfloor \sqrt{n} \rfloor$, we put

$$\vartheta'_{h,k} := \frac{1}{k(k_1 + k)} \quad \text{and} \quad \vartheta''_{h,k} := \frac{1}{k(k_2 + k)}.$$

Splitting the path of integration along the Farey arcs $-\vartheta'_{h,k} \leq \Phi \leq \vartheta''_{h,k}$, where $\Phi := t - \frac{h}{k}$ and $0 \leq h < k \leq N$ with $(h,k) = 1$, we have

$$A\left(\frac{a}{c}; n\right) = \sum_{h,k} e^{-\frac{2\pi i n h}{k}} \int_{-\vartheta'_{h,k}}^{\vartheta''_{h,k}} \mathcal{O}\left(\frac{a}{c}; e^{\frac{2\pi i}{k}(h+iz)}\right) \cdot e^{\frac{2\pi n z}{k}} d\Phi, \quad (4.2.1)$$

where $z = \frac{k}{n} - k\Phi i$.

The reader familiar with some basics from Farey theory might immediately recognize the inequality

$$\frac{1}{k+k_j} \leq \frac{1}{N+1}$$

for $j = 1, 2$, together with several other known facts (which are otherwise very easy to prove) such as

$$\mathrm{Re}(z) = \frac{k}{n},\ \mathrm{Re}\left(\frac{1}{z}\right) > \frac{k}{2},\ |z|^{-\frac{1}{2}} \leq n^{\frac{1}{2}} \cdot k^{-\frac{1}{2}} \text{ or } \vartheta'_{h,k} + \vartheta''_{h,k} \leq \frac{2}{k(N+1)}.$$

For a nice introduction to Farey fractions, one can consult [Apo90, Ch. 5.4].

4.2.2 Modular transformation laws

Our next step in the proof of Theorem 4.1 requires the modular transformations[3] for $\mathcal{O}\left(\frac{a}{c}; q\right)$ established by Bringmann and Lovejoy [BL07], the proof of which can be found in [BL07, pp. 11–17]. For $0 < a < c$ coprime with $c > 2$, and $s = s(b,c)$ and $t = t(b,c)$ as in Chapter 4.1.3, let

$$\mathcal{U}\left(\frac{a}{c}; q\right) = \mathcal{U}\left(\frac{a}{c}; z\right) := \frac{\eta\left(\frac{z}{2}\right)}{\eta^2(z)} \sin\left(\frac{\pi a}{c}\right) \sum_{n \in \mathbb{Z}} \frac{(1+q^n) q^{n^2+\frac{n}{2}}}{1 - 2\cos\left(\frac{2\pi a}{c}\right) q^n + q^{2n}},$$

$$\mathcal{U}(a,b,c;q) = \mathcal{U}(a,b,c;z) := \frac{\eta\left(\frac{z}{2}\right)}{\eta^2(z)} e^{\frac{\pi i a}{c}\left(\frac{4b}{c} - 1 - 2s\right)} q^{\frac{sb}{c} + \frac{b}{2c} - \frac{b^2}{c^2}}$$

[3]In passing, we correct the definitions of $\mathcal{U}(a,b,c;q)$ and $\mathcal{V}(a,b,c;q)$, as most likely some misprints occurred in their original expressions from [BL07, p. 8]. The necessary changes become clear on consulting the proof, see [BL07, pp. 11–17].

$$
\begin{aligned}
&\times \sum_{m\in\mathbb{Z}} \frac{q^{\frac{m}{2}(2m+1)+ms}}{1-e^{-\frac{2\pi ia}{c}}q^{m+\frac{b}{c}}},\\
\mathcal{V}\left(a,b,c;q\right) = \mathcal{V}(a,b,c;z) :=\ & \frac{\eta\left(\frac{z}{2}\right)}{\eta^2(z)} e^{\frac{\pi ia}{c}\left(\frac{4b}{c}-1-2s\right)} q^{\frac{sb}{c}+\frac{b}{2c}-\frac{b^2}{c^2}}\\
&\times \sum_{m\in\mathbb{Z}} \frac{q^{\frac{m(2m+1)}{2}+ms}\left(1+e^{-\frac{2\pi ia}{c}}q^{m+\frac{b}{c}}\right)}{1-e^{-\frac{2\pi ia}{c}}q^{m+\frac{b}{c}}},\\
\mathcal{O}\left(a,b,c;q\right) = \mathcal{O}(a,b,c;z) :=\ & \frac{\eta(2z)}{\eta^2(z)} e^{\frac{\pi ia}{c}\left(\frac{4b}{c}-1-t\right)} q^{\frac{tb}{2c}+\frac{b}{2c}-\frac{b^2}{c^2}}\\
&\times \sum_{m\in\mathbb{Z}} (-1)^m \frac{q^{\frac{m}{2}(2m+1)+\frac{mt}{2}}}{1-e^{-\frac{2\pi ia}{c}}q^{m+\frac{b}{c}}},\\
\mathcal{V}\left(\frac{a}{c};q\right) = \mathcal{V}\left(\frac{a}{c};z\right) :=\ & \frac{\eta(2z)}{\eta^2(z)} q^{\frac{1}{4}} \sum_{m\in\mathbb{Z}} \frac{q^{m^2+m}\left(1+e^{-\frac{2\pi ia}{c}}q^{m+\frac{1}{2}}\right)}{1-e^{-\frac{2\pi ia}{c}}q^{m+\frac{1}{2}}}.
\end{aligned}
$$

Furthermore, if

$$
H_{a,c}(x) := \frac{e^x}{1-2\cos\left(\frac{2\pi a}{c}\right)e^x+e^{2x}}, \tag{4.2.2}
$$

we consider, for $\nu \in \mathbb{Z}$, $k \in \mathbb{N}$ and $\widetilde{k}$ as defined in Chapter 4.1.3, the Mordell-type integral

$$
I_{a,c,k,\nu} := \int_{\mathbb{R}} e^{-\frac{2\pi zx^2}{k}} H_{a,c}\left(\frac{2\pi i\nu}{k} - \frac{2\pi zx}{k} - \frac{\widetilde{k}\pi i}{2k}\right) dx.
$$

If k is even and $c \mid k$, or if k is odd, $a = 1$ and $c = 4k$, there might be a pole at $x = 0$. In these cases we need to take the *Cauchy principal value* of the integral. We will make this precise at a later stage.

By using Poisson summation and proceeding similarly to Andrews [And66], Bringmann and Lovejoy [BL07] proved the following transformation laws[4].

[4]Some further corrections are in order; namely, the "$-$" sign in front of the expressions from (3)–(6) in their original formulation [BL07, Th. 2.1] should be a "+", and the other way around for (1) and (2), the reason being that the "$\pm$" sign from the expression of the residues $\lambda^{\pm}_{n,m}$ (see [BL07, p. 13]) is meant to be a "$\mp$". All necessary modifications are made here.

Theorem 4.5 ([BL07, Th. 2.1]). *Assume the notation above and let* $q = e^{\frac{2\pi i}{k}(h+iz)}$ *and* $q_1 = e^{\frac{2\pi i}{k}\left(h'+\frac{i}{z}\right)}$, *with* $z \in \mathbb{C}$ *and* $\mathrm{Re}(z) > 0$.

(1) *If* $c \mid k$ *and* $2 \mid k$, *then*

$$\begin{aligned}\mathcal{O}\left(\frac{a}{c};q\right) &= (-1)^{k_1+1} i e^{-\frac{2\pi a^2 h' k_1}{c}} \tan\left(\frac{\pi a}{c}\right)\cot\left(\frac{\pi a h'}{c}\right)\\ &\quad\times \frac{\omega_{h,k}^2}{\omega_{h,k/2}} z^{-\frac{1}{2}} \mathcal{O}\left(\frac{ah'}{c};q_1\right)\\ &\quad+ \frac{4\sin^2\left(\frac{\pi a}{c}\right)\cdot\omega_{h,k}^2}{\omega_{h,k/2}\cdot k} z^{-\frac{1}{2}} \sum_{\nu=0}^{k-1}(-1)^\nu e^{-\frac{2\pi i h'\nu^2}{k}} I_{a,c,k,\nu}(z).\end{aligned}$$

(2) *If* $c \mid k$ *and* $2 \nmid k$, *then*

$$\begin{aligned}\mathcal{O}\left(\frac{a}{c};q\right) &= -\sqrt{2} i e^{\frac{\pi i h'}{8k} - \frac{2\pi i a^2 h' k_1}{c}} \tan\left(\frac{\pi a}{c}\right) \frac{\omega_{h,k}^2}{\omega_{2h,k}} z^{-\frac{1}{2}} \mathcal{U}\left(\frac{ah'}{c};q_1\right)\\ &\quad+ \frac{4\sqrt{2}\sin^2\left(\frac{\pi a}{c}\right)\cdot\omega_{h,k}^2}{\omega_{2h,k}\cdot k} z^{-\frac{1}{2}} \sum_{\nu=0}^{k-1} e^{-\frac{\pi i h'}{k}(2\nu^2-\nu)} I_{a,c,k,\nu}(z).\end{aligned}$$

(3) *If* $c \nmid k$, $2 \mid k$ *and* $c_1 \neq 2$, *then*

$$\begin{aligned}\mathcal{O}\left(\frac{a}{c};q\right) &= 2e^{-\frac{2\pi i a^2 h' k_1}{c_1 c}} \tan\left(\frac{\pi a}{c}\right) \frac{\omega_{h,k}^2}{\omega_{h,k/2}} z^{-\frac{1}{2}}\\ &\quad\times (-1)^{c_1(\ell+k_1)} \mathcal{O}\left(ah', \frac{\ell c}{c_1}, c; q_1\right)\\ &\quad+ \frac{4\sin^2\left(\frac{\pi a}{c}\right)\cdot\omega_{h,k}^2}{\omega_{h,k/2}\cdot k} z^{\frac{1}{2}} \sum_{\nu=0}^{k-1}(-1)^\nu e^{-\frac{2\pi i h'\nu^2}{k}} I_{a,c,k,\nu}(z).\end{aligned}$$

(4) *If* $c \nmid k$, $2 \mid k$ *and* $c_1 = 2$, *then*

$$\begin{aligned}\mathcal{O}\left(\frac{a}{c};q\right) &= e^{-\frac{\pi i a^2 h' k_1}{c}} \tan\left(\frac{\pi a}{c}\right) \frac{\omega_{h,k}^2}{\omega_{h,k/2}\cdot k} z^{-\frac{1}{2}} \mathcal{V}\left(\frac{ah'}{c};q_1\right)\\ &\quad+ \frac{4\sin^2\left(\frac{\pi a}{c}\right)\cdot\omega_{h,k}^2}{\omega_{h,k/2}\cdot k} z^{\frac{1}{2}} \sum_{\nu=0}^{k-1}(-1)^\nu e^{-\frac{2\pi i h'\nu^2}{k}} I_{a,c,k,\nu}(z).\end{aligned}$$

(5) *If $c \nmid k$, $2 \nmid k$ and $c_1 \neq 4$, then*

$$\mathcal{O}\left(\frac{a}{c};q\right) = \sqrt{2}e^{\frac{\pi i h'}{8k} - \frac{2\pi i a^2 h' k_1}{c_1 c}} \tan\left(\frac{\pi a}{c}\right) \frac{\omega_{h,k}^2}{\omega_{2h,k}} \times z^{-\frac{1}{2}} \mathcal{U}\left(ah', \frac{\ell c}{c_1}, c; q_1\right)$$

$$+ \frac{4\sqrt{2}\sin^2\left(\frac{\pi a}{c}\right)\cdot \omega_{h,k}^2}{\omega_{2h,k}\cdot k} z^{\frac{1}{2}} \sum_{\nu=0}^{k-1} e^{-\frac{\pi i h'}{k}(2\nu^2-\nu)} I_{a,c,k,\nu}(z).$$

(6) *If $c \nmid k$, $2 \nmid k$ and $c_1 = 4$, then*

$$\mathcal{O}\left(\frac{a}{c};q\right) = e^{\frac{\pi i h'}{8k} - \frac{2\pi i a^2 h' k_1}{c_1 c}} \tan\left(\frac{\pi a}{c}\right) \frac{\omega_{h,k}^2}{\sqrt{2}\cdot\omega_{2h,k}} \times z^{-\frac{1}{2}} \mathcal{V}\left(ah', \frac{\ell c}{c_1}, c; q_1\right)$$

$$+ \frac{4\sqrt{2}\sin^2\left(\frac{\pi a}{c}\right)\cdot \omega_{h,k}^2}{\omega_{2h,k}\cdot k} z^{\frac{1}{2}} \sum_{\nu=0}^{k-1} e^{-\frac{\pi i h'}{k}(2\nu^2-\nu)} I_{a,c,k,\nu}(z).$$

In addition to these modular transformations, we need some further estimates.

4.2.3 The Mordell integral $I_{a,c,k,\nu}$

In the previous subsection we introduced

$$I_{a,c,k,\nu} = \int_{\mathbb{R}} e^{-\frac{2\pi z x^2}{k}} H_{a,c}\left(\frac{2\pi i \nu}{k} - \frac{2\pi z x}{k} - \frac{\widetilde{k}\pi i}{2k}\right) dx. \tag{4.2.3}$$

Recalling the definition (4.2.2), it is easy to see that

$$H_{a,c}(x) = \frac{1}{4\sinh\left(\frac{x}{2}+\frac{\pi i a}{c}\right)\sinh\left(\frac{x}{2}-\frac{\pi i a}{c}\right)},$$

and so $H_{a,c}(x)$ can only have poles in points of the form

$$x = 2\pi i\left(n \pm \frac{a}{c}\right)$$

with $n \in \mathbb{Z}$.

For $2 \mid k$, $c \mid k$ and $\nu = \frac{ka}{c}$ or $\nu = k\left(1 - \frac{a}{c}\right)$, there may be a pole at $x = 0$. The same is true if $2 \nmid k$, $\nu = 0$, $a = 1$ and $c = 4k$. In both cases we must consider the Cauchy principal value of the integral $I_{a,c,k,\nu}$.

The following[5] is adapted after [Bri09, Lem. 3.1].

Lemma 4.1. *Let $n \in \mathbb{N}$, $N = \lfloor \sqrt{n} \rfloor$ and $z = \frac{k}{n} - k\Phi i$, where $-\frac{1}{k(k+k_1)} \leq \Phi \leq \frac{1}{k(k+k_2)}$ and $\frac{h_1}{k_1} < \frac{h}{k} < \frac{h_2}{k_2}$ are adjacent Farey fractions in the Farey sequence of order N. If*

$$g_{a,c,k,\nu} := \begin{cases} \left(\min\left\{\left|\frac{\nu}{k} - \frac{1}{4k} + \frac{a}{c}\right|, \left|\frac{\nu}{k} - \frac{1}{4k} - \frac{a}{c}\right|\right\}\right)^{-1} & \text{if } k \text{ is odd, } \nu \neq 0 \\ & \text{and } \frac{a}{c} \neq \frac{1}{4k}; \\ \left(\min\left\{\left|\frac{\nu}{k} + \frac{a}{c}\right|, \left|\frac{\nu}{k} - \frac{a}{c}\right|\right\}\right)^{-1} & \text{if } k \text{ is even and} \\ & \nu \neq \frac{ka}{c}, \ \frac{k(c-a)}{c}; \\ \frac{c}{a} & \text{otherwise,} \end{cases}$$

and $\{x\} = x - \lfloor x \rfloor$ is the fractional part of $x \in \mathbb{R}$, then

$$z^{\frac{1}{2}} \cdot I_{a,c,k,\nu} \ll_c k^{-\frac{1}{2}} \cdot n^{\frac{1}{2}} \cdot g_{a,c,k,\nu}.$$

Proof. Let us first treat the case when k is odd and we encounter no poles. We have $\widetilde{k} = 1$ and

$$I_{a,c,k,\nu} = \int_{\mathbb{R}} e^{-\frac{2\pi z x^2}{k}} H_{a,c}\left(\frac{2\pi i \nu}{k} - \frac{2\pi z x}{k} - \frac{\pi i}{2k}\right) dx.$$

If we write $\frac{\pi z}{k} = re^{i\phi}$ with $r > 0$, then $|\phi| < \frac{\pi}{2}$ since $\mathrm{Re}(z) > 0$. The substitution $\tau = \frac{\pi z x}{k}$ yields

$$z^{\frac{1}{2}} \cdot I_{a,c,k,\nu}(z) = \frac{k}{\pi z^{\frac{1}{2}}} \int_L e^{-\frac{2k\tau^2}{\pi z}} H_{a,c}\left(\frac{2\pi i \nu}{k} - \frac{\pi i}{2k} - 2\tau\right) d\tau, \quad (4.2.4)$$

where L is the line passing through 0 at an angle of argument $\pm\phi$. One easily sees that, for $0 \leq t \leq \phi$,

$$\left| e^{-\frac{2kR^2 e^{2it}}{\pi z}} H_{a,c}\left(\frac{2\pi i \nu}{k} - \frac{\pi i}{2k} \pm 2Re^{it}\right) dx \right| \to 0 \quad \text{as } R \to \infty.$$

[5]Note that there are a few typos in the formulation of the original result from which this lemma is inspired. More precisely, in the statement of [Bri09, Lem. 3.1], $n^{\frac{1}{4}}$ should read $n^{\frac{1}{2}}$, k should read $k^{-\frac{1}{2}}$ and the $6kc$ factor from the definition of $g_{a,c,k,\nu}$ should be removed. These changes, however, do not affect the proof.

As the integrand from (4.2.4) has no poles, we can shift the path L of integration to the real line and obtain

$$z^{\frac{1}{2}} \cdot I_{a,c,k,\nu}(z) = \frac{k}{\pi z^{\frac{1}{2}}} \int_{\mathbb{R}} e^{-\frac{2kt^2}{\pi z}} H_{a,c}\left(\frac{2\pi i \nu}{k} - \frac{\pi i}{2k} - 2t\right) dt.$$

The inequality

$$\left|\sinh\left(\frac{\pi i \nu}{k} - \frac{\pi i}{4k} - t \pm \frac{\pi i a}{c}\right)\right| \geq \left|\sin\left(\frac{\pi \nu}{k} - \frac{\pi}{4k} \pm \frac{\pi a}{c}\right)\right|$$

follows immediately for $t \in \mathbb{R}$ from the definition of sinh and some easy manipulations, while the estimate

$$\begin{aligned}&\left|\sin\left(\frac{\pi \nu}{k} - \frac{\pi}{4k} - \frac{\pi a}{c}\right)\right| \left|\sin\left(\frac{\pi \nu}{k} - \frac{\pi}{4k} + \frac{\pi a}{c}\right)\right| \\ \gg_c &\min\left\{\left|\frac{\nu}{k} - \frac{1}{4k} + \frac{a}{c}\right|, \left|\frac{\nu}{k} - \frac{1}{4k} - \frac{a}{c}\right|\right\}\end{aligned}$$

is clear. Therefore we have

$$z^{\frac{1}{2}} I_{a,c,k,\nu}(z) \ll_c \frac{k|z|^{-\frac{1}{2}}}{\min\left\{\left\{\frac{\nu}{k} - \frac{1}{4k} + \frac{a}{c}\right\}, \left\{\frac{\nu}{k} - \frac{1}{4k} - \frac{a}{c}\right\}\right\}} \int_{\mathbb{R}} e^{-\frac{2k}{\pi}\operatorname{Re}\left(\frac{1}{z}\right)t^2} dt,$$

and, noting that

$$\left|e^{-\frac{2kt^2}{\pi z}}\right| = e^{-\frac{2k}{\pi}\operatorname{Re}\left(\frac{1}{z}\right)t^2}, \quad \operatorname{Re}\left(\frac{1}{z}\right)^{-\frac{1}{2}} \cdot |z|^{-\frac{1}{2}} \leq \sqrt{2} \cdot \sqrt{n} \cdot k^{-1},$$

the claim follows on making the substitution $t \mapsto \sqrt{\frac{2k\operatorname{Re}\left(\frac{1}{z}\right)}{\pi}} \cdot t$.

If k is even and $c \nmid k$, then we proceed similarly as above. If, however, the integrand in (4.2.3) has a pole at $x = 0$, in both of the cases $c \mid k$ and $c \nmid k$ we must consider the principal value of the integral, defined over the standard rectangular contour in the upper-half plane containing the real line indented over a semicircle around 0 as its basis.

For simplicity, let us present the case when $2 \mid k$, as the case $2 \nmid k$ is completely analogous. After doing the same change of variables as before and (if needed) shifting the path of integration (which will now consist of a straight line passing through 0 at an angle $\pm\phi$ with a small segment centered at 0 removed and replaced by a semicircle inclined also at an angle $\pm\phi$), the new path of integration will be given by $\gamma_{R,\varepsilon} = [-R, -\varepsilon] \cup C_\varepsilon \cup [R, \varepsilon]$, where C_ε is the positively oriented semicircle of radius ε around 0 and

$$\begin{aligned} I_{a,c,k,\nu} &= \frac{k}{\pi z} \int_{\gamma_{R,\varepsilon}} e^{-\frac{2kt^2}{\pi z}} H_{a,c}\left(\frac{2\pi i \nu}{k} - 2t\right) dt \\ &= \frac{k}{4\pi z} \int_{\gamma_{R,\varepsilon}} \frac{e^{-\frac{2kt^2}{\pi z}}}{\sinh(t)\sinh\left(t - \frac{2\pi i a}{c}\right)} dt. \end{aligned}$$

If we let $D_{R,\varepsilon}$ be the enclosed path of integration $\gamma_{R,\varepsilon} \cup [R, R + \pi ia/c] \cup [R + \pi ia/c, -R + \pi ia/c] \cup [-R + \pi ia/c, -R]$ and we set

$$f(w) := \frac{e^{-\frac{2kw^2}{\pi z}}}{\sinh(w)\sinh\left(w - \frac{2\pi i a}{c}\right)},$$

then by the Residue Theorem we obtain

$$\frac{4\pi z}{k} I_{a,c,k,\nu} = -\frac{2\pi}{\sin\left(\frac{2\pi a}{c}\right)} + \left(\int_{-R+\pi ia/c}^{-R} + \int_{-R+\pi ia/c}^{R+\pi ia/c} + \int_{R+\pi ia/c}^{R}\right) f(w) dw,$$

since inside and on $D_{R,\varepsilon}$ the only pole of f is at $w = 0$, with residue

$$\mathrm{Res}_{w=0} f(w) = \frac{i}{\sin\left(\frac{2\pi a}{c}\right)}.$$

On $[-R + \pi ia/c, -R]$ and $[R + \pi ia/c, R]$ we have

$$\left|\sinh(w)\sinh\left(w - \frac{2\pi i a}{c}\right)\right| \geq \sinh^2 R \quad \text{and} \quad \left|e^{-\frac{2kw^2}{\pi z}}\right| = e^{-\frac{2k}{\pi}\operatorname{Re}\left(\frac{1}{z}\right)R^2},$$

thus the two corresponding integrals tend to 0 as $R \to 0$, whereas on $[-R + \pi ia/c, R + \pi ia/c]$ we have, after a change of variables,

$$\int_{-R+\pi ia/c}^{R+\pi ia/c} f(w) dw = \int_{-R}^{R} \frac{e^{-\frac{2k\left(t+\frac{\pi i a}{c}\right)^2}{\pi z}}}{\sinh\left(t + \frac{\pi i a}{c}\right)\sinh\left(t - \frac{\pi i a}{c}\right)} dt.$$

Proceeding now along the same lines as before, we obtain

$$z^{\frac{1}{2}} \cdot I_{a,c,k,\nu}(z) \ll \left(\frac{\pi a}{c}\right)^{-1} \cdot \frac{k}{|z|^{\frac{1}{2}}} \int_{\mathbb{R}} e^{-\frac{2k}{\pi}\operatorname{Re}\left(\frac{1}{z}\right)t^2} dt,$$

and the proof is complete. $\square$

4.2.4 Kloosterman sums

The following is a variation of [And66, Lem. 4.1], cf. Bringmann [Bri09, Lem. 3.2].

Lemma 4.2. *Let $m, n \in \mathbb{Z}$, $0 \le \sigma_1 < \sigma_2 \le k$ and $D \in \mathbb{Z}$ with $(D, k) = 1$.*

(i) *We have*

$$\sideset{}{'}\sum_{\substack{h \\ \sigma_1 \le Dh' \le \sigma_2}} \frac{\omega_{h,k}^2}{\omega_{2h,k}} e^{\frac{2\pi i}{k}(hn+h'm)} \ll (24n+1, k)^{\frac{1}{2}} \cdot k^{\frac{1}{2}+\varepsilon}. \tag{4.2.5}$$

(ii) *If $c \mid k$, we have*

$$\tan\left(\frac{\pi a}{c}\right) \sideset{}{'}\sum_{\substack{h \\ \sigma_1 \le Dh' \le \sigma_2}} \frac{\omega_{h,k}^2}{\omega_{2h,k}} \frac{1}{\sin\left(\frac{\pi a h'}{c}\right)} \times e^{-\frac{2\pi i h' a^2 k_1}{c}} e^{\frac{2\pi i}{k}(nh+mh')} \ll (24n+1, k)^{\frac{1}{2}} \cdot k^{\frac{1}{2}+\varepsilon}. \tag{4.2.6}$$

(iii) *If $c \mid k$, we have*

$$\tan\left(\frac{\pi a}{c}\right) \sideset{}{'}\sum_{\substack{h \\ \sigma_1 \le Dh' \le \sigma_2}} \frac{\omega_{h,k}^2}{\omega_{2h,k}} (-1)^{k_1+1} \cot\left(\frac{\pi a h'}{c}\right) \times e^{-\frac{2\pi i h' a^2 k_1}{c}} e^{\frac{2\pi i}{k}(nh+mh')} \ll (24n+1, k)^{\frac{1}{2}} \cdot k^{\frac{1}{2}+\varepsilon}. \tag{4.2.7}$$

The implied constants are independent of a and k, and $\varepsilon > 0$ can be taken arbitrarily.

Proof. Part (i) follows simply on replacing $\omega_{h,k}$ by $\frac{\omega_{h,k}^2}{\omega_{2h,k}}$ in the proof of Andrews [And66, Lem. 4.1]. As the proof of (4.2.7) is completely analogous to that of (4.2.6), we deal only with part (ii). We set $\widetilde{c} = c$ if k is odd, and $\widetilde{c} = 2c$ if k is even. Since $e^{-\frac{2\pi i h' a^2 k_1}{c}}$ depends only on the residue class of h' modulo $\widetilde{c}$, the left-hand side of (4.2.6) can be rewritten as

$$\tan\left(\frac{\pi a}{c}\right) \sum_{c_j} \frac{e^{-\frac{2\pi i a^2 k_1 c_j}{c}}}{\sin\left(\frac{\pi a c_j}{c}\right)} \sideset{}{'}\sum_{\substack{h \\ \sigma_1 \le Dh' \le \sigma_2 \\ h' \equiv c_j \pmod{\widetilde{c}}}} \frac{\omega_{h,k}^2}{\omega_{2h,k}} \cdot e^{\frac{2\pi i}{k}(nh+mh')}, \tag{4.2.8}$$

where c_j runs over a set of primitive residues modulo $\widetilde{c}$. Furthermore, if S denotes the most right-hand sum in (4.2.8), we have

$$\begin{aligned} S &= \frac{1}{\widetilde{c}} \sideset{}{'}\sum_{\substack{h \\ \sigma_1 \le Dh' \le \sigma_2}} \frac{\omega_{h,k}^2}{\omega_{2h,k}} \cdot e^{\frac{2\pi i}{k}(nh+mh')} \sum_{r \pmod{\widetilde{c}}} e^{\frac{2\pi i r}{\widetilde{c}}(h'-c_j)} \\ &= \frac{1}{\widetilde{c}} \sum_{r \pmod{\widetilde{c}}} e^{-\frac{2\pi i r c_j}{\widetilde{c}}(h'-c_j)} \sideset{}{'}\sum_{\substack{h \\ \sigma_1 \le Dh' \le \sigma_2 \\ h' \equiv c_j \pmod{\widetilde{c}}}} \frac{\omega_{h,k}^2}{\omega_{2h,k}} \cdot e^{\frac{2\pi i}{k}\left(nh+\left(m+\frac{kr}{\widetilde{c}}\right)h'\right)} \end{aligned}$$

and the proof is concluded on invoking (i) and noting that $\frac{kr}{\widetilde{c}} \in \mathbb{Z}$. $\square$

4.3 Asymptotics for $A\left(\frac{a}{c}; n\right)$ and $N(a, c, n)$

We turn our focus now to the proof of Theorem 4.1 and proceed as described in the strategy outlined in Chapter 4.2, the whole section being dedicated to this purpose.

Proof of Theorem 4.1. On using Cauchy's Theorem and splitting the path of integration into Farey arcs as explained in Chapter 4.2.1, we can rewrite, on invoking (4.2.1) and Theorem 4.5,

$$A\left(\frac{a}{c}; n\right) = \sum_1 + \sum_2 + \sum_3 + \sum_4 + \sum_5 + \sum_6 + \sum_7 + \sum_8,$$

where

$$\sum_1 := i \tan\left(\frac{\pi a}{c}\right) \sum_{\substack{h,k \\ 2|k,\ c|k}} \frac{\omega_{h,k}^2}{\omega_{h,k/2}} (-1)^{k_1+1} \cot\left(\frac{\pi a h'}{c}\right)$$
$$\times e^{-\frac{2\pi i a^2 h' k_1}{c} - \frac{2\pi i n h}{k}} \int_{-\vartheta'_{h,k}}^{\vartheta''_{h,k}} z^{-\frac{1}{2}} e^{\frac{2\pi n z}{k}} \mathcal{O}\left(\frac{ah'}{c}; q_1\right) d\Phi,$$

$$\sum_2 := -\sqrt{2} i \tan\left(\frac{\pi a}{c}\right) \sum_{\substack{h,k \\ 2\nmid k,\ c|k}} \frac{\omega_{h,k}^2}{\omega_{2h,k}}$$
$$\times e^{\frac{\pi i h'}{8k} - \frac{2\pi i a^2 h' k_1}{c} - \frac{2\pi i n h}{k}} \int_{-\vartheta'_{h,k}}^{\vartheta''_{h,k}} z^{-\frac{1}{2}} e^{\frac{2\pi n z}{k}} \mathcal{U}\left(\frac{ah'}{c}; q_1\right) d\Phi,$$

$$\sum_3 := 2\tan\left(\frac{\pi a}{c}\right) \sum_{\substack{h,k \\ 2|k,\ c\nmid k,\ c_1\neq 2}} \frac{\omega_{h,k}^2}{\omega_{h,k/2}} (-1)^{c_1(\ell+k_1)}$$

$$\times e^{-\frac{2\pi i a^2 h' k_1}{c_1 c} - \frac{2\pi i n h}{k}} \int_{-\vartheta'_{h,k}}^{\vartheta''_{h,k}} z^{-\frac{1}{2}} e^{\frac{2\pi n z}{k}} \mathcal{O}\left(ah', \frac{\ell c}{c_1}, c; q_1\right) d\Phi,$$

$$\sum_4 := \tan\left(\frac{\pi a}{c}\right) \sum_{\substack{h,k \\ 2|k,\ c\nmid k,\ c_1=2}} \frac{\omega_{h,k}^2}{\omega_{h,k/2}}$$

$$\times e^{-\frac{\pi i a^2 h' k_1}{c} - \frac{2\pi i n h}{k}} \int_{-\vartheta'_{h,k}}^{\vartheta''_{h,k}} z^{-\frac{1}{2}} e^{\frac{2\pi n z}{k}} \mathcal{V}\left(\frac{ah'}{c}; q_1\right) d\Phi,$$

$$\sum_5 := \sqrt{2}\tan\left(\frac{\pi a}{c}\right) \sum_{\substack{h,k \\ 2\nmid k,\ c\nmid k,\ c_1\neq 4}} \frac{\omega_{h,k}^2}{\omega_{2h,k}}$$

$$\times e^{\frac{\pi i h'}{8k} - \frac{2\pi i a^2 h' k_1}{c_1 c} - \frac{2\pi i n h}{k}} \int_{-\vartheta'_{h,k}}^{\vartheta''_{h,k}} z^{-\frac{1}{2}} e^{\frac{2\pi n z}{k}} \mathcal{U}\left(ah', \frac{\ell c}{c_1}, c; q_1\right) d\Phi,$$

$$\sum_6 := \frac{1}{\sqrt{2}}\tan\left(\frac{\pi a}{c}\right) \sum_{\substack{h,k \\ 2\nmid k,\ c\nmid k,\ c_1=4}} \frac{\omega_{h,k}^2}{\omega_{h,k/2}}$$

$$\times e^{\frac{\pi i h'}{8k} - \frac{2\pi i a^2 h' k_1}{c_1 c} - \frac{2\pi i n h}{k}} \int_{-\vartheta'_{h,k}}^{\vartheta''_{h,k}} z^{-\frac{1}{2}} e^{\frac{2\pi n z}{k}} \mathcal{V}\left(ah', \frac{\ell c}{c_1}, c; q_1\right) d\Phi,$$

$$\sum_7 := 4\sin^2\left(\frac{\pi a}{c}\right) \sum_{\substack{h,k \\ 2|k}} \frac{\omega_{h,k}^2}{\omega_{h,k/2}\cdot k} e^{-\frac{2\pi i n h}{k}} \sum_{\nu=0}^{k-1} (-1)^\nu e^{-\frac{2\pi i h' \nu^2}{k}}$$

$$\times \int_{-\vartheta'_{h,k}}^{\vartheta''_{h,k}} z^{\frac{1}{2}} e^{\frac{2\pi n z}{k}} I_{a,c,k,\nu}(z) d\Phi,$$

$$\sum_8 := 4\sqrt{2}\sin^2\left(\frac{\pi a}{c}\right) \sum_{\substack{h,k \\ 2\nmid k}} \frac{\omega_{h,k}^2}{\omega_{2h,k}\cdot k} e^{-\frac{2\pi i n h}{k}} \sum_{\nu=0}^{k-1} e^{-\frac{\pi i h'}{k}(2\nu^2-\nu)}$$

$$\times \int_{-\vartheta'_{h,k}}^{\vartheta''_{h,k}} z^{\frac{1}{2}} e^{\frac{2\pi n z}{k}} I_{a,c,k,\nu}(z) d\Phi.$$

For the reader's convenience, we divide our proof into several steps, and we start by estimating the sums $\sum_2, \sum_5$ and $\sum_6$, which, as we shall see, will give the main contribution.

The sums $\sum_1, \sum_3, \sum_4, \sum_7$ and $\sum_8$ will go into an error term and will be dealt with at the end. Here the analysis will also split, as the latter two sums can be treated together.

4.3.1 Estimates for the sums $\sum_2, \sum_5$ and $\sum_6$

To estimate $\sum_2$, notice that we can write

$$\begin{aligned}
\sum_{n\in\mathbb{Z}} \frac{(1+q^n)q^{n^2+\frac{n}{2}}}{1-2\cos\left(\frac{2\pi a}{c}\right)q^n+q^{2n}} &= \frac{1}{2\sin^2\left(\frac{\pi a}{c}\right)} + 2\sum_{n\geq 1} \frac{(1+q^n)q^{n^2+\frac{n}{2}}}{1-2\cos\left(\frac{2\pi a}{c}\right)q^n+q^{2n}} \\
&= \frac{1}{2\sin^2\left(\frac{\pi a}{c}\right)} + 2\sum_{2|n} \frac{(1+q^n)q^{n^2+\frac{n}{2}}}{1-2\cos\left(\frac{2\pi a}{c}\right)q^n+q^{2n}} \\
&\quad + 2q^{\frac{1}{2}} \sum_{2\nmid n} \frac{(1+q^n)q^{n^2+\frac{n-1}{2}}}{1-2\cos\left(\frac{2\pi a}{c}\right)q^n+q^{2n}} \\
&= \frac{1}{2\sin^2\left(\frac{\pi a}{c}\right)} + \sum_{r\geq 1} a_2(r) e^{\frac{2\pi i m_r h'}{k} - \frac{2\pi r}{kz}} \\
&\quad + q^{\frac{1}{2}} \sum_{r\geq 1} b_2(r) e^{\frac{2\pi i n_r h'}{k} - \frac{2\pi r}{kz}},
\end{aligned}$$

where $m_r, n_r \in \mathbb{Z}$ and the coefficients $a_2(r)$ and $b_2(r)$ are independent of k and h. On replacing z by $z_1 = z/2$ in (4.1.1), we have

$$\begin{aligned}
\mathcal{U}\left(\frac{ah'}{c}; q_1\right) &= \sin\left(\frac{\pi a h'}{c}\right) \frac{\eta\left(\frac{z_1}{2}\right)}{\eta^2(z_1)} \sum_{n\in\mathbb{Z}} \frac{(1+q_1^n)q_1^{n^2+\frac{n}{2}}}{1-2q_1^n\cos\left(\frac{2\pi a h'}{c}\right)+q_1^{2n}} \\
&= \sin\left(\frac{\pi a h'}{c}\right) \frac{\eta\left(\frac{z_1}{2}\right)}{\eta^2(z_1)} \\
&\quad \times \left(\frac{1}{2\sin^2\left(\frac{\pi a h'}{c}\right)} + 2\sum_{n\geq 1} \frac{(1+q_1^n)q_1^{n^2+\frac{n}{2}}}{1-2q_1^n\cos\left(\frac{2\pi a h'}{c}\right)+q_1^{2n}}\right) \\
&= 2\sin\left(\frac{\pi a h'}{c}\right) q_1^{-\frac{1}{16}} \overline{P}(q_1) \sum_{n\geq 1} \frac{(1+q_1^n)q_1^{n^2+\frac{n}{2}}}{1-2q_1^n\cos\left(\frac{2\pi a h'}{c}\right)+q_1^{2n}}
\end{aligned}$$

$$+\frac{\overline{P}(q_1)}{2q_1^{\frac{1}{16}}\sin\left(\frac{\pi a h'}{c}\right)},$$

where we write $q_1 = e^{2\pi i z_1}$. It follows that

$$\begin{aligned}\sum_2 &= -\sqrt{2}i\tan\left(\frac{\pi a}{c}\right)\sum_{\substack{h,k\\ 2\nmid k,\ c|k}}\frac{\omega_{h,k}^2}{\omega_{2h,k}}e^{\frac{\pi i h'}{8k}-\frac{2\pi i a^2 h' k_1}{c}-\frac{2\pi i n h}{k}}\\ &\quad\times\int_{-\vartheta'_{h,k}}^{\vartheta''_{h,k}} z^{-\frac{1}{2}}e^{\frac{2\pi n z}{k}}\mathcal{U}\left(\frac{ah'}{c};q_1\right)d\Phi\\ &= -\frac{i}{\sqrt{2}}\tan\left(\frac{\pi a}{c}\right)\sum_{2\nmid k,\ c|k}\sum_h{}' \frac{\omega_{h,k}^2}{\omega_{2h,k}}\frac{1}{\sin\left(\frac{\pi a h'}{c}\right)}e^{-\frac{2\pi i a^2 h' k_1}{c}-\frac{2\pi i n h}{k}}\\ &\quad\times\int_{-\vartheta'_{h,k}}^{\vartheta''_{h,k}} z^{-\frac{1}{2}}e^{\frac{2\pi n z}{k}+\frac{\pi}{8kz}}\widetilde{\mathcal{U}}\left(\frac{ah'}{c};q_1\right)d\Phi,\end{aligned}$$

with

$$\begin{aligned}\widetilde{\mathcal{U}}\left(\frac{ah'}{c};q_1\right) &= 1+4\sin^2\left(\frac{\pi a h'}{c}\right)\sum_{r\geq 1}a_2(r)e^{\frac{2\pi i m_r h'}{k}-\frac{2\pi r}{kz}}\\ &\quad+4\sin^2\left(\frac{\pi a h'}{c}\right)\sum_{r\geq 1}b_2(r)q^{\frac{1}{2}}e^{\frac{2\pi i n_r h'}{k}-\frac{2\pi r}{kz}}.\end{aligned}$$

We treat the sum coming from the constant term and the two sums coming from the case $r \geq 1$ separately. The former will contribute to the main term, while the latter two sums will contribute to the error term. We denote the associated sums by S_1, S_2 and S_3 and we first estimate S_2 (S_3 is dealt with in a similar manner).

We recall, from Chapter 4.2.1, the easy facts that

$$\mathrm{Re}(z) = \frac{k}{n},\ \mathrm{Re}\left(\frac{1}{z}\right) > \frac{k}{2},\ |z|^{-\frac{1}{2}} \leq n^{\frac{1}{2}}\cdot k^{-\frac{1}{2}},\ \vartheta'_{h,k}+\vartheta''_{h,k} \leq \frac{2}{k(N+1)}. \tag{4.3.1}$$

We write

$$\int_{-\vartheta'_{h,k}}^{\vartheta''_{h,k}} = \int_{-\frac{1}{k(N+k)}}^{\frac{1}{k(N+k)}} + \int_{-\frac{1}{k(k_1+k)}}^{-\frac{1}{k(N+k)}} + \int_{\frac{1}{k(N+k)}}^{\frac{1}{k(k_2+k)}} \tag{4.3.2}$$

and denote the associated sums by S_{21}, S_{22} and S_{23}. This way of splitting the integral is motivated by the Farey dissection used by Rademacher [Rad38, pp. 504–509]. It allows us to interchange summation with the integral and yields a so-called *complete* Kloosterman sum and two *incomplete* Kloosterman sums. Lemma 4.2 applies to both types of sums.

We first consider S_{21}. As we have already seen,

$$\overline{p}(n) \sim \frac{1}{8n} e^{\pi\sqrt{n}},$$

thus $\overline{p}(n) < e^{\pi\sqrt{n}}$ as $n \to \infty$. Clearly, the coefficients of $\mathcal{O}(u;q)$, regarded as a series in q when evaluated at a root of unity $u = \zeta_c^a$, satisfy

$$\left| A\left(\frac{a}{c};n\right)\right| \leq \sum_{m=-\infty}^{\infty} |\overline{N}(m,n)\zeta_c^{am}| \leq \sum_{m=-\infty}^{\infty} \overline{N}(m,n) = \overline{p}(n),$$

thus, in light of the transformation behavior shown in Theorem 4.5, the coefficients $a_2(r)$ and $b_2(r)$ satisfy

$$|a_2(r)|,\ |b_2(r)| \leq e^{\pi\sqrt{r}} \quad \text{as } r \to \infty. \tag{4.3.3}$$

As the integral that appears inside the sum does not depend on h, in evaluating S_{21} we can perform summation with respect to h. Using, in turn, the bound (4.3.3), Lemma 4.2, the estimates from (4.3.1), and the well-known bound $\sigma_0(n) = o(n^\epsilon)$ for all $\epsilon > 0$, we obtain

$$\begin{aligned}
S_{21} \ll &\left| \sum_{r=1}^{\infty} a_2(r) \sum_{c|k} \tan\left(\frac{\pi a}{c}\right) \sum_h{}' \frac{\omega_{h,k}^2}{\omega_{2h,k}} \frac{1}{\sin\left(\frac{\pi a h'}{c}\right)} \right. \\
&\left. \times e^{-\frac{2\pi i h' a^2 k_1}{c} - \frac{2\pi i n h}{k} + \frac{2\pi i m_r h'}{k}} \int_{-\frac{1}{k(N+k)}}^{\frac{1}{k(N+k)}} z^{-\frac{1}{2}} \cdot e^{-\frac{2\pi}{kz}\left(r-\frac{1}{16}\right)+\frac{2\pi z n}{k}} d\Phi \right| \\
&\ll \sum_{r=1}^{\infty} |a_2(r)| e^{-\pi r} \sum_k k^{-1+\varepsilon}(24n+1,k)^{\frac{1}{2}} \ll \sum_{\substack{d|24n+1 \\ d\leq N}} d^{\frac{1}{2}} \sum_{k\leq\frac{N}{d}} (dk)^{-1+\varepsilon} \\
&\ll \sum_{\substack{d|24n+1 \\ d\leq N}} d^{-\frac{1}{2}+\varepsilon} \int_1^{N/d} x^{-1+\varepsilon} dx = \sum_{\substack{d|24n+1 \\ d\leq N}} d^{-\frac{1}{2}} \cdot d^{\varepsilon} \cdot \left(\frac{N}{d}\right)^{\varepsilon}
\end{aligned}$$

$$\ll \sum_{d|24n+1} d^{-\frac{1}{2}} \cdot n^{\frac{\varepsilon}{2}} \ll n^{\epsilon+\frac{\varepsilon}{2}} \ll n^{\varepsilon}.$$

For a proof of the fact that $\sigma_0(n) = o(n^{\epsilon})$ see, e.g., [Apo76, p. 296]. Here we bound trivially

$$\sum_{d|24n+1} d^{-\frac{1}{2}} < \sum_{d|24n+1} 1 = \sigma_0(24n+1) = o(n^{\epsilon})$$

and choose $0 < \epsilon < \varepsilon/2$, where $\sigma_0(n)$ denotes, as usual, the number of divisors of n.

Since S_{22} and S_{23} are treated in the exact same way, we only consider S_{22}. Writing

$$\int_{-\frac{1}{k(k_1+k)}}^{-\frac{1}{k(N+k)}} = \sum_{\ell=k_1+k}^{N+k-1} \int_{-\frac{1}{k\ell}}^{-\frac{1}{k(\ell+1)}}$$

we see that

$$S_{22} \ll \Bigg| \sum_{r=1}^{\infty} a_2(r) \sum_{c|k} \sum_{\ell=k_1+k}^{N+k-1} \int_{-\frac{1}{k\ell}}^{-\frac{1}{k(\ell+1)}} z^{-\frac{1}{2}} \cdot e^{-\frac{2\pi}{kz}\left(r-\frac{1}{16}\right)+\frac{2\pi z n}{k}} d\Phi$$
$$\times \tan\left(\frac{\pi a}{c}\right) \sum_{\substack{h \\ N<k+k_1\leq \ell}}{}' \frac{\omega_{h,k}^2}{\omega_{2h,k}} \frac{1}{\sin\left(\frac{\pi a h'}{c}\right)} e^{-\frac{2\pi i h' a^2 k_1}{c} - \frac{2\pi i n h}{k} + \frac{2\pi i m_r h'}{k}} \Bigg|.$$

Again, from basic facts of Farey theory, it follows that

$$N-k < k_1, k_2 \leq N \quad \text{and} \quad k_1 \equiv -k_2 \equiv -h' \pmod{k},$$

conditions which imply the restriction of h' to one or two intervals in the range $0 \leq h' < k$. Therefore we can use Lemma 4.2 to estimate the above expression just as in the case of S_{21}.

As for the estimation of S_1, we can split the integration path into

$$\int_{-\vartheta'_{h,k}}^{\vartheta''_{h,k}} = \int_{-\frac{1}{kN}}^{\frac{1}{kN}} - \int_{-\frac{1}{kN}}^{-\frac{1}{k(k_1+k)}} - \int_{\frac{1}{k(k_2+k)}}^{\frac{1}{kN}}$$

and denote the associated sums by S_{11}, S_{12} and S_{13}. The sums S_{12} and S_{13} contribute to the error term and, since they are of the same shape, we only consider S_{12}. Further, decomposing

$$\int_{-\frac{1}{kN}}^{-\frac{1}{k(k_1+k)}} = \sum_{\ell=N}^{k_1+k-1} \int_{-\frac{1}{k\ell}}^{-\frac{1}{k(\ell+1)}}$$

gives

$$S_{12} \ll \Bigg| \sum_{c|k} \sum_{\ell=N}^{k_1+k-1} \int_{-\frac{1}{k\ell}}^{-\frac{1}{k(\ell+1)}} z^{-\frac{1}{2}} \cdot e^{\frac{\pi}{8kz}+\frac{2\pi zn}{k}} d\Phi$$

$$\times \tan\left(\frac{\pi a}{c}\right) \sideset{}{'}\sum_{\substack{h \\ \ell<k_1+k-1\leq N+k-1}} \frac{\omega_{h,k}^2}{\omega_{2h,k}} \frac{1}{\sin\left(\frac{\pi a h'}{c}\right)} e^{-\frac{2\pi i h' a^2 k_1}{c}-\frac{2\pi i n h}{k}} \Bigg| .$$

Using the facts that

$$\mathrm{Re}(z) = \frac{k}{n}, \quad \mathrm{Re}\left(\frac{1}{z}\right) < k \quad \text{and} \quad |z|^2 \geq \frac{k^2}{n^2},$$

this sum can be estimated as before against $O(n^\varepsilon)$. Thus,

$$\sum_2 = i \sum_{c|k} B_{a,c,k}(-n,0) \int_{-\frac{1}{kN}}^{\frac{1}{kN}} z^{-\frac{1}{2}} \cdot e^{\frac{2\pi zn}{k}+\frac{\pi}{8kz}} d\Phi + O(n^\varepsilon). \quad (4.3.4)$$

We stop here for the moment with the estimation of $\sum_2$ and turn our attention to $\sum_5$. This sum is treated in a similar manner, but some comments regarding necessary modifications are in order. On noting that

$$\begin{aligned} \mathcal{U}(a,b,c;q) = &\frac{\eta\left(\frac{z}{2}\right)}{\eta^2(z)} e^{\frac{\pi i a}{c}\left(\frac{4b}{c}-2s\right)} q^{\frac{sb}{c}-\frac{b^2}{c^2}} \sum_{m\geq 0} \frac{e^{-\frac{\pi i a}{c}} q^{\frac{m(2m+1)}{2}+ms+\frac{b}{2c}}}{1-e^{-\frac{2\pi i a}{c}} q^{m+\frac{b}{c}}} \\ &- \frac{\eta\left(\frac{z}{2}\right)}{\eta^2(z)} e^{\frac{\pi i a}{c}\left(\frac{4b}{c}-2s\right)} q^{\frac{sb}{c}-\frac{b^2}{c^2}} \sum_{m\geq 1} \frac{e^{\frac{\pi i a}{c}} q^{\frac{m(2m+1)}{2}-ms-\frac{b}{2c}}}{1-e^{\frac{2\pi i a}{c}} q^{m-\frac{b}{c}}}, \end{aligned} \quad (4.3.5)$$

we see that

$$\begin{aligned} e^{\frac{\pi i h'}{8k}-\frac{2\pi i a^2 h' k_1}{c_1 c}} \mathcal{U}\left(ah', \frac{\ell c}{c_1}, c; q_1\right) = &\sum_{r\geq r_0} a_5(r) e^{\frac{2\pi i m_r h'}{k}} e^{-\frac{\pi r}{k c_1^2 z}} \\ &+ e^{\frac{\pi i h'}{k}} \sum_{r\geq r_1} b_5(r) e^{\frac{2\pi i n_r h'}{k}} e^{-\frac{\pi r}{k c_1^2 z}}, \end{aligned} \quad (4.3.6)$$

where $m_r, n_r, r_0, r_1 \in \mathbb{Z}$. By the same argument as for S_{21}, one sees immediately that the part which might contribute to the main term can come only from those terms with $r < 0$. A straightforward, but rather tedious computation shows that such terms can arise only for $s = 0$, $m = 0$ in the first sum, respectively for $s = 2$, $m = 1$ in the second sum obtained by expressing $\mathcal{U}\left(ah', \frac{\ell c}{c_1}, c; q_1\right)$ as shown in (4.3.5). In the former case, the contribution is given by

$$e^{-\frac{2\pi i a^2 h' k_1}{c_1 c} + \frac{4\pi i a h' \ell}{c_1 c} - \frac{\pi i a h'}{c}} \cdot q_1^{-\frac{1}{16} - \frac{\ell^2}{c_1^2} + \frac{\ell}{2c_1}} \sum_{\substack{r \\ \delta_{c,k,r} > 0}} e^{-\frac{2\pi i a h' r}{c}} \cdot q_1^{\frac{\ell r}{c_1}},$$

and, in the latter, by

$$-e^{-\frac{2\pi i a^2 h' k_1}{c_1 c} + \frac{4\pi i a h' \ell}{c_1 c} - \frac{3\pi i a h'}{c}} \cdot q_1^{-\frac{1}{16} - \frac{\ell^2}{c_1^2} + \frac{3\ell}{2c_1} - \frac{1}{2}} \sum_{\substack{r \\ \delta_{c,k,r} > 0}} e^{\frac{2\pi i a h' r}{c}} \cdot q_1^{\left(1 - \frac{\ell}{c_1}\right) r}.$$

To evaluate $\sum_5$, note that one can split the sum over k into groups based on the value k_1, which is defined in terms of c_1 and ℓ. In each such group, the value of $\delta_{c,k,r}$ (hence the condition $\delta_{c,k,r} > 0$) is independent of k, and the number of terms satisfying $\delta_{a,c,k,r} > 0$ is finite and bounded in terms of c_1 (hence of c). Moreover, the coefficients $a_5(r)$ and $b_5(r)$ are independent of k in any such fixed group. Since the terms with $r < 0$ from (4.3.6) can be estimated as in the case of S_2, we obtain

$$\sum_5 = \sqrt{2} \tan\left(\frac{\pi a}{c}\right) \sum_{\substack{k,r \\ c \nmid k,\ 2 \nmid k,\ c_1 \neq 4 \\ \delta_{c,k,r} > 0}} \sideset{}{'}\sum_h \frac{\omega_{h,k}^2}{\omega_{2h,k}} e^{\frac{2\pi i}{k}(-nh + m_{a,c,k,r} h')}$$
$$\times \int_{-\vartheta'_{h,k}}^{\vartheta''_{h,k}} z^{-\frac{1}{2}} \cdot e^{\frac{2\pi n z}{k} + \frac{2\pi}{kz}\delta_{c,k,r}} d\Phi + O(n^{\varepsilon}), \tag{4.3.7}$$

with $\delta_{c,k,r}$ and $m_{a,c,k,r}$ as defined in (4.1.4) and (4.1.5). In a completely similar way, we compute

$$\sum_6 = \frac{1}{\sqrt{2}} \tan\left(\frac{\pi a}{c}\right) \sum_{\substack{k,r \\ c \nmid k,\ 2 \nmid k,\ c_1 = 4 \\ \delta'_{c,k,r} > 0}} \sideset{}{'}\sum_h \frac{\omega_{h,k}^2}{\omega_{2h,k}} e^{\frac{2\pi i}{k}(-nh + m_{a,c,k,r} h')}$$

$$
\begin{aligned}
&\times \int_{-\vartheta'_{h,k}}^{\vartheta''_{h,k}} z^{-\frac{1}{2}} e^{\frac{2\pi n z}{k}+\frac{2\pi}{kz}\delta_{c,k,r}} d\Phi \\
&+ \frac{1}{\sqrt{2}} \tan\left(\frac{\pi a}{c}\right) \sum_{\substack{k,r \\ c\nmid k,\ 2\nmid k,\ c_1=4 \\ \delta''_{c,k,r}>0}} \sum_h{}' \frac{\omega_{h,k}^2}{\omega_{2h,k}} e^{\frac{2\pi i}{k}(-nh+m'_{a,c,k,r}h')} \\
&\times \int_{-\vartheta'_{h,k}}^{\vartheta''_{h,k}} z^{-\frac{1}{2}} e^{\frac{2\pi n z}{k}+\frac{2\pi}{kz}\delta'_{c,k,r}} d\Phi \\
&+ O(n^{\varepsilon}),
\end{aligned}
$$

where we define

$$
\delta'_{c,k,r} := \begin{cases} \frac{1}{16} - \frac{3\ell}{2c_1} + \frac{\ell^2}{c_1^2} - r\frac{\ell}{c_1} & \text{if } 0 < \frac{\ell}{c_1} \leq \frac{1}{4}, \\ \frac{1}{16} - \frac{3\ell}{2c_1} + \frac{\ell^2}{c_1^2} + \frac{1}{2} - r\left(1 - \frac{\ell}{c_1}\right) & \text{if } \frac{1}{4} < \frac{\ell}{c_1} \leq \frac{3}{4}, \\ \frac{1}{16} - \frac{5\ell}{2c_1} + \frac{\ell^2}{c_1^2} + \frac{3}{2} - r\left(1 - \frac{\ell}{c_1}\right) & \text{if } \frac{3}{4} < \frac{\ell}{c_1} < 1, \end{cases}
$$

and

$$
m'_{a,c,k,r} := \begin{cases} -\frac{2(ak_1-\ell)^2+c_1(2r+3)(ak_1-\ell)}{2c_1^2} & \text{if } 0 < \frac{\ell}{c_1} \leq \frac{1}{4}, \\ -\frac{2(ak_1-\ell)^2-c_1(2r-3)(ak_1-\ell)-c_1^2(2r-1)}{2c_1^2} & \text{if } \frac{1}{4} < \frac{\ell}{c_1} \leq \frac{3}{4}, \\ -\frac{2(ak_1-\ell)^2-c_1(2r-5)(ak_1-\ell)-c_1^2(2r-3)}{2c_1^2} & \text{if } \frac{3}{4} < \frac{\ell}{c_1} < 1. \end{cases}
$$

An easy computation shows that if $c_1 = 4$, then $\delta_{a,c,k,r} \leq 0$ for all $r \geq 0$, and that $\delta'_{a,c,k,r} > 0$ if and only if $r = 0$, $m = 1$, $s = 1$ and $\ell = 2$, case which is impossible as it leads to $ak_1 \equiv 2 \pmod 4$, and by assumption k is odd, while the condition $(a, c) = 1$ implies that a is odd as well. Therefore $\sum_6$ will only contribute to the error term.

To finish the estimation of these sums, we are only left with computing integrals of the form

$$
\mathcal{I}_{k,v} = \int_{-\frac{1}{kN}}^{\frac{1}{kN}} z^{-\frac{1}{2}} \cdot e^{\frac{2\pi}{k}\left(nz+\frac{v}{z}\right)} d\Phi,
$$

which, upon substituting $z = \frac{k}{n} - ik\Phi$, are equal to

$$
\mathcal{I}_{k,v} = \frac{1}{ki} \int_{\frac{k}{n}-\frac{i}{N}}^{\frac{k}{n}+\frac{i}{N}} z^{-\frac{1}{2}} \cdot e^{\frac{2\pi}{k}\left(nz+\frac{v}{z}\right)} dz. \tag{4.3.8}
$$

To compute these integrals, we proceed in the way described by Dragonette [Dra52, p. 492] and made more precise by Bringmann [Bri09, p. 3497]. In doing so, we enclose the path of integration by including the smaller arc of the circle through $\frac{k}{n} \pm \frac{i}{N}$ and tangent to the imaginary axis at 0, which we denote by Γ. If $z = x + iy$, then Γ is given by $x^2 + y^2 = wx$, with $w = \frac{k}{n} + \frac{n}{N^2 k}$. Using the fact that $2 > w > \frac{1}{k}$, $\mathrm{Re}(z) \leq \frac{k}{n}$ and $\mathrm{Re}\left(\frac{1}{z}\right) < k$ on the smaller arc, the integral along this arc is seen to be of order $O\big(n^{-\frac{1}{4}}\big)$. By Cauchy's Theorem, the path of integration in (4.3.8) can be further changed into the larger arc of Γ, hence

$$\mathcal{I}_{k,v} = \frac{1}{ki} \int_{\frac{k}{n} - \frac{i}{N}}^{\frac{k}{n} + \frac{i}{N}} z^{-\frac{1}{2}} \cdot e^{\frac{2\pi}{k}\left(nz + \frac{v}{z}\right)} dz + O\big(n^{-\frac{1}{8}}\big).$$

Making the substitution $t = \frac{2\pi v}{kz}$ gives

$$\mathcal{I}_{k,v} = \frac{2\pi}{k} \left(\frac{2\pi v}{k}\right)^{\frac{1}{2}} \frac{1}{2\pi i} \int_{\gamma - i\infty}^{\gamma + i\infty} t^{-\frac{3}{2}} \cdot e^{t + \frac{\alpha}{t}} dt + O\big(n^{-\frac{1}{8}}\big),$$

where $\gamma \in \mathbb{R}$ and $\alpha = \frac{4\pi^2 vn}{k^2}$. Using the Hankel integral formula, we compute (see, e.g, [Wat44, §3.7 and §6.2])

$$\frac{1}{2\pi i} \int_{\gamma - i\infty}^{\gamma + i\infty} t^{-\frac{3}{2}} \cdot e^{t + \frac{\alpha}{t}} dt = \frac{1}{\sqrt{\pi\alpha}} \cdot \sinh(2\sqrt{\alpha}),$$

hence

$$\mathcal{I}_{k,v} = \sqrt{\frac{2}{kn}} \cdot \sinh\left(\frac{4\pi\sqrt{vn}}{k}\right) + O\big(n^{-\frac{1}{8}}\big). \tag{4.3.9}$$

On applying (4.3.9) to (4.3.4) and (4.3.7) for $v = \frac{1}{16}$ and $v = \delta_{c,k,r}$ respectively, we have

$$\sum_2 + \sum_5 + \sum_6 = i\sqrt{\frac{2}{n}} \sum_{\substack{1 \leq k \leq \sqrt{n} \\ 2 \nmid k,\ c \mid k}} \frac{B_{a,c,k}(-n,0)}{\sqrt{k}} \sinh\left(\frac{\pi\sqrt{n}}{k}\right)$$

$$+ 2\sqrt{\frac{2}{n}} \sum_{\substack{1 \leq k \leq \sqrt{n} \\ c \nmid k,\ 2 \nmid k,\ c_1 \neq 4 \\ r \geq 0,\ \delta_{c,k,r} > 0}} \frac{D_{a,c,k}(-n, m_{a,c,k,r})}{\sqrt{k}}$$

$$\times \sinh\left(\frac{4\pi\sqrt{\delta_{c,k,r} n}}{k}\right) + O(n^{\varepsilon}).$$

4.3.2 Estimates for the sums $\sum_1, \sum_3$ and $\sum_4$

We show that these sums contribute only to the error term. Let us start our discussion with $\sum_1$, which equals

$$\sum_1 = i\tan\left(\frac{\pi a}{c}\right) \sum_{\substack{h,k \\ 2|k,\ c|k}} \frac{\omega_{h,k}^2}{\omega_{h,k/2}} (-1)^{k_1+1} \cot\left(\frac{\pi a h'}{c}\right) e^{-\frac{2\pi i a^2 h' k_1}{c} - \frac{2\pi i n h}{k}}$$
$$\times \int_{-\vartheta'_{h,k}}^{\vartheta''_{h,k}} z^{-\frac{1}{2}} e^{\frac{2\pi n z}{k}} \mathcal{O}\left(\frac{ah'}{c}; q_1\right) d\Phi.$$

Although not written down explicitly in [BL07], one can readily see, e.g., by inspecting the proof of Theorem 2.1 from [BL07, pp. 11–17], that

$$\begin{aligned}
\mathcal{O}\left(\frac{ah'}{c}; q_1\right) &= 4\sin^2\left(\frac{\pi a h'}{c}\right) \frac{\eta(2z_1)}{\eta(z_1^2)} \sum_{n\in\mathbb{Z}} \frac{(-1)^n q_1^{n^2+n}}{1 - 2q_1^n \cos\left(\frac{2\pi a h'}{c}\right) + q_1^{2n}} \\
&= \frac{\eta(2z_1)}{\eta(z_1^2)} \left(1 + 8\sin^2\left(\frac{\pi a h'}{c}\right) \sum_{n\geq 1} \frac{(-1)^n q_1^{n^2+n}}{1 - 2q_1^n \cos\left(\frac{2\pi a h'}{c}\right) + q_1^{2n}}\right) \\
&= \overline{P}(q_1) \left(1 + 8\sin^2\left(\frac{\pi a h'}{c}\right) \sum_{n\geq 1} \frac{(-1)^n q_1^{n^2+n}}{1 - 2q_1^n \cos\left(\frac{2\pi a h'}{c}\right) + q_1^{2n}}\right),
\end{aligned}$$

where we set $q_1 = e^{2\pi i z_1}$. We can rewrite this as

$$\mathcal{O}\left(\frac{ah'}{c}; q_1\right) = 1 + \sum_{r\geq 1} a_1(r) \cdot e^{\frac{2\pi i m_r h'}{k}} \cdot e^{-\frac{2\pi r}{kz}},$$

with $m_r \in \mathbb{Z}$ and the coefficients $a_1(r)$ being independent of k and h. Now the sum coming from $r \geq 1$ will go, as we have seen in the case of S_2, into an error term of the form $O(n^{\varepsilon})$, hence

$$\sum_1 = i\tan\left(\frac{\pi a}{c}\right) \sum_{\substack{h,k \\ 2|k,\ c|k}} \frac{\omega_{h,k}^2}{\omega_{h,k/2}} (-1)^{k_1+1} \cot\left(\frac{\pi a h'}{c}\right) e^{-\frac{2\pi i a^2 h' k_1}{c} - \frac{2\pi i n h}{k}}$$

$$\times \int_{-\vartheta'_{h,k}}^{\vartheta''_{h,k}} z^{-\frac{1}{2}} \cdot e^{\frac{2\pi n z}{k}} d\Phi + O(n^{\varepsilon}).$$

As for the sum coming from the constant term, let us denote it simply by S, on splitting the path of integration exactly as in the case of S_1 and working out the estimates in a similar manner, we obtain

$$S = i \sum_{c|k,\ 2|k} A_{a,c,k}(-n,0) \int_{-\frac{1}{kN}}^{\frac{1}{kN}} z^{-\frac{1}{2}} \cdot e^{\frac{2\pi n z}{k}} d\Phi + O(n^{\varepsilon}).$$

By applying part (iii) of Lemma 4.2 and arguing as in the case of S_{21} (except that now $m_r = 0$), we get

$$\begin{aligned}
&\left| i \sum_{c|k,\ 2|k} A_{a,c,k}(-n,0) \int_{-\frac{1}{kN}}^{\frac{1}{kN}} z^{-\frac{1}{2}} \cdot e^{\frac{2\pi n z}{k}} d\Phi \right| \\
&\ll \sum_{k} k^{\frac{1}{2}+\varepsilon} \cdot (24n+1,k)^{\frac{1}{2}} \cdot \frac{1}{k(N+1)} n^{\frac{1}{2}} k^{-\frac{1}{2}} \\
&\ll \sum_{k} k^{-1+\varepsilon} \cdot (24n+1,k)^{\frac{1}{2}} \\
&\ll \sum_{\substack{d|24n+1 \\ d\le N}} d^{-\frac{1}{2}+\varepsilon} \int_{1}^{N/d} x^{-1+\varepsilon} dx \\
&\ll \sum_{\substack{d|24n+1 \\ d\le N}} d^{-\frac{1}{2}} \cdot d^{\varepsilon} \cdot \left(\frac{N}{d}\right)^{\varepsilon} \\
&\ll n^{\varepsilon},
\end{aligned}$$

proving the claim.

We next deal with $\sum_3$ and $\sum_4$. The reader interested in writing down the computations explicitly will see that the two sums can be expressed as

$$\mathcal{O}\left(ah', \frac{\ell c}{c_1}, c; q_1\right) = \sum_{r\ge 0} a_3(r) \cdot e^{\frac{2\pi i m_r h'}{k}} \cdot e^{-\frac{\pi r}{k c_1^2 z}}$$

and

$$\mathcal{V}\left(\frac{ah'}{c}; q_1\right) = \sum_{r\ge 0} a_4(r) \cdot e^{\frac{2\pi i n_r h'}{k}} \cdot e^{-\frac{(2r+1)\pi}{4kz}},$$

where $m_r, n_r \in \mathbb{Z}$ and the coefficients $a_3(r)$ and $a_4(r)$ are independent of k and h. Since $r \geq 0$, it is obvious that both sums will be of order $O(n^\varepsilon)$, the argument being the same as for S_2.

4.3.3 Estimates for the sums $\sum_7$ and $\sum_8$

The estimation of the remaining sums $\sum_7$ and $\sum_8$ is not difficult and is inspired by Bringmann [Bri09, p. 3497]. Let us, however, elaborate a bit more here. Again, we split the path of integration as in (4.3.2). The resulting sums can each be bounded on the various intervals of integration by

$$\left(\sum_k k^{-1}\right)\left(\sum_h{}' 1\right) \cdot \sum_{\nu=0}^{k-1} k^{-1} N^{-1} z^{\frac{1}{2}} I_{a,c,k,v}(z) \ll N^{-1} n^{\frac{1}{2}} k^{-\frac{1}{2}} g_{a,c,k,\nu} \\ \ll k^\varepsilon \ll n^\varepsilon,$$

for any $\varepsilon > 0$. Here we have used, in turn, a trivial bound for the Kloosterman sums appearing in front of the integrals from $\sum_7$ and $\sum_8$, Lemma 4.1, and the easy estimate

$$\sum_{1 \leq \nu \leq k} g_{a,c,k,\nu} \ll \sum_{1 \leq \nu \leq 4ck} \frac{1}{\nu} \ll k^\varepsilon.$$

By this we conclude the rather lengthy proof of Theorem 4.1. $\square$

Proof of Corollary 4.1. Let us first assume $c > 2$. On combining Theorem 4.1 and identity (4.1.8), we obtain

$$\begin{aligned}\overline{N}(a,c,n) = \frac{1}{c}\sum_{j=1}^{c-1} \zeta_c^{-aj} \Bigg(& i\sqrt{\frac{2}{n}} \sum_{c'|k,\ 2\nmid k} \frac{B_{j',c',k}(-n,0)}{\sqrt{k}} \sinh\left(\frac{\pi\sqrt{n}}{k}\right) \\ & + 2\sqrt{\frac{2}{n}} \sum_{\substack{c'\nmid k,\ 2\nmid k,\ \tilde{c}\neq 4 \\ r\geq 0,\ \delta_{c',k,r}>0}} \frac{D_{j',c',k}(-n, m_{j',c',k,r})}{\sqrt{k}} \\ & \times \sinh\left(\frac{4\pi\sqrt{\delta_{c',k,r} n}}{k}\right)\Bigg) \\ & + \frac{\overline{p}(n)}{c} + O_c(n^\varepsilon), \end{aligned} \tag{4.3.10}$$

where $c' = \frac{c}{(c,j)}$, $j' = \frac{j}{(c,j)}$, $\widetilde{c} = \frac{c'}{(c',k)}$ and $\varepsilon > 0$ is arbitrary. As $n \to \infty$, we know that

$$\overline{p}(n) \sim \frac{1}{8n} e^{\pi\sqrt{n}}.$$

Since $c' \geq 2$ (as $j \leq c-1$), summation of the $B_{j',c',k}$ terms in (4.3.10) can only start from $k = 3$, meaning that the asymptotic contribution of these sums is (at most) of order $\sinh\left(\frac{\pi\sqrt{n}}{3}\right)$, thus dominated by $\overline{p}(n)$.

We claim that the same is true for the contribution coming from the $D_{j',c',k}$ sums. For this, note that, directly from the definition (4.1.4), it follows that $\delta_{c,k,r} \leq \frac{1}{16}$, therefore

$$\sinh\left(\frac{4\pi\sqrt{\delta_{c,k,r}n}}{k}\right) \leq \sinh\left(\frac{\pi\sqrt{n}}{k}\right).$$

If summation of the $D_{j',c',k}$ terms in (4.3.10) starts from $k = 3$ (note that $2 \nmid k$), then there is nothing to prove; so assume $k = 1$. It is an easy exercise to prove that equality above cannot be, in fact, obtained, and that, since $c_1 = c$, we have $\delta_{c,k,r} \leq \frac{1}{16} - \frac{1}{2c} + \frac{1}{c^2} = \frac{1}{16} - \frac{c-2}{2c^2}$, with $c \geq 3$, thereby proving the claim.

In case $c = 2$, we leave it as an exercise for the interested reader to prove that the coefficients of $\mathcal{O}(-1;q)$ are of order $O(n^{\varepsilon})$ and are thus dominated by $\overline{p}(n)$. This can be done by using the transformation behavior described in [BL07, Cor. 4.2] and carrying out estimates similar to those from the proof of Theorem 4.1. $\square$

4.4 A few inequalities

In this section we prove the inequalities stated in Theorems 4.2–4.4. We will elaborate more on Theorem 4.2, while only sketching the main steps in the proofs of Theorems 4.3 and 4.4, as the ideas are similar.

Before giving the proof of Theorem 4.2, we must establish some identities. The following is an easy generalization of [JZZ18, Lem. 3.1].

Lemma 4.3. *If $a \in \mathbb{N}$ is odd and $5 \nmid a$, then*

$$\mathcal{O}(\zeta_{10}^a; q) = \sum_{n=0}^{\infty} (\overline{N}(0,10,n) + \overline{N}(1,10,n) - \overline{N}(4,10,n) - \overline{N}(5,10,n)) q^n$$

$$+(\zeta_{10}^{2a}-\zeta_{10}^{3a})\sum_{n=0}^{\infty}(\overline{N}(1,10,n)+\overline{N}(2,10,n)-\overline{N}(3,10,n)-\overline{N}(4,10,n))q^n.$$

Proof. Plugging $u=\zeta_{10}^a$ into (4.1.2) gives

$$\begin{aligned}\mathcal{O}(\zeta_{10}^a;q) &= \sum_{n=0}^{\infty}\overline{N}(m,n)\zeta_{10}^{am}q^n \\ &= \frac{(-q)_\infty}{(q)_\infty}\sum_{n=-\infty}^{\infty}\frac{(1-\zeta_{10}^a)(1-\zeta_{10}^{-a})(-1)^n q^{n^2+n}}{(1-\zeta_{10}^a q^n)(1-\zeta_{10}^{-a}q^n)}.\end{aligned} \tag{4.4.1}$$

Using the fact that $\overline{N}(a,m,n)=\overline{N}(m-a,m,n)$, which can be easily deduced from $\overline{N}(m,n)=\overline{N}(-m,n)$ (see, e.g., [Lov05, Prop. 1.1]), and noting that $\zeta_{10}^{5a}=-1$ and $1-\zeta_{10}^a+\zeta_{10}^{2a}-\zeta_{10}^{3a}+\zeta_{10}^{4a}=0$ for $5\nmid a$ odd, we can rewrite (4.4.1) as

$$\begin{aligned}\mathcal{O}(\zeta_{10}^a;q) &= \sum_{n=0}^{\infty}\sum_{\ell=0}^{9}\overline{N}(\ell,10,n)\zeta_{10}^{\ell a}q^n \\ &= \sum_{n=0}^{\infty}(\overline{N}(0,10,n)+(\zeta_{10}^a-\zeta_{10}^{4a})\overline{N}(1,10,n) \\ &\quad +(\zeta_{10}^{2a}-\zeta_{10}^{3a})\overline{N}(2,10,n)+(\zeta_{10}^{3a}-\zeta_{10}^{2a})\overline{N}(3,10,n) \\ &\quad +(\zeta_{10}^{4a}-\zeta_{10}^{a})\overline{N}(4,10,n)-\overline{N}(5,10,n))q^n \\ &= \sum_{n=0}^{\infty}(\overline{N}(0,10,n)+(1+\zeta_{10}^{2a}-\zeta_{10}^{3a})\overline{N}(1,10,n) \\ &\quad +(\zeta_{10}^{2a}-\zeta_{10}^{3a})\overline{N}(2,10,n)+(\zeta_{10}^{3a}-\zeta_{10}^{2a})\overline{N}(3,10,n) \\ &\quad -(1+\zeta_{10}^{2a}-\zeta_{10}^{3a})\overline{N}(4,10,n)-\overline{N}(5,10,n))q^n \\ &= \sum_{n=0}^{\infty}(\overline{N}(0,10,n)+\overline{N}(1,10,n)+\overline{N}(4,10,n)-\overline{N}(5,10,n)q^n \\ &\quad +(\zeta_{10}^{2a}-\zeta_{10}^{3a})\sum_{n=0}^{\infty}(\overline{N}(1,10,n)+\overline{N}(2,10,n))q^n \\ &\quad -(\zeta_{10}^{2a}-\zeta_{10}^{3a})\sum_{n=0}^{\infty}(\overline{N}(3,10,n)+\overline{N}(4,10,n))q^n,\end{aligned}$$

which concludes the proof. $\square$

In a similar fashion, we have the following result. For a proof of the case $a = 1$, see [JZZ18, Lem. 2.1].

Lemma 4.4. *If $a \in \mathbb{N}$ is odd and $3 \nmid a$, then*

$$\mathcal{O}(\zeta_6^a; q) = \sum_{n=0}^{\infty} (\overline{N}(0,6,n) + \overline{N}(1,6,n) - \overline{N}(2,6,n) - \overline{N}(3,6,n))q^n.$$

Proof of Theorem 4.2. Setting $a = 1$ and $a = 3$ in Lemma 4.3, we obtain

$$\begin{aligned}\mathcal{O}(\zeta_{10}; q) &= \sum_{n=0}^{\infty} (\overline{N}(0,10,n)+\overline{N}(1,10,n)-\overline{N}(4,10,n)-\overline{N}(5,10,n))q^n \\ &+(\zeta_{10}^2-\zeta_{10}^3) \sum_{n=0}^{\infty} (\overline{N}(1,10,n)+\overline{N}(2,10,n)-\overline{N}(3,10,n)-\overline{N}(4,10,n))q^n,\end{aligned} \tag{4.4.2}$$

and

$$\begin{aligned}\mathcal{O}(\zeta_{10}^3; q) &= \sum_{n=0}^{\infty} (\overline{N}(0,10,n)+\overline{N}(1,10,n)-\overline{N}(4,10,n)-\overline{N}(5,10,n))q^n \\ &+(\zeta_{10}^4-\zeta_{10}) \sum_{n=0}^{\infty} (\overline{N}(1,10,n)+\overline{N}(2,10,n)-\overline{N}(3,10,n)-\overline{N}(4,10,n))q^n.\end{aligned} \tag{4.4.3}$$

Subtracting (4.4.3) from (4.4.2) yields

$$\begin{aligned}&\sum_{n=0}^{\infty} (\overline{N}(1,10,n) + \overline{N}(2,10,n) - \overline{N}(3,10,n) - \overline{N}(4,10,n))q^n \\ &= \frac{\mathcal{O}(\zeta_{10}; q) - \mathcal{O}(\zeta_{10}^3; q)}{\zeta_{10} + \zeta_{10}^2 - \zeta_{10}^3 - \zeta_{10}^4} \\ &= \frac{\mathcal{O}(\zeta_{10}; q) - \mathcal{O}(\zeta_{10}^3; q)}{1 + 4\cos\left(\frac{2\pi}{5}\right)},\end{aligned}$$

thus proving (4.1.12) is equivalent to showing that, for $n \geq 0$,

$$A\left(\frac{1}{10}; n\right) \geq A\left(\frac{3}{10}; n\right).$$

For $a = 1$, $c = 10$ we have $m_{1,10,1,0} = 0$ and $\delta_{c,k,r} > 0$ if and only if $r = 0$, in which case $\delta_{c,k,r} = \frac{9}{400}$, hence

$$A\left(\frac{1}{10};n\right) = 2\sqrt{\frac{2}{n}} \sum_{\substack{1\leq k\leq\sqrt{n} \\ k\equiv 1,9 \pmod{10}}} \frac{D_{a,c,k}(-n, m_{1,10,k,0})}{\sqrt{k}} \sinh\left(\frac{3\pi\sqrt{n}}{5k}\right) + O_c(n^{\varepsilon}), \tag{4.4.4}$$

whereas, for $a = 3$ and $c = 10$, we have $\delta_{c,k,r} > 0$ if and only if $r = 0$, in which case $\delta_{c,k,r} = \frac{9}{400}$, thus

$$A\left(\frac{3}{10};n\right) = 2\sqrt{\frac{2}{n}} \sum_{\substack{1\leq k\leq\sqrt{n} \\ k\equiv 3,7 \pmod{10}}} \frac{D_{a,c,k}(-n, m_{3,10,k,0})}{\sqrt{k}} \sinh\left(\frac{3\pi\sqrt{n}}{5k}\right) + O_c(n^{\varepsilon}). \tag{4.4.5}$$

We further compute

$$D_{1,10,1}(-n, 0) = \frac{1}{\sqrt{2}} \tan\left(\frac{\pi}{10}\right),$$

and so the term corresponding to $k = 1$ in the sum from (4.4.4) is given by

$$\frac{2}{\sqrt{n}} \tan\left(\frac{\pi}{10}\right) \sinh\left(\frac{3\pi\sqrt{n}}{5}\right).$$

Using a trivial bound for the Kloosterman sum from (4.4.5) and taking into account the contributions coming from the various error terms involved, estimates which we make explicit at the end of this section, we see that this term is dominant for $n \geq 1030$, hence

$$A\left(\frac{1}{10};n\right) \geq A\left(\frac{3}{10};n\right)$$

for $n \geq 1030$. In Mathematica we see that the inequality is true for $n < 1030$ as well.

To prove (4.1.13), we set $a = 1$ and $a = 3$ in Lemma 4.3 and obtain

$$\mathcal{O}(\zeta_{10};q) = \sum_{n=0}^{\infty} \mathcal{S}_1(n)q^n + (1 + \zeta_{10}^2 - \zeta_{10}^3) \sum_{n=0}^{\infty} \mathcal{S}_2(n)q^n, \tag{4.4.6}$$

where

$$\mathcal{S}_1(n) := \overline{N}(0,10,n) + \overline{N}(3,10,n) - \overline{N}(2,10,n) - \overline{N}(5,10,n),$$
$$\mathcal{S}_2(n) := \overline{N}(1,10,n) + \overline{N}(2,10,n) - \overline{N}(3,10,n) - \overline{N}(4,10,n)$$

and

$$\mathcal{O}(\zeta_{10}^3; q) = \sum_{n=0}^{\infty} \mathcal{S}_3(n) q^n + (1 - \zeta_{10} + \zeta_{10}^4) \sum_{n=0}^{\infty} \mathcal{S}_4(n) q^n, \tag{4.4.7}$$

where

$$\mathcal{S}_3(n) := \overline{N}(0,10,n) + \overline{N}(3,10,n) - \overline{N}(2,10,n) - \overline{N}(5,10,n),$$
$$\mathcal{S}_4(n) := \overline{N}(1,10,n) + \overline{N}(2,10,n) - \overline{N}(3,10,n) - \overline{N}(4,10,n).$$

Combining (4.4.6) and (4.4.7) and setting $\alpha = \frac{1+\zeta_{10}^2-\zeta_{10}^3}{1-\zeta_{10}+\zeta_{10}^4}$, we obtain

$$\mathcal{O}(\zeta_{10}; q) - \alpha \mathcal{O}(\zeta_{10}^3; q) = (1-\alpha) \sum_{n=0}^{\infty} \mathcal{S}_5(n) q^n,$$

with

$$\mathcal{S}_5(n) := \overline{N}(0,10,n) + \overline{N}(3,10,n) - \overline{N}(2,10,n) - \overline{N}(5,10,n),$$

hence, as it is easy to see that $\alpha = -(1 + 2\cos(\pi/5))$, proving the claim amounts to showing

$$A\left(\frac{1}{10}; n\right) + \left(1 + 2\cos\frac{\pi}{5}\right) A\left(\frac{3}{10}; n\right) \geq 0$$

for all $n \geq 0$, which follows from the estimates used for proving (4.1.12). The proof of (4.1.14) follows simply on adding the inequalities (4.1.12) and (4.1.13). □

We can also sketch now the proofs of Theorems 4.3 and 4.4.

Proof of Theorem 4.3 (Sketch). Reasoning along the same lines, on setting $a = 1$ in Lemma 4.4 and recalling (4.1.3), the claim is equivalent to proving

$$A\left(\frac{1}{6}; n\right) \geq 0$$

for $n \geq 0$. It is easy to see that, for $a = 1$ and $c = 6$, we have $m_{1,6,1,0} = 0$ and $\delta_{c,k,r} > 0$ if and only if $r = 0$, in which case $\delta_{c,k,r} = \frac{1}{144}$, thus the dominant term of

$$A\left(\frac{1}{6}; n\right) = 2\sqrt{\frac{2}{n}} \sum_{\substack{1 \leq k \leq \sqrt{n} \\ k \equiv 1,5 \pmod 6}} \frac{D_{1,6,k}(-n, m_{1,6,k,0})}{\sqrt{k}} \cdot \sinh\left(\frac{3\pi\sqrt{n}}{5k}\right)$$

is given by

$$\frac{2}{\sqrt{n}} \tan\left(\frac{\pi}{6}\right) \sinh\left(\frac{\pi\sqrt{n}}{3}\right).$$

By working out similar bounds as in the proof of (4.1.12) and checking numerically for the small values of n, the proof of (4.1.15) is concluded.

The inequalities (4.1.16)–(4.1.18) are equivalent to (4.1.19)–(4.1.21). The proof relies on the identity

$$\begin{aligned}
\mathcal{O}(\zeta_6^2; q) &= \sum_{n=0}^{\infty} (\overline{N}(0,6,n) - \overline{N}(1,6,n) - \overline{N}(2,6,n) + \overline{N}(3,6,n))q^n \\
&= \sum_{n=0}^{\infty} (\overline{N}(0,6,n) - \overline{N}(1,6,n) - \overline{N}(4,6,n) + \overline{N}(3,6,n))q^n \\
&= \sum_{n=0}^{\infty} (\overline{N}(0,3,n) - \overline{N}(1,3,n))q^n
\end{aligned}$$

and details are left to the interested reader. The fact that $\overline{N}(1,3,n) = \overline{N}(2,3,n)$ follows easily from adding the identities $\overline{N}(1,6,n) = \overline{N}(5,6,n)$ and $\overline{N}(2,6,n) = \overline{N}(4,6,n)$. $\square$

Proof of Theorem 4.4 (Sketch). By using either [WZ20, Lem. 5.1] (on identifying the notation $\overline{R}(u;q) = \mathcal{O}(u;q)$) or identity (4.1.8) (which, in combination with (4.1.2), amounts to the same result), we have

$$\sum_{n=0}^{\infty} \overline{N}(0,6,n)q^n = \frac{1}{6}(\mathcal{O}(1;q) + 2\mathcal{O}(\zeta_6;q) + 2\mathcal{O}(\zeta_6^2;q) + \mathcal{O}(\zeta_6^3;q)), \tag{4.4.8}$$

$$\sum_{n=0}^{\infty} \overline{N}(1,6,n)q^n = \frac{1}{6}(\mathcal{O}(1;q) + \mathcal{O}(\zeta_6;q) - \mathcal{O}(\zeta_6^2;q) - \mathcal{O}(\zeta_6^3;q)), \tag{4.4.9}$$

$$\sum_{n=0}^{\infty} \overline{N}(2,6,n)q^n = \frac{1}{6}(\mathcal{O}(1;q) - \mathcal{O}(\zeta_6;q) - \mathcal{O}(\zeta_6^2;q) + \mathcal{O}(\zeta_6^3;q)), \tag{4.4.10}$$

$$\sum_{n=0}^{\infty} \overline{N}(3,6,n)q^n = \frac{1}{6}(\mathcal{O}(1;q) - 2\mathcal{O}(\zeta_6;q) + 2\mathcal{O}(\zeta_6^2;q) - \mathcal{O}(\zeta_6^3;q)). \tag{4.4.11}$$

In light of Remark 4.8, to prove the inequalities (4.1.22)–(4.1.24) it suffices to show that, for $n \geq 11$,

$$\overline{N}(1,6,n) \geq \overline{N}(2,6,n),$$

$$\overline{N}(0,6,3n) \geq \overline{N}(1,6,3n), \quad \overline{N}(0,6,3n+1) \geq \overline{N}(1,6,3n+1),$$
$$\overline{N}(0,6,3n+2) \leq \overline{N}(1,6,3n+2).$$

Therefore, on combining (4.4.9) and (4.4.10), the first inequality above is equivalent to

$$A\left(\frac{1}{6};n\right) \geq 0, \tag{4.4.12}$$

whereas, for $i = 0, 1$, the second and third are equivalent, on combining (4.4.8) and (4.4.9), to

$$A\left(\frac{1}{6};3n+i\right) + 3A\left(\frac{1}{3};3n+i\right) \geq 0, \tag{4.4.13}$$

$$A\left(\frac{1}{6};3n+2\right) + 3A\left(\frac{1}{3};3n+2\right) \leq 0. \tag{4.4.14}$$

Again, the attentive reader might wonder what happens with the term $\mathcal{O}(-1;q)$ (coming from the case $j = c/2$ in (4.1.8)), to which Theorem 4.5 does not apply, as its statement is formulated under the assumption $c > 2$. However, while working out the transformations found by Bringmann and Lovejoy in this case, see [BL07, Cor. 4.2], and doing the same estimates as in the proof of Theorem 4.1, one can easily infer that the sums involved are of order $O(n^{\varepsilon})$. Therefore, as n grows large, we only need to prove (4.4.12)–(4.4.14), which follow immediately from Theorem 4.1. Again, explicit bounds can be provided just as described in the next subsection, and a numerical check for the small values of n concludes the proof. $\square$

4.4.1 Some explicit computations

As we have mentioned earlier, we will now fill in the missing details from the proof of (4.1.12), by explaining how to bound the different sums and error terms appearing in (4.4.4) and (4.4.5). The same arguments apply for all the other inequalities. We have already seen that

$$A\left(\frac{1}{10};n\right)=2\sqrt{\frac{2}{n}}\sum_{\substack{1\le k\le\sqrt{n}\\ k\equiv 1,9 \pmod{10}}}\frac{D_{a,c,k}(-n,m_{1,10,k,0})}{\sqrt{k}}\sinh\left(\frac{3\pi\sqrt{n}}{5k}\right),$$

and that the term corresponding to $k=1$ in (4.4.4) equals

$$\frac{2}{\sqrt{n}}\tan\left(\frac{\pi}{10}\right)\sinh\left(\frac{3\pi\sqrt{n}}{5}\right). \tag{4.4.15}$$

By using a trivial bound for the Kloosterman sums involved, the remaining terms can be estimated against

$$\sum_{2\le k\le \frac{N-1}{10}}\frac{4\sqrt{k}}{\sqrt{n}}\sinh\left(\frac{3\pi\sqrt{n}}{5(10k+1)}\right)+\sum_{1\le k\le \frac{N-9}{10}}\frac{4\sqrt{k}}{\sqrt{n}}\sinh\left(\frac{3\pi\sqrt{n}}{5(10k+9)}\right), \tag{4.4.16}$$

and the contribution coming from $\mathcal{U}\left(h',\frac{\ell}{10},10;q_1\right)$ is less than

$$\sqrt{2}e^{2\pi}\sum_{r=1}^{\infty}|a_5(r)|e^{-\frac{\pi r}{50}}\sum_{\substack{1\le k\le N\\ k\equiv 1,9 \pmod{10}}}k^{-\frac{1}{2}}$$
$$+\sqrt{2}e^{2\pi}\sum_{r=1}^{\infty}|b_5(r)|e^{-\frac{\pi r}{50}}\sum_{\substack{1\le k\le N\\ k\equiv 1,9 \pmod{10}}}k^{-\frac{1}{2}}. \tag{4.4.17}$$

Making the integration path in (4.3.4) symmetric gives an error of order

$$2e^{2\pi+\frac{\pi}{8}}\cdot n^{-\frac{1}{2}}\sum_{\substack{1\le k\le N\\ k\equiv 1,9 \pmod{10}}}k^{\frac{1}{2}}, \tag{4.4.18}$$

and integrating along the small arc of Γ gives an error not bigger than

$$8\pi e^{2\pi+\frac{\pi}{16}}\cdot n^{-\frac{3}{4}}\sum_{\substack{1\le k\le N\\ k\equiv 1,9 \pmod{10}}}k. \tag{4.4.19}$$

The sums $\sum_2, \sum_4$ and $\sum_6$ do not contribute in the case $c = 10$, whereas $\sum_1, \sum_3$ can be treated simultaneously. The contribution coming from $\mathcal{O}\left(\frac{h'}{10}; q_1\right)$ can be estimated against

$$\frac{2e^{2\pi}}{\sqrt{10}} \sum_{1 \le k \le \frac{N}{10}} k^{-\frac{1}{2}} + \frac{2e^{2\pi}}{\sqrt{10}} \sum_{r=1}^{\infty} |a_1(r)| e^{-\pi r} \sum_{1 \le k \le \frac{N}{10}} k^{-\frac{1}{2}}, \tag{4.4.20}$$

and that coming from $\mathcal{O}\left(h', \frac{\ell}{2}, 10; q_1\right)$ against

$$2e^{2\pi} \sum_{r=1}^{\infty} |a_3(r)| e^{-\frac{\pi r}{50}} \sum_{\substack{1 \le k \le N \\ k \equiv 1,9 \ (\mathrm{mod}\ 10)}} k^{-\frac{1}{2}}. \tag{4.4.21}$$

Applying (4.3.3) to $|a_3(r)|, |a_5(r)|, |b_5(r)|$ gives $\sum_{r=1}^{\infty} |a_3(r)| e^{-\frac{\pi r}{50}} < 1.17944$, $\sum_{r=1}^{\infty} |a_3(r)| e^{-\frac{\pi r}{50}} < 4.01014 \cdot 10^{19}$, and similar bounds for $a_5(r), b_5(r)$. Making the estimates in Lemma 4.1 explicit, we have

$$\sum_7 \le \frac{2e^{2\pi}\sqrt{\pi}}{5} \sum_{2 \nmid k} k^{-\frac{3}{2}} \sum_{\nu=1}^{k} \left(\min\left\{ \left| \frac{\nu}{k} - \frac{1}{4k} + \frac{1}{10} \right|, \left| \frac{\nu}{k} - \frac{1}{4k} - \frac{1}{10} \right| \right\} \right)^{-1}, \tag{4.4.22}$$

$$\sum_8 \le \frac{2e^{2\pi}\sqrt{2\pi}}{5} \left(\sum_{2 \mid k,\ 5 \nmid k} k^{-\frac{3}{2}} \sum_{\nu=1}^{k} \left(\min\left\{ \left| \frac{\nu}{k} + \frac{1}{10} \right|, \left| \frac{\nu}{k} - \frac{1}{10} \right| \right\} \right)^{-1} + \frac{1}{10\sqrt{10}} \sum_{1 \le k \le \frac{N}{10}} k^{-\frac{1}{2}} \right). \tag{4.4.23}$$

For $a = 3$ and $c = 10$, we proceed just like in (4.4.16) to get

$$\sum_{1 \le k \le \frac{N-3}{10}} \frac{4\sqrt{k}}{\sqrt{n}} \sinh\left(\frac{3\pi\sqrt{n}}{5(10k+3)} \right) + \sum_{1 \le k \le \frac{N-7}{10}} \frac{4\sqrt{k}}{\sqrt{n}} \sinh\left(\frac{3\pi\sqrt{n}}{5(10k+7)} \right)$$

as an overall bound for the main contribution in (4.4.5) and we use the same estimates from (4.4.17)–(4.4.23) on changing whatever necessary, e.g., the sums will now run over $k \equiv 3 \pmod{10}$ and $k \equiv 7 \pmod{10}$. Putting all estimates together, we see that the term in (4.4.15) is dominant for $n \ge 1030$. The inequality (4.1.12) can be checked numerically in Mathematica to hold true also for $n < 1030$.

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References

[AIS17] E. Alwaise, E. Iannuzzi and H. Swisher, A proof of some conjectures of Mao on partition rank inequalities, *Ramanujan J.* **43** (2017), no. 3, 633–648.

[And66] G. E. Andrews, On the theorems of Watson and Dragonette for Ramanujan's mock theta functions, *Amer. J. Math.* **88** (1966), no. 2, 454–490.

[Apo76] T. M. Apostol, *Introduction to analytic number theory*, Undergraduate Texts in Mathematics, Springer-Verlag, New York-Heidelberg, 1976.

[Apo90] T. M. Apostol, *Modular functions and Dirichlet series in number theory*, Second edition, Graduate Texts in Mathematics, 41. Springer-Verlag, New York, 1990.

[AS54] A. O. L. Atkin and P. Swinnerton-Dyer, Some properties of partitions, *Proc. London Math. Soc. (3)* **4** (1954), 84–106.

[Bri09] K. Bringmann, Asymptotics for rank partition functions, *Trans. Amer. Math. Soc.* **361** (2009), no. 7, 3483–3500.

[BL07] K. Bringmann and J. Lovejoy, Dyson's rank, overpartitions, and weak Maass Forms, *Int. Math. Res. Not. IMRN* (2007), no. 19, Art. ID rnm063, 34 pp.

[BO10] K. Bringmann and K. Ono, Dyson's ranks and Maass forms, *Ann. of Math. (2)* **171** (2010), no. 1, 419–449.

[CL04] S. Corteel and J. Lovejoy, Overpartitions, *Trans. Amer. Math. Soc.* **356** (2004), no. 4, 1623–1635.

[Dew10] M. Dewar, The nonholomorphic parts of certain weak Maass forms, *J. Number Theory* **130** (2010), no. 3, 559–573.

[Dra52] L. A. Dragonette, Some asymptotic formulae for the mock theta series of Ramanujan, *Trans. Amer. Math. Soc.* **72** (1952), 474–500.

[Dys44] F. J. Dyson, Some guesses in the theory of partitions, *Eureka* (1944), no. 8, 10–15.

[HR18] G. H. Hardy and S. Ramanujan, Asymptotic formulae for the distribution of integers of various types, *Proc. London Math. Soc. Series (2)* **16** (1918), 112–132.

[Jen16] C. Jennings-Shaffer, Overpartition rank differences modulo 7 by Maass forms, *J. Number Theory* **163** (2016), 331–358.

[JR] C. Jennings-Shaffer and D. Reihill, Asymptotic formulas related to the M_2-rank of partitions without repeated odd parts, available as preprint at `https://arxiv.org/abs/1805.11319`.

[JZZ18] K. Q. Ji, H. W. J. Zhang and A. X. H. Zhao, Ranks of overpartitions modulo 6 and 10, *J. Numbery Theory* **184** (2018), 235–269.

[Lov05] J. Lovejoy, Rank and conjugation for the Frobenius representation of an overpartition, *Ann. Comb.* **9** (2005), no. 3, 321–334.

[LO08] J. Lovejoy and R. Osburn, Rank differences for overpartitions, *Q. J. Math.* **59** (2008), no. 2, 257–273.

[Mal20] J. Males, Asymptotic equidistribution and convexity for partition ranks, *Ramanujan J.* (2020), available online at `https://doi.org/10.1007/s11139-019-00202-8`.

[Mao13] R. Mao, Ranks of partitions modulo 10, *J. Number Theory* **133** (2013), no. 11, 3678–3702.

[Mao15] R. Mao, The M_2-rank of partitions without repeated odd parts modulo 6 and 10, *Ramanujan J.* **37** (2015), no. 2, 391–419.

[Pak06] I. Pak, Partition bijections, a survey, *Ramanujan J.* **12** (2006), no. 1, 5–75.

[Rad38] H. Rademacher, The Fourier coefficients of the modular invariant $J(\tau)$, *Amer. J. Math.* **60** (1938), no. 2, 501–512.

[Ram21] S. Ramanujan, Congruence properties of partitions, *Math. Z.* **9** (1921), 147–153. University Press, 1927, 355 pp.

[Wat44] G. N. Watson, *A treatise on the theory of Bessel functions*, Second edition, Cambridge University Press, Cambridge, 1944.

[WZ20] B. Wei and H. W. J. Zhang, Generalized Lambert series identities and applications in rank differences, *Proc. Amer. Math. Soc.* (2020), available online at `https://doi.org/10.1090/proc/15075`, 17 pp.

[Zuc39] H. S. Zuckerman, On the coefficients of certain modular forms belonging to subgroups of the modular group, *Trans. Amer. Math. Soc.* **45** (1939), no. 2, pp. 298–321.

Chapter 5

Summary and discussion

In this chapter, we give a short recapitulation of the results of this thesis and we discuss possible future research directions. Throughout we use the same notation from Chapters 2–4.

5.1 Recapitulation of results

5.1.1 Partitions into powers

We started this thesis by studying a conjecture formulated by Bringmann and Mahlburg (2012). Based on the initial work done in their unpublished notes [BM], we managed to prove in Chapter 2 (cf. [Cio19]) that the conjecture holds true.

Conjecture 5.1 (Bringmann–Mahlburg, 2012)**.**

(i) *As $n \to \infty$, we have*

$$p_2(0,2,n) \sim p_2(1,2,n).$$

(ii) *We have*

$$\begin{cases} p_2(0,2,n) > p_2(1,2,n) & \text{if } n \text{ is even,} \\ p_2(0,2,n) < p_2(1,2,n) & \text{if } n \text{ is odd.} \end{cases}$$

Theorem 5.1. *Conjecture 5.1 is true as $n \to \infty$.*

We did so by combining Wright's transformations for

$$H_r(q) = \sum_{n=0}^{\infty} p_r(n) q^n,$$

the generating function of partitions into rth powers, with an application of the circle and the saddle-point method. An essential step of the proof was to show that the bound

$$\mathrm{Re}\left(\frac{\lambda_{a,b}}{\sqrt{\tau'}}\right) \leq \frac{\lambda_{0,1} - c}{\sqrt{y}} \tag{5.1.1}$$

holds for some constant $c > 0$, with

$$\lambda_{a,b} = \Lambda_{a,b} + \frac{1}{2\sqrt{2}}\Lambda_{\frac{8a}{(b,8)},\frac{b}{(b,8)}} - \Lambda_{\frac{4a}{(b,4)},\frac{b}{(b,4)}},$$

and

$$\Lambda_{a,b} = \frac{\Gamma\left(\frac{3}{2}\right)}{b}\sum_{m=1}^{\infty}\frac{S_2(ma,b)}{m^{\frac{3}{2}}} = \frac{\Gamma\left(\frac{3}{2}\right)}{b}\sum_{m=1}^{\infty}\sum_{n=1}^{b}\exp\left(\frac{2\pi i amn^2}{b}\right)$$

an infinite sum of quadratic Gauss sums. In order to prove (5.1.1) we needed several ingredients. First, we used explicit formulas for $S_2(a,b)$ to turn this sum into something more tractable. By using a bound on reciprocal sums of divisors, originally posed by Tóth as problem E3432 in the *American Mathematical Monthly* in 1991, bound which was proven to be sharp by Wei and Yang [WY97], and Wu and Yang [WY99], we managed to show that inequality (5.1.1) holds for $b \geq 527$. The smaller values of b were shown to satisfy the same bound by rewriting our expression for $\lambda_{a,b}$ in terms of Hurwitz zeta functions, which, in turn, made a numerical check finite and computationally possible.

As suggested by Kathrin Bringmann and Ef Sofos, we studied [Cio] the same problem for partitions into any powers instead of squares. As presented in detail in Chapter 3, we proved that the same alternating inequalities and equidistribution behavior hold.

Theorem 5.2. *For any $r \geq 2$ and n sufficiently large, we have*

$$p_r(0,2,n) \sim p_r(1,2,n) \sim \frac{p_r(n)}{2} \tag{5.1.2}$$

and

$$\begin{cases} p_r(0,2,n) > p_r(1,2,n) & \text{if } n \text{ is even,} \\ p_r(0,2,n) < p_r(1,2,n) & \text{if } n \text{ is odd.} \end{cases} \tag{5.1.3}$$

In order to prove Theorem 5.2 we had to find a different argument than that used to show (5.1.1). On one hand, the reader is perhaps aware of the fact that explicit formulas for the general rth order Gauss sums

$$S_r(a,b) = \sum_{n=1}^{b} \exp\left(\frac{2\pi i a n^r}{b}\right),$$

with which we need to deal now, are not available for values $r > 2$. In fact, studying estimates for Gauss sums has proven to be a difficult and long studied problem; see, for instance, the contributions of Montgomery, Vaughan, Wooley, Heath-Brown and Konyagin [HK00, MVW95]. On the other hand, applying the same philosophy as in the case $r = 2$ would have required a computer check which was not feasible for all values $r \geq 2$. However, through correspondence with Igor Shaprlinski, the author got to learn about a relatively recent bound due to Banks and Shparlinski [BS15] which says that

$$|S_r(a,b)| \leq \mathcal{A} b^{1-\frac{1}{r}}, \tag{5.1.4}$$

where $\mathcal{A} = 4.709236\ldots$ is Stechkin's constant. A careful inspection of the proof of Lemma 2.2 from Chapter 2 ([Cio19, Lem. 2]) shows, however, that we need an inequality such as (5.1.1) to hold only as $y \to 0$. A modification in the proof of Lemma 2.2, combined with the inequality (5.1.4), allows us then to generalize to any $r \geq 2$ and conclude the proof of Theorem 5.2. This is explained in the commentary at the end of Chapter 3.4.5.

5.1.2 Overpartition ranks

In Chapter 4 we moved our attention to overpartitions and studied their rank generating functions. Using the circle method and the modular transformations found by Bringmann and Lovejoy [BL07] for the generating function

$$\mathcal{O}(u;q) = \sum_{m=-\infty}^{\infty} \sum_{n=0}^{\infty} \overline{N}(m,n) u^m q^n,$$

we computed asymptotics for the coefficients $A\left(\frac{a}{c};q\right)$ and we deduced the following.

Corollary 5.1. *If $c \geq 2$, then for any $0 \leq a \leq c-1$ we have, as $n \to \infty$,*

$$\overline{N}(a,c,n) \sim \frac{\overline{p}(n)}{c} \sim \frac{1}{c} \cdot \frac{e^{\pi\sqrt{n}}}{8n}.$$

It is perhaps worth mentioning that Corollary 5.1 was rather an unexpected byproduct of Theorem 4.1, as the original goal of [Cio20] was to prove the rank inequalities conjectured by Ji, Wei, Zhang and Zhao [JZZ18, WZ20]. Indeed, this was accomplished by using the asymptotics found for $A\left(\frac{a}{c};q\right)$.

Conjecture 5.2 (Ji–Zhang–Zhao, 2018)**.**

(i) *For $n \geq 0$ and $1 \leq i \leq 4$, we have*

$$\overline{N}(0,10,5n+i)+\overline{N}(1,10,5n+i) \geq \overline{N}(4,10,5n+i)+\overline{N}(5,10,5n+i).$$

(ii) *For $n \geq 0$, we have*

$$\overline{N}(1,10,n)+\overline{N}(2,10,n) \geq \overline{N}(3,10,n)+\overline{N}(4,10,n).$$

Conjecture 5.3 (Wei–Zhang, 2018)**.** *For $n \geq 11$, we have*

$$\overline{N}(0,6,3n) \geq \overline{N}(1,6,3n) = \overline{N}(3,6,3n) \geq \overline{N}(2,6,3n),$$

$$\overline{N}(0,6,3n+1) \geq \overline{N}(1,6,3n+1) = \overline{N}(3,6,3n+1) \geq \overline{N}(2,6,3n+1),$$

$$\overline{N}(1,6,3n+2) \geq \overline{N}(2,6,3n+2) \geq \overline{N}(0,6,3n+2) \geq \overline{N}(3,6,3n+2).$$

Theorem 5.3. *Conjectures 5.2 and 5.3 are true.*

5.2 Further questions

While answering the open problems that motivated this thesis, our research also led to many new and, we believe, interesting questions. We discuss some of them in what follows.

5.2.1 Partitions into powers

It would be of interest to see how far do results like Theorem 5.2 go. It was suggested by Kathrin Bringmann to study what happens, for instance, in the case of partitions into triangular numbers, or for the distribution of partitions into powers modulo 3. Another conjecture formulated in the unpublished manuscript of Bringmann and Mahlburg [BM] stated the following.

Conjecture 5.4. *For all n, we have*

$$p_2(0,3,n) \sim p_2(1,3,n) \sim p_2(2,3,n) \sim \frac{p_2(n)}{3} \tag{5.2.1}$$

and

$$\begin{cases} p_2(0,3,n) > p_2(1,3,n) > p_2(2,3,n) & \text{if } n \equiv 0 \pmod 3, \\ p_2(1,3,n) > p_2(2,3,n) > p_2(0,3,n) & \text{if } n \equiv 1 \pmod 3, \\ p_2(2,3,n) > p_2(0,3,n) > p_2(1,3,n) & \text{if } n \equiv 2 \pmod 3. \end{cases} \tag{5.2.2}$$

Indeed, in a very recent preprint [Zho], Zhou proved the equidistribution of partitions into parts that are certain polynomial functions. Under several conditions imposed on m, Zhou showed that, as $n \to \infty$, for non-constant polynomials $f \in \mathbb{Q}[x]$ taking as values (coprime) positive integers, we have

$$p_f(a,m,n) \sim \frac{p_f(n)}{m}.$$

A quick check of the assumptions formulated in [Zho, Cor. 1.2] shows that powers and triangular numbers fall into the category of polynomial functions studied by Zhou. As particular cases of his result, (5.1.2) and (5.2.1) hold true as $n \to \infty$.

However, Zhou's arguments do not shed any light on why, or if alternating (or circular) inequalities such as (5.1.3) or (5.2.2) should hold in general. This motivates the following open questions.

Problem 5.1. Do alternating or cyclic inequalities like (5.1.3) and (5.2.2) hold for partitions into polynomial functions satisfying properties similar to (or less restrictive than) those given by Zhou?

Problem 5.2. If yes, do the inequalities depend on the parity of n in case we look at partitions with an even and with an odd number of parts or, more generally, on the residue class of n modulo m?

Problem 5.3. In particular, do cyclic inequalities hold between the various $p_r(a,m,n)$ quantities and, if so, do they depend on the residue class of n modulo m?

Finally, we note that, although we could not directly apply Meinardus' Theorem to our problem, we did end up, however, with the same two estimates as he did. Thus, we obtained the asymptotic formulas that his theorem would have heuristically predicted, which leads to a natural question.

Problem 5.4. Can Meinardus' Theorem be strengthened in any way so as to deal with a class of infinite product generating functions more general than that studied in [Mei54]?

5.2.2 Overpartition ranks

Using a different method, Males recently obtained the same equidistribution result as in Corollary 5.1 for ranks of partitions and proved a convexity conjecture formulated by Hou and Jagadeesan [HJ18], see [Mal20]. In doing so, Males employed the level ℓ Appell functions introduced by Zwegers [Zwe19] as

$$A_\ell(u,v;\tau) := e^{\pi i \ell u} \sum_{n\in\mathbb{Z}} \frac{(-1)^{\ell n} q^{\frac{\ell n(n+1)}{2}} e^{2\pi i n v}}{1-e^{2\pi i u} q^n}.$$

As suggested by Kathrin Bringmann, it seems very likely that the same tool can be used, for instance, to recover the modular transformations from [BL07, Th. 2.1].

Problem 5.5. Can higher level Appell functions be used to recover, or to give shorter proofs for the transformation laws of $\mathcal{O}\left(\frac{a}{c};q\right)$?

Problem 5.6. Can higher level Appell functions be used to give alternative proofs for inequalities such as those formulated in Conjectures 5.2 and 5.3?

Problem 5.7. Do inequalities for other moduli, such as those given in [JR, pp. 40–41] for M_2-ranks of partitions without repeated odd parts, hold for overpartition ranks?

Bibliography

[AIS17] E. Alwaise, E. Iannuzzi and H. Swisher, A proof of some conjectures of Mao on partition rank inequalities, *Ramanujan J.* **43** (2017), no. 3, 633–648.

[And66] G. E. Andrews, On the theorems of Watson and Dragonette for Ramanujan's mock theta functions, *Amer. J. Math.* **88** (1966), no. 2, 454–490.

[And98] G. E. Andrews, *The theory of partitions*, Reprint of the 1976 original, Cambridge Mathematical Library. Cambridge University Press, Cambridge, 1998.

[AAR99] G. E. Andrews, R. Askey and R. Roy, *Special functions*, Encyclopedia of Mathematics and its Applications 71. Cambridge University Press, Cambridge, 1999.

[AG88] G. E. Andrews and F. Garvan, Dyson's crank of a partition, *Bull. Amer. Math. Soc.* **18** (1988), 167–171

[AL00] G. E. Andrews and R. P. Lewis, The ranks and cranks of partitions moduli 2, 3 and 4, *J. Number Theory* **85** (2000), 74–84.

[Apo76] T. M. Apostol, *Introduction to analytic number theory*, Undergraduate Texts in Mathematics, Springer-Verlag, New York-Heidelberg, 1976.

[Apo90] T. M. Apostol, *Modular functions and Dirichlet series in number theory*, Second edition, Graduate Texts in Mathematics, 41. Springer-Verlag, New York, 1990.

[AS54] A. O. L. Atkin and P. Swinnerton-Dyer, Some properties of partitions, *Proc. London Math. Soc. (3)* **4** (1954), 84–106.

[BS15] W. Banks and I. E. Shparlinski, On Gauss sums and the evaluation of Stechkin's constant, *Math. Comp.* **85** (2015), no. 301, 2569–2581.

[BEW98] B. C. Berndt, R. J. Evans and K. S. Williams, *Gauss and Jacobi sums*, Canadian Mathematical Society Series of Monographs and Advanced Texts. A Wiley-Interscience Publication. John Wiley & Sons, Inc., New York, 1998.

[CP04] C. Bessenrodt and I. Pak, Partition congruences by involutions, *European J. Combin.* **25** (2004), no. 8, 1139–1149.

[Bre93] F. Brenti, Determinants of Super-Schur Functions, Lattice Paths, and Dotted Plane Partitions, *Adv. Math.* **98** (1993), no. 1, 27–64.

[Bri09] K. Bringmann, Asymptotics for rank partition functions, *Trans. Amer. Math. Soc.* **361** (2009), no. 7, 3483–3500.

[BL07] K. Bringmann and J. Lovejoy, Dyson's rank, overpartitions, and weak Maass Forms, *Int. Math. Res. Not. IMRN* (2007), no. 19, Art. ID rnm063, 34 pp.

[BM] K. Bringmann and K. Mahlburg, Transformation laws and asymptotics for nonmodular products, unpublished preprint.

[BO10] K. Bringmann and K. Ono, Dyson's ranks and Maass forms, *Ann. of Math. (2)* **171** (2010), no. 1, 419–449.

[Cio19] A. Ciolan, Ranks of overpartitions: Asymptotics and inequalities, *J. Math. Anal. Appl.* **480** (2019), no. 2, Art. ID 123444, 28 pp.

[Cio20] A. Ciolan, Asymptotics and inequalities for partitions into squares, *Int. J. Number Theory* **16** (2020), no. 1, 121–143.

[Cio] A. Ciolan, Equidistribution and inequalities for partitions into powers, available as preprint at `https://arxiv.org/abs/2002.05682`.

[CP11] T. Cochrane and C. Pinner, Explicit bounds on monomial and binomial exponential sums, *Q. J. Math.* **62** (2011), no. 2, 323–349.

[CE03] H. Cohn and N. Elkies, New upper bounds on sphere packings. I, *Ann. of Math. (2)* **157** (2003), no. 2, 689–714.

[Cor03] S. Corteel, Particle seas and basic hypergeometric series, *Adv. Appl. Math.* **31** (2003), no. 1, 199–214.

[CH04] S. Corteel and P. Hitczenko, Multiplicity and number of parts in overpartitions, *Ann. Comb.* **8** (2004), no. 3, 287–301.

[CL02] S. Corteel and J. Lovejoy, Frobenius partitions and Ramanujan's ${}_1\psi_1$ summation, *J. Combin. Theory Ser. A* **97** (2002), no. 1, 177–183.

[CL04] S. Corteel and J. Lovejoy, Overpartitions, *Trans. Amer. Math. Soc.* **356** (2004), no. 4, 1623–1635.

[DLM03] P. Desrosiers, L. Lapointe and P. Mathieu, Jack polynomials in superspace, *Comm. Math. Phys.* **242** (2003), no. 1–2, 331–360.

[Dew10] M. Dewar, The nonholomorphic parts of certain weak Maass forms, *J. Number Theory* **130** (2010), no. 3, 559–573.

[Dra52] L. A. Dragonette, Some asymptotic formulae for the mock theta series of Ramanujan, *Trans. Amer. Math. Soc.* **72** (1952), 474–500.

[Dys44] F. J. Dyson, Some guesses in the theory of partitions, *Eureka* (1944), no. 8, 10–15.

[FJM05a] J.-F. Fortin, P. Jacob and P. Mathieu, Generating function for K-restricted jagged partitions, *Electron. J. Combin.* **12** (2005), Research paper 12, 17 pp.

[FJM05b] J.-F. Fortin, P. Jacob and P. Mathieu, Jagged Partitions, *Ramanujan J.* **10** (2005), no. 2, 215–235.

[Fra81] F. Franklin, Sur le développement du produit infini $(1-x)(1-x^2)(1-x^3)\ldots$, *Comptes Rendus* **82** (1881), 448–450.

[Gla76] J. W. L. Glaisher, On formulae of verification in the partition of numbers, *Proc. Royal Soc. London* **24** (1876), 250–259.

[Gla83] J. W. L. Glaisher, A theorem in partitions, *Messenger of Math.* **12** (1883), 158–170.

[HR18] G. H. Hardy and S. Ramanujan, Asymptotic formulae for the distribution of integers of various types, *Proc. London Math. Soc. Series (2)* **16** (1918), 112–132.

[HK00] D. R. Heath-Brown and S. Konyagin, New bounds for Gauss sums derived from kth powers, and for Heilbronn's exponential sum, *Q. J. Math.* **51** (2000), no. 2, 221–235.

[Hil09] D. Hilbert, Beweis für die Darstellbarkeit der ganzen Zahlen durch eine feste Anzahl n-ter Potenzen (Waringsches Problem), *Math. Ann.* **67** (1909), no. 3, 281–300.

[HJ18] E. Hou and M. Jagadeesan, Dyson's partition ranks and their multiplicative extensions, *Ramanujan J.* **45** (2018), no. 3, 817–839.

[Jac29] C. G. J. Jacobi, *Fundamenta nova theoriae functionum ellipticarum*, in *Gesammelte Werke*, Band I (1829), 49–239.

[Jen16] C. Jennings-Shaffer, Overpartition rank differences modulo 7 by Maass forms, *J. Number Theory* **163** (2016), 331–358.

[JR] C. Jennings-Shaffer and D. Reihill, Asymptotic formulas related to the M_2-rank of partitions without repeated odd parts, available as preprint at `https://arxiv.org/abs/1805.11319`.

[JZZ18] K. Q. Ji, H. W. J. Zhang and A. X. H. Zhao, Ranks of overpartitions modulo 6 and 10, *J. Numbery Theory* **184** (2018), 235–269.

[JS87] J. T. Joichi and D. Stanton, Bijective proofs of basic hypergeometric series identities, *Pacific J. Math.* **127** (1987), 103–120.

[KK04] S.-J. Kang and J.-H. Kwon, Crystal bases of the Fock space representations and string functions, *J. Algebra* **280** (2004), no. 1, 313–349.

[Lew97] R. P. Lewis, The ranks of partitions modulo 2. 15th British Combinatorial Conference (Stirling, 1995), *Discrete Math.* **167/168** (1997), 445–449.

[Lov04a] J. Lovejoy, Overpartitions and real quadratic fields, *J. Number Theory* **106** (2004), no. 1, 178–186.

[Lov04b] J. Lovejoy, Overpartition theorems of the Rogers-Ramanujan type, *J. London Math. Soc. (2)* **69** (2004), no. 3, 562–574.

[Lov05] J. Lovejoy, Rank and conjugation for the Frobenius representation of an overpartition, *Ann. Comb.* **9** (2005), no. 3, 321–334.

[LM08] J. Lovejoy and O. Mallet, n-color overpartitions, twisted divisor functions, and Rogers-Ramanujan identities, *South East Asian J. Math. Math. Sci.* **6** (2008), no. 2, 23–36.

[LO08] J. Lovejoy and R. Osburn, Rank differences for overpartitions, *Q. J. Math.* **59** (2008), no. 2, 257–273.

[Mal20] J. Males, Asymptotic equidistribution and convexity for partition ranks, *Ramanujan J.* (2020), available online at `https://doi.org/10.1007/s11139-019-00202-8`.

[Mao13] R. Mao, Ranks of partitions modulo 10, *J. Number Theory* **133** (2013), no. 11, 3678–3702.

[Mao15] R. Mao, The M_2-rank of partitions without repeated odd parts modulo 6 and 10, *Ramanujan J.* **37** (2015), no. 2, 391–419.

[Mei54] G. Meinardus, Asymptotische aussagen über Partitionen, *Math. Z.* **59** (1954), 388–398.

[MVW95] H. L. Montgomery, R. C. Vaughan and T. D. Wooley, Some remaks on Gauss sums associated with kth powers, *Math. Proc. Comb. Phil. Soc.* **118** (1995), no. 21, 21–33.

[Pak06] I. Pak, Partition bijections, a survey, *Ramanujan J.* **12** (2006), no. 1, 5–75.

[Rad38] H. Rademacher, The Fourier coefficients of the modular invariant $J(\tau)$, *Amer. J. Math.* **60** (1938), no. 2, 501–512.

[Ram21] S. Ramanujan, Congruence properties of partitions, *Math. Z.* **9** (1921), 147–153.

[Ser76] J.-P. Serre, Divisibilité de certaines fonctions arithmétiques, *Enseignement Math.* **22** (1976), 227–260.

[Shp91] I. E. Shparlinskiĭ, Estimates for Gauss sums (Russian), *Mat. Zametki* **50** (1991), no. 1, 122–130; English transl., *Math. Notes* **50** (1991), no. 1–2, 740–746 (1992).

[Ste75] S. B. Stečkin, An estimate for Gaussian sums (Russian), *Mat. Zametki* **17** (1975), no. 4, 579–588.

[Swi73] H. P. F. Swinnerton-Dyer, On ℓ-adic representations and congruences for coefficients of modular forms, in *Modular Functions of One Variable III*, Eds.: W. Kuijk, J.-P. Serre, *Lecture Notes in Mathematics* **350**, Springer, 1973.

[Swi87] H. P. F. Swinnerton-Dyer, Congruence properties of $\tau(n)$, in *Ramanujan Revisited* (Urbana-Champaign, Ill., 1987), 289–311, Academic Press, Boston, MA, 1988.

[Tit86] E. C. Titchmarsh, *The theory of the Riemann zeta-function*, Second edition, The Clarendon Press, Oxford University Press, New York, 1986.

[Tót91] L. Tóth, Elementary Problems: E 3432, *Amer. Math. Monthly* **98** (1991), no. 3, 263–264.

[VW02] R. C. Vaughan and T. D. Wooley, *Waring's problem: a survey,* in *Number theory for the millennium, III* (Urbana, Ill., 2000), 301–340, A K Peters, Natick, MA, 2002.

[Wat44] G. N. Watson, *A treatise on the theory of Bessel functions*, Second edition, Cambridge University Press, Cambridge, 1944.

[WZ20] B. Wei and H. W. J. Zhang, Generalized Lambert series identities and applications in rank differences, *Proc. Amer. Math. Soc.* (2020), available online at `https://doi.org/10.1090/proc/15075`, 17 pp.

[WY99] Sh.-R. Wei and B.-Ch. Yang, An improvement of Franel's inequality (in Chinese), *J. Central Univ. Nationalities* **8** (1999), no. 1, 66–68.

[Wri34] E. M. Wright, Asymptotic partition formulae. III. Partitions into k-th powers, *Acta Math.* **63** (1934), no. 1, 143–191.

[WY97] K. Wu and B.-Ch. Yang, Some refinements of Franel's inequality (in Chinese), *J. South China Normal Univ. Natur. Sci. Ed.* (1997), no. 3, 5–8.

[Yee04] A.-J. Yee, Combinatorial proofs of Ramanujan's ${}_1\psi_1$ summation and the q-Gauss summation, *J. Combin. Theory Ser. A* **105** (2004), no. 1, 63–77.

[Zho] N. H. Zhou, Note on partitions into polynomials with number of parts in an arithmetic progression, available as preprint at `https://arxiv.org/abs/1909.13549`.

[Zuc39] H. S. Zuckerman, On the coefficients of certain modular forms belonging to subgroups of the modular group, *Trans. Amer. Math. Soc.* **45** (1939), no. 2, 298–321.

[Zwe19] S. Zwegers, Multivariable Appell functions and nonholomorphic Jacobi forms, *Res. Math. Sci.* **6** (2019), Art. 16, 15 pp.

Index of notation

$\mathbb{N}$	the set of positive integers
$\mathbb{Z}$	the ring of integers
$\mathbb{Q}$	the field of rational numbers
$\mathbb{R}$	the field of real numbers
$\mathbb{C}$	the field of complex numbers
(a, b)	the greatest common divisor of a and b
$\Gamma(s)$	Gamma function
$\zeta(s)$	Riemann zeta function
$\zeta(s, q)$	Hurwitz zeta function
$\lfloor x \rfloor$	integral part of x
$\{x\}$	fractional part of x
$O;\ O_c$	$f(x) = O(g(x))$ if there exists $C > 0$ such that $\lvert f(x)\rvert \leq Cg(x)$ as $x \to \infty$; the implied constant depends on c
o	$f(x) = o(g(x))$ if for any $\varepsilon > 0$ there is x_0 so that $\lvert f(x)\rvert \leq \varepsilon g(x)$ for $x \geq x_0$
$\ll;\ \ll_c$	$f(x) \ll g(x)$ if $f(x) = O(g(x))$; the implied constant depends on c
$\sim$	$f(x) \sim g(x)$ as $x \to \infty$ if $f(x) = g(x)(1+o(1))$ as $x \to \infty$
$\exp(z)$	is sometimes used to denote e^z
$\sigma_k(n)$	sum of the kth powers of divisors of n, $\sigma_k(n) = \sum_{d\mid n} d^k$
ζ_n	the primitive nth root of unity $e^{\frac{2\pi i}{n}}$
$\mathrm{Re}(z)$	real part of $z \in \mathbb{C}$
$\mathrm{Im}(z)$	imaginary part of $z \in \mathbb{C}$

Erklärung

Hiermit versichere ich an Eides statt, dass ich die vorliegende Dissertation selbstständig und ohne die Benutzung anderer als der angegebenen Hilfsmittel und Literatur angefertigt habe. Alle Stellen, die wörtlich oder sinngemäß aus veröffentlichten und nicht veröffentlichten Werken dem Wortlaut oder dem Sinn nach entnommen wurden, sind als solche kenntlich gemacht. Ich versichere an Eides statt, dass diese Dissertation noch keiner anderen Fakultät oder Universität zur Prüfung vorgelegen hat; dass sie – abgesehen von unten angegebenen Teilpublikationen und eingebundenen Artikeln und Manuskripten – noch nicht veröffentlicht worden ist sowie, dass ich eine Veröffentlichung der Dissertation vor Abschluss der Promotion nicht ohne Genehmigung des Promotionsausschusses vornehmen werde. Die Bestimmungen dieser Ordnung sind mir bekannt. Darüber hinaus erkläre ich hiermit, dass ich die Ordnung zur Sicherung guter wissenschaftlicher Praxis und zum Umgang mit wissenschaftlichem Fehlverhalten der Universität zu Köln gelesen und sie bei der Durchführung der Dissertation zugrundeliegenden Arbeiten und der schriftlich verfassten Dissertation beachtet habe und verpflichte mich hiermit, die dort genannten Vorgaben bei allen wissenschaftlichen Tätigkeiten zu beachten und umzusetzen. Ich versichere, dass die eingereichte elektronische Fassung der eingereichten Druckfassung vollständig entspricht.

[1] A. Ciolan, Ranks of overpartitions: Asymptotics and inequalities, *J. Math. Anal. Appl.* **480** (2019), no. 2, Art. ID 123444, 28 pp.

[2] A. Ciolan, Asymptotics and inequalities for partitions into squares, *Int. J. Number Theory* **16** (2020), no. 1, 121–143.

[3] A. Ciolan, Equidistribution and inequalities for partitions into powers, submitted for publication; available as preprint at `https://arxiv.org/abs/2002.05682`.

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Emil-Alexandru Ciolan