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Weighted Multipolar Hardy Inequalities in $\mathbb{R}^N$ and Kolmogorov Type Operators

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Contents

Abstract

The main purpose of the thesis, which describes the topics I was involved and the results achieved so far, is to introduce the multipolar weighted Hardy inequalities in $\mathbb{R}^N$ in the context of the study of Kolmogorov type operators perturbed by singular potentials and of the related evolution problems.

From the mathematical point of view, the interest in inverse square potentials of type $V \sim \frac{c}{|x|^2}$ relies in the criticality: they have the same homogeneity as the Laplacian and do not belong to the Kato's class, then they cannot be regarded as a lower order perturbation term. Furthermore the study of such singular potentials is motived by applications to many fields. We deal with the evolution problem

$$(P) \qquad \begin{cases} \partial_t u(x,t) = Lu(x,t) + V(x)u(x,t), & x \in \mathbb{R}^N, t > 0, \\ u(\cdot, 0) = u_0 \geq 0 \in L^2_\mu, \end{cases}$$

where L is the Kolmogorov operator

$$Lu = \Delta u + \frac{\nabla \mu}{\mu} \cdot \nabla u, \tag{0.1}$$

$\mu \in C^{1,\alpha}_{loc}(\mathbb{R}^N)$, $\mu > 0$, a probability density on $\mathbb{R}^N$, $N \geq 3$, perturbed by a multipolar inverse square potential of the type

$$V(x) = \sum_{i=1}^{n} \frac{c}{|x - a_i|^2}, \quad x \in \mathbb{R}^N, \quad c > 0, \quad a_1, \ldots, a_n \in \mathbb{R}^N$$

and $L^2_\mu := L(\mathbb{R}^N, d\mu)$, with $d\mu(x) = \mu(x)dx$.

The operator L defined in (0.1) can also be defined via the bilinear form

$$a_\mu(u,v) = \int_{\mathbb{R}^N} \nabla u \cdot \nabla v \, d\mu = -\int_{\mathbb{R}^N} (Lu) v \, d\mu.$$

We state existence and nonexistence results following the approach of X. Cabré and Y. Martel in [11] and using some results stated in [33, 12].

There exists a relation between the weak solution of (P) and the *bottom of the spectrum* of the operator $-(L+V)$

$$\lambda_1(L+V) := \inf_{\varphi \in H^1_\mu \setminus \{0\}} \left(\frac{\int_{\mathbb{R}^N} |\nabla \varphi|^2 \, d\mu - \int_{\mathbb{R}^N} V \varphi^2 \, d\mu}{\int_{\mathbb{R}^N} \varphi^2 \, d\mu} \right)$$

with H^1_μ a suitable weighted Sobolev space.

The estimate of the bottom of the spectrum $\lambda_1(L+V)$ is equivalent to the weighted Hardy inequality with $V(x) = \sum_{i=1}^n \frac{c}{|x-a_i|^2}$, $c \leq c_o = c_o(N) = \left(\frac{N-2}{2}\right)^2$,

$$\int_{\mathbb{R}^N} V \, \varphi^2 \, d\mu \leq \int_{\mathbb{R}^N} |\nabla \varphi|^2 d\mu + K \int_{\mathbb{R}^N} \varphi^2 d\mu, \quad \varphi \in H^1_\mu, \qquad K > 0, \tag{0.2}$$

and to the sharpness of the best possible constant.

Then the existence of positive solutions to (P) is related to the Hardy inequality (0.2) and the nonexistence is due to the optimality of the constant c_o.

Our results about Hardy-type inequalities fit into the context of the so-called *multipolar Hardy inequalities*.

The main problem to get the estimate with the best possibile constant is due to the mutual interaction among the poles. It is not easy to overcome the difficulties related to this aspect and until now the proof based on IMS method is the unique which allows us to achieve the optimal constant. The proof of the optimality is another crucial point in the proof.

As far as we know there are no results in literature about the weighted multipolar Hardy inequalities.

The thesis describes, in the first part (Chapter 1), the reference results we can find in literature about the behaviour of the operators with inverse square potentials in the unipolar and multipolar case (existence and nonexistence of positive solutions to evolution problems with Schrödinger and Kolmogorov type operators and positivity of the quadratic form associated with Schrödinger operators). Furthermore we recall the Hardy inequalities in the case of Lebesgue measure and in the weighted case.

In the second part (Chapters 2 and 3) we report our results about Kolmogorov type operators and weighted Hardy inequalities.

Introduction

The main purpose of the thesis, which describes the topics I was involved and the results achieved so far, is to introduce the multipolar weighted Hardy inequalities in $\mathbb{R}^N$ in the context of the study of Kolmogorov type operators perturbed by singular potentials and of the related evolution problems.

From the mathematical point of view, the interest in inverse square potentials of type $V \sim \frac{c}{|x|^2}$ relies in the criticality: they have the same homogeneity as the Laplacian and do not belong to the Kato's class, then they cannot be regarded as a lower order perturbation term. Furthermore interest in singular potentials is due to the applications to many fields, for example in many physical contexts as molecular physics [40], quantum cosmology (see e.g. [6]) and combustion models [29]. Multipolar potentials are associated with the interaction of a finite number of electric dipoles as, for example, in molecular systems consisting of n nuclei of unit charge located in a finite number of points $a_1, \dots, a_n$ and of n electrons. The Hartree-Fock model describes these systems (see [16]). This type of potentials also appears in some models of physical chemistry (see [7]).

Elliptic operators with bounded coefficients have been widely studied in literature both in $\mathbb{R}^N$ and in open subsets of $\mathbb{R}^N$ since the 1950s, and nowadays they are well understood. The interest in elliptic operators with unbounded coefficients in $\mathbb{R}^N$ has grown considerably as a consequence of their numerous applications in many fields of science and economics.

Due to their importance, the literature on these operators has recently spread out considerably and now we are able to treat uniformly elliptic operators of the type

$$\mathcal{A}u(x) = \sum_{i,j=1}^{N} q_{ij}(x) D_{ij} u(x) + \sum_{i=1}^{N} b_i(x) D_i u(x) + c(x) u(x), \quad x \in \mathbb{R}^N,$$

under rather weak assumptions on the coefficients, both with analytic and probabilistic methods.

If we assume that $(q_{ij}(x))$ satisfies the ellipticity condition and q_{ij}, b_i $(i, j = 1, \dots, N)$ and c belong to $C^{0,\alpha}_{loc}(\mathbb{R}^N)$ for some $\alpha \in (0,1)$, it is possible to prove that the parabolic problem

$$\begin{cases} u_t(t,x) = \mathcal{A}u(t,x), & t > 0,\, x \in \mathbb{R}^N, \\ u(0,x) = u_0(x), \end{cases}$$

admits a classical solution for any $u_0 \in C_b(\mathbb{R}^N)$. Moreover, there exists a semigroup $\{T(t)\}_{t\geq 0}$ defined in $C_b(\mathbb{R}^N)$ such that the solution of the problem is given by $u(t,x) = (T(t)u_0)(x)$ for any $u_0 \in C_b(\mathbb{R}^N)$ (see [42]).

The operator $\mathcal{A}$ is not in general the infinitesimal generator of this semigroup, but it generates $\{T(t)\}_{t\geq 0}$ in a weak sense.

In the thesis we deal with the evolution problem

$$(P) \quad \begin{cases} \partial_t u(x,t) = Lu(x,t) + V(x)u(x,t), & x \in \mathbb{R}^N, t > 0, \\ u(\cdot, 0) = u_0 \geq 0 \in L^2_\mu, \end{cases}$$

where L is the Kolmogorov operator

$$Lu = \Delta u + \frac{\nabla \mu}{\mu} \cdot \nabla u, \tag{0.3}$$

$\mu \in C^{1,\alpha}_{loc}(\mathbb{R}^N)$, $\mu > 0$, a probability density on $\mathbb{R}^N$, $N \geq 3$, perturbed by a multipolar inverse square potential of the type

$$V(x) = \sum_{i=1}^{n} \frac{c}{|x - a_i|^2}, \quad x \in \mathbb{R}^N, \quad c > 0, \quad a_1, \dots, a_n \in \mathbb{R}^N \tag{0.4}$$

and $L^2_\mu := L(\mathbb{R}^N, d\mu)$, with $d\mu(x) = \mu(x)dx$.

The operator L defined in (0.3) can also be defined via the bilinear form

$$a_\mu(u,v) = \int_{\mathbb{R}^N} \nabla u \cdot \nabla v \, d\mu = -\int_{\mathbb{R}^N} (Lu)v \, d\mu.$$

We can derive properties for L and generation results through properties of the form a_μ.

It is well known that the semigroup $\{T(t)\}_{t\geq 0}$ weakly generated by the operator L in $C_b(\mathbb{R}^N)$ can be extended to a positivity preserving and analytic strongly continuous semigroup on the weighted space L^2_μ. This is possible since $d\mu$ is the invariant measure for $\{T(t)\}_{t\geq 0}$ in $C_b(\mathbb{R}^N)$ (see [42]).

We state existence and nonexistence results using the relation between the weak solution of (P) and the *bottom of the spectrum* of the operator $-(L+V)$

$$\lambda_1(L+V) := \inf_{\varphi \in H^1_\mu \setminus \{0\}} \left(\frac{\int_{\mathbb{R}^N} |\nabla \varphi|^2 \, d\mu - \int_{\mathbb{R}^N} V\varphi^2 \, d\mu}{\int_{\mathbb{R}^N} \varphi^2 \, d\mu} \right)$$

with H^1_μ suitable weighted Sobolev space.

When $\mu = 1$ X. Cabré and Y. Martel in [11] showed that the boundedness of $\lambda_1(\Delta + V)$, $0 \le V \in L^1_{loc}(\mathbb{R}^N)$, is a necessary and sufficient condition for the existence of positive exponentially bounded in time solutions to the associated initial value problem. Later G. R. Goldstein, J. A. Goldstein, A. Rhandi in [33] and A. Canale, F. Gregorio, A. Rhandi, C. Tacelli in [12] extended the result to the case of Kolmogorov operators.

The estimate of the bottom of the spectrum $\lambda_1(L+V)$ is equivalent to the weighted Hardy inequality with $V(x) = \sum_{i=1}^n \frac{c}{|x-a_i|^2}$, $c \le c_o$,

$$\int_{\mathbb{R}^N} V \varphi^2 \, d\mu \le \int_{\mathbb{R}^N} |\nabla \varphi|^2 d\mu + K \int_{\mathbb{R}^N} \varphi^2 d\mu, \quad \varphi \in H^1_\mu, \qquad K > 0, \tag{0.5}$$

and to the sharpness of the best possible constant.

Then the existence of positive solutions to (P) is related to the Hardy inequality (0.5) and the nonexistence is due to the optimality of the constant c_o.

Our results about Hardy-type inequalities fit into the context of the so-called *multipolar Hardy inequalities*.

The main problem to get the estimate with the best possibile constant is due to the mutual interaction among the poles. It has been not easy to overcome the difficulties related to this aspect and until now the proof based on IMS method is the unique which allows us to achieve the optimal constant. The proof of the optimality is another crucial point in the proof.

As far as we know there are no results in literature about the weighted multipolar Hardy inequalities.

The thesis describes, in the first part (Chapter 1), the reference results we can find in literature about the behaviour of the operators with inverse square potentials in the unipolar and multipolar case (existence and nonexistence of positive solutions to evolution problems with Schrödinger and Kolmogorov type operators and positivity of the quadratic form associated with Schrödinger operators). Furthermore we recall the Hardy inequalities in the case of Lebesgue measure and in the weighted case.

In the second part (Chapters 2 and 3) we report our results about Kolmogorov type operators and weighted Hardy inequalities.

In particular the thesis is structured as follows.

Chapter 1 deals with Schrödinger operators with singular potentials. The simplest examples of operators belonging to the class we consider are the Schrödinger operators acting on $L^2(\mathbb{R}^N)$, which correspond to the case $\mu = 1$ in (0.3)

$$\mathcal{L}u(x) := -\Delta u(x) - V(x)u(x).$$

We introduce the classical Hardy inequality on $\mathbb{R}^N$, $N \geq 3$,

$$c_o \int_{\mathbb{R}^N} \frac{\varphi^2}{|x|^2}\, dx \leq \int_{\mathbb{R}^N} |\nabla \varphi|^2\, dx, \quad \forall\, \varphi \in H^1(\mathbb{R}^N),$$

where $c \leq c_o = c_o(N) = \left(\frac{N-2}{2}\right)^2$, with c_o optimal constant. For the inequality, we refer to the proof given by E. Mitidieri in [47], which makes use of the *vector field method*, while to prove the optimality of the constant we adapt the technique used in [12].

Afterwards we consider the initial value problem corresponding to the operator $\Delta + V$

$$\begin{cases} u_t = \Delta u + Vu & t > 0,\, x \in \mathbb{R}^N, \\ u(0, \cdot) = u_0 \geq 0 \in L^2(\mathbb{R}^N), \end{cases} \tag{0.6}$$

where $0 \leq V \in L^1_{loc}(\mathbb{R}^N)$. It is well known that if $V \leq \frac{c}{|x|^{2-\varepsilon}}$, $c > 0$, $\varepsilon > 0$, then the problem is well-posed. But for $\varepsilon = 0$ it may not have positive solutions.

We recall a remarkable result stated in 1984 by P. Baras and J. A. Goldstein in [4]. The authors showed that in the case $V(x) = \frac{c}{|x|^2}$ the evolution problem (0.6) admits a unique positive solution if $c \leq c_o = \left(\frac{N-2}{2}\right)^2$ and no positive solutions exist if $c > c_o$. When it exists, the solution is exponentially bounded, on the contrary, if $c > c_o$, there is the so called instantaneous blowup phenomenon (cf. [44]).

An analogous result has been obtained in 1999 by X. Cabré and Y. Martel in [11] for general potentials $0 \leq V \in L^1_{loc}(\mathbb{R}^N)$ with a different approch.

Their approach is based on the estimate of the first eigenvalue (or bottom of the spectrum) λ_1 of the operator $-\Delta - V$, defined as

$$\lambda_1 = \inf_{\varphi \in H^1(\mathbb{R}^N)\setminus\{0\}} \left(\frac{\int_{\mathbb{R}^N} |\nabla \varphi|^2\, d\mu - \int_{\mathbb{R}^N} V\varphi^2\, d\mu}{\int_{\mathbb{R}^N} \varphi^2\, d\mu} \right).$$

In particular they showed that, if $\lambda_1 > -\infty$, then the problem (0.6) admits a positive and exponentially bounded in time weak solution. Conversely, if $\lambda_1 = -\infty$, there exist no positive solutions.

The classical Hardy inequality plays a crucial role in the proof of the statement.

In the context of Schrödinger operators with multipolar inverse square potentials of the type

$$\mathcal{L} = -\Delta - \sum_{i=1}^{n} \frac{c_i^+}{|x-a_i|^2}, \qquad a_1, \dots, a_n \in \mathbb{R}^N,$$

$n \geq 2$, $c_i \in \mathbb{R}$, $c_i^+ = \max\{c_i, 0\}$, for any $i \in \{1, \dots, n\}$. V. Felli, E. M. Marchini and S. Terracini in [25] proved that the associated quadratic form

$$Q(\varphi) := \int_{\mathbb{R}^N} |\nabla \varphi|^2 \, dx - \sum_{i=1}^{n} c_i \int_{\mathbb{R}^N} \frac{\varphi^2}{|x-a_i|^2} \, dx$$

is positive if $\sum_{i=1}^n c_i^+ < \frac{(N-2)^2}{4}$, conversely if $\sum_{i=1}^n c_i^+ > \frac{(N-2)^2}{4}$ there exists a configuration of poles such that Q is not positive.

In the Chapter we describe some results in literature on multipolar Hardy inequalities on $\mathbb{R}^N$ when the measure is the Lebesgue measure.

In the case of the potential $V(x) = \sum_{i=1}^n \frac{c}{|x-a_i|^2}$, with $c > 0$, R. Bosi, J. Dolbeaut and M. J. Esteban in [10] proved that there exists a positive constant K such that the following multipolar Hardy inequality holds

$$c \int_{\mathbb{R}^N} V \varphi^2 \, dx \leq \int_{\mathbb{R}^N} |\nabla \varphi|^2 \, dx + K \int_{\mathbb{R}^N} \varphi^2 \, dx, \quad \forall \, \varphi \in H^1(\mathbb{R}^N) \qquad (0.7)$$

for any $c \in \left(0, \left(\frac{N-2}{2}\right)^2\right]$. The estimate is obtained using the so-called *IMS* (Ismagilov, Morgan, Morgan-Simon, Sigal) truncation method, which consists in localizing the wave functions around the singularities by using a partition of unity of $\mathbb{R}^N$.

This method allows to get the inequality and to achieve the constant c_o thanks to the well-known unipolar inequality. So the authors can avoid the problems related to the mutual interaction among the poles.

We report also a Hardy-type inequality as (0.7) with $K = 0$ and $V = c \sum_{1 \leq i < j \leq n} \frac{|a_i - a_j|^2}{|x-a_i|^2 |x-a_j|^2}$, stated by C. Cazacu and E. Zuazua in [18], which improves a result stated in [10].

We then consider the class Ornstein-Uhlenbeck type operator

$$Lu = \Delta u - Ax \cdot \nabla u,$$

where A is a positive definite real Hermitian $N \times N$-matrix, perturbed by the unipolar inverse square potential $V = \frac{c}{|x|^2}$.

We present the results stated in [33], an extension of the Cabré and Martel's result to the case of Kolmogorov type operators and the estimate

$$c_o \int_{\mathbb{R}^N} \frac{\varphi^2}{|x|^2}\, d\mu \leq \int_{\mathbb{R}^N} |\nabla \varphi|^2\, d\mu + K \int_{\mathbb{R}^N} \varphi^2\, d\mu, \quad \forall\, \varphi \in H^1_\mu \tag{0.8}$$

with respect to the Gaussian measure $d\mu = Ce^{-\frac{1}{2}\langle Ax,x\rangle}dx$, with C normalization constant, which is the invariant measure for L. Here H^1_μ denotes the set of all the functions in L^2_μ having distributional derivative in $(L^2_\mu)^N$. The proof of the estimate (0.8) is based on the vector field method.

Furthermore the authors proved that the constant $c_o = \left(\frac{N-2}{2}\right)^2$ in the inequality is optimal. This allowed them to characterize the existence of the semigroup solution of the parabolic problem (P) corresponding to $L + V$ in L^2_μ. In particular they obtained nonexistence of positive exponentially bounded solutions if the coefficient of the inverse square potential is greater than c_o.

Finally we report recent results in [12] about weighted unipolar Hardy inequalities and Kolmogorov operators with more general drift term with respect to the paper [33].

Chapter 2 is dedicated to the results stated in [13]. We consider the generalized Ornstein-Uhlenbeck operator

$$Lu = \Delta u - \sum_{i=1}^{n} A(x - a_i) \cdot \nabla u, \qquad a_1, \ldots, a_n \in \mathbb{R}^N \tag{0.9}$$

where A is a positive definite real Hermitian $N \times N$-matrix, and the associated evolution problem (P) with the multipolar singular potential V defined in (0.4).

We are motivated to consider the Gaussian measure

$$d\mu(x) = \mu(x)dx = Ce^{-\frac{1}{2}\sum_{i=1}^{n}\langle A(x-a_i),(x-a_i)\rangle}dx,$$

with C normalization constant, which is the unique invariant measure for the Ornstein-Uhlenbeck type operator (0.9).

The main result in Chapter 2 is the following weighted multipolar Hardy inequality

$$c\int_{\mathbb{R}^N}\sum_{i=1}^{n}\frac{\varphi^2}{|x-a_i|^2}\,d\mu\le\int_{\mathbb{R}^N}|\nabla\varphi|^2\,d\mu + \left[\frac{k+(n+1)c}{r_0^2}+\frac{n}{2}\operatorname{Tr}A\right]\int_{\mathbb{R}^N}\varphi^2\,d\mu \tag{0.10}$$

which holds for all $\varphi\in H^1_\mu$, where $r_0=\min_{i\neq j}|a_i-a_j|/2$, $i,j=1,\dots,n$, $k\in[0,\pi^2)$ and $c\in(0,c_o]$ with $c_o=c_o(N)=\left(\frac{N-2}{2}\right)^2$ optimal constant.

Our technique, unlike the vector field method used in the case $n=1$ in [33], allow us to overcome the difficulties due to the mutual interaction among the poles and to achieve the constant c_o in the left-hand side in the inequality.

We obtain the estimate (0.10) using a way which allows us to get it starting from the result obtained in [10] in the case of Lebesgue measure and exploiting a suitable bound that the function μ satisfies.

The optimality of the constant c_o is less immediate to obtain. The crucial points to estimate the bottom of the spectrum are the choice of a suitable function φ which involves only one pole and the connection we state between the weight functions in the case of one pole and in the case of multiple poles.

In the paper [13] we prove also in a different way the weighted inequality through the IMS method arguing as in [10]. To this aim we need to use a Hardy inequality in the case $n=1$ which we need to prove. In fact, we recall that in the IMS method a fundamental tool is an estimate with a single pole which allows us to achieve the optimal constant c_o in the inequality.

Furthermore, we state an existence and nonexistence result putting together our weighted Hardy inequality and Goldstein-Goldstein-Rhandi's result.

We conclude Chapter 2 by setting semigroup generation results via the bilinear form technique. To this aim we consider the bilinear form associated to the operator $-(L+V)$

$$a_c(u,v):=\int_{\mathbb{R}^N}\nabla u\cdot\nabla v\,d\mu - c\sum_{i=1}^{n}\int_{\mathbb{R}^N}\frac{uv}{|x-a_i|^2}\,d\mu$$

with domain H^1_μ. Studying the form we state the positivity of the solution.

Chapter 3 is devoted to some results which extend the previous ones to the case of Kolmogorov type operators ([14, 15]). The work [15] is in progress for some aspects. The basic idea is to extend the multipolar Hardy inequalities when the measure is different from the Gaussian measure, then of more general type. We use the IMS method adapted to the weighted case to get the result. So we need a weighted unipolar Hardy inequality proved in [14] which can be used in the proof. This motivated the interest for this type of inequality in relation to the multipolar case.

The proof of the unipolar inequality is different from the others in literature. To state the optimality, which represents in general a delicate point in the proof, we introduce a suitable function. This inequality allow us to get existence and nonexistence results for solutions to parabolic problem associated to Kolmogorov operators.

In the Chapter we also state weighted multipolar Hardy inequalities using vector field method. We state a preliminary inequality which represent an extension of the result obtained for Lebesgue measure (see Chapter 1). Then we are able to prove the Hardy inequality and to overcome the difficulties due to the mutual interaction among the poles but we do not achieve the best constant.

To complete the work of the thesis we include two appendices where we summarize some topics related to the Chapters.

In Appendix A we recall some results and terminology on sesquilinear forms and their associated operators. Appendix B deals with Semigroup Theory and invariant measures.

Chapter 1

Schrödinger and Kolmogorov operators with inverse square potentials

In this Chapter we present some results we can find in literature about unipolar and multipolar Hardy inequalities in $\mathbb{R}^N$, Schrödinger and Kolmogorov type operators and related evolution problems. We focus on weighted Hardy inequalities and existence and nonexistence results for solutions to the evolution problems.

The Chapter is structured as follows.

In Section 1.1 we deal with the classical Hardy inequality in $\mathbb{R}^N$, $N \geq 3$. We present the proof given by E. Mitidieri in [47], while for the optimality of the constant we adapt the proof in [12].

Section 1.2 concerns the initial value problem corresponding to the Schrödinger operators with singular potentials $-\Delta - V$. We recall remarkable results stated by P. Baras and J. A. Goldstein in [4] in the case of the potential $V(x) = \frac{c}{|x|^2}$, $c > 0$, and by X. Cabré and Y. Martel in [11] in the general case $0 \leq V \in L^1_{loc}(\mathbb{R}^N)$.

In Section 1.3 we consider Schrödinger operators with multipolar inverse square potentials.

For these operators we report a necessary and sufficient condition for the positivity of the associated quadratic form stated by V. Felli, E. M. Marchini and S. Terracini in [25].

Then we focus in Section 1.4 on multipolar Hardy inequalities and related matters about the optimality of the constant in the estimates. A notable achievement is the inequality with optimal constant stated by R. Bosi, J. Dolbeaut and M. J. Esteban in [10].

In Section 1.5 we deal with Kolmogorov type operators

$$Lu = \Delta u + \frac{\nabla \mu}{\mu} \cdot \nabla u.$$

with weight function $\mu \in C^{1,\alpha}_{loc}(\mathbb{R}^N)$, $\mu > 0$, in the drift term. We report an extension of Cabré-Martel's results for the parabolic problem corresponding to the perturbed operator $L+V$, where $V \in L^1_{loc}(\mathbb{R}^N)$, stated in [33].

In Section 1.6 we deal with an Ornstein-Uhlenbeck type operator perturbed by the unipolar inverse square potential $V(x) = \frac{c}{|x|^2}$. We present a weighted Hardy inequality with optimal constant, stated in [33], which allows to state the existence and nonexistence of the weak solutions to the parabolic problem corresponding to $L+V$ is in terms of the constant c in the potential. It has been initially reference paper of our work on the multipolar case.

In Section 1.7 a more general result about the weighted Hardy inequality stated in [12] is given. It is related to Kolmogorov type operators with more general drift term. The authors get existence and nonexistence conditions stating an extension of the result in [33], given in Section 1.6, to the case of more general measures.

1.1 The classical Hardy inequality in $\mathbb{R}^N$

The classical Hardy inequality was originally introduced in [35] in the one dimensional case as the attempt to simplify the proof of Hilbert's double series theorem (see [36, Theorem 315]).

We present the estimate in $\mathbb{R}^N$, $N \geq 3$. The proof we report is due to Mitidieri [47] and it is based on the so-called *vector field method*. In literature there are alternative techniques to prove the result, see for example [3, 28].

Theorem 1.1. *For every $\varphi \in H^1(\mathbb{R}^N)$, $N \geq 3$, the following inequality holds*

$$\left(\frac{N-2}{2}\right)^2 \int_{\mathbb{R}^N} \frac{\varphi^2}{|x|^2}\, dx \leq \int_{\mathbb{R}^N} |\nabla \varphi|^2\, dx. \tag{1.1}$$

Proof. First we observe that it is enough to assume that $\varphi \in C_c^\infty(\mathbb{R}^N)$. We define the vector field F_ε as

$$F_\varepsilon(x) = \left(\frac{x_1}{\varepsilon + |x|^2}\varphi^2, \cdots, \frac{x_N}{\varepsilon + |x|^2}\varphi^2 \right), \quad \varepsilon > 0.$$

By the divergence theorem we have

$$\begin{aligned}\int_{\mathbb{R}^N} \frac{N\varepsilon + (N-2)|x|^2}{(\varepsilon + |x|^2)^2}\,\varphi^2\,dx = \\ &= -2\int_{\mathbb{R}^N} \frac{x\cdot\nabla\varphi}{(\varepsilon+|x|^2)^2}\,\varphi\,dx \\ &\le 2\int_{\mathbb{R}^N} \frac{|x||\varphi|}{(\varepsilon+|x|^2)^2}\,|\nabla\varphi|\,dx \\ &\le 2\left(\int_{\mathbb{R}^N} \frac{|x|^2}{(\varepsilon+|x|^2)^2}\,\varphi^2\,dx\right)^{\frac{1}{2}} \left(\int_{\mathbb{R}^N} |\nabla\varphi|^2\,dx\right)^{\frac{1}{2}},\end{aligned}$$

and then

$$\left(\frac{N-2}{2}\right)^2 \int_{\mathbb{R}^N} \frac{|x|^2}{(\varepsilon+|x|^2)^2}\,\varphi^2\,dx \le \int_{\mathbb{R}^N} |\nabla\varphi|^2\,dx.$$

Letting $\varepsilon \to 0^+$ in the above inequality we obtain the claim. □

In the following Theorem we show that the constant in (1.1) is optimal adapting the proof given in [12] to the case of Lebesgue measure (see also [32, Theorem 2.3]).

Theorem 1.2. *There exists a function $\varphi \in H^1(\mathbb{R}^N)$, $N \ge 3$, such that inequality (1.1) does not hold if $c > \left(\frac{N-2}{2}\right)^2$.*

Proof. Let γ be such that $\max\left\{-\sqrt{c}, -\frac{N}{2}\right\} < \gamma \le -\frac{N-2}{2}$, so that $|x|^{2\gamma} \in L^1_{loc}(\mathbb{R}^N)$ and $|x|^{2\gamma-2} \notin L^1_{loc}(\mathbb{R}^N)$ and $\gamma^2 < c$.

Let $n \in \mathbb{N}$ and $\vartheta \in C_c^\infty(\mathbb{R}^N)$, $0 \le \vartheta \le 1$, $\vartheta = 1$ in $B_1(a)$ and $\vartheta = 0$ in $B_2^c(a)$. Set $\varphi_n(x) = \min\{|x|^\gamma\vartheta(x), n^{-\gamma}\}$. We observe that

$$\varphi_n(x) = \begin{cases} \left(\frac{1}{n}\right)^\gamma & \text{if } |x| < \frac{1}{n}, \\ |x|^\gamma & \text{if } \frac{1}{n} \le |x| < 1, \\ |x|^\gamma\vartheta(x) & \text{if } 1 \le |x| < 2, \\ 0 & \text{if } |x| \ge 2. \end{cases}$$

The functions $\varphi_n(x)$ are in $H^1(\mathbb{R}^N)$.

Let us assume $c > \left(\frac{N-2}{2}\right)^2$. We want to show that $\lambda_1 = -\infty$.

We get

$$
\begin{aligned}
&\int_{\mathbb{R}^N} \left(|\nabla \varphi_n|^2 - \frac{c}{|x|^2}\varphi_n^2 \right) dx = \\
&\quad = \int_{B_1 \setminus B_{\frac{1}{n}}} \left(|\nabla |x|^\gamma|^2 - \frac{c}{|x|^2}|x|^{2\gamma} \right) dx + \int_{B_1^c} |\nabla |x|^\gamma \vartheta(x)|^2 dx \\
&\qquad - \int_{B_1^c} \frac{c}{|x|^2} \left(|x|^\gamma \vartheta(x) \right)^2 dx - c \int_{B_{\frac{1}{n}}} n^{-2\gamma} \frac{1}{|x|^2} dx \\
&\quad \le (\gamma^2 - c) \int_{B_1 \setminus B_{\frac{1}{n}}} |x|^{2\gamma-2} dx + 2 \int_{B_1^c} \left(|x|^{2\gamma} |\nabla \vartheta|^2 + \gamma^2 \vartheta^2 |x|^{2\gamma-2} \right) dx \\
&\quad \le (\gamma^2 - c) \int_{B_1 \setminus B_{\frac{1}{n}}} |x|^{2\gamma-2} dx + 2(\|\nabla \vartheta\|_\infty^2 + \gamma^2) \int_{B_1^c} dx \\
&\quad = (\gamma^2 - c) \int_{B_1 \setminus B_{\frac{1}{n}}} |x|^{2\gamma-2} dx + C_1 \ . \qquad (1.2)
\end{aligned}
$$

On the other hand,

$$
\int_{\mathbb{R}^N} \varphi_n^2(x)\, dx \ge \int_{B_2 \setminus B_1} |x|^{2\gamma} \vartheta^2(x)\, dx = C_2. \qquad (1.3)
$$

From (1.2) and (1.3) one obtains

$$
\lambda_1 \le \frac{\int_{\mathbb{R}^N} \left(|\nabla \varphi_n|^2 - \frac{c}{|x|^2}\varphi_n^2 \right) dx}{\int_{\mathbb{R}^N} \varphi_n^2 dx} \le \frac{(\gamma^2 - c) \int_{B_1 \setminus B_{\frac{1}{n}}} |x|^{2\gamma-2}\, dx + C_1}{C_2}.
$$

Taking into account that $\gamma^2 - c < 0$ and

$$
\lim_{n \to +\infty} \int_{B_1 \setminus B_{\frac{1}{n}}} |x|^{2\gamma-2}\, dx = +\infty,
$$

we get $\lambda_1 = -\infty$.

□

1.2 Schrödinger operators with inverse square potentials

In this Section we consider the parabolic problem corresponding to the Schrödinger operator $\mathcal{L} = -\Delta - V$, where $0 \leq V \in L^1_{loc}(\mathbb{R}^N)$,

$$\begin{cases} u_t = \Delta u + Vu & t > 0, \quad x \in \mathbb{R}^N, \\ u(0,\cdot) = u_0 \geq 0 \in L^2(\mathbb{R}^N). \end{cases} \tag{1.4}$$

It is well known that if $V \leq \frac{c}{|x|^{2-\varepsilon}}$, $c > 0$, $\varepsilon > 0$, then the problem is well-posed. But for $\varepsilon = 0$ it may not have positive solution.

In 1984 P. Baras and J. A. Goldstein in [4] stated the following necessary and sufficient conditions for the existence of positive solutions in the case of the potential $V(x) = \frac{c}{|x|^2}$, $c > 0$.

Theorem 1.3. *Let us assume $V(x) = \frac{c}{|x|^2}$, $c > 0$. The problem (1.4) has a unique positive solution for each $u_0 \in L^2(\mathbb{R}^N)$ if $c \leq \left(\frac{N-2}{2}\right)^2$ and no positive solution if $c > \left(\frac{N-2}{2}\right)^2$. When it exists, the solution is exponentially bounded, on the contrary, if $c > \left(\frac{N-2}{2}\right)^2$, there is the so-called blowup phenomenon.*

Afterwards, in 1999 X. Cabré and Y. Martel in [11] stated analogous conditions for the existence of weak solutions to (1.4) in the more general case of potentials $0 \leq V \in L^1_{loc}(\mathbb{R}^N)$.

We say that $u \geq 0$ is a weak solution to (1.4) if, for each $T, R > 0$ we have $u \in C([0,T], L^2(\mathbb{R}^N))$, $Vu \in L^1((0,T) \times B_R, dtdx)$, and

$$\int_0^T \int_{\mathbb{R}^N} u(-\phi_t - \Delta\phi)\, dxdt - \int_{\mathbb{R}^N} u_0 \phi(0,\cdot)\, dx = \int_0^T \int_{\mathbb{R}^N} V u \phi\, dxdt$$

for all $\phi \in W^{2,1}_{loc}([0,T] \times \mathbb{R}^N)$. If $T = \infty$ we say that u is a global weak solution to (1.4).

Moreover, we define the *first eigenvalue* (or *bottom of the spectrum*) of the operator $-\Delta - V$ in $\mathbb{R}^N$, as

$$\lambda_1(\Delta + V) := \inf_{\varphi \in H^1(\mathbb{R}^N)\setminus\{0\}} \frac{\int_{\mathbb{R}^N} |\nabla\varphi|^2\, dx - \int_{\mathbb{R}^N} V\varphi^2\, dx}{\int_{\mathbb{R}^N} \varphi^2\, dx}.$$

Note that the case $\lambda_1(\Delta + V) = -\infty$ is allowed.

We report below the statement of Cabré-Martel's Theorem.

Theorem 1.4. *Let us assume* $0 \leq V \in L^1_{loc}(\mathbb{R}^N)$. *The following statements hold.*

(i) If $\lambda_1(\Delta + V) > -\infty$, *then there exists a global weak solution to (1.4) such that*

$$\|u(t)\|_{L^2(\mathbb{R}^N)} \leq Me^{\omega t}\|u_0\|_{L^2(\mathbb{R}^N)}, \tag{1.5}$$

for $t \geq 0$ *and some constants* $M \geq 1$, $\omega \in \mathbb{R}$;

(ii) If $\lambda_1(\Delta + V) = -\infty$, *then for any* $0 \leq u_0 \in L^2(\mathbb{R}^N) \setminus \{0\}$ *the problem (1.4) has no positive solution such that satisfying (1.5).*

As remarked in [11], in the case of the potential $V(x) = \frac{c}{|x|^2}$ the existence of positive solutions to (1.4) is related to the classical Hardy inequality. The nonexistence of solutions is due to the optimality of the constant in the inequality. Therefore, studying the bottom of the spectrum is equivalent to studying the Hardy inequality and the sharpness of the best possible constant.

1.3 Schrödinger operator with multipolar inverse square potentials

Let us consider the Schrödinger operators with multipolar inverse square potentials

$$\mathcal{L} = -\Delta - \sum_{i=1}^{n} \frac{c_i}{|x - a_i|^2}, \tag{1.6}$$

where $a_i \in \mathbb{R}^N$ for $i \in \{1, \ldots, n\}$, $a_i \neq a_j$ for $i \neq j$, $c_i \in \mathbb{R}$ for all $i \in \{1, \ldots, n\}$.

The interest in Schrödinger operators with multipolar potentials (see e.g. [22, 26, 27, 25, 18, 17]) is motivated by applications to Physics (see e.g. [7, 16]).

An interesting problem concerns the positivity of the quadratic form associated with the operator (1.6), defined as

$$Q(u) = Q_{c_1,\ldots,c_n,a_1,\ldots,a_n}(u) := \int_{\mathbb{R}^N} |\nabla u(x)|^2\, dx - \sum_{i=1}^{n} c_i \frac{u^2(x)}{|x - a_i|^2}\, dx.$$

We say that the form Q is *positive definite* if there exists a positive constant

$$\varepsilon = \varepsilon(c_1, \dots, c_n, a_1, \dots, a_n)$$

such that

$$Q(u) \geq \varepsilon \int_{\mathbb{R}^N} |\nabla u(x)|^2 \, dx, \qquad \text{for all } u \in \mathcal{D}^{1,2}(\mathbb{R}^N),$$

where the functional space $\mathcal{D}^{1,2}(\mathbb{R}^N)$ is the completion of $C_c^\infty(\mathbb{R}^N)$ with respect to the Dirichlet norm

$$\|u\|_{\mathcal{D}^{1,2}(\mathbb{R}^N)} := \left(\int_{\mathbb{R}^N} |\nabla u(x)|^2 \, dx \right)^{\frac{1}{2}}.$$

When we have a single pole a complete answer to the question of the positivity is provided by the classical Hardy inequality (1.1).

In the case of the operator with multiple singularities, V. Felli, E. M. Marchini and S. Terracini in [25] stated the following necessary and sufficient condition for the positivity of the form Q for at least a configuration of poles.

Theorem 1.5. *Let $c_1, \dots, c_n \in \mathbb{R}$. Then*

$$c_i < \left(\frac{N-2}{2} \right)^2, \quad \textit{for every } i = 1, \dots, n, \quad \textit{and} \quad \sum_{i=1}^{n} c_i < \left(\frac{N-2}{2} \right)^2,$$

is a necessary and sufficient condition for the existence of a configuration of poles $a_1, \dots, a_n$ such that the quadratic form associated to the operator $\mathcal{L} = -\Delta - \sum_{i=1}^n \frac{c_i}{|x-a_i|^2}$ is positive definite.

1.4 Multipolar Hardy inequalities

Due to the connection between Schrödinger operators with singular potentials and Hardy inequalities, it is of great interest for us the case of inequalities involving multiple singularities. These are known in literature as *multipolar Hardy inequalities.*

The first example of inequality of this type is the following

$$c\int_{\mathbb{R}^N}\sum_{i=1}^{n}\frac{\varphi^2}{|x-a_i|^2}\,dx \le \int_{\mathbb{R}^N}|\nabla\varphi|^2\,dx, \quad c>0, \quad \varphi\in H^1(\mathbb{R}^N), \quad (1.7)$$

which is the natural generalization of the classical Hardy inequality (1.1) to the case of n poles, with $n \ge 2$.

Unlike the unipolar case, finding the optimal value c_o of c in the inequality (1.7) where $n \ge 2$, is still an open problem (cf. [30, Section 9.5]). However, the best constant in (1.7) satisfies

$$\frac{(N-2)^2}{4n} \le c_o \le \left(\frac{N-2}{2}\right)^2. \quad (1.8)$$

The lower bound for c_o is given by the following Proposition. As an exercise we adapt the proof based on the vector field method to the multipolar case.

Proposition 1.6. *The following inequality holds*

$$\frac{(N-2)^2}{4n}\int_{\mathbb{R}^N}\sum_{i=1}^{n}\frac{\varphi^2}{|x-a_i|^2}\,dx \le \int_{\mathbb{R}^N}|\nabla\varphi|^2\,dx, \quad (1.9)$$

for all $\varphi\in H^1(\mathbb{R}^N)$.

Proof. By density, we consider functions $\varphi\in C_c^\infty(\mathbb{R}^N)$. Let us define the vector field $F(x)=c\sum_{i=1}^n\frac{x-a_i}{|x-a_i|^2}$, we get

$$\begin{aligned}
c(N-2)\int_{\mathbb{R}^N}\sum_{i=1}^{n}\frac{\varphi^2}{|x-a_i|^2}dx &= \\
&= \int_{\mathbb{R}^N}\varphi^2\operatorname{div}F\,dx \\
&\le -2\int_{\mathbb{R}^N}\varphi F\cdot\nabla\varphi\,dx \\
&= -2c\sum_{i=1}^{n}\int_{\mathbb{R}^N}\varphi\nabla\varphi\cdot\frac{(x-a_i)}{|x-a_i|^2}\,dx \\
&\le 2c\sum_{i=1}^{n}\left(\int_{\mathbb{R}^N}|\nabla\varphi|^2\,dx\right)^{\frac{1}{2}}\left(\int_{\mathbb{R}^N}\frac{\varphi^2}{|x-a_i|^2}\,dx\right)^{\frac{1}{2}} \\
&\le n\int_{\mathbb{R}^N}|\nabla\varphi|^2\,dx + c^2\int_{\mathbb{R}^N}\sum_{i=1}^{n}\frac{\varphi^2}{|x-a_i|^2}\,dx.
\end{aligned}$$

Then we get

$$\left[c(N-2)-c^2\right]\int_{\mathbb{R}^N}\sum_{i=1}^{n}\frac{\varphi^2}{|x-a_i|^2}dx \le n\int_{\mathbb{R}^N}|\nabla\varphi|^2\,dx$$

Taking the maximum of the function $c(N-2)-c^2$ we get the inequality. □

In the following Proposition we show that the constant $\left(\frac{N-2}{2}\right)^2$ cannot be optimal in (1.7) if $n \ge 2$. This result is known and we prove it as an exercise.

Proposition 1.7. *The constant* $\left(\frac{N-2}{2}\right)^2$ *is optimal in (1.7) if and only if* $n = 1$.

Proof. If $n = 1$ the classical Hardy inequality ensures the optimality $\left(\frac{N-2}{2}\right)^2$ in (1.7).

Now assume that the best constant in (1.7) is $c_o = \left(\frac{N-2}{2}\right)^2$, i.e. the same of the classical Hardy inequality

$$c_o\int_{\mathbb{R}^N}\frac{\varphi^2}{|x-a_i|^2}\,dx \le \int_{\mathbb{R}^N}|\nabla\varphi|^2\,dx,$$

which holds for every $i \in \{1,\ldots,n\}$. If it were $n > 1$, from the optimality of c_o in (1.7), we get

$$c_o\sum_{i=1}^{n}\int_{\mathbb{R}^N}\frac{\varphi^2}{|x-a_i|^2}\,dx \le \int_{\mathbb{R}^N}|\nabla\varphi|^2\,dx \tag{1.10}$$

Then, multiplying by n, we get

$$n\,c_o\sum_{i=1}^{n}\int_{\mathbb{R}^N}\frac{\varphi^2}{|x-a_i|^2}\,dx \le n\int_{\mathbb{R}^N}|\nabla\varphi|^2\,dx. \tag{1.11}$$

From the optimality of c_o in (1.10), being $n\,c_o > c_o$, we obtain

$$n\,c_o\int_{\mathbb{R}^N}\frac{\varphi^2}{|x-a_i|^2}\,dx > \int_{\mathbb{R}^N}|\nabla\varphi|^2\,dx,$$

for all $i = 1,\ldots,n$, and then

$$n\,c_o\int_{\mathbb{R}^N}\sum_{i=1}^{n}\frac{\varphi^2}{|x-a_i|^2}\,dx > n\int_{\mathbb{R}^N}|\nabla\varphi|^2\,dx. \tag{1.12}$$

From (1.11) and (1.12) we get

$$n \int_{\mathbb{R}^N} |\nabla\varphi|^2 \, dx < n \, c_o \int_{\mathbb{R}^N} \sum_{i=1}^{n} \frac{\varphi^2}{|x - a_i|^2} \, dx \le n \int_{\mathbb{R}^N} |\nabla\varphi|^2 \, dx.$$

This leads to the contradiction $n < n$. Therefore $n = 1$. □

The following multipolar Hardy inequality (1.13), which is the main result we report in this Section, has been stated by R. Bosi, J. Dolbeault and M. J. Esteban in [10]. The proof of the inequality is based on the so-called *IMS truncation method*, which consists in localizing the wave functions around the singularities by using a partition of unity in $\mathbb{R}^N$. We will describe the method in Chapter 3, where we will adapt it to the weighted case.

The result emphasizes that constant $c_o = c_o(N) := \left(\frac{N-2}{2}\right)^2$ can be optimal to the price of adding a lower order term in L^2-norm on the right-hand side in the inequality.

Theorem 1.8. *Assume $N \ge 3$, $n \ge 2$ and let $r_0 = \min_{i \ne j} |a_i - a_j|/2$, $i, j = 1, \dots, n$. Then there exists a constant $k \in [0, \pi^2)$ such that*

$$c \sum_{i=1}^{n} \int_{\mathbb{R}^N} \frac{\varphi^2}{|x - a_i|^2} \, dx \le \int_{\mathbb{R}^N} |\nabla\varphi|^2 \, dx + \left[\frac{k + (n+1)c_o}{r_0^2} \right] \int_{\mathbb{R}^N} \varphi^2 \, dx \tag{1.13}$$

for all $\varphi \in H^1(\mathbb{R}^N)$, where $c \in (0, c_o]$ with $c_o = c_o(N) := \left(\frac{N-2}{2}\right)^2$.

Another result in the context of multipolar inequalities with optimal constant is the following multipolar Hardy-type inequality recently stated by C. Cazacu and E. Zuazua in [18].

Theorem 1.9. *Let $N \ge 3$, $a_1, \dots, a_n$, $n \ge 2$, such that $a_i \ne a_j$. The following inequality holds*

$$\frac{(N-2)^2}{n^2} \sum_{\substack{i,j=1 \\ i<j}}^{n} \int_{\mathbb{R}^N} \frac{|a_i - a_j|^2}{|x - a_i|^2 |x - a_j|^2} \varphi^2 \, dx \le \int_{\mathbb{R}^N} |\nabla\varphi|^2 \, dx,$$

for all $\varphi \in H^1(\mathbb{R}^N)$, Moreover, the constant $\frac{(N-2)^2}{n^2}$ is optimal.

1.5 Kolmogorov operators perturbed by singular potentials

In this Section we deal with the class of the Kolmogorov type operators

$$Lu = \Delta u + \frac{\nabla \mu}{\mu} \cdot \nabla u, \tag{1.14}$$

defined on smooth functions. Here μ is a probability density on $\mathbb{R}^N$, $N \geq 3$, satisfying $\mu \in C^{1,\alpha}_{loc}(\mathbb{R}^N)$ for some $\alpha \in (0,1)$, $\mu(x) > 0$ for all $x \in \mathbb{R}^N$.

The operator L arises from the Dirichlet form

$$a_\mu(u,v) = \int_{\mathbb{R}^N} \nabla u \cdot \nabla v \, d\mu \qquad u, v \in C_c^\infty(\mathbb{R}^N). \tag{1.15}$$

Indeed by integrating by parts in (1.15) we get

$$a_\mu(u,v) = -\int_{\mathbb{R}^N} Luv \, d\mu, \qquad u, v \in C_c^\infty(\mathbb{R}^N). \tag{1.16}$$

It is known that the operator L with domain

$$D_{max}(L) = \{u \in C_b(\mathbb{R}^N) \cap W^{2,p}_{loc}(\mathbb{R}^N) \text{ for all } 1 < p < \infty, Lu \in C_b(\mathbb{R}^N)\}$$

is the weak generator of a not necessarily C_0-semigroup $\{T(t)\}_{t\geq 0}$ in $C_b(\mathbb{R}^N)$ (see Appendix B).

Moreover, if we set $d\mu := \mu(x)dx$, for any $u \in C_c^\infty(\mathbb{R}^N)$ we have

$$\begin{aligned}\int_{\mathbb{R}^N} Lu \, d\mu &= \int_{\mathbb{R}^N} \Delta u \, d\mu + \int_{\mathbb{R}^N} \frac{\nabla \mu}{\mu} \nabla u \, d\mu \\ &= \int_{\mathbb{R}^N} \Delta u \, \mu \, dx + \int_{\mathbb{R}^N} \nabla u \nabla \mu \, dx \\ &= -\int_{\mathbb{R}^N} \nabla u \nabla \mu \, dx + \int_{\mathbb{R}^N} \nabla u \nabla \mu \, dx = 0.\end{aligned}$$

Then $d\mu$ is the invariant measure for the semigroup $\{T(t)\}_{t\geq 0}$ in $C_b(\mathbb{R}^N)$. So we can extend it to a positivity preserving and analytic C_0-semigroup on $L^2_\mu := L^2(\mathbb{R}^N, d\mu)$, whose generator is still denoted by L.

Furthermore we denote by H^1_μ the set of all the functions $f \in L^2_\mu$ having distributional derivative ∇f in $(L^2_\mu)^N$. In Proposition B.26 in

Appendix B we list some general results about the semigroup $\{T(t)\}_{t\geq 0}$ and weighted space H^1_μ. In particular, it holds that $C_c^\infty(\mathbb{R}^N)$ is densely embedded in H^1_μ.

In the following we consider the Kolmogorov operator L defined in (1.14) perturbed by a singular potential $0 \leq V \in L^1_{loc}(\mathbb{R}^N)$.

As observed in [12], the operator $L+V$ in L^2_μ is equivalent to the Schrödinger operator $H = \Delta + (U_\mu + V)$ in $L^2(\mathbb{R}^N)$, where

$$U_\mu := \frac{1}{4}\left|\frac{\nabla\mu}{\mu}\right|^2 - \frac{1}{2}\frac{\Delta\mu}{\mu}.$$

Indeed, taking the transformation $T\varphi = \frac{1}{\sqrt{\mu}}\varphi$ one has

$$\begin{aligned}
THT^{-1}\varphi =& TH(\sqrt{\mu}\varphi) \\
=& T(\sqrt{\mu}\Delta\varphi + 2(\nabla\sqrt{\mu})\cdot(\nabla\varphi) + (\Delta\sqrt{\mu})\varphi) + U\mu\sqrt{\mu}\varphi + V\sqrt{\mu}\varphi \\
=& \frac{1}{\sqrt{\mu}}\left(\sqrt{\mu}\Delta\varphi + \frac{\nabla\varphi}{\sqrt{\mu}}\cdot\nabla\varphi + \frac{1}{2}\frac{\Delta\mu}{\mu}\sqrt{\mu}\varphi - \frac{1}{4}\frac{|\nabla\mu|^2}{\mu\sqrt{\mu}}\varphi \right.\\
&\left. + \frac{1}{4}\left|\frac{\nabla\mu}{\mu}\right|^2\sqrt{\mu}\varphi - \frac{1}{2}\frac{\Delta\mu}{\mu}\sqrt{\mu}\varphi + V\sqrt{\mu}\varphi\right) \\
=& \Delta\varphi + \frac{\nabla\mu}{\mu}\cdot\nabla\varphi + V\varphi.
\end{aligned}$$

Then $L + V = THT^{-1}$.

As seen in Section 1.2, if $L = \Delta$ Cabré-Martel's Theorem provides conditions for the existence of positive weak solutions to the corresponding initial value problem. A natural question is if similar conditions also hold for the problem associated with operator $L + V$

$$(P) \quad \begin{cases} \partial_t u(x,t) = Lu(x,t) + V(x)u(x,t), & t > 0,\, x\in\mathbb{R}^N, \\ u(\cdot,0) = u_0 \geq 0 \in L^2_\mu. \end{cases}$$

The problem has been dealt by G. R. Goldstein, J. A. Goldstein and A. Rhandi in [33].

We say that u is a weak solution to (P) if, for each $T, R > 0$, we have

$$u \in C([0,T]\,, L^2_\mu), \quad Vu \in L^1(B_R\times(0,T)\,, d\mu dt)$$

and

$$\int_0^T \int_{\mathbb{R}^N} u(-\partial_t\phi - L\phi)\, d\mu dt - \int_{\mathbb{R}^N} u_0\phi(\cdot,0)\, d\mu = \int_0^T \int_{\mathbb{R}^N} Vu\phi\, d\mu dt$$

for all $\phi \in W_2^{2,1}(\mathbb{R}^N \times [0,T])$ having compact support with $\phi(\cdot,T) = 0$.

For any $\Omega \subset \mathbb{R}^N$, $W_2^{2,1}(\Omega \times (0,T))$ is the parabolic Sobolev space of the functions $u \in L^2(\Omega \times (0,T))$ having weak space derivatives $D_x^\alpha u \in L^2(\Omega \times (0,T))$ for $|\alpha| \le 2$ and weak time derivative $\partial_t u \in L^2(\Omega \times (0,T))$ equipped with the norm

$$\|u\|_{W_2^{2,1}(\Omega\times(0,T))} := \Bigg(\|u\|^2_{L^2(\Omega\times(0,T))} + \|\partial_t u\|^2_{L^2(\Omega\times(0,T))} + \sum_{1\le|\alpha|\le 2} \|D^\alpha u\|^2_{L^2(\Omega\times(0,T))} \Bigg)^{\frac{1}{2}}.$$

We define the *bottom of the spectrum* of $-(L+V)$ to be

$$\lambda_1(L+V) := \inf_{\varphi\in H^1_\mu\setminus\{0\}} \left(\frac{\int_{\mathbb{R}^N} |\nabla\varphi|^2\, d\mu - \int_{\mathbb{R}^N} V\varphi^2\, d\mu}{\int_{\mathbb{R}^N} \varphi^2\, d\mu} \right).$$

Note that the case of $\lambda_1(L+V) = -\infty$ is allowed.

In order to investigate on the existence of positive weak solution to (P), the authors in [33] considered the approximate problem

$$(P_n) \quad \begin{cases} \partial_t u_n(x,t) = Lu_n(x,t) + V_n(x)u_n(x,t), & t > 0,\, x \in \mathbb{R}^N, \\ u_n(\cdot,0) = u_0 \in L^2_{\mu_+}, \end{cases}$$

where $V_n = \min(V,n)$, and stated the following Lemma (see [33, Appendix]).

Lemma 1.10. *If u is a positive weak solution of (P) and $0 \le u_0 \in L^2_\mu$, $u_0 \ne 0$, then*

$$0 < u_n(x,t) \le u(x,t)$$

for all $t > 0$ and a.e. $x \in \mathbb{R}^N$, where u_n is the positive solution of (P_n).

Then, following the approach used in [11], they stated the next Theorem.

Theorem 1.11. *Assume that* $0 < \mu \in C^{1,\alpha}_{loc}(\mathbb{R}^N)$ *is a probability density on* $\mathbb{R}^N$ *and* $0 \le V \in L^1_{loc}(\mathbb{R}^N)$. *Then the following hold:*

(i) If $\lambda_1(L+V) > -\infty$, *then there exists a positive weak solution* $u \in C([0,\infty), L^2_\mu)$ *of* (P) *satisfying*

$$\|u(t)\|_{L^2_\mu} \le M e^{\omega t} \|u_0\|_{L^2_\mu}, \quad t \ge 0 \tag{1.17}$$

for some constants $M \ge 1$ *and* $\omega \in \mathbb{R}$.

(ii) If $\lambda_1(L+V) = -\infty$, *then for any* $0 \le u_0 \in L^2_\mu \setminus \{0\}$, *there is no positive weak solution of* (P) *satisfying (1.17).*

Proof. (i) Let us consider the approximate problem (P_n). Since L generates a positivity preserving analytic semigroup on L^2_μ and V_n is bounded and nonnegative, it follows that $L + V_n$ generates a positivity preserving analytic semigroup $S_n(\cdot)$ on L^2_μ. Assume u_0 is not the zero function. Hence (P_n) admits a unique positive classical solution $u_n = S_n(\cdot)u_0$, i.e. $u_n \in C^1((0,\infty), D(L))$ and satisfies (P_n). Moreover,

$$0 < u_n(x,t) \le u_{n+1}(x,t), \quad n = 1, 2, 3, \dots$$

holds on $\mathbb{R}^N \times (0,\infty)$. Multiplying (P_n) by u_n and integrating we obtain, by using (1.16),

$$\begin{aligned}
\frac{1}{2}\int_{\mathbb{R}^N} \partial_t (u_n)^2 \, d\mu &= \int_{\mathbb{R}^N} L u_n . u_n \, d\mu + \int_{\mathbb{R}^N} V_n u_n^2 \, d\mu \\
&= -\int_{\mathbb{R}^N} |\nabla u_n|^2 \, d\mu + \int_{\mathbb{R}^N} V_n u_n^2 \, d\mu \\
&\le -\int_{\mathbb{R}^N} |\nabla u_n|^2 \, d\mu + \int_{\mathbb{R}^N} V u_n^2 \, d\mu.
\end{aligned}$$

Hence,

$$\frac{1}{2}\int_{\mathbb{R}^N} \partial_t (u_n)^2 \, d\mu \le -\lambda_1(L+V) \int_{\mathbb{R}^N} u_n^2 \, d\mu.$$

Thus,

$$\|u_n(t)\|_{L^2_\mu} \le e^{-\lambda_1(L+V)t} \|u_0\|_{L^2_\mu}, \quad t \ge 0. \tag{1.18}$$

This implies that

$$\|S_n(t)\| \leq e^{-\lambda_1(L+V)t}, \quad t \geq 0.$$

Therefore there is a C_0-semigroup $S(\cdot)$ on L^2_μ satisfying $\lim_{n\to\infty} S_n(t)f = S(t)f$ for all $f \in L^2_\mu$, $t \geq 0$ (cf. [3, Proposition 3.6]). Set $u(t) := S(t)u_0$, $t \geq 0$. It follows from the Trotter-Neveu-Kato Theorem (see Appendix B) that $u_n(t)$ converges to $u(t)$ in L^2_μ uniformly for $t \in [0,T]$. Since $u_n \in C([0,\infty), L^2_\mu)$ is a weak solution of (P_n), it follows that $u \in C([0,\infty), L^2_\mu)$ is a weak solution of (P). The estimate (1.17) follows from (1.18) and it holds with $M = 1$.

(ii) Suppose that there is a positive weak solution u of (P) with initial data $u_0 \in L^2_{\mu_+}$ satisfying (1.17). Let u_n be the unique positive solution of (P_n). Then, by Proposition 1.10,

$$0 < u_n(x,t) \leq u(x,t)$$

for all $t > 0$ and a.e. x. By the monotone convergence theorem, there is a positive weak solution $\widetilde{u}(t) = \lim_{n\to\infty} u_n(t)$ of (P), called the minimal solution. Let $\varphi \in C_c^\infty(\mathbb{R}^N)$ with $\int_{\mathbb{R}^N} \varphi^2\, d\mu = 1$. Let us consider $u_0 \in L^2_{\mu_+}$, $u_0 \neq 0$. Since V_n is nonnegative and bounded, it follows that

$$u_n(t) \geq T(t)u_0, \quad t \geq 0,$$

where $T(\cdot)$ is the semigroup generated by L in L^2_μ. It is known that for $f \in C_b(\mathbb{R}^N)$,

$$T(t)f(x) = \int_{\mathbb{R}^N} p(t,x,y) f(y)\, dy, \quad t > 0,\, x \in \mathbb{R}^N,$$

with $0 < p \in C((0,\infty) \times \mathbb{R}^N \times \mathbb{R}^N)$ (cf. [45, Proposition 2.1] and Theorem B.18 in Appendix B). Hence, for a fixed $r > 0$ and $t > 0$,

$$c_r(t) := \min_{(x,y)\in \operatorname{supp}\varphi \times B_r} p(t,x,y) > 0.$$

Thus, for a.e. $x \in \operatorname{supp}\varphi$,

$$\begin{aligned} T(t)u_0(x) &= \lim_{m\to\infty} T(t)u_{0,m}(x) \\ &= \lim_{m\to\infty} \int_{\mathbb{R}^N} p(t,x,y) u_{0,m}(y)\, dy \\ &\geq c_r(t) \lim_{m\to\infty} \int_{B_r} u_{0,m}(y)\, dy \\ &= c_r(t) \int_{B_r} u_0(y)\, dy =: c_r(t;u_0), \end{aligned}$$

where $u_{0,m} \in C_b(\mathbb{R}^N)$ is such that $\lim_{m\to\infty} \|u_0 - u_{0,m}\|_{L^2_\mu} = 0$. Therefore,

$$u_n(t) \geq T(t)u_0 \geq c_r(t, u_0) > 0, \quad t > 0, \tag{1.19}$$

a.e. on $\operatorname{supp}\varphi$.

Multiplying (P_n) by $\frac{\varphi^2}{u_n}$, integrating and taking into account (1.19) we obtain

$$\begin{aligned}
\int_{\mathbb{R}^N} V_n\varphi^2\,d\mu &= \partial_t\left(\int_{\mathbb{R}^N}(\log u_n)\varphi^2\,d\mu\right) + \int_{\mathbb{R}^N}\nabla u_n\cdot\nabla\left(\frac{\varphi^2}{u_n}\right)d\mu \\
&= \partial_t\left(\int_{\mathbb{R}^N}(\log u_n)\varphi^2\,d\mu\right) + 2\int_{\mathbb{R}^N}(\nabla u_n\cdot\nabla\varphi)\frac{\varphi}{u_n}\,d\mu \\
&\quad -\int_{\mathbb{R}^N}|\nabla u_n|^2\frac{\varphi^2}{u_n^2}\,d\mu \\
&\leq \partial_t\left(\int_{\mathbb{R}^N}(\log u_n)\varphi^2\,d\mu\right) \\
&\quad +2\left(\int_{\mathbb{R}^N}|\nabla u_n|^2\frac{\varphi^2}{u_n^2}\,d\mu\right)^{\frac{1}{2}}\left(\int_{\mathbb{R}^N}|\nabla\varphi|^2\,d\mu\right)^{\frac{1}{2}} \\
&\quad -\int_{\mathbb{R}^N}|\nabla u_n|^2\frac{\varphi^2}{u_n^2}\,d\mu \\
&\leq \partial_t\left(\int_{\mathbb{R}^N}(\log u_n)\varphi^2\,d\mu\right) + \int_{\mathbb{R}^N}|\nabla\varphi|^2\,d\mu.
\end{aligned}$$

Integrating with respect to $t \in (1, \infty)$ we get

$$(t-1)\int_{\mathbb{R}^N} V_n\varphi^2\,d\mu \leq \int_{\mathbb{R}^N}\log\left(\frac{u_n(t)}{u_n(1)}\right)\varphi^2\,d\mu + (t-1)\int_{\mathbb{R}^N}|\nabla\varphi|^2\,d\mu$$

for every $t > 1$. Letting $n \to \infty$, by the monotone convergence theorem and (1.19), we obtain

$$\begin{aligned}
\int_{\mathbb{R}^N} V\varphi^2\,d\mu - \int_{\mathbb{R}^N}|\nabla\varphi|^2\,d\mu \leq \frac{1}{t-1}\Bigg[&\int_{\mathbb{R}^N}\log(\widetilde{u}(t))\varphi^2\,d\mu \\
&-\int_{\mathbb{R}^N}\log(\widetilde{u}(1))\varphi^2\,d\mu\Bigg]
\end{aligned}$$

for every $t > 1$. Using Jensen's and Hölder's inequalities and (1.17) we deduce

$$
\begin{aligned}
\int_{\mathbb{R}^N} V\varphi^2\,d\mu - \int_{\mathbb{R}^N} |\nabla\varphi|^2\,d\mu
&\leq \frac{1}{t-1}\left\{\log\left[\left(\int_{\mathbb{R}^N} \widetilde{u}(t)^2\,d\mu\right)^{\frac{1}{2}}\left(\int_{\mathbb{R}^N} \varphi^4\,d\mu\right)^{\frac{1}{2}}\right]\right. \\
&\qquad\qquad \left. - \int_{\mathbb{R}^N} \log(\widetilde{u}(1))\varphi^2\,d\mu\right\} \\
&\leq \frac{1}{2(t-1)}\left\{\log\left(\int_{\mathbb{R}^N} \widetilde{u}(t)^2\,d\mu\right) + \log\left(\int_{\mathbb{R}^N} \varphi^4\,d\mu\right)\right. \\
&\qquad\qquad \left. -2\int_{\mathbb{R}^N} \log(\widetilde{u}(1))\varphi^2\,d\mu\right\} \\
&\leq \frac{1}{2(t-1)}\left\{\log\left(\int_{\mathbb{R}^N} u(t)^2\,d\mu\right) + \log\left(\int_{\mathbb{R}^N} \varphi^4\,d\mu\right)\right. \\
&\qquad\qquad \left. -2\int_{\mathbb{R}^N} \log(\widetilde{u}(1))\varphi^2\,d\mu\right\} \\
&\leq \frac{1}{2(t-1)}\left\{2\log(M\|u_0\|_{L^2_\mu}) + 2\omega t + 2\log\|\varphi\|_\infty\right. \\
&\qquad\qquad \left. -2\int_{\mathbb{R}^N} \log(\widetilde{u}(1))\varphi^2\,d\mu\right\}.
\end{aligned}
$$

Now, by letting $t \to +\infty$, we obtain

$$\int_{\mathbb{R}^N} V\varphi^2\,d\mu - \int_{\mathbb{R}^N} |\nabla\varphi|^2\,d\mu \leq \omega$$

for any $\varphi \in C_c^\infty(\mathbb{R}^N)$ with $\int_{\mathbb{R}^N}\varphi^2\,d\mu = 1$. Thus, by density of $C_c^\infty(\mathbb{R}^N)$ in H^1_μ we obtain that $\lambda_1(L+V) > -\infty$, and this ends the proof of the theorem. □

1.6 Ornstein-Uhlenbeck type operators perturbed by unipolar inverse square potentials

In this Section we focus on the Ornstein-Uhlenbeck type operator

$$Lu = \Delta u - Ax\cdot\nabla u \tag{1.20}$$

defined on smooth functions, with A a positive definite real Hermitian $N \times N$-matrix.

We can regard the operator defined in (1.20) as a Kolmogorov operator with $\frac{\nabla \mu}{\mu} = Ax$ in the drift term.

The Gaussian measure

$$d\mu = \mu(x)dx = Ce^{-\frac{1}{2}\langle Ax,x\rangle}\,dx, \quad x \in \mathbb{R}^N,$$

where

$$C = \left(\int_{\mathbb{R}^N} e^{-\frac{1}{2}\langle Ax,x\rangle}\,dx\right)^{-1},$$

is the invariant measure for the operator L. Indeed, if $\mu = Ce^{-\frac{1}{2}\langle Ax,x\rangle}$, then we get

$$\begin{aligned}
\int_{\mathbb{R}^N} Lu\,d\mu &= \int_{\mathbb{R}^N} (\Delta u - Ax\nabla u)\;d\mu \\
&= \int_{\mathbb{R}^N} \Delta u\mu\,dx - \int_{\mathbb{R}^N} Ax\nabla u\mu\,dx \\
&= -\int_{\mathbb{R}^N} \nabla u\nabla\mu\,dx - \int_{\mathbb{R}^N} Ax\nabla u\mu\,dx \\
&= C\int_{\mathbb{R}^N} \nabla uAxe^{-\frac{1}{2}\langle Ax,x\rangle}\,dx - C\int_{\mathbb{R}^N} Ax\nabla ue^{-\frac{1}{2}\langle Ax,x\rangle}\,dx = 0.
\end{aligned}$$

As in the more general case, L is the generator of a not necessarily strongly continuous semigroup, which can be extended to a positive and analytic strongly continuous semigroup $\{T(t)\}$ on $L^2_\mu = L^2(\mathbb{R}^N, d\mu)$. The generator of $\{T(t)\}$ has domain $H^2_\mu := \{u \in H^1_\mu : D_k u \in H^1_\mu \text{ for each } 1 \le k \le N\}$ (cf. [20, 43, 46]) and it still denoted by L.

Now we perturb the operator L by the inverse square potential

$$V(x) = \frac{c}{|x|^2},$$

where $x \in \mathbb{R}^N$, $c > 0$, and consider the initial value problem (P) corresponding to $L + V$.

In [33] the authors proved the following weighted Hardy inequality with optimal constant. As we will see, the result allows to state existence and nonexistence conditions via the Theorem 1.11.

Theorem 1.12. *Assume $N \geq 3$ and A a positive definite real Hermitian $N \times N$-matrix. One has*

$$c\int_{\mathbb{R}^N} \frac{\varphi^2}{|x|^2}\,d\mu \leq \int_{\mathbb{R}^N} |\nabla\varphi|^2\,d\mu + \|A\|\sqrt{c}\int_{\mathbb{R}^N} \varphi^2\,d\mu \qquad (1.21)$$

for all $\varphi \in H^1_\mu$, where $c \in (0, c_o]$ with $c_o = c_o(N) := \left(\frac{N-2}{2}\right)^2$ optimal constant.

Proof. Let us consider the function $F(x) := C\frac{cx}{|x|^2}e^{-\frac{1}{2}\langle Ax,x\rangle}$. By density, it suffices to prove (1.21) for all $\varphi \in C_c^\infty(\mathbb{R}^N)$. Then

$$\begin{aligned} C\int_{\mathbb{R}^N}\varphi^2\left(c\frac{(N-2)}{|x|^2} - c\frac{\langle Ax,x\rangle}{|x|^2}\right)e^{-\frac{1}{2}\langle Ax,x\rangle}\,dx \\ &= \int_{\mathbb{R}^N}\varphi^2 \mathrm{div} F\,dx \\ &= -2\int_{\mathbb{R}^N}\varphi F\cdot\nabla\varphi\,dx \\ &= -2c\int_{\mathbb{R}^N}\varphi\frac{x}{|x|^2}\cdot\nabla\varphi\,d\mu \\ &\leq 2c\left(\int_{\mathbb{R}^N}|\nabla\varphi|^2\,d\mu\right)^{\frac{1}{2}}\left(\int_{\mathbb{R}^N}\frac{\varphi^2}{|x|^2}\,d\mu\right)^{\frac{1}{2}} \\ &\leq \int_{\mathbb{R}^N}|\nabla\varphi|^2\,d\mu + c^2\int_{\mathbb{R}^N}\frac{\varphi^2}{|x|^2}\,d\mu. \end{aligned}$$

Hence,

$$\begin{aligned} \int_{\mathbb{R}^N}\frac{\varphi^2}{|x|^2}\left(c(N-2)-c^2\right)d\mu &\leq \int_{\mathbb{R}^N}|\nabla\varphi|^2\,d\mu + c\int_{\mathbb{R}^N}\varphi^2\frac{\langle Ax,x\rangle}{|x|^2}\,d\mu \\ &\leq \int_{\mathbb{R}^N}|\nabla\varphi|^2\,d\mu + c\|A\|\int_{\mathbb{R}^N}\varphi^2\,d\mu. \end{aligned}$$

Now, the maximum of the function $c \mapsto c(N-2) - c^2$ is attained at $c_{max} = \frac{N-2}{2}$ and this proves (1.21). For the optimality take $c > c_o$ and $\varphi(x) := |x|^\gamma$ with $\gamma > 1 - \frac{N}{2}$. Then $\varphi \in H^1_\mu$ and

$$\int_{\mathbb{R}^N}\left(|\nabla\varphi|^2 - \frac{c}{|x|^2}\varphi^2\right)d\mu = (\gamma^2 - c)\int_{\mathbb{R}^N}|x|^{2(\gamma-1)}\,d\mu.$$

Hence the bottom of the spectrum $\lambda_1(L+V)$ satisfies

$$\lambda_1(L+V) \leq (\gamma^2 - c)\frac{\int_{\mathbb{R}^N} |x|^{2(\gamma-1)}\, d\mu}{\int_{\mathbb{R}^N} |x|^{2\gamma}\, d\mu}.$$

Since, for every $x \in \mathbb{R}^N$ and some constants α_1, $\alpha_2 > 0$

$$\alpha_1 |x|^2 \leq |A^{\frac{1}{2}} x|^2 \leq \alpha_2 |x|^2,$$

it follows that

$$\begin{aligned}\alpha_2^{-(\frac{N}{2}+\beta)} \int_{\mathbb{R}^N} |x|^{2\beta} e^{-\frac{|x|^2}{2}}\, dx &\leq \int_{\mathbb{R}^N} |x|^{2\beta} e^{-\frac{|A^{\frac{1}{2}}x|^2}{2}}\, dx \\ &\leq \alpha_1^{-(\frac{N}{2}+\beta)} \int_{\mathbb{R}^N} |x|^{2\beta} e^{-\frac{|x|^2}{2}}\, dx.\end{aligned}$$

Hence, by using

$$\int_{\mathbb{R}^N} |x|^{2\beta} e^{-\frac{|x|^2}{2}}\, dx = \sigma_N 2^{\beta+\frac{N}{2}-1}\Gamma(\beta+\frac{N}{2}), \quad \beta + \frac{N}{2} > 0, \tag{1.22}$$

we see that

$$\frac{\int_{\mathbb{R}^N} |x|^{2(\gamma-1)}\, d\mu}{\int_{\mathbb{R}^N} |x|^{2\gamma}\, d\mu} \geq \frac{\alpha_2^{-(\frac{N}{2}+\gamma-1)}}{\alpha_1^{-(\frac{N}{2}+\gamma)}(2\gamma+N-2)}.$$

Now, since $c > c_o = \left(\frac{N-2}{2}\right)^2$, it follows that

$$\lambda_1(L+V) \leq \lim_{\gamma \to \left(1-\frac{N}{2}\right)^+} \frac{(\gamma^2 - c)\alpha_2^{-(\frac{N}{2}+\gamma-1)}}{\alpha_1^{-(\frac{N}{2}+\gamma)}(2\gamma+N-2)} = -\infty.$$

Thus, for any $M > 0$, there is $\varphi \in H^1_\mu$ such that

$$\int_{\mathbb{R}^N} |\nabla\varphi|^2\, d\mu - c\int_{\mathbb{R}^N} \frac{\varphi^2}{|x|^2}\, d\mu < -M\int_{\mathbb{R}^N} \varphi^2\, d\mu.$$

By taking $M := \sqrt{c}\|A\|$ we find $\varphi \in H^1_\mu$ such that

$$c\int_{\mathbb{R}^N} \frac{\varphi^2}{|x|^2}\, d\mu > \int_{\mathbb{R}^N} |\nabla\varphi|^2\, d\mu + \sqrt{c}\|A\| \int_{\mathbb{R}^N} \varphi^2\, d\mu.$$

This proves the optimality of c_o. $\square$

Putting together Theorem 1.11 and Theorem 1.12, it is possible to state existence and nonexistence conditions for the solution to the problem (P) in terms of the constant c in $V(x) = \frac{c}{|x|^2}$.

Theorem 1.13. *Assume that $N \geq 3$, A a positive definite real Hermitian $N \times N$-matrix and $0 \leq V(x) \leq \frac{c}{|x|^2}$, with $c > 0$, $x \in \mathbb{R}^N$. Let L the Ornstein-Uhlenbeck type operator (1.20). Then the following assertions hold:*

(i) If $c \leq c_o$, then there exists a weak solution $u \in C([0,\infty), L^2_\mu)$ of

$$\begin{cases} \partial_t u(x,t) = L + V(x)u(x,t), & x \in \mathbb{R}^N, t > 0, \\ u(\cdot,t) = u_0 \in L^2_\mu, \end{cases} \tag{1.23}$$

satisfying

$$\|u(t)\|_{L^2_\mu} \leq Me^{\omega t}\|u_0\|_{L^2_\mu}, \quad t \geq 0 \tag{1.24}$$

for some constants $M \geq 1$, $\omega \in \mathbb{R}$, and any $u_0 \in L^2_\mu$.

(ii) If $c > c_o$, then for any $0 \leq u_0 \in L^2_\mu$, $u_0 \neq 0$, there is no positive weak solution of (1.23) with $V(x) = \frac{c}{|x|^2}$ satisfying (1.24).

Similar conditions have been stated in [34, 37], where the authors replaced the Ornstein-Uhlenbeck operator (1.20) with the nonlinear p-Kolmogorov operator

$$K_p u := \operatorname{div}(|\nabla u|^{p-2}\nabla u) + |\nabla u|^{p-2}\nabla u \cdot \frac{\nabla\mu}{\mu},$$

for $1 < p < \infty$, with density function

$$\mu(x) = Ce^{-\frac{1}{p}\langle Ax,x\rangle^{\frac{p}{2}}},$$

where A is a positive definite real Hermitian $N \times N$-matrix and C is the normalization constant.

1.7 Kolmogorov operators with general drift term

In the context of Kolmogorov type operators, we report the unipolar weighted Hardy inequality with optimal constant stated by A. Canale, F. Gregorio, A. Rhandi and C. Tacelli in [12].

The inequality holds with respect to general measures $d\mu = \mu(x)dx$, including the ones which allow degeneracy at one point.

In order to get the result, in the paper they considered weighted functions μ satisfying the following hypotheses:

i) $\mu \geq 0$, $\mu^{\frac{1}{2}} \in H^1_{loc}(\mathbb{R}^N)$, $\Delta\mu \in L^1_{loc}(\mathbb{R}^N)$;

ii) the constant

$$c_{o,\mu} := \liminf_{x \to 0} (c_o(N) - |x|^2 U_\mu)$$

is finite, where $c_o(N) = \left(\frac{N-2}{2}\right)^2$ and $U_\mu := \frac{1}{4}\left|\frac{\nabla\mu}{\mu}\right|^2 - \frac{1}{2}\frac{\Delta\mu}{\mu}$;

iii) for every $R > 0$ the function

$$U := U_\mu - \frac{1}{|x|^2} \limsup_{x \to 0} |x|^2 U_\mu$$

is bounded from above in $\mathbb{R}^N \setminus B_R$;

iv) there exists a $R_0 > 0$ such that

$$|x|^2 U(x) \leq \frac{1}{4} \frac{1}{|\log|x||^2}, \quad \forall\, x \in B_{R_0};$$

v) there exists $\sup_{\delta \in \mathbb{R}} \left\{ \frac{1}{|x|^\delta} \in L^1_{loc}(\mathbb{R}^N, d\mu) \right\} =: N_0$.

Under these conditions the authors stated the following Theorem.

Theorem 1.14. *Assume hypotheses* $i) - iv)$*. Then for any* $\varphi \in H^1_\mu$ *the following inequality holds*

$$c_{0,\mu} \int_{\mathbb{R}^N} \frac{\varphi^2}{|x|^2} d\mu \leq \int_{\mathbb{R}^N} |\nabla\varphi|^2 d\mu + C_\mu \int_{\mathbb{R}^N} \varphi^2 d\mu.$$

The proof makes use of improved Hardy inequalities (see [1, 49]) to get the result.

Under assumption v) the authors proved the optimality in the case $c_{0,\mu} = c_0(N_0) = \left(\frac{N_0-2}{2}\right)^2$. The technique has been described in Theorem 1.2, in Section 1.1, in the case of Lebesgue measure.

The above inequality is related to the initial value problem corresponding to the Kolmogorov type operators

$$Lu = \Delta u + \frac{\nabla \mu}{\mu} \cdot \nabla u. \tag{1.25}$$

μ is a probability density on $\mathbb{R}^N$ satisfying $0 < \mu \in C^{1,\alpha}_{loc}(\mathbb{R}^N \setminus \{0\})$ for some $\alpha \in (0,1)$, perturbed by the potential $V(x) = \frac{c}{|x|^2}$, $c > 0$. Indeed in [12] the authors showed that it is possible to get existence and nonexistence conditions for positive solutions to the problem

$$(P) \quad \begin{cases} \partial_t u(x,t) = Lu(x,t) + V(x)u(x,t), & t > 0,\, x \in \mathbb{R}^N, \\ u(\cdot, 0) = u_0 \geq 0 \in L^2_\mu, \end{cases}$$

under the further hypothesis

vi) $\mu \in H^1_{loc}(\mathbb{R}^N)$, $\frac{\nabla \mu}{\mu} \in L^r_{loc}(\mathbb{R}^N)$ for some $r > N$, and $\inf_{x \in K} \mu(x) > 0$ for any compact set $K \subset \mathbb{R}^N$.

Theorem 1.11 in Section 1.5 is based on Cabré-Martel's idea and it was proved in [33, Theorem 2.1] for measures μ belonging to $C^{1,\alpha}_{loc}(\mathbb{R}^N)$. The proof relies on certain properties of the operator L and of its corresponding semigroup $T(t)$ in L^2_μ. Furthermore, the strict positivity on compact sets of $T(t)u_0$, if $0 \leq u_0 \in L^2_\mu \setminus \{0\}$, is required.

Hence, in order to claim that Theorem 1.11 holds in this more general context, the authors in [12] had to ensure that these properties hold.

By [2, Corollary 3.7], the closure of $(L, C_c^\infty(\mathbb{R}^N))$ on L^2_μ generates a strongly continuous Markov semigroup $T(\cdot)$ on L^2_μ, which is also analytic.

Moreover, the authors stated the following result about the self-adjoint operator $(L, D(L))$, defined by the closure of $(L, C_c^\infty(\mathbb{R}^N))$ on L^2_μ.

Proposition 1.15. *Assume that μ satisfies hypothesis vi). Then the following assertions hold.*

1) $D(L) \subset H^1_\mu$.

2) For every $f \in D(L)$, $g \in H^1_\mu$ we have

$$\int Lf g \, d\mu = -\int \nabla f \cdot \nabla g \, d\mu.$$

3) $T(t)L^2_\mu \subset D(L)$ for all $t > 0$.

These properties allow to state the following Theorem.

Theorem 1.16. *Let $0 \leq V \in L^1_{loc}(\mathbb{R}^N)$. Assume that $0 < \mu \in C^{1,\alpha}_{loc}(\mathbb{R}^N \setminus \{0\})$ is a probability density satisfying hypothesis vi). Then the following assertions hold:*

1) If $\lambda_1(L+V) > -\infty$, then there exists a positive weak solution $u \in C([0,\infty), L^2_\mu)$ of (P) satisfying

$$\|u(t)\|_{L^2_\mu} \leq M e^{\omega t} \|u_0\|_{L^2_\mu}, \quad t \geq 0 \tag{1.26}$$

for some constants $M \geq 1$ and $\omega \in \mathbb{R}$.

2) If $\lambda_1(L+V) = -\infty$, then for any $0 \leq u_0 \in L^2_\mu \setminus \{0\}$, there is no positive weak solution of (P) satisfying (1.26).

Finally, using the weighted Hardy inequality (1.14) and Theorem 1.16, the authors stated the following existence and nonexistence conditions.

Theorem 1.17. *Let $0 \leq V(x) \leq \frac{c}{|x|^2}$, $c > 0$. Assume that the weight function μ satisfies hypotheses i)–vi). Then the following assertions hold:*

1) If $0 \leq c \leq c_0(N_0)$, then there exists a weak solution $u \in C([0,\infty), L^2_\mu)$ of (P) satisfying

$$\|u(t)\|_{L^2_\mu} \leq M e^{\omega t} \|u_0\|_{L^2_\mu}, \quad t \geq 0 \tag{1.27}$$

for some constants $M \geq 1$, $\omega \in \mathbb{R}$, and any $u_0 \in L^2_\mu$.

2) If $c > c_0(N_0)$, then for any $0 \leq u_0 \in L^2_\mu$, $u_0 \neq 0$, there is no positive weak solution of (P) with $V(x) = \frac{c}{|x|^2}$ satisfying (1.27).

Chapter 2

Weighted multipolar Hardy inequalities and Ornstein-Uhlenbeck type operators

This Chapter is dedicated to our results stated in [13]. In the paper we consider Ornstein-Uhlenbeck type operators

$$Lu = \Delta u - \sum_{i=1}^{n} A(x - a_i) \cdot \nabla u,$$

where A is a positive definite real Hermitian $N \times N$-matrix, $a_i \in \mathbb{R}^N$ for $i = 1, \dots, n$, perturbed by the multipolar singular potential

$$V(x) = \sum_{i=1}^{n} \frac{c}{|x - a_i|^2}, \quad c > 0.$$

The main results we state are the following multipolar weighted Hardy inequality

$$c \int_{\mathbb{R}^N} \sum_{i=1}^{n} \frac{\varphi^2}{|x - a_i|^2} \, d\mu \le \int_{\mathbb{R}^N} |\nabla \varphi|^2 \, d\mu + \left[\frac{k + (n+1)c}{r_0^2} + \frac{n}{2} \operatorname{Tr} A \right] \int_{\mathbb{R}^N} \varphi^2 \, d\mu$$

which holds for all $\varphi \in H^1_\mu$, where $r_0 = \min_{i\neq j} |a_i - a_j|/2$, $i, j = 1, \ldots, n$, $k \in [0, \pi^2)$ and $c \leq c_o = \left(\frac{N-2}{2}\right)^2$, and the optimality of the constant c_o. The measure $d\mu$ in the inequality is the invariant measure for L.

The estimate, which we present in Section 2.1, allows us to get necessary and sufficient conditions for the existence of the solution to the initial value problem associated to $L + V$, following the same approach used in Section 1.6, Chapter 1.

Our technique, unlike the vector field method used in the case $n = 1$ in [33], allow us to overcome the difficulties due to the mutual interaction among the poles and to achieve the constant c_o in the left-hand side in the inequality.

We prove the estimate using an idea which allows us to get it in a direct way starting from the result obtained in [10] in the case of the Lebesgue measure and exploiting a suitable bound which the function μ we consider satisfies.

The optimality of the constant c_o is less immediate to obtain. The crucial points to estimate the bottom of the spectrum are the choice of a suitable function φ which involves only one pole and the connection we state between the weight functions in the case of one pole and in the case of multiple poles.

In Section 2.2 we state a semigroup generation result via the bilinear form technique. Studying the form associated with the operator $-(L+V)$ we state the positivity of the solution.

2.1 Ornstein-Uhlenbeck type operators perturbed by a multipolar inverse square potential

Let us consider the following Ornstein-Uhlenbeck type operator

$$Lu = \Delta u - \sum_{i=1}^{n} A(x - a_i) \cdot \nabla u. \tag{2.1}$$

on smooth functions, with A a positive definite real Hermitian $N \times N$-matrix.

The Gaussian measure

$$d\mu = \mu(x)dx = C\, e^{-\frac{1}{2}\sum_{i=1}^{n}\langle A(x-a_i),x-a_i\rangle}\, dx, \tag{2.2}$$

with C is the normalization constant, is the invariant measure for L (the proof of this is analogous to that given in the unipolar case in Section 1.6, Chapter 1). Then the semigroup generated by L can be extended to a positive and analytic strongly continuous semigroup $\{T(t)\}_{t\geq 0}$ on L^2_μ.

As in the unipolar case, the generator of $\{T(t)\}_{t\geq 0}$, still denoted by L, has domain $H^2_\mu := \{u \in H^1_\mu : D_k u \in H^1_\mu \text{ for each } 1 \leq k \leq N\}$, where the space H^1_μ is defined as in Section 1.5 in Chapter 1. We recall that $C_c^\infty(\mathbb{R}^N)$ is densely embedded in H^1_μ (see Appendix B).

We consider the operator (2.1) perturbed by the multipolar inverse square potential

$$V(x) = \sum_{i=1}^{n} \frac{c}{|x-a_i|^2} = c\, V_n, \tag{2.3}$$

where $x \in \mathbb{R}^N$, $c > 0$, $a_i \in \mathbb{R}^N$, $i = 1, \ldots, n$.

The main result in this Section is the weighted multipolar Hardy inequality stated in [13]. Before stating the estimate, we consider the following Lemma, which will be useful in our proof.

Lemma 2.1. *The following estimate hold:*

$$\begin{aligned} -\sum_{j\neq i} |a_i - a_j|^2 + \frac{n+1}{2}|x-a_i|^2 &\leq \sum_{i=1}^{n} |x-a_i|^2 \\ &\leq (2n-1)|x-a_i|^2 + 2\sum_{j\neq i}|a_i-a_j|^2 \end{aligned} \tag{2.4}$$

for any $i, j \in \{1, \ldots, n\}$.

Proof. Starting from the inequalities

$$|x-a_j|^2 = |x-a_i+a_i-a_j|^2 \leq 2|x-a_i|^2 + 2|a_i-a_j|^2$$

$$|x-a_j|^2 \geq \frac{|x-a_i|^2}{2} - |a_i-a_j|^2,$$

as a consequence we obtain

$$\sum_{i=1}^{n}|x-a_i|^2 = |x-a_i|^2+\sum_{j\neq i}|x-a_j|^2$$
$$\leq |x-a_i|^2+2(n-1)|x-a_i|^2+2\sum_{i\neq j}^{n}|a_i-a_j|^2$$

and

$$\sum_{i=1}^{n}|x-a_i|^2 \geq |x-a_i|^2+\frac{n-1}{2}|x-a_i|^2-\sum_{i\neq j}^{n}|a_i-a_j|^2.$$

□

Now we are able to state our weighted Hardy inequality.

Theorem 2.2. *Assume $N\geq 3$, $n\geq 2$, A a positive definite real Hermitian $N\times N$-matrix and let $r_0=\min_{i\neq j}|a_i-a_j|/2$, $i,j=1,\dots,n$. Then there exists a constant $k\in[0,\pi^2)$ such that*

$$c\int_{\mathbb{R}^N}\sum_{i=1}^{n}\frac{\varphi^2}{|x-a_i|^2}\,d\mu \leq \int_{\mathbb{R}^N}|\nabla\varphi|^2\,d\mu + \left[\frac{k+(n+1)c}{r_0^2}+\frac{n}{2}\operatorname{Tr}A\right]\int_{\mathbb{R}^N}\varphi^2\,d\mu \tag{2.5}$$

for all $\varphi\in H^1_\mu$, where $c\in(0,c_o]$ with $c_o=c_o(N):=\left(\frac{N-2}{2}\right)^2$ optimal constant.

Proof.
Step 1 (Inequality)

By density we can consider functions $\varphi\in C_c^\infty(\mathbb{R}^N)$.
The starting point is the multipolar Hardy inequality stated by Bosi, Dolbeault and Esteban in [10, Theorem 1] presented in Chapter 2:

$$c\int_{\mathbb{R}^N}\sum_{i=1}^{n}\frac{\varphi^2}{|x-a_i|^2}\,dx \leq \int_{\mathbb{R}^N}|\nabla\varphi|^2\,dx+\left[\frac{k+(n+1)c}{r_0^2}\right]\int_{\mathbb{R}^N}\varphi^2\,dx \tag{2.6}$$

for all $\varphi\in H^1(\mathbb{R}^N)$, with $n\geq 2$, $k\in[0,\pi^2)$ and $c\in(0,c_o]$.

Applying (2.6) to the function $\varphi\sqrt{\mu}$, we have

$$c\int_{\mathbb{R}^N}\sum_{i=1}^{n}\frac{\varphi^2}{|x-a_i|^2}\,d\mu \le \int_{\mathbb{R}^N}|\nabla(\varphi\sqrt{\mu})|^2\,dx + \left[\frac{k+(n+1)c}{r_0^2}\right]\int_{\mathbb{R}^N}\varphi^2\,d\mu.$$

By means the easy calculation

$$\begin{aligned}\int_{\mathbb{R}^N}|\nabla(\varphi\sqrt{\mu})|^2\,dx &= \int_{\mathbb{R}^N}\left|(\nabla\varphi)\sqrt{\mu}+\varphi\frac{\nabla\mu}{2\sqrt{\mu}}\right|^2\,dx\\ &= \int_{\mathbb{R}^N}|\nabla\varphi|^2\,d\mu + \int_{\mathbb{R}^N}\left(\frac{1}{4}\left|\frac{\nabla\mu}{\mu}\right|^2-\frac{1}{2}\frac{\Delta\mu}{\mu}\right)\varphi^2\,d\mu,\end{aligned}$$

and observing that we can estimate the last integral above taking into account that

$$\begin{aligned}\frac{1}{4}\left|\frac{\nabla\mu}{\mu}\right|^2-\frac{1}{2}\frac{\Delta\mu}{\mu} &= \frac{1}{4}\left|\sum_{j=1}^{n}A(x-a_j)\right|^2\\ &\quad -\frac{1}{2}\left[-n\,\mathrm{Tr}\,A+\left|\sum_{j=1}^{n}A(x-a_j)\right|^2\right]\le\frac{n}{2}\,\mathrm{Tr}\,A\,,\end{aligned}\tag{2.7}$$

we get the result.

Step 2 (*Optimality*)

To state the optimality of the constant c_o we suppose that $c > c_o$.

Let us fix i and consider the function $\varphi = |x-a_i|^{\gamma}$, $\gamma\in(1-\frac{N}{2},0)$. The function φ belongs to H^1_μ and

$$\int_{\mathbb{R}^N}\left(|\nabla\varphi|^2-c\,\frac{\varphi^2}{|x-a_i|^2}\right)\,d\mu=(\gamma^2-c)\int_{\mathbb{R}^N}|x-a_i|^{2(\gamma-1)}\,d\mu.$$

Hence the bottom of the spectrum λ_1 of the operator $-(L+V)$ satisfies

$$\lambda_1\le(\gamma^2-c)\frac{\int_{\mathbb{R}^N}|x-a_i|^{2(\gamma-1)}\,d\mu}{\int_{\mathbb{R}^N}|x-a_i|^{2\gamma}\,d\mu}\tag{2.8}$$

since

$$\int_{\mathbb{R}^N}\left(|\nabla\varphi|^2-V\varphi^2\right)\,d\mu\le\int_{\mathbb{R}^N}\left(|\nabla\varphi|^2-c\,\frac{\varphi^2}{|x-a_i|^2}\right)\,d\mu.$$

We are able to state that for any $i \in \{1, \dots, n\}$ it holds

$$C_1\, e^{-\alpha_2(2n-1)\frac{|x-a_i|^2}{2}} \le e^{-\sum_{i=1}^n \frac{|A^{\frac12}(x-a_i)|^2}{2}} \le C_2\, e^{-\alpha_1 \frac{n+1}{2}\frac{|x-a_i|^2}{2}} \tag{2.9}$$

with $C_1 = e^{-\alpha_2 \sum_{i\neq j}|a_i-a_j|^2}$ and $C_2 = e^{\frac{\alpha_1}{2}\sum_{i\neq j}|a_i-a_j|^2}$ which is a consequence of the inequalities (2.4) and

$$\alpha_1 \sum_{i=1}^n |x-a_i|^2 \le \sum_{i=1}^n |A^{\frac12}(x-a_i)|^2 \le \alpha_2 \sum_{i=1}^n |x-a_i|^2, \qquad \alpha_1\,, \alpha_2 > 0.$$

For simplicity in the following we place $\tilde\alpha_1 = \alpha_1 \frac{n+1}{2}$ and $\tilde\alpha_2 = \alpha_2(2n-1)$.

The equivalence between the weight functions in the case of one pole and in the case of multiple poles allows us to calculate integrals in (2.8). Indeed, by a change of variables and by (2.9)

$$\begin{aligned}\int_{\mathbb{R}^N} &|x-a_i|^{2\beta} e^{-\sum_{i=1}^n \frac{|A^{\frac12}(x-a_i)|^2}{2}}\,dx\\ &\le C_2 \int_{\mathbb{R}^N} |x-a_i|^{2\beta} e^{-\tilde\alpha_1 \frac{|x-a_i|^2}{2}}\,dx\\ &= C_2\, 2^{\beta+\frac N2}\, \tilde\alpha_1^{-\beta-\frac N2} \int_{\mathbb{R}^N} |x-a_i|^{2\beta} e^{-\frac{|x-a_i|^2}{2}}\,dx.\end{aligned} \tag{2.10}$$

Taking in mind the definition of Gamma integral function

$$\int_{\mathbb{R}^N} |x|^{2\beta} e^{-\frac{|x|^2}{2}}\,dx = \sigma_N\, 2^{\beta+\frac N2 -1}\Gamma\left(\beta+\frac N2\right), \quad \beta + \frac N2 > 0,$$

we get from (2.10)

$$\begin{aligned}\int_{\mathbb{R}^N} &|x-a_i|^{2\beta} e^{-\sum_{i=1}^n \frac{|A^{\frac12}(x-a_i)|^2}{2}}\,dx \le\\ &\le C_2\, 2^{2\beta+N-1}\tilde\alpha_1^{-\beta-\frac N2}\sigma_N \Gamma\left(\beta+\frac N2\right).\end{aligned} \tag{2.11}$$

Reasoning as above we obtain the estimate

$$
\begin{aligned}
\int_{\mathbb{R}^N} |x-a_i|^{2\beta} e^{-\sum_{i=1}^n \frac{|A^{\frac{1}{2}}(x-a_i)|^2}{2}}\,dx \\
&\geq C_1 \int_{\mathbb{R}^N} |x-a_i|^{2\beta} e^{-\tilde{\alpha}_2 \frac{|x-a_i|^2}{2}}\,dx \\
&= C_1 \tilde{\alpha}_2^{-\beta-\frac{N}{2}} \int_{\mathbb{R}^N} |x-a_i|^{2\beta} e^{-\frac{|x-a_i|^2}{2}}\,dx \\
&= C_1\, 2^{\beta+\frac{N}{2}-1} \tilde{\alpha}_2^{-\beta-\frac{N}{2}} \sigma_N \Gamma\left(\beta+\frac{N}{2}\right).
\end{aligned}
\tag{2.12}
$$

Therefore, using (2.11) and (2.12), we get

$$
\begin{aligned}
\frac{\int_{\mathbb{R}^N} |x-a_i|^{2(\gamma-1)}\,d\mu}{\int_{\mathbb{R}^N} |x-a_i|^{2\gamma}\,d\mu} &\geq \frac{C_1\, 2^{\gamma+\frac{N}{2}-2} \tilde{\alpha}_2^{-\gamma-\frac{N}{2}+1} \sigma_N \Gamma(\gamma+\frac{N}{2}-1)}{C_2\, 2^{2\gamma+N-1} \tilde{\alpha}_1^{-\gamma-\frac{N}{2}} \sigma_N \Gamma(\gamma+\frac{N}{2})} \\
&= \frac{C_1\, 2^{\gamma+\frac{N}{2}-2} \tilde{\alpha}_2^{-\gamma-\frac{N}{2}+1}}{C_2\, 2^{2\gamma+N-1} \tilde{\alpha}_1^{-\gamma-\frac{N}{2}} (\gamma+\frac{N}{2}-1)}.
\end{aligned}
$$

Then

$$
\lambda_1 \leq \lim_{\gamma \to \left(1-\frac{N}{2}\right)^+} (\gamma^2 - c) \frac{C_1\, 2^{\gamma+\frac{N}{2}-2} \tilde{\alpha}_2^{-\gamma-\frac{N}{2}+1}}{C_2\, 2^{2\gamma+N-1} \tilde{\alpha}_1^{-\gamma-\frac{N}{2}} (\gamma+\frac{N}{2}-1)} = -\infty.
$$

Thus, for any $M > 0$, there is $\varphi \in H^1_\mu$ such that

$$
\int_{\mathbb{R}^N} |\nabla \varphi|^2\,d\mu - c \int_{\mathbb{R}^N} \frac{\varphi^2}{|x-a_i|^2}\,d\mu < -M \int_{\mathbb{R}^N} \varphi^2\,d\mu.
$$

By taking $M := \frac{k+(n+1)c}{r_0^2} + \frac{n}{2}\operatorname{Tr} A$ we find $\varphi \in H^1_\mu$ such that

$$
c \int_{\mathbb{R}^N} \frac{\varphi^2}{|x-a_i|^2}\,d\mu > \int_{\mathbb{R}^N} |\nabla \varphi|^2\,d\mu + \left[\frac{k+(n+1)c}{r_0^2} + \frac{n}{2}\operatorname{Tr} A\right] \int_{\mathbb{R}^N} \varphi^2\,d\mu
$$

which leads to a contradiction with respect the weighted Hardy inequality (2.5) because, of course,

$$
c \int_{\mathbb{R}^N} \frac{\varphi^2}{|x-a_i|^2}\,d\mu \leq c \int_{\mathbb{R}^N} \sum_{i=1}^n \frac{\varphi^2}{|x-a_i|^2}\,d\mu.
$$

This proves the optimality of c_o. □

We remark that in the above proof we get the weighted Hardy inequality in a direct way, but, as we show in [13], it is possible to prove the inequality also by using the IMS method. We will present the method in Chapter 3, in a more general context.

Moreover in [13] we observe that when $c \in (0, \frac{c_o}{n}]$ the constant on the right-hand side of (2.5) can be improved using a different proof based on the multipolar Hardy inequality in the case of Lebesgue measure. The inequality (2.13) we state below holds also in the case $n = 1$.

Theorem 2.3. *Assume $N \geq 3$ and $n \geq 1$. Then we get*

$$\frac{c_o}{n} \int_{\mathbb{R}^N} \sum_{i=1}^{n} \frac{\varphi^2}{|x - a_i|^2} \, d\mu \leq \int_{\mathbb{R}^N} |\nabla \varphi|^2 \, d\mu + \frac{n}{2} \operatorname{Tr} A \int_{\mathbb{R}^N} \varphi^2 \, d\mu \qquad (2.13)$$

for any $\varphi \in H^1_\mu$, where $c_o = c_o(N) := \left(\frac{N-2}{2}\right)^2$.

Proof. We start from the known inequality stated in Proposition 1.6 in Chapter 1

$$\frac{c_o}{n} \int_{\mathbb{R}^N} \sum_{i=1}^{n} \frac{\varphi^2}{|x - a_i|^2} \, dx \leq \int_{\mathbb{R}^N} |\nabla \varphi|^2 \, dx \qquad (2.14)$$

for all $\varphi \in H^1(\mathbb{R}^N)$, where $c_o = c_o(N) := \left(\frac{N-2}{2}\right)^2$, which we can get immediately by using the Hardy inequality with one pole.

Then we apply the inequality (2.14) to the function $\varphi\sqrt{\mu}$ and reason as in the proof of Theorem 2.2. □

The potential $V(x) = \sum_{i=1}^{n} \frac{c}{|x-a_i|^2}$ and the Gaussian density $\mu(x)$ satisfy the hypotheses of the Theorem 1.11 in Chapter 1. We can therefore state the following existence and nonexistence result of positive solutions for the evolution problem associated with the perturbed operator $L + V$ as a consequence of the weighted Hardy inequality (2.5) and Theorem 1.11 in Chapter 1.

Theorem 2.4. *Assume $N \geq 3$, A a positive definite real Hermitian $N \times N$-matrix and $0 \leq V(x) \leq \sum_{i=1}^{n} \frac{c}{|x-a_i|^2}$, with $c > 0$, $x, a_i \in \mathbb{R}^N$, $i \in \{1, \ldots, n\}$. Let L the Ornstein-Uhlenbeck type operator (2.1). Then the following assertions hold:*

i) If $c \leq c_o$ there exists a positive weak solution $u \in C\left([0,\infty), L^2_\mu\right)$ of

$$\begin{cases} \partial_t u(x,t) = L + V(x)u(x,t), & x \in \mathbb{R}^N, t > 0, \\ u(\cdot,t) = u_0 \geq 0 \in L^2_\mu, \end{cases} \tag{2.15}$$

satisfying

$$\|u(t)\|_{L^2_\mu} \leq M e^{\omega t} \|u_0\|_{L^2_\mu}, \qquad t \geq 0 \tag{2.16}$$

for some constants $M \geq 1$, $\omega \in \mathbb{R}$, and any $u_0 \in L^2_\mu$.

ii) If $c > c_o$ there exists no positive weak solution of (2.15) with $V(x) = \sum_{i=1}^n \frac{c}{|x-a_i|^2}$ satisfying (2.16) for any $0 \leq u_0 \in L^2_\mu$, $u_0 \neq 0$.

2.2 Existence of solutions via the bilinear form technique

Following a different approach based on the bilinear form associated to the operator $-(L+V)$, we are able to state the generation of an analytic C_0-semigroup. This will allow us to get an existence of solution result in a more classical way.

We consider the case of the Ornstein-Uhlenbeck type operator (2.1). However, it is clear that the same reasoning also holds for operators involving only one pole.

Let us define the bilinear form

$$a_c(u,v) := \int_{\mathbb{R}^N} \nabla u \cdot \nabla v \, d\mu - c \sum_{i=1}^n \int_{\mathbb{R}^N} \frac{uv}{|x-a_i|^2} \, d\mu \tag{2.17}$$

for $u, v \in D(a_c) = H^1_\mu$, $N \geq 3$ and $c > 0$.

Arguing as in [3, Propositions 2.2 and 2.3], we are able to get the next result.

Proposition 2.5. *The following statements hold:*

i) a_c is closed if $c < c_o$;

ii) a_{c_o} is closable;

iii) a_c is quasi-accretive for all $c \in (0, c_o]$.

Proof.

i) Let $q = c\left(\frac{N-2}{2}\right)^{-2} < 1$. By the weighted Hardy inequality (2.5) we get

$$\begin{aligned} a_c(u,u) \geq & q\left[\int_{\mathbb{R}^N} |\nabla u|^2 d\mu - \left(\frac{N-2}{2}\right)^2 \sum_{i=1}^n \int_{\mathbb{R}^N} \frac{u^2\,d\mu}{|x-a_i|^2}\right] \\ & + (1-q)\int_{\mathbb{R}^N} |\nabla u|^2\,d\mu \\ & \geq -qK\int_{\mathbb{R}^N} u^2\,d\mu + (1-q)\int_{\mathbb{R}^N} |\nabla u|^2\,d\mu, \end{aligned} \tag{2.18}$$

where $K = \frac{k+(n+1)c_o}{r_0^2} + \frac{n}{2}\operatorname{Tr} A$, From (2.18) we get

$$a_c(u,u) + (qK + 1 - q)\|u\|^2_{L^2_\mu} \geq (1-q)\,\|u\|^2_{H^1_\mu},$$

and then

$$a_c(u,u) + \|u\|^2_{L^2_\mu} \geq \left(\frac{1-q}{qK+1}\right)\|u\|^2_{H^1_\mu}.$$

This shows that the norm $\|\cdot\|_{a_c}$ associated to the bilinear form is equivalent to $\|\cdot\|_{H^1_\mu}$.

ii) Let $c = c_o$. Consider the symmetric operator B defined as

$$Bu = Lu + c_o\sum_{i=1}^n \frac{u}{|x-a_i|^2}$$

with domain $D(B) = C_c^\infty(\mathbb{R}^N)$. It is well known (cf. [51, Chapter 1]) that the bilinear form b given by $b(u,v) = -(Bu,v)_{L^2(\mu)}$, with domain $D(b) = D(B)$, is closable (and the operator associated with $\bar{b}$ is called the *Friedrichs extension* of B). Moreover it holds that $b(u,v) = a_{c_o}(u,v)$ for $u, v \in C_c^\infty(\mathbb{R}^N)$. Since $C_c^\infty(\mathbb{R}^N)$ is dense in H^1_μ, it follows that $H^1_\mu \subset D(\bar{b})$ and $a_{c_o} = \bar{b}$ on $H^1_\mu \times H^1_\mu$. This implies that a_{c_o} is closable and $\overline{a_{c_o}} = b$.

iii) By the weighted Hardy inequality (2.5) we immediately get

$$a_c(u,u) \geq -K\,(u,u)_{H^1_\mu}$$

for all $u \in H^1_\mu$, where $K = \frac{k+(n+1)c_o}{r_0^2} + \frac{n}{2}\operatorname{Tr} A$.

□

The associated operator $\mathcal{A}$ with a_c on L^2_μ is defined by

$$D(\mathcal{A}) = \left\{ u \in D(a_c) : \exists\, v \in L^2_\mu \;\; s.\;\; t.\;\; a_c(u,\phi) = \int_{\mathbb{R}^N} v\phi\, d\mu \;\; \forall \phi \in D(a_c) \right\},$$

$$\mathcal{A}u = v.$$

Now we can state the generation result.

Theorem 2.6. *The operator $-\mathcal{A}$ generates a positive C_0-semigroup on L^2_μ satisfying*

$$\|S(t)\| \le e^{Kt}, \quad t \ge 0,$$

where $K = \frac{k+(n+1)c_o}{r_0^2} + \frac{n}{2}\operatorname{Tr} A$.

Proof. Let us consider the case $c < c_o$.

Since the bilinear form a_c is quasi-accretive, the form $a_c + K$, with $K = \frac{k+(n+1)c_o}{r_0^2} + \frac{n}{2}\operatorname{Tr} A$, is accretive. Then also its associate operator $\mathcal{A} + KI$ is accretive (see Appendix A). From [51, Chapter 1, Proposition 1.51] it follows that $-(\mathcal{A}+KI)$ generates a contraction C_0-semigroup $\{T(t)\}_{t\ge0}$ on L^2_μ. This means that $\{T(t)\}_{t\ge0}$ satisfies the condition $\|T(t)\| \le Me^{\omega t}$ for $M = 1$ and $\omega = 0$. From the bounded perturbation theorem (see e.g. [23, Chapter III]) it follows that the operator $-\mathcal{A} - KI + KI = -\mathcal{A} = L + V$ generates a C_0-semigroup $\{S(t)\}_{t\ge0}$ on L^2_μ satisfying

$$\|S(t)\| \le Me^{(\omega + M\|KI\|)t} = e^{Kt}, \quad t \ge 0.$$

For the case $c = c_o$ the same conclusion holds taking the closure $\overline{a_{c_o}}$ instead of a_{c_o} in the definition of $\mathcal{A}$.

The positivity of the semigroup can be obtained as in [3, Section 2]. Indeed, we can regard $\{S(t)\}_{t\ge0}$ as the limit of positivity preserving semigroups described by cut-off potentials.

Let $-\mathcal{A}_k = L + \min(V,k)$, $k \in \mathbb{N}$. Since L is the generator of a positivity preserving semigroup on L^2_μ and $\min(V,k)$ is bounded and non-negative, $\mathcal{A}_k$ generates a positivity preserving semigroup (see Appendix B), denoted by $\{S_k(t)\}_{t\ge0}$. Moreover $-\mathcal{A}_k \le -\mathcal{A}_{k+1}$, then

$$0 \le S_k(t) \le S_{k+1}(t).$$

The operator $\mathcal{A}_k$ is associated to the bilinear form

$$a_{c,k}(u,v) := \int_{\mathbb{R}^N} \nabla u \cdot \nabla v \, d\mu - c \int_{\mathbb{R}^N} uv \min(V,k) \, d\mu$$

for all $u, v \in H^1_\mu$, with $c > 0$.

Then we have that the corresponding quadratic forms satisfies

$$a_{c,k}(u) \geq a_{c,k+1}(u), \quad a_{c,k}(u) \to a_c(u),$$

for all $u \in H^1_\mu$.

If $c \leq c_o$ it follows from the monotone convergence theorem for forms (cf. [53, Theorem 7.5.18]) that $\mathcal{A}_k$ converges to $\mathcal{A}$ in strong resolvent sense, and then, as a consequence,

$$\lim_{k \to \infty} S_k(t) = S(t)$$

strongly in L^2_μ (see e.g. [52, Chapter 3, Lemma 4.1]).

Finally, similar to [3, Proposition 2.5] we can observe that if $c > c_o$ then

$$\lim_{k \to \infty} \|S_k(t)\| = \infty, \quad t > 0.$$

In fact, for a self-adjoint operator A on a Hilbert space H one has $\|e^{tA}\|_{\mathcal{L}(H)} = e^{ts(A)}$ where $s(A) = \sup\{(Au,u)_H : u \in D(A), \|u\|_H = 1\}$ is the spectral bound of A. In our case

$$s(-\mathcal{A}_k) = \sup \Bigg\{ - \int_{\mathbb{R}^N} |\nabla u|^2 d\mu + c \int_{\mathbb{R}^N} |u|^2 \min(V,k) \, d\mu : \\ u \in D(-\mathcal{A}_k), \, \|u\|_{L^2_\mu} = 1 \Bigg\}.$$

From the optimality of the constant c_o in the weighted Hardy inequality it follows that $s(-\mathcal{A}_k) \to \infty$ as $k \to \infty$ if $c > c_o$.

□

Chapter 3

Weighted Hardy inequalities with measures of more general type and Kolmogorov operators

We focus on weighted Hardy inequalities and present some recent results stated in [14, 15]. The work [15] is in progress for some aspects. The basic idea is to state multipolar Hardy inequalities when the measure $d\mu = \mu(x)dx$ is different from the Gaussian measure, then of more general type. To this aim we need the weighted unipolar Hardy inequality stated in [14]. These inequalities can be applied to the study of the Kolmogorov type operators with more general drift terms

$$Lu = \Delta u + \frac{\nabla \mu}{\mu} \cdot \nabla u \tag{3.1}$$

and of the related evolution problems.

In Section 3.1 we prove a unipolar weighted Hardy inequality

$$c_\mu \int_{\mathbb{R}^N} \frac{\varphi^2}{|x|^2}\, d\mu \le \int_{\mathbb{R}^N} |\nabla \varphi|^2\, d\mu + C_\mu \int_{\mathbb{R}^N} \varphi^2\, d\mu, \quad \forall\, \varphi \in H^1_\mu,$$

with respect to measures which satisfy some general conditions. The proof is different from the others in literature, in particular is different from the one stated in [12]. Under further suitable hypotheses we are able to state the optimality of the constant in the inequality using a technique close to [12] but introducing a different function φ. We give some examples of weight functions which satisfy our assumptions.

In Section 3.2 we prove a weighted multipolar Hardy inequality for more general weight functions with respect to the Gaussian measure using the vector field method. We are able to overcome the difficulties related to the poles but we do not achieve the best constant in the estimate.

In Section 3.3 we focus on a weighted multipolar Hardy inequality

$$c_\mu \int_{\mathbb{R}^N} \sum_{i=1}^{n} \frac{\varphi^2}{|x-a_i|^2}\, d\mu \le \int_{\mathbb{R}^N} |\nabla \varphi|^2\, d\mu + C_\mu \int_{\mathbb{R}^N} \varphi^2\, d\mu, \quad \forall\, \varphi \in H^1_\mu,$$

by adapting the IMS method to the weighted case. So we can achieve the best constant. The estimate stated in Section 3.1 plays a crucial role in the proof of the result.

3.1 Weighted unipolar Hardy inequalities

Let μ a weight function in $\mathbb{R}^N$. We define the weighted Sobolev space $H^1_\mu = H^1(\mathbb{R}^N, \mu(x)dx))$ as the space of functions in $L^2_\mu := L^2(\mathbb{R}^N, \mu(x)dx)$ whose weak derivatives belong to $(L^2_\mu)^N$.

As first step we consider the following conditions on μ which we need to state a preliminary weighted Hardy inequality.

$H_1)$ $\mu \ge 0$, $\mu \in L^1_{loc}(\mathbb{R}^N)$;

$H_2)$ $\nabla \mu \in L^1_{loc}(\mathbb{R}^N)$;

$H_3)$ there exist constants $k_1, k_2 \in \mathbb{R}$, $k_2 > 2-N$, such that if

$$f_\varepsilon = (\varepsilon + |x|^2)^{\frac{\alpha}{2}}, \quad \alpha < 0, \quad \varepsilon > 0,$$

it holds

$$\frac{\nabla f_\varepsilon}{f_\varepsilon} \cdot \nabla \mu = \frac{\alpha x}{\varepsilon + |x|^2} \cdot \nabla \mu \le \left(k_1 + \frac{k_2 \alpha}{\varepsilon + |x|^2} \right) \mu$$

for any $\varepsilon > 0$.

The condition $H_3)$ contains the requirement that the scalar product $\alpha x \cdot \frac{\nabla\mu}{\mu}$ is bounded in B_R, $R>0$, while $\frac{\alpha x}{\varepsilon+|x|^2}\cdot\frac{\nabla\mu}{\mu}$ is bounded in $\mathbb{R}^N\setminus B_R$, where B_R is a ball of radius R centered in zero.

The reason we use the function f_ε will be clear in the proof of the weighted Hardy inequality (*wHi*) which we will state below.

The idea to introduce the functions f_ε is due to [21] but our proof is different. Finally we observe that we need the condition $k_2>2-N$ to apply Fatou's lemma in the proof of Theorem 3.1.

Theorem 3.1. *Under conditions $H_1)$–$H_3)$ there exists a positive constant c such that*

$$c\int_{\mathbb{R}^N}\frac{\varphi^2}{|x|^2}\,d\mu\le\int_{\mathbb{R}^N}|\nabla\varphi|^2\,d\mu+k_1\int_{\mathbb{R}^N}\varphi^2\,d\mu, \tag{3.2}$$

for any function $\varphi\in C_c^\infty(\mathbb{R}^N)$, where $c\in(0,c_o(N+k_2)]$ with $c_o(N+k_2)=\left(\frac{N+k_2-2}{2}\right)^2$.

Proof. As first step we start from the integral of the square of the gradient of the function φ. Then we introduce $\psi=\frac{\varphi}{f_\varepsilon}$, with f_ε defined in $H_3)$, and integrate by parts taking in mind $H_1)$ and $H_2)$.

$$\begin{aligned}
\int_{\mathbb{R}^N}|\nabla\varphi|^2\,d\mu&=\int_{\mathbb{R}^N}|\nabla(\psi f_\varepsilon)|^2\,d\mu\\
&=\int_{\mathbb{R}^N}|\nabla\psi f_\varepsilon+\nabla f_\varepsilon\psi|^2\,d\mu\\
&=\int_{\mathbb{R}^N}|\nabla\psi|^2f_\varepsilon^2\,d\mu+\int_{\mathbb{R}^N}\psi^2|\nabla f_\varepsilon|^2\,d\mu+2\int_{\mathbb{R}^N}f_\varepsilon\psi\nabla\psi\cdot\nabla f_\varepsilon\,d\mu\\
&=\int_{\mathbb{R}^N}|\nabla\psi|^2f_\varepsilon^2\,d\mu+\int_{\mathbb{R}^N}\psi^2|\nabla f_\varepsilon|^2\,d\mu\\
&-\int_{\mathbb{R}^N}\psi^2|\nabla f_\varepsilon|^2\,d\mu-\int_{\mathbb{R}^N}f_\varepsilon^2\psi^2\frac{\Delta f_\varepsilon}{f_\varepsilon}\,d\mu-\int_{\mathbb{R}^N}f_\varepsilon^2\psi^2\frac{\nabla f_\varepsilon}{f_\varepsilon}\cdot\nabla\mu\,dx.
\end{aligned} \tag{3.3}$$

Observing that

$$\Delta f_\varepsilon=\frac{\alpha(N-2+\alpha)|x|^2+\alpha\varepsilon N}{(\varepsilon+|x|^2)^{2-\frac{\alpha}{2}}}$$

and using hypothesis $H_3)$ we deduce that

$$\begin{aligned}\int_{\mathbb{R}^N} |\nabla\varphi|^2\,d\mu &\geq -\int_{\mathbb{R}^N} \frac{\Delta f_\varepsilon}{f_\varepsilon}\varphi^2\,d\mu - \int_{\mathbb{R}^N} \frac{\nabla f_\varepsilon}{f_\varepsilon}\cdot\nabla\mu\,\varphi^2\,dx \\ &\geq -\big[\alpha(N-2)+\alpha^2\big]\int_{\mathbb{R}^N} \frac{|x|^2}{(\varepsilon+|x|^2)^2}\varphi^2\,d\mu - \varepsilon\alpha N\int_{\mathbb{R}^N} \frac{\varphi^2}{(\varepsilon+|x|^2)^2}\,d\mu \\ &- k_1\int_{\mathbb{R}^N}\varphi^2\,d\mu - k_2\alpha\int_{\mathbb{R}^N}\frac{\varphi^2}{\varepsilon+|x|^2}\,d\mu \\ &= [-\alpha(N-2+k_2)-\alpha^2]\int_{\mathbb{R}^N}\frac{|x|^2}{(\varepsilon+|x|^2)^2}\varphi^2\,d\mu \\ &- \varepsilon\alpha(N+k_2)\int_{\mathbb{R}^N}\frac{\varphi^2}{(\varepsilon+|x|^2)^2}\,d\mu - k_1\int_{\mathbb{R}^N}\varphi^2\,d\mu.\end{aligned}$$

The constant $-\alpha(N-2+k_2)-\alpha^2$ is greater than zero since $\alpha<0$ and $k_2>2-N$, so by Fatou's lemma we state the following estimate letting $\varepsilon\to 0$

$$\int_{\mathbb{R}^N}|\nabla\varphi|^2\,d\mu + k_1\int_{\mathbb{R}^N}\varphi^2\,d\mu \geq c\int_{\mathbb{R}^N}\frac{\varphi^2}{|x|^2}\,d\mu,$$

with $c=-\alpha(N-2+k_2)-\alpha^2$. Finally we observe that

$$\max_\alpha[-\alpha(N+k_2-2)-\alpha^2] = \left(\frac{N+k_2-2}{2}\right)^2 := c_o(N+k_2),$$

attained for $\alpha_o = -\frac{N+k_2-2}{2}$. □

We observe that in the case $\mu=1$ we obtain the classical Hardy inequality.

Remark 3.2. In an alternative way we can define f_ε in $H_3)$ setting $\alpha=\alpha_o$ and get the estimate (3.2) with $c=c_o(N+k_2)$. Although the result it goes in the same direction, in the proof we point out that $c_o(N+k_2)$ is the maximum value of the constant c.

Now we suppose that

$H_4)$ $\mu\geq 0$, $\sqrt{\mu}\in H^1_{loc}(\mathbb{R}^N)$;

$H_5)$ $\mu^{-1}\in L^1_{loc}(\mathbb{R}^N)$.

Let us observe that in the hypotheses H_4)-H_5) the space $C_c^\infty(\mathbb{R}^N)$ is dense in H^1_μ and H^1_μ is the completion of $C_c^\infty(\mathbb{R}^N)$ with respect to the Sobolev norm

$$\|\cdot\|^2_{H^1_\mu} := \|\cdot\|^2_{L^2_\mu} + \|\nabla\cdot\|^2_{L^2_\mu}$$

(see [55]). For some interesting papers on density of smooth functions in weighted Sobolev spaces and related questions we refer, for example, to [38, 24, 8, 39, 19, 56, 9].

So we can deduce the following result from Theorem 3.1 by density argument.

Theorem 3.3. *Under conditions H_2)–H_5) there exists a positive constant c such that*

$$c\int_{\mathbb{R}^N}\frac{\varphi^2}{|x|^2}\,d\mu \le \int_{\mathbb{R}^N}|\nabla\varphi|^2\,d\mu + k_1\int_{\mathbb{R}^N}\varphi^2\,d\mu, \tag{3.4}$$

for any function $\varphi \in H^1_\mu$, where $c \in (0, c_o(N+k_2)]$ with $c_o(N+k_2) = \left(\frac{N+k_2-2}{2}\right)^2$.

Remark 3.4. We remark that our proof of *wHi* is different from the others in literature, in particular it is different from the one stated in [12] for the weighted case (see Section 1.7, Chapter 1). Furthermore some hypotheses are different. In Theorem 3.3 we achieve the best constant $c_0(N+k_2)$ (cf. [12, Theorem 1.3]) without requiring $\Delta\mu \in L^1_{loc}(\mathbb{R}^N)$ and without the condition *iv*) in Section 1.7, Chapter 1, which allows the authors in [12] to use improved Hardy inequalities (see [1, 49]) to get the result.

We present some examples of functions μ which satisfy the hypotheses of Theorem 3.3.

We remark that, in the hypotheses $\mu = \mu(|x|) \in C^1$ for $|x| \in [r_0, +\infty[$, $r_0 \ge 0$, a class of weight functions μ which satisfies H_3) is the following

$$\mu(x) \ge C e^{-\frac{k_1}{2|\alpha|}|x|^2}|x|^{k_2 - \frac{k_1}{|\alpha|}\varepsilon}, \quad \text{for} \quad |x| \ge r_0, \tag{3.5}$$

where C is a constant depending on $\mu(r_0)$ and on r_0.

In the case of radial functions, $\mu(x) = \mu(|x|)$, if we set $|x| = \rho$ the condition H_3) states that μ satisfies the following inequality

$$\frac{\alpha\rho}{\varepsilon+\rho^2}\mu'(\rho) \le \left(k_1 + \frac{k_2\alpha}{\varepsilon+\rho^2}\right)\mu(\rho),$$

which implies

$$\mu'(\rho) \geq a(\rho)\mu(\rho)$$

with

$$a(\rho) = \frac{k_1}{\alpha}\left(\frac{\varepsilon + \rho^2}{\rho}\right) + \frac{k_2}{\rho}.$$

Integrating in $[r_0, r]$ we get

$$\mu(r) \geq \mu(r_0)e^{\int_{r_0}^{r} a(s)ds} = \mu(r_0)\left(\frac{r}{r_0}\right)^{k_2 - \frac{k_1}{|\alpha|}\varepsilon} e^{-\frac{k_1}{2|\alpha|}(r^2 - r_0^2)} \qquad \text{for} \quad r \geq r_0,$$

from which

$$\mu(r) \geq \frac{\mu(r_0)}{r_0^{k_2 - \frac{k_1}{|\alpha|}\varepsilon}} e^{\frac{k_1}{2|\alpha|}r_0^2} r^{k_2 - \frac{k_1}{|\alpha|}\varepsilon} e^{-\frac{k_1}{2|\alpha|}r^2} \qquad \text{for} \quad r \geq r_0.$$

Example 3.5. Another class of weight functions satisfying H_3), when $k_1 = k_2 = 0$, consists of the bounded increasing functions, as, for example, $\cos e^{-|x|^2}$. Such a function verifies the requirements in the Theorem 3.3.

In the following example we consider a wide class of functions which contains the Gaussian measure and polynomial type measures, a class of functions which behaves as $\frac{1}{|x|^\gamma}$ when $|x|$ goes to zero.

Example 3.6. We consider the following weight functions

$$\mu(x) = \frac{1}{|x|^\gamma} e^{-\delta|x|^m}, \quad \delta \geq 0, \quad \gamma < N - 2 \tag{3.6}$$

and state for which values of γ and m the functions in (3.6) are "good" functions to get *wHi*.

The weight μ satisfies H_2), H_4) and H_5) if $\gamma > -N$. The condition H_3)

$$\frac{\alpha(-\gamma - \delta m|x|^m)}{\varepsilon + |x|^2} \leq k_1 + \frac{\alpha k_2}{\varepsilon + |x|^2}$$

is fulfilled if

$$-(\alpha\gamma + \alpha k_2 + k_1\varepsilon) - \alpha\delta m|x|^m - k_1|x|^2 \leq 0. \tag{3.7}$$

In the case $\delta = 0$ we only need to require that $\gamma \leq -k_2 - \frac{k_1}{\alpha}\varepsilon$ and we are able to get the Caffarelli-Niremberg inequality

$$\left(\frac{N-2-\gamma}{2}\right)^2 \int_{\mathbb{R}^N} \frac{\varphi^2}{|x|^2}|x|^{-\gamma}\,dx \leq \int_{\mathbb{R}^N} |\nabla\varphi|^2|x|^{-\gamma}\,dx \qquad \forall\varphi \in H^1_\mu.$$

While if $\gamma = 0$ the inequality (3.7) holds if $m \leq 2$ and k_1 is large enough.

In general to get (3.7) we need the following conditions on parameters and on the constant k_1

i) $\gamma \in (-N, -k_2]$, $\delta = 0$, $k_1 = 0$,

ii) $\gamma \in (-N, -k_2]$, $k_1 \geq -2\alpha\delta$, $m = 2$,

iii) $\gamma \in (-N, -k_2)$, $k_1 \geq \tilde{k}_1$, $m < 2$,

where $\tilde{k}_1 = \frac{\frac{m}{2}\left(1-\frac{m}{2}\right)^{\frac{2}{m}-1}(-\alpha\delta m)^{\frac{2}{m}}}{[\alpha(\gamma+k_2)]^{\frac{2}{m}-1}}$, to get the inequality (3.4).

Example 3.7. The function $\mu(x) = [\log(1+|x|)]^{-\gamma}$, for γ as in *i)*, behaves as $\frac{1}{|x|^\gamma}$ when $|x|$ goes to 0. So we can state the weighted Hardy inequality (3.4) with $k_1 = 0$ as in the previous case.

To state the optimality of the constant $c_o(N+K_2)$ in the estimate (3.4) we need further assumptions on μ as usually it is done. We remark that in the proof of optimality the choice of the function φ plays a fundamental role. The technique we use is close to the one used in [12] but the function φ we use is different (see Theorem 1.2 in Chapter 1 for the proof in [12] adapted to the case of Lebesgue measure).

We suppose

$H_6)$ there exists $\sup_{\delta\in\mathbb{R}}\left\{\frac{1}{|x|^\delta} \in L^1_{loc}(\mathbb{R}^N, d\mu)\right\} := N_0$ and $k_2 = N_0 - N$.

We observe that the condition $H_6)$ is necessary for the technique used in the proof of the optimality. In [12] there is a similar condition to get the optimality of the constant when $c_\mu = \left(\frac{N_0-2}{2}\right)^2$. In our case the requirement $H_6)$ involves the constant k_2.

For example the functions μ such that

$$\lim_{x\to 0}\frac{\mu}{|x|^{k_2}}=l, \qquad l>0,$$

verify H_6).

The result below states the optimality of the constant $c_o(N+k_2)$ in the Hardy inequality.

Theorem 3.8. *In the hypotheses of Theorem 3.3 and if H_6) holds, for $c>c_o(N+k_2)=\left(\frac{N+k_2-2}{2}\right)^2$ the inequality (3.4) doesn't hold for any $\varphi\in H^1_\mu$.*

Proof. Let $\theta\in C_c^\infty(\mathbb{R}^N)$ a cut-off function, $0\le\theta\le 1$, $\theta=1$ in B_1 and $\theta=0$ in B_2^c. We introduce the function

$$\varphi_\varepsilon(x)=\begin{cases}(\varepsilon+|x|)^\eta & \text{if } |x|\in[0,1[,\\ (\varepsilon+|x|)^\eta\theta(x) & \text{if } |x|\in[1,2[,\\ 0 & \text{if } |x|\in[2,+\infty[,\end{cases}$$

where $\varepsilon>0$ and the exponent η is such that

$$\max\left\{-\sqrt{c},-\frac{N+k_2}{2}\right\}<\eta<\min\left\{-\frac{N+k_2-2}{2},0\right\}.$$

The function φ_ε belongs to H^1_μ for any $\varepsilon>0$.

For this choice of η we obtain $\eta^2<c$, $|x|^{2\eta}\in L^1_{loc}(\mathbb{R}^N,d\mu)$ and $|x|^{2\eta-2}\notin L^1_{loc}(\mathbb{R}^N,d\mu)$.

Let us assume that $c>c_o(N+k_2)$. Our aim is to prove that the bottom of the spectrum of the operator $-(L+V)$, with L the operator defined in (3.1) and $V(x)=\frac{c}{|x|^2}$,

$$\lambda_1=\inf_{\varphi\in H^1_\mu\setminus\{0\}}\left(\frac{\int_{\mathbb{R}^N}|\nabla\varphi|^2\,d\mu-\int_{\mathbb{R}^N}\frac{c}{|x|^2}\varphi^2\,d\mu}{\int_{\mathbb{R}^N}\varphi^2\,d\mu}\right). \tag{3.8}$$

is $-\infty$. For this purpose we estimate at first the numerator in (3.8).

$$
\begin{aligned}
&\int_{\mathbb{R}^N} \left(|\nabla \varphi_\varepsilon|^2 - \frac{c}{|x|^2}\varphi_\varepsilon^2 \right) d\mu = \\
&= \int_{B_1} \left[|\nabla(\varepsilon + |x|)^\eta|^2 - \frac{c}{|x|^2}(\varepsilon + |x|)^{2\eta} \right] d\mu \\
&\quad + \int_{B_1^c} \left[|\nabla(\varepsilon + |x|)^\eta \theta|^2 - \frac{c}{|x|^2}(\varepsilon + |x|)^{2\eta}\theta^2 \right] d\mu \\
&= \int_{B_1} \left[\eta^2(\varepsilon + |x|)^{2\eta-2} - \frac{c}{|x|^2}(\varepsilon + |x|)^{2\eta} \right] d\mu \\
&\quad + \eta^2 \int_{B_1^c} (\varepsilon + |x|)^{2\eta-2}\theta^2 \, d\mu + \int_{B_1^c} (\varepsilon + |x|)^{2\eta} |\nabla\theta|^2 \, d\mu \qquad (3.9)\\
&\quad + 2\eta \int_{B_1^c} \theta(\varepsilon + |x|)^{2\eta-1} \frac{x}{|x|} \cdot \nabla\theta \, d\mu \\
&\le \int_{B_1} (\varepsilon + |x|)^{2\eta} \left[\frac{\eta^2}{(\varepsilon + |x|)^2} - \frac{c}{|x|^2} \right] d\mu \\
&\quad + 2\eta^2 \int_{B_1^c} (\varepsilon + |x|)^{2\eta-2}\theta^2 \, d\mu + 2\int_{B_1^c} (\varepsilon + |x|)^{2\eta} |\nabla\theta|^2 \, d\mu \\
&\le \int_{B_1} (\varepsilon + |x|)^{2\eta} \left[\frac{\eta^2}{(\varepsilon + |x|)^2} - \frac{c}{|x|^2} \right] d\mu + C_1,
\end{aligned}
$$

where $C_1 = (2\eta^2 + 2\|\nabla\theta\|_\infty) \int_{B_1^c} d\mu$.

Furthermore

$$
\int_{\mathbb{R}^N} \varphi_\varepsilon^2 \, d\mu \ge \int_{B_2 \setminus B_1} (\varepsilon + |x|)^{2\eta}\theta^2 \, d\mu = C_{2,\varepsilon}. \tag{3.10}
$$

Put together (3.9) and (3.10) from (3.8) we get

$$
\lambda_1 \le \frac{\int_{B_1} (\varepsilon + |x|)^{2\eta} \left(\frac{\eta^2}{(\varepsilon+|x|)^2} - \frac{c}{|x|^2} \right) d\mu + C_1}{C_{2,\varepsilon}}.
$$

Letting $\varepsilon \to 0$ in the numerator above, taking in mind that $|x|^{2\eta} \in L^1_{loc}(\mathbb{R}^N, d\mu)$ and Fatou's lemma, we obtain

$$
\lim_{\varepsilon\to 0} \int_{B_1} (\varepsilon+|x|)^{2\eta} \left(\frac{\eta^2}{(\varepsilon + |x|)^2} - \frac{c}{|x|^2} \right) d\mu \le -(c-\eta^2) \int_{B_1} |x|^{2\eta-2} \, d\mu = -\infty
$$

and, then, $\lambda_1 = -\infty$. $\square$

In order to get necessary and sufficient conditions for the existence and nonexistence of positive solutions to the evolution problem corresponding to $L+V$, where L is the operator defined in (3.1) and $V(x) = \frac{c}{|x|^2}$, $c > 0$, we need a weighted Hardy inequality.

In the standard setting one considers probability densities $0 < \mu \in C^{1,\lambda}_{loc}(\mathbb{R}^N)$ for some $\lambda \in (0,1)$ while in the more general case $0 < \mu \in C^{1,\lambda}_{loc}(\mathbb{R}^N \setminus \{0\})$.

Then arguing as in [12], putting together Theorem 3.8 and Theorem 1.16 in Section 1.7, Chapter 1, we get the following result.

Theorem 3.9. *Let* $0 \le V(x) \le \frac{c}{|x|^2}$, $c > 0$. *Assume the hypotheses of Theorem 3.8 and the condition vi) in Section 1.7, Chapter 1. The following assertions hold.*

1) If $c \le c_o(N+k_2)$, *then there exists a positive weak solution* $u \in C([0,\infty), L^2_\mu)$ *of*

$$(P) \quad \begin{cases} \partial_t u(x,t) = Lu(x,t) + V(x)u(x,t), & t > 0,\, x \in \mathbb{R}^N, \\ u(\cdot,0) = u_0 \ge 0 \in L^2_\mu, \end{cases}$$

satisfying

$$\|u(t)\|_{L^2_\mu} \le M e^{\omega t} \|u_0\|_{L^2_\mu}, \quad t \ge 0 \tag{3.11}$$

for some constants $M \ge 1$, $\omega \in \mathbb{R}$, *and any* $u_0 \in L^2_\mu$.

2) If $c > c_o(N+k_2)$, *then for any* $0 \le u_0 \in L^2_\mu$, $u_0 \ne 0$, *there is no positive weak solution of* (P) *satisfying (3.11).*

In the standard setting we argue as in [33], putting together inequality (3.4) and Theorem 1.11 in Section 1.5, Chapter 1.

We observe that the functions $e^{-\delta|x|^m}$ in Example 3.6 fully satisfies the condition $vi)$ in Section 1.7, Chapter 1, while $\cos e^{-|x|^2}$ in Example 3.5 is $(1,\lambda)$-Hölderian in $\mathbb{R}^N$.

3.2 Weighted multipolar Hardy inequalities via the vector field method

The vector field method suggests us to consider the vectorial function

$$F(x) = \sum_{i=1}^{n} \beta \frac{x - a_i}{|x - a_i|^2}\mu, \qquad \beta > 0.$$

The hypotheses on weights μ are $H_2)$, $H_4)$, $H_5)$ in Section 3.1 and the following

$H_3')$ there exists constants $k_1, k_2 \in \mathbb{R}$, such that

$$\beta \sum_{i=1}^{n} \frac{(x - a_i)}{|x - a_j|^2} \cdot \nabla\mu \geq \left(-k_1 + \sum_{i=1}^{n} \frac{k_2\beta}{|x - a_i|^2}\right)\mu.$$

The next Theorem states a preliminary weighted Hardy inequality which extends Proposition 1.6 in Section 1.4, Chapter 1, to the weighted case.

Theorem 3.10. *Let $r_0 = \min_{i\neq j} |a_i - a_j|/2$, $N \geq 3$, $n \geq 1$. Under hypotheses $H_2)$, $H_3')$, $H_4)$ and $H_5)$ we get*

$$\begin{aligned} \frac{c_o(N + k_2)}{n} \int_{\mathbb{R}^N} \sum_{i=1}^{n} \frac{\varphi^2}{|x - a_i|^2}\, d\mu & \\ + \frac{\beta^2}{2} \int_{\mathbb{R}^N} \sum_{\substack{i,j=1 \\ i\neq j}}^{n} \frac{|a_i - a_j|^2}{|x - a_i|^2|x - a_j|^2}\varphi^2\, d\mu & \\ \leq \int_{\mathbb{R}^N} |\nabla\varphi|^2\, d\mu + k_1 \int_{\mathbb{R}^N} \varphi^2 d\mu, & \end{aligned} \tag{3.12}$$

for any $\varphi \in H^1_\mu$, where $c_o(N + k_2) := \left(\frac{N+k_2-2}{2}\right)^2$.

Proof. By density, it is enough to prove (3.12) for every $\varphi \in C_c^\infty(\mathbb{R}^N)$. It is immediate to verify that

$$\int_{\mathbb{R}^N} \varphi^2 \text{div} F\, dx = \beta \int_{\mathbb{R}^N} \sum_{i=1}^{n} \left[\frac{N-2}{|x - a_i|^2}\mu + \frac{(x - a_i)}{|x - a_i|^2} \cdot \nabla\mu\right] \varphi^2 dx. \tag{3.13}$$

On the other hand, integrating by parts and using Hölder and Young inequalities, we get

$$\begin{aligned}\int_{\mathbb{R}^N} \varphi^2 \mathrm{div} F \, dx &= -2 \int_{\mathbb{R}^N} \varphi F \cdot \nabla \varphi \, dx \\ &\le 2 \left[\int_{\mathbb{R}^N} |\nabla \varphi|^2 \, d\mu \right]^{\frac{1}{2}} \left[\int_{\mathbb{R}^N} \left(\sum_{i=1}^{n} \frac{\beta \, (x-a_i)}{|x-a_i|^2} \right)^2 \varphi^2 \, d\mu \right]^{\frac{1}{2}} \\ &\le \int_{\mathbb{R}^N} |\nabla \varphi|^2 \, d\mu + \int_{\mathbb{R}^N} \left(\sum_{i=1}^{n} \frac{\beta \, (x-a_i)}{|x-a_i|^2} \right)^2 \varphi^2 \, d\mu. \end{aligned} \tag{3.14}$$

From (3.13) and (3.14) we deduce

$$\begin{aligned}\int_{\mathbb{R}^N} \sum_{i=1}^{n} \frac{\beta(N-2)}{|x-a_i|^2} \varphi^2 d\mu \le \int_{\mathbb{R}^N} |\nabla \varphi|^2 \, d\mu & \\ + \int_{\mathbb{R}^N} \sum_{i=1}^{n} \frac{\beta^2}{|x-a_i|^2} \, \varphi^2 \, d\mu & \\ + \int_{\mathbb{R}^N} \sum_{\substack{i,j=1 \\ i \ne j}}^{n} \frac{\beta^2 \, (x-a_i)(x-a_j)}{|x-a_i|^2 |x-a_j|^2} \, \varphi^2 \, d\mu & \\ - \beta \int_{\mathbb{R}^N} \sum_{i=1}^{n} \frac{(x-a_i)}{|x-a_i|^2} \cdot \nabla \mu \, \varphi^2 dx. & \end{aligned} \tag{3.15}$$

Now we observe that

$$
\begin{aligned}
&\sum_{\substack{i,j=1\\ i\neq j}}^{n}\frac{(x-a_i)(x-a_j)}{|x-a_i|^2|x-a_j|^2}=\\
&\qquad=\sum_{\substack{i,j=1\\ i\neq j}}^{n}\frac{|x|^2-xa_i-xa_j+a_ia_j}{|x-a_i|^2|x-a_j|^2}\\
&\qquad=\sum_{\substack{i,j=1\\ i\neq j}}^{n}\frac{\frac{|x-a_i|^2}{2}+\frac{|x-a_j|^2}{2}-\frac{|a_i-a_j|^2}{2}}{|x-a_i|^2|x-a_j|^2}\\
&\qquad=\sum_{\substack{i,j=1\\ i\neq j}}^{n}\frac{1}{2}\left(\frac{1}{|x-a_i|^2}+\frac{1}{|x-a_j|^2}-\frac{|a_i-a_j|^2}{|x-a_i|^2|x-a_j|^2}\right)\\
&\qquad=(n-1)\sum_{i=1}^{n}\frac{1}{|x-a_i|^2}-\frac{1}{2}\sum_{\substack{i,j=1\\ i\neq j}}^{n}\frac{|a_i-a_j|^2}{|x-a_i|^2|x-a_j|^2}.
\end{aligned}
\tag{3.16}
$$

Then, by hypothesis H_3') and by (3.16) it follows that

$$
\begin{aligned}
&\left[(N+k_2-2)\beta-n\beta^2\right]\sum_{i=1}^{n}\int_{\mathbb{R}^N}\frac{\varphi^2}{|x-a_i|^2}\,d\mu\\
&\qquad\qquad+\frac{\beta^2}{2}\int_{\mathbb{R}^N}\sum_{\substack{i,j=1\\ i\neq j}}^{n}\frac{|a_i-a_j|^2}{|x-a_i|^2|x-a_j|^2}\varphi^2\,d\mu\\
&\qquad\qquad\leq\int_{\mathbb{R}^N}|\nabla\varphi|^2\,d\mu+k_1\int_{\mathbb{R}^N}\varphi^2d\mu.
\end{aligned}
\tag{3.17}
$$

The Theorem is proved observing that

$$
\max_{\beta}[(N+k_2-2)\beta-n\beta^2]=\frac{(N+k_2-2)^2}{4n}
$$

□

Now our aim is to estimate the second term on the left hand side in (3.17) to get a more general Hardy inequality. From a mathematical point of view the principal problem is due to the square of the sum on the right hand-side in (3.14). To overcome the difficulties we are able to isolate singularities but we can not achieve the constant $c_o(N+k_2)$.

We get the following result.

Theorem 3.11. *Let* $r_0 = \min_{i\neq j} |a_i - a_j|/2$, $N \geq 3$, $n \geq 1$. *Then if conditions* $H_2)$, $H_3')$, $H_4)$ *and* $H_5)$ *hold, we get*

$$c\int_{\mathbb{R}^N}\sum_{i=1}^{n}\frac{\varphi^2}{|x-a_i|^2}\,d\mu \leq \int_{\mathbb{R}^N}|\nabla\varphi|^2\,d\mu + K\int_{\mathbb{R}^N}\varphi^2\,d\mu \tag{3.18}$$

for any $\varphi \in H^1_\mu$, *where* $c \in]0, c_o(N+k_2)[$ *and* $K = K(n, c, r_0)$.

Proof. Arguing as in the proof of Theorem 3.10 we get

$$\begin{aligned}
\int_{\mathbb{R}^N}\sum_{i=1}^{n}\frac{\beta(N-2)}{|x-a_i|^2}\varphi^2 d\mu \leq & \int_{\mathbb{R}^N}|\nabla\varphi|^2\,d\mu \\
& + \int_{\mathbb{R}^N}\sum_{i=1}^{n}\frac{\beta^2}{|x-a_i|^2}\,\varphi^2\,d\mu \\
& + \int_{\mathbb{R}^N\setminus\bigcup_{k=1}^n B(a_k,r_0)}\sum_{\substack{i,j\\ i\neq j}}\frac{\beta^2\,(x-a_i)(x-a_j)}{|x-a_i|^2|x-a_j|^2}\,\varphi^2\,d\mu \\
& + \int_{\bigcup_{k=1}^n B(a_k,r_0)}\sum_{\substack{i,j\\ i\neq j}}\frac{\beta^2\,(x-a_i)(x-a_j)}{|x-a_i|^2|x-a_j|^2}\,\varphi^2\,d\mu \\
& - \beta\int_{\mathbb{R}^N}\sum_{i=1}^{n}\frac{(x-a_i)}{|x-a_i|^2}\cdot\nabla\mu\,\varphi^2 dx \\
& = I_1 + I_2 + I_3 + I_4 + I_5.
\end{aligned} \tag{3.19}$$

Let us estimate I_3 and I_4. The first integral can be estimate as follows

$$I_3 \leq \frac{\beta^2}{r_0^2}n(n-1)\int_{\mathbb{R}^N\setminus\bigcup_{k=1}^n B(a_k,r_0)}\varphi^2\,d\mu. \tag{3.20}$$

For the second integral we isolate the singularities and then, using again Young inequality, we get

$$
\begin{aligned}
I_4 \leq & \sum_{k=1}^{n}\Bigg(\int_{B(a_k,r_0)} \sum_{\substack{j=1\\ j\neq k}}^{n} \frac{\beta^2}{|x-a_k||x-a_j|}\,\varphi^2\,d\mu + \\
& + \int_{B(a_k,r_0)} \sum_{\substack{i,j=1\\ j\neq i\neq k}}^{n} \frac{\beta^2}{|x-a_i||x-a_j|}\,\varphi^2\,d\mu\Bigg) \\
& \leq \sum_{k=1}^{n}\Bigg\{\frac{\epsilon}{2}\int_{B(a_k,r_0)} \frac{\beta^2}{|x-a_k|^2}\,\varphi^2\,d\mu + \\
& + \frac{1}{2\epsilon}\int_{B(a_k,r_0)} \sum_{\substack{j\\ j\neq k}} \frac{\beta^2}{|x-a_j|^2}\,\varphi^2\,d\mu \\
& + \frac{\beta^2}{r_0^2}(n-1)^2\int_{B(a_k,r_0)} \varphi^2\,d\mu\Bigg\} \\
& \leq \sum_{k=1}^{n}\Bigg\{\frac{\epsilon}{2}\int_{B(a_k,r_0)} \frac{\beta^2}{|x-a_k|^2}\,\varphi^2\,d\mu + \Bigg[\frac{\beta^2(n-1)}{2\epsilon\, r_0^2} + \\
& + \frac{\beta^2(n-1)^2}{r_0^2}\Bigg]\int_{B(a_k,r_0)} \varphi^2\,d\mu\Bigg\} \\
& \leq \sum_{k=1}^{n}\Bigg\{\frac{\epsilon}{2}\int_{B(a_k,r_0)} \sum_{i=1}^{n}\frac{\beta^2}{|x-a_i|^2}\,\varphi^2\,d\mu + \\
& + \frac{\beta^2(n-1)}{r_0^2}\Bigg[\frac{1}{2\epsilon} + (n-1)\Bigg]\int_{B(a_k,r_0)} \varphi^2\,d\mu\Bigg\}.
\end{aligned}
\tag{3.21}
$$

The integral I_5 can be estimate applying H_3').

Taking into account (3.19) and using (3.20), (3.21) we deduce that

$$
\begin{aligned}
\int_{\mathbb{R}^N} \sum_{i=1}^{n} & \frac{\beta(N+k_2-2)-\beta^2(1+\frac{\epsilon}{2})}{|x-a_i|^2}\varphi^2\,d\mu \\
& \leq \int_{\mathbb{R}^N} |\nabla\varphi|^2 d\mu + K\int_{\mathbb{R}^N} \varphi^2\,d\mu,
\end{aligned}
\tag{3.22}
$$

where

$$
K = \frac{\beta^2}{r_0^2}(n-1)\left(n-1+\frac{1}{2\epsilon}\right) + k_1.
$$

The maximum of the function $\beta \mapsto (N+k_2-2)\beta - \beta^2(1+\frac{\epsilon}{2})$ is $\frac{c_o(N+k_2)}{1+\frac{\epsilon}{2}}$ attained in $\beta_{max} = \frac{\sqrt{c_o(N+k_2)}}{1+\frac{\epsilon}{2}}$. So, if we set

$$c = (N+k_2-2)\beta - \beta^2\left(1+\frac{\epsilon}{2}\right) \tag{3.23}$$

we deduce from (3.23) that for $c \in \left(0, \frac{c_o(N+k_2)}{1+\frac{\epsilon}{2}}\right]$, for any $\epsilon > 0$, it holds

$$c\int_{\mathbb{R}^N}\sum_{i=1}^{n}\frac{\varphi^2}{|x-a_i|^2}\,d\mu \le \int_{\mathbb{R}^N}|\nabla\varphi|^2\,d\mu + K\int_{\mathbb{R}^N}\varphi^2 d\mu.$$

The relation (3.23) between β and c allow us to write β in the following form

$$\beta_\epsilon^\pm = \frac{\sqrt{c_o(N+k_2)} \pm \sqrt{c_o(N+k_2) - c[1+\frac{\epsilon}{2}]}}{1+\frac{\epsilon}{2}}.$$

□

3.3 Weighted multipolar Hardy inequalities via the IMS method

In this Section we state the inequality using the so-called IMS truncation method (for Ismagilov, Morgan, Morgan-Simon, Sigal, see [48, 54]), which consists in localizing the wave functions around the singularities by using a partition of unity. This method, unlike the vector field one, allows us to achieve the constant $c_o(N+k_2)$.

We argue as in [10] adapting the proof to the weighted case.

The hypotheses on the weight functions μ are $H_2)$, $H_4)$, $H_5)$ in Section 3.1 and the following

$H_3'')$ there exist constants $k_1, k_2 \in \mathbb{R}$, $k_2 > 2-N$, such that if

$$f_{\varepsilon,i} = (\varepsilon + |x-a_i|^2)^{\frac{\alpha}{2}}, \quad \alpha < 0, \quad \varepsilon > 0,$$

it holds

$$\frac{\nabla f_{\varepsilon,i}}{f_{\varepsilon,i}}\cdot\nabla\mu = \frac{\alpha(x-a_i)}{\varepsilon+|x-a_i|^2}\cdot\nabla\mu \le \left(k_1 + \frac{k_2\alpha}{\varepsilon+|x-a_i|^2}\right)\mu$$

for any $i = 1,\dots,n$, and for any $\varepsilon > 0$.

These conditions allow us to consider the weighted unipolar Hardy inequality wHi with respect any single pole a_i, $i = 1, \dots, n$,

$$c \int_{\mathbb{R}^N} \frac{\varphi^2}{|x - a_i|^2} \, d\mu \leq \int_{\mathbb{R}^N} |\nabla \varphi|^2 \, d\mu + k_1 \int_{\mathbb{R}^N} \varphi^2 \, d\mu,$$

for any function $\varphi \in H^1_\mu$, where $c \in (0, c_o(N + k_2)]$ with $c_o(N + k_2) = \left(\frac{N+k_2-2}{2}\right)^2$. Such an estimate plays a fundamental role in the proof of the multipolar Hardy inequality.

The statement of our inequality is the following.

Theorem 3.12. *Assume hypotheses* $H_2)$, $H_3'')$, $H_4)$ *and* $H_5)$. *Let* $N \geq 3$, $n \geq 2$ *and* $r_0 = \min_{i \neq j} |a_i - a_j|/2$, $i, j = 1, \dots, n$. *Then there exists a constant* $k_0 \in [0, \pi^2)$ *such that*

$$\begin{aligned} c \int_{\mathbb{R}^N} \sum_{i=1}^n \frac{\varphi^2}{|x - a_i|^2} \, d\mu \leq & \int_{\mathbb{R}^N} |\nabla \varphi|^2 \, d\mu \\ & + \left[\frac{k + (n+1)c}{r_0^2} + k_1 \right] \int_{\mathbb{R}^N} \varphi^2 \, d\mu, \end{aligned} \tag{3.24}$$

for all $\varphi \in H^1_\mu$, *where* $c \in (0, c_o(N + k_2)]$ *with* $c_o(N + k_2) = \left(\frac{N+k_2-2}{2}\right)^2$.

In order to prove the Theorem via the IMS method, we need to recall the notion of partition of unity and some related lemmas.

We say that a finite family $\{J_i\}_{i=1}^{n+1}$ of real valued functions $J_i \in W^{1,\infty}(\mathbb{R}^N)$ is a *partition of unity* in $\mathbb{R}^N$ if $\sum_{i=1}^{n+1} J_i^2 = 1$.
Any family of this type has the following properties:

(a) $\sum_{i=1}^{n+1} J_i \partial_\alpha J_i = 0$ for any $\alpha = 1, \dots, N$;

(b) $J_{n+1} = \sqrt{1 - \sum_{i=1}^n J_i^2}$;

(c) $\sum_{i=1}^{n+1} |\nabla J_i|^2 \in L^\infty(\mathbb{R}^N)$.

Furthermore we require that

$$\Omega_i \cap \Omega_j = \emptyset \quad \text{for any } i, j = 1, \dots, n, \ i \neq j, \tag{3.25}$$

where $\overline{\Omega}_i = \text{supp}(J_i)$, $i = 1, \dots, n$. By the property (a) we get

$$\sum_{\alpha=1}^N |J_{n+1} \partial_\alpha J_{n+1}|^2 = \sum_{\alpha=1}^N \left| \sum_{j=1}^n J_j \partial_\alpha J_j \right|^2 = \sum_{\alpha=1}^N \sum_{j=1}^n |J_j \partial_\alpha J_j|^2,$$

from which

$$|\nabla J_{n+1}|^2 = \sum_{i=1}^{n} \frac{J_i^2}{1 - J_i^2} |\nabla J_i|^2.$$

As a consequence we obtain an explicit formula for the sum of the gradients:

(d) $\sum_{i=1}^{n+1} |\nabla J_i|^2 = \sum_{i=1}^{n} |\nabla J_i|^2 + \sum_{i=1}^{n} \frac{J_i^2}{1-J_i^2} |\nabla J_i|^2 = \sum_{i=1}^{n} \frac{|\nabla J_i|^2}{1-J_i^2}$.

Note that to avoid a singularity for the gradient of J_{n+1} at the points where $1 - J_i^2 = 0$, from (d) we shall assume the additional constraint $|\nabla J_i|^2 = F(x)(1 - J_i^2)$, for $i = 1, \dots, n$ and for some $F \in L^\infty(\mathbb{R}^N)$.

By proceeding as in [10, Lemma 2], we are able to state the following result.

Lemma 3.13. *Let $\{J_i\}_{i=1}^{n+1}$ be a partition of unity satisfying (3.25). For any $\varphi \in H^1_\mu$ and any $V \in L^1_{loc}(\mathbb{R}^N)$ we get*

$$\int_{\mathbb{R}^N} \left(|\nabla \varphi|^2 - V\varphi^2\right) d\mu = \sum_{i=1}^{n+1} \int_{\mathbb{R}^N} (|\nabla (J_i \varphi)|^2 - V(J_i\varphi)^2) d\mu - \int_{\mathbb{R}^N} \sum_{i=1}^{n+1} |\nabla J_i|^2 \varphi^2 \, d\mu.$$

Proof. We can immediately observe that

$$\int_{\mathbb{R}^N} V \left(\sum_{i=1}^{n+1} (J_i\varphi)^2\right) d\mu = \int_{\mathbb{R}^N} V \left(\sum_{i=1}^{n+1} J_i^2\right) \varphi^2 \, d\mu = \int_{\mathbb{R}^N} V\varphi^2 \, d\mu. \tag{3.26}$$

On the other hand,

$$
\begin{aligned}
\sum_{i=1}^{n+1} |\nabla (J_i\varphi)|^2 &= \sum_{i=1}^{n+1} |(\nabla J_i)\varphi + (\nabla\varphi)J_i|^2 \\
&= \sum_{i=1}^{n+1} |\nabla J_i|^2\varphi^2 + \sum_{i=1}^{n+1} |\nabla\varphi|^2 J_i^2 \\
&\quad + 2\sum_{i=1}^{n+1}(J_i\nabla J_i)(\varphi\nabla\varphi) \\
&= \sum_{i=1}^{n+1} |\nabla J_i|^2\varphi^2 + |\nabla\varphi|^2 + \left(\sum_{i=1}^{n+1} J_i\nabla J_i\right)\nabla\varphi^2.
\end{aligned} \tag{3.27}
$$

By property (a) it follows that $\left(\sum_{i=1}^{n+1} J_i\nabla J_i\right)\nabla\varphi^2 = 0$, then by integrating (3.27) on $\mathbb{R}^N$ we obtain

$$
\int_{\mathbb{R}^N} |\nabla\varphi|^2\, d\mu = \int_{\mathbb{R}^N} \sum_{i=1}^{n+1} |\nabla (J_i\varphi)|^2\, d\mu - \int_{\mathbb{R}^N} \sum_{i=1}^{n+1} |\nabla J_i|^2\varphi^2\, d\mu. \tag{3.28}
$$

From (3.26) and (3.28) we get the result. □

In the following we set

$$
V_n(x) = \sum_{i=1}^{n} \frac{1}{|x-a_i|^2}.
$$

We recall a preliminary Lemma, stated in [10], about the case $n=2$, with $a_1 = a$, $a_2 = -a$ and $0 < r_0 \le |a|$.

Lemma 3.14. *There is a partition of the unity $\{J_i\}_{i=1}^3$ satisfying (3.25) with $J_1 \equiv 1$ on $B(a, \frac{r_0}{2})$, $J_1 \equiv 0$ on $B(a, r_0)^c$, $J_2(x) = J_1(-x)$ for any $x \in \mathbb{R}^N$, $0 < r_0 \le |a|$, such that, for any $c > 0$, there exists a constant $k_0 \in [0, \pi^2)$ for which, almost everywhere for all $x \in \Omega := \mathrm{supp}(J_1) \cup \mathrm{supp}(J_2)$, we have*

$$
\sum_{i=1}^{3} |\nabla J_i|^2 + c\, J_3^2\, V_2(x) = \sum_{i=1,2} \frac{|\nabla J_i|^2}{1-J_i^2} + c\, J_3^2\, V_2(x) \le \frac{k_0 + 2c}{r_0^2}. \tag{3.29}
$$

As observed in [10], a partition of unity satisfying the hypotheses of Lemma 3.14 is given by setting

$$J(t) := \begin{cases} 1 & \text{if } t \leq 1/2 \\ \sin(\pi t) & \text{if } 1/2 \leq t \leq 1 \\ 0 & \text{if } t \geq 1 \end{cases} \tag{3.30}$$

and defining $J_1(x) := J(|x-a|/r_0)$, $J_2(x) := J(|x+a|/r_0)$, and $J_3(x) := \sqrt{1 - J_1^2 - J_2^2}$.

Now we are able to proceed with the proof of inequality (3.24).

Proof of Theorem 3.12. Let us define the following quadratic form

$$Q[\varphi] := \int_{\mathbb{R}^N} \left(|\nabla \varphi|^2 - cV_n(x)\varphi^2 \right) d\mu, \qquad \varphi \in H^1_\mu, \tag{3.31}$$

where $V_n(x) = \sum_{i=1}^n \frac{1}{|x-a_i|^2}$.

Consider a partition of unity $\{J_i\}_{i=1}^{n+1}$ satisfying (3.25) such that $J_i(x) = J(|x-a_i|/r_0)$ for all $x \in \mathbb{R}^N$, $i = 1, \dots, n$, with J as in (3.30), $\text{supp}(J_i) = \overline{\Omega}_i$, where $\Omega_i = B(a_i, r_0)$. Moreover set $\Omega := \cup_{i=1}^n \overline{\Omega}_i$ and $\Gamma := \mathbb{R}^N \setminus \Omega$. Then $|x - a_i| \geq r_0$ in $\overline{\Omega}_j$ for $i \neq j$, and $V_n(x) \leq \frac{n}{r_0^2}$ on Γ.

By virtue of Lemma 3.13 we are able to write (3.31) as follows

$$Q[\varphi] = \sum_{i=1}^n Q[J_i \varphi] + R_n, \qquad \varphi \in H^1_\mu, \tag{3.32}$$

where

$$R_n = \int_{\mathbb{R}^N} |\nabla(J_{n+1}\varphi)|^2 \, d\mu - c \int_{\mathbb{R}^N} V_n |J_{n+1}\varphi|^2 \, d\mu - \sum_{i=1}^{n+1} \int_{\mathbb{R}^N} |\nabla J_i|^2 \varphi^2 \, d\mu.$$

Thanks to the property (d) we have

$$\begin{aligned} R_n = & \int_{\mathbb{R}^N} |\nabla(J_{n+1}\varphi)|^2 \, d\mu - c \int_{\mathbb{R}^N} V_n \left(1 - \sum_{i=1}^n J_i^2 \right) \varphi^2 \, d\mu \\ & - \sum_{i=1}^n \int_{\mathbb{R}^N} \frac{|\nabla J_i|^2}{1 - J_i^2} \varphi^2 \, d\mu \\ \geq & -c \int_{\mathbb{R}^N} V_n \left(1 - \sum_{i=1}^n J_i^2 \right) \varphi^2 \, d\mu - \sum_{i=1}^n \int_{\mathbb{R}^N} \frac{|\nabla J_i|^2}{1 - J_i^2} \varphi^2 \, d\mu. \end{aligned}$$

Moreover, using the condition (3.25) we get

$$R_n \geq -\sum_{i=1}^{n} \int_{\Omega_i} \left[\frac{|\nabla J_i|^2}{1-J_i^2} + c\left(1-J_i^2\right) V_n(x) \right] \varphi^2 \, d\mu - \frac{c\, n}{r_0^2} \int_{\Gamma} \varphi^2 \, d\mu.$$

For every $i = 1, \ldots, n$ we can apply Lemma 3.14 on Ω_i with $(a_i, a_j) = (-a, a)$ up to a change of coordinates for some $j \neq i$. Considering the partition $\left\{ J_i, J_j, \sqrt{1 - J_i^2 - J_j^2} \right\}$ and taking into account that $J_j \equiv 0$ on Ω_i, we get

$$\begin{aligned} R_n \geq& -\sum_{i=1}^{n} \int_{\Omega_i} \left[\frac{k_0 + 2c}{r_0^2} + c(1 - J_i^2) \left(\sum_{k \neq i,j} \frac{1}{|x - a_k|^2} \right) \right] \varphi^2 \, d\mu \\ &- \frac{c\, n}{r_0^2} \int_{\Gamma} \varphi^2 \, d\mu \\ \geq& -\sum_{i=1}^{n} \int_{\Omega_i} \left[\frac{k_0 + 2c}{r_0^2} + \frac{(n-2)c}{r_0^2} (1 - J_i^2) \right] \varphi^2 \, d\mu \\ &- \frac{c\, n}{r_0^2} \int_{\Gamma} \varphi^2 \, d\mu, \end{aligned} \tag{3.33}$$

where $k_0 \in [0, \pi^2)$ such that, since we can bound $\frac{1}{|x-a_k|^2}$ by $\frac{1}{r_0^2}$ for all $k \neq i, j$. Taking into account (3.31) and using the unipolar Hardy inequality (3.3), which holds under our assumptions with respect to each pole $a_i \in \mathbb{R}^N$, $i = 1, \ldots, n$, we obtain

$$\begin{aligned} Q[J_i \varphi] &= \int_{\mathbb{R}^N} |\nabla J_i \varphi|^2 \, d\mu - c \int_{\mathbb{R}^N} \left(\frac{1}{|x - a_i|^2} + \sum_{\substack{j=1 \\ j \neq i}}^{n} \frac{1}{|x - a_j|^2} \right) |J_i \varphi|^2 \, d\mu \\ &\geq - \left[k_1 + \frac{(n-1)c}{r_0^2} \right] \int_{\Omega_i} |J_i \varphi|^2 \, d\mu, \end{aligned}$$

from which

$$\sum_{i=1}^{n} Q[J_i \varphi] \geq -k_1 \sum_{i=1}^{n} \int_{\Omega_i} \varphi^2 \, d\mu - \frac{(n-1)c}{r_0^2} \sum_{i=1}^{n} \int_{\Omega_i} J_i^2 \varphi^2 \, d\mu \tag{3.34}$$

From (3.32), (3.33) and (3.34) we deduce

$$\begin{aligned} Q[\varphi] \geq& -\sum_{i=1}^{n} \int_{\Omega_i} \left[\frac{k_0 + 2c}{r_0^2} + \frac{(n-2)c}{r_0^2} (1 - J_i^2) + k_1 + \frac{(n-1)c}{r_0^2} J_i^2 \right] \varphi^2 \, d\mu \\ &- \frac{c\, n}{r_0^2} \int_{\Gamma} \varphi^2 \, d\mu. \end{aligned}$$

Since

$$k_0 + 2c + c(n-2)(1 - J_i^2) + c(n-1)J_i^2 = k_0 + cn + cJ_i^2 \leq k_0 + c(n+1),$$

we finally obtain

$$\begin{aligned} Q[\varphi] &\geq -\left[\frac{k_0 + (n+1)c}{r_0^2} + k_1\right] \int_\Omega \varphi^2 \, d\mu - \frac{c\, n}{r_0^2} \int_\Gamma \varphi^2 \, d\mu \\ &\geq -\left[\frac{k_0 + (n+1)c}{r_0^2} + k_1\right] \int_{\mathbb{R}^N} \varphi^2 \, d\mu, \end{aligned}$$

from which we get inequality (3.24). □

Appendix A

Sesquilinear forms and associated operators

We recall some results and terminology on sesquilinear forms and their associated operators on Hilbert spaces. The reference for this Appendix is [51].

A.1 Bounded sesquilinear forms

Let H be a Hilbert space over $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$ endowed with the inner product $(\cdot, \cdot)$. Let us denote with $\| \cdot \|$ the norm on H corresponding to the inner product. A *sesquilinear form* a on H is an application from $H \times H$ into $\mathbb{K}$ such that for avery $\alpha \in \mathbb{K}$ and $u, v \in H$

$$
\begin{aligned}
a(\alpha u + v, h) &= \alpha\, a(u, h) + a(v, h), \\
a(u, \alpha v + h) &= \bar{\alpha}\, a(u, v) + a(u, h).
\end{aligned}
$$

Here $\bar{\alpha}$ denotes the conjugate number of α. Of course, if $\mathbb{K} = \mathbb{R}$ then $\bar{\alpha} = \alpha$ and a is a bilinear form.

Definition A.1. A sesquilinear form $a : H \times H \to \mathbb{K}$ is *continuous* if there exists a constant M such that

$$|a(u, v)| \le M \|u\| \|u\| \text{ for all } u, v \in H.$$

Every continuous sesquilinear form can be represented by a unique bounded linear operators, as stated by the following Proposition.

Proposition A.2. *Let $a : H \times H \to \mathbb{K}$ be a continuous sesquilinear form. There exists a unique bounded linear operator T acting on H such that*

$$a(u, v) = (Tu, v) \ \textit{for all} \ u, v \in H.$$

The bounded operator T is called *the operator associated with a.*

A.2 Unbounded sesquilinear forms and their associated operators

Let us consider a sesquilinear form a which do not act on the whole space H, but only on a linear subspace $D(a)$ of H. The map $a : D(a) \times D(a) \to \mathbb{K}$ is called *unbounded sesquilinear form*, and $D(a)$ is called *the domain of a.*

Definition A.3. Let $a : D(a) \times D(a) \to \mathbb{K}$ be a sesquilinear form. We say that:

(i) a is *densely defined* if $D(a)$ is dense in H;

(ii) a is *accretive* if $\Re a(u, u) \geq 0$ for all $u \in D(a)$;

(iii) a is *continuous* if there exists a non-negative constant M such that

$$|a(u, v)| \leq M\|u\|_a\|v\|_a \quad \text{for all} \ u, v \in D(a),$$

where $\|u\|_a := \sqrt{\Re a(u, u) + \|u\|^2}$;

(iv) a is *closed* if $(D(a), \|\cdot\|_a)$ is a complete space

If a satisfies $(i) - (iv)$, one checks easily that $\|\cdot\|_a$ is a norm on $D(a)$. It is called the *norm associated with the form a.*

Moreover, we could consider forms that are merely bounded from below, that is,

$$\Re a(u, u) \geq -\gamma\,(u, u)\,\text{for all} \ u, v \in D(a)$$

for some positive constant γ. The general theory of these forms (sometimes called *quasi-accretive* forms) does not differ much from that of accretive ones. A simple perturbation argument (which consists of considering the form $a+\gamma$, defined as $(a+\gamma)(u,v) := a(u,u)+\gamma\,(u,v))$ for all $u,v \in D(a))$ allows us, for simplicity, to consider only accretive forms.

It may happen in some in some situations that a sesquilinear form a satisfies $(i)-(iii)$ but not (iv). In this case, one can try to find an extension of a which is a closed form and acts on a subspace of H.

Definition A.4. A densely defined accretive sesquilinear form a is called *closable* if there exists a closed accretive form c, acting on a subspace $D(c)$ of H, such that $D(a) \subseteq D(c)$ and $a(u,v) = c(u,v)$ for all $u,v \in D(a)$.

A closed extension, when it exists, is not unique in general. Nevertheless, in that case, one can define the smallest closed extension $\bar{a}$. One way to do this is to define $\bar{a}$ as follows:

$$D(\bar{a}) := \{u \in H \text{ s.t. } \exists u_n \in D(a) : u_n \to u \text{ (in } H) \text{ and } \\ a(u_n - u_m, u_n - u_m) \to 0 \text{ as } n,m \to \infty\}$$

and

$$\bar{a}(u,v) := \lim_{n\to\infty} a(u_n, v_n) \tag{A.1}$$

for $u,v \in D(\bar{a})$, where $(u_n)_n$ and $(v_n)_n$ are sequences of elements of $D(a)$ which converge respectively to u and v (with respect the norm of H) and satisfy $a(u_n - u_m, u_n - u_m) \to 0$ and $a(v_n - u_m, v_n - u_m) \to 0$ as $n,m \to \infty$.

One can prove that if a densely defined, accretive and continuous sesquilinear form a is closable, then $\bar{a}$ is well defined and and satisfies $(i)-(iv)$. In addiction, every closed extension of a is also an extension of $\bar{a}$.

Definition A.5. If the form a is closable, then $\bar{a}$ define in (A.1) with domain $D(\bar{a})$ is called the *closure* of the form a.

Let a be a densely defined, accretive, continuous, and closed sesquilinear form on H. One can define an unbounded operator A, called the *operator associate with* a with domain the following linear subspace of H:

$$D(A) = \{u \in H \text{ s. t. } \exists v \in H : a(u,\phi) = (v,\phi) \quad \forall\, \phi \in D(a)\}.$$

There are several important properties of operators which are associated with sesquilinear forms. For example, if A is the operator associated with a densely defined, accretive, continuous, and closed sesquilinear form a, then A is densely defined and for every $\lambda > 0$ the operator $\lambda I + A$ is invertible (from $D(A)$ into H) and its inverse $(\lambda I + A)^{-1}$ is a bounded operator on H (here I is the identity operator). Moreover one can see that if a is symmetric then A is self-adjoint.

We conclude this Appendix by giving the definition of *sectorial form*.

Definition A.6. A sesquilinear form $a : D(a) \times D(a) \mapsto \mathbb{C}$, acting on a complex Hilbert space H, is called *sectorial* if there exists a non-negative constant C such that

$$\Im a(u,u) \leq C \Re(u,u) \quad \text{for all} \;\; u \in D(a).$$

One can show that every sectorial form acting on a complex Hilbert space H is continuous.

Appendix B

Operator semigroups and invariant measures

Here we give some basics of the theory of operator semigroups on Banach spaces: we introduce the definitions of strongly continuous semigroup, infinitesimal generator, positivity on Banach spaces and the main results of the theory, as the Hille-Yosida Theorem. Notions, results and notation we lists are taken from [5, 23, 31, 50, 51].

Moreover, at the end of this Appendix we deal with a more advanced tool: the theory of invariant measures. The references for this part are [42, 41].

B.1 Strongly continuous operator semigroups

We report below the main definitions and the fundamental theorems of the Semigroup Theory. In particular, we focus on *strongly continuous semigroup.*

Definition B.1. A family $\{T(t)\}_{t\geq 0}$ of bounded linear operators on a Banach space X is called a *(one-parameter) semigroup* on X, if

$$T(0) = I \quad \text{and} \quad T(t+s) = T(t)T(s) \quad \text{for all } t, s \geq 0.$$

A semigroup $\{T(t)\}_{t\geq 0}$ is called *strongly continuous*, or a C_0*-semigroup*, if, for every $u \in X$ it holds $t \mapsto T(t)u$ is a continuous map from $[0, +\infty)$ to X.

Moreover, a strongly continuous semigroup $\{T(t)\}_{t\geq 0}$ is called a *contraction semigroup* if

$$\|T(t)\|_{\mathcal{L}(X)} \leq 1 \quad \text{for all } t \geq 0,$$

or, equivalently, if

$$\|T(t)u\| \leq \|u\| \quad \text{for all } t \geq 0,\ u \in X.$$

Strongly continuous semigroups satisfy the following property.

Proposition B.2. *Let $\{T(t)\}_{t\geq 0}$ a strongly continuous semigroup on X. Then there are $M \geq 1$ and $\omega \in \mathbb{R}$ such that*

$$\|T(t)\|_{\mathcal{L}(X)} \leq Me^{\omega t} \quad \textit{for all } t \geq 0.$$

Definition B.3. Let $\{T(t)\}_{t\geq 0}$ a strongly continuous semigroup on X. The *(infinitesimal) generator* of $\{T(t)\}_{t\geq 0}$ is the operator $A : D(A) \mapsto X$ defined as follows:

$$Au := \lim_{t\to 0^+} \frac{T(t)u - u}{t}, \quad \text{for all } u \in D(A),$$

where

$$D(A) = \{u \in X : \exists \lim_{t\to 0^+} \frac{T(t)u - u}{t} \text{ in } X\}.$$

The following Proposition state some important properties of the generator of a strongly continuous semigroup.

Proposition B.4. *Let A the generator of a strongly continuous semigroup $\{T(t)\}_{t\geq 0}$ on X. Then the domain $D(A)$ satisfies*

$$T(t)D(A) \subseteq D(A) \quad \textit{and} \quad AT(t)u = T(t)Au, \quad \textit{for all } t \geq 0, u \in D(A).$$

Moreover, for all $t \geq 0$ and $u \in D(A)$ the map $t \mapsto T(t)u$ is differentiable and

$$\frac{d}{dt}T(t)u = AT(t)u.$$

Generators of strongly continuous semigroups play an important role in the context of abstract Cauchy problems, as stated in the following Theorem.

Theorem B.5. *Let A be the generator of a strongly continuous semigroup $\{T(t)\}_{t\geq 0}$. Then, for any $u_0 \in D(A)$, the function*

$$u : t \in [0, +\infty) \mapsto u(t) = T(t)u_0 \in D(A),$$

is the unique classical solution of the abstract Cauchy problem

$$\begin{cases} \frac{du}{dt} = Au(t), & t \geq 0, \\ u(0) = u_0, \end{cases}$$

which belongs to $C([0, +\infty); D(A)) \cap C^1([0, +\infty); X)$.

Another important property of the generator is the following.

Theorem B.6. *The generator of a strongly continuous semigroup is a closed and densely defined linear operator that determines the semigroup uniquely.*

We define the *resolvent set* $\rho(A)$ and the *resolvent operator* $R(\lambda, A)$ of a linear operator $(A, D(A))$ as follows:

$$\rho(A) := \{\lambda \in \mathbb{C} : (\lambda I - A)^{-1} \in \mathcal{L}(X)\},$$
$$R(\lambda, A) := (\lambda I - A)^{-1}.$$

The next result is known as the *Hille-Yosida Generation Theorem* (1948).

Theorem B.7. *For a linear operator $(A, D(A))$ on a Banach space X, the following properties are equivalent.*

(i) $(A, D(A))$ *generates a strongly continuous contraction semigroup.*

(ii) $(A, D(A))$ *is closed, densely defined, and for every $\lambda > 0$ one has*

$$\lambda \in \rho(A) \quad \textit{and} \quad \|\lambda R(\lambda, A)\|_{\mathcal{L}(X)} \leq 1.$$

Moreover, as a consequence of the Hille-Yosida Theorem, if the operator A is bounded on X, then it generates a strongly continuous semigroup given by

$$e^{tA} = \sum_{n=0}^{\infty} \frac{t^n}{n!} A^n.$$

By analogy to this case, the strongly continuous semigroup generated by the operator A is denoted by $\{e^{tA}\}_{t\geq 0}$ even when A is not bounded.

The following result is known as the *Trotter-Neveu-Kato Theorem.*

Theorem B.8. *Let A_n, $n \in \mathbb{N}_0$, generate a strongly continuous semigroup $\{T_n(t)\}$ on a Banach space X such that*

$$\|T_n(t)\|_{\mathcal{L}(X)} \leq Me^{\omega t}, \quad n \in \mathbb{N}_0,\, t \geq 0$$

with M, ω are independent of n, t. Then

(i) $\lim_{n\to\infty} T_n(t)x = T_0(t)x$ *for each* $t \geq 0$, $x \in X$ *implies* $\lim_{n\to\infty}(\lambda - A_n)^{-1}x = (\lambda - A_0)^{-1}x$ *for each* $x \in X$, *uniformly for* λ *in compact subsets of* (ω, ∞)*;*

(ii) $\lim_{n\to\infty}(\lambda - A_n)^{-1}x = (\lambda - A_0)^{-1}x$ *for each* $x \in X$, $\lambda > \omega$ *implies* $\lim_{n\to\infty} T_n(t)x = T_0(t)x$ *for each* $x \in X$, *uniformly for* t *in compact subsets of* $[0, \infty)$.

Now we introduce a particular type of operator semigroups, the *analytic semigroups*. To this aim we need to give the definition of *sectorial operator*.

Definition B.9. Let X be a complex Banach space. A linear operator $A : D(A) \subset X \mapsto X$ is called *sectorial* (of angle δ) if there exists $\delta \in (0, \frac{\pi}{2}]$ such that

(i) $\rho(A) \supset \Sigma_{\frac{\pi}{2}+\delta} := \{\lambda \in \mathbb{C} : |\arg\lambda| < \frac{\pi}{2} + \delta\} \setminus \{0\}$;

(ii) for each $\varepsilon \in (0, \delta)$ there exists $M_\varepsilon \geq 1$ such that $\|R(\lambda, A)\|_{\mathcal{L}(X)} < \frac{M_\varepsilon}{|\lambda|}$, for all $0 \neq \lambda \in \overline{\Sigma}_{\frac{\pi}{2}+\delta-\varepsilon}$.

Analytic semigroups are defined as follows.

Definition B.10. A family of operators $\{T(z)\}_{z\in\Sigma_\delta\cup\{0\}} \subset \mathcal{L}(X)$ is called an *analytic semigroup* (of angle $\delta \in (0, \frac{\pi}{2})$) if

(i) $T(0) = I$ and $T(z_1 + z_2) = T(z_1)T(z_2)$ for all z_1, $z_2 \in \Sigma_\delta$;

(ii) The map $z \mapsto T(z)$ is analytic in Σ_δ;

(iii) $\lim_{\Sigma_{\delta'}\ni z\to 0} T(z)x = x$ for all $x \in X$ and $0 < \delta' < \delta$.

If, in addition,

(iv) $T(z)$ is bounded in $\Sigma_{\delta'}$ for every $0 < \delta' < \delta$, we call $\{T(z)\}_{z\in\Sigma_\delta\cup\{0\}}$ a *bounded analytic semigroup.*

One of the properties which characterize the generator of an analytic semigroup is stated in the following Theorem.

Theorem B.11. *For an operator* $(A, D(A))$ *on a Banach space* X*, the following statements are equivalent.*

(i) A *generates a bounded analytic semigroup* $\{T(z)\}_{z\in\Sigma_\delta\cup\{0\}}$ *on* X*;*

(ii) A *is densely defined and sectorial.*

Moreover, the analyticity of a semigroup is preserved under bounded perturbation.

B.2 Sesquilinear forms and generation of semigroups

The operator associated with a sectorial form is a sectorial operator and the converse holds too, as stated in the following Proposition.

Proposition B.12. *Let* a *be a densely defined, accretive, continuous, and closed sesquilinear form acting on a complex Hilbert space* H*. Denote by* A *the operator associated with* a *on* H*. The following assertions are equivalent:*

(i) a *is a sectorial form.*

(ii) A *is a sectorial operator.*

We recall that an unbounded linear operator $A : D(A) \to H$ on an Hilbert space H is *accretive* if $\Re(Au, u) \geq 0$ for every $u \in D(A)$. An accretive operator is called *m-accretive* if the range of $I + A$ is the whole space H.

The following result is a particular case of the well-known Lumer-Phillips Theorem for generators of contraction semigroups.

Theorem B.13. *Let* A *be a densely defined operator on* H*. The following assertions are equivalent.*

1. *The operator A is closable and $-\bar{A}$ is the generator of a strongly continuous contraction semigroup on H.*

2. *$\bar{A}$ is m-accretive.*

3. *A is accretive and there exists $\lambda > 0$ such that $\lambda I + A$ has dense range.*

One can easily see that the operator associated with a densely defined, accretive, continuous, and closed sesquilinear form is m-accretive. As a consequence, we get the following result.

Corollary B.14. *Let A be the operator associated with a densely defined, accretive, continuous, and closed sesquilinear form a on H, then $-A$ is the generator of a strongly continuous contraction semigroup on H.*

Therefore, we can associate to the form a the semigroup $\{e^{-tA}\}_{t\geq 0}$.

The following result states that, under appropriate conditions, the vice versa also holds.

Proposition B.15. *Let A be an m-accretive operator on a complex Hilbert space H. If the $I + A$ is sectorial then there exists a unique sesquilinear form a which is densely defined, accretive, continuous, and closed and such that A is the operator associated with a.*

B.3 Positive semigroups

Let us consider the Banach space $X := L^p(\Omega, d\mu)$, for $1 \leq p < \infty$, where Ω is an open subset of $\mathbb{R}^N$. A function f in X is called *positive* (in symbols: $0 \leq f$) if

$$0 \leq f(s) \quad \text{for (almost) all } \; s \in \Omega.$$

For real-valued functions $f, g \in X$ we write if $0 \leq g - f$ and obtain an ordering making (the real part of) X into a vector lattice. To indicate that $0 \leq f$ and $0 \neq f$ we use the notation $0 < f$. Moreover, for an arbitrary (complex-valued) function $f \in X$ we define its absolute value $|f|$ as

$$|f|(s) = |f(s)| \quad \text{for } s \in \Omega.$$

Recalling the definition of the norm on X, we see that

$$|f| \leq |g| \quad \text{implies} \quad \|f\| \leq \|g\| \quad \text{for all } f, g \in X.$$

These properties make the space X a *Banach lattice*.

Definition B.16. A strongly continuous semigroup $\{T(t)_{t\geq 0}\}$ on a Banach lattice X is called *positive* (or *positivity preserving*) if each operator $T(t)$ is positive, i.e., if

$$0 \leq f \in X \quad \text{implies} \quad 0 \leq T(t)f \qquad \text{for each } t \geq 0,$$

or equivalently, if

$$|T(t)f| \leq T(t)|f| \quad \text{holds for each } f \in X,\, t \geq 0.$$

Various characterizations of generators of positive semigroups can be found in [5], [23] and [50]. Moreover we recall the following result, known as the *bounded perturbation theorem for positive semigroup.*

Theorem B.17. *Let A be the generator of a positive strongly continuous semigroup $\{T(t)\}_{t\geq 0}$ and let $B \in \mathcal{L}(X)$ be a positive operator on the Banach lattice X. Then $A + B$ generates a positive semigroup $\{S(t)\}_{t\geq 0}$ such that $0 \leq T(t) \leq S(t)$ for all $t \geq 0$.*

B.4 Invariant measures

Let $\mathcal{A}$ be a differential operator defined on smooth functions by

$$\mathcal{A}u(x) = \sum_{i,j=1}^{N} q_{ij}(x) D_{ij} u(x) + \sum_{i=1}^{N} b_i(x) D_i u(x) + c(x)u(x), \quad x \in \mathbb{R}^N. \tag{B.1}$$

We assume the following hypotheses on the coefficients of $\mathcal{A}$:

(i) $q_{ij} \equiv q_{ji}$ for any $i, j = 1, \ldots, N$ and

$$\sum_{i,j=1}^{N} q_{ij}(x)\xi_i\xi_j \geq \kappa(x)|\xi|^2, \quad \kappa(x) > 0, \xi, x \in \mathbb{R}^N;$$

(ii) q_{ij}, b_i $(i, j = 1, \ldots, N)$ and c belong to $C^{0,\alpha}_{loc}(\mathbb{R}^N)$ for some $\alpha \in (0, 1)$;

(iii) there exists $c_0 \in \mathbb{R}^N$ such that

$$c(x) \leq c_0, \quad x \in \mathbb{R}^N.$$

Here $C^{0,\alpha}_{loc}(\mathbb{R}^N)$ denotes the set of the α-Hölder continuous functions in any compact subset of $\mathbb{R}^N$, for some $\alpha \in (0, 1)$.

We introduce a realization A of $\mathcal{A}$ in $C_b(\mathbb{R}^N)$, the space of bounded and continuous functions in $\mathbb{R}^N$, with domain $D_{max}(\mathcal{A})$, defined as follows:

$$D_{max}(\mathcal{A}) := \{u \in C_b(\mathbb{R}^N) \cap W^{2,p}_{loc}(\mathbb{R}^N) \text{ for all } 1 < p < \infty : \mathcal{A}u \in C_b(\mathbb{R}^N)\},$$

$$Au = \mathcal{A}u.$$

The following Theorem shows how it is possible to associate a semigroup of linear operators in $C_b(\mathbb{R}^N)$ to $\mathcal{A}$.

Theorem B.18. *For any $f \in C_b(\mathbb{R}^N)$ there exists a classical solution to the problem*

$$\begin{cases} u_t(t, x) = \mathcal{A}u(t, x), & t > 0,\, x \in \mathbb{R}^N, \\ u(0, x) = f(x). \end{cases}$$

Moreover, there exists a semigroup $\{T(t)\}_{t\geq 0}$ defined in $C_b(\mathbb{R}^N)$ such that, for any $f \in C_b(\mathbb{R}^N)$, the solution of the problem is represented by

$$u(t, x) = (T(t)f)(x), \quad t \geq 0,\, x \in \mathbb{R}^N.$$

In general, the semigroup $\{T(t)\}_{t\geq 0}$ is not strongly continuous. As a consequence, we cannot define its generator in the usual sense.

Nevertheless, we can associate to this semigroup the so-called *weak generator*, defined as follows

$$(\hat{A}f)(x) = \lim_{t\to 0^+} \frac{T(t)f(x) - f(x)}{t}, \quad x \in \mathbb{R}^N, \quad f \in D(\hat{A})$$

where

$$D(\hat{A}) = \left\{ f \in C_b(\mathbb{R}^N) : \sup_{t\in(0,1)} \frac{\|T(t)f(x) - f(x)\|_\infty}{t} < \infty \text{ and } \exists g \in C_b(\mathbb{R}^N) : \right.$$
$$\left. \lim_{t\to 0^+} \frac{T(t)f(x) - f(x)}{t} = g(x)\, \forall x \in \mathbb{R}^N \right\}.$$

In general $D(\hat{A})$ is a proper subset of $D_{max}(\mathcal{A})$. It can be shown that $(A, D(\mathcal{A})) = (\hat{A}, D(\hat{A}))$ if and only if $(c_0, +\infty) \subset \rho(A)$, or, equivalently, if and only if there exists $\lambda > c_0$ such that $\lambda \in \rho(A)$.

Henceforth we consider, for simplicity, $c(x) \equiv 0$ in (B.1). Under this assumption we give the definition of *invariant measure*.

Definition B.19. Let $B_b(\mathbb{R}^N)$ the set of the bounded Borel measurable functions. We say that a Borel probability measure $d\mu$ is an *invariant measure* for $\{T(t)\}_{t\geq 0}$ if

$$\int_{\mathbb{R}^N} T(t)f\, d\mu = \int_{\mathbb{R}^N} f\, d\mu \quad \forall\, f \in B_b(\mathbb{R}^N).$$

The following Lemma give us an equivalent definition of invariant measure.

Lemma B.20. *A Borel probability measure $d\mu$ such that*

$$\int_{\mathbb{R}^N} T(t)f\, d\mu = \int_{\mathbb{R}^N} f\, d\mu, \quad t > 0,$$

for any $f \in C_c^\infty(\mathbb{R}^N)$ is an invariant measure for $\{T(t)\}_{t\geq 0}$.

The following result provides a characterization of the invariant measures.

Proposition B.21. *A Borel probability measure $d\mu$ is an invariant measure for $\{T(t)\}_{t\geq 0}$ if and only if*

$$\int_{\mathbb{R}^N} \hat{A}f\, d\mu = 0 \quad \forall f \in D(\hat{A}).$$

Another interesting property is stated in the following Theorem.

Theorem B.22. *If $d\mu$ is an invariant measure for $\{T(t)\}_{t\geq 0}$, then $C_c^\infty(\mathbb{R}^N)$ is dense in $L^p_\mu = L^p(\mathbb{R}^N, d\mu)$ for any $p \in [1, +\infty)$.*

Moreover, the following uniqueness result holds.

Theorem B.23. *There exists at most one invariant measure for $\{T(t)\}_{t\geq 0}$.*

The result we report below represents one of the main tools provided by the theory of invariant measures.

Proposition B.24. *Let $d\mu$ be the invariant measure of the semigroup $\{T(t)\}_{t\geq 0}$ in $C_b(\mathbb{R}^N)$. For any $p \in [1, +\infty)$, $\{T(t)\}_{t\geq 0}$ extends to a C_0-semigroup of contractions in L^p_μ.*

The extended semigroup is still denoted by $\{T(t)\}_{t\geq 0}$, while its generator is denoted with A_p. We will simply write A for A_2.

Henceforth we consider the operator $\mathcal{A}$ given by

$$\mathcal{A}u(x) = \Delta u(x) - \langle \nabla U(x) + G(x), \nabla u(x)\rangle, \quad x \in \mathbb{R}^N, \tag{B.2}$$

on smooth functions, where U belongs to $C^{1,\alpha}_{loc}(\mathbb{R}^N)$, the set of the functions in $C^1(\mathbb{R}^N)$ with first-order derivatives α-Hölder continuous in any compact subset of $\mathbb{R}^N$, for some $\alpha \in (0,1)$, and G belongs to $C^1(\mathbb{R}^N, \mathbb{R}^N)$.

The following Theorem holds.

Theorem B.25. *Assume that the functions U and G in (B.2) satisfy the following conditions:*

(i) $e^{-U} \in L^1(\mathbb{R}^N)$;

(ii) $\mathrm{div} G = \langle G, \nabla U\rangle$ *and* $\int_{\mathbb{R}^N} |G(x)| e^{-U(x)}\, dx < -\infty$.

Then the probability measure

$$d\mu = K^{-1} e^{-U(x)}\, dx, \quad K = \int_{\mathbb{R}^N} e^{-U(x)}\, dx$$

is the invariant measure of $\{T(t)\}_{t\geq 0}$ in L^2_μ.

The following Proposition lists important properties of the extended semigroup and of its generator.

Proposition B.26. *Let H^1_μ the set of all the functions $f \in L^2_\mu$ having distributional derivative ∇f in $(L^2_\mu)^N$. Under the hypotheses of Theorem B.25 the following assertions hold:*

(a) $C_c^\infty(\mathbb{R}^N)$ is a core for A in L^2_μ;

(b) $D(A)$ is continuously and densely embedded in H^1_μ;

(c) $\int_{\mathbb{R}^N} \nabla f \cdot \nabla g \, d\mu = -\int_{\mathbb{R}^N} (Af) g \, d\mu$, for all $f \in D(A)$, $g \in H^1_\mu$;

(d) $T(t)L^2_\mu \subset H^1_\mu$, for all $t > 0$.

Regarding the assertion (a), we recall that a subspace D of the domain $D(A)$ of a linear operator A acting on a Banach space X, $A : D(A) \subseteq X \mapsto X$, is a *core* for A if D is dense in $D(A)$ for the graph norm $\|u\|_A = \|u\| + \|Au\|$, $u \in D(A)$.

From (a) and (b) it follows that $C_c^\infty(\mathbb{R}^N)$ is densely embedded in H^1_μ.

Moreover, we observe that the space H^1_μ is also the completion of $C_c^\infty(\mathbb{R}^N)$ in the norm

$$\|f\|^2_{H^1_\mu} := \|f\|^2_{L^2_\mu} + \||\nabla f|\|^2_{L^2_\mu}.$$

Finally, we consider the case when the operator $\mathcal{A}$ is given by

$$\mathcal{A}u(x) = \Delta u(x) - \langle \nabla U(x), \nabla u(x) \rangle, \quad x \in \mathbb{R}^N, \tag{B.3}$$

i.e. $G \equiv 0$ in (B.2). It is possible to prove that if $U \in C^2(\mathbb{R}^N)$ and

$$\sum_{i,j=1}^N D_{ij}U(x)\xi_i\xi_j \geq 0 \quad x, \xi \in \mathbb{R}^N,$$

then the domain of the generator of $\{T(t)\}_{t\geq 0}$ in L^2_μ is

$$D(A) = \left\{ u \in H^2_\mu := H^2(\mathbb{R}^N, d\mu) : \langle \nabla U, \nabla u \rangle \in L^2_\mu \right\}.$$

Moreover, if U is a convex function which goes to $+\infty$ as $|x|$ tends to $+\infty$, then the semigroup $\{T(t)\}_{t\geq 0}$ in L^2_μ is analytic.

Another useful result is the following.

Theorem B.27. *Let $U \in C^2(\mathbb{R}^N)$ and $e^{-U} \in L^1(\mathbb{R}^N)$. If the function U satisfies the condition*

$$\Delta U(x) \leq \delta |\nabla U(x)|^2 + M,$$

for some $\delta \in (0,1)$ and $M > 0$, then the domain of the operator defined in (B.3) is $D(A) = H^2_\mu$ and the graph-norm of $D(A)$ is equivalent to the H^2_μ-norm.

List of symbols

$\mathbb{N}$	set of natural numbers
$\mathbb{R}$	set of real numbers
$\mathbb{C}$	set of all complex numbers
$\Re$	real part
$\Im$	imaginary part
a^+	positive part of $a \in \mathbb{R}$
$\mathbb{R}^N$	Euclidean N-dimensional space
$x \cdot y$, $\langle x, y\rangle$	inner Euclidean product between the vectors $x, y \in \mathbb{R}^N$
$\|x\|$	euclidean norm of $x \in \mathbb{R}^N$
$\\|f\\|_\infty$	the sup-norm of $f : \Omega \subset \mathbb{R}^N \to \mathbb{R}$, i.e., $\\|f\\|_\infty := \sup_\Omega \|f\|$ (whenever is finite)
$B(x, r)$	open ball in $\mathbb{R}^N$ with centre at x and radius $r > 0$
$\overline{\Omega}$	closure of $\Omega \subset \mathbb{R}^N$

$\mathrm{Tr}A$	trace of the matrix A
Δ	Laplace operator
$D(L)$	domain of the operator L
$\mathcal{L}(X)$	space of the bounded linear operators in the Banach space X
$\Vert T\Vert_{\mathcal{L}(X)}$	operator norm of T in $\mathcal{L}(X)$
$\{T(t)\}_{t\geq 0}$	semigroup of linear operators in a Banach space X
Γ	Euler-Gamma function
$\mathrm{supp}\, f$	support of the function f, i.e. the set closure of the set of arguments for which f is not zero
$C_b(\mathbb{R}^N)$	space of the bounded and continuous functions defined in $\mathbb{R}^N$
$C_c^\infty(\mathbb{R}^N)$	space of infinitely many time derivable functions with compact support in $\mathbb{R}^N$
$C_{loc}^{k,\alpha}(\mathbb{R}^N)$	space of functions which are continuously differentiable in $\mathbb{R}^N$ whose kth derivatives are α-Hölder continuous in any compact subset of $\mathbb{R}^N$, where $\alpha \in (0,1]$
L^p_μ	space of measurable functions u in $\mathbb{R}^N$ with respect to the measure $d\mu$, with $\Vert u\Vert^p_{L^p_\mu} = \int_{\mathbb{R}^N} \vert u(x)\vert^p d\mu < +\infty$
H^k_μ	space of the functions $u \in L^2_\mu$ having distributional derivative up to order k in $(L^2_\mu)^N$

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