COLLOQUIA: VILASIFEST

WDVV equations

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Summary. — The paper aims to point out a novel geometric characterization of the WDVV equations of 2D topological field theory.

1. – Introduction

The Witten-Dijkgraaf-Verlinde-Verlinde equations are a remarkable set of nonlinear partial differential equations discovered in 2D topological field theory at the end of 80's [1]. Afterwards they have found numerous and interesting applications in many areas. I simply quote the 2D SUSY Yang-Mills theory, the theory of Seiberg-Witten systems, the theory of Whitham equations, and the theory of integrable systems [2].

The mathematical structure of the WDVV equations has already been thoroughly studied, in the 90's, by Boris Dubrovin, who has invented the beautiful and far-reaching concept of Frobenius manifold to give the WDVV equations a geometric interpretation. Since then, the theory of Frobenius manifolds has become a subject of interest in itself [3-5]. The purpose of the present paper is to consider again the question of the mathematical structure of the WDVV equations. I wish to argue that there are two distinct ways of dealing with these equations. On the one hand, they can be seen as the equations defining a special class of associative and commutative algebras. This is the point of view followed by Boris Dubrovin, leading to the theory of Frobenius manifolds. On the other hand, the WDVV equations may be red as the equations defining special arrangements of 1-forms on a manifold, called Lenard complexes. This is the point of view worked out in this paper, leading to the theory of Haantjes manifolds.

The paper is rather concise and direct, and the references to the theory of WDVV equations and to the theory of Frobenius manifolds are reduced to the bare essential. The WDVV equations are defined in sect. **2**. In sect. **3** I remind the concept of Frobenius manifold and its relationship to the WDVV equations. In sect. **4** I present the concept of "Lenard complex", and I explain why it is related to the WDVV equations.

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2. – WDVV equations

The WDVV equations are an overdetermined system of nonlinear PDE's on a single function $F(x_1, x_2, \ldots, x_n)$ of n coordinates. The equations are constructed in two steps. First one considers the Hessian matrix of the function F

$$(\mathbf{2}.1) \qquad\qquad\qquad h = \text{Hessian}(F),$$

and then the components

$$(\mathbf{2}.2) \qquad\qquad\qquad c_j = \frac{\partial h}{\partial x_j}.$$

of the gradient of this matrix with respect to the given coordinates $(x_1, x_2, \ldots, x_n)$. One of these matrices, say c_1, is assumed to be invertible. The WDVV equations are hence written in the matrix form

$$(\mathbf{2}.3) \qquad \frac{\partial h}{\partial x_j} \cdot \left(\frac{\partial h}{\partial x_1} \right)^{-1} \cdot \frac{\partial h}{\partial x_l} = \frac{\partial h}{\partial x_l} \cdot \left(\frac{\partial h}{\partial x_1} \right)^{-1} \cdot \frac{\partial h}{\partial x_j}.$$

Often a second assumption is made on the matrix c_1, by requiring that it does not depend on the coordinates x_j:

$$(\mathbf{2}.4) \qquad\qquad\qquad \frac{\partial c_1}{\partial x_j} = 0.$$

The acceptance of this assumption, however, depends on the field of interest. For instance, it is accepted in 2D topological field theory and in the theory of Whitham equations, but it is rejected in the theory of Seiberg-Witten systems [6]. Accordingly I will consider eq. ($\mathbf{2}$.4) as an auxiliary assumption, useful but not indispensable to define the WDVV equations. In other words, in this paper I consider the "generalized" WDVV equations, which abstract from condition ($\mathbf{2}$.4), contrary to the "ordinary" WDVV equations which require it [7,8].

There is another form of the WDVV equations which must be quoted. It is

$$(\mathbf{2}.5) \qquad\qquad\qquad C_j \cdot C_l - C_l \cdot C_j = 0,$$

and concerns the matrices

$$(\mathbf{2}.6) \qquad\qquad\qquad C_j = \left(\frac{\partial h}{\partial x_1} \right)^{-1} \cdot \frac{\partial h}{\partial x_j}.$$

In this form the WDVV equations are more specifically called "associativity equations", since they entail that the entries $C_{jk}^l(x_1, \ldots, x_n)$ of the matrices C_j are the "structure constants" of an associative commutative algebra with unity.

The WDVV equations are clearly non tensorial. Any change of coordinates, different from an affine transformation, destroys the form of the equations. The coordinates $(x_1, x_2, \ldots, x_n)$ are therefore a basic constituent of the theory. This remark serves to

clarify that the problem of giving the WDVV equations a geometric (or intrinsic) interpretation has two complementary aspects. On one side, one would write these equations in a form that does not depend on the choice of the coordinates. On the other side, one must demand that the geometrical structure used to attain this result should be capable to select a class of coordinates, affinely related, in which the intrinsic equations take the specific form (**2**.3). The rule for the selection of the distinguished coordinates $(x_1, x_2, \ldots, x_n)$ is the key point of the process of geometrization of the WDVV equations. It is also the point which will mark the difference between the two approaches discussed in this paper.

Before leaving the subject of WDVV equations, let me show an amazing example. It has been worked out by M. Kontsevich in 1994 [9]. The equation

$$f^2_{x_2 x_2 x_3} = f_{x_3 x_3 x_3} + f_{x_2 x_2 x_2} \cdot f_{x_2 x_3 x_3}$$

is one of the most simple instances of WDVV equations in $\mathbb{R}^3$. It comes from the standard WDVV equation (**2**.3) by choosing a function F of the form

$$F(x_1 x_2 x_3) = \frac{1}{2}(x_1^2 x_3 + x_1 x_2^2) + f(x_2, x_3).$$

Albeit the equation is nonlinear, one may look for a solution in form of series

$$F = \sum_k \frac{N_k}{(3k-1)} x_3^{3k-1} e^{kx_2}.$$

The insertion of this series into the equation gives a recursive relation on the coefficients:

$$N_1 = 1, \qquad N_k = \sum_{p+q=k\geq 2} N_p N_q p^2 q \left[q \binom{3k-4}{3k-2} - p \binom{3k-4}{3k-1} \right].$$

The first coefficients are

$$N_2 = 1,$$
$$N_3 = 12,$$
$$N_4 = 620.$$

They are integers. These integers are of interest for the enumerative geometry. Indeed, N_1 is the number of straight lines passing through two distinct points; N_2 is the number of conics passing through five points in generic positions; N_3 is the number of cubics passing through eight points, and so on [10]. As proved by Kontsevich, all these coefficients are of interest to the enumerative geometry.

3. – Frobenius manifolds

In the early 90's, Boris Dubrovin has been the first to tackle the problem of giving the WDVV equations an intrinsic form, namely a form which does not depend on the choice of the coordinates. His idea has been to focus the attention on the matrices c_1

and C_j which enter into the WDVV equations in the way explained before. He considers the symmetric matrix c_1 as the matrix of the components of a semi-Riemannian metric, in the distinguished coordinate system $(x_1, \ldots, x_n)$. Since he uses the additional assumption

$$(\mathbf{3}.1) \qquad \frac{\partial c_1}{\partial x_j} = 0,$$

characteristic of 2D topological field theory , this metric is flat. Furthermore, he considers the entries $C_{jk}^l(x_1, \ldots, x_n)$ of the matrices $(C_1 = Id, C_2, \ldots, C_n)$ as the components of a third-order tensor field of type (1,2). This tensor field defines a product on the tangent bundle. The product is first defined on the basis associated with the coordinates $(x_1, \ldots, x_n)$

$$(\mathbf{3}.2) \qquad \partial_j \circ \partial_k = C_{jk}^l \partial_l,$$

and then extended by linearity. It is known under the name of "multiplicative structure on the tangent bundle". If the matrices C_j verify the associativity equations (**2**.5), this product is clearly associative, commutative, and with unity. The unity is the distinguished vector field $\frac{\partial}{\partial x_1}$.

These preliminary remarks have led Boris Dubrovin to choose, as the proper setting where to frame the geometric study of the WDVV equations, a class of manifolds endowed with three geometric structures:

1. a flat semi-Riemannian metrics $\qquad\qquad g : TM \times TM \to \mathbb{R}$
2. an associative and commutative product $\qquad \circ : TM \times TM \to TM$
   on the tangent bundle
3. a distinguished vector field $\qquad\qquad\qquad e : M \to TM.$

The conditions which must relate these structures in order to reproduce the WDVV equations are specified by Dubrovin in his definition of Frobenius manifold.

Definition 1. *A Frobenius manifold is a semi-Riemannian manifold (M,g), endowed with an associative and commutative product on the tangent bundle, and with a distinguished vector field $e : M \to TM$, which verify the following five conditions:*

1. $Riemann(g) = 0$

2. $g(X \circ Y, Z) = g(X, Y \circ Z)$

3. $e \circ X = X$

4. $\nabla_\bullet e = 0$

5. $\nabla_W(g(X \circ Y, Z)) = \nabla_Z(g(X \circ Y, W))$

for any set of four covariantly constant vector fields (X,Y,Z,W).

The first four conditions are simple and natural: the metric is flat; the product is symmetric with respect to the metric; the distinguished vector field e is the unity of the

product; it is covariantly constant with respect to the Levi-Civita connection induced by the metric. The fifth condition, on the contrary, is much less intuitive. It demands that the covariant derivative of the symmetric tensor field $g(X \circ Y, Z)$ be still a symmetric tensor field of the fourth-order. Notwithstanding, this fifth condition is the key to arrive to the WDVV equations.

The return to the WDVV equations proceeds as follows. The metric g gives the flat coordinates x_j; the product gives the structure constants $C_{jk}^l(x_1, \ldots, x_n)$; the structure constants and the metric give the third-order symmetric tensor field

$$(\mathbf{3}.3) \qquad c_{jkm} = g_{ml} C_{jk}^l(x_1, \ldots, x_n).$$

The fifth condition implies that the components of this tensor field, in the flat coordinates, are the third-order derivatives of a function $F(x_1, x_2, \ldots, x_n)$. The associativity property of the product, finally, entails that this function satisfies the ordinary WDVV equations.

This is the essence of the geometric interpretation of the WDVV equations in the language of the theory of Frobenius manifolds.

4. – Lenard complexes

There is another way of giving the WDVV equations a geometrical interpretation. It appears on the stage when the attention is focused on the Hessian matrix of the function F

$$h = \text{Hessian}(F),$$

rather than on its derivatives

$$c_1 = \frac{\partial h}{\partial x_1} \qquad C_j = \left(\frac{\partial h}{\partial x_1} \right)^{-1} \cdot \frac{\partial h}{\partial x_j}.$$

This shift of focus has the effect to bring into action a novel geometrical structure, called a Lenard complex on a Haantjes manifold.

The most convenient way to discover the novel geometrical structure is to consider the problem of the characterization of the Hessian matrices on a manifold M.

Problem (geometrical characterization of the Hessian matrices). *Given n^2 scalar-valued functions $A_{jl} : M \to \mathbb{R}$ satisfying the symmetry condition*

$$(\mathbf{4}.1) \qquad A_{jl} = A_{lj},$$

the problem is to know if there exist a distinguished coordinate system x_j, on M, such that the functions A_{jl}, written in these coordinates, are the entries of the Hessian matrix of a suitable function $F(x_1, \ldots, x_n)$. Stated in a different form, the problem is to work out a criterion that guarantees that the system of partial differential equations

$$(\mathbf{4}.2) \qquad \frac{\partial^2 F}{\partial x_j \partial x_l}(x_1, \ldots, x_n) = A_{jl}(x_1, \ldots, x_n),$$

where both the coordinates x_j and the function F are unknown, admits a solutions.

To state this criterion, I need a few new objects. The first is the square of 1-forms whose entries are the differentials of the functions A_{jl}. The second is a specific column of this square, chosen according to the criterion that the differentials sitting on this column are linearly independent. The third is the special differential of this column which belongs to the diagonal. The fourth is the family of tensor fields of type (1,1) which intertwine the remaining columns of the square with the distinguished column which has been selected. To be specific, let me assume that the distinguished column is the first column of the square of 1-forms. I use special symbols to denote the entries of the first column of the square and, in particular, the entry sitting on the diagonal. I set

$$(\mathbf{4}.3) \qquad\qquad\qquad \mathrm{d}a_l = \mathrm{d}A_{1l}$$

$$(\mathbf{4}.4) \qquad\qquad\qquad \mathrm{d}A = \mathrm{d}A_{11}.$$

The tensor fields $K_j : TM \to TM$ which intertwine the columns are accordingly defined by the relations

$$(\mathbf{4}.5) \qquad\qquad\qquad K_j \mathrm{d}a_l = \mathrm{d}A_{jl}.$$

I noticed that $K_1 = Id$, and that

$$(\mathbf{4}.6) \qquad\qquad\qquad K_j K_l \mathrm{d}A = \mathrm{d}A_{jl},$$

by the symmetry condition (**4**.1).

Definition 2. *The 1-form* $\mathrm{d}A$, *the functions* a_l, *and the tensor fields* K_j *are called the pivot, the a-coordinates, and the recursion operators attached to the square of 1-forms* $\mathrm{d}A_{jl}$ *respectively.*

Let me now consider any vector field X on the manifold M. The recursion operators K_j allow to generate the chain of vector fields

$$X_j = K_j X.$$

Of particular interest, for our problem, are the derivatives of the functions A_{jl} along the vector fields of the chain. They will be denoted by the symbols

$$c_{jlm} = X_m(A_{jl}) = \mathrm{d}A_{jl}(X_m).$$

Definition 3. *The chain of vectors fields* X_j *is called the Lenard chain generated by* X *and by the recursion operators* K_j. *The functions* c_{jlm} *are called the "3-points correlation functions", relating the chain of vector fields to the square of 1-forms.*

The chain of vector fields and the associated correlation functions are the tools which allow to answer the question set initially.

Lemma 1. *The system of partial differential equations (**4**.2) has a solution if and only if the following two conditions hold true:*

    I. *There exists a vector field X on M which generates a Lenard chain of linearly independent commuting vector fields*

$$(\mathbf{4}.7) \qquad\qquad\qquad [X_j, X_l] = 0.$$

    II. *The 3-points correlation functions of the Lenard chain of vector fields are symmetric in (j,l,m):*

$$(\mathbf{4}.8) \qquad\qquad\qquad \mathrm{d}A_{jl}(X_m) = \mathrm{d}A_{jm}(X_l).$$

Proof. ***If:*** By assumption (I) there exists a coordinate system x_j such that

$$X_j = \frac{\partial}{\partial x_j}.$$

By assumption (II), in these coordinates the partial derivatives of the functions A_{jl} are symmetric in the indexes (j, l, m):

$$\frac{\partial A_{jl}}{\partial x^m} = \frac{\partial A_{jm}}{\partial x^l}.$$

Therefore, these functions are the second-order derivatives of a single function $F(x_1, \ldots, x_n)$.

    Only if: Let me assume that the functions A_{jl} are the entries of the Hessian matrix of a function F, in a distinguished coordinate system x_j. I set $X = \frac{\partial}{\partial x_1}$. To complete the proof of the Lemma, I have to show that

$$(4.9) \qquad\qquad\qquad K_j \frac{\partial}{\partial x_1} = \frac{\partial}{\partial x_j},$$

namely that the basis $\frac{\partial}{\partial x_i}$, associated with the distinguished coordinates, is the Lenard chain generated by the recursion operators K_j and by the vector field $\frac{\partial}{\partial x_1}$ corresponding to the pivot of the square of 1-forms. I notice that

$$\mathrm{d}a_l \left(K_j \frac{\partial}{\partial x^1} \right) = \mathrm{d}A_{jl} \left(\frac{\partial}{\partial x^1} \right)$$
$$= \frac{\partial A_{jl}}{\partial x^1}$$
$$= \frac{\partial A_{1l}}{\partial x^j}$$
$$= \mathrm{d}a_l \left(\frac{\partial}{\partial x_j} \right).$$

This shows that condition I is true. Since condition II is obviously satisfied, the proof is complete. □

After this preliminary study of the Hessian matrices in general, the return to the WDVV equations is rather simple. As the reader may expect, the solutions of the "generalized" WDVV equations are characterized by the property that the recursion operators attached to their Hessian matrices, in the sense explained above, commute in pairs. More precisely, one may prove the following claim.

Lemma 2. *The recursion operators attached to the Hessian matrix of any solution of the WDVV equations commute in pairs, and they identify the basis $\frac{\partial}{\partial x_j}$, of the distinguished coordinates used to write the WDVV equations, with the Lenard chain generated by $\frac{\partial}{\partial x_1}$. Viceversa, assume that the recursion operators K_j attached to any set of functions A_{jl}, verifying the symmetry condition $A_{jl} = A_{lj}$, commute in pairs. Assume, furthermore, that there exist a vector field X, on M, which satisfies the condition I of Lemma 1. Then the functions A_{jl} are the entries of the Hessian matrix of a function F which satisfies the WDVV equations.*

Proof. The proof follows from the simple remark that the matrices representing the recursion operators K_j, on the basis $\frac{\partial}{\partial x_j} = K_j X$ defined by the chain of vector fields, are

$$(\mathbf{4}.10) \qquad K_j = \left(\frac{\partial h}{\partial x_1} \right)^{-1} \cdot \frac{\partial h}{\partial x_j},$$

where h denotes the symmetric matrix whose entries are the functions A_{jl}. Thus, the operators K_j commute if and only if the matrices $\frac{\partial h}{\partial x_j}$ obey the WDVV equations (**2**.3). $\square$

The above two Lemmas suggest to consider the following composite geometric structure, called a "Lenard complex on a Haantjes manifold". It is a mild extension of the concept of "Lenard chain on a bi-Hamiltonian manifolds" [11,12]. To define the complex one needs:

    1. a vector field, $X : M \to TM,$

    2. an exact 1-form, $\mathrm{d}A : M \to T^*M,$

    3. a family of tensor field of type (1,1), $K_j : TM \to TM,$

in number equal to the dimension of the manifold. By assumption, they pairwise commute

$$K_j K_l - K_l K_j = 0.$$

Their action on X and $\mathrm{d}A$ gives rise to the usual chains of vectors fields

$$X_j = K_j X$$

and of 1-forms

$$\theta_j = K_j \mathrm{d}A.$$

More importantly they also give rise to the (symmetric) square of 1-forms

$$\theta_{jl} = K_j K_l \mathrm{d}A.$$

This square of forms is the main novelty of Lenard complexes with respect to the old theory of Lenard chains. Another difference is that the recursion operator K_j are not the powers of a single operator K. A third difference is that nothing is assumed, *a priori*, on the torsion of the recursion operators K_j. They may have torsion. Notwithstanding, it can be shown that the recursion operators of a Lenard complex have always vanishing Haantjes torsion. This is the ultimate reason to call Haantjes manifolds the manifolds supporting a Lenard complex.

Definition 4. *The composite structure formed by the chain of vector fields X_j, by the chain of 1-forms θ_j, and by the square of 1-forms θ_{jl} is a Lenard complex on the Haantjes manifold M if*

$$[X_j, X_l] = 0,$$
$$\mathrm{d}\theta_j = 0,$$
$$\mathrm{d}\theta_{jl} = 0,$$

that is if the vector fields commute and if the 1-forms both of the chain and of the square are closed and, therefore, locally exact. If, furthermore, $K_1 = \mathrm{Id}$ the Lenard complex is said to admit a unity.

By adopting this language, the main content of Lemma 2 may be finally stated in the following geometric form.

Proposition. *There exists a one-to-one correspondence between the solutions of the generalized WDVV equations and the Lenard complexes with unity on a Haantjes manifold. In other words: the Hessian matrix of any solution of the WDVV equations is the matrix of the potentials of the square of 1-forms of a Lenard complex with unity; viceversa, the potentials of the square of 1-forms of a Lenard complex with unity are the entries of the Hessian matrix of a function which satisfies the WDVV equations.*

This Proposition is a simple restatement of the previous two Lemmas, and does not require accordingly an independent proof. Allowing to identify the solutions of the WDVV equations with the Lenard complexes (with unity) on a Haantjes manifold, it provides the second geometric interpretation of the WDVV equations announced at the beginning of this paper.

5. – Concluding remarks

In this paper I have compared two different geometrical interpretations of the WDVV equations of 2D topological field theory. The first is the classical interpretation proposed by Boris Dubrovin, based on the concept of Frobenius manifold. The second is a novel interpretation, based on the concept of Lenard complex on a Haantjes manifold. The geometric scheme of Frobenius manifolds suggests to interpret the derivatives of the Hessian matrix of a solution of the WDVV equations

$$c_1 = \frac{\partial h}{\partial x_1} \qquad C_j = \left(\frac{\partial h}{\partial x_1} \right)^{-1} \cdot \frac{\partial h}{\partial x_j}$$

as a flat semi-Riemannian metric and as a multiplicative structure on the tangent bundle respectively. The geometric scheme of Haantjes manifolds suggests to interpret the Hessian matrix itself, rather than its derivatives, as the matrix of the potentials of the square of 1-forms of a Lenard complex. By the first approach the theory of WDVV equations is framed into the geometry of semi-Riemannian flat manifolds. By the second approach the same theory is framed within the geometry of bi-Hamiltonian manifolds. The main difference is in the way of introducing the distinguished coordinates x_j that are used to write the WDVV equations. In the scheme of Frobenius manifolds they are the flat coordinates of the flat semi-Riemannian metric. This point of view demands the supplementary condition

$$\frac{\partial c_1}{\partial x_j} = 0$$

and, therefore, restrict the attention to the ordinary WDVV equations. In the scheme of Haantjes manifolds, instead, the coordinates x_j are the coordinates defined by a Lenard chain of commuting vector fields. It does not demand the supplementary condition. Thus it allows to deal with the generalized WDVV equations as well.

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REFERENCES

[1] DIJKGRAAF R., VERLINDE E. and VERLINDE H., *Nucl. Phys. B*, **352** (1991) 59.

[2] MARSHAKOV A., *Seiberg-Witten theory and integrable systems* (Word Scientific, Singapore) 1999.

[3] DUBROVIN B., *Geometry of 2D topological field theory, Integrable systems and Quantum Groups* (Montecatini Terme 1993), *Lect. Notes Math.*, **1620** (1996) 120.

[4] MANIN Y., *Frobenius manifolds quantum cohomology and moduli spaces*, Vol. **47**, AMS Colloquium publications, 1999.

[5] HERTLING C., *Frobenius manifolds and moduli spaces for singularities* (Cambridge University Press) 2002.

[6] MARSHAKOV A., MIRONOV A. and MOROZOV A., *Phys. Lett. B*, **389** (1996) 43.

[7] MOROZOV A., *Phys. Lett. B*, **427** (1998) 93.

[8] FEIGIN M. V. and VESELOV A. P., *Amer. Math. Soc. Transl.(2)*, **224** (2008) 111.

[9] KONTSEVICH M. and MANIN YU., *Commun. Math. Phys.*, **164** (1994) 525.

[10] STRACHAN I., *Philos. Trans. R. Soc. London A*, **361** (2003) 2633.

[11] MAGRI F., *Recursion operator and Frobenius manifolds*, in SIGMA 8 (2012), Paper 076.

[12] MAGRI F., *Haantjes Manifolds and Veselov Systems*, in *Proceedings of the Conference "Physics and Mathematics of Nonlinear Phenomena", 20-27 June 2015, Gallipoli Italy*, arXiv:1510.07951, to be published in *Theor. Math. Phys.*