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RESEARCH ARTICLE

The energy of four graphs of some metacyclic 2-groups

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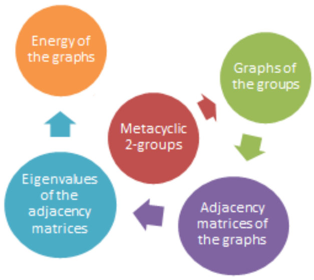
Graphical abstract



Abstract

Let G be a metacyclic 2-group and Γ_G is the graph of G. The adjacency matrix of Γ_G is a matrix $A = [a_{ij}]$ consisting of $0's$ and $1's$ in which the entry a_{ij} is 1 if there is an edge between the i^{th} and j^{th} vertices and 0 otherwise. The energy of a graph is the sum of all absolute values of the eigenvalues of the adjacency matrix of the graph. In this paper, the energy of commuting graph, non-commuting graph, conjugate graph and conjugacy class graph of some metacyclic 2-groups are presented. The results show that the energy of these graphs of the groups must be an even integer.

Keywords: Energy of graph, adjacency matrix, conjugacy class, metacyclic group.

INTRODUCTION

Ivan Gutman has first defined the energy of a graph in 1978 motivated by Hückel's theory in 1930's [1]. Hückel Molecular Orbital Theory has been used by chemists in approximating the energies related to p-electron orbitals in conjugated hydrocarbon [2]. In mathematics, the energy of a graph of a group is basically the sum of the absolute values of the eigenvalues. The eigenvalues are determined based on the adjacency matrix of the related graph.

This paper consists of three sections. The first section is the introduction section, followed by the second section, namely the preliminaries where some basic concepts, definitions and previous results on group and graph theory are stated. In the third section, the main results on computing the energies of four graphs of some metacyclic 2-groups are presented. These graphs are the commuting graph, non-commuting graph, conjugate graph, and conjugacy class graph.

PRELIMINARIES

Group theory is widely used in many branches of physical sciences. It is also used in solving Rubik's cube and to study the shape of viruses. When an operation like multiplication or composition is applied to a set or system, then the group is formed [2]. The following are some definitions in group theory that are used in this research.

Definition 1.1 [3] Metacyclic Group
A group is metacyclic if it has a cyclic normal subgroup H such that G/H is cyclic.
The following is the definition of the conjugate between two elements of a group G.

Definition 1.2 [4] Conjugate
Let a and b be two elements in finite group G, then a and b are called conjugate if there exist an element g in G such that $gag^{-1} = b$.

Definition 1.3 [5] Conjugacy Class
Let $x \in G$. The conjugacy class of g is the set $cl(g) = \{aga^{-1} | a \in G\}$ for all a in G.

Definition 1.4 [6] Center of a Group
The center, $Z(G)$ of a group G is the subset if elements in G that commute with every element of G, written as $Z(G) = \{a \in G \mid ax = xa, \forall x \in G\}$.

Graph theory has a wide range of applications in numerous areas such as engineering, biological sciences and computer sciences [7]. A graph of a group, $\Gamma(G)$ is a graph which consists of a finite set of vertices and edges where the vertices can consist of the elements of the group or based on the properties of group while K_n denotes a complete graph of n vertices in which all vertices are connected to each other. The following are some basic concepts on graph theory which will be frequently used in the later sections.

Definition 1.5 [8] Commuting Graph
Let G be a finite group. The commuting graph of G, denoted by Γ_G^{comm}, is the graph whose vertex set is $G - Z(G)$ and whose edges are pairs $\{h, g\} \subseteq G - Z(G)$, such that $h \neq g$ and $[h, g] \in Z(G)$.

Definition 1.6 [9] Non-commuting Graph
Let G be a finite group. The non-commuting graph of G, denoted by Γ_G^{nc}, is the graph of vertex set $G - Z(G)$ and two distinct vertices x and y are joined by an edge whenever $xy \neq yx$.

Definition 1.7 [10] Conjugate Graph
A nonabelian group G with vertex set $G\backslash Z(G)$ such that two distinct vertices are joined by an edge if they are conjugate is said to be a conjugate graph, denoted by Γ_G^{conj}.

The conjugate graph, denoted by, Γ_G^{conj}, has been introduced by Erfanian and Toule [10] in 2012.

Definition 1.8 [11] Conjugacy Class Graph
Let G be a finite group. A conjugacy class graph, denoted as Γ_G^{CC}, is a graph with vertices $V = \{v_1, \dots, v_n\}$ represented by the non-central conjugacy classes of G. Two vertices v_1 and v_2 are connected if $|v_1|$ and $|v_2|$ have a common prime divisor.

The main idea to compute the energy of graph is by calculating the eigenvalues of the adjacency matrix. Hence, the characteristic polynomial need to be obtained first in order to find the eigenvalues.

Definition 1.9 [5] Adjacency Matrix
The adjacency matrix is also called a connection matrix of a graph of group G, Γ_G with n vertices and no parallel edges which is defined as the following :

$$A(\Gamma_G) = \begin{cases} x_{ij} = 1, & \text{if } V_i \to V_j, \\ x_{ij} = 0, & \text{otherwise,} \end{cases}$$

where $V_i \to V_j$ represents the edge between i^{th} and j^{th} vertices.

Definition 1.10 [12] The Characteristic Polynomial
The equation $\det(A - \lambda I) = 0$ is called the characteristic equation of A where A is the adjacency matrix, λ is a scalar and I is the identity matrix. If $f(\lambda) = \det(\lambda I - A)$, then f is called a characteristic polynomial of A.

Definition 1.11 [1] Energy of Graph
The energy of a graph of a group G, Γ_G, denoted by $\varepsilon = \varepsilon(\Gamma_G)$ is the sum of absolute values of all eigenvalues of a graph, written as

$$\varepsilon(\Gamma_G) = \sum_{i=1}^{n} |\lambda_i|$$

where λ_i are the eigenvalues of the graph which $i = 1, \dots, n$.

In 2005, Beuerle [13] separated the classification of metacyclic p-groups into two parts, namely for the non-abelian metacyclic p-groups of class two and class three. Based on [13], the metacyclic p-groups of nilpotency class two are then partitioned into two families of non-isomorphic p-groups stated as follows :

1. $G \cong < a, b : a^{2^\alpha} = 1, b^{2^\beta} = 1, [a, b] = a^{2^{\alpha-\lambda}} >$,
   where $\alpha, \beta, \lambda \in \mathbb{N}, \alpha \geq 2\lambda, \beta \geq \lambda \geq 1$.
2. $G \cong < a, b : a^4 = 1, b^2 = [a, b] = a^{-2} >$,
   a quaternion group of order 8, Q_8.

The research considers all groups in the above classification up to order 32 in which $p = 2$, which gives the following :

$G_1 \cong < a, b : a^4 = b^2 = 1, bab = a^{-1} >$, the dihedral group of order 8.
$G_2 \cong < a, b : a^4 = 1, b^2 = [b, a] = a^{-2} >$, the quaternion group of order 8.
$G_3 \cong < a, b : a^8 = b^2 = [a, b] = a^4 >$, modular-16.
$G_4 \cong < a, b : a^{16} = b^2 = 1, [a, b] = a^8 >$, modular-32.

Now, some works related to commuting graph, non-commuting graph, conjugate graph and conjugacy class graph are stated.

In 2016, the conjugacy classes, conjugate graph and conjugacy class graph of G_2, G_3 and G_4 have been determined by Bilhikmah *et al.* in [14]. Recently, Alimon *et al.* [15] have extended the findings by determining the adjacency matrices of the conjugate graph for some metacyclic 2-groups in 2017. The following are some related theorems and the proofs can be found in [14].

Theorem 2.1 [14] Let G_2 be the quaternion group of order 8, $G_2 \cong < a, b : a^4 = 1, b^2 = [b, a] = a^{-2} >$. Then, the conjugate graph of G_2 is $\Gamma_{G_2}^{conj} = \bigcup_{i=1}^{3} K_2$, i.e the union of three complete components K_2 and the conjugacy class graph of G_2 is $\Gamma_{G_2}^{CC} = K_3$.

Theorem 2.2 [14] Let G_3 be a metacyclic 2-group of order 16, $G_3 \cong < a, b : a^8 = b^2 = 1, [a, b] = a^4 >$. Then, the conjugate graph of G_3 is $\Gamma_{G_3}^{conj} = \bigcup_{i=1}^{6} K_2$ while the conjugacy class graph of G_3 is $\Gamma_{G_3}^{CC} = K_6$.

Theorem 2.3 [14] Let G_4 be a metacyclic 2-group of order 32, $G_4 \cong < a, b : a^{16} = b^2 = 1, [a, b] = a^8 >$. Then, the conjugate graph of G_4 is $\Gamma_{G_4}^{conj} = \bigcup_{i=1}^{12} K_2$ and the conjugacy class graph of G_4 is $\Gamma_{G_3}^{CC} = K_{12}$.

In 2010, Bapat has shown that if the energy of a graph is a rational number, then it must be an even integer [16].

RESULTS AND DISCUSSION

In this section, the adjacency matrices of the commuting graphs, non-commuting graphs, conjugate graphs and conjugacy class graphs of dihedral group of order 8, G_1, quaternion group of order 8, G_2, a metacyclic 2-group of order 16 and 32 i.e G_3 and G_4, respectively, are given. Then, the energy of the commuting graphs, non-commuting graphs, conjugate graphs, and conjugacy class graphs of G_1, G_2, G_3, and G_4, respectively are determined.

The adjacency matrices of commuting graphs of four non-abelian metacyclic 2-groups are given in the following lemmas.

Lemma 3.1 Let $G_1 \cong < a, b : a^4 = b^2 = 1, bab = a^{-1} >$. Then, the adjacency matrix of the commuting graph of G_1 is stated as follows:

$$A\left(\Gamma_{G_1}^{comm}\right) = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Proof The dihedral group of order eight, G_1 has eight elements where it has two central elements. Thus, the number of vertices of G_1 is 6 and by Definition 1.5, the commuting graph of G_1, $\Gamma_{G_1}^{comm}$ consists of three components of K_2, as represented in Figure 3.1.

Figure 3.1 The commuting graph of G_1

In the adjacency matrix, the connected vertices are represented as 1 and 0, otherwise. □

Lemma 3.2 Let $G_2 \cong < a, b : a^4 = 1, b^2 = [a, b] = a^{-2} >$. Then, the adjacency matrix of the commuting graph of G_2 is stated as follows:

$$A\left(\Gamma_{G_2}^{comm}\right)=\begin{bmatrix} 0&0&1&0&0&0\\ 0&0&0&0&0&1\\ 1&0&0&0&0&0\\ 0&0&0&0&1&0\\ 0&0&0&1&0&0\\ 0&1&0&0&0&0 \end{bmatrix}.$$

Proof The quaternion group of order eight, G_2 has eight elements where it has two central elements. Thus, the number of vertices of G_2 is 6 and by Definition 1.5, the commuting graph of G_2, $\Gamma_{G_2}^{comm}$ consists of three components of K_2, as represented in Figure 3.2.

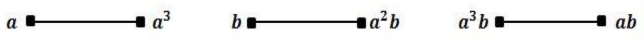


Figure 3.2 The commuting graph of G_2

In the adjacency matrix, the connected vertices are represented as 1 and 0, otherwise. □

Lemma 3.3 Let $G_3 \cong < a, b : a^8 = b^2 = 1, [a, b] = a^4 >$. Then, the adjacency matrix of the commuting graph of G_3 is stated as follows:

$$A\left(\Gamma_{G_3}^{comm}\right)=\begin{bmatrix} 0&0&0&1&0&1&0&0&1&0&0&0\\ 0&0&1&0&0&0&0&0&0&0&1&1\\ 0&1&0&0&0&0&0&0&0&0&1&1\\ 1&0&0&0&0&1&0&0&1&0&0&0\\ 0&0&0&0&0&0&1&1&0&1&0&0\\ 1&0&0&1&0&0&0&0&1&0&0&0\\ 0&0&0&0&1&0&0&1&0&1&0&0\\ 0&0&0&0&1&0&1&0&0&1&0&0\\ 1&0&0&1&0&1&0&0&0&0&0&0\\ 0&0&0&0&1&0&1&1&0&0&0&0\\ 0&1&1&0&0&0&0&0&0&0&0&1\\ 0&1&1&0&0&0&0&0&0&0&1&0 \end{bmatrix}.$$

Proof The metacyclic 2-group of order 16, G_3 has 16 elements where it has four central elements. Thus, the number of vertices of G_3 is 12 and by Definition 1.5, the commuting graph of G_3 , $\Gamma_{G_3}^{comm}$ consists of three components of K_4, as represented in Figure 3.3.

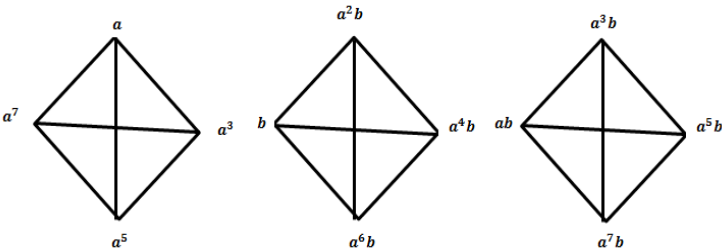


Figure 3.3 The commuting graph of G_3

In the adjacency matrix, the connected vertices are represented as 1 and 0, otherwise. □

Lemma 3.4 Let $G_4 \cong < a, b : a^{16} = b^2 = 1, [a, b] = a^8 >$. Then, the adjacency matrix of the commuting graph of G_4, $A(\Gamma_{G_4}^{comm})$ is stated as follows:

$$A(\Gamma_{G_4}^{comm})=\begin{bmatrix} 0&0&1&0&0&1&0&0&1&0&0&1&0&0&1&0&0&1&0&0&1&0&0&0\\ 0&0&0&0&1&0&0&1&0&0&1&0&0&1&0&0&1&0&0&1&0&0&0&1\\ 1&0&0&0&0&1&0&0&1&0&0&1&0&0&1&0&0&1&0&0&1&0&0&0\\ 0&0&0&0&0&0&1&0&0&1&0&0&1&0&0&1&0&0&1&0&0&1&1&0\\ 0&1&0&0&0&0&0&1&0&0&1&0&0&1&0&0&1&0&0&1&0&0&0&1\\ 1&0&1&0&0&0&0&0&1&0&0&1&0&0&1&0&0&1&0&0&1&0&0&0\\ 0&0&0&1&0&0&0&0&0&1&0&0&1&0&0&1&0&0&1&0&0&1&1&0\\ 0&1&0&0&1&0&0&0&0&0&1&0&0&1&0&0&1&0&0&1&0&0&0&1\\ 1&0&1&0&0&1&0&0&0&0&0&1&0&0&1&0&0&1&0&0&1&0&0&0\\ 0&0&0&1&0&0&1&0&0&0&0&0&1&0&0&1&0&0&1&0&0&1&1&0\\ 0&1&0&0&1&0&0&1&0&0&0&0&0&1&0&0&1&0&0&1&0&0&0&1\\ 1&0&1&0&0&1&0&0&1&0&0&0&0&0&1&0&0&1&0&0&1&0&0&0\\ 0&0&0&1&0&0&1&0&0&1&0&0&0&0&0&1&0&0&1&0&0&1&1&0\\ 0&1&0&0&1&0&0&1&0&0&1&0&0&0&0&0&1&0&0&1&0&0&0&1\\ 1&0&1&0&0&1&0&0&1&0&0&1&0&0&0&0&0&1&0&0&1&0&0&0\\ 0&0&0&1&0&0&1&0&0&1&0&0&1&0&0&0&0&0&1&0&0&1&1&0\\ 0&1&0&0&1&0&0&1&0&0&1&0&0&1&0&0&0&0&0&1&0&0&0&1\\ 1&0&1&0&0&1&0&0&1&0&0&1&0&0&1&0&0&0&0&0&1&0&0&0\\ 0&0&0&1&0&0&1&0&0&1&0&0&1&0&0&1&0&0&0&0&0&1&1&0\\ 0&1&0&0&1&0&0&1&0&0&1&0&0&1&0&0&1&0&0&0&0&0&0&1\\ 1&0&1&0&0&1&0&0&1&0&0&1&0&0&1&0&0&1&0&0&0&0&0&0\\ 0&0&0&1&0&0&1&0&0&1&0&0&1&0&0&1&0&0&1&0&0&0&1&0\\ 0&0&0&1&0&0&1&0&0&1&0&0&1&0&0&1&0&0&1&0&0&1&0&0\\ 0&1&0&0&1&0&0&1&0&0&1&0&0&1&0&0&1&0&0&1&0&0&0&0 \end{bmatrix}.$$

Proof The metacyclic 2-group of order 32, G_4 has 32 elements where it has eight central elements. Thus, the number of vertices of G_3 is 24 and by Definition 1.5, the commuting graph of G_4, $\Gamma_{G_4}^{comm}$ consists of three components of K_8, as represented in Figure 3.4.

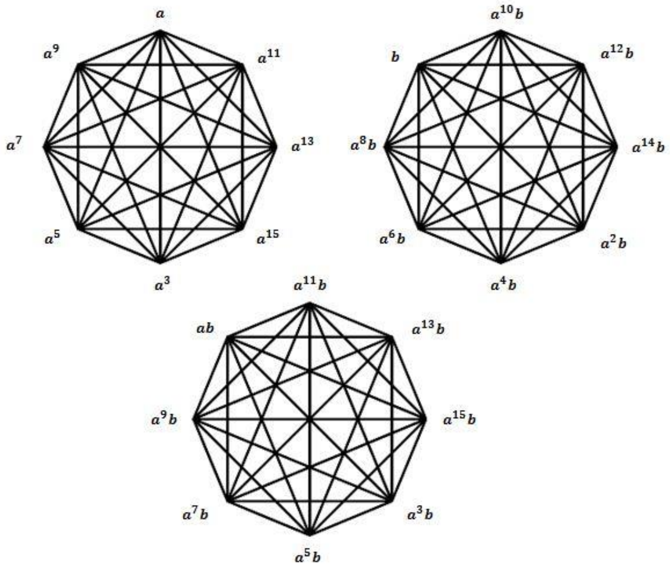


Figure 3.4 The commuting graph of G_4

In the adjacency matrix, the connected vertices are represented as 1 and 0, otherwise. □

Next, the adjacency matrices of non-commuting graphs of G_1, G_2, G_3, and G_4 are stated in the following lemmas.

Lemma 3.5 Let $G_1 \cong < a, b : a^4 = b^2 = 1, bab = a^{-1} >$. Then, the adjacency matrix of the non-commuting graph of G_1 is stated as follows:

$$A\left(\Gamma_{G_1}^{nc}\right)=\begin{bmatrix} 0&1&0&1&1&1\\ 1&0&1&1&1&0\\ 0&1&0&1&1&1\\ 1&1&1&0&0&1\\ 1&1&1&0&0&1\\ 1&0&1&1&1&0 \end{bmatrix}.$$

Proof Since the non-commuting graph is a complement of commuitng graph, then the non-commuting graph of G_1 is illustrated in Figure 3.5.

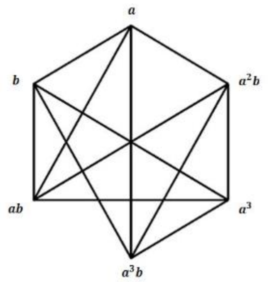


Figure 3.5 The non-commuting graph of G_1

In the adjacency matrix, the connected vertices are represented as 1 and 0, otherwise. □

Lemma 3.6 Let $G_2 \cong < a, b : a^4 = 1, b^2 = [a, b] = a^{-2} >$. Then, the adjacency matrix of the non-commuting graph of G_2 is stated as follows:

$$A\left(\Gamma_{G_2}^{nc}\right) = \begin{bmatrix} 0 & 1 & 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 & 1 & 0 \end{bmatrix}.$$

Proof Since the non-commuting graph is a complement of commuitng graph, then the non-commuting graph of G_2 is illustrated in Figure 3.6.

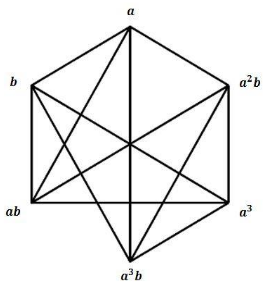


Figure 3.6 The non-commuting graph of G_2

In the adjacency matrix, the connected vertices are represented as 1 and 0, otherwise. □

Lemma 3.7 Let $G_3 \cong < a, b : a^8 = b^2 = 1, [a, b] = a^4 >$. Then, the adjacency matrix of the non-commuting graph of G_3, $A(\Gamma_{G_3}^{nc})$ is stated as follows:

$$A\left(\Gamma_{G_3}^{nc}\right) = \begin{bmatrix} 0 & 1 & 0 & 1 & 1 & 0 & 1 & 1 & 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 0 & 1 & 1 & 0 & 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 1 & 0 & 1 & 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 1 & 1 & 0 & 1 & 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 & 0 & 1 & 1 & 0 & 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 1 & 0 & 1 & 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 1 & 1 & 0 & 1 & 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 & 0 & 1 & 1 & 0 & 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 1 & 0 & 1 & 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 1 & 1 & 0 & 1 & 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 & 1 & 1 & 0 & 1 & 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 & 0 & 1 & 1 & 0 & 1 & 1 & 1 & 0 \end{bmatrix}.$$

Proof Since the non-commuting graph is a complement of commuitng graph, then the non-commuting graph of G_3 is illustrated in Figure 3.7.

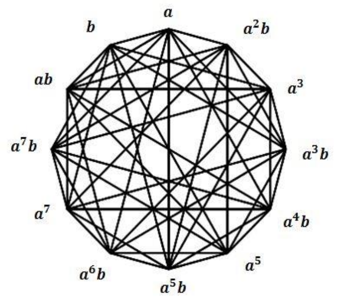


Figure 3.7 The non-commuting graph of G_3

In the adjacency matrix, the connected vertices are represented as 1 and 0, otherwise. □

Lemma 3.8 Let $G_4 \cong < a, b : a^{16} = b^2 = 1, [a, b] = a^8$. Then, the adjacency matrix of the non-commuting graph of G_4, $A(\Gamma_{G_4}^{nc})$ is stated as follows:

$$A\left(\Gamma_{G_4}^{nc}\right) = \begin{bmatrix} 0&1&0&1&1&0&1&1&0&1&1&0&1&1&0&1&1&0&1&1&0&1&1&1 \\ 1&0&1&1&0&1&1&0&1&1&0&1&1&0&1&1&0&1&1&0&1&1&1&0 \\ 0&1&0&1&1&0&1&1&0&1&1&0&1&1&0&1&1&0&1&1&0&1&1&1 \\ 1&1&1&0&1&1&0&1&1&0&1&1&0&1&1&0&1&1&0&1&1&0&0&1 \\ 1&0&1&1&0&1&1&0&1&1&0&1&1&0&1&1&0&1&1&0&1&1&1&0 \\ 0&1&0&1&1&0&1&1&0&1&1&0&1&1&0&1&1&0&1&1&0&1&1&1 \\ 1&1&1&0&1&1&0&1&1&0&1&1&0&1&1&0&1&1&0&1&1&0&0&1 \\ 1&0&1&1&0&1&1&0&1&1&0&1&1&0&1&1&0&1&1&0&1&1&1&0 \\ 0&1&0&1&1&0&1&1&0&1&1&0&1&1&0&1&1&0&1&1&0&1&1&1 \\ 1&1&1&0&1&1&0&1&1&0&1&1&0&1&1&0&1&1&0&1&1&0&0&1 \\ 1&0&1&1&0&1&1&0&1&1&0&1&1&0&1&1&0&1&1&0&1&1&1&0 \\ 0&1&0&1&1&0&1&1&0&1&1&0&1&1&0&1&1&0&1&1&0&1&1&1 \\ 1&1&1&0&1&1&0&1&1&0&1&1&0&1&1&0&1&1&0&1&1&0&0&1 \\ 1&0&1&1&0&1&1&0&1&1&0&1&1&0&1&1&0&1&1&0&1&1&1&0 \\ 0&1&0&1&1&0&1&1&0&1&1&0&1&1&0&1&1&0&1&1&0&1&1&1 \\ 1&1&1&0&1&1&0&1&1&0&1&1&0&1&1&0&1&1&0&1&1&0&0&1 \\ 1&0&1&1&0&1&1&0&1&1&0&1&1&0&1&1&0&1&1&0&1&1&1&0 \\ 0&1&0&1&1&0&1&1&0&1&1&0&1&1&0&1&1&0&1&1&0&1&1&1 \\ 1&1&1&0&1&1&0&1&1&0&1&1&0&1&1&0&1&1&0&1&1&0&0&1 \\ 1&0&1&1&0&1&1&0&1&1&0&1&1&0&1&1&0&1&1&0&1&1&1&0 \\ 0&1&0&1&1&0&1&1&0&1&1&0&1&1&0&1&1&0&1&1&0&1&1&1 \\ 1&1&1&0&1&1&0&1&1&0&1&1&0&1&1&0&1&1&0&1&1&0&0&1 \\ 1&1&1&0&1&1&0&1&1&0&1&1&0&1&1&0&1&1&0&1&1&0&0&1 \\ 1&0&1&1&0&1&1&0&1&1&0&1&1&0&1&1&0&1&1&0&1&1&1&0 \end{bmatrix}.$$

Proof Since the non-commuting graph is a complement of commuitng graph, then the non-commuting graph of G_4 is illustrated in Figure 3.8.

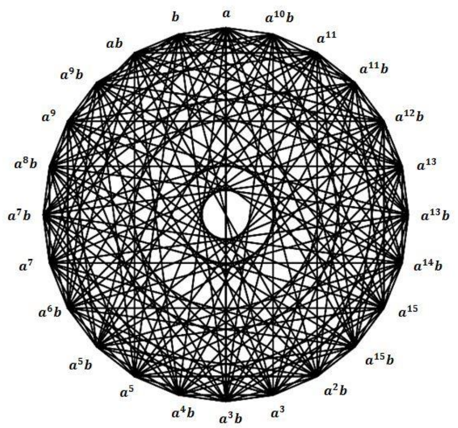


Figure 3.8 The non-commuting graph of G_4

In the adjacency matrix, the connected vertices are represented as 1 and 0, otherwise. □

Next, the adjacency matrices of the conjugate graph of G_1 is given in the following lemma while the adjacency matrices of the conjugate graphs of G_2, G_3 and G_4 have been determined in [14].

Lemma 3.9 Let $G_1 \cong < a, b : a^4 = b^2 = 1, bab = a^{-1} >$. Then, the adjacency matrix of the conjugate graph of G_1 is stated as follows:

$$A\left(\Gamma_{G_1}^{conj}\right) = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Proof In [17], Sarmin *et al.* determined the conjugacy classes of G_1 which are $cl(e) = \{e\}$, $cl(a) = \{a, a^3\}$, $cl(b) = \{b, a^2b\}$, $cl(a^2) = \{a^2\}$ and $cl(a^3b) = \{ab, a^3b\}$. Then, by Definition 1.7, the vertex set of the conjugate graph of G_1 is the set $V\left(\Gamma_{G_1}^{conj}\right) = \{a, a^3, b, a^2b, ab, a^3b\}$, while the edge set is the set of pairs of elements that conjugate to each other in G_1 which is $E\left(\left(\Gamma_{G_1}^{conj}\right) = \{\{a, a^3\}, \{b, a^2b\}, \{ab, a^3b\}\}$. Therefore, $\Gamma_{G_1}^{conj} = \cup_{i=1}^{3} K_2$, as illustrated in Figure 3.9.

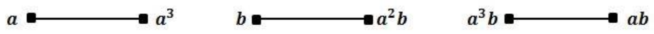


Figure 3.9 The conjugate graph of G_1

In the adjacency matrix, the connected vertices are represented as 1 and 0, otherwise. □

Then, the adjacency matrices of the conjugacy class graphs of all four metacyclic 2-groups are given in the following lemmas.

Lemma 3.10 Let $G_1 \cong < a, b : a^4 = b^2 = 1, bab = a^{-1} >$. Then, the adjacency matrix of the conjugacy class graph of G_1 is stated as follows:

$$A\left(\Gamma_{G_1}^{CC}\right) = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}.$$

Proof In [17], Sarmin *et al.* determined the conjugacy classes of G_1 which are $cl(e) = \{e\}$, $cl(a) = \{a, a^3\}$, $cl(b) = \{b, a^2b\}$, $cl(a^2) = \{a^2\}$ and $cl(a^3b) = \{ab, a^3b\}$. By Definition 1.8, the two vertices are connected if they have a common prime divisor. Hence, the vertex set is $V\left(\Gamma_{G_1}^{CC}\right) = \{cl(a), cl(b), cl(a^3b)\}$ and the edge set is $E\left(\Gamma_{G_1}^{CC}\right) = \{\{cl(a), cl(b)\}, \{cl(a), cl(a^3b)\}, \{cl(b), cl(a^3b)\}\}$. The conjugacy class graph of G_1 is $\Gamma_{G_1}^{CC} = K_3$, as illustrated in Figure 3.10.

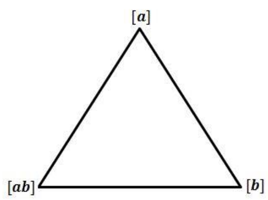


Figure 3.10 The conjugacy class graph of G_1

In the adjacency matrix, the connected vertices are represented as 1 and 0, otherwise. □

Lemma 3.11 Let $G_2 \cong < a, b : a^4 = 1, b^2 = [a, b] = a^{-2} >$. Then, the adjacency matrix of the conjugacy class graph of G_2 is stated as follows:

$$A\left(\Gamma_{G_2}^{CC}\right) = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}.$$

Proof In [14], Bilhikmah *et al.* determined the conjugacy class graph of G_2 as stated in Theorem 2.1 which given in Figure 3.11.

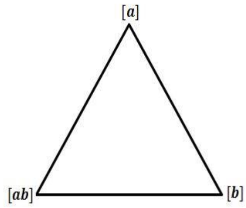


Figure 3.11 The conjugacy class graph of G_2

By Definition 1.9, in the adjacency matrix, the connected vertices are represented as 1 and 0, otherwise. □

Lemma 3.12 Let $G_3 \cong < a, b : a^8 = b^2 = 1, [a, b] = a^4 >$. Then, the adjacency matrix of the conjugacy class graph of G_3 is stated as follows:

$$A\left(\Gamma_{G_3}^{CC}\right) = \begin{bmatrix} 0 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 1 & 0 \end{bmatrix}.$$

Proof In [14], Bilhikmah *et al.* determined the conjugacy class graph of G_3 as stated in Theorem 2.2 which given in Figure 3.12.

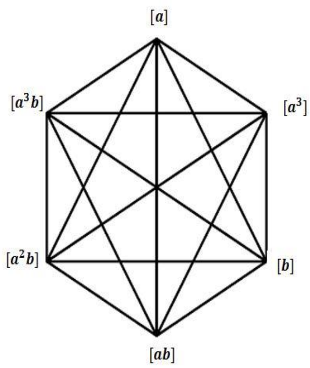


Figure 3.12 The conjugacy class graph of G_3

By Definition 1.9, in the adjacency matrix, the connected vertices are represented as 1 and 0, otherwise. □

Lemma 3.13 Let $G_4 \cong < a, b : a^{16} = b^2 = 1, [a, b] = a^8 >$. Then, the adjacency matrix of the conjugacy class graph of G_4 is stated as follows:

$$A\left(\Gamma_{G_4}^{CC}\right)=\begin{bmatrix} 0&1&1&1&1&1&1&1&1&1&1&1\\ 1&0&1&1&1&1&1&1&1&1&1&1\\ 1&1&0&1&1&1&1&1&1&1&1&1\\ 1&1&1&0&1&1&1&1&1&1&1&1\\ 1&1&1&1&0&1&1&1&1&1&1&1\\ 1&1&1&1&1&0&1&1&1&1&1&1\\ 1&1&1&1&1&1&0&1&1&1&1&1\\ 1&1&1&1&1&1&1&0&1&1&1&1\\ 1&1&1&1&1&1&1&1&0&1&1&1\\ 1&1&1&1&1&1&1&1&1&0&1&1\\ 1&1&1&1&1&1&1&1&1&1&0&1\\ 1&1&1&1&1&1&1&1&1&1&1&0 \end{bmatrix}.$$

Proof In [14], Bilhikmah *et al.* determinde the conjugacy class graph of G_4 as stated in Theorem 2.3 which given in Figure 3.13.

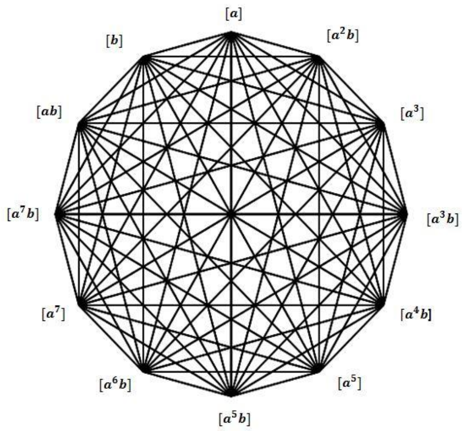


Figure 3.13 The conjugacy class graph of G_4

By Definition 1.9, in the adjacency matrix, the connected vertices are represented as 1 and 0, otherwise. □

Now, the energy of commuting graphs of all four groups are presented in the following theorems.

Theorem 3.1 Let $G_1 \cong < a,b : a^4 = b^2 = 1, bab = a^{-1} >$. Then, the energy of the commuting graph of G_1, $\varepsilon\left(\Gamma_{G_1}^{comm}\right) = 6$.

Proof Based on Lemma 3.1, the eigenvalues of the adjacency matrix of the commuting graph of G_1 are $\lambda_1 = \lambda_3 = \lambda_5 = 1$ and $\lambda_2 = \lambda_4 = \lambda_6 = -1$. Then, by Definition 1.11,

$$\varepsilon\left(\Gamma_{G_1}^{comm}\right) = 3|1| + 3|-1| = 6. \quad \square$$

Theorem 3.2 Let $G_2 \cong < a,b : a^4 = 1, b^2 = [a,b] = a^{-2} >$. Then, the energy of the commuting graph of G_2, $\varepsilon\left(\Gamma_{G_2}^{comm}\right) = 6$.

Proof Based on Lemma 3.2, the eigenvalues of the adjacency matrix of the commuting graph of G_2 are $\lambda_1 = \lambda_3 = \lambda_5 = 1$ and $\lambda_2 = \lambda_4 = \lambda_6 = -1$. Then, by Definition 1.11,

$$\varepsilon\left(\Gamma_{G_2}^{comm}\right) = 3|1| + 3|-1| = 6. \quad \square$$

Theorem 3.3 Let $G_3 \cong < a,b : a^8 = b^2 = 1, [a,b] = a^4 >$. Then, the energy of the commuting graph of G_3, $\varepsilon\left(\Gamma_{G_3}^{comm}\right) = 6$.

Proof Based on Lemma 3.3, the eigenvalues of the adjacency matrix of the commuting graph of G_3 are $\lambda_1 = \lambda_2 = \lambda_3 = 3$ and $\lambda_4 = \lambda_5 = \dots = \lambda_{12} = -1$. Then, by Definition 1.11,

$$\varepsilon\left(\Gamma_{G_3}^{comm}\right) = 3|3| + 9|-1| = 18. \quad \square$$

Theorem 3.4 Let $G_4 \cong < a,b : a^{16} = b^2 = 1, [a,b] = a^8 >$. Then, the energy of the commuting graph of G_4, $\varepsilon\left(\Gamma_{G_4}^{comm}\right) = 42$.

Proof Based on Lemma 3.4, the eigenvalues of the adjacency matrix of the commuting graph of G_4 are $\lambda_1 = \lambda_2 = \lambda_3 = 7$ and $\lambda_4 = \lambda_5 = \cdots = \lambda_{24} = -1$. Then, using Definition 1.11,

$$\varepsilon\left(\Gamma_{G_4}^{comm}\right) = 3|7| + 21|-1| = 42. \quad \square$$

It is shown that the energies of the commuting graphs of the metacyclic 2-groups are even integer.

Next, the energy of non-commuting graphs of all four groups are presented in the following.

Theorem 3.5 Let $G_1 \cong < a,b : a^4 = b^2 = 1, bab = a^{-1} >$. Then, the energy of the non-commuting graph of G_1, $\varepsilon\left(\Gamma_{G_1}^{nc}\right) = 8$.

Proof Based on Lemma 3.5, the eigenvalues of the adjacency matrix of the non-commuting graph of G_1 are $\lambda_1 = 4$, $\lambda_2 = \lambda_3 = -2$ and $\lambda_4 = \lambda_5 = \lambda_6 = 0$. Then, by Definition 1.11,

$$\varepsilon\left(\Gamma_{G_1}^{nc}\right) = |4| + 2|-2| + 3|0| = 8. \quad \square$$

Theorem 3.6 Let $G_2 \cong < a,b : a^4 = 1, b^2 = [b,a] = a^{-2} >$. Then, the energy of the non-commuting graph of G_2, $\varepsilon\left(\Gamma_{G_2}^{nc}\right) = 8$.

Proof Based on Lemma 3.6, the eigenvalues of the adjacency matrix of the non-commuting graph of G_2 are $\lambda_1 = 4$, $\lambda_2 = \lambda_3 = -2$ and $\lambda_4 = \lambda_5 = \lambda_6 = 0$. Then, by Definition 1.11,

$$\varepsilon\left(\Gamma_{G_2}^{nc}\right) = |4| + 2|-2| + 3|0| = 8. \quad \square$$

Theorem 3.7 Let $G_3 \cong < a,b : a^8 = b^2 = 1, [a,b] = a^4 >$. Then, the energy of the non-commuting graph of G_3, $\varepsilon\left(\Gamma_{G_3}^{nc}\right) = 16$.

Proof Based on Lemma 3.7, the eigenvalues of the adjacency matrix of the non-commuting graph of G_3 are $\lambda_1 = 8$, $\lambda_2 = \lambda_3 = -4$ and $\lambda_4 = \cdots = \lambda_{12}=0$. Then, by Definition 1.11,

$$\varepsilon\left(\Gamma_{G_3}^{nc}\right) = |8| + 2|-4| = 16. \quad \square$$

Theorem 3.8 Let $G_4 \cong < a,b : a^{16} = b^2 = 1, [a,b] = a^8 >$. Then, the energy of the non-commuting graph of G_4, $\varepsilon\left(\Gamma_{G_4}^{nc}\right) = 32$.

Proof Based on Lemma 3.8, the eigenvalues of the adjacency matrix of the non-commuting graph of G_4 are $\lambda_1 = 16$, $\lambda_2 = \lambda_3 = -8$, and $\lambda_4 = \lambda_5 = \cdots = \lambda_{24} = 0$. Then, by Definition 1.11,

$$\varepsilon\left(\Gamma_{G_4}^{nc}\right) = |16| + 2|-8| + 0 = 32. \quad \square$$

It is found that the energies of non-commuting graphs of the metacyclic 2-groups are even integer.

In the following theorems, the energy of the conjugate graphs of all four groups considered in this paper are presented.

Theorem 3.9 Let $G_1 \cong < a,b : a^4 = b^2 = 1, bab = a^{-1} >$. Then, the energy of the conjugate graph of G_1, $\varepsilon\left(\Gamma_{G_1}^{conj}\right) = 6$.

Proof Based on Lemma 3.9, the eigenvalues of the adjacency matrix of the conjugate graph of G_1 are $\lambda_1 = \lambda_3 = \lambda_5 = 1$ and $\lambda_2 = \lambda_4 = \lambda_6 = -1$. Then, by Definition 1.11,

$$\varepsilon\left(\Gamma_{G_1}^{conj}\right) = 3|1| + 3|-1| = 6. \quad \square$$

Theorem 3.10 Let $G_2 \cong < a, b : a^4 = 1, b^2 = [b, a] = a^{-2} >$. Then, the energy of the conjugate graph of G_2, $\varepsilon\left(\Gamma_{G_2}^{conj}\right) = 6$.

Proof Based on Theorem 2.1, $\Gamma_{G_2}^{conj} = \bigcup_{i=1}^{3} K_2$. Then, based on the adjacency matrix of $\Gamma_{G_2}^{conj}$ found in [14], the eigenvalues of the adjacency matrix of the conjugate graph of G_2 are $\lambda_1 = \lambda_3 = \lambda_5 = 1$ and $\lambda_2 = \lambda_4 = \lambda_6 = -1$. Then, by Definition 1.11,

$$\varepsilon\left(\Gamma_{G_2}^{conj}\right) = 3|1| + 3|-1| = 6. \quad \square$$

Theorem 3.11 Let $G_3 \cong < a, b : a^8 = b^2 = 1, [a, b] = a^4 >$. Then, the energy of the conjugate graph of G_3, $\varepsilon\left(\Gamma_{G_3}^{conj}\right) = 12$.

Proof Based on Theorem 2.2, the conjugate graph of G_3 is $\Gamma_{G_3}^{conj} = \bigcup_{i=1}^{6} K_2$. Then, based on the adjacency matrix of $\Gamma_{G_3}^{conj} = \bigcup_{i=1}^{6} K_2$ found in [14], the eigenvalues of the adjacency matrix of the conjugate graph of G_3 are $\lambda_1 = \lambda_3 = \cdots = \lambda_9 = \lambda_{11} = 1$, and $\lambda_2 = \lambda_4 = \cdots = \lambda_{12} = -1$. Then, by Definition 1.11,

$$\varepsilon\left(\Gamma_{G_3}^{conj}\right) = 6|1| + 6|-1| = 12. \quad \square$$

Theorem 3.12 Let $G_4 \cong < a, b : a^{16} = b^2 = 1, [a, b] = a^8 >$. Then, the energy of the conjugate graph of G_4, $\varepsilon\left(\Gamma_{G_4}^{conj}\right) = 24$.

Proof Based on Theorem 2.3, the conjugate graph of G_4 is $\Gamma_{G_4}^{conj} = \bigcup_{i=1}^{12} K_2$. Then, based on the adjacency matrix of $\Gamma_{G_4}^{conj} = \bigcup_{i=1}^{12} K_2$ found in [14], the eigenvalues of the adjacency matrix of the conjugate graph of G_4 are $\lambda_1 = \lambda_3 = \cdots = \lambda_{21} = \lambda_{23} = 1$, and $\lambda_2 = \lambda_4 = \cdots = \lambda_{24} = -1$. Then, by Definition 1.11,

$$\varepsilon\left(\Gamma_{G_4}^{conj}\right) = 12|1| + 12|-1| = 24. \quad \square$$

We can see that the energies of the conjugate graphs of the non-abelian metacyclic 2-groups are even integer.

The energy of conjugacy class graphs of all four groups are presented in theorems below.

Theorem 3.13 Let $G_1 \cong < a, b : a^4 = b^2 = 1, bab = a^{-1} >$. Then, the energy of the conjugacy class graph of G_1, $\varepsilon\left(\Gamma_{G_1}^{CC}\right) = 4$.

Proof Based on Lemma 3.10, the eigenvalues of the adjacency matrix of the conjugacy class graph of G_1 are $\lambda_1 = 2$ and $\lambda_2 = \lambda_3 = -1$. Then,by Definition 1.11,

$$\varepsilon\left(\Gamma_{G_1}^{CC}\right) = |2| + 2|-1| = 4. \quad \square$$

Theorem 3.14 Let $G_2 \cong < a, b : a^4 = 1, b^2 = [b, a] = a^{-2} >$. Then, the energy of the conjugacy class graph of G_2, $\varepsilon\left(\Gamma_{G_2}^{CC}\right) = 4$.

Proof Based on Theorem 2.1, the conjugacy class graph of G_2 is $\Gamma_{G_2}^{CC} = K_3$. Then, based on the adjacency matrix of $\Gamma_{G_2}^{CC}$ in Lemma 3.11, the eigenvalues of the adjacency matrix of the conjugacy class graph of G_2 are $\lambda_1 = 2$ and $\lambda_2 = \lambda_3 = -1$. Then, by Definition 1.11,

$$\varepsilon\left(\Gamma_{G_2}^{CC}\right) = |2| + 2|-1| = 4. \quad \square$$

Theorem 3.15 Let $G_3 \cong < a, b : a^8 = b^2 = 1, [a, b] = a^4 >$. Then, the energy of the conjugacy class graph of G_3, $\varepsilon\left(\Gamma_{G_3}^{CC}\right) = 10$.

Proof Based on Theorem 2.2, the conjugacy class graph of G_3 is $\Gamma_{G_3}^{CC} = K_6$. Based on the adjacency matrix of $\Gamma_{G_3}^{CC}$ in Lemma 3.12, the eigenvalues of the adjacency matrix of the conjugacy class graph of G_3 are $\lambda_1 = 5$, and $\lambda_2 = \cdots = \lambda_6 = -1$. Then, by Definition 1.11,

$$\varepsilon\left(\Gamma_{G_3}^{CC}\right) = |5| + 5|-1| = 10. \quad \square$$

Theorem 3.16 Let $G_4 \cong < a, b : a^{16} = b^2 = 1, [a, b] = a^8 >$. Then, the energy of the conjugacy class graph of G_4, $\varepsilon\left(\Gamma_{G_4}^{CC}\right) = 22$.

Proof Based on Theorem 2.3, the conjugacy class graph of G_4 is $\Gamma_{G_4}^{CC} = K_{12}$. Then, based on the adjacency matrix of $\Gamma_{G_4}^{CC}$ in Lemma 3.13, the eigenvalues of the adjacency matrix of the conjugacy class graph of G_4 are $\lambda_1 = 11$, and $\lambda_2 = \cdots = \lambda_{12} = -1$. Then, by Definition 1.11,

$$\varepsilon\left(\Gamma_{G_4}^{CC}\right) = |11| + 11|-1| = 22. \quad \square$$

It is shown that the energies of the conjugacy class graphs of the non-abelian metacyclic 2-groups are even integer.

CONCLUSION

In this paper, the commuting and non-commuting graphs of four metacyclic 2-groups are determined by using their definitions. Then, the adjacency matrices of all four types of graphs including the conjugate graph and conjugacy class graph of four metacyclic 2-groups are formed and their eigenvalues are found. Next, the energies of the graphs are computed based on the eigenvalues. It has been shown in this paper that the energies of these graphs of the groups must are even integer.

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