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Recursion Operators for CBC system with reductions. Geometric theory

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Summary. — We discuss some recent developments of the geometric theory of the Recursion Operators (Generating Operators) for Caudrey-Beals-Coifman systems (CBC systems) on semisimple Lie algebras. As is well known the essence of this interpretation is that the Recursion Operators could be considered as adjoint to Nijenhuis tensors on certain infinite-dimensional manifolds. In particular, we discuss the case when there are $\mathbb{Z}_p$ reductions of Mikhailov type.

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1. – Introduction

The present article is about the theory of the soliton equations and the Recursion Operators as one of the principal tools in this theory, in particular about the gauge-covariant theory of these operators. The concept of the gauge-equivalent soliton equations originates from the paper of Zakharov and Mikhailov [1], where it has been applied to reveal some connections in the integrable equations in field theory, but the most popular is the example of the gauge equivalence between the nonlinear Schrödinger equation (NLS) $\mathrm{i}\varphi_t + \varphi_{xx} + 2\varphi|\varphi|^2 = 0$, $\lim_{x\to\pm\infty} \varphi(x) = 0$ and the Heisenberg Ferromagnet (HF) equation: $S_t = -\frac{1}{2\mathrm{i}}[S, S_{xx}]$ with $S(x)$ being a $\mathfrak{sl}(2,\mathbb{C})$-valued function such that $S^\dagger = -S$, $\lim_{x\to\pm\infty} S(x) = \sigma_3 = \mathrm{diag}\,(1,-1)$, $S^2(x) = \mathbf{1}$ and "$\dagger$" means Hermitian conjugation. As is well known, a characteristic property of the soliton equations is that they have Lax

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representation, that is they can be represented as $[L, A] = 0$, where L and A are linear operators in x and t, respectively:

$$L = \mathrm{i}\partial_x + U(q, q_x, \dots, \lambda), \quad A = \mathrm{i}\partial_t + V(q, q_x, \dots, \lambda).$$

In the above U, V are matrix functions depending on a "potential" q in terms of which the corresponding soliton equation is written and λ a spectral parameter. The NLS equation has Lax representation $[L, A] = 0$, where the operator L is defined by the system $L\psi = (\mathrm{i}\partial_x + q - \lambda\sigma_3)\psi = 0$ known as Zakharov-Shabat (ZS) system. Here the "potential" $q(x)$ is a smooth function of the type

$$\begin{pmatrix} 0 & q_+(x) \\ q_-(x) & 0 \end{pmatrix}, \quad \lim_{x\to\pm\infty} q_\pm(x) = 0.$$

To get the NLS we put $q_+(x) = \varphi(x)$, $q_-(x) = \varphi^*(x)$, where * means complex conjugation and A has the form $\mathrm{i}\partial_t + V$, where V is matrix polynomial of degree 2 in λ with first coefficient $\lambda^2\sigma_3$, depending on q and q_x. From its side HF equations has a Lax representation $[\tilde{L}, \tilde{A}] = 0$ with $\tilde{L}$ being defined by the linear system $(\mathrm{i}\partial_x - \lambda S(x))\tilde{\psi} = \tilde{L}\tilde{\psi} = 0$. Zakharov and Takhtadjan [2] showed that the operators L, A and $\tilde{L}, \tilde{A}$ are gauge equivalent, that is $\tilde{L} = \psi_0^{-1} L \psi_0$, $\tilde{A} = \psi_0^{-1} A \psi_0$ where ψ_0 satisfies $L\psi_0 = 0$ for $\lambda = 0$ and $\lim_{x\to\infty} \psi_0(x) = \mathbf{1}$, $\lim_{x\to-\infty} \psi_0(x) = T^{-1}(0)$ where $T(\lambda)$ is the so-called transition matrix. In particular,

$$\tilde{L} = \mathrm{i}\partial_x - \lambda S(x), \quad S = \psi_0^{-1}\sigma_3\psi_0. \tag{1}$$

The above shows that the gauge-equivalence is a kind of changing the variables transformation which takes NLS into HF. This result has been extended in two directions:

First direction. A correspondence has been established between the hierarchies of soliton systems associated with L and $\tilde{L}$, their conservation laws, Hamiltonian structures, etc. This has been achieved through the theory of Recursion Operators Λ and $\tilde{\Lambda}$ related to L and $\tilde{L}$, respectively, [3].

Second Direction. The Recursion Operator theory was generalized for the auxiliary linear problems of the type

$$\begin{aligned} &(\mathrm{i}\partial_x + q(x) - \lambda J)\,\psi = L\psi = 0, \\ &(\mathrm{i}\partial_x - \lambda S(x))\,\tilde{\psi} = \tilde{L}\tilde{\psi} = 0. \end{aligned} \tag{2}$$

In (2) L is the so-called generalized Zakharov-Shabat system (GZS) (when J is real) and Caudrey-Beals-Coifman (CBC) system [4, 5] when J is complex. The potential $q(x)$ and J belong to a fixed simple Lie algebra $\mathfrak{g}$ in some finite-dimensional irreducible representation. The element J is regular, that is the kernel of ad_J is a Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g}$. $q(x)$ belongs to the orthogonal completion $\mathfrak{h}^\perp = \bar{\mathfrak{g}}$ of $\mathfrak{h}$ with respect to the Killing form $\langle X, Y\rangle = \mathrm{tr}\,(\mathrm{ad}_X\,\mathrm{ad}_Y)$, $X, Y \in \mathfrak{g}$ and $\lim_{x\to\pm\infty} q(x) = 0$. The system $\tilde{L}$ is referred as GZS system (or CBC) system in pole gauge in contrast to L which is referred as GZS (CBC) system in canonical gauge. In $\tilde{L}$ the potential $S(x)$ takes values in the orbit of J in the adjoint representation of $\mathfrak{g}$ and satisfies $\lim_{x\to\pm\infty} S(x) = J_\pm$. By analogy with $\mathfrak{sl}(2)$ the first nonlinear systems in the hierarchies of soliton equations corresponding to L and $\tilde{L}$ are called NLS and HF type equations, respectively.

Finally, the gauge-equivalence between soliton equations hierarchies has been given a geometric interpretation [6, 7] using the theory of the Poisson-Nijenhuis manifolds.

Since the theory of the Recursion Operators for the system $\tilde{L}$ could be obtained with a gauge transformation from that for L we speak about gauge-covariant theory of Recursion Operators for the CBC system, [3, 8, 9, 7]. A complete bibliography about the above topics one can find in the monograph book [7]. However, it reflects the situation prior to 2008. Since then there appeared a new trend, namely to incorporate in the theory of the Recursion Operators the Mikhailov-type reductions, [10-12]. Let us briefly introduce them. Suppose that we have Lax representation $[L(\lambda), M(\lambda)] = 0$, where L and M are given below and U, V are matrix functions:

$$
\begin{aligned}
&L(\lambda)\psi(x,t,\lambda) = (\mathrm{i}\partial_x + U(x,t,\lambda))\,\psi(x,t,\lambda) = 0,\\
&M(\lambda)\psi(x,t,\lambda) = (\mathrm{i}\partial_t + V(x,t,\lambda))\,\psi(x,t,\lambda) = 0.
\end{aligned}
\tag{3}
$$

The Mikhailov's reduction group G_R [10-12] is a finite group which preserves the Lax representation. G_R has two realizations: G_R acts on the algebra through automorphisms, or through complex conjugations and on $\mathbb{C}$ as subgroup of the group of the conformal maps of $\mathbb{C}$. For example, for Mikhailov reduction groups G_R generated by an automorphisms for each element $g \in G_R$ we have an automorphism or a complex conjugation $\mathcal{A}$, (we denote by the same letter the automorphism of the Lie group and the automorphism of the corresponding Lie algebra) and a holomorphic map γ. Suppose

$$
\mathcal{A}(U(\gamma(\lambda))) = U(\lambda), \qquad \mathcal{A}(V(\gamma(\lambda))) = V(\lambda).
\tag{4}
$$

Then one can see that if $\psi(x,\lambda)$ is a solution of (3), $\mathcal{A}(\psi(x,\gamma(\lambda))$ also is. Alternatively, one can say that the reduction group acts on the fundamental solutions of the above system as just described, then as easily seen we must have the relations (4). Since Mikhailov's reductions group preserves the Lax representation, it is a powerful tool for obtaining algebraic constraints of $U(x,t.\lambda)$ and $V(x,t.\lambda)$ which are automatically compatible with the evolution given by $[L, M] = 0$, see [10-13] and [14, 15]. The reductions defined by complex conjugations are perhaps the easiest to study, one puts then $\gamma(\lambda) = \lambda^*$ and they lead to $\mathbb{Z}_2$ reductions. The effect of such a reduction is that we restrict ourselves to the corresponding real form of the initial Lie algebra $\mathfrak{g}$. We of course can have two (or more) complex conjugations etc. More interesting things occur when we have reductions defined by an automorphism $\mathcal{K}$ of $\mathfrak{g}$ having finite order p, the effect of which on the Recursion Operators spectral theory for CBC in canonical gauge which has been considered recently in [15]. So below we shall concentrate on the case. Let us also remark that the geometric theory in case of some particular choices of the reductions has been also studied, in this respect we would like to mention the works [16, 17] (for system L) and [18-20] (for the system $\tilde{L}$). However, the works treating systems with reductions concentrate either on L [15] or on $\tilde{L}$ and until now there is little about treating L and $\tilde{L}$ simultaneously. In [21] we made an effort to do it for the so-called GMV system, [22, 23], using the spectral approach but the issue deserves to be considered in more generality. Now we want to discuss the possibilities that exist using the geometric approach.

2. – Preliminaries

Now let us describe the things more precisely. We assume that Mikhailov reduction group G_M is generated by one element, which we denote by g_0. We define its action on the fundamental solutions ψ of the CBC system to be

$$g_0(\psi(x,\lambda)) = \mathcal{K}(\psi(x,\omega^{-1}\lambda)), \tag{5}$$

where $\mathcal{K}$ will be the automorphism of the connected Lie group G corresponding to automorphism of finite order p of the algebra $\mathfrak{g}$, denoted by the same letter $\mathcal{K}$. Important class of such automorphisms are the Coxeter automorphisms, then p is the so-called Coxeter number. We also assume that in the above $\omega = \exp(\frac{2\pi \mathrm{i}}{p})$. Since $g_0^p = \mathrm{id}$ the reduction group G_M is isomorphic to $\mathbb{Z}_p$ and the reduction defined by (5) immediately leads to $\mathcal{K}J = \omega J$ and $\mathcal{K}q = q$. Thus the automorphism $\mathcal{K}$ preserves the Cartan subalgebra $\mathfrak{h}$, recall that $\mathfrak{h} = \ker \mathrm{ad}_J$.

When there is automorphism $\mathcal{K}$ of $\mathfrak{g}$ finite order p leaving $\mathfrak{h}$ invariant its possible eigenvalues are ω^s, $s = 0, 1, \ldots, p-1$, $\omega = \exp\frac{2\pi \mathrm{i}}{p}$, denote its eigenspaces by $\mathfrak{g}_K^{[s]}$. (If ω^s is not an eigenvalue we assume that $\mathfrak{g}_K^{[s]} = 0$). Since for $\mathcal{K}$ the spaces $\mathfrak{h}$ and $\bar{\mathfrak{g}}$ are invariant the eigenspace $\mathfrak{g}_K^{[s]}$ splits into $\bar{\mathfrak{g}}_K^{[s]} \oplus \mathfrak{h}_K^{[s]}$, $\bar{\mathfrak{g}}_K^{[s]} \subset \bar{\mathfrak{g}}_K$, $\mathfrak{h}_K^{[s]} \subset \mathfrak{h}$ so that

$$\bar{\mathfrak{g}} = \oplus_{s=0}^{p-1} \bar{\mathfrak{g}}_K^{[s]}, \qquad \mathfrak{h} = \oplus_{s=0}^{p-1} \mathfrak{h}_K^{[s]}. \tag{6}$$

We have that $J \in \mathfrak{h}^{[1]}$ and the assumption on the potential q means that q takes values in $\bar{\mathfrak{g}}_K^{[0]}$. In particular, this means that

$$\mathrm{ad}_J(\bar{\mathfrak{g}}_K^{[s]}) \subset \bar{\mathfrak{g}}_K^{[s+1]},\ \mathrm{ad}_J^{-1}(\bar{\mathfrak{g}}_K^{[s]}) \subset \bar{\mathfrak{g}}_K^{[s-1]},\ \mathrm{ad}_q(\mathfrak{g}_K^{[s]}) \subset \mathfrak{g}_K^{[s]} \tag{7}$$

(the superscripts are understood modulo p).

The main result obtained in [15] for the reductions of the above type for the Recursion Operators related to the CBC system in could be formulated as follows: *In case we have $\mathbb{Z}_p$ reductions defined by an automorphism, for the expansions of functions taking values in a fixed space $\bar{\mathfrak{g}}^{[s]}$ the role of the Recursion Operators are played by the p-th powers of the operators $\Lambda_\pm$.*

3. – Recursion Operators. Geometric approach

The key ingredients for the geometric picture are the notion of a Poisson tensor, Nijenhuis tensor and a coupled Poisson and Nijenhuis tensors, so called P-N structure. As known if $\mathcal{M}$ is a manifold, then a Poisson tensor (Poisson structure) on $\mathcal{M}$ is a field of skew-symmetric linear maps $m \mapsto P_m : T_m^* \mapsto T_m$ (T_m and T_m^* are the cotangent and the tangent spaces at $m \in \mathcal{M}$) such that the so-called Schouten bracket $[P,P]_S$ vanishes. This ensures that on the space of closed forms (and as a consequence on the functions on $\mathcal{M}$ we have a Poisson bracket. If α, β are closed forms, it is defined as $\{\alpha,\beta\}_P = -d(\alpha, P\beta)$ where (,) is the canonical pairing between vectors and co-vectors. A Nijenhujis tensor (Nijenhuis structure) on $\mathcal{M}$ is a field of linear maps $m \mapsto N_m : T_m \mapsto T_m$ such that the so-called Nijenhuis bracket $[N,N]$ is zero. In such a short review we cannot give references and definitions about all the objects since this means to mention hundreds of works and we want just to outline the geometric theory of the Recursion Operators. Let us only mention that it has been first suggested by F. Magri [24,25], see also [26]. Later, a slightly different approach to integrabilitys has been developed in [27-32], underlying that the properties of the Nijenhuis tensor are essential. One can find comprehensive bibliography of the theory prior to 2005 in [7]. We refer to that book also for the properties of Poisson tensor and Nijenhuis tensor. Finally, we say that on the manifold $\mathcal{M}$ is defined P-N structure (Poisson-Nijenhuis structure) if on $\mathcal{M}$ are defined simultaneously Poisson tensor P and Nijenhuis tensor N which satisfy the following coupling conditions:

$$\begin{aligned} &\text{a)} \qquad NP = PN^*, \\ &\text{b)} \quad PL_{N(X)}(\alpha) - PL_X(N^*\alpha) + L_{P(\alpha)}(N)(X) = 0, \end{aligned} \tag{8}$$

for arbitrary choice of the vector field X and the 1-form α. By L_X is denoted the Lie derivative with respect the vector field X. The structure we described seems rather exotic, but in fact in the theory of the soliton equations it is common. As is known in almost every approach to the theory of completely integrable systems one can notice that a crucial role is played by the so-called compatible Poisson tensors, also called Hamiltonian pairs, see for example [33]. They can be used to construct P-N manifolds. Two Poisson tensors P and Q are compatible if the tensor $P + Q$ is Poisson tensor too. The following construction is basic for the construction of P-N manifolds: Let P and Q are Poisson tensors on the manifold $\mathcal{M}$. Let Q^{-1} exists and is smooth field of continuous linear mappings $m \to Q_m^{-1}$. Then the tensor fields $N = P \circ Q^{-1}$ and Q endow the manifold with P-N structure. Conversely, if the N is Nijenhuis tensor field, satisfying the coupling conditions with the Poisson tensor Q then Q and $N \circ Q \equiv NQ$ are compatible Poisson tensors on $\mathcal{M}$.

The application of the P-N structures are motivated by the interesting features of their fundamental fields and the fact that they define also hierarchies of Poisson tensors. The field X is called fundamental for the P-N structure if $L_X N = 0$, $L_X P = 0$. In the following theorem are collected the most essential properties of the fundamental fields. In fact this theorem gives the geometric interpretation of the properties of the Generating Operators.

Theorem 3.1. Let $\mathcal{M}$ be P-N manifold. Let χ_N^* be the set of 1-forms α satisfying the conditions $d\alpha = 0$, $dN^*\alpha = 0$. $\chi_N^*(\mathcal{M})$ is called the set of fundamental forms. Then the set of vector fields $\chi_{PN}(\mathcal{M})$ consisting of $X_\alpha = P(\alpha)$ such that $d\alpha = 0$, $dN^*\alpha = 0$ are fundamental for the P-N structure. The vector spaces $\chi_{PN}(\mathcal{M})$ and $\chi_N^*(\mathcal{M})$ are Lie algebras (with respect to the Lie bracket and Poisson bracket respectively) and P is homomorphism between these algebras:

$$[X_\alpha, X_\beta] = -P\{\alpha, \beta\}_P. \tag{9}$$

The above algebras are invariant with respect of the action of N (N^*) and N (N^*) commute with the Lie algebra operation:

$$\begin{aligned} &N^*\{\alpha,\beta\}_P = \{N^*\alpha,\beta\}_P = \{\alpha, N^*\beta\}_P; \qquad \alpha,\beta \in \chi_N^*(\mathcal{M}), \\ &N[X_\alpha, X_\beta] = [NX_\alpha, X_\beta] = [X_\alpha, NX_\beta]; \qquad X_\alpha, X_\beta \in \chi_{PN}(\mathcal{M}). \end{aligned} \tag{10}$$

The most remarkable properties of the P-N structure are referred as “hereditary” properties, it means that N moves fundamental fields into fundamental fields and generates also a hierarchy P-N structures. More precisely, if $\mathcal{M}$ is P-N manifold, we have

1. On $\mathcal{M}$ there is infinite number of P-N structures defined by the pairs $(N^k P, N^s)$, $k, s = 1, 2, \ldots$.
2. If $\alpha, \beta \in \chi^*_N(\mathcal{M})$ are in involution (X_α and X_β commute) then for arbitrary natural numbers m and n the forms $(N^*)^m\alpha$, $(N^*)^n\beta$ are also in involution (the fields $(N)^m X_\alpha$ and $(N)^n X_\beta$ commute).
3. The fields of the type $(N)^k X_\alpha$ are Hamiltonian with respect to the hierarchy of symplectic forms $P, NP, \ldots N^k P$. If N^{-1} exists the hierarchy is infinite and consists of the Poisson tensors of the type $N^r P$ where r is integer.

One can see that in order to obtain a P-N structure from a pair of compatible Poisson tensors one of them should be invertible. It can happen that it is not the case so the way out of this problem is to restrict the Poisson tensors on some submanifold where one will have the desired property. This is not automatic and in fact one has restriction theorems for Poisson structures and for P-N structures. We just mention that since we cannot go into more details here, for such results see [7, 34, 35].

4. – Geometric theory for the $\mathfrak{g}$-CBC Recursion Operators in canonical gauge

For a subspace $\mathfrak{a} \subset \mathfrak{g}$ let us denote by $\mathfrak{F}(\mathfrak{a})$ the set of smooth, fast decreasing functions: $f : \mathbb{R} \to \mathfrak{a}$. Clearly, $\mathfrak{F}(\mathfrak{g})$ and $\mathfrak{F}(\mathfrak{h})$ are Lie algebras too if we define the Lie bracket of two functions f, g point-wise. Also, admitting some lack of rigor we shall identify $\mathfrak{F}(\mathfrak{g})$ and $\mathfrak{F}(\mathfrak{g})^*$ using the bilinear form $\langle\langle X, Y\rangle\rangle = \int_{-\infty}^{+\infty} \langle X(x), Y(x)\rangle \mathrm{d}x$, $X, Y \in \mathfrak{F}(\mathfrak{g})$. A general fact from the theory of the Poisson structures used for the equations that can be solved via CBC linear problem is that on the infinite-dimensional manifold $\mathcal{M} = \mathfrak{F}(\mathfrak{g})$, we have the following compatible Poisson tensors (after identifying $\mathfrak{F}(\mathfrak{g})$ and with $\mathfrak{F}(\mathfrak{g})^*$):

$$Q^0_q(\xi) = -\mathrm{ad}_{\,\xi} J, \quad P^0_q(\xi) = -\mathrm{ad}_{\,\xi} q + \mathrm{i}\partial_x \xi; \quad q(x), \xi(x) \in \mathfrak{F}(\mathfrak{g}). \tag{11}$$

The tensor Q^0 is not kernel free and therefore we cannot find $(Q^0)^{-1}$. Fortunately one can restrict Q^0 on some integral leaf of the distribution $\mathrm{im}\,(Q^0)$ and then the restricted tensor will be nondegenerate. One of these leafs is the manifold $\mathcal{M}_0 = \mathfrak{F}(\bar{\mathfrak{g}})$ to which the "potential" of the CBC linear problem belongs. Then one can find that the restriction Q of Q^0 has the same form, that is $Q = \mathrm{ad}_{\,J}$. As to P^0, it can be shown that it also admits restriction P to $\mathcal{M}_0$, if $\pi_0(\alpha) = \alpha \in T^*_q(\mathcal{M}_0)$ then

$$P(\alpha) = \mathrm{i}\partial_x \alpha + \pi_0([q, \alpha]) + [q, \mathrm{i}(\mathbf{1} - \pi_0) \int_{-\infty}^{x} [q, \alpha](y) \mathrm{d}y\ ], \tag{12}$$

where by π_0 is denoted the projection on the orthogonal complement of the Cartan subalgebra.

Now it is possible to obtain the Nijenhuis tensor $N = P \circ Q^{-1} = PQ^{-1}$ and it turns out that we have $N^* = \Lambda_\pm$, where

$$\Lambda_\pm(X(x)) = \mathrm{ad}_{\,J}^{-1} \left(\mathrm{i}\partial_x X + \pi_0[q, X] + \mathrm{i}\mathrm{ad}_{\,q} \int_{\pm\infty}^{x} (\mathrm{id} - \pi_0)[q(y), X(y)]\mathrm{d}y \right). \tag{13}$$

In other words the adjoint of N is exactly the Recursion Operator we have for L, [7]. (During the calculations we must restrict to forms on which the action of the two operators $\Lambda_\pm$ is the same). Finally, one finds that the fields $X_B : q \mapsto X_B(q) = [B, q]$, $B \in \mathfrak{h}$ and the forms $\alpha_B : q \mapsto \alpha_B(q) = \mathrm{ad}\,_J^{-1}[B, q]$ are fundamental for the P-N structure. To these fields (forms) correspond the soliton equations solvable through the CBC system. The above, together with the properties of the P-N manifolds explains geometrically the remarkable properties of the Recursion Operators rigorously obtained using expansions over adjoint solutions.

When there are reductions defined by an automorphism $\mathcal{K}$ as described in Preliminaries, the manifold of potentials is restricted to the submanifold $\mathcal{N} = \mathfrak{F}(\bar{\mathfrak{g}}_K^{[0]})$. Since $\mathcal{N}$ is a vector space we identify its tangent space at some q with $\mathfrak{F}(\bar{\mathfrak{g}}_K^{[0]})$ and using the non degeneracy of the form $\langle\langle\ ,\ \rangle\rangle$ we identify with $\mathfrak{F}(\bar{\mathfrak{g}}_K^{[0]})$ also the cotangent space at q.

From the expression for N we see that if $X \in \mathfrak{F}(\bar{\mathfrak{g}}_K^{[s]})$

$$
\begin{aligned}
(14)\qquad & NX = [\mathrm{i}\partial_x X + \pi_0 \mathrm{ad}\,_q + \mathrm{i}\mathrm{ad}\,_q(\mathbf{1} - \pi_0)\partial_x^{-1}\mathrm{ad}\,_q]\mathrm{ad}\,_J^{-1} X \in \mathfrak{F}(\bar{\mathfrak{g}}_K^{[s-1]}),\\
& N\mathfrak{F}(\bar{\mathfrak{g}}_K^{[s]}) \subset \mathfrak{F}(\bar{\mathfrak{g}}_K^{[s-1]}), \quad \mathrm{ad}\,_J \mathfrak{F}(\bar{\mathfrak{g}}_K^{[s]}) = \mathfrak{F}(\bar{\mathfrak{g}}_K^{[s-1]}).
\end{aligned}
$$

In particular, $N(T_q(\mathcal{N})) = N\mathfrak{F}(\bar{\mathfrak{g}}_K^{[0]}) \subset \mathfrak{F}(\bar{\mathfrak{g}}_K^{[p-1]})$ so N does not allow restriction on $\mathcal{N}$. However, as easily seen, N^p allows restriction and is of course a Nijenhuis tensor on $\mathcal{N}$. Similarly, one sees that the Poisson tensor field $Q = \mathrm{ad}\,_J$ restricted on $\mathcal{N}$ reduces to zero. Then of course one can try to restrict P since, as easily seen $P\mathfrak{F}(\bar{\mathfrak{g}}_K^{[s]}) \subset \mathfrak{F}(\bar{\mathfrak{g}}_K^{[s]})$. One finds that P can be restricted and the restriction $\bar{P}$ has the same form, so we shall still denote it by P.

Since P allows a restriction on $\mathcal{N}$ let us recall that if $\mathcal{M}$ is endowed with P-N structure, where P is a Poisson tensor and N is a Nijenhuis tensor, then for natural k and s each pair $(N^k P = (N^*)^k P, N^s)$ endows $\mathcal{M}$ with a P-N structure. Thus we conclude that now $\mathcal{N}$ is endowed with the P-N structure given by P and N^p. Since $N^* = \Lambda_\pm$ we see that the geometric picture also gives that in the case of $\mathbb{Z}_p$ reduction the role of the Recursion Operator is played by $\Lambda_\pm^p$.

The last thing that remains to be done is to calculate the integrable equations (fundamental fields) related to the P-N structure we just introduced. Clearly not all the fields from the hierarchies generated by the fields $X_B = [B, q]$, $B \in \mathfrak{h}$ will be tangent to the submanifold $\mathcal{N}$ so we must find which of the fields from the hierarchies "survive" the reduction. We arrive therefore to the result that the hierarchies of fundamental fields are:

$$
(15)\qquad N^{kp+s} X_B, \quad B \in \mathfrak{g}_K^{[s]},\ s = 0, 1, \dots p-1; \quad k = 0, 1, 2, \dots ,
$$

(in fact the general hierarchies are obtained taking finite linear combinations of these fields with constant coefficients but this is the usual way the things are referred to).

5. – Geometric theory for the $\mathfrak{g}$-CBC Recursion Operators in pole gauge

Now we are going to consider the pole gauge situation. Fixing the element J for the GZS $\mathfrak{g}$-system take the manifold $\mathcal{N}$ of all the smooth function $S(x)$ with domain $\mathbb{R}$, taking values in the orbit $\mathcal{Q}_J$. Also, $S(x)$ tends fast enough to some constant values when $x \mapsto \pm\infty$. Naturally, $\mathcal{N}$ can be considered as submanifold of $\mathcal{M}_0$ —the manifold of smooth functions with values in $\mathfrak{g}$ tending fast enough to constant values when $x \to \pm\infty$.

Then it is reasonable to assume that the tangent space $T_S(\mathcal{M}_0) = \mathfrak{F}(\mathfrak{g})$ and the tangent space $T_S(\mathcal{N})$ at S consists of all the smooth functions $X : \mathbb{R} \mapsto \mathfrak{g}$ that tend to zero fast enough when $x \mapsto \pm\infty$ and such that at each point they are orthogonal to $S(x)$. We denote that space by $\mathfrak{F}(\mathfrak{h}_S^{\perp})$. We shall also assume that the dual space $T_S^*(\mathcal{M}_0)$ is identified by $T_S^*(\mathcal{M}_0)$ via the inner product $\langle\langle\ ,\ \rangle\rangle$.

The first fact with which we start is that the fields of operators

$$\alpha \mapsto P'(\alpha) = \mathrm{i}\partial_x \alpha, \qquad \alpha \mapsto Q(\alpha) = \mathrm{ad}_S(\alpha), \tag{16}$$
$$S \in \mathcal{M}_0 = \mathfrak{F}(\mathfrak{g}), \qquad \alpha \in T_S^*(\mathcal{M})$$

could be interpreted as Poisson tensors on the manifold $\mathcal{M}$. Let us also mention the tensor Q is the canonical Kirillov tensor which acquires this form because coadjoint and adjoint representation are equivalent.

We are not going to repeat in detail all the procedure of reducing these tensors on $\mathcal{N}$, it is very much the same as in the canonical gauge. The Poisson tensors P' and Q could be restricted from the manifold $\mathcal{M}$ to the manifold $\mathcal{N}$ using some restriction theorems, [34,35]. The restriction is easy to do in abstract terms but explicit expressions through S and its derivatives are of course something different. For the lowest rank case (rank 1 or $\mathfrak{g} = \mathrm{sl}(2,\mathbb{C})$) the formula for N (or Λ) has been obtained in various works quite long ago, in the 80-ties, while for the case $\mathfrak{g} = \mathrm{sl}(3,\mathbb{C})$ it has been obtained in [18] and for $\mathfrak{g} = \mathrm{sl}(n,\mathbb{C})$ in 2012, [20]. As before one can check that $\tilde{N}^* = \tilde{\Lambda}_\pm$ where $\tilde{\Lambda}_\pm = \mathrm{Ad}(\psi_0)\Lambda_\pm \mathrm{Ad}(\psi_0)$ is exactly the operator suggested by the spectral approach. In fact this is the story for the general case also. Then the rest of the theory is developed just as in case of canonical gauge CBC. One could also continue and investigate what happens with the P-N structure when $\tilde{L}$ with a $\mathbb{Z}_p$ reduction defined by an inner authomorphism $\mathcal{H}$ of order p ($X \mapsto \mathcal{H}(X) = HXH^{-1}$) and such that $\mathcal{H}(S) = HSH^{-1} = \omega S$. We shall not repeat here these considerations, everything is just the same as in the case of the canonical gauge but there is a way to do this faster in another way, considering L and $\tilde{L}$ simultaneously, that is, in a gauge-covariant way. We are going to do this in the next section.

6. – The map $q \mapsto S[q]$

Let $q(x)$ be the potential in the CBC linear problem and let $q \mapsto \Psi[q]$ be a function depending on $q(x)$, it could be for example a Jost solution $\psi(x,\lambda)$ for fixed λ (in particular ψ_0), an asymptotic of such solution, the transition matrix or some of its components, $S[q]$, etc. Suppose X be a vector field on the space of potential, we shall denote it δq, when we need to mention X explicitly we shall write $\delta_X q$. Then the Gateau derivative of Ψ in the direction of X will be denoted by $\delta\Psi[q]$ or simply by $\delta\Psi$. Our first goal will be to calculate some useful derivatives. To this end we start with $F([q]) = S = \psi_0^{-1} J \psi_0$ and we readily get

$$\delta S = [S, \psi_0^{-1}\delta\psi_0] = \psi_0^{-1}[J, \delta\psi_0\psi_0^{-1}]\psi_0. \tag{17}$$

Next we calculate the derivative $\delta\psi_0$. Taking into account that $\mathrm{i}\partial_x\psi_0 + q\psi_0 = 0$ we obtain $\mathrm{i}\partial_x(\delta\psi_0) + \delta q\psi_0 + q\delta\psi_0 = 0$ or equivalently

$$\mathrm{i}\partial_x(\delta\psi_0)\hat{\psi}_0 + \delta q + q\delta\psi_0\hat{\psi}_0 = 0,$$

where the notation $\hat{\psi}_0 = \psi_0^{-1}$ is used. The above can be transformed into

$$\mathrm{i}\partial_x(\delta\psi_0\hat{\psi}_0) - \delta\psi_0(\mathrm{i}\partial_x\hat{\psi}_0) + \delta q + q\delta\psi_0\hat{\psi}_0 = 0,$$

which taking into account the identity $\partial_x\psi_0^{-1} = -\psi_0^{-1}\partial_x\psi_0\psi_0^{-1}$ gives

$$\mathrm{i}\partial_x(\delta\psi_0\hat{\psi}_0) + [q, \delta\psi_0\hat{\psi}_0] = -\delta q. \tag{18}$$

Taking the projections of this equation on the subalgebra $\mathfrak{h}$ and its orthogonal complement $\mathfrak{h}^\perp = \bar{\mathfrak{g}}$ we get

$$\begin{aligned}
&\mathrm{i}\partial_x(\delta\psi_0\hat{\psi}_0)^d + [q, (\delta\psi_0\hat{\psi}_0)^d]^a = 0, \\
&\mathrm{i}\partial_x(\delta\psi_0\hat{\psi}_0)^a + [q, (\delta\psi_0\hat{\psi}_0)^d] + [q, (\delta\psi_0\hat{\psi}_0)^a]^d = -\delta q,
\end{aligned} \tag{19}$$

where for given $X \in \mathfrak{g}$ we denote by X^d the projection $(\mathbf{1} - \pi_0)X$ and by X^a the projection $\pi_0 X$. Integrating the first equation we obtain

$$(\delta\psi_0\hat{\psi}_0)^d = \mathrm{i}\int_{\pm\infty}^{x} (\mathbf{1} - \pi_0)[q, \pi_0(\delta\psi_0\hat{\psi}_0)]\mathrm{d}y + (\delta\psi_0\hat{\psi}_0)^d_\pm \tag{20}$$

where $(\delta\psi_0\hat{\psi}_0)^d_\pm = \lim_{x\to\pm\infty}(\delta\psi_0\hat{\psi}_0)^d$ and introducing the above into the second equation gives

$$\mathrm{ad}_J\Lambda_\pm(\delta\psi_0\hat{\psi}_0)^a = -[q, (\delta\psi_0\hat{\psi}_0)^d_\pm] - \delta q. \tag{21}$$

If we have two bi-orthogonal bases $\{B_i\}_{i=1}^r$ and $\{B^i\}_{i=1}^r$ of $\mathfrak{h}$, that is two bases such that $\langle B_i, B^j\rangle = \delta_i^j$, we can cast the above formula into the form

$$\delta q = -\mathrm{ad}_J\Lambda_\pm(\delta\psi_0\hat{\psi}_0)^a + [B_j, q]\langle B^j, (\delta\psi_0\hat{\psi}_0)_\pm\rangle, \tag{22}$$

where $(\delta\psi_0\hat{\psi}_0)_\pm = \delta\psi_0\hat{\psi}_0(\pm\infty)$ and summation over repeated indexes is assumed. Using (17) the above gives

$$\delta q = -\mathrm{ad}_J\Lambda_\pm\mathrm{ad}_J^{-1}\pi_0(\psi_0\delta S\hat{\psi}_0) + [B_j, q]\langle B^j, (\delta\psi_0\hat{\psi}_0)_\pm\rangle \tag{23}$$

and applying to the both sides $\mathrm{Ad}(\hat{\psi}_0)$ we get

$$\widetilde{\delta q} = -\mathrm{ad}_S\tilde{\Lambda}_\pm\mathrm{ad}_S^{-1}\delta S + [\widetilde{B_j}, \tilde{q}]\langle B^j, (\delta\psi_0\hat{\psi}_0)_\pm\rangle$$

where if $Z(x)$ is a function with values in $\mathfrak{g}$ we denote by $\tilde{Z}(x)$ the function $\hat{\psi}_0 Z\psi_0$. This formula could be transformed further if we express $\tilde{q}$ through S. The last can be achieved if we differentiate $S = \hat{\psi}_0 J\psi_0$. Then we get

$$\mathrm{i}S_x = -[S, \hat{\psi}_0 q\psi_0] = -[S, \tilde{q}] \Rightarrow \tilde{q} = -\mathrm{i}\,\mathrm{ad}_S^{-1}S_x.$$

Thus finally

$$\widetilde{\delta q} = -\mathrm{ad}\,_S\tilde{\Lambda}_{\pm}\mathrm{ad}\,_S^{-1}\delta S + \mathrm{i}[\mathrm{ad}\ _S^{-1}S_x, \widetilde{B_j}]\langle B^j, (\delta\psi_0\hat{\psi}_0)_{\pm}\rangle \tag{24}$$

or equivalently

$$\widetilde{\delta q} = -\tilde{N}_{\pm}\delta S + \mathrm{i}[\mathrm{ad}\,_S^{-1}S_x, \widetilde{B_j}]\langle B^j, (\delta\psi_0\hat{\psi}_0)_{\pm}\rangle \tag{25}$$

where $\tilde{N}_{\pm} = \tilde{\Lambda}^*_{\pm}$. In fact if we denote by F the map taking q to S then the formulae (23),(24),(25) give the Gateau derivative of F^{-1}.

Let us make one final remark. In the applications that are known the components of $\psi_0(\pm\infty)$ are either constant or are integrals of motion for the hierarchies of integrable equations that are related to L, so in fact in all the calculations one can assume that $\delta\psi_0(\pm\infty) = 0$ which simplifies these calculations considerably, in fact we can write

$$\widetilde{\delta q} = -\tilde{N}_{\pm}\delta S = -\mathrm{ad}\,_S\tilde{\Lambda}_{\pm}\mathrm{ad}\,_S^{-1}\delta S. \tag{26}$$

Let us find the correspondence between gauge-equivalent equations. In order to simplify our task note that the operators $\Lambda_{\pm}$ could be made to act formally on functions that take values not only in $\bar{\mathfrak{g}}$ but in $\mathfrak{g}$. In particular, taking a constant element $B \in \mathfrak{h}$ we immediately get $\Lambda_{\pm}B = \mathrm{ad}\,_J^{-1}[q, B] = -\mathrm{ad}\,_J^{-1}[B, q]$. Applying to this expression $\mathrm{Ad}\,(\hat{\psi}_0)$ we get

$$\hat{\psi}_0(\mathrm{ad}\,_J^{-1}[B, q])\psi_0 = -\mathrm{iad}\,_S^{-1}[\tilde{B}, \mathrm{ad}\,_S^{-1}S_x] = -\tilde{\Lambda}_{\pm}\tilde{B}. \tag{27}$$

The formulae obtained in the above permit to obtain in a purely geometric way the transformation from the hierarchy of NLEEs associated with L to the hierarchy associated with $\tilde{L}$. Indeed, as well known, the hierarchy associated with L has the form

$$\mathrm{iad}\,_J^{-1}q_t + \Lambda^n_{\pm}\left(\mathrm{ad}\,_J^{-1}[B, q]\right) = 0 \tag{28}$$

(B here has the same meaning as in the above). Let us apply to both sides the transformation $\mathrm{Ad}\,(\hat{\psi}_0)$. Assume that $\delta q = q_t$ where the evolution is taken along one the vector fields defined by the hierarchy written above. Then we can assume that $(\delta\psi_0)_{\pm} = 0$ and we finally get

$$-\mathrm{iad}\,_S^{-1}\tilde{\Lambda}_{\pm}\frac{\partial S}{\partial t} + (\tilde{\Lambda}_{\pm})^n\tilde{\Lambda}_{\pm}\tilde{\pi}_0 B = 0 \tag{29}$$

or simply

$$-\mathrm{iad}\,_S^{-1}\frac{\partial S}{\partial t} + (\tilde{\Lambda}_{\pm})^n\tilde{\pi}_0 B = 0. \tag{30}$$

As to the relation between the P-N structures related with L and $\tilde{L}$ respectively, it is well known that if we use the map F in order to obtain from the P-N structure for L the P-N structure for $\tilde{L}$ we have, see [7]

Theorem 6.1. With the notation we had the above

$$\tilde{N} = [dF] \circ N \circ [dF]^{-1}, \tag{31}$$
$$\tilde{Q} = ad_S = [dF] \circ N^2 \circ Q \circ ([dF])^* = \tilde{N}^2 [dF] \circ Q \circ ([dF])^*.$$

(In these formulae the terms containing $\psi_0(\pm\infty)$ have been disregarded). From the above equations easily follows that

$$\tilde{N}^s \tilde{Q} = \tilde{N}^{s+2} [dF] \circ Q \circ ([dF])^*. \tag{32}$$

giving the famous "shift by 2" in the Hamiltonian structures related to L and $\tilde{L}$. Note also that we obviously have

$$\tilde{P} = \tilde{N}\tilde{Q} = \tilde{N}^2 [dF] \circ NQ \circ ([dF])^* = \tilde{N}^2 [dF] \circ P \circ ([dF])^*. \tag{33}$$

6˙1. $\mathbb{Z}_p$ *reductions.* – Consider now the impact of reductions of $\mathbb{Z}_p$ type defined by an inner automorphism $X \mapsto \mathcal{K}(X) = KXK^{-1}$ ($K^p = \mathbf{1}$), that is, for the CBC system $L = \mathrm{i}\partial_x + q - \lambda J$ we have $\mathcal{K}q = q$, $\mathcal{K}J = \omega J$ where $\omega = \frac{2\pi \mathrm{i}}{p}$. Since ψ_0 a solution of the equation $\mathrm{i}\partial_x \psi_0 + q\psi_0 = 0$ and since $\mathcal{K}q = KqK^{-1} = q$ one has that $K\psi_0 = \psi_0 H$ with some constant matrix H. Thus $\hat{\psi}_0 K \psi_0 = H$ and $B^p = \mathbf{1}$. An important observation is that we can assume here H not only independent of x, but also independent of q. Indeed, usually ψ_0 is the Jost solution for $\lambda = 0$ or it is a Jost solution multiplied by a constant matrix. In both cases $\psi_0(-\infty) = R$ and R does not depend on q since $\hat{R}KR = H$. As a consequence, from the above we have

$$HSH^{-1} = H\hat{\psi}_0 J \psi_0 H^{-1} = \hat{\psi}_0 KJK^{-1} \psi_0 = \hat{\psi}_0 \omega J K^{-1} \psi_0 = \omega S.$$

Then we obtain that $\tilde{L} = \mathrm{i}\partial_x - \lambda S$ allows reduction of $\mathbb{Z}_p$ type defined by an inned automorphism $X \mapsto \mathcal{H}(X) = HXH^{-1}$ ($H^p = \mathbf{1}$). Note also that

$$\mathrm{Ad}\,(\hat{\psi}_0(x))\mathcal{K} = \mathcal{H}\mathrm{Ad}\,(\hat{\psi}_0(x)). \tag{34}$$

Denote the eigenspaces corresponding to eigenvalues ω^s (as usual s is understood modulo p) for the automorphisms $\mathcal{K}$ and $\mathcal{H}$ by $\mathfrak{g}_K^{[s]}$ and $\mathfrak{g}_H^{[s]}$, respectively.

Using the properties of the automorphism $\mathcal{K}$ (the fact that it commutes with the projection π_0 on the Cartan subalgebra $\mathfrak{h}$) and the facts that $\mathcal{K}q = q$ and $\mathcal{K}J = \omega J$ we easily get

Lemma 6.1. If $\mathcal{K}$ is an automorphism of order p defining the $\mathbb{Z}_p$ reduction then

$$\Lambda_\pm \circ \mathcal{K} = \omega \mathcal{K} \circ \Lambda_\pm. \tag{35}$$

But then, taking into account that $\tilde{\Lambda}_\pm = \mathrm{Ad}(\hat{\psi}_0(x))\Lambda_\pm \mathrm{Ad}(\psi_0(x))$, we get also that

$$\tilde{\Lambda}_\pm \circ \mathcal{H} = \omega \mathcal{H} \circ \tilde{\Lambda}_\pm. \tag{36}$$

Since the Killing form is invariant under automorphisms, we get immediately that

$$N \circ \mathcal{K} = \omega \mathcal{K} \circ N, \quad \tilde{N} \circ \mathcal{H} = \omega \mathcal{H} \circ \tilde{N}. \tag{37}$$

Now, $q \mapsto S(q)$ maps functions taking values in $\mathfrak{g}_K^{[s]}$ to functions taking values in $\mathfrak{g}_H^{[1]}$, so in fact dF is identically zero restricted to $\mathfrak{g}_K^{[s]}$ with $s \neq 0 \ (\mathrm{mod}\,(p))$. Since $(\mathfrak{g}_K^{[s]})^* = \mathfrak{g}_K^{[p-s]}$ and $(\mathfrak{g}_H^{[s]})^* = \mathfrak{g}_H^{[p-s]}$ we see immediately that Q and $\tilde{Q}$ trivialize when restricted to the new manifolds of potentials. Also, the first equation in (31) is still true, but the second trivializes to give $0 = 0$. However, the relation (33) do not trivialize and one sees that it is compatible with the algebraic properties of the operators involved in it. Moreover, considering the following relations, which are also a consequence from (32)

$$\tilde{N}^s(\tilde{N}\tilde{Q}) = \tilde{N}^2[dF] \circ N^s(NQ) \circ ([dF])^* \tag{38}$$

we see that they trivialize exactly when $s \neq 0 \ \mathrm{mod}\,(p)$. But this means that on the new manifolds of potentials “survive” only the P-N structures (NQ, N^{pk}) (respectively $(\tilde{N}\tilde{Q}, \tilde{N}^{pk})$) and therefore, the Nijenhuis tensor when we have $\mathbb{Z}_p$ reduction will be N^p, $\tilde{N}^p$ just as we have seen in the particular cases we have described in the above. As for the NLEEs (or the fundamental fields of the above P-N structures), we see that for the system $\tilde{L}$, after the $\mathbb{Z}_p$ reduction, just as it was the case for the system L, “survive” the fields:

$$\tilde{N}^{kp+s}\tilde{H}, \quad H \in \mathfrak{g}_H^{[s]}, \ s = 0, 1, \ldots p-1; \quad k = 0, 1, 2, \ldots \tag{39}$$

Let us mention that we have developed the theory starting from the system L and passing to the system $\tilde{L}$. Starting from $\tilde{L}$ and passing to L is also possible and in fact has been performed in the case of the system we call the GMV system. In this case one has the system $\tilde{L} = \mathrm{i}\partial_x - \lambda S$ where $S(x)$ takes values in the orbit $\mathcal{O}_J$ of some element $J \in \mathfrak{g}$. In this case we need to “undress” S, that is, to find g such that $S = \hat{g}Jg$. This g plays the role of ψ_0 and the things unroll in the opposite direction the same way as when we pass from L to $\tilde{L}$.

7. – Comments and conclusions

We have shown that the effect of the Mikhailov-type reductions of the Generating Operators associated with Caudrey-Beals-Coifman systems in canonical and in pole gauge have a transparent geometric interpretation, confirming the discoveries of the spectral approach to the theory of the Generating Operators, namely that if we had at the beginning a Poisson-Nijenhuis structure (N, P) then after a $\mathbb{Z}_p$ reduction defined by an automorphism of order p we have a Poisson-Nijenhuis structure (N^p, P) defined on some submanifold. Our considerations also show that the relations between the Poisson-Nijenhuis structures is not destroyed by the reductions and therefore we can again speak about gauge-covariant theory of these operators.

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