FOUR APPROACHES TO SUPPOSITION

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1. Introduction

Suppositions—or propositions provisionally accepted *for the sake of argument*—afford us a distinctive set of tools for deliberation. We use these tools to guide activities that are essential to intelligent behaviour, such as making predictions, forming plans, regretting past decisions, and determining our preferences about possible consequences of our actions. Russell (1904, p. 343) even once wrote that, without supposition, "inference would be inexplicable".

Legend has it there are two basic modes of supposition, corresponding to those expressed in the indicative and subjunctive grammatical moods. When a supposition is introduced in the indicative, we assess propositions relative to what we would expect upon learning that the supposition were true. When it is introduced in the subjunctive, our evaluations align with our judgments about how things would be if the supposition were *in fact* true (independent of whether we were aware of it). But suppositional judgments may be partitioned along another axis. In some suppositional contexts, we offer coarse-grained qualitative judgments about *whether or not* propositions are acceptable. In others, we give finer-grained quantitative judgments reflecting *how acceptable* we find propositions. In sum, this leaves us with four types of suppositional judgments to accommodate. Accordingly, there are four varieties of normative theories of suppositional judgement that have been developed:

(a) qualitative indicative theories,
(b) qualitative subjunctive theories,
(c) quantitative indicative theories, and
(d) quantitative subjunctive theories.

The accounts given by (a) and (b) respectively specify norms for rational qualitative judgments under indicative and subjunctive suppositions, while those in (c) and (d) respectively offer norms governing quantitative judgments under indicative and subjunctive suppositions.

The primary purpose of this paper is to shed light on the structure of these four varieties of normative theories of supposition by systematically explicating the relationships between canonical representatives of each. We approach this project by treating supposition as a form of 'provisional belief revision' in which a person temporarily accepts the supposition as true and makes some appropriate changes to her other opinions so as to accommodate their supposition. The idea is that suppositional judgments are supposed to reflect an agent's judgments about how things would be in some hypothetical state of affairs satisfying the supposition. Following this approach, theories of supposition are formalised in terms of functions mapping some representation of the agent's epistemic state along with a supposition to a hypothetical epistemic state representing their suppositional judgments.

Theories of indicative and subjunctive supposition are thus characterised using different functions, while qualitative and quantitative theories differ in their respective representation of epistemic

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states. Qualitative approaches are articulated in terms of coarse-grained full/categorical/outright belief, while quantitative ones rely on finer-grained partial beliefs represented by numerical credences. As we will look at both types of theories, our agents' epistemic states will consist of both qualitative beliefs and numerical credences. Given a set of possible worlds W and an agenda $\mathcal{A}$ comprising an algebra of subsets of W corresponding to propositions expressible in the finite propositional language $\mathcal{L}$, we let their beliefs be represented by a corpus comprising the set $\mathbf{B} \subseteq \mathcal{A}$ containing each proposition believed by the agent. The set $\mathscr{B}$ will denote the set of all possible corpora so that $\mathscr{B} = \wp(\mathcal{A})$. An agent's credences will be represented by a credence function $c : \mathcal{A} \mapsto [0, 1]$ satisfying the Kolmogorov axioms of probability, and $\mathscr{C}$ will denote the set of all probability functions over $\mathcal{A}$. When convenient, we will abuse our formalism by confusing sentences $X \in \mathcal{L}$ with their truth-sets $[\![X]\!] := \{w \in W : w \models X\}$. We will also introduce analogous notation for sets of sentences $\Gamma \subseteq \mathcal{L}$, by defining $[\![\Gamma]\!] := \{w \in W : w \models \bigwedge \Gamma\}$.[1] With this minimal formalism in hand we can be more precise about our four representative theories, depicted in the table below.

		Judgment	
		Qualitative	*Quantitative*
Supposition	*Indicative*	AGM Revision: $\mathbf{B}^*_S$	Conditionalization: $c(\cdot \mid S)$
	Subjunctive	KM Update: $\mathbf{B}^\diamond_S$	Imaging: $c_S(\cdot)$

Table 1. Four Theories of Supposition

The two qualitative theories listed in the first column of the table (AGM revision and KM update) are influential in the artificial intelligence and computer science communities. In those contexts, they are regarded as procedures for keeping databases up to date with newly received information. Each is characterised by a set of rationality postulates that together axiomatise their own functions mapping a corpus together with a proposition to a new corpus. Given such a function, $\circ : \mathscr{B} \times \mathcal{A} \mapsto \mathscr{B}$, the set $\mathbf{B}^\circ_S$ consists of the propositions that $\circ$ recommends an agent with the corpus $\mathbf{B}$ accept under the supposition that S. Our representative qualitative indicative theory is given by the postulates describing *AGM revision* operations ($*$), introduced in the seminal (Alchourrón et al., 1985).[2] For our qualitative subjunctive theory, we will consider the *KM update* operations ($\diamond$) characterised by the postulates proposed in Katsuno and Mendelzon (1992).[3]

The need to distinguish between these two types of belief change was first noted by Keller and Winslett Wilkins (1985), who proposed that "knowledge-adding" revisions are appropriate when

[1] Note that while individual propositions can be unproblematically identified with their corresponding truth sets, the same is not true for sets of propositions, since there can exist $\Gamma \neq \Gamma'$ such that $[\![\Gamma]\!] = [\![\Gamma']\!]$.

[2] While the seminal 1985 paper cited above is was the first full characterisation of AGM's revision operator, this work was the fusion of two independent projects. Alchourrón and Makinson (1981, 1982) had previously been investigating the derogation and revision of legal codes, while Gärdenfors (1978, 1981) had done considerable work on conditionals and belief change.

[3] It is worth mentioning that KM is not normally presented as a theory of subjunctive supposition. One of this paper's main contributions is a novel argument for viewing the KM axioms as qualitative rationality norms for subjunctive supposition.

new information is acquired about a static world, while "change-recording" updates are appropriate when learning that the world has changed in some way.[4] Interestingly, both operations can be characterised as making the minimal change to the agent's corpus needed to consistently accommodate new information, but each corresponds to a different interpretation of what constitutes 'minimal change'. It is often said that while revision corresponds to a 'global' interpretation of minimality on which minimal change returns a corpus whose overall global structure is as similar as possible to that of the original belief set, update corresponds to a 'local' interpretation on which minimal change is achieved by applying local operations to the possible worlds that are consistent with the original corpus, and then constructing the new corpus from the worlds yielded by those operations.

The quantitative theories that we will consider are defined in similar fashion. Each is specified by a function $f : \mathscr{C} \times \mathcal{A} \mapsto \mathscr{C}$ mapping each credence function and proposition to a new credence function. Our representative theory of indicative supposition will be given by *conditionalization*, where $c(\cdot|S)$ denotes the credence function $c(\cdot)$ conditional on S. Lastly, our quantitative subjunctive theory will be given by the *imaging* rule introduced by Lewis (1976), where $c_S(\cdot)$ is understood as the result of imaging $c(\cdot)$ by S. There are some deep parallels between, on the one hand, the relationship between conditionalization and imaging and, on the other, the relationship between revision and update. Conditionalization returns the globally most similar credence function that represents the new information as certain, while imaging shifts the probability mass from each world that is inconsistent with the new information to the locally most similar world that is not. On the basis of these similarities, Katsuno and Mendelzon (1992, p. 184) note that imaging can be regarded "as a probabilistic version of update, and conditionalization as a probabilistic version of revision". Despite the prevalence of remarks to this effect in the literature, we are unaware of any attempts to systematically investigate how this plays out at the operational level. One way to understand the purpose of this paper is as an effort to make this claim precise, then to systematically explicate in what sense, if any, it is actually true.

We proceed as follows: section 2 briefly sets the stage with further discussion of the distinction between indicative and subjunctive supposition. Section 3 introduces our representative quantitative accounts and explains our approach for comparing their recommendations with those provided by qualitative theories. In section 4, we compare the theories of indicative supposition listed on the first row of table 1 by drawing on (and extending) results established by Shear and Fitelson (2019). In section 5 we turn to the theories of subjunctive supposition from the second row of the table (KM and imaging), and systematically taxonomise the conditions under which they cohere with one another. Section 6 then addresses the remaining two diagonal comparisons suggested by Table 1 (LIS *vs.* KM and LSS *vs.* AGM). Finally, section 7 summarises the key findings of the analysis and outlines some prospects and remaining issues for future work. (A summary of all results from this paper is also available in an appendix.)

2. Two Modes of Supposition

On the standard story, the grammatical distinction between the indicative and subjunctive moods in a supposition aligns with a semantic difference between 'epistemic' or 'ontic' shifts in the modal

[4]Although this motivation for update as a distinct process from revision is *prima facie* plausible, it is only satisfactory for limited applications. Friedman and Halpern (1999) have persuasively argued that there are no deep difference between these two types of operations. In particular, they show that the apparent difference between revisions and updates can be recast as a relic of the chosen language. What may be described as a dynamically changing world in one language can be redescribed as a static world using appropriate temporal indices. It may be useful to retain the distinction between revision and update in areas like computer science where there is genuine import to the language in which a database management procedure is implemented. However, in epistemology, where questions are less bound to syntactic matters, other motivation is needed. Still, we see value in the distinction when these operations are understood in terms of supposition rather than belief change.

base used for subsequent evaluations.[5] In ordinary (non-suppositional) contexts, we assess propositions by the lights of our current opinions. In general, once we have supposed that S for the sake of argument, we are to temporarily shift those opinions to match some hypothetical alternative epistemic state that represents S as true. When the supposition is offered in the indicative mood, that shift is epistemic in the sense that it accords with the change of opinions that we would have undergone upon simply learning S. Contrastively, when put forth in the subjunctive mood, the shift of our opinions is ontic, since we are to adopt opinions that coincide with those that we would come to hold if we were to learn that S had suddenly been *made true* by some 'local miracle' or 'ideal intervention'.

To see how this works, it will be instructive to look at an example. Adapting the classic case from Ernest Adams (1970), consider the indicative supposition in (1) and the subjunctive supposition in (2) along with the propositions expressed by (3) and (4):

(1) Suppose that Oswald *didn't* shoot Kennedy...
(2) Suppose that Oswald *hadn't* shot Kennedy...
(3) Someone else shot Kennedy.
(4) Kennedy would have left Dallas unharmed.

Provided the indicative supposition in (1), the proposition expressed by (3) will no doubt seem acceptable. This is because learning that Oswald did not shoot Kennedy would not lead any reasonable person to give up the belief that Kennedy was shot; instead, the natural inference is to conclude that someone else was the assassin. In contrast, given the subjunctive supposition in (2), (4) seems appropriate. Here, we are to assess propositions relative to the most similar counterfactual world to the actual one in which Kennedy was never shot by Oswald. Since a world in which Oswald took but missed his shot is more similar to the actual one than one in which there was a second shooter, we judge that (4) is acceptable.

This clearly illustrates that the way in which rational agents adjust their epistemic states upon indicatively supposing a proposition will generally be radically different to the way in which they adjust those states upon supposing the same proposition in the subjunctive mood. We turn now to introducing the most salient quantitative theories for how one should adjust their judgments under indicative and subjunctive suppositions.

3. Quantitative Theories of Supposition and their Lockean Counterparts

3.1. **Quantitative Theories of Supposition.** Bayesian conditionalization is most commonly understood as a diachronic norm governing the update of probabilistic credence functions. Under that interpretation, when an agent with a prior credence function c learns that some event E has occurred, she should adopt the posterior c' matching c conditioned on E so that $c'(X) = c(X \mid E)$ for all X. Conditionalization is standardly defined in the following way.

Conditionalization: Given a credence function $c \in \mathscr{C}$ and any propositions $S \in \mathcal{A}$ with $c(s) > 0$, conditioning c by S results in the credence function $c(\cdot \mid S)$ such that, for all $X \in \mathcal{A}$,

$$c(X \mid S) =_{df} \frac{c(X \wedge S)}{c(S)}.$$

[5] The "epistemic"/"ontic" terminology was introduced in a series of papers by Lindström and Rabinowicz (1992a,b, 1998) discussing indicative and subjunctive conditionals. It is widely acknowledged that the correspondence between indicative/subjunctive conditionals and epistemic/ontic conditionals is not perfect—there are a number of cases where the two come apart, see Rott (1999a). The same is true for supposition. Still, for the purposes of this paper, we will ignore these imperfections and rely on the indicative/subjunctive terminology to capture the epistemic/ontic distinction.

Given the Bayesian understanding of conditionalization as an account of learning, and the close relationship between rational learning and indicative supposition, it is no surprise that conditionalization has also been understood as a normative quantitative model of indicative supposition. Interestingly, such an interpretation was first suggested by Rev. Thomas Bayes, who wrote, "The probability that two subsequent events will both happen is compounded of the probability of the first and the probability of the second *on the supposition the first happens*" (1763, p. 379). There are also more recent examples of this interpretation in the literature. For instance, this interpretation is explicitly endorsed by evidential decision theorists in their account of *ex ante* evaluations of option-outcomes.

The most popular alternative to evidential decision theory—causal decision theory—replaces the use of indicative suppositions in the calculation with subjunctive suppositions. The debate between evidentialists and causalists in decision theory boils down to a dispute about which type of supposition is relevant for *ex ante* evaluations of options.[6] The standard treatments of quantitative subjunctive supposition derive from the imaging rule mentioned in the previous section. Although a number of different versions of imaging have been developed in the literature, we will focus on its best known (and simplest) version, first proposed by Lewis. On an intuitive level, the difference between conditionalization and imaging can be understood in terms of the type of minimal change they encode. We mentioned earlier that conditionalization relies on a global measure of similarity, where imaging uses a local one. This point is elegantly explained by Lewis (1976, p. 311):

> "Imaging P on A gives a minimal revision in this sense: unlike all other revisions of P to make A certain, it involves no gratuitous movement of probability from worlds to dissimilar worlds. Conditionalizing P on A gives a minimal revision in this different sense: unlike an other revisions of P to make A certain, it does not distort the profile of probability ratios, equalities, and inequalities among sentences that imply A."

To introduce the details of imaging, we will need to impose some extra structure on the space of possible worlds. Specifically, we assume that, for any proposition X and possible world w, there is a unique "closest" world at which the sentence X is true. This notion is captured by using a *selection function*, $\sigma : W \times \mathcal{A} \mapsto W$. Intuitively, $\sigma(w, X)$ picks out the "closest" or "most similar" possible world to w that satisfies X. Our selection function will be subject to two basic conditions.

Centering: If $w \models X$, then $\sigma(w, X) = w$.

This first condition requires that each world is the unique closest world to itself, *i.e.* if X is true at w, then there is no closer world where X is true.

Uniformity: If $\sigma(w, X) \models Y$ and $\sigma(w, Y) \models X$, then $\sigma(w, X) = \sigma(w, Y)$.

This second condition says that whenever the closest X-world satisfies Y and the closest Y-world satisfies X, they are one and the same. In order to illustrate the conceptual motivation for this constraint, we will take a brief but necessary detour into an important philosophical application of selection functions—namely, the semantics of subjunctive conditionals.

Under what conditions are subjunctive conditionals such as "If Richard Nixon had pressed the button, there would have been a nuclear war" true? According to the proposal by Stalnaker (1968), this question is best answered in a semantics that utilises selection functions of the kind described above. The idea, roughly put, is that the subjunctive conditional in the example above is true just in

[6] Ahmed (2014) provides further explanation of the difference between evidential and causal decision theory from the perspective of an evidentialist, while Joyce (1999) does so from the point of view of a causalist.

case the closest possible world in which Richard Nixon *did* push the button is one where there was a nuclear war. The suggestion is that the truth value of the subjunctive conditional 'if X were true, Y would be true' at a world w is given by the following definition:

Stalnaker conditional (→): The truth-conditions for the Stalnaker conditional, $X \to Y$, are given by the semantic clause below.

$$w \models X \to Y \iff \sigma(w, X) \models Y$$

As should be clear from its definition, the Stalnaker conditional is non-truth functional, since the truth-value of $X \to Y$ at a world w does not supervene on the truth-values of its components at w. Rather, it is true at w just in case the closest world to w at which its antecedent is true is also one at which its consequent is true. For present illustrative purposes, we take subjunctive conditionals such as 'If Richard Nixon had pressed the button, there would have been a nuclear war' to be adequately modelled using the Stalnaker conditional.

Given this semantics for subjunctive conditionals, the motivation for **Uniformity** becomes very clear. When $\sigma(w, X) \models Y$ and $\sigma(w, Y) \models X$, the subjunctives 'if X were true, Y would be true' and 'if Y were true, X would be true' are both true on the semantics. Now imagine that $\sigma(w, X) \neq \sigma(w, Y)$. This implies that there is some Z such that the subjunctive 'If X were true, Z would be true' is true, but the subjunctive 'If Y were true, Z would be true' is false. Thus, the following sentence comes out as true:

$$(X \leftrightarrow Y) \wedge (X \to Z) \wedge (Y \to \neg Z)$$

Clearly, this would be a deeply strange and counterintuitive result. For this reason, we assume that our selection function satisfies the **Uniformity** condition.[7]

We are now ready to introduce Lewis's imaging rule, which will serve as our representative quantitative theory of subjunctive supposition. Stated formally:

Imaging: The result c_S of imaging a credence function c on a proposition S is defined as follows.

$$c_S(w) = \begin{cases} c(w) + \sum\limits_{\substack{w' \in [\![\neg S]\!]: \\ \sigma(w', S) = w}} c(w') & \text{if } w \in [\![S]\!] \\ 0 & \text{if } w \in [\![\neg S]\!] \end{cases}$$

Intuitively, when c is imaged on S, each world w consistent with S keeps all of its original probability, while the prior probability assigned to each world that is inconsistent with S is transferred to the closest world satisfying S.[8]

As suggested earlier, conditionalization and imaging differ in whether their recommendations are driven by global or local considerations. Conditionalization recommends the closest credence function that accommodates S where the distance between credence functions is interpreted in terms of their global behaviour. In contrast, imaging operates at the local level by shifting credence from each world to the closest world satisfying S.

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[7]Note that the **Uniformity** condition can also be directly motivated in terms of subjunctive supposition (and without reference to subjunctive conditionals), since failures of **Uniformity** imply that it is sometimes rational to (i) believe X upon subjunctively supposing Y, (ii) believe Y upon subjunctively supposing X, (iii) believe Z upon subjunctively supposing X, (iv) believe $\neg Z$ upon subjunctively supposing Y. Although this is less obviously bizarre than the problems that **Uniformity** violations create for the semantics of subjunctive conditionals, it is also a puzzling and intuitively irrational form of suppositional reasoning.

[8]For generalisations of Lewis' imaging rule that allow for more than one closest world, see e.g. (Gärdenfors, 1982), (Joyce, 1999). For a generalisation of imaging to the context of *partial supposition* (analogous to Jeffrey's generalisation of Bayesian conditionalisation), see (Eva and Hartmann, 2020).

3.2. **Lockean Theories of Supposition.** With our quantitative accounts of indicative and subjunctive supposition in hand, we will now outline our approach to comparing them with the qualitative theories we will introduce later. As mentioned earlier, qualitative and quantitative theories articulate the norms of suppositional judgement in terms of different kinds of doxastic attitude. Qualitative theories rely on agents' belief corpora to offer binary judgements about *whether* they should regard propositions as acceptable under a supposition. Quantitative theories on the other hand use an agent's credences to generate numerical judgments corresponding to *how* acceptable agents ought to find each proposition under any given supposition. To directly compare the two we need a way to bridge the gap between qualitative and quantitative attitudes.

To do so, we apply a suitably adapted version of the *Lockean Thesis*, so-called by Foley (1993). As it is traditionally understood, the Lockean Thesis provides a normative bridge principle between beliefs and credences, which requires that an agent believes that X just in case she has "sufficiently high" credence in X. This is standardly understood as saying that an agent should believe a proposition X if and only if her credence in X is at least as great as some *Lockean threshold*, $t \in (1/2, 1]$. Put formally:

Lockean Thesis ($\mathbf{LT}^t$): For some $t \in (1/2, 1]$: $X \in \mathbf{B} \Longleftrightarrow c(X) \geq t$.

This principle will be presupposed as a synchronic coherence requirement used to specify the beliefs that are coherent with an agent's credences. So, when we are talking about Lockean agents, we will presuppose that they have beliefs and credences satisfying $\mathbf{LT}^t$ for some $t \in (1/2, 1]$. There is an extensive literature on the Lockean Thesis and its motivations.[9] Featured prominently in that literature is the Lottery Paradox, first discussed by Kyburg (1961), and the tension it brings to the surface between $\mathbf{LT}^t$ and the popular normative requirements that beliefs be logically consistent and deductively closed. Primarily for space considerations, we will only briefly engage with that literature at a few points in the next section. Instead, we will unreflectively adopt $\mathbf{LT}^t$ as a technical tool to aid in our comparative project.

But the Lockean Thesis will play another role in our exploration beyond being a standing synchronic coherence requirement. It will also be used together with the quantitative theories of supposition introduced earlier to construct qualitative suppositional judgments that can be directly compared with the representative qualitative theories of supposition. We begin by introducing the Lockean theory of indicative supposition (LIS). LIS is specified in terms of an operation, $\circledast : \mathscr{B}\times\mathcal{A} \mapsto \mathscr{B}$, defined as follows:

$$\mathbf{B}^{\circledast}_S =_{df} \{X : c(X \mid S) \geq t\}$$

Where $\mathbf{B}$ and b are respectively a corpus of beliefs and credence function satisfying $\mathbf{LT}^t$ and S is any proposition, $\mathbf{B}^{\circledast}_S$ denotes the set of acceptable propositions under the supposition S. The Lockean theory of subjunctive supposition (LSS) is similarly characterised in terms of the operator $\blacklozenge : \mathscr{B} \times \mathcal{A} \mapsto \mathscr{B}$, defined as follows:

$$\mathbf{B}^{\blacklozenge}_S =_{df} \{X : c_S(X) \geq t\}$$

Strictly speaking, the two Lockean operations ($\circledast$ and $\blacklozenge$) are not singular operations, but rather characterise families of operations—one for each $t \in (1/2, 1]$. When it is useful, we will restrict our attention to certain subsets of Lockean thresholds by letting $\circledast^{[t,t']}$ ($\blacklozenge^{[t,t']}$) denote the family of operators bounded by the closed interval $[t, t']$. Analogous conventions will be adopted for the open and half-open intervals.

[9]For some discussion of these matters see, Easwaran (2016), Leitgeb (2017), Dorst (2019), Douven and Rott (2018), Schurz (2019) and Jackson (2020).

4. Indicative Supposition

In their seminal (1985) paper, Alchourrón, Gärdenfors, and Makinson introduced their revision operation ($*$). Aside from being the now orthodox account of belief revision, the AGM theory has been understood as an account of indicative supposition. Even Isaac Levi, who was highly critical of AGM as a theory of belief revision, acknowledged that "the AGM approach fares better as an account of suppositional reasoning for the sake of the argument" (1996, p. 290). We follow suit and present the theory as a normative theory of indicative supposition.

The AGM theory relies on the syntactic representation of epistemic states as "belief sets", which comprise deductively closed sets of sentences. Formally, this means that $\mathbf{B}$ is taken to be $Cn(\mathbf{B})$, where $Cn(\Gamma) =_{df} \{X : \bigwedge \Gamma \vdash X\}$.[10] Revising $\mathbf{B}$ by a sentence S delivers the new belief set $\mathbf{B}^*_S$, understood as the set of sentences that are acceptable under the supposition S for an agent with the corpus $\mathbf{B}$. This reflects AGM's presupposition of **Cogency** as a synchronic coherence requirement on admissible beliefs and suppositional judgments. This requirement, stated below, says that belief corpora and suppositional judgements must be logically consistent and closed under deductive consequence.

Cogency: A set $\mathbf{B}$ is *cogent* just in case:

(a) $\mathbf{B}$ logically consistent, *i.e.* $\mathbf{B} \nvdash \bot$, and
(b) $\mathbf{B}$ is deductively closed, *i.e.* $\mathbf{B} = Cn(\mathbf{B})$.

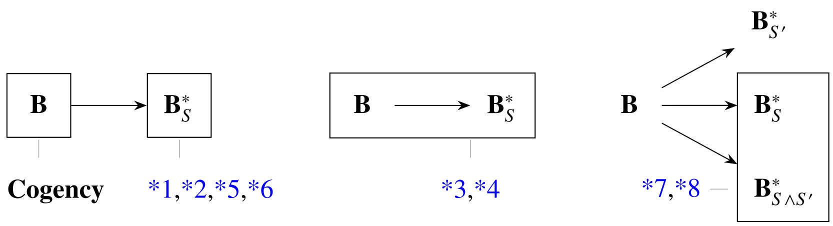
This results in a coarse-grained representation of epistemic states/suppositional judgments that comes with certain definite costs. For one, since there is just one inconsistent belief set ($\mathbf{B}_\bot = \mathcal{L}$), AGM leaves no room to distinguish between agents with inconsistent beliefs/suppositional judgments. This same belief set represents both an agent who believes, as in the Lottery paradox from Kyburg (1961), each of $P_1, \ldots, P_n$ and also that $\neg(P_1 \wedge \cdots \wedge P_n)$ and another who believes the outright contradiction $P \wedge \neg P$. Similarly, Nebel (1989) observes that the reasons *why* beliefs are held are not reflected in this representation. An agent who independently believes that P and Q is represented in the same way as another who believes that Q on the basis of their beliefs that P and $P \supset Q$. Such dependencies may be important for belief dynamics as seen by considering the possibility that these agents lose their beliefs that P. We will not dwell on this point further and simply note that AGM's **Cogency** assumption will result in some important divergences between AGM and the Lockean accounts.

The AGM revision operation ($*$) is axiomatised by the six "basic Gärdenfors postulates", (*1) – (*6), together with the two "supplementary postulates", (*7) and (*8).

(*1)	$\mathbf{B}^*_S = Cn(\mathbf{B}^*_S)$	**(Closure)**
(*2)	$S \in \mathbf{B}^*_S$	**(Success)**
(*3)	$\mathbf{B}^*_S \subseteq Cn(\mathbf{B} \cup \{S\})$	**(Inclusion)**
(*4)	If $\mathbf{B} \nvdash \neg S$, then $\mathbf{B} \subseteq \mathbf{B}^*_S$	**(Preservation)**
(*5)	If $\nvdash \neg S$, then $\mathbf{B}^*_S \nvdash \bot$	**(Consistency)**
(*6)	If $\vdash S \equiv S'$, then $\mathbf{B}^*_S = \mathbf{B}^*_{S'}$	**(Extensionality)**
(*7)	$\mathbf{B}^*_{S \wedge S'} \subseteq Cn(\mathbf{B}^*_S \cup \{S'\})$	**(Superexpansion)**
(*8)	If $\mathbf{B}^*_S \nvdash \neg S'$, then $\mathbf{B}^*_{S \wedge S'} \supseteq Cn(\mathbf{B}^*_S \cup \{S'\})$	**(Subexpansion)**

[10]For present purposes, we assume that $\vdash$ is the classical consequence relation, however, this is strictly speaking more than is required. In the theory's original formulation, $\vdash$ can be any consistent, compact, and supraclassical consequence relation satisfying *modus ponens* and the deduction theorem.

To explain these postulates, it will be instructive to take a brief detour to discuss the types of coherence requirements they encode. For this, we follow Rott (1999b, 2001) in thinking that they include requirements of three distinct types: *synchronic*, *diachronic*, and *dispositional*. While synchronic coherence provides us with conditions under which a single set of judgments (either a corpus or a set of judgments under a single supposition) hangs together, diachronic coherence accounts for the constraints that the agent's corpus places on individual sets of suppositional judgments. Lastly, dispositional coherence involves constraints that may be imposed across different sets of suppositional judgments. A visual explanation is provided by the figure below adapted from (Rott, 1999b, p. 404, Fig. 1).



(A) Synchronic (B) Diachronic (C) Dispositional

FIGURE 1. The relata of the three types of coherence

Whereas **Cogency** is taken as a background synchronic coherence requirement on belief sets, **Closure** (*1) and **Consistency** (*5) ensure that suppositional judgments also satisfy **Cogency**. Since the agent's beliefs do not play any role in determining the content of these constraints, both postulates are straightforwardly seen as purely synchronic requirements on suppositional judgments. For the same reason, **Success** (*2) and **Extensionality** (*6) may also be regarded as synchronic requirement on suppositional judgments. Unlike the standing synchronic requirements embodied by *1 and *5, the motivations for *2 and *6 are grounded in constitutive or theoretical considerations about the nature of supposition. We take *2 to be a constitutive requirement of supposition. If supposing that S did not result in S being accepted, then this would hardly seem like S had been supposed at all. On the other hand, *6 captures a theoretical commitment that surface grammar or intensional considerations should play no role in determining which propositions are acceptable under a supposition.[11]

The next two postulates, **Inclusion** (*3) and **Preservation**[12] (*4), provide AGM's diachronic coherence requirements. Respectively, these impose upper and lower bounds on the set of suppositional judgments. The restriction imposed by *3 ensures that only propositions that are logically related to **B** or S are acceptable under the supposition that S. On the other hand, *4 requires that beliefs should not fail to be acceptable under the supposition S unless S is logically inconsistent with the agent's corpus. It is worth noting that this places no restrictions on suppositional judgments when the supposition is inconsistent with the agent's belief set.

[11] While we embrace this commitment for present purposes, it should be acknowledged that there is room to disagree here. One might think that suppositional judgments should be hyperintensional due to considerations of topic-sensitivity or relevance. A recent discussion of these matters in the context of AGM is available in (Berto, 2019).

[12] The original formulation of these postulates do not include **Preservation** and, instead, include the stronger **Vacuity** principle requiring that if $\mathbf{B} \nvdash \neg S$, then $\mathbf{B} * S \supseteq Cn(\mathbf{B} \cup \{S\})$. However, **Preservation** implies **Vacuity** in the context of **Closure** and **Success** and is preferable for both aesthetic and conceptual reasons.

Lastly, we have the dispositional coherence requirements given by the two supplementary postulates, **Superexpansion** (*7) and **Subexpansion** (*8), which respectively generalise *3 and *4. Indeed, when combined with the eminently plausible principle **Idempotence** ($*\top$), which requires that $\mathbf{B}^*_\top = \mathbf{B}$, *7 and *8 respectively imply *3 and *4. Since they provide AGM's dispositional coherence requirements, it should be no surprise that the supplementary postulates have been widely discussed in the literature on iterated belief revision.

4.1. **LIS and the AGM Postulates.** The question now arises: how do the suppositional judgments recommended by LIS relate to those given under the qualitative account based on AGM? A partial answer to this question is given by previously established results. We will complete this picture after surveying the extant results from the literature.

Beginning with their synchronic requirements, there is an immediate tension between $\mathbf{LT}^t$ and **Cogency** that has been extensively discussed in the literatures on the Preface and Lottery Paradoxes—these same issues straightforwardly apply to the synchronic requirements imposed by *1 and *5. The remaining basic Gärdenfors postulates have been considered from a Lockean perspective by Shear and Fitelson (2019).[13] LIS satisfies both of the remaining AGM synchronic coherence requirements, *2 and *6. Neither result is surprising: LIS satisfies *2 in virtue of the fact that $c(S \mid S) = 1$, while the satisfaction of *6 is secured by the extensional character of conditionalization.

The situation is more interesting for the diachronic requirements given by *3 and *4. Interestingly, *3 is satisfied by LIS in full generality. The reason why is relatively easy to see. It is a theorem of the probability calculus that $c(S \supset X) \geq c(X \mid S)$. Thus, whenever $X \in \mathbf{B}^{\circledast}_S$ it follows that $S \supset X \in \mathbf{B}$, and so $\mathbf{B}^{\circledast}_S \subseteq Cn(\mathbf{B} \cup \{S\})$. Turning to the final basic postulate, *4, we see that in general LIS can violate this requirement. The basic reason why is relatively clear though there are some subtleties that we will discuss. As the characteristic postulate of AGM, *4 says that an agent's beliefs should remain acceptable under any supposition that is logically consistent with her corpus. However, when an agent is not *fully certain* of one of her beliefs (say X), it is possible for that some supposition (S) might be logically consistent with her corpus but still count as counter-evidence to X in the sense that $c(X \mid S) < c(X)$. This allows for the possibility that $c(X) \geq t$ even though $c(X \mid S) < t$ and, thus, that $\mathbf{B} \nvdash \neg S$ but $\mathbf{B} \not\subseteq \mathbf{B}^{\circledast}_S$. Still, there are some further constraints that can be imposed under which LIS can be made to satisfy *4.

The explanation immediately above is suggestive of the first situation in which LIS will be guaranteed to satisfy *4. Indeed, Gärdenfors (1988) established a result, which implies that when belief is taken to imply certainty (*i.e.* when $t = 1$), LIS will satisfy *4. Moreover, Gärdenfors' result actually implies that LIS will satisfy *all* of the AGM postulates. One might wonder then: is the resulting satisfaction of *4 a consequence of the fact that *1 and *5 are satisfied when $t = 1$?

Shear and Fitelson show that the answer to this question is *no*, LIS can violate *4 even under the further assumption of **Cogency**. However, they establish the more surprising result that, assuming **Cogency**, LIS can only violate *4 when the Lockean threshold is relatively high. In particular, such violations are only possible when the Lockean threshold is at least the inverse of the Golden ratio (*i.e.* when $t \in (\phi^{-1}, 1)$, where $\phi^{-1} \approx 0.618$). As an immediate corollary, assuming both **Cogency** and that $t \in (1/2, \phi^{-1}]$, LIS satisfies all of the basic Gärdenfors postulates, *1 – *6.

[13] Their investigations into the contrasting diachronic coherence requirements of Lockeanism and AGM explored a "Lockean revision" operation, which is formally identical to the operation characterising LIS. For an alternative presentation of their results and some discussion, see Genin (2019).

But this only tells part of the story about the import of the "Golden threshold" at ϕ^{-1}.[14] This is because LIS exhibits interesting behaviour relative to the two weakened variants of **Preservation** provided below.

(*4^v) If $S, X \in \mathbf{B}$, then $\neg X \notin \mathbf{B}^*_S$ **(Very Weak Preservation)**

(*4^w) If $S \in \mathbf{B}$, then $\mathbf{B} \subseteq \mathbf{B}^*_S$ **(Weak Preservation)**

The first of these postulates, **Very Weak Preservation** (*4^v), requires that taking something that you already believe as a supposition for the sake of argument should not lead you to *reject* any of your other beliefs under that supposition. The second, **Weak Preservation** (*4^w), says that under the same conditions, you should accept anything that you believe.

Although imposing the assumption of **Cogency** on LIS was *not* sufficient to guarantee the satisfaction of full **Preservation** (*4), it turns out that it *is* sufficient to ensure that LIS will satisfy both of the weaker requirements, *4^v and *4^w. However, there is another way to guarantee that LIS will satisfy **Very Weak Preservation**: if the Lockean threshold is at least ϕ^{-1}, then LIS will satisfy *4^v (even without the help of **Cogency**). These results are summarised in the table below.

	Cogency?	*4	*4^w	*4^v
⋇$^{[\phi,1)}$	N			✓
⋇	Y		✓	✓
⋇$^{(1/2,\phi)}$	Y	✓	✓	✓

TABLE 2. LIS and Some Variants of **Preservation**

The import of these results will depend on how you regard *4^v, *4^w, *4, and **Cogency**. We regard *4^v as eminently reasonable: it would seem very strange to believe both P and Q, but reject Q under the supposition that P. After all, that would mean that P's certain truth would provide sufficient evidence to accept $\neg Q$—that would seem to be ruled out by your concurrent beliefs that P and that Q. For the die-hard Lockeans who reject **Cogency**, this gives reason to maintain that the Lockean threshold must be a sufficiently high ($t > \phi^{-1}$) so as to rule out this possibility. The import of the remaining results is up for debate. A Lockean who finds *4^w plausible will be forced into adopting **Cogency**. However, this would be harder to motivate for a Lockean since once we accept that rational belief need not require certainty, there is no obvious argument in favour of *4^w. Still, proponents of AGM who find LIS attractive may take solace in the realisation that their preferred account can be reconciled with LIS through the acceptance of a sufficiently low threshold.[15]

Thus far, we have presented a number of results concerning LIS and the basic Gärdenfors postulates, *1 – *6, but have not addressed two remaining supplementary postulates, *7 and *8. Shear and Fitelson only mention these postulates in passing, since their primary concern was with the diachronic requirements governing single-step belief change rather than the dispositional requirements that provide bridges between different potential revisions. However, in the context of supposition, dispositional requirements are more obviously relevant. Accordingly, we will now complete the picture by reporting some new results establishing that the relationship between LIS and *3 and *4 carries over to their generalisations given by *7 and *8.

[14]For further results illustrating the significance of ϕ^{-1} for conditional reasoning in Lockean agents, see Eva (Forthcoming).

[15]This is not the only way of reconciling Lockeanism with AGM. Building on his Stability Theory of Belief, Leitgeb (2013, 2017) has recently proposed a belief revision operator satisfying the Lockean thesis and all of the AGM postulates. However, that approach comes with certain definitive costs that have been discussed in the literature; see Titelbaum (Forthcoming) for an overview of these issues.

Proposition 1 *LIS must satisfy *7. That is, the following is satisfied for any* $\mathbf{B}$, S, S', *and* $t \in (1/2, 1]$*:*

$$\mathbf{B}^{\circledast}_{S\wedge S'} \subseteq Cn(\mathbf{B}^{\circledast}_S \cup \{S'\})$$

Proof. Let $X \in \mathbf{B}^{\circledast}_{S\wedge S'}$, *i.e.* $c(X \mid S \wedge S') \geq t$. Then, letting $c^S(\cdot) := c(\cdot \mid S)$, we get:

$$c(S' \supset X \mid S) = c^S(S' \supset X) \geq c^S(X \mid S') = c^{S\wedge S'}(X) = c(X \mid S \wedge S')$$

Thus, $c(S' \supset X \mid S) \geq t$ and so $S' \supset X \in \mathbf{B}^{\circledast}_S$. From this we conclude $X \in Cn(\mathbf{B}^{\circledast}_S \cup \{S'\})$. □

Proposition 2 *In the absence of* ***Cogency****, LIS can violate *8 for any* $t \in (1/2, 1)$*. That is, if* $t \in (1/2, 1)$*, it is possible that:*

$$\mathbf{B}^{\circledast}_S \nvdash \neg S', \textit{ but } \mathbf{B}^{\circledast}_{S\wedge S'} \not\supseteq Cn(\mathbf{B}^{\circledast}_S \cup \{S'\})$$

Proof. Let c be any credence function satisfying the conditions below, where $\varepsilon > 0$ is arbitrarily small:

$$c(S \wedge S' \wedge X) = \varepsilon \qquad c(S \wedge S' \wedge \neg X) = 1 - t \qquad c(S \wedge \neg S' \wedge X) = t - \varepsilon$$

It is simple to see that case provides the basis for a counterexample to *8 for any threshold $t \in (1/2, 1)$ in the absence of **Cogency**. □

Proposition 3 *The twin requirements of* ***Cogency*** *and* $t \in (1/2, \phi^{-1}]$ *are necessary and sufficient to guarantee that LIS satisfies *8.*

Proof. Supposing **Cogency**, we let S' be consistent with $\mathbf{B}^{\circledast}_S$ and $X \in Cn(\mathbf{B}^{\circledast}_S \cup \{S'\})$, and define c as a vector on the assignments below.

$$c(S \wedge S' \wedge X) = \alpha \quad c(S \wedge S' \wedge \neg X) = \beta \quad c(S \wedge \neg S' \wedge X) = \gamma \quad c(S \wedge \neg S' \wedge \neg X) = \delta$$

We start by showing $t \in (1/2, \phi^{-1}]$ only if $X \in \mathbf{B}^{\circledast}_{S\wedge S'}$, and hence that *8 is satisfied. For contradiction, suppose that $t \in (1/2, \phi^{-1}]$, but $c(X \mid S \wedge S') < t$. Since $X \in Cn(\mathbf{B}^{\circledast}_S \cup \{S'\})$ implies $S' \supset X \in \mathbf{B}^{\circledast}_S$, our assumptions imply

$$\frac{\alpha}{\alpha + \beta} < t, \text{ and} \tag{1}$$

$$\frac{\alpha + \gamma + \delta}{\alpha + \beta + \gamma + \delta} \geq t. \tag{2}$$

First, note that since $S' \supset X \in \mathbf{B}^{\circledast}_S$, by **Cogency** $S' \supset \neg X \in \mathbf{B}^{\circledast}_S$ would imply that $(S' \supset X) \wedge (S' \supset \neg X) \in \mathbf{B}^{\circledast}_S$. This is equivalent to $\neg S' \in \mathbf{B}^{\circledast}_S$ thus contradicting our assumption that S' is consistent with $\mathbf{B}^{\circledast}_S$. So, $S' \supset \neg X \notin \mathbf{B}^{\circledast}_S$ (*i.e.* $c(S' \supset \neg X \mid S) < t$) which gives us

$$\frac{\alpha}{\alpha + \beta + \gamma + \delta} > 1 - t. \tag{3}$$

Next, observe that $S' \in \mathbf{B}^{\circledast}_S$ would imply by **Cogency** that $S' \wedge X \in \mathbf{B}^{\circledast}_S$, since $S' \supset X \in \mathbf{B}^{\circledast}_S$. But then $c(X \mid S' \wedge S) \geq c(X \wedge S' \mid S) \geq t$, which contradicts 1. So $c(S' \mid S) < t$, which implies that

$$\frac{\alpha + \beta}{\alpha + \beta + \gamma + \delta} < t. \tag{4}$$

Taken together, 3 and 4 give us 5, which combined with 3 lets us infer 6.

$$\frac{\beta}{\alpha+\beta+\gamma+\delta} < 2t-1. \tag{5}$$

$$\frac{\alpha}{\alpha+\beta} = \frac{\frac{\alpha}{\alpha+\beta+\gamma+\delta}}{\frac{\alpha+\beta}{\alpha+\beta+\gamma+\delta}} > \frac{1-t}{t}. \tag{6}$$

Now, since $t \in (1/2, \phi^{-1}]$, we can use the special fact about the Golden Ratio that $t \leq \phi^{-1}$ iff $t^2 \leq 1-t$ to infer $\frac{\alpha}{\alpha+\beta} \geq t$, which contradicts our assumption 1. Thus, our initial assumptions were inconsistent and we infer that assuming $t \in (1/2, \phi^{-1}]$ together with **Cogency** suffices to guarantee that LIS satisfies *8.

To see that LIS can violate *8 for any $t \in (\phi^{-1}, 1)$ even when **Cogency** is assumed, consider any credence function c satisfying the following constraints, where $\varepsilon > 0$ is arbitrarily small:

$$c(S \wedge S' \wedge X) = 1 - t + \varepsilon$$

$$c(S \wedge S' \wedge \neg X) = (\frac{1}{t} - 1 + \varepsilon)(1 - t + \varepsilon)$$

$$c(S \wedge \neg S' \wedge X) = 1 - (1 - t + \varepsilon) - (\frac{1}{t} - 1 + \varepsilon)(1 - t + \varepsilon)$$

By construction, we have that $c(S' \supset X) > t$ and $c(X|S \wedge S') < t$, which shows that $X \in Cn(\mathbf{B}^{⁎}_{S} \cup \{S'\})$ but $X \notin \mathbf{B}^{⁎}_{S \wedge S'}$, as desired. Note also that since $c(S) = 1$, $\mathbf{B}^{⁎}_{S} = \mathbf{B}$. Furthermore, it can be verified that $\mathbf{B} = Cn(S \wedge X)$ holds for every (and only) $t > \phi^{-1}$, which establishes **Cogency** and confirms that S' is consistent with $\mathbf{B}^{⁎}_{S}$. □

This completes our assessment of the relationship between the theories of suppositions provided by LIS and AGM. A full summary of the results from this section is given in Table 3 below.

	Cogency?	*1	*2	*3	*4	*4^w	*4^v	*5	*6	*7	*8
$⁎^1$	Y	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓
$⁎$	N		✓	✓					✓	✓	
$⁎$	Y	✓	✓	✓		✓	✓	✓	✓	✓	
$⁎^{(\phi^{-1},1)}$	N		✓	✓			✓		✓	✓	
$⁎^{(1/2,\phi^{-1}]}$	Y	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓

TABLE 3. LIS and the AGM Postulates

In the next section, we turn our attention to the relationship between the subjunctive theories. Our approach will proceed in similar fashion by assessing LSS relative to the postulates of Katsuno and Mendelzon's theory of belief update.

5. Subjunctive Supposition

To begin, it will be worthwhile to see why AGM revision would inappropriate to use as a theory of subjunctive supposition. Consider the following version of the widely discussed adaptation from Peppas (2008) of a classic case from Ginsberg (1986):

> *Philippa is looking through an open door into a room containing a table, a magazine and a book. One of the two items is on the table and the other is on the floor, but because of poor lighting, Philippa cannot distinguish which is which.*

Now, imagine that Philippa thinks to herself, "Suppose that the book were on the floor." Under this (subjunctive) supposition, what should she accept regarding the location of the magazine? Well, if some 'local miracle' occurred that resulted in the book being on the floor, this would not result in a change regarding the location of the magazine. Thus, her judgment regarding the magazine's location in the suppositional context should remain unchanged from in the categorical one and she should accept that it is either on the table or the floor without accepting either individual disjunct. But, this is not what AGM would recommend. Let B and M respectively be the propositions 'the book is on the floor' and 'the magazine is on the floor'. For simplicity, let Philippa's beliefs include only $\mathbf{B} = Cn(B \equiv \neg M)$ to capture her belief that only that one of the two is on the table. Then, since $\mathbf{B} \nvdash B$, we get $\neg M \in \mathbf{B}^*_B$ and so AGM revision would recommend that she accept that the magazine is not on the floor. This is clearly the wrong result.

Cases like these motivated computer science and artificial intelligence researchers to develop alternative belief change operations, known as *updates*.[16] Katsuno and Mendelzon (1992) introduced postulates axiomatising their update operation in similar fashion to the AGM postulates for revision.[17] These postulates are formulated below, where saying that $\mathbf{B}$ is *complete* means that $[\![\mathbf{B}]\!]$ is a singleton (or equivalently that either $X \in \mathbf{B}$ or $\neg X \in \mathbf{B}$ for any sentence X).

($\diamond$0) $\mathbf{B}^\diamond_S = Cn(\mathbf{B}^\diamond_S)$ **(Closure)**
($\diamond$1) $S \in \mathbf{B}^\diamond_S$ **(Success)**
($\diamond$2) If $S \in \mathbf{B}$ then $\mathbf{B}^\diamond_S = \mathbf{B}$ **(Stability)**
($\diamond$3) If $\mathbf{B} \nvdash \bot$ and $\nvdash \neg S$, then $\mathbf{B}^\diamond_S \nvdash \bot$ **(Consistency Preservation)**
($\diamond$4) If $\vdash S \equiv S'$, then $\mathbf{B}^\diamond_S = \mathbf{B}^\diamond_{S'}$ **(Extensionality)**
($\diamond$5) $\mathbf{B}^\diamond_{S \wedge S'} \subseteq Cn(\mathbf{B}^\diamond_S \cup \{S'\})$ **(Chernoff)**
($\diamond$6) If $S \in \mathbf{B}^\diamond_{S'}$ and $S' \in \mathbf{B}^\diamond_S$, then $\mathbf{B}^\diamond_S = \mathbf{B}^\diamond_{S'}$ **(Reciprocity)**
($\diamond$7) If $\mathbf{B}$ is complete, then $\mathbf{B}^\diamond_{S \vee S'} \subseteq Cn(\mathbf{B}^\diamond_S \cup \mathbf{B}^\diamond_{S'})$ **(Primeness)**
($\diamond$8) If $[\![\mathbf{B}]\!] = [\![\mathbf{B}']\!] \cup [\![\mathbf{B}'']\!]$, then $[\![\mathbf{B}^\diamond_S]\!] = [\![\mathbf{B}'^\diamond_S]\!] \cup [\![\mathbf{B}''^\diamond_S]\!]$ **(Compositionality)**

Some of these postulates are familiar from the AGM postulates, while some are new. **Closure** ($\diamond$0), **Success** ($\diamond$1), **Extensionality** ($\diamond$4), and **Chernoff**[18] ($\diamond$5) are respectively identical to $*1$, $*2$, $*6$, and $*7$ from earlier. **Stability** ($\diamond$2) and **Consistency Preservation** ($\diamond$3) are each weakened versions of requirements familiar from AGM. **Stability** ($\diamond$2) says that whenever an agent takes one of their beliefs as a supposition, the set of suppositionally acceptable propositions should just be comprised of their beliefs. This is equivalent to $*4^w$ together with a version of $*3$ weakened to only apply when $S \in \mathbf{B}$. Just as we think that $*3$ is unimpeachable, so too is its weakened version. On the other hand, $*4^w$ is not on such firm footing. We already saw that this can fail for LIS.[19] **Consistency Preservation** ($\diamond$3) offers a weaker consistency requirement than is imposed by $*5$ and only applies when both the corpus and the supposition are each individually consistent.

[16]The first account of update was given by Winslett (1988) with her "Possible Models Approach", which built on earlier work from Ginsberg (1986) and Ginsberg and Smith (1988, 1987). Notable subsequent offerings are available in (Winslett, 1990), (Dalal, 1988), (Forbus, 1989), (Zhang and Foo, 1996), and (Herzig, 1996). A systematic comparison of how these operations relate to the KM postulates, introduced below, is provided by Herzig and Rifi (1999).

[17]These postulates were originally stated in a more semantic formalism. For continuity with the AGM postulates, we provide them using an equivalent syntactic formulation.

[18]We follow Jessica Collins in adopting the alternative name in place of **Superexpansion** to honour of Hermann Chernoff, who first formulated the principle in 1954 in the context of the theory of finite choice functions.

[19]Moreover, Herzig and Rifi (1999) show that this postulate is not satisfied by many of the competing update operators to KM update mentioned in footnote 16.

The next two postulates are new. **Reciprocity** ($\diamond 6$) corresponds to the widely discussed (CSO) axiom of conditional logics. This requirement says that if S' is acceptable under the supposition that S and *vice versa*, then S and S' generates the same suppositional judgments. Herzig (1998, p. 127–128) show that, given $\diamond 1$, $\diamond 5$, and $*\top$, $\diamond 6$ implies $\diamond 2$. Since these three postulates are relatively innocuous, any reservations about $\diamond 2$ carry over to $\diamond 6$. **Primeness** ($\diamond 7$) can be seen as the requirement that when an opinionated agent supposes a disjunction, then their suppositional judgements should satisfy one of its disjuncts. This principle seems appropriate when using a finite language (as in the present case) when we are guaranteed a witness for the truth of a disjunction. It may be less desirable when the language is infinite and there is no such guarantee.

This brings us to KM update's characteristic postulate, **Compositionality**[20] ($\diamond 8$), which provides the basis for regarding update as an operation of 'local belief change'. This is made perspicuous by considering the limiting case in which $[\![\mathbf{B}]\!] = \{w_1, w_2, \ldots, w_n\}$ and $[\![\mathbf{B}_i]\!] = \{w_i\}$ where we see that $\diamond 8$ implies that

$$[\![\mathbf{B}^{\diamond}_S]\!] = \bigcup_{1 \le i \le n} [\![\mathbf{B}_i{}^{\diamond}_S]\!].$$

Thus, when an agent supposes that S, she should thereby accept each sentence that would be common to the suppositional judgements recommended for each of the opinionated (*viz.* complete) belief sets that are consistent with her beliefs. Just as we saw with imaging, the overall set of suppositional judgments is defined as a function of the suppositional judgments that would be given at each world consistent with the agent's opinions. This point has been made in slightly different terms by Pearl (2000, p. 242). He observes a parallel between $\diamond 8$ and the fact—established by (Gärdenfors, 1988, p. 113)—that imaging "preserves mixtures". That is, if a probability function Pr is a mixture of $\Pr'$ and $\Pr''$, then $\Pr_S$ is a mixture of $\Pr'_S$ and $\Pr''_S$. Put more carefully, Gärdenfors' result shows us that every imaging operator satisfies the condition that if $\Pr(X) = [\alpha \Pr'(X) + (1-\alpha)\Pr''(X)]$, then $\Pr_S(X) = [\alpha \Pr'_S(X) + (1-\alpha)\Pr''_S(X)]$. The structural similarity between this condition and $\diamond 8$ help further reinforce the connection between update and imaging.

Lastly, observe that, as with AGM's postulates, the KM postulates include synchronic, diachronic, and dispositional coherence requirements. The synchronic requirements are given by $\diamond 0$ and $\diamond 1$; the diachronic requirements are provided by $\diamond 2$ and $\diamond 3$; and, the dispositional requirements are found in the remaining $\diamond 4$ – $\diamond 8$.

5.1. LSS and the KM Postuates.

We now proceed to consider how LSS relates to the KM postulates from above. Beginning with the general case where no further constraints are imposed on LSS, we establish which of the KM postulates are satisfied by LSS. As recorded in the proposition below, LSS is guaranteed to satisfy five of the KM postulates: **Success** ($\diamond 1$), **Consistency Preservation** ($\diamond 3$), **Extensionality** ($\diamond 4$), **Chernoff** ($\diamond 5$), and **Primeness** ($\diamond 7$).

Proposition 4 *LSS must satisfy $\diamond 1$, $\diamond 3$, $\diamond 4$, $\diamond 5$ and $\diamond 7$. That is, each of the following is satisfied for any* $\mathbf{B}$, S *and* $t \in (1/2, 1]$*:*

- (a) $S \in \mathbf{B}^{\blacklozenge}_S$
- (b) *If* $\mathbf{B} \nvdash \bot$ *and* $\nvdash \neg S$*, then* $\mathbf{B}^{\blacklozenge}_S \nvdash \bot$
- (c) *If* $\vdash S \equiv S'$*, then* $\mathbf{B}^{\blacklozenge}_S = \mathbf{B}^{\blacklozenge}_{S'}$
- (d) $\mathbf{B}^{\blacklozenge}_{S \wedge S'} \subseteq Cn(\mathbf{B}^{\blacklozenge}_S \cup \{S'\})$
- (e) *If* $\mathbf{B}$ *is complete, then* $\mathbf{B}^{\blacklozenge}_{S \vee S'} \subseteq Cn(\mathbf{B}^{\blacklozenge}_S \cup \mathbf{B}^{\blacklozenge}_{S'})$

[20] Katsuno and Mendelzon call this the "Disjunction Rule". Again, we choose to follow the terminology used by Collins (1991), which we feel better captures the intuitive content of the postulate.

Proof. Proceeding sequentially:

(a) Simply recall that $c_S(S) = 1$ to infer $S \in \mathbf{B}^\blacklozenge_S$ and, thus, conclude that LIS must satisfy ◇1. □

(b) First, suppose that $\mathbf{B} \nvdash \bot$ and $\nvdash \neg S$. Next, note that whenever $\mathbf{B}$ is consistent, if $w \in [\![\mathbf{B}]\!]$, then $b(w) > 1 - t$. We prove the contrapositive by first supposing that $\mathbf{B}^\blacklozenge_S$ is inconsistent, *i.e.* $\mathbf{B}^\blacklozenge_S \vdash \bot$. That implies that for any $w \in W$, $\neg w \in \mathbf{B}^\blacklozenge_S$ and hence $S \to \neg w \in \mathbf{B}$. But, since S is consistent, there is no w^* such that $w^* \models S \to \neg w$ for every $w \in W$, and therefore $\mathbf{B}$ is also inconsistent. □

(c) Suppose that $\vdash S \equiv S'$. This implies that $\sigma(w, S) \models S'$ just in case $\sigma(w, S') \models S$. By **Uniformity** we get $\sigma(w, S) = \sigma(w, S')$ and conclude $\mathbf{B}^\blacklozenge_S = \mathbf{B}^\blacklozenge_{S'}$. So, LSS must satisfy ◇4. □

(d) To show that LSS must satisfy ◇5, first suppose $X \in \mathbf{B}^\blacklozenge_{Y \wedge Z}$ so that $c_{Y\wedge Z}(X) \geq t$. Now, we show that if $\sigma(w, Y \wedge Z) \models X$, then $\sigma(w, Y) \models Z \supset X$. To do so, we assume that $\sigma(w, Y \wedge Z) \models X$. Then, either $\sigma(w, Y) \models Z$ or $\sigma(w, Y) \models \neg Z$. In the first case, we may infer $\sigma(w, Y) = \sigma(w, Y \wedge Z)$, and by our assumption that $\sigma(w, Y) \models X$, we conclude $\sigma(w, Y) \models Z \supset X$. In the second case, $\sigma(w, Y) \models \neg Z$ and so $\sigma(w, Y) \models Z \supset X$. So either way $\sigma(w, Y) \models Z \supset X$ as desired. Applying the definition of imaging gives us

$$c_{Y\wedge Z}(X) = \sum_{\substack{w \in W: \\ \sigma(w, Y\wedge Z) \models X}} c(w) \qquad \text{and} \qquad c_Y(Z \supset X) = \sum_{\substack{w \in W: \\ \sigma(w, Y) \models Z \supset X}} c(w),$$

which imply $c_Y(Z \supset X) \geq c_{Y\wedge Z}(X) \geq t$. From this we may then infer $Z \supset X \in \mathbf{B}^\blacklozenge_Y$ and thus conclude that $X \in Cn(\mathbf{B}^\blacklozenge_Y \cup \{Z\})$. □

(e) We begin by supposing that $\mathbf{B}$ is complete, which means that there is a unique world satisfying all propositions in $\mathbf{B}$—call this $w_\mathbf{B}$. This implies that $c(w_\mathbf{B}) \geq t > 1/2$. Now, let $X \in \mathbf{B}^\blacklozenge_{S \vee S'}$ and infer $c_{(S\vee S')}(X) \geq t$. Since $c(w_\mathbf{B}) \geq t$ and $c_{S\vee S'}(X) \geq t$, it must be that $\sigma(w_\mathbf{B}, S \vee S') \models X$. Clearly either $\sigma(w_\mathbf{B}, S \vee S') \models S$ must satisfy either S or S'. Assuming the former, we infer $\sigma(w_\mathbf{B}, S) = \sigma(w_\mathbf{B}, S \vee S') \models X$ and thus $c_S(X) \geq t$ and so $X \in \mathbf{B}^\blacklozenge_S$. The same reasoning suffices for the latter. Thus, we infer $X \in Cn(\mathbf{B}^\blacklozenge_S \cup \mathbf{B}^\blacklozenge_{S'})$ to conclude that LSS must satisfy ◇7. □

Most of these results will not be unexpected. **Success** (◇1) should be validated by any plausible account of supposition, while **Extensionality** (◇4) will hold in any non-hyperintensional account like LSS. The generalisation of (*3) given by **Chernoff** (◇5) holds in virtue of the fact that the probability of a material conditional cannot be less than the probability of its consequent. The satisfaction of **Primeness** (◇7) is intuitive, since if $\mathbf{B}$ is complete it should already decide either S or S' and updating by their disjunction should not result in more propositions being accepted than by either disjunct. The only result that is remotely surprising is that LSS satisfies **Consistency Preservation** (◇3). Lockean accounts typically struggle to satisfy consistency requirements. So, it is interesting to note that LSS will not lead you to an inconsistent set of suppositional judgments when your beliefs are consistent.

We turn now to the remaining KM postulates: **Closure** (◇0), **Stability** (◇2), **Reciprocity** (◇6) and **Compositionality** (◇8). When no additional restrictions are imposed, LSS can violate each as shown below.

Proposition 5 *LSS can violate ◇0, ◇2, ◇6, and ◇8. That is, each of the following is possible:*

(a) $\mathbf{B}^\blacklozenge_S \neq Cn(\mathbf{B}^\blacklozenge_S)$

(b) $S \in \mathbf{B}$, *but* $\mathbf{B}^\blacklozenge_S \neq \mathbf{B}$

(c) $S \in \mathbf{B}^\blacklozenge_{S'}$, *and* $S' \in \mathbf{B}^\blacklozenge_S$, *but* $\mathbf{B}^\blacklozenge_S \neq \mathbf{B}^\blacklozenge_{S'}$

(d) $[\![\mathbf{B}]\!] = [\![\mathbf{B}']\!] \cup [\![\mathbf{B}'']\!]$, *but* $[\![\mathbf{B}^\blacklozenge_S]\!] \neq [\![\mathbf{B}'^\blacklozenge_S]\!] \cup [\![\mathbf{B}''^\blacklozenge_S]\!]$

Proof. Proceeding sequentially:

(a) To see that LSS can violate $\diamond 0$, simply recall that Lockean accounts generally permit violations of deductive closure, as demonstrated in the Lottery Paradox. □

(b) A counterexample showing that LSS can violate $\diamond 2$ for any $t \in (1/2, 1)$ is generated by the assignments provided on the table below, where $\varepsilon > 0$ is arbitrarily small.

W	φ	$c(\varphi)$	$c_S(\varphi)$	
w_1	$S \wedge X$	$t - \varepsilon$	$1 - \varepsilon$	$\sigma(w_1, S) = w_1$
w_2	$S \wedge \neg X$	ε	ε	$\sigma(w_2, S) = w_2$
w_3	$\neg S \wedge X$	0	0	$\sigma(w_3, S) = w_2$
w_4	$\neg S \wedge \neg X$	$1 - t$	0	$\sigma(w_4, S) = w_1$

It is easy to see that $X \notin \mathbf{B}$, but $X \in \mathbf{B}^{\blacklozenge}_S$. □

(c) Our counterexample showing that LSS can violate $\diamond 6$ proceeds by assuming that W contains the following six possible worlds.

$$w_1 \models S \wedge S' \wedge X \qquad w_2 \models S \wedge S' \wedge \neg X \qquad w_3 \models S \wedge \neg S' \wedge X$$
$$w_4 \models S \wedge \neg S' \wedge \neg X \qquad w_5 \models \neg S \wedge S' \wedge X \qquad w_6 \models \neg S \wedge S' \wedge \neg X$$

Now, let c be such that $c(w_3) = c(w_4) = c(w_5) = c(w_6) = 1/4$ and select any σ such that

$$\sigma(w_3, S') = \sigma(w_4, S') = w_1 \qquad \sigma(w_5, S) = \sigma(w_6, S) = w_2.$$

This gives us $c_S(S') = c_{S'}(S) = 1$, $c_S(X) = 0$ and $c_{S'}(X) = 1$, which implies that $S \in \mathbf{B}^{\blacklozenge}_{S'}$ and $S' \in \mathbf{B}^{\blacklozenge}_S$, but $X \notin \mathbf{B}^{\blacklozenge}_S$ and $X \in \mathbf{B}^{\blacklozenge}_{S'}$. Note that the choice of t played no role here and this suffices as a counterexample to the postulate for any $t \in (1/2, 1]$. □

(d) To build a counterexample showing that LSS can violate $\diamond 8$, fix some threshold t, let $\varepsilon > 0$ be arbitrarily small, and let n be such that $\frac{t-\varepsilon}{n-3} \leq 1 - t$. Then where $W = \{w_1, ..., w_n\}$, let $\sigma(w_i, w_{n-1} \vee w_n) = w_{n-1}$ for $i \leq n - 3$ and $\sigma(w_{n-2}, w_{n-1} \vee w_n) = w_n$. The credence functions c, c', and c'' are defined piecewise below.

$$c(w_i) := \begin{cases} \frac{t-\varepsilon}{n-3} & i \leq n-3 \\ 1 - t + \varepsilon & i = n-2 \\ 0 & \text{otherwise} \end{cases} \qquad c'(w_i) := \begin{cases} 1 & i = n-2 \\ 0 & \text{otherwise} \end{cases} \qquad c''(w_i) := \begin{cases} 1-t & i = 1 \\ t & i = n-2 \\ 0 & \text{otherwise} \end{cases}$$

Let $\mathbf{B}, \mathbf{B}', \mathbf{B}''$ be the Lockean belief sets corresponding to c, c', c'', respectively. It is easy to see that $[\![\mathbf{B}]\!] = [\![\mathbf{B}']\!] = [\![\mathbf{B}'']\!] = [\![\mathbf{B}']\!] \cup [\![\mathbf{B}'']\!] = \{w_{n-2}\}$. Imaging each of these credence functions on $w_{n-1} \vee w_n$ results in the following assignments.

$$c_{w_{n-1} \vee w_n}(w_{n-1}) = t - \varepsilon \qquad c'_{w_{n-1} \vee w_n}(w_{n-1}) = 0 \qquad c''_{w_{n-1} \vee w_n}(w_{n-1}) = 1 - t$$
$$c_{w_{n-1} \vee w_n}(w_n) = 1 - t + \varepsilon \qquad c'_{w_{n-1} \vee w_n}(w_n) = 1 \qquad c''_{w_{n-1} \vee w_n}(w_n) = t$$

Thus, we see that $[\![\mathbf{B}^{\blacklozenge}_S]\!] = \{w_{n-1}, w_n\} \neq [\![\mathbf{B}'^{\blacklozenge}_S]\!] \cup [\![\mathbf{B}''^{\blacklozenge}_S]\!] = \{w_n\}$. □

The first three of these results are expected. As Lockean accounts generally fail to require **Cogency**, we find that LSS similarly may violate $\diamond 0$. We also see that LSS can violate $\diamond 2$. This postulate is equivalent to the conjunction of $*3$ and $*4$. Recall that LIS violated the latter and we find similar behaviour with LSS. Next, the fact that LSS can violate $\diamond 6$ is somewhat obvious. The violation of $\diamond 8$ is somewhat more surprising. As we briefly discussed earlier, $\diamond 8$ is deeply connected to the idea that update proffers a form of 'local belief change'; and, as we have mentioned, Lewis presents imaging as a method for updating credences by a local dynamics. But, as we will see in the next section, all is not lost.

5.2. **Closure under the Stalnaker Conditional and Convergence between LSS and KM.** When we considered the relationship between the indicative theories given by LIS and AGM, we also saw divergences in the general case—most notably, LIS could violate AGM's characteristic postulate *4. However, we also saw that the two could be made to converge so long as we assume **Cogency** and a sufficiently low Lockean threshold. We might then wonder whether there is a similar path towards convergence between LSS and KM.

As we will soon see, there is such a path. However, the requirements involved in establishing convergence between LSS and KM are different. In this case, neither restrictions on the Lockean threshold nor standard **Cogency** will suffice. Instead, we will augment **Cogency** with the additional requirement that **B** is closed under the Stalnaker conditional ($\rightarrow$). But this will take some work since our language does not officially include $\rightarrow$. To deal with this, we will augment our finite propositional language $\mathcal{L}$ to the "flat fragment" of $\mathcal{L}$ extended with the Stalnaker conditional. That is, we introduce $\rightarrow$ into the language's signature to generate $\mathcal{L}^+$, which only adds conditional sentences of the form $X \rightarrow Y$, where $X, Y \in \mathcal{L}$. The statement of **Cogency** remains unchanged from earlier. However, the type of logical consequence used in the expression of its requirements (Cn) is richer. We let '**Cogency**$^{\rightarrow}$' refer to the stronger requirement that results from imposing **Cogency** with the richer language $\mathcal{L}^+$. At this stage, there are two important observations to make. Firstly, it is well known that the probability of the Stalnaker conditional $X \rightarrow Y$ is given by the probability of Y after imaging on X, $c_X(Y)$. Thus, the conditions under which Stalnaker conditionals are believed are clear: $X \rightarrow Y \in \mathbf{B}$ iff $c_X(Y) \geq t$. Second, observe that the Stalnaker conditional satisfies modus ponens, *i.e.* $X \rightarrow Y, X \vdash Y$. This means that **Cogency**$^{\rightarrow}$ requires that $X \rightarrow Y \in \mathbf{B}$ and $X \in \mathbf{B}$ imply $Y \in \mathbf{B}$.

Surprisingly, we find that in this richer environment where we have **Cogency**$^{\rightarrow}$, LSS satisfies all of the KM postulates. We have already shown that LSS will always satisfy ◇1, ◇4, ◇5, and ◇7; it is straightforward to see that Propositions 4 and 5 will carry over to this richer environment. So, it remains only to show that, given **Cogency**$^{\rightarrow}$, the remaining postulates are all satisfied.

Proposition 6 *Assuming* ***Cogency***$^{\rightarrow}$*, LSS must satisfy ◇0, ◇2, ◇6, and ◇8. That is, for any c and $t \in (1/2, 1]$, if* **B** *satisfies* ***Cogency***$^{\rightarrow}$*, then:*

(a) $\mathbf{B}^{\blacklozenge}_S = Cn(\mathbf{B}^{\blacklozenge}_S)$
(b) *if* $S \in \mathbf{B}$ *then* $\mathbf{B} = \mathbf{B}^{\blacklozenge}_S$
(c) *if* $S \in \mathbf{B}^{\blacklozenge}_{S'}$ *and* $S' \in \mathbf{B}^{\blacklozenge}_S$*, then* $\mathbf{B}^{\blacklozenge}_S = \mathbf{B}^{\blacklozenge}_{S'}$
(d) *if* $[\![\mathbf{B}]\!] = [\![\mathbf{B}']\!] \cup [\![\mathbf{B}'']\!]$*, then* $[\![\mathbf{B}^{\blacklozenge}_S]\!] = [\![\mathbf{B}'^{\blacklozenge}_S]\!] \cup [\![\mathbf{B}''^{\blacklozenge}_S]\!]$

Proof. As before, we proceed sequentially, where **Cogency**$^{\rightarrow}$ is taken as a standing assumption:

(a) It is an immediate consequence of assuming **Cogency**$^{\rightarrow}$ that LSS satisfies ◇0. □
(b) Suppose that $S \in \mathbf{B}$ to show that $\mathbf{B} \subseteq \mathbf{B}^{\blacklozenge}_S$. Let $X \in \mathbf{B}$ and by **Cogency**$^{\rightarrow}$ infer $S \wedge X \in \mathbf{B}$. This implies $c(S \wedge X) \geq t$. Since imaging on S won't lower the probability of any $S \wedge X$ worlds, it follows that $c_S(X) \geq t$ and thus $X \in \mathbf{B}^{\blacklozenge}_S$. For the other direction, let $X \in \mathbf{B}^{\blacklozenge}_S$ so that $c_S(X) \geq t$ and hence $S \rightarrow X \in \mathbf{B}$. By **Cogency**$^{\rightarrow}$, we get $X \in \mathbf{B}$ as desired and thus conclude that LSS now satisfies ◇2. □
(c) Suppose that $S \in \mathbf{B}^{\blacklozenge}_{S'}$ and $S' \in \mathbf{B}^{\blacklozenge}_S$. This gives us $c_S(S') \geq t$ and $c_{S'}(S) \geq t$, from which we infer that $S \rightarrow S' \in \mathbf{B}$ and $S' \rightarrow S \in \mathbf{B}$ and hence $S \leftrightarrow S' \in \mathbf{B}$. Now, letting $X \in \mathbf{B}^{\blacklozenge}_S$, we infer $S \rightarrow X \in \mathbf{B}$. By **Uniformity** and **Cogency**$^{\rightarrow}$, $S \rightarrow X \in \mathbf{B}$ and $S \leftrightarrow S' \in \mathbf{B}$ jointly entail $S' \rightarrow X \in \mathbf{B}$. Thus we infer $X \in \mathbf{B}^{\blacklozenge}_{S'}$ and hence $\mathbf{B}^{\blacklozenge}_S \subseteq \mathbf{B}^{\blacklozenge}_{S'}$. The same argument shows the converse. Thus, given **Cogency**$^{\rightarrow}$, LSS will satisfy ◇6. □
(d) To show that LSS will now satisfy ◇8, let $[\![\mathbf{B}]\!] = [\![\mathbf{B}']\!] \cup [\![\mathbf{B}'']\!]$, and suppose that **B**, **B**′ and **B**″ are all cogent$^{\rightarrow}$, and satisfy $\mathbf{LT}^t$ with respect to the credence functions c, c' and c''. Let

$w \in [\![\mathbf{B}^{\blacklozenge}_S]\!]$, $w \notin [\![\mathbf{B}'^{\blacklozenge}_S]\!] \cup [\![\mathbf{B}''^{\blacklozenge}_S]\!]$. This implies that $S \rightarrow \neg w \notin \mathbf{B}$, $S \rightarrow \neg w \in \mathbf{B}', \mathbf{B}''$, and hence that $\forall w' \in [\![\mathbf{B}']\!] \cup [\![\mathbf{B}'']\!]$, $w' \vdash S \rightarrow \neg w$, *i.e.* $\forall w' \in [\![\mathbf{B}]\!]$, $w' \vdash S \Rightarrow \neg w$. Since $\mathbf{B}$ is cogent, we have

$$t \leq c(\bigwedge \mathbf{B}) = c(\bigvee [\![\mathbf{B}]\!]) \leq c(S \rightarrow \neg w).$$

This implies that $S \rightarrow \neg w \in \mathbf{B}$, which is a contradiction. So $w \in [\![\mathbf{B}^{\blacklozenge}_S]\!]$ implies $w \in [\![\mathbf{B}'^{\blacklozenge}_S]\!] \cup [\![\mathbf{B}''^{\blacklozenge}_S]\!]$, *i.e.* $[\![\mathbf{B}^{\blacklozenge}_S]\!] \subseteq [\![\mathbf{B}'^{\blacklozenge}_S]\!] \cup [\![\mathbf{B}''^{\blacklozenge}_S]\!]$. Conversely, let $w \notin [\![\mathbf{B}^{\blacklozenge}_S]\!]$, $w \in [\![\mathbf{B}'^{\blacklozenge}_S]\!] \cup [\![\mathbf{B}''^{\blacklozenge}_S]\!]$. For argument's sake, let $w \in [\![\mathbf{B}'^{\blacklozenge}_S]\!]$. This implies that $S \rightarrow \neg w \in \mathbf{B}$, $S \rightarrow \neg w \notin \mathbf{B}'$. and hence that $\forall w' \in [\![\mathbf{B}]\!]$, $w' \vdash S \rightarrow \neg w$, *i.e.* $\forall w' \in [\![\mathbf{B}']\!]$, $w' \vdash S \rightarrow \neg w$. Since $\mathbf{B}'$ is cogent, we have

$$t \leq c'(\bigwedge \mathbf{B}') = c'(\bigvee [\![\mathbf{B}']\!]) \leq c'(S \rightarrow \neg w)$$

This implies that $S \rightarrow \neg w \in \mathbf{B}'$, which is a contradiction. So $w \in [\![\mathbf{B}'^{\blacklozenge}_S]\!] \cup [\![\mathbf{B}''^{\blacklozenge}_S]\!]$ implies $w \in [\![\mathbf{B}^{\blacklozenge}_S]\!]$, i.e. $[\![\mathbf{B}'^{\blacklozenge}_S]\!] \cup [\![\mathbf{B}''^{\blacklozenge}_S]\!] \subseteq [\![\mathbf{B}^{\blacklozenge}_S]\!]$. □

The results established in this section are summarised below in Table 4, where we see that once **Cogency**$^{\rightarrow}$ is imposed LSS satisfies all of the KM postulates.

	Cogency$^{\rightarrow}$?	$\diamond$0	$\diamond$1	$\diamond$2	$\diamond$3	$\diamond$4	$\diamond$5	$\diamond$6	$\diamond$7	$\diamond$8
$\blacklozenge$	N		✓		✓	✓	✓		✓	
$\blacklozenge$	Y	✓	✓	✓	✓	✓	✓	✓	✓	✓

TABLE 4. LSS and the KM Postulates

Perhaps the most important observation is that, in the presence of **Cogency**$^{\rightarrow}$, the quantitative norms of subjunctive supposition specified by LSS coheres perfectly with the qualitative norms provided by KM. This is in stark contrast to the vexed relationship between LIS and AGM, which falls short of perfect coherence, even when all relevant cogency constraints are imposed.

6. LIS vs KM and LSS vs AGM

We have now compared the most prominent extant quantitative theories of indicative and subjunctive supposition to their qualitative counterparts, and identified conditions under which the respective qualitative and qualitative accounts cohere with one another. In this section, we turn to the two further comparisons between (i) the judgments given by LIS that are based on our quantitative indicative theory, and the qualitative subjunctive theory based on KM update, and (ii) those given by LSS that are based on our quantitative subjunctive theory and the qualitative subjunctive theory based on AGM revision. Our strategy will remain the same from before. We will consider how LIS fares by the lights of the KM postulates and how LSS holds up with respect to the AGM postulates. Of course, these comparisons are less philosophically salient than those in sections 4, 5 (since there is no reason to expect quantitative norms of subjunctive (indicative) supposition to cohere with qualitative norms of indicative (subjunctive) supposition). Nonetheless, there are still a couple of reasons why they are worth exploring. One is simply a matter of completeness and technical interest. But, they offer a certain dialectical benefit as well. As we will see, the contrasts between how LIS and LSS behave with respect to the AGM and KM postulates will help reinforce our understanding of the relative importance of certain postulates to indicative and subjunctive supposition.

6.1. LIS vs KM.

We begin by cataloguing the relationship between LIS and KM. In the next two propositions, we consider the general case and establish which of the KM postulates are universally satisfied by LIS and which can be violated. In proposition 7, we see that LIS must satisfy **Success** ($\diamond$1), **Extensionality** ($\diamond$4), and **Chernoff** ($\diamond$5).

Proposition 7 *LIS must satisfy $\diamond 1$, $\diamond 4$ and $\diamond 5$. That is, each of the following is satisfied for any* $\mathbf{B}, S, S'$, *and* $t \in (1/2, 1]$*:*

(a) $S \in \mathbf{B}^{\circledast}_S$
(b) *If* $\vdash S \equiv S'$, *then* $\mathbf{B}^{\circledast}_S = \mathbf{B}^{\circledast}_{S'}$
(c) $\mathbf{B}^{\circledast}_{S \wedge S'} \subseteq Cn(\mathbf{B}^{\circledast}_S \cup \{S'\})$

Proof. Since these principles are identical to $*2$, $*6$, and $*7$, respectively, and (as we saw in section 4) LIS satisfies each of these postulates, LIS must then also satisfy $\diamond$1, $\diamond$4, and $\diamond$5. □

Turning now to the postulates, **Closure** ($\diamond$0), **Stability** ($\diamond$2), **Consistency Preservation** ($\diamond$3), **Reciprocity** ($\diamond$6), **Primeness** ($\diamond$7), and **Compositionality** ($\diamond$8), the following proposition establishes that each can be violated by LIS.

Proposition 8 *LIS can violate $\diamond 0$, $\diamond 2$, $\diamond 3$, $\diamond 6$, $\diamond 7$ and $\diamond 8$. That is, each of the following is possible:*

(a) $\mathbf{B}^{\circledast}_S \neq Cn(\mathbf{B}^{\circledast}_S)$
(b) $S \in \mathbf{B}$, *but* $\mathbf{B}^{\circledast}_S \neq \mathbf{B}$
(c) $\mathbf{B} \nvdash \bot$ *and* $\nvdash \neg S$, *but* $\mathbf{B}^{\circledast}_S \vdash \bot$
(d) $S \in \mathbf{B}^{\circledast}_{S'}$, *and* $S' \in \mathbf{B}^{\circledast}_S$, *but* $\mathbf{B}^{\circledast}_S \neq \mathbf{B}^{\circledast}_{S'}$
(e) $\mathbf{B}$ *is complete, but* $\mathbf{B}^{\circledast}_{S \vee S'} \nsubseteq Cn(\mathbf{B}^{\circledast}_S \cup \mathbf{B}^{\circledast}_{S'})$
(f) $[\![\mathbf{B}]\!] = [\![\mathbf{B}']\!] \cup [\![\mathbf{B}'']\!]$, *but* $[\![\mathbf{B}^{\circledast}_S]\!] \neq [\![\mathbf{B}'^{\circledast}_S]\!] \cup [\![\mathbf{B}''^{\circledast}_S]\!]$

Proof. Proceeding sequentially:

(a) This is immediate since $\diamond$0 is identical to $*1$, which can be violated by LIS. □
(b) This follows from the fact that LIS can violate $*4^w$ and that $\diamond$2 implies $*4^w$. □
(c) To show that LIS can violate $\diamond$3, consider the following counterexample. For arbitrary $t \in (1/2, 1)$, let n be such that $\frac{1}{n-1} \leq 1 - t$, let $W = \{w_1, ..., w_n\}$, and let $\varepsilon > 0$ be arbitrarily small. Finally, let c be given by $c(w_1) = 1 - \varepsilon$ and $c(w_i) = \frac{\varepsilon}{n}$ for $i > 1$. Then $\mathbf{B} = Cn(\{w_1\})$, which is consistent. However, $\mathbf{B}^{\circledast}_{\neg w_1}$ is inconsistent since for any $w \in W$, $\neg w \in \mathbf{B}^{\circledast}_{\neg w_1}$. □
(d) To see that LIS can violate $\diamond$6, first recall that LIS can violate $*4^w$ (so, it is possible that $S, X \in \mathbf{B}$, but that there is some $X \in \mathbf{B}$ such that $X \notin \mathbf{B}^{\circledast}_S$) and that LIS must satisfy $*\top$ (*i.e.* $\mathbf{B}^{\circledast}_\top = \mathbf{B}$). Now, to find a counterexample to $\diamond$6, simply find a counterexample to $*4^w$ and consider the two revisions: $\mathbf{B}^{\circledast}_\top$ and $\mathbf{B}^{\circledast}_S$. By $*\top$, we know that $S \in \mathbf{B}^{\circledast}_\top$. And, it is trivial that $\top \in \mathbf{B}^{\circledast}_S$. But, we also know that $\mathbf{B} \nsubseteq \mathbf{B}^{\circledast}_S$ and, thus, $\mathbf{B}^{\circledast}_\top \neq \mathbf{B}^{\circledast}_S$. □
(e) For our counterexample to $\diamond$7, set $t = 17/20$ and let c be defined as in the table below.

W	φ	$c(\varphi)$	$c(\varphi \mid S)$	$c(\varphi \mid S')$	$c(\varphi \mid S \vee S')$
w_1	$S \wedge S'$	9/1480	3/20	3/20	3/37
w_2	$S \wedge \neg S'$	51/1480	17/20	0	17/37
w_3	$\neg S \wedge S'$	51/1480	0	17/20	17/37
w_4	$\neg S \wedge \neg S'$	37/40	0	0	0

It is straightforward to see that $\mathbf{B}$, $\mathbf{B}^{\circledast}_S$, and $\mathbf{B}^{\circledast}_{S'}$ all satisfy **Cogency** and constitute a violation of $\diamond$7: first, note that $[\![\mathbf{B}]\!] = \{w_4\}$ and so $\mathbf{B}$ is *complete*, then observe that $\neg(S \wedge S') \in \mathbf{B}^{\circledast}_{S \vee S'}$, but $\neg(S \wedge S') \notin Cn(\mathbf{B}^{\circledast}_S \cup \mathbf{B}^{\circledast}_{S'})$. □

(f) For our counterexample to $\diamond 8$, fix a threshold $t \in (1/2, 1)$ and let c, c', and c'' be defined as in the table below, where $\varepsilon > 0$ arbitrarily small.

W	φ	$c(\varphi)$	$c'(\varphi)$	$c''(\varphi)$
w_1	$X \wedge Y$	ε	$1-t-\varepsilon$	0
w_2	$X \wedge \neg Y$	$1-t-\varepsilon$	ε	ε
w_3	$\neg X \wedge Y$	0	0	$1-t-\varepsilon$
w_4	$\neg X \wedge \neg Y$	t	t	t

Since $[\![\mathbf{B}]\!] = [\![\mathbf{B}']\!] = [\![\mathbf{B}'']\!] = \{w_4\}$, we see that all three are complete (thus satisfying **Cogency**) and that $[\![\mathbf{B}]\!] = [\![\mathbf{B}']\!] \cup [\![\mathbf{B}'']\!]$. Now, let $S := X \vee Y$ and inspect the table below.

W	φ	$c(\varphi \mid S)$	$c'(\varphi \mid S)$	$c''(\varphi \mid S)$
w_1	$X \wedge Y$	$\frac{\varepsilon}{1-t}$	$\frac{1-t-\varepsilon}{1-t}$	0
w_2	$X \wedge \neg Y$	$\frac{1-t-\varepsilon}{1-t}$	$\frac{\varepsilon}{1-t}$	$\frac{\varepsilon}{1-t}$
w_3	$\neg X \wedge Y$	0	0	$\frac{1-t-\varepsilon}{1-t}$
w_4	$\neg X \wedge \neg Y$	0	0	0

We see that $[\![\mathbf{B}^{\circledast}_S]\!] = \{w_2\}$, $[\![\mathbf{B}'^{\circledast}_S]\!] = \{w_1\}$, and $[\![\mathbf{B}''^{\circledast}_S]\!] = \{w_3\}$. Thus, $[\![\mathbf{B}^{\circledast}_S]\!] \neq [\![\mathbf{B}'^{\circledast}_S]\!] \cup [\![\mathbf{B}''^{\circledast}_S]\!]$. □

Unsurprisingly, these results show that in general LIS may significantly diverge from the KM postulates. However, we might wonder whether additional constraints can be imposed to bring them closer together. Although we will see that they can become much closer in their behaviour, there is no obvious way to get LIS to satisfy all of the KM postulates. In the postulate below, we show that assuming **Cogency** recovers $\diamond 0$, $\diamond 2$, $\diamond 3$, and $\diamond 6$. Nonetheless, as foreshadowed in the proofs above for $\diamond 7$ and $\diamond 8$, **Cogency** is not sufficient to ensure that they are satisfied by LIS.

Proposition 9 *Assuming* ***Cogency****, LIS must satisfy* $\diamond 0$*,* $\diamond 2$*,* $\diamond 3$*, and* $\diamond 6$*. That is, assuming* ***Cogency****, all of the following are satisfied for any* $\mathbf{B}, S, S'$*, and* $t \in (1/2, 1]$*:*

(a) $\mathbf{B}^{\circledast}_S = Cn(\mathbf{B}^{\circledast}_S)$
(b) *If* $S \in \mathbf{B}$*, then* $\mathbf{B}^{\circledast}_S = \mathbf{B}$
(c) *If* $\mathbf{B} \nvdash \bot$ *and* $\nvdash \neg S$*, then* $\mathbf{B}^{\circledast}_S \nvdash \bot$
(d) *If* $S \in \mathbf{B}^{\circledast}_S$ *and* $S' \in \mathbf{B}^{\circledast}_S$*, then* $\mathbf{B}^{\circledast}_S = \mathbf{B}^{\circledast}_{S'}$

Proof. Proceeding sequentially, where **Cogency** is taken as a standing assumption:

(a) Immediate from the assumption of **Cogency**. □
(b) Here, the satisfaction of $\diamond 2$ follows from its equivalence with the conjunction of $*3$ and $*4^w$. As we saw earlier, LIS always satisfies $*3$, while **Cogency** suffices for LIS to satisfy $*4^w$. □
(c) Immediate from the assumption of **Cogency**. □
(d) Let $\mathbf{B}$ be cogent and let $S \in \mathbf{B}^{\circledast}_{S'}$, $S' \in \mathbf{B}^{\circledast}_S$ and $X \in \mathbf{B}^{\circledast}_S$. Since $\mathbf{B}^{\circledast}_S$ is cogent, it follows that $S' \wedge X \in \mathbf{B}^{\circledast}_S$, and hence that $c(S' \wedge X \mid S) \geq t$. It is easy to see that $c(S' \wedge X \mid S) \geq t$ implies $c(\neg S \vee X \mid S') \geq t$. Therefore, from $S' \wedge X \in \mathbf{B}^{\circledast}_S$, it follows that $\neg S \vee X \in \mathbf{B}^{\circledast}_{S'}$. Now, from **Cogency** and $S \in \mathbf{B}^{\circledast}_{S'}$, it follows that $X \in \mathbf{B}^{\circledast}_{S'}$, and hence that $\mathbf{B}^{\circledast}_{S'} \subseteq \mathbf{B}^{\circledast}_S$. The other direction can be proved in analogous fashion. □

Interestingly, the following proposition demonstrates that by further adopting a sufficiently low threshold of $t \in (1/2, \phi^{-1}]$, we are able to recover $\diamond 7$ (though it is insufficient to recover $\diamond 8$).

Proposition 10 *Assuming* ***Cogency****, LIS must satisfy* $\diamond 7$ *just in case* $t \in (1/2, \phi^{-1}]$*.*

Proof. Assume **Cogency** and that $t \in (1/2, \phi^{-1}]$. We begin by observing that $\diamond 7$ holds where $S \vee S'$ is consistent with **B**: If **B** is cogent and complete, then $S \vee S'$ is consistent with **B** iff $S \in \mathbf{B}$ or $S' \in \mathbf{B}$. But, since LIS satisfies $*4^w$ provided **Cogency** and $t \in (1/2, \phi^{-1}]$, this means that $\mathbf{B}^{\circledast}_{S \vee S'} = \mathbf{B}^{\circledast}_{S}$ or $\mathbf{B}^{\circledast}_{S \vee S'} = \mathbf{B}_{S'}$. Either way, LIS satisfies $\diamond 7$. So, it remains to check the case where $S \vee S'$ is inconsistent with **B**. For this case, let our algebra contain the following worlds:

$$w_1 \models S \wedge S' \qquad w_2 \models S \wedge \neg S' \qquad w_3 \models \neg S \wedge S \qquad w_4 \models \neg S \wedge \neg S'$$

Assuming the antecedent that **B** is complete (together with our assumption that $S \vee S'$ is inconsistent with **B**) gives us $[\![\mathbf{B}]\!] = \{w_4\}$, which in turn implies

$$b(w_1) + b(w_2) + b(w_3) \leq 1 - t. \tag{1}$$

Now, suppose for reductio that $\mathbf{B}^{\circledast}_{S \vee S'} \nsubseteq Cn(\mathbf{B}^{\circledast}_{S} \cup \mathbf{B}^{\circledast}_{S'})$. This implies that $[\![\mathbf{B}^{\circledast}_{S \vee S'}]\!] \nsupseteq [\![\mathbf{B}^{\circledast}_{S}]\!] \cap [\![\mathbf{B}^{\circledast}_{S'}]\!]$. But, that can only be the case when $w_1 \notin [\![\mathbf{B}^{\circledast}_{S \vee S'}]\!]$. Thus, we infer

$$\frac{b(w_1)}{b(w_1) + b(w_2) + b(w_3)} \leq 1 - t. \tag{2}$$

Using 1 and simplifying, we get $b(w_1) \leq 1 - 2t + t^2$. Recalling that $t \leq \phi^{-1}$ iff $t^2 \leq 1 - t$, we infer $b(w_1) \leq 2 - 3t$. Plugging this value back into 1 gives us $b(w_2) + b(w_3) \leq 2t - 2$. With 1 and 2, this gives us $\frac{2t-2}{1-t} \geq t$, which simplifies to $t^2 + t \geq 2$. But, since $t^2 + t \leq 1$ iff $t \leq \phi^{-1}$, this contradicts our assumption that $t \in (1/2, \phi^{-1}]$. □

At this stage, we would like to direct the reader's attention to a few salient aspects of the results presented in this section. First, it is noteworthy that the conditions which ensure that LIS satisfies $\diamond 7$ are exactly the conditions which ensure LIS satisfies $*4$ and $\diamond 8$ (and, thus, all of the AGM postulates). On the face of it, this may seem surprising. However, those familiar with the literature may recall that $\diamond 7$ stands in a special relationship to $*7$ and $*8$. Gärdenfors (1988, p. 57) showed that given the basic postulates, $*1$ – $*6$, the conjunction of the two supplementary postulates, $*7$ and $*8$, is equivalent to the 'factoring' condition stated below.

(*V) Either (i) $\mathbf{B}^{*}_{A \vee B} = \mathbf{B}^{*}_{A}$ or (ii) $\mathbf{B}^{*}_{A \vee B} = \mathbf{B}^{*}_{B}$ or (iii) $\mathbf{B}^{*}_{A \vee B} = \mathbf{B}^{*}_{A} \cap \mathbf{B}^{*}_{B}$ **(Factoring)**

It is simple to see that $*\mathrm{V}$ implies $\mathbf{B}^{*}_{S \vee S'} \subseteq Cn(\mathbf{B}^{*}_{S} \cup \mathbf{B}^{*}_{S'})$ and, thus, as a corollary we see that taken together $*1$ – $*8$ imply ($\diamond 7$). Since LIS satisfies all of the AGM postulates provided **Cogency** and $t \in (1/2, \phi^{-1}]$, it follows that LIS satisfies ($\diamond 7$) under the same conditions.

Secondly, it is worth noting explicitly that $\diamond 8$ is the only KM postulate that LIS can violate for any choice of Lockean threshold even under the **Cogency** assumption.[21] This reinforces the already prevalent impression that $\diamond 8$ is in some sense the most distinctive and characteristic KM postulate when it comes to distinguishing between the kinds of belief change embodied by the KM and AGM postulates, respectively.

Finally, it is also worth making explicit the observation, entailed by the preceding analysis, that while there *are* certain (highly restrictive) conditions under which LIS perfectly coheres with the qualitative norms given by AGM belief revision, there are *no* similar conditions which ensure coherence of LIS with the qualitative norms given by the KM theory of belief update.

[21] To verify this, see the counterexample to $\diamond 8$ provided in Proposition 8.

	Cogency?	◇0	◇1	◇2	◇3	◇4	◇5	◇6	◇7	◇8
$\divideontimes$	N		✓			✓	✓			
$\divideontimes$	Y	✓	✓	✓	✓	✓	✓	✓		
$\divideontimes^{(1/2,\phi^{-1}]}$	Y	✓	✓	✓	✓	✓	✓	✓	✓	

TABLE 5. LIS and the KM Postulates

6.2. **LSS vs AGM.** We turn now to the second 'diagonal' comparison between the theories featuring in Table 1. Specifically, we focus now on identifying points of coherence and divergence between the quantitative norms of subjunctive supposition enshrined in LSS and the qualitative norms of indicative supposition encoded in the AGM postulates. Again, we begin with the most general case. Proposition 11 establishes which of the AGM postulates are universally satisfied by LSS, while Proposition 12 reports the divergences.

Proposition 11 *LSS must satisfy *2, *3, *6, and *7. That is, each of the following is satisfied for any* $\mathbf{B}$*,* S *and* $t \in (1/2, 1]$*:*

(a) $S \in \mathbf{B}^{\blacklozenge}_{S}$
(b) $\mathbf{B}^{\blacklozenge}_{S} \subseteq Cn(\mathbf{B}^{\blacklozenge} \cup \{S\})$
(c) *If* $\vdash S \equiv S'$*, then* $\mathbf{B}^{\blacklozenge}_{S} = \mathbf{B}^{\blacklozenge}_{S'}$
(d) $\mathbf{B}^{\blacklozenge}_{S \wedge S'} \subseteq Cn(\mathbf{B}^{\blacklozenge}_{S} \cup \{S'\})$

Proof. Proceeding sequentially:

(a) In Proposition 4, we saw that LSS satisfies ◇1, which is identical to *2. □
(b) Let $X \in \mathbf{B}^{\blacklozenge}_{S}$. Then $S \rightarrow X \in \mathbf{B}$, which implies $X \in Cn(B \cup \{S\})$, and thus LSS satisfies *3. □
(c) In Proposition 4, we saw that LSS satisfies ◇4, which is identical to *6. □
(d) Let $X \in \mathbf{B}^{\blacklozenge}_{S \wedge S'}$. Then $c_{S \wedge S'}(X) \geq t$, *i.e.*

$$c_{S \wedge S'}(X) = \sum_{\substack{w \in W \\ \sigma(w, S \wedge S') \models X}} c(w) \geq t.$$

Next, note that

$$c_S(S' \supset X) = \sum_{\substack{w \in W \\ \sigma(w, S) \models S' \supset X}} c(w).$$

Furthermore, for any $w \in W$, $\sigma(w, S) \models \neg(S' \supset X)$ if and only if $\sigma(w, S) \models S' \wedge \neg X$. This in turn entails by **Uniformity** that $\sigma(w, S) = \sigma(w, S \wedge S')$, and hence that $\sigma(w, S \wedge S') \models \neg X$. So $c_S(\neg(S' \supset X)) \leq c_{S \wedge S'}(\neg X) \leq 1 - t$. So $c_S(S' \supset X) \geq t$ and $S' \supset X \in \mathbf{B}^{\blacklozenge}_{S}$, as desired. □

Proposition 12 *LSS can violate *1, *4, *5, and *8. That is, each of the following is possible:*

(a) $\mathbf{B}^{\blacklozenge}_{S} \neq Cn(\mathbf{B}^{\blacklozenge}_{S})$
(b) $\mathbf{B}^{\blacklozenge}_{S} \nvdash \neg S$*, but* $\mathbf{B} \nsubseteq \mathbf{B}^{\blacklozenge}_{S}$
(c) $\nvdash \neg S$*, but* $\mathbf{B}^{\blacklozenge}_{S} \vdash \bot$
(d) $\mathbf{B}^{\blacklozenge}_{S} \nvdash \neg S'$*, but* $\mathbf{B}^{\blacklozenge}_{S \wedge S'} \nsupseteq Cn(\mathbf{B}^{\blacklozenge}_{S} \cup \{S'\})$

Proof. Proceeding sequentially:

(a) This is immediate from the fact that Lockean agents can violate closure requirements. □

(b) To show that LSS can violate Preservation for any threshold $t \in (1/2, 1]$, even when we assume **Cogency**$^{\rightarrow}$, let $t \in (1/2, 1]$ and suppose that $\sigma(w_4, S) = w_1$ and that c is as defined below.

W	φ	$c(\varphi)$	$c_S(\varphi)$
w_1	$S \wedge X$	0	$1/2$
w_2	$S \wedge \neg X$	$1/2$	$1/2$
w_3	$\neg S \wedge X$	0	0
w_4	$\neg S \wedge \neg X$	$1/2$	0

This yields the prior belief set $\mathbf{B} = Cn(\{\neg X\})$ and the suppositional judgement set $\mathbf{B}^{\blacklozenge}_S = Cn(S)$, both satisfying **Cogency**$^{\rightarrow}$.[22] But then we see that $\mathbf{B} \nvDash \neg S$, $\neg X \in \mathbf{B}$ and $\neg X \notin \mathbf{B}^{\blacklozenge}_S$, since $c_S(\neg X) = 1/2 < t$. □

(c) This is immediate from the fact that Lockean agents can violate consistency requirements. □

(d) To see this, let W contain the following worlds.

$w_1 \vDash \neg S \wedge S' \wedge \neg X$ $w_2 \vDash \neg S \wedge \neg S' \wedge \neg X$ $w_3 \vDash S \wedge S' \wedge X$

$w_4 \vDash S \wedge \neg S' \wedge X$ $w_5 \vDash S \wedge S' \wedge \neg X$

Now, let c be such that $c(w_1) = c(w_2) = 1/2$ so we have $\mathbf{B} = Cn(\neg S \wedge \neg X)$. Now, let σ satisfy the conditions below.

$\sigma(w_1, S) = \sigma(w_1, S \wedge S') = w_3$ $\sigma(w_2, S) = w_4$ $\sigma(w_2, S \wedge S') = w_5$

This gives us $c_S(w_3) = c_S(w_4) = 1/2$ and $c_{S\wedge S'}(w_3) = c_{S\wedge S'}(w_5) = 1/2$, which respectively yield $\mathbf{B}^{\blacklozenge}_S = Cn(S \wedge X)$ and $\mathbf{B}^{\blacklozenge}_{S\wedge S'} = Cn(S \wedge S')$. All three belief sets, $\mathbf{B}$, $\mathbf{B}^{\blacklozenge}_S$, and $\mathbf{B}^{\blacklozenge}_{S\wedge S'}$, are cogent. Thus, it's clear that (i) S' is consistent with $\mathbf{B}^{\blacklozenge}_S$, (ii) $X \in Cn(\mathbf{B}^{\blacklozenge}_S \cup \{S'\})$ and (iii) $X \notin \mathbf{B}^{\blacklozenge}_{S\wedge S'}$, which gives us the desired counterexample to *8. □

Clearly, the violations of *1 and *5 noted in Proposition 12 are straightforwardly remedied by the assumption of **Cogency**$^{\rightarrow}$. However, the violation of AGM's distinctive *4 postulate does not disappear under the **Cogency**$^{\rightarrow}$ assumption, and limiting the range of available Lockean thresholds doesn't help either. Thus, just as ⋄8 is the one KM postulate that is universally violated by LIS (for all thresholds, and even given the relevant cogency assumption), *4 is the one AGM postulate that is universally violated by LSS (for all thresholds, and even given the relevant cogency assumption). Again, this reinforces the already prevalent impression that just as ⋄8 is the most characteristic norm of subjunctive supposition encoded in the KM postulates, *4 is the most characteristic norm of indicative supposition encoded in the AGM postulates.

Before concluding, we turn briefly to investigating whether, and under what conditions, LSS satisfies the weakenings of *4 discussed in section 4.

Proposition 13 *LSS can violate* $*4^w$ *for any Lockean threshold* $t \in (1/2, 1)$. *That is for any* $t \in (1/2, 1)$, *it is possible to have* $S \in \mathbf{B}$ *even though* $\mathbf{B} \not\subseteq \mathbf{B}^{\blacklozenge}_S$.

[22] Of course, these belief sets will also contain some Stalnaker conditionals, but we can define the selection functions to ensure the satisfaction of **Cogency**$^{\rightarrow}$.

Proof. To see this, consider the following credence function, where $\varepsilon > 0$ is arbitrarily small, and let $\sigma(w_3, S) = \sigma(w_4, S) = w_2$.

W	φ	$c(\varphi)$	$c_S(\varphi)$
w_1	$S \wedge X$	$t - \varepsilon$	$t - \varepsilon$
w_2	$S \wedge \neg X$	ε	$1 - t + \varepsilon$
w_3	$\neg S \wedge X$	ε	0
w_4	$\neg S \wedge \neg X$	$1 - t - \varepsilon$	0

Then $c(S), c(X) \geq t$ and $c_S(X) < t$, which gives us $S, X \in \mathbf{B}$ but $X \notin \mathbf{B}^{\blacklozenge}_S$. □

Proposition 14 *Assuming* **Cogency**, *LSS satisfies* *4^w, *i.e.* $S \in \mathbf{B}$ *entails* $\mathbf{B} \subseteq \mathbf{B}^{\blacklozenge}_S$ *when we assume* **Cogency**.

Proof. To see this, let $S, X \in \mathbf{B}$. By **Cogency**, $S \wedge X \in \mathbf{B}$ and hence $c(S \wedge X) \geq t$, which entails $c_S(X) \geq t$ and hence $X \in \mathbf{B}^{\blacklozenge}_S$. □

So just as LIS can, in general, violate *4^w, but satisfies it in the presence of **Cogency**, LSS does the same. Turning to its weaker cousin (*4^v) where we saw some interesting behaviour from LIS with respect to the Golden Threshold, we also find some interesting threshold related beahviour. Specifically, the proposition below establishes that just as LIS satisfies *4^v when $t > \phi^{-1}$, LSS satisfies *4^v when $t > 2/3$.

Proposition 15 *LSS satisfies* *4^v, *for all and only Lockean thresholds* $t \in (2/3, 1)$, *i.e. it is possible to have* $S, X \in \mathbf{B}$ *with* $\neg X \in \mathbf{B}^{\divideontimes}_S$ *if and only if* $t \geq 2/3$.

Proof. Let $t > 2/3$ and assume that $X, S \in \mathbf{B}$. By the assumption, we know that $c(S)$, $c(X) > 2/3$, which implies $c(S \wedge X) > 1/3$. Since imaging by S does not decrease the probability of any S worlds, we infer $c_S(S \wedge X) > 1/3$ and, thus, $c_S(X) > 1/3$, which in turn implies $c_S(\neg X) \leq 2/3$. So $\neg X \notin \mathbf{B}^{\blacklozenge}_S$, as desired. Now set $t \leq 2/3$ and let $W = \{w_1, w_2, w_3, w_4\}$ as given below.

$$w_1 \models S \wedge X \qquad w_2 \models S \wedge \neg X \qquad w_3 \models \neg S \wedge X \qquad w_4 \models \neg S \wedge \neg X$$

Since $t \leq 2/3$, the credence function defined so that $c(w_1) = c(w_2) = c(w_3) = \frac{t}{2}$ and $c(w_4) = 1 - \frac{3t}{2}$ is probabilistic. Since $c(S), c(X) \geq t$, we have $S, X \in \mathbf{B}$. Now suppose that $\sigma(w_3, S) = \sigma(w_4, S) = w_2$. Then $c_S(\neg X) = c(w_2) + c(w_3) + c(w_4) \geq t$. So $\neg X \in \mathbf{B}^{\blacklozenge}_S$, which is a violation of *4^v. □

The results established in this section are summarised in Table 6

	Cogency$^{\rightarrow}$?	*1	*2	*3	*4	*4^w	*4^v	*5	*6	*7	*8
$\blacklozenge$	N		✓	✓					✓	✓	
$\blacklozenge$	Y	✓	✓	✓		✓	✓	✓	✓	✓	
$\blacklozenge^{(2/3,1)}$	N	✓	✓	✓			✓	✓	✓	✓	

TABLE 6. LSS and the AGM Postulates

7. Conclusion and Future Work

Recall that one of the basic aims of this paper was to systematically evaluate the claim that 'imaging is to KM as conditionalization is to AGM' from the perspective of a Lockean theory of belief and supposition. Below is a summary of the most significant implications of our analysis for this evaluation and an overview of all results from this paper is found in an appendix.

1: Firstly, there is a significant sense in which our analysis has vindicated the popular analogy between the relationship of imaging to KM on the one hand, and the relationship of conditionalization to AGM, on the other. Specifically, we have shown that while there are conditions—namely, $t \in (1/2, \phi^{-1}]$ and **Cogency**—under which LIS coheres perfectly with AGM, there are similarly conditions—**Cogency**$^{\rightarrow}$—under which LSS coheres perfectly with KM. However, no combination of similar conditions ensures coherence between LSS and AGM or between LIS and KM.

2: We have also identified the characteristic postulates that prevent LIS/LSS from cohering perfectly with KM/AGM, namely **Compositionality** ($\diamond$8)/**Preservation** (*4) and **Subexpansion** (*8). Apart from these postulates, LIS can be made to cohere perfectly with KM, and LSS can be made to cohere perfectly with AGM. This goes some way towards formally justifying the intuitive claim that Compositionality and Preservation are the most distinctive qualitative norms of indicative and subjunctive suppositional reasoning, respectively.

3: Finally, it is worth emphasising that in the presence of the relevant cogency assumptions, LIS and LSS actually coincide on every KM/AGM postulate other than Compositionality, Preservation and the supplementary AGM postulate (*8), i.e. cogency assumptions largely obscure the most salient differences between LIS and LSS when it comes to qualitative norms of suppositional judgement. In the absence of cogency assumptions, the differences between LIS and LSS are far greater.

One major problem that arises from our analysis is to find sets of qualitative suppositional reasoning norms that precisely axiomatise LIS and LSS respectively. Such axiomatisations would allow us to pinpoint the qualitative norms that are characteristic of the suppositional reasoning practices of all Lockean agents, and would constitute potentially compelling competitors to the AGM/KM postulates, which have dominated the discussion of qualitative belief change norms ever since their formulation.

Appendix A. Summary of Results

Synchronic: ☐ Diachronic: (light grey) Dispositional: (dark grey)

			AGM Postulates										*KM Postulates*								
			$*1$	$*2$	$*3$	$*4$	$*4^w$	$*4^v$	$*5$	$*6$	$*7$	$*8$	$\diamond 0$	$\diamond 1$	$\diamond 2$	$\diamond 3$	$\diamond 4$	$\diamond 5$	$\diamond 6$	$\diamond 7$	$\diamond 8$
LIS	$\divideontimes^1$		✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	
	$\divideontimes$			✓	✓					✓	✓			✓			✓	✓			
	$\divideontimes$	**Cogency**	✓	✓	✓		✓	✓	✓	✓	✓		✓	✓	✓	✓	✓	✓	✓		
	$\divideontimes^{(\phi^{-1},1)}$			✓	✓			✓		✓	✓			✓			✓	✓			
	$\divideontimes^{(1/2,\phi^{-1}]}$	**Cogency**	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	
LSS	$\blacklozenge$			✓	✓					✓	✓			✓		✓	✓	✓		✓	
	$\blacklozenge^{(2/3,1)}$		✓	✓	✓			✓	✓	✓	✓			✓		✓	✓	✓		✓	
	$\blacklozenge^1$		✓	✓	✓		✓	✓	✓	✓	✓		✓	✓	✓	✓	✓	✓	✓	✓	✓
	$\blacklozenge$	**Cogency**$^{\rightarrow}$	✓	✓	✓		✓	✓	✓	✓	✓		✓	✓	✓	✓	✓	✓	✓	✓	✓

Table 7. Overview of All Relationships between LIS/LSS and AGM/KM

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