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Aggregation in Game Theoretical Situations

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Contents

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Chapter 1

Introduction

In this thesis some aspects of the class of Aggregative Games are investigated.

Aggregative games are strategic form games where each payoff function depends on the corresponding player's strategy and some aggregation among strategies of all players involved. Classical examples of aggregation are the unweighted sum and the mean.

The concept of aggregative games goes back to Selten (see [92]) who considers as aggregation function the summation of players' strategies. Later, this concept has been studied in the case of other aggregation functions and it has been generalized to the concept of quasi-aggregative games (see [2], [18], [24], [41], [54], [98]).

In literature, there are many games that present an aggregative structure: among them, we mention Cournot and Bertrand games, patent races, models of contests of fighting and model with aggregate demand externalities. Many common games in industrial organization, public economics and macroeconomics are aggregative games. Computational results for the class of aggregative games have been also investigated (see, for example, [36], [45]).

The thesis proceeds to provide existence results of several innovative equilibrium solutions in aggregative games.

In Game Theory the well known Nash equilibrium concept is a solution concept used in a non-cooperative situation where all the players act simultaneously, optimizing their own payoff taking into account the decisions of the opponents. There is also the possibility that players do not act simultaneously: for example, in the classical Stackelberg leader–follower model ([100]), a player, called the leader, acts first, anticipating the strategy of the opponent, known as the follower, who reacts optimally to the leader's decision. In this case, the leader's optimization problem contains a nested optimization task that corresponds to the follower's optimization problem.

In the case of multiple players, more than two, it is possible to have a hierarchy between two groups of players: a group, acting as a leader, decides first and the other group reacts to the leaders' decision. Now, it is necessary to determine the behavior within each group. In several applications, players at the same hierarchical level decide in a non-cooperative way (see [21], [46]): each player in the group knows that any other group member optimizes his own payoff taking into account the behavior of the rest. Thus, a Nash equilibrium problem is solved within each group and a Stackelberg model is assumed between the two groups. This leads to the multi-leader multi-follower equilibrium concept.

In literature, this model appeared in the context of oligopolistic markets (see [94]) with one leader firm and several follower firms acting as Cournot competitors. Other applications can be found, for example, in transportation (see [68]), in the analysis of deregulated electricity markets (see [39]), in water management systems (see [89]) and in wireless networks (see [43]). See [40] for a survey on the topic.

As it happens in concrete situations, in various contexts, such as in economics, evolutionary biology and computer networks, some uncertainty may appear in the data and a stochastic model can be formulated [93]. Usually, a random variable may affect the payoffs, and then one can consider the expected payoffs with respect to its probability distribution. Then, the players optimize the expected payoffs according to the considered solution concept. De Miguel and Xu (see [24]) extend the multiple-leader Stackelberg–Nash–Cournot model studied in [94] to the stochastic case assuming uncertainty in the demand function: leader firms choose their supply levels first, knowing the demand function only in distribution and followers make their decisions after observing the leader supply levels and the realized demand function.

As mentioned before, the Cournot game presents an aggregative structure thus, considering as starting point the paper by De Miguel and Xu and referring to [54], the first part of this thesis is devoted to provide a general framework for dealing with a hierarchy between two groups of players involved in an aggregative game, i.e. whose payoff functions depend on aggregation of strategies, in the case in which there is some uncertainty that hits each player's payoff. More precisely, following the existing literature on aggregative games ([2], [19], [20], [41]) and in line with [24], in Chapter 3 we focus on a new equilibrium solution concept, namely the multi-leader multi-follower equilibrium concept for the class of stochastic aggregative games. The considered game presents asymmetry between two groups of players, acting non-cooperatively within the group and one group is the leader in a leader-follower hierarchical model. Moreover, the model is affected by risk and the game is considered in a stochastic context. Thus, assuming an exogenous uncertainty affecting the aggregator, the multi-leader multi-follower equilibrium model under uncertainty is presented and existence results for the stochastic resulting game are obtained in the smooth case of nice aggregative games as well as in the general case of aggregative games with strategic substitutes.

The second part of this thesis is devoted to give existence results of equilibrium solutions in some applications to environmental and resource economics, that are concerned with the economic aspects of the utilization of natural renewable resources (fisheries, forests), natural exhaustible resources (oil, minerals) and environmental resources (water, air). A lot of environmental and resource games can be cast as aggregative games since they depend, for example, on the summation of emissions in the case of pollution games (see [7], [13], [42]) or on the summation of irrigation water that each farmer pumps from the groudwater resource (see [27]) and so on.

Thus, in line with [31], [32] and referring to [57], the first application that we examine is on Common-Pool Resources. A common-pool resource (CPR) is a natural or human-made resource, like open-seas fisheries, unfenced grazing range and groundwater basins, from which a group of people can benefit. A CPR consists of two kinds of variables: the stock variable, given by the core resource (for example, in the open-seas fisheries it's the total quantity of fish available in nature), and the flow variable, given by the limited quantity of fringe units that can be extracted (for example, the limited quantity of units that can be fished).

A problem which a CPR copes is the overuse: in fact, a CPR is a subtractable resource i.e., since its supply is limited, if the quantity that can be restored is used more and more there will be a shortage of it. This problem can lead to the destruction of the CPR (e.g. the CPR is destroyed if, in a short range of period, all the fish of a certain species are taken). Historically, in 1833 the economist William Forster Llyod published a pamphlet which included an example of overuse of a common parcel of land shared by cattle herders (see [49]).

Figure 1 shows the annual change in groundwater storage in the 37 largest groundwater basins in the world: in particular, the basins colored with shades of brown have had more water extracted that the quantity that could be naturally replenished while the ones in blue have had an increase of the level of the water maybe due to precipitation, ice melting and so on.

In this context, the economic investigation is about how to make a trade-off

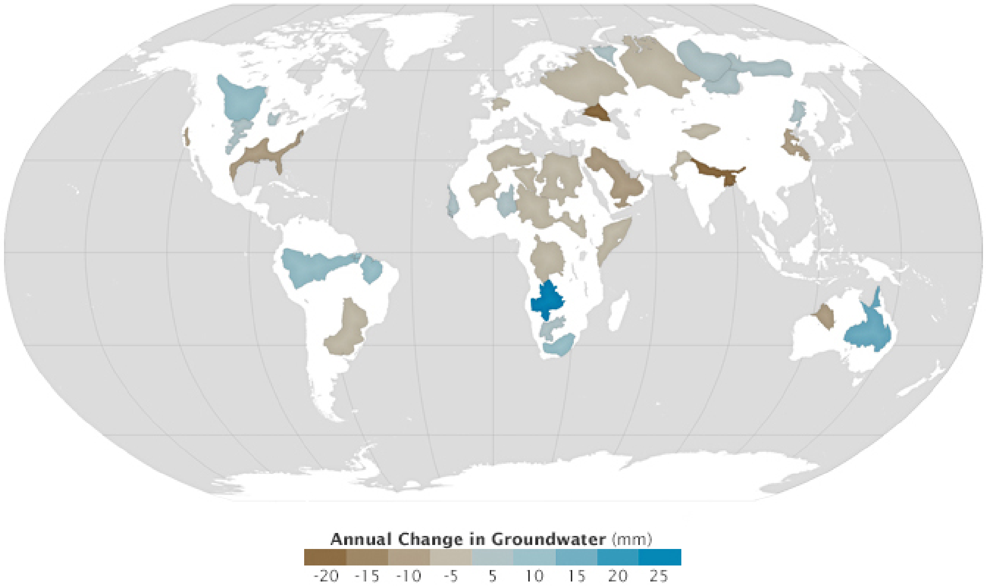

Figure 1.1: Acquired January 2003-December 2013. NASA Earth Observatory images by Joshua Stevens using GRACE global groundwater data courtesy of Jay Famiglietti NASA JPL/University of California Irvine and Richey et al. (2015). Caption assembled by Mike Carlowicz, based on stories by Alan Buis (JPL) and Janet Wilson (UCI).

between preserve and, at the same time, consume the resource and benefit from it.

The concept of CPR goes back to Gordon in 1954 (see [34]) who examines the economic theory of natural resource utilization in the context of fishing industry. In order to understand which is the better way to manage a CPR, Hardin in 1968 claims that if all members in a group act just according to their own-self interest with no regards for others, all resources would eventually be destroyed and this leads to the so called *tragedy of commons* (see [38], [76]), that is a central concept in the study of human ecology and of the environment in general (see [87]).

In 1990, Ostrom lists eight design principles which are prerequisites for a stable CPR management (see [86]). This management is highly dependent on the type of resource considered and in [88] Ostrom points out that the adaptive governance is the best method to obtain a good management of a CPR.

Applications of the CPR concept have been developed later for example in the case of renewable energy (see [102]).

It is important to point out that a CPR is not a Public Good (PG) (see [8]). In fact, they are both non-excludable goods but they differs on the rivalry property in the sense that a PG can be consumed without reducing the availability for others (for example the air or the national defense), while consuming a CPR will decrease the available resource for others (see [4]).

Let us explicitely note that aggregative games represent a fundamental instrument to model a game that involves a CPR. This is because a CPR is, as the name itself suggests, a resource that is common for a group of people that benefit from it. Thus, everything that is linked to a CPR depends on the aggregation of strategies and, since the payoff in aggregative games depends explicitely on the aggregation of strategies (see [2]), it is clear that an aggregative game is an appropriate tool in order to model a CPR game and to obtain more sophisticated results. As consequence, the novelty of this part of the thesis is about modeling a CPR game as an aggregative game and doing an equilibrium analysis on it, providing existence results in both deterministic case and stochastic case in which the possibility of a natural disaster is considered in the model.

Although a CPR game could be studied in a dynamic framework, since each CPR is a subtractable resource and so a priori players should take into account the total quantity of resource available in nature that varies time by time, in this setting we suppose that the players involved in the CPR game are myopic in the sense that they do not account for the dynamics in their equilibrium decision problem i.e. they optimize their individual payoff without considering the impact of their decisions on the stock of the resource.

Sometimes environmental games present not only a generic aggregative structure but an additively separable aggregative one.

In additively separable aggregative games, each payoff function is a sum of a function that depends on an aggregation of strategies and a function that depends on player's own strategy.

The model of additively separable aggregative games appeared in literature, among others, in the context of International Environmental Agreements (IEA)(see [25]), studying the formation of stable IEA in the case in which each country's choice variable is emission and then extending the results to the dual case i.e. the case where the choice variable is abatement effort.

Also Public Good Provision games are in the context of additively separable aggregative games (see [9]) where each player consumes a certain amount of a private good and donates a certain other amount to the supply of the public good. Thus the payoff function of each player turns to depend not only on the quantity of private good that he consumes but also on all the gifts to the public good made by all individuals.

McGuinty and Milam (see [70]) investigate the impact of asymmetry in a voluntary public goods environment by proposing an improved design that explicitely isolates individual incentives, without assuming a dominant strategy.

In the framework of additively separable aggregative games and in line with the asymmetry considered in [70], in the last part of the thesis we introduce and study a class of non-cooperative games, called Social Purpose Games, where the payoffs depend separately on a player's own strategy (individual benefit) and on a function of the strategy profile which is common to all players (social benefit) weighted by an individual benefit parameter which allows for asymmetry of public benefit across agent. We apply the general results obtained for this class of games to a water resource game, doing an equilibrium analysis and giving a stability result for the coalition arising from a partial cooperative equilibrium study.

The novelty about the presence of the weights, given by the individual benefit parameters, underlines how the collective part of the payoff is important for each player with respect to the individual part. This novelty is in the spirit of hedonic games, i.e. games where each player's payoff is completely determined by the identity of other members of his coalition, in which, as showed in [10], for achieving either individual stability or efficiency, symmetry across agents must be sacrificed, prefering asymmetry.

After the Introduction, in Chapter 2 we introduce some fundamental preliminaries for a better understanding of all the work done, in Chapter 3 we present the multi-leader multi-follower aggregative game under uncertainty, exploring existence results of the equilibrium in a regular case and then in a more general case, and in Chapter 4 we show applications of aggregative game theory to an investment in Common-Pool Resouces game and, after introducing the class of Social Purpose Games, to a withdrawal water resource game. Some concluding remarks and some directions of further research are presented in Chapter 5.

Chapter 2

Preliminaries

2.1 Non-cooperative games: simultaneous vs sequential games

One of the most important feature of the decisional interdependence among more than one person is the so called *strategic interaction*: the result gained by an individual does not depend just only on his own actions but also on the actions made by the other people that interact with him. A *game* is the formal representation of a strategic interaction between people with different but not necessarily opposite interests.

A classical distinction in game theory is made between non-cooperative and cooperative game.

The focus of non-cooperative games is the individual behavior and, even if in some situations preplay communication between agents is allowed, they cannot make agreements except for those which are established by rules of the game. On the contrary, the focus of cooperative games is the behavior of players' coalitions, since cooperative game theory studies negotiations among rational agents who can make agreements about how to play the game.

From now on let us deal with non-cooperative games.

In order to describe a non-cooperative game we need the following essential concepts:

- *players* or *decision makers* that are the agents who partecipate in the game, where a player can be an individual or a set of individuals;
- *actions* or *strategies* that are the choices that a decision maker can take. Actually an action and a strategy are not the same concept. In fact a strategy is a *decisional plan* that specifies *a priori* how a player will behave in each situation in which he has to play. Any consequence of such a strategy is called an action;
- *payoff functions* that measure desirability of the possible outcomes of the game and that mathematically are real valued function defined on the Cartesian product of the action spaces.

In the context of non-cooperative games, in which a set of agents interact and choose a strategy according to a given set of rules, we can distinguish between simultaneous and sequential games, depending on whether the decision is taken simultaneously or sequentially.
The most familiar representation of strategic interactions in game theory is given by the so called *normal form game* (or *strategic form game*).

Definition 2.1 (Strategic form or normal form game)**.** An *N-person normal-form game* ($N \in \mathbb{N}$, where $\mathbb{N}$ is the set of natural numbers) is a tuple

$$\Gamma =< \mathcal{N}; U_1, ..., U_N; f_1, .., f_N >,$$

where:
• $\mathcal{N} = \{1, 2, ..., N\}$ is the finite set of the N players, with N the fixed number of players that are involved in the game and indexed by i;
• U_i is the set of possible actions or strategies for player $i \in \mathcal{N}$; we denote by $U = U_1 \times U_2 \times ... \times U_N$ and each vector $x = (x_1, ..., x_N) \in U$ is called an action profile;

• $f_i : U \to \mathbb{R}$ is called the objective or payoff function of player $i \in \mathcal{N}$ ($\mathbb{R}$ is the set of real numbers); it can represent a cost (to minimize) or a profit (to maximize).

The normal form allows us to represent every player's utility for every state of the world in the case where states of the world depend only on the players' combined actions.
Note that if player 1 chooses $x_1 \in U_1$, ..., player N chooses $x_N \in U_N$, then each player i obtains a cost or a profit $f_i(x_1, ..., x_N)$. We define for $i \in \mathcal{N}$ the vector $x_{-i} = (x_1, .., x_{i-1}, x_{i+1}, .., x_N)$ and the set $U_{-i} = \Pi_{j \in \mathcal{N} \setminus \{i\}} U_j$. Let $x \in U$ and $i \in \mathcal{N}$. Sometimes we denote $x = (x_i, x_{-i})$. We suppose in the following that all the players are profit maximizing, except when explicitly specified.

Definition 2.2 (Finite game)**.** If each player has a finite number of available strategies, we say that the game is a *finite game.*

Two classical examples are the following.

Example 2.1 (**Prisoner dilemma**)**.** Two suspects are arrested by the police for the same crime and the two prisoners have been separated. The police have insufficient evidence so they cannot convict them if none of the two prisoners confesses. Thus the police visit each of them to offer the same deal: if one testifies for the prosecution against the other and the other remains silent, the former goes free and the latter receives the full 30 year sentence. If both remain silent, both prisoners are sentenced to only 2 years in jail for a minor charge. If each betrays the other, each receives a 8 year sentence. Each prisoner must make the choice of whether to betray the other or to remain silent. How should the prisoners act?

The situation can be modeled as a two-person finite game

$$\Gamma =< 2; U_1, U_2; f_1, f_2 >,$$

where the strategy sets are $U_1 = U_2 = \{NC, C\}$ where the choice C means "confess", the choice NC means "not confess" the payoffs represent the years sentence and each player wants to minimize the years in jail .

Table 2.1: Prisoner dilemma game matrix

	C	NC
C	8,8	0,30
NC	30,0	2, 2

Example 2.2 (**Linear Cournot duopoly**). There are two firms with identical products, which operate in a market where the market demand function is known. Let us denote the production level of firm i by q_i . The demand function p relates the unit price of the product to the total quantity offered by the firms. Let us assume a linear structure for the market demand curve

$$p = a - (q_1 + q_2)$$

and linear production costs to both firms cq_i $(i = 1, 2)$ with a, c positive constants and $a > c$. Then the situation can be modeled by a two-person game

$$\Gamma =< 2; U_1, U_2; f_1, f_2 >,$$

where 1 and 2 are the two firms profit maximizing, the strategy sets are $U_1 = U_2 = [0, +\infty[$ and the profit functions are given by:

$$f_i(q_1, q_2) = q_i(a - (q_1 + q_2)) - cq_i, \; i = 1, 2.$$

Each firm has to choose the optimal quantity to produce in order to maximize his own profit, given the choice of the opponent firm.

Let us consider simultaneous-move games, in which all players move only once and at the same time.

Let us introduce the concept of dominant and dominated strategies. These notions and the concept of iterated dominance provide a first restriction on the strategies that rational players should choose to play.
From now on, let us suppose that the cardinality of the strategy set U_i, for any $i \in \mathcal{N}$, is greater or equal than 2.

Definition 2.3 (Dominated strategies)**.** Let $\Gamma =< \mathcal{N}; U_1, .., U_N; f_1, .., f_N >$ be a strategic form game and $x_i', x_i'' \in U_i$ for $i \in \mathcal{N}$. The strategy x_i' is *dominated* (in a maximizing problem) for player i if there exists x_i'' such that

$$f_i(x_i', x_{-i}) < f_i(x_i'', x_{-i}), \ \forall x_{-i} \in U_{-i}.$$

Sometimes dominated strategies are called *strictly* dominated strategies. The strategy x_i' is weakly dominated for player i if there exists x_i'' such that the above inequality holds with weak inequality and it is strict for at least one x_{-i}.
We should expect that a player will not play dominated strategies, those for which there is an alternative strategy that yields him a higher profit regardless of what the other players do.
Let us analyse the dominated strategies in the prisoner dilemma: for player 2 the strategy NC is strictly dominated by C then player 2 will not choose NC since it is rational and player 1 knows that player 2 is rational (common knowledge). So from the initial game we can consider the reduced game in Table 2.2 where the strategy NC for player 2 is eliminated (grey column).

Table 2.2: Prisoner dilemma: dominated strategies - step 1

	C	NC
C	8,8	0,30
NC	30,0	2, 2

The same holds for player 1 in this reduced game: the strategy NC is strictly dominated by C, so that the profile (C, C) is an obvious candidate for the prediction of how the game will be played. We can consider the reduced game in Table 2.3 where the strategy NC for player 1 is eliminated.

Table 2.3: Prisoner dilemma: dominated strategies - step 2

	C	NC
C	8,8	0,30
NC	30,0	2, 2

The final result of the game has given by this last reduced game that is (C, C): we say that we solve the game by *iterated strict dominance*.
Let us formalize the iterated strict dominance concept. Given $U = U_1 \times \cdots \times U_N$, we define the set of player i's *undominated responses* by

$$R_i(U) = \{x_i \in U_i : \forall x'_i \in U_i \exists x = (x_1, \ldots, x_N) \in U s.t. f_i(x_i, x_{-i}) \geq f_i(x'_i, x_{-i})\}.$$

Let $R(U) = (R_i(U) : i \in \mathcal{N})$ be the list of undominated responses for each player. In order to represent the process of iterated elimination of strict dominated strategies, define $U^0 = U$ the full set of strategy profiles. For $s \geq 1$, let us define $U^s = R(U^{s-1})$. A strategy x_i is *serially undominated* if $x_i \in R_i(U^s)$ for all s: these are the strategies that survive to the iterated strict dominance process.
Let us note that in the prisoner dilemma case although cooperating would give each player a payoff of 2, self interest leads to an inefficient outcome with payoff $8 > 2$ for both players. Thus this procedure not always offers efficient outcomes.

Moreover, in other games, like in the other two classical examples i.e. battle of the sexes and matching pennies games, no (strict) dominated strategy can

be found for each player. Thus we can face also the problem of non existence of (strict) dominated strategies.
Thus, let us introduce one of the most used solution concepts in Game Theory i.e. the *Nash equilibrium* (see [78]) that gives more accurate predictions about the solution of the game than other solution concepts.
The fundamental feature of Nash equilibrium is the *strategic stability*: none of the players, knowing the strategies chosen by the others, wants to deviate from the one that he has chosen. In other words, every component of a Nash equilibrium vector is the *optimal response*, for the related player, to the other components.

Definition 2.4 (Nash equilibrium)**.** A *Nash equilibrium* for Γ is a strategy profile $\hat{x} = (\hat{x}_1, \hat{x}_2, ..., \hat{x}_N) \in U$ such that for any $i \in \mathcal{N}$ and for any $x_i \in U_i$ we have that

$$f_i(\hat{x}) \geq f_i(x_i, \hat{x}_{-i}).$$

Such a solution is self-enforcing in the sense that once the players are playing such a solution, it is in every player's interest to remain in his strategy. We denote by $NE(\Gamma)$ the set of the Nash equilibrium strategy profiles of the game Γ.

Any $\hat{x} = (\hat{x}_1, \hat{x}_2, ..., \hat{x}_N) \in NE(\Gamma)$ is a vector such that for any $i \in \mathcal{N}$, $\hat{x}_i$ is solution to the optimization problem

$$\max_{x_i \in U_i} f_i(x_i, \hat{x}_{-i}).$$

We can restate the definition of a Nash equilibrium through the introduction of the concept of a player's best reply correspondence.

Definition 2.5 (Best reply)**.** For any $i \in \mathcal{N}$, for any $x_{-i} \in U_{-i}$, the set

$$B_i(x_{-i}) = \{x_i \in U_i : f_i(x_i, x_{-i}) \geq f_i(x_i', x_{-i}) \ \forall x_i' \in U_i\}$$

is called player i 's best reply to the opponents' decision x_{-i}. The correspondence $B : x = (x_1, ..., x_N) \in U \to (B_1(x_{-1}), ..., B_n(x_{-N})) \in U$ is called the *best reply correspondence.*

Note that the strategy profile $\hat{x}$ is a Nash equilibrium of the game Γ if and only if $\hat{x}_i \in B_i(\hat{x}_{-i})$ for $i = 1, ..., N$ and $\hat{x} \in NE(\Gamma)$ if and only if $\hat{x} \in U$ is a fixed point of B, i.e.

$$\hat{x} \in B(\hat{x}).$$

Nash equilibrium solutions in normal form games can be obtained as the intersection points of the best reply curves (or correspondences) of the players.

Example 2.3 (**Prisoner dilemma**)**.** The players have to minimize the years sentences. We can compute the best reply functions (Table 2.4) and see that the unique Nash equilibrium is the profile (C, C).

Table 2.4: Prisoner dilemma - best replies

	NC	C
NC	2, 2	30, $\underline{0}$
C	$\underline{0}$,30	$\underline{8}$, $\underline{8}$

Example 2.4 (**Linear Cournot duopoly**)**.** In this case the firms are profit maximizing and the best reply functions are simply computed (the profit functions are strictly concave) by means of the first order conditions:

$$b_1(q_2) = \{(a - q_2 - c)/2\}$$

$$b_2(q_1) = \{(a - q_1 - c)/2\}$$

The Nash equilibrium solution (also called Nash-Cournot equilibrium) is the intersection of the two best replies, $(\hat{q}_1, \hat{q}_2) = ((a - c)/3, (a - c)/3)$.

Example 2.5 (**Linear Cournot duopoly with a leader firm**)**.** As in Example 2.2, there are two firms with identical products, which operate in a market in which the market demand function is known. Now they act sequentially: firm 2 reacts to the firm 1's decision.

Firm 1 acts as the leader and announces $q_1 \geq 0$; firm 2, the follower, observes q_1 and firm 2 reacts by choosing $q_2 \in B_2(q_1)$, $B_2(q_1) = \{(a - q_1 - c)/2\}$, that is (for $q_1 < a - c$) a solution of

$$\max_{q_2 \geq 0} f_2(q_1, q_2) = \max_{q_2 \geq 0} q_2(a - q_1 - q_2 - c).$$

Firm 1 knows that for any q_1 firm 2 will choose $q_2 = B_2(q_1)$ (that is unique) and solves

$$\max_{q_1 \geq 0} f_1(q_1, B_2(q_1)) = \max_{q_1 \geq 0} q_1(a - q_1 - c)/2.$$

The equilibrium strategy, called either Stackelberg equilibrium strategy or, in this particular example, Stackelberg-Cournot equilibrium strategy, is $q_1^* = (a - c)/2$ and firm 2 will choose $q_2^* = B_2(q_1^*) = (a - c)/4$.

Starting from Example 2.5, let us now focus on a solution concept for two-person games that involves a hierarchical structure in decision making: one of the players (called *leader*) declares and announces his strategy before the other player (called *follower*). The follower observes this and in equilibrium picks the optimal strategy as a response. Players may engage in a Stackelberg competition if one has some sort of advantage enabling it to move first. In other words, the leader must have commitment power. Such games are called *Stackelberg games* and have been introduced in the context of duopoly problems by H. von Stackelberg, who published “Market Structure and Equilibrium” (Marktform und Gleichgewicht) in 1934 ([100]).

This game displays sequential moves: players choose at different stages of the game taking their own decision. Player 1 behaves as a leader and plays first anticipating the reactions of the rival and takes them into account before choosing his strategy. Player 2 behaves as a follower answering to player 1 in an optimal way.

Let $\Gamma =< 2; U_1, U_2; f_1, f_2 >$ be a two-person game, where player 1 is the leader and both players are profit maximizers. For any $x_1 \in U_1$, let $B_2(x_1)$ be the follower's best reply to the leader's decision x_1. At this point, suppose that for any $x_1 \in U_1$ the best reply is a singleton denoted by $B_2(x_1) = \{\tilde{x}_2(x_1)\}$, so that B_2 is a function.

The case in which there are multiple follower best replies will be explicitely illustrated in Section 2.2.

Definition 2.6. (Two-person Stackelberg equilibrium) In a two-person game Γ with player 1 as leader, a strategy $\bar{x}_1 \in U_1$ is called a *Stackelberg equilibrium strategy* for the leader if

$$f_1(\bar{x}_1, \tilde{x}_2(\bar{x}_1)) \geq f_1(x_1, \tilde{x}_2(x_1)), \forall x_1 \in U_1 \qquad \mathcal{S}$$

where $\tilde{x}_2(x_1)$, for any $x_1 \in U_1$, is the unique solution of the problem

$$f_2(x_1, \tilde{x}_2(x_1)) \geq f_2(x_1, x_2), \forall x_2 \in U_2 \qquad P(x_1).$$

The Stackelberg equilibrium strategy $\bar{x}_1 \in U_1$ is a solution to the upper level problem $\mathcal{S}$, and $\tilde{x}_2(\bar{x}_1)$ is the optimal choice for player 2, the follower. The pair $(\bar{x}_1, \tilde{x}_2(\bar{x}_1))$ will be called *Stackelberg equilibrium.*

Sometimes $P(x_1)$ is called the lower level problem and corresponds to the follower's optimization problem[1]. The game is solved by backward induction, i.e. analyzing the game from back to front.

Possible extentions of Definition 2.6 concern removing the uniqueness of the best reply and the limitation to two-person games, as we are going to illustrate in Section 2.2.

[1]Remark that a Stackelberg game can also be seen as a subgame-perfect Nash equilibrium of a two-stage game (i.e. the strategy profile that corresponds to a Nash equilibrium of every subgame of the original game) with complete information.

2.2 Leader/Follower Model

Let us briefly introduce some of the different aspects of the Stackelberg Leader/Follower model, enlightening some generalizations of the model introduced in Definition 2.6. For more details and discussion on open problems see [56] and the literature therein.

Case 1: multiple follower reaction. As it happens in many cases, the lower level problem $P(x_1)$ may have more than one solution for at least one $x_1 \in U_1$. Let us consider for any $x_1 \in U_1$ the best reply $B_2(x_1)$ of the follower player, that is a correspondence defined on U_1 and valued in U_2 mapping to any $x_1 \in U_1$ the subset $B_2(x_1) \subseteq U_2$ of all possible solutions to the problem $P(x_1)$. In this case the best reply is a multi-valued function and the upper level problem has to be formulated depending on the leader's behavior. The leader has to optimize the updated profit function, but he does not know what the follower's choice in the set $B_2(x_1)$ is. So, a possible approach is that the leader supposes that the follower's choice is the best for himself and solves the following upper level problem: find $\bar{x}_1 \in U_1$ s.t.

$$\max_{x_2 \in B_2(\bar{x}_1)} f_1(\bar{x}_1, x_2) = \max_{x_1 \in U_1} \max_{x_2 \in B_2(x_1)} f_1(x_1, x_2) \qquad \mathcal{S}^s$$

where $B_2(x_1)$, for any $x_1 \in U_1$, is the set of all possible solutions to the problem $P(x_1)$. Any pair $(\bar{x}_1, \bar{x}_2)$ with $\bar{x}_2 \in B_2(\bar{x}_1)$ is referred to as a *strong Stackelberg solution* for the two-person game with player 1 as the leader.

This solution concept corresponds to an optimistic leader's point of view ([11],[48]).

A very common feature in applications is the so called weak Stackelberg strategy or security strategy. We now suppose that the leader prevents the worst that can happen when the follower chooses his decision in the set of the best replies. So he minimizes the worst and solves the following upper level problem: find $\bar{x}_1 \in U_1$ s.t.

$$\min_{x_2 \in B_2(\bar{x}_1)} f_1(\bar{x}_1, x_2) = \max_{x_1 \in U_1} \min_{x_2 \in B_2(x_1)} f_1(x_1, x_2) \qquad \mathcal{S}^w$$

where $B_2(x_1)$, for any $x_1 \in U_1$, is the set of all possible solutions to the problem $P(x_1)$. Any pair $(\bar{x}_1, \bar{x}_2)$ with $\bar{x}_2 \in B_2(\bar{x}_1)$ is referred to as a *weak Stackelberg solution* for the two-person game with player 1 as the leader ([11],[48]).

Differently from the strong solution concept, the weak one corresponds to a pessimistic leader's point of view.

Existence of weak Stackelberg strategies is a difficult task from mathematical point of view, because it may not exist even in smooth examples. An existence theorem guarantees the existence of weak Stackelberg strategies under assumptions on the structure of the best reply set ([5]). Existence results of solutions as well as of approximated solutions can be found in [50], [51] for the strong and the weak Stackelberg problem under general assumptions. Existence of solutions and approximations in the context of mixed strategies for the follower as well as for both players are in [58], [59].

A more general definition is the so-called *intermediate Stackelberg strategy* (see [60], [62]) where the leader has some probabilistic information about the choice of the follower in the optimal reaction set.

Case 2: multiple follower case. A more general case, dealing with one leader and multiple followers, is the so-called *Stackelberg-Nash problem.*

Let us consider a $N+1$-person game $\Gamma =< \mathcal{N}; U_0, U_1, ..., U_N; f_0, f_1, ..., f_N >$, where $\mathcal{N} = \{0, 1, ..., N\}$, one player 0 is the leader and the rest of them $1, ..., N$ are followers in the sense that players $1, \ldots, N$ act simultaneously and answer to 0's strategy in optimal way. It is supposed that the N followers are engaged in a non-cooperative competition corresponding to a Nash equilibrium problem. Let

$U_0, U_1, ..., U_N$ be the leader's and the followers' strategy sets, respectively. Let $f_0, f_1, ..., f_N$ be real valued functions defined on $U_0 \times U_1 \times .... \times U_N$ representing the leader's and the followers' profit functions.

The leader is assumed to announce his strategy $x_0 \in U_0$ in advance and commit himself to it. For a given $x_0 \in U_0$ the followers select $(x_1, ..., x_N) \in R(x_0)$ where $R(x_0)$ is the set of the Nash equilibria of the N-person game with players $1, ..., N$, strategy sets $U_1, ..., U_N$ and profit functions $f_1, ..., f_N$. For each $x_0 \in U_0$, that is the leader's decision, the followers solve the following lower level Nash equilibrium problem $NE(x_0)$:

$$\text{find } (\bar{x}_1, ..., \bar{x}_N) \in U_1 \times .... \times U_N \text{ such that}$$

$$f_i(x_0, \bar{x}_1, ..., \bar{x}_N) = \max_{x_i \in U_i} f_i(x_0, \bar{x}_1, ..., \bar{x}_{i-1}, x_i, \bar{x}_{i+1}, ..., \bar{x}_N) \quad \forall i = 1, ..., N$$

The non-empty set $R(x_0)$ of the solutions to the problem $NE(x_0)$ is called the followers' reaction set. The leader takes into account the followers Nash equilibrium, that we assume to be unique, and solves an optimization problem in a backward induction scheme.

Let $(\tilde{x}_1(x_0), ..., \tilde{x}_N(x_0)) \in U_1 \times .... \times U_N$ be the unique solution of the problem $NE(x_0)$, the map

$$x_0 \in U_0 \to R(x_0) = (\tilde{x}_1(x_0), ..., \tilde{x}_N(x_0))$$

is called the followers' best reply (or response). The leader has to compute a solution of the following upper level problem $\mathcal{S}^{SN}$: find $\bar{x}_0 \in U_0$ such that

$$f_0(\bar{x}_0, \tilde{x}_1(\bar{x}_0), ..., \tilde{x}_N(\bar{x}_0)) = \max_{x \in U_0} f_0(x, \tilde{x}_1(x), ..., \tilde{x}_N(x)) \qquad \mathcal{S}^{SN}$$

Any solution $\bar{x}_0 \in U_0$ to the problem $\mathcal{S}^{SN}$ is called a *Stackelberg-Nash equilibrium strategy.*

The given definition for $N = 1$ is nothing but the classical Stackelberg equilibrium solution. This model, for $N > 1$, has been introduced in the oligopolistic market context in [94] and studied from a computational point of view in [52]. Existence of solutions and approximate solutions under general assumptions are in [74]. Existence of solutions in mixed strategies has been given in [61], [63] for two followers playing a zero-sum or a non-zero sum game, respectively.

An example dealing with communication networks is studied in [6]: the problem is formulated as a leader-follower game, with a single leader (the service provider, who sets the price) and a large number of Nash followers (the users, who decide on their flow rates), and the asymptotical behavior with an infinite number of followers is discussed.

Case 3: multiple player games with hierarchy. It is possible to extend the Stackelberg Leader/Follower model also in the case of multiple players: it is necessary to fix the hierarchical level of each player and precise his behavior as leader as well as follower.

A possible definition is the generalization of the Stackelberg-Nash problem to a $M+N$-person game with M players acting as leaders and the rest of them behave as N followers. It is assumed a non-cooperative behavior between the leaders and between the followers, so the model can be written by considering a Nash equilibrium problem at the lower level with the follower players and another Nash equilibrium problem at the upper level with the leader players.

An existence result for equilibria for Stackelberg games where a collection of leaders compete in a Nash game constrained by the equilibrium conditions of another Nash game amongst the followers, imposing no single-valuedness assumption on the equilibrium of the follower-level game, has been given in [46] under the assumption that the objectives of the leaders admit a quasi-potential function and applied in communication networks.

Case 4: inverse Stackelberg game. In the two-person *inverse Stackelberg* *game* the leader does not announce the strategy x_1, as in the Stackelberg one, but a function $g_L(\cdot)$, which maps x_2 into x_1. Given the function g_L, the follower's optimal choice x_2^* satisfies

$$f_2(g_L(x_2^*), x_2^*) \geq f_2(g_L(x_2), x_2), \forall x_2 \in X_2.$$

The leader, before announcing the function g_L, realizes how the follower will play, and he should exploit this knowledge in order to choose the best possible g_L-function, such that his own profit becomes as big as possible, i.e.

$$g_L^*(\cdot) = \text{argmax}_{g_L(\cdot)}\ f_1(g_L(x_2(g_L(\cdot))), x_2(g_L(\cdot)))$$

The problem is in general very difficult to solve. However, if the leader knows what he can achieve (in terms of maximal profit) and what has to be done by all players to reach this outcome, the leader may be able to persuade other players to help him to reach this goal (i.e., the value of the leader's profit function obtained if all players maximize it). If it is unknown what the leader can achieve in terms of maximal profits, finding the leader's optimal g_L-strategy is generally very difficult.

The problem has been studied for special classes of payoff and applied to an electricity market problem ([84]).

2.3 Supermodular Games

Supermodular games are characterized by "strategic complementaries" in the sense that the marginal utility of increasing a player's strategy increases when the other players' strategies increase. This particular class of games is interesting for several reasons. Firstly, it includes a lot of applied models. Secondly, existence results of a pure strategy equilibrium hold without the quasi-concavity assumption of the payoff functions. Finally, they have nice comparative statics properties. The theory of supermodular optimization has been developed by Topkis (see [96], [97]), Vives (see [98], [99]) and by Granot and Veinott (see [37]).

Let us consider a non-cooperative N-person game in normal form

$$\Gamma = < \mathcal{N}; U_1, ..., U_N; f_1, .., f_N > .$$

In order to introduce the notion of supermodular games, let us give some mathematical preliminary tools, considering generic sets $(U_i)_{i=1}^N$ and generic function f_i defined on U_i for any $i \in \{1, \ldots, N\}$.

Let us suppose U_i equipped with a partial order $\geq$ that is transitive, reflexive and antisymmetric[2] (see [72]). Given $U_i' \subset U_i$, $\overline{x} \in U_i$ is called an *upper bound* for U_i' if $\overline{x} \geq x$ for all $x \in U_i'$; $\overline{x}$ is called the *supremum* of U_i' (denoted by $sup(U_i')$) if for all upper bounds x of U_i', $x \geq \overline{x}$. Lower bounds and infimums are defined analogously. A point x is a *maximal* element of U_i if there is no $y \in U_i$ such that $y > x$ (that is, no y such that $y \geq x$ but not $x \geq y$); it is the *largest* element of U_i if $x \geq y$ for all $y \in U_i$. *Minimal* and *smallest* elements are defined similarly. A set may have many maximal and minimal elements, but it can have at most one largest and one smallest element.

The set U_i is a *lattice* if $x, x' \in U_i$ implies $x \wedge x', x \vee x' \in U_i$, where $x \wedge x'$ and $x \vee x'$ denote, respectively, the infimum and supremum between x and x'. The lattice is *complete* if for all nonempty subsets $U_i' \subset U_i$, $inf(U_i') \in U_i$ and $sup(U_i') \in U_i$.

The real line (with the usual order) is a lattice and any compact subset of it is, in fact, a complete lattice, as any set in $\mathbb{R}^N$ formed as the product of N compact sets (with the product order).

A *sublattice* U_i' of a lattice U_i is a subset of U_i that is closed under $\vee$ and $\wedge$. A *complete sublattice* U_i' is a sublattice such that the infimum and supremum

[2] Recall that *transitive* means that $x \geq y$ and $y \geq z$ implies $x \geq z$; *reflexive* means that $x \geq x$; *antisymmetric* means that $x \geq y$ and $y \geq x$ implies $x = y$.

of every subset of U_i' are in U_i'.

A *chain* $C_i \subset U_i$ is a totally ordered subset of U_i that is, for any $x, x' \in C_i$, $x \geq x'$ or $x \leq x'$. Given a complete lattice U_i, a function $f_i : U_i \to \mathbb{R}$ is *order continuous* if it converges along every chain C_i in both increasing and decreasing directions i.e. if $lim_{x_i \in C_i, x_i \to inf(C_i)} f_i(x_i) = f_i(inf(C_i))$ and $lim_{x_i \in C_i, x_i \to sup(C_i)} f_i(x_i) = f_i(sup(C_i))$. It is *order upper semi-continuous* if $limsup_{x_i \in C_i, x_i \to inf(C_i)} f_i(x_i) \leq f_i(inf(C_i))$ and $limsup_{x_i \in C_i, x_i \to sup(C_i)} f_i(x_i) \leq f_i(sup(C_i))$.

Let us suppose that in the game Γ each strategy set U_i is a partially ordered lattice. A function $f_i : U_i \times U_{-i} \to \mathbb{R}$ is *supermodular* in x_i if for all fixed $x_{-i} \in U_{-i}$,

$$f_i(x_i \vee x_i', x_{-i}) - f_i(x_i, x_{-i}) \geq f_i(x_i', x_{-i}) - f_i(x_i \wedge x_i', x_{-i})$$

for all $x_i, x_i' \in U_i$.

Supermodularity represents the economic notion of complementary inputs. The following result is a characterization of supermodularity for twice continuously differentiable functions with Euclidean domains. The standard order on such domains is the so-called "product order" i.e., $x \leq y$ if and only if $x_i \leq y_i$ for all i, for each $x = (x_1, \ldots, x_N), y = (y_1, \ldots, y_N) \in \mathbb{R}^N$.

Topkis's Characterization Theorem *Let $I = [\underline{x}, \overline{x}]$ be an interval in $\mathbb{R}^N$. Suppose that $f : \mathbb{R}^N \to \mathbb{R}$ is twice continuously differentiable on some open set containing I. Then, f is supermodular on I if and only if for all $x \in I$ and all $i \neq j$, $\partial^2 f / \partial x_i \partial x_j \geq 0$.*

In general, the order structure of the lattice is the only instrument that supermodularity uses. It does not involve assumptions of convexity or connectedness of the domain or convexity, concavity or differentiability of the function itself. However, as the theorem suggests, it is easy to check the supermodularity of smooth functions on Euclidean intervals.

Let us suppose that U_i for any $i \in \{1, \ldots, N\}$ is a lattice. A function

$f_i : U_i \times U_{-i} \to \mathbb{R}$ exhibits *increasing differences* in x_i and x_{-i} if for all $x_i' > x_i$ the function $f_i(x_i', x_{-i}) - f_i(x_i, x_{-i})$ is nondecreasing in x_{-i} [3]. Let us explicitely note that the assumption of increasing differences is nothing but the assumption of strategic complementarity. In fact it means that it becomes more profitable for player i to increase his choice variable when the other players increase their choice variables as well.

We are now ready to introduce the notion of supermodular games. Let us suppose that in the game Γ, considered above, each strategy set U_i is a partially ordered lattice, with relation $\geq$, and the strategy profiles are endowed with the product order i.e. if $x = (x_1, \dots, x_N)$ and $x' = (x_1', \dots, x_N')$ are two strategy profiles and $x \geq x'$, it means that $x_i \geq x_i'$ for any $i \in \{1, \dots, N\}$.

Definition 2.7. The game $\Gamma =< \mathcal{N}; U_1, ..., U_N; f_1, .., f_N >$ is a supermodular game if, for each $i \in \mathcal{N}$:

(1) U_i is a complete lattice;

(2) $f_i : U \to \mathbb{R} \cup \{-\infty\}$ is order upper semicontinuous in x_i, with x_{-i} fixed, order continuous in x_{-i}, with x_i fixed, and has a finite upper bound;

(3) f_i is supermodular in x_i, for fixed x_{-i};

(4) f_i has increasing differences in x_i and x_{-i}.

From Topkis's Characterization Theorem we can prove the following:

Theorem 2.1 ([72]). *The game $\Gamma =< \mathcal{N}; U_1, ..., U_N; f_1, .., f_N >$ is a supermodular game if, for each $i \in \mathcal{N}$:*

(1') U_i is an interval in $\mathbb{R}^{k_i}$;

(2') f_i is twice continuously differentiable on U_i;

[3] Similarly $f_i : U_i \times U_{-i} \to \mathbb{R}$ exhibits *decreasing differences* in x_i and x_{-i} if for all $x_i' > x_i$ the function $f_i(x_i', x_{-i}) - f_i(x_i, x_{-i})$ is nonincreasing in x_{-i}

(3') $\frac{\partial^2 f_i}{\partial x_{i_j} \partial x_{i_h}} \geq 0$ *for all* i *and all* $1 \leq j \leq h \leq k_i$*;*

(4') $\frac{\partial^2 f_i}{\partial x_{i_j} \partial x_{l_h}} \geq 0$ *for all* $i \neq l$*,* $1 \leq j \leq k_i$ *and* $1 \leq h \leq k_l$*.*

Games that satisfy conditions *(1')-(4')* are called *smooth supermodular games.*

Referring to the definition of strategy serially undominated, given in the previous section, let us give the following theorem, that is crucial in the analysis of a supermodular game:

Theorem 2.2 ([72])**.** *Let* Γ *be a supermodular game. For each player* i*, there exist largest and smallest serially undominated strategies,* $\overline{x}_i$ *and* $\underline{x}_i$ *respectively. Moreover, the strategy profiles* $(\overline{x}_i)_{i \in \mathcal{N}}$ *and* $(\underline{x}_i)_{i \in \mathcal{N}}$ *are pure Nash equilibrium profiles.*

For further details and results on supermodular games see [37], [72], [96], [97], [98] and [99].

2.4 Potential Games

In some situations a Nash equilibrium may not exists, may not be unique and also may not be optimal for the players. It turns out to be very hard sometimes to check if the game has a Nash equilibrium and, even if a Nash equilibrium exists, it could be very hard to compute it by using a fixed point theorem (see [17], [30], [80]).

When one has to face one of these situations, the class of *potential games* could be very useful. Namely, in games with a potential function, known as potential games, the problem of finding a Nash equilibrium is a simple minimization or maximization problem rather than a fixed point problem. Thus, potential games appear the natural link between optimization and game theory: by solving an optimization problem the players find a Nash equilibrium strategy.

Potential games have been introduced by Monderer and Shapley (see [73]): the idea is that a game is said potential if the information that is sufficient to determine Nash equilibria can be summarized in a single function on the strategy space, the potential function.

Definition 2.8. A game $\Gamma = \langle \mathcal{N}; U_1, \dots U_N; f_1, \dots, f_N \rangle$ is an *exact potential* game (or simply *potential* game) if there exists a function $P : U \to \mathbb{R}$ such that for each player $i \in \mathcal{N} = \{1, \dots, N\}$, each strategy profile $x_{-i} \in U_{-i}$ of i's opponents, and each pair $x_i, y_i \in U_i$ of strategies of player i:

$$f_i(y_i, x_{-i}) - f_i(x_i, x_{-i}) = P(y_i, x_{-i}) - P(x_i, x_{-i}).$$

The function P is called an exact potential (or, in short, a *potential*) of the game Γ.

In words, if P is a potential function of Γ, the difference induced by a single deviation is equal to that of the deviator's payoff function.
Clearly, by definition, the strategy equilibrium set of the game Γ coincides with the strategy equilibrium set of the game $\Gamma_P = \langle \mathcal{N}; U_1, \dots U_N; P \rangle$ that is the game in which every player's payoff is given by P. In fact, $\overline{x} \in NE(\Gamma)$ if and only if for any $i = 1, \dots, N$ we have that

$$P(\overline{x}) \geq P(x_i, \overline{x}_{-i})$$

for every $x_i \in U_i$.
As immediate consequences of Definition 2.8, we have the following propositions.

Proposition 2.1. *Let Γ be a potential game with potential P. Then*

- $NE(\Gamma) = NE(\Gamma_P)$;
- $argmax_{x \in U} P(x) \subseteq NE(\Gamma)$;
- *if P admits a maximal value (maximizer), then Γ has a (pure) Nash equilibrium.*

From Propositions 2.1 we can ensure that if $\overline{x} \in argmax_{x \in U} P(x)$ then $\overline{x}$ is a Nash equilibrium of Γ. The converse does not hold true in general. However, adding some assumptions, we can prove the following:

Theorem 2.3 ([91]). *Let Γ be a potential game with potential function P. If $U = U_1 \times \cdots \times U_N$ is a convex set and P is a continuously differentiable function on the interior of U, then*

- *if $\overline{x}$ is a NE of Γ, then $\overline{x}$ is a stationary point of P;*
- *if P is concave on U and if $\overline{x}$ is a NE of Γ, then $\overline{x} \in argmax_{x \in U} P(x)$. If P is strictly concave, such NE must be unique.*

For example, a finite game Γ that turns out to be potential, has a Nash equilibrium strategy. The following proposition is proved in [73].

Proposition 2.2. *Let Γ be a potential game with potential P. Then P is uniquely defined up to an additive constant.*

Example 2.6. In the case of Cournot duopoly of Example 2.2 if we define a function

$$P(q_1, q_2) = (a - c)(q_1 + q_2) - (q_1^2 + q_2^2) - q_1 q_2$$

we can verify that for every firm i and for every $q_{-i} \in \mathbb{R}_+$,

$$f_i(q_i, q_{-i}) - f_i(x_i, q_{-i}) = P(q_i, q_{-i}) - P(x_i, q_{-i})$$

for every $q_i, x_i \in \mathbb{R}_+$, so the Cournot duopoly with linear inverse demand function is a potential game.

Let us give another definition similar to Definition 2.8:

Definition 2.9. A game $\Gamma = \langle \mathcal{N}; U_1, \ldots U_N; f_1, \ldots, f_N \rangle$ is a *weighted potential* game if there exists a function $P : U \to \mathbb{R}$ (the potential) such that for each player $i \in \mathcal{N}$, each strategy profile $x_{-i} \in U_{-i}$ of i's opponents, and each pair $x_i, y_i \in U_i$ of strategies of player i and for fixed weights $\omega_i > 0$:

$$f_i(y_i, x_{-i}) - f_i(x_i, x_{-i}) = \omega_i(P(y_i, x_{-i}) - P(x_i, x_{-i})).$$

In weighted potential games, a change in payoff due to a unilaterally deviating player matches the sign in the value of P but scaled by a weight factor. Clearly, all exact potential games are weighted potential games with all players' weights equal to 1.
Although defined separately, exact potential games and weighted potential games are equivalent if we scale appropriately the payoff functions.

Lemma 2.1. $\Gamma = \langle \mathcal{N}; U_1, \dots U_N; f_1, \dots, f_N \rangle$ *is a weighted potential game with potential function* P *and weights* $(\omega_i)_{i=1}^N$ *if and only if*

$$\Gamma' = \left\langle \mathcal{N}; U_1, \dots U_N; g_1 = \frac{1}{\omega_1} f_1, \dots, g_N = \frac{1}{\omega_N} f_N \right\rangle$$

is a potential game with potential function P*.*

More general is the following definition:

Definition 2.10. A game $\Gamma = \langle \mathcal{N}; U_1, \dots U_N; f_1, \dots, f_N \rangle$ is an *ordinal potential* game if there exists a function $P : U \to \mathbb{R}$ such that for each player $i \in \mathcal{N} = \{1, \dots, N\}$, each strategy profile $x_{-i} \in U_{-i}$ of i's opponents, and each pair $x_i, y_i \in U_i$ of strategies of player i:

$$f_i(y_i, x_{-i}) - f_i(x_i, x_{-i}) > 0 \iff P(y_i, x_{-i}) - P(x_i, x_{-i}) > 0.$$

The function P is called an ordinal potential of the game Γ.

In words, if P is an ordinal potential of Γ, the sign of the change in payoff to a unilaterally deviating player matches the sign of the change in the value of P.

Example 2.7. If we consider a symmetric duopoly Cournot competition with linear cost function $c_i(q_i) = cq_i$ with $i \in \{1, 2\}$ and with generic positive inverse demand function $F(q_1 + q_2)$, the profit function of firm i is defined on $\mathbb{R}^2_+$ as

$$f_i(q_1, q_2) = F(q_1 + q_2)q_i - cq_i.$$

Define a function $P : \mathbb{R}^2_{++} \to \mathbb{R}$:

$$P(q_1, q_2) = q_1 q_2 (F(q_1 + q_2) - c).$$

For every firm i and for every $q_{-i} \in \mathbb{R}_{++}$,

$$f_i(q_i, q_{-i}) - f_i(x_i, q_{-i}) > 0 \iff P(q_i, q_{-i}) - P(x_i, q_{-i}) > 0,$$

for every $q_i, x_i \in \mathbb{R}_{++}$, so the Cournot duopoly with generic inverse demand function is an ordinal potential game.

As announced at the beginning of this section, potential games (exact and ordinal ones) are really useful in order to show the existence of Nash equilibria as shown in the following:

Theorem 2.4 ([91])**.** *Let Γ be a potential game with a potential function P. If we assume that the set of maxima of P is non empty, then Γ admits at least one NE.*
If, in addition, U is compact, convex set, and P is a continuously differentiable function on the interior of U and strictly concave on U, then the NE of Γ is unique.

Corollary 2.5 ([73])**.** *Every finite potential game admits at least one NE.*

Corollary 2.6. *Let Γ be an infinite potential game with a potential function P. If U is a compact set and P is upper semicontinuous on U then there exists at least one NE of Γ.*

For further details and properties see [53] and [73].

2.5 Best-reply Potential Games and Quasi aggregative Games

In potential games introduced in the previous section, information concerning Nash equilibria can be incorporated into a single real-valued function on the strategy space. All classes of potential games that Monderer and Shapley ([73]) defined share the finite improvement property: start with an arbitrary strategy profile that can be improved deviating to a better strategy. Under this property, this process eventually ends in a Nash equilibrium. Voorneveld in [101] introduces a new class of potential games namely the best-reply potential games. The main distinctive feature is that it allows infinite improvement paths, by imposing restrictions only on paths in which players that can improve actually deviate to a best reply.

Definition 2.11. A game

$$\Gamma = \langle \mathcal{N}, (U_i)_{i\in\mathcal{N}}, (f_i)_{i\in\mathcal{N}} \rangle$$

is called *best-reply potential* game if there exists a function $P : U \to \mathbb{R}$ such that for any $i \in \mathcal{N}$ and for any $x_{-i} \in U_{-i}$

$$argmax_{x_i\in U_i} f_i(x_i, x_{-i}) = argmax_{x_i\in U_i} P(x_i, x_{-i}).$$

The function P is called a (best-reply) *potential* of the game Γ.

In words, a game $\Gamma = \langle \mathcal{N}, (U_i)_{i\in\mathcal{N}}, (f_i)_{i\in\mathcal{N}} \rangle$ is a best-reply potential game if there exists a coordination game $\langle \mathcal{N}, (U_i)_{i\in\mathcal{N}}, P \rangle$ where the payoff of each player is given by function P such that the best-reply correspondence of each player $i \in \mathcal{N}$ in Γ coincides with his best-reply correspondence in the coordination game.

Analogous to what obtained in [73], the following holds:

Proposition 2.3. *Let* $\Gamma = \langle \mathcal{N}, (U_i)_{i\in\mathcal{N}}, (f_i)_{i\in\mathcal{N}} \rangle$ *be a best-reply potential game with best-reply potential* P*:*

- *the Nash equilibria of* $\Gamma = \langle \mathcal{N}, (U_i)_{i\in\mathcal{N}}, (f_i)_{i\in\mathcal{N}} \rangle$ *and* $\langle \mathcal{N}, (U_i)_{i\in\mathcal{N}}, P \rangle$ *coincide;*

- *if* P *has a maximum over* U *(e.g. if* U *is finite),* Γ *has a Nash equilibrium.*

For more details, see [101].
In [41] the class of quasi-aggregative games is introduced and conditions under which such games admit a best-reply potential are established, implying existence of a pure strategy Nash equilibrium without any convexity or quasi concavity assumptions.

Definition 2.12. A game $\Gamma = \langle \mathcal{N}, (U_i)_{i\in\mathcal{N}}, (f_i)_{i\in\mathcal{N}} \rangle$ is called a *quasi-aggregative* game with *aggregator* function $g : U \to \mathbb{R}$ if there exist continuous functions $F_i : \mathbb{R} \times U_i \to \mathbb{R}$ (the *shift functions*) and $\sigma_i : U_{-i} \to Y_{-i} \subseteq \mathbb{R}$, $i \in \mathcal{N}$, (the *interaction functions*) such that each of the payoff functions f_i with $i \in \mathcal{N}$ can be written:

$$f_i(x) = \overline{f}_i(\sigma_i(x_{-i}), x_i)$$

where $\overline{f}_i : Y_{-i} \times U_i \to \mathbb{R}$ and

$$g(x) = F_i(\sigma_i(x_{-i}), x_i)$$

for all $x \in U$ and $i \in \mathcal{N}$.

In order to prove that a quasi-aggregative game admits a best-reply potential, the following assumptions are required:

Assumption 2.1. Each of the correspondences $R_i : Y_{-i} \to 2^{U_i}$ is strictly decreasing (i.e. every selection from R_i is decreasing).

Assumption 2.2. The shift-functions $F_i : Y_{-i} \times U_i \to \mathbb{R}$, $i \in \mathcal{N}$, all exhibit increasing differences in y_{-i} and x_i.

Theorem 2.7 ([41]). *Let Γ be a quasi-aggregative game with compact strategy sets and upper semicontinuous payoff functions. Then if Assumptions 2.1 and 2.2 are satisfied, the game is a best-reply potential game. Moreover, associated potential function may be found which is upper semicontinuous.*

2.6 Aggregative Games

Let us consider a non-cooperative game in normal form $\Gamma = \langle \mathcal{N}, (U_i)_{i\in\mathcal{N}}, (f_i)_{i\in\mathcal{N}} \rangle$ where

- $\mathcal{N} = \{1, \ldots, N\}$ is the finite set of players ($N \in \mathbb{N}$ is a natural number);
- for any $i \in \mathcal{N}$, $U_i \subseteq \mathbb{R}^N$ is the finite-dimensional strategy set and $f_i : U \to \mathbb{R}$ the payoff function of player i.

As usual, let us denote by $U = \prod_{i=1}^{N} U_i$ and $U_{-i} = \prod_{j\neq i} U_j$, $i \in \mathcal{N}$.

Definition 2.13. The game $\Gamma = \langle \mathcal{N}, (U_i)_{i\in\mathcal{N}}, (f_i)_{i\in\mathcal{N}} \rangle$ is called *aggregative* if there exists a continuous and additively separable function $g : U \to \mathbb{R}$ (the aggregator) and functions $F_i : U_i \times \mathbb{R} \to \mathbb{R}$ (the reduced payoff functions) such that for each player $i \in \mathcal{N}$:

$$f_i(x_i, x_{-i}) = F_i(x_i, g(x))$$

for $x_i \in U_i$ and for all $x \in U$ (see [2], [3], [18]).

The game $\Gamma = \langle \mathcal{N}, (U_i)_{i\in\mathcal{N}}, (f_i)_{i\in\mathcal{N}} \rangle$ is called *additively separable aggregative* if there exists a continuous and additively separable function $g : U \to \mathbb{R}$ (the aggregator) and functions $l_i : \mathbb{R} \to \mathbb{R}$ and functions $m_i : U_i \to \mathbb{R}$ such that for each player $i \in \mathcal{N}$:

$$f_i(x_i, x_{-i}) = l_i(g(x)) + m_i(x_i)$$

for $x_i \in U_i$ and for all $x \in U$ (see [2]).

Recall that a function $g : U \to \mathbb{R}$ is *additively separable* if there exist strictly increasing functions $H : \mathbb{R} \to \mathbb{R}$ and $h_1, \dots h_N : U_i \to \mathbb{R}$ such that $g(x) = H\Big(\sum_{i\in\mathcal{N}} h_i(x_i)\Big)$ for all $x \in U$ (see [35]). Obvious examples of additively separable functions are given by the sum $g(x) = \sum_{i\in\mathcal{N}} x_i$ and the mean $g(x) = N^{-1}\sum_{i\in\mathcal{N}} x_i$. Moreover, Constant Elasticity of Substitution (CES) function, i.e. $g(x) = (\alpha_1 x_1^\beta + \cdots + \alpha_N x_N^\beta)^{\frac{1}{\beta}}$ with $U \subseteq \mathbb{R}^N_+$ and $\beta, \alpha_1, \dots \alpha_N > 0$, and Cobb-Douglas function, i.e. $g(x) = \prod_{i\in\mathcal{N}} x_i^{\alpha_i}$ with $\alpha_1, \dots, \alpha_N > 0$, are additively separable functions [4].

We deal in the following with aggregative games

$$\Gamma = \langle \mathcal{N}, (U_i)_{i\in\mathcal{N}}, (F_i)_{i\in\mathcal{N}}, g\rangle$$

where g is the aggregator that is additively separable and $F_i(x_i, g(x))$ are real valued functions defined on $U_i \times \mathbb{R}$ for any $i \in \mathcal{N}$, which are the reduced payoffs.

Let us give the definition of equilibrium:

Definition 2.14. Let

$$\Gamma = \langle \mathcal{N}, (U_i)_{i\in\mathcal{N}}, (F_i)_{i\in\mathcal{N}}, g\rangle$$

be an aggregative game. Then $x^* = (x_1^*, \dots, x_N^*)$ is a pure Nash equilibrium if for any $i \in \mathcal{N}$,

$$x_i^* \in argmax_{x_i\in U_i} F_i(x_i, g(x_i, x_{-i}^*)).$$

If x^* is an equilibrium, then an *equilibrium aggregate* is $Q \equiv g(x^*)$. If there exist the smallest and the largest equilibrium aggregate, these are denoted by Q_* and Q^*, respectively.

Let us introduce the *reduced best-reply correspondence*

$$BR_i(x_{-i}) = \tilde{BR}_i(\sum_{j\neq i} h_j(x_j))$$

[4] For the CES function $h_i(x_i) = \alpha_i x_i^\beta$(with $x_i \geq 0$) and $H(z) = z^{\frac{1}{\beta}}$. For the Cobb-Douglas function $h_i(x_i) = \alpha_i log(x_i)$ and $H(z) = exp(z)$ (with $x_i > 0$).

that is the best reply correspondence of player i that depends on the *aggregate of the other players* $\sum_{j\neq i} h_j(x_j)$. Let us now fix an *aggregate* i.e. a value in the aggregator's domain, $Q \in G \equiv \{g(x) : x \in U\}$. Since $Q = g(x) \iff \sum_{j\neq i} h_j(x_j) = H^{-1}(Q) - h_i(x_i)$, we can find the set of best-replies for player i as

$$B_i(Q) \equiv \{x_i \in U_i : x_i \in \tilde{BR}_i(H^{-1}(Q) - h_i(x_i))\}.$$

$B_i : G \to 2^{U_i}$ is called *backward reply correspondence* of player i. If we consider a value Q for which $Q = g(x^*)$ and $x^* \in B_i(Q)$ for all i, lead to an equilibrium. Thus, introducing $Z : G \to 2^G$ such that $Z(Q) \equiv \{g(x) \in G : x_i \in B_i(Q) \forall i \in \mathcal{N}\}$, Q is an equilibrium aggregate if and only if $Q \in Z(Q)$.
Given these notions, it is useful to recall two existence results obtained in [2]. In order to give the first result, let us introduce the following definition:

Definition 2.15. An aggregative game $\Gamma = \langle \mathcal{N}, (U_i)_{i\in\mathcal{N}}, (F_i)_{i\in\mathcal{N}}, g\rangle$ is said to be a *nice aggregative game* if:

- the aggregator g is twice continuously differentiable;
- each strategy set U_i is compact and convex, and every payoff function $f_i(x) = F_i(x_i, g(x))$ is twice continuously differentiable and pseudo-concave in the player's own strategies[5];
- for each player, the first-order conditions hold whenever a boundary strategy is a best response, i.e., $D_{x_i}F_i(x_i, g(x)) = 0$ whenever $x_i \in \partial U_i$ and $(v - x_i)^T D_{x_i}F_i(x_i, g(x)) \leq 0$ for all $v \in U_i$.

Thus, under convexity assumptions, the following holds:

[5] A differentiable function $f_i(x_i, x_{-i})$ is *pseudo-concave* in x_i if for all $x_i, x_i' \in U_i$

$$(x_i' - x_i)^T D_{x_i} f_i(x_i, x_{-i}) \leq 0 \Rightarrow f_i(x_i', x_{-i}) \leq f_i(x_i, x_{-i}),$$

(see [67]).

Theorem 2.8 ([2]). *Let $\Gamma = \langle \mathcal{N}, (U_i)_{i\in\mathcal{N}}, (F_i)_{i\in\mathcal{N}}, g\rangle$ be a nice aggregative game. Then, there exists an equilibrium $x^* \in U$ and also the smallest and largest equilibrium aggregates Q_* and Q^*.*

The second useful result is another existence result, obtained without any assumption of convexity, but with assumption of supermodularity and decreasing differences.

Definition 2.16. The game $\Gamma = \langle \mathcal{N}, (U_i)_{i\in\mathcal{N}}, (F_i)_{i\in\mathcal{N}}, g\rangle$ is an *aggregative game with strategic substitutes* if it is aggregative, strategy sets are lattices and each player's payoff function $f_i(x_i, x_{-i})$ is supermodular in x_i and exhibits decreasing differences in x_i and x_{-i}.

Theorem 2.9 ([2]). *Let $\Gamma = \langle \mathcal{N}, (U_i)_{i\in\mathcal{N}}, (F_i)_{i\in\mathcal{N}}, g\rangle$ be an aggregative game with strategic substitutes. Then, there exists an equilibrium $x^* \in U$ and also the smallest and largest equilibrium aggregates Q_* and Q^*.*

If moreover, we consider a parametric aggregative game

$$\Gamma_t = \langle \mathcal{N}, (U_i)_{i\in\mathcal{N}}, (F_i)_{i\in\mathcal{N}}, g, t\rangle$$

where, denoting by $\mathcal{T}$ a state space, $t \in \mathcal{T}$ is a particular state of the world, which is an exogenous parameter, and, for any $i \in \mathcal{N}$, $f_i : U \times \mathcal{T} \to \mathbb{R}$ is the payoff function of player i, we can give the following definition:

Definition 2.17. The game $\Gamma_t = \langle \mathcal{N}, (U_i)_{i\in\mathcal{N}}, (f_i)_{i\in\mathcal{N}}, t\rangle$ is called *aggregative* if there exists a continuous and additively separable function $g : U \to \mathbb{R}$ (the aggregator) and functions $F_i : U_i \times \mathbb{R} \times \mathcal{T} \to \mathbb{R}$ (the reduced payoff functions) such that for each player $i \in \mathcal{N}$:

$$f_i(x_i, x_{-i}, t) = F_i(x_i, g(x), t)$$

for $x_i \in U_i$, for all $x \in U$ and for all $t \in \mathcal{T}$ (see [2], [3], [18]).

In this parametric context, if $x^*(t)$ is an equilibrium, then an *equilibrium aggregate*, given t, is $Q(t) \equiv g(x^*(t))$. If there exist the smallest and the largest equilibrium aggregate, these are denoted by $Q_*(t)$ and $Q^*(t)$, respectively. Thus Theorems 2.8, 2.9 can be restated in the following way:

Theorem 2.10 ([2]). *Let $\Gamma_t = \langle \mathcal{N}, (U_i)_{i\in\mathcal{N}}, (F_i)_{i\in\mathcal{N}}, g, t\rangle$ be a nice aggregative game for any $t \in \mathcal{T}$. Then, there exists an equilibrium $x^*(t) \in U$ and also the smallest and largest equilibrium aggregates $Q_*(t)$ and $Q^*(t)$. Moreover, $Q_* : \mathcal{T} \to \mathbb{R}$ is a lower semicontinuous function and $Q^* : \mathcal{T} \to \mathbb{R}$ is an upper semicontinuous function.*

Theorem 2.11 ([2]). *Let $\Gamma_t = \langle \mathcal{N}, (U_i)_{i\in\mathcal{N}}, (F_i)_{i\in\mathcal{N}}, g, t\rangle$ be an aggregative game with strategic substitutes for any $t \in \mathcal{T}$. Then, there exists an equilibrium $x^*(t) \in U$ and also the smallest and largest equilibrium aggregates $Q_*(t)$ and $Q^*(t)$. Moreover, $Q_* : \mathcal{T} \to \mathbb{R}$ is a lower semicontinuous function and $Q^* : \mathcal{T} \to \mathbb{R}$ is an upper semicontinuous function.*

2.7 Partial Cooperative Games

In various practical situations the interaction between agents can be a mixture of non-cooperative and cooperative behaviour. Considering a N-person game, partial cooperation between a portion of players that sign a cooperative agreement is firstly studied in [64] for symmetric potential games, assuming that the non-cooperators' reaction set, given by the solution of non-cooperators' Nash equilibrium problem, is a singleton.

In [65] and [66] an extention of the partial cooperation framework to certain games is presented. In these games the non-cooperating players select from multiple best replies. In [66] symmetric aggregative games are considered and it is assumed that the non-cooperators coordinate on the symmetric Nash equilibrium that yields the highest payoff to them, and, thus, do not consider any non-symmetric Nash equilibrium that, possibly, might result in higher payoff for all. In [65], on the other hand, it is assumed that the non-cooperators coordinate on the Nash equilibrium with the greatest or lowest strategy vector, irrespective of the payoff attained by the non-cooperators in this equilibrium. Mallozzi and Tijs′ assumptions are inapplicable to most strategic situations. However, at the same time they are hard to do away with. As long as the game is symmetric, identical strategies will lead to identical payoffs. Hence, the choice of a strategy that maximizes individual payoffs in the coalition of cooperators also maximizes joint payoffs if we assume that all members of the coalition of cooperators select the same strategy. Hence, cooperation is sustainable without payoff sharing as long as cooperation yields higher payoffs compared to the purely non-cooperative situation in which a Nash equilibrium outcome is attained, provided it exists. If the game is not symmetric, however, the selection of the same strategy by the coalition of cooperators need not confer identical payoffs to all members of the group. In [15] the symmetric assumption is bypassed assuming that the coalition of cooperators can choose a strategy that maximizes its joint payoffs. Moreover they assume that the coalition of cooperators is risk-averse and chooses a maximin strategy. Hence, if there are multiple best replies given a strategic agreement of the coalition of cooperators, then the coalition of cooperators takes into account only the worst possible outcome.

Let us remark that in the context of partial cooperative games, two kinds of solution concept can be considered: the partial cooperative Nash equilibrium and the partial cooperative leadership equilibrium.

For more details on the partial cooperative Nash equilibrium, see [15].

Here, we focus on the second kind of equilibrium founded on the idea that the coalition of cooperators has a strategic leadership position with respect to the non-cooperators. Thus, the non-cooperators find the best reply to the other players' actions and the coalition of cooperators anticipates the non-cooperators' best reply selection.

Let us formalize the definition of partial cooperative leadership equilibrium, recalling also an existence result for it (see [15]), using similar conditions to the Nash theorem in [79] .

Given a N-person normal form game $\Gamma = \langle \mathcal{N}, (U_i)_{i=1}^N, (f_i)_{i=1}^N \rangle$, let us fix $k \in \{0, \dots, N\}$, called *level of cooperation*, let us suppose that k players cooperate and let us denote $C = \{N-k+1, \dots, N\}$ the set of cooperators and $\overline{C} = \{1, \dots, N-k\}$ the complement of C i.e. the set of non-cooperators. Denote by $x^C = (x_{N-k+1}, \dots, x_N) \in \prod_{i=N-k+1}^N U_i$ and $x^{\overline{C}} = (x_1, \dots, x_{N-k}) \in \prod_{j=1}^{N-k} U_j$. The game $\overline{\Gamma} = \left\langle C, \overline{C}, (U_i)_{i\in C}, (U_j)_{j\in\overline{C}}, (f_i)_{i\in C}, (f_j)_{j\in\overline{C}} \right\rangle$ is called *partial cooperative game.*

In order to explicitely introduce the leader-follower equilibrium concept, for any $x^C \in \prod_{i=N-k+1}^N U_i$ denote $\Gamma_{N-k}^{x^C} = \langle \overline{C}, \prod_{j=1}^{N-k} U_j, \omega^{x^C} \rangle$ the normal form game, called *conditional partial cooperative game*, given by player set $\overline{C}$ of non-cooperators whose strategy set is U_j and who have conditional payoff function $\omega_j^{x^C}(x^{\overline{C}}) : \prod_{j=1}^{N-k} U_j \to \mathbb{R}$ defined as

$$\omega_j^{x^C}(x^{\overline{C}}) = f_j(x^{\overline{C}}, x^C).$$

The set of the conditional game's Nash equilibria is denoted by $NE_{x^C} \subset \prod_{j=1}^{N-k} U_j$.

Assuming that $NE_{x^C} \neq \emptyset$, in order to select one Nash equilibrium among followers, as previously anticipated, we assume that cooperators are pessimistic in the sense that the coalition of the cooperators, the leader, supposes that the followers' (non-cooperators) choice is the worst for herself and select a maxmin strategy. Thus, let us introduce

$$f(x^C) = \min_{x^{\overline{C}} \in NE_{x^C}} \sum_{i=N-k+1}^{N} f_i(x^{\overline{C}}, x^C)$$

and

$$\tilde{U} = \{\tilde{x}^C \in \prod_{i=N-k+1}^{N} U_i : f(\tilde{x}^C) = \max_{x^C \in \prod_{i=N-k+1}^{N} U_i} f(x^C)\}.$$

Definition 2.18. An action tuple $(x_*^{\overline{C}}, x_*^C) \in \prod_{j=1}^{N-k} f_j \times \prod_{i=N-k+1}^{N} f_i$ is a *partial cooperative leadership equilibrium* for the game $\overline{\Gamma}$ if $x_*^C \in \tilde{U}$ and

$$x_*^{\overline{C}} \in arg \min_{x^{\overline{C}} \in NE_{x_*^C}} \sum_{i=N-k+1}^{N} f_i(x^{\overline{C}}, x_*^C).$$

In this context the following holds:

Theorem 2.12 ([15]). *Assume that*

- *all action sets U_i for $i \in C$ and U_j for $j \in \overline{C}$ are non-empty, compact and convex subset of an Euclidean space;*
- *all payoff functions f_i for $i \in C$ and f_j for $j \in \overline{C}$ are continuous on the space of action tuple $\prod_{i \in C} U_i \times \prod_{j \in \overline{C}} U_j$;*
- *for every non-cooperator $j \in \overline{C}$ the payoff function f_j is quasi-concave on U_j.*

Then the game $\overline{\Gamma}$ admits at least one partial cooperative leadership equilibrium.

Chapter 3

Multi-Leader Multi-Follower Aggregative Uncertain Games

In this chapter, we generalize the multi-leader multi-follower equilibrium concept for the class of aggregative games (see [18], [41], [98]), namely games where each player's payoff depends on his own actions and an aggregate of the actions of all the players.

We present the multi-leader multi-follower equilibrium model under uncertainty, assuming an exogenous uncertainty affecting the aggregator, and some existence results for the stochastic resulting game are obtained in the smooth case of nice aggregative games, where payoff functions are continuous and concave in own strategies, as well as in the general case of aggregative games with strategic substitutes. Applicative examples, such as the global emission game and the teamwork project game, are illustrated. For more details see [54].

We point out that our results hold for the general class of aggregative games and generalize the ones obtained by De Miguel and Xu (see [24]) and by Nakamura (see [77]) for the Cournot oligopoly games. Moreover, we briefly discuss the experimental evaluation based on the Sample Average Approximation (SAA) method (see [44]) for the global emission game.

We present the model in Section 3.1, then we study it in the smooth case in Section 3.2 and in the strategic substitutes case in Section 3.3, providing existence theorems and examples.

3.1 The model

We consider a $M+N$-player aggregative game where M players (leaders) have the leadership in the decision process: they commit a strategy knowing the best reply of other N players (followers) who are involved in a non-cooperative Nash equilibrium problem. Here, $M, N \in \mathbb{N}_0 = \{0, 1, 2, \dots\}$; if M or N is equal to zero, the game turns out to be a non-cooperative Nash equilibrium problem.

We suppose that there is a shock in the game that hits the payoff functions of both leaders and followers that is represented by a continuous random variable $\xi : \Omega \to \mathbb{R}$ on a probability state space $(\Omega, \Sigma, \mathcal{P})$. Of course, we obtain different payoffs for every realization of the random variable ξ and, thus, by the distribution of ξ, we can characterize the uncertainty in the payoff functions.

Thus, let us consider the following aggregative normal form game

$$\Gamma = \left\langle \mathcal{M} + \mathcal{N}, (U_i)_{i=1}^M, (V_j)_{j=1}^N, (l_i)_{i=1}^M, (f_j)_{j=1}^N, g, (\Omega, \Sigma, \mathcal{P}), \xi \right\rangle$$

where:

- $\mathcal{M} = \{1, \dots, M\}$, with M fixed number of leaders and $\mathcal{N} = \{1, \dots, N\}$ with N fixed number of followers;
- for every $i \in \mathcal{M}$, $U_i \subseteq \mathbb{R}^+$ is the finite-dimensional strategy set of leader i and, for every $j \in \mathcal{N}$, $V_j \subseteq \mathbb{R}^+$ is the finite-dimensional strategy set of follower j;
- denoted by $U = \prod_{i=1}^M U_i$ and $V = \prod_{j=1}^N V_j$, $g : U \times V \to \mathbb{R}$ is the

aggregator function which is additively separable i.e.

$$g(x,y) = H\left(\sum_{i=1}^{M} x_i + \sum_{j=1}^{N} y_j\right) = H(X+Y),$$

where $x = (x_1, \dots, x_M)$ with $x_i \in U_i$, $\forall i = 1, \dots, M$ and $y = (y_1, \dots, y_N)$ with $y_j \in V_j$, $\forall j = 1, \dots, N$ and $H : \mathbb{R} \to \mathbb{R}$ is a strictly increasing function;

- $(\Omega, \Sigma, \mathcal{P})$ is a probability state space, where any $\omega \in \Omega$ represents some state of the world;
- $\xi : \Omega \to \mathbb{R}$ is a stochastic variable on the probability state space $(\Omega, \Sigma, \mathcal{P})$;
- for every $i \in \mathcal{M}$ and for every $j \in \mathcal{N}$, $l_i : U_i \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ and $f_j : V_j \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ are payoff functions respectively of leader i and follower j, tipically described as $l_i(x_i, g(x,y), \xi(\omega)) = l_i(x_i, H(X+Y), \xi(\omega))$ and $f_j(y_j, g(x,y), \xi(\omega)) = f_j(y_j, H(X+Y), \xi(\omega))$.

The jth follower chooses his strategy after observing the realization of the shock and the strategies chosen by all the leaders and he will keep the aggregate quantity of the leaders and the quantity of the other followers fixed. Thus, for a fixed $x \in U$ and $\xi(\omega) \in \mathbb{R}$, the followers solve a Nash equilibrium problem:

$$max_{y_j \in V_j} f_j(y_j, H(X + y_j + Y_{-j}), \xi(\omega)) \tag{3.1}$$

for any $j = 1, \dots, N$, where $Y_{-j} = \sum_{k \neq j} y_k$ and $X = \sum_{i=1}^{M} x_i$ is the aggregate leaders' committed strategy.

The ith leader chooses his strategy knowing the payoff function only in distribution since the shock $\xi(\omega)$ is not realized yet. Moreover, since he acts simultaneously with all other leaders, he must take into account that the strategies of other leaders, $x_{-i} \in U_{-i} = \prod_{k \neq i} U_k$, are fixed and, since he acts before every follower, he must also consider the reaction of the followers to the aggregate leaders' strategy that is a solution to problem 3.1, i.e. $y_1(H(X, \xi(\omega))), \dots, y_N(H(X, \xi(\omega)))$.

Then, if $Y(H(X,\xi(\omega))) = \sum_{j=1}^{N} y_j(H(X,\xi(\omega)))$, any leader considers the expectation with respect to $\xi(\omega)$ of his profit $l_i(x_i, H(X+Y(H(X,\xi(\omega)))), \xi(\omega))$ and solves the problem:

$$max_{x_i\in U_i}\mathbb{E}[l_i(x_i, H(x_i+X_{-i}+Y(H(x_i+X_{-i},\xi(\omega)))), \xi(\omega))], \tag{3.2}$$

where $\mathbb{E}$ denotes expectation with respect to the random variable ξ and $X_{-i} = \sum_{k\neq i} x_k$.

In the following, U and V are assumed to be compact and each $f_j : V_j \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is assumed to be upper semicontinuous on $V_j \times \mathbb{R} \times \mathbb{R}$ and continuous in $\mathbb{R}\times\mathbb{R}$, and analogously each $l_i : U_i \times \mathbb{R}\times\mathbb{R} \to \mathbb{R}$ is assumed to be upper semicontinuous on $U_i \times \mathbb{R}\times\mathbb{R}$ and continuous in $\mathbb{R}\times\mathbb{R}$.

Suppose now that the followers' problem 3.1 has a unique solution.

Definition 3.1. A *multi-leader multi-follower equilibrium with aggregate uncertainty* (MLMFA equilibrium) is an $M+N$-tuple

$$(x_1^\star, \ldots, x_M^\star, y_1(H(X^\star,\cdot)), \ldots, y_N(H(X^\star,\cdot))),$$

such that

$$\mathbb{E}[l_i(x_i^\star, H(X^\star+Y(H(X^\star,\xi(\omega)))), \xi(\omega))] = \tag{3.3}$$

$$max_{x_i\in U_i}\mathbb{E}[l_i(x_i, H(x_i+X_{-i}^\star+Y(H(x_i+X_{-i}^\star,\xi(\omega)))), \xi(\omega))]$$

for any $i = 1, \ldots, M$, where

(3.4)
$$y_j(H(X^\star,\xi(\omega))) \in argmax_{y_j\in V_j} f_j(y_j, H(X^\star+y_j+Y_{-j}(H(X^\star,\xi(\omega)))), \xi(\omega))$$

for any $j = 1, \ldots, N$ and $(y_1(H(X^\star,\xi(\omega))), \ldots, y_N(H(X^\star,\xi(\omega))))$ is the Nash equilibrium among followers given the aggregate leaders' strategy and the realized shock $\xi(\omega)$.

The given definition generalizes to aggregative games the multiple-leader Stackelberg equilibrium given by De Miguel and Xu in the case of a Nash–Cournot oligopoly game (see [24]).

Herein, we suppose that $\xi(\omega)$ is a continuous random variable on the state space Ω with density function $\rho(t)$, with $\mathcal{T}$ supporting set. Thus we can rewrite, for $t \in \mathcal{T}$, the normal form game Γ as

$$\langle \mathcal{M} + \mathcal{N}, (U_i)_{i=1}^M, (V_j)_{j=1}^N, (l_i)_{i=1}^M, (f_j)_{j=1}^N, g, t \rangle$$

where the leaders' and the followers' payoff functions are rewritten in the following way:

$$\mathbb{E}[l_i(x_i, H(X + Y(H(X, \xi(\omega)))), \xi(\omega)) = \int_{\mathcal{T}} l_i(x_i, H(x_i + X_{-i} + Y(H(x_i + X_{-i}, t))), t)\rho(t)dt \tag{3.5}$$

for any $i \in \mathcal{M}$ and for $t \in \mathcal{T}$ and

$$f_j(y_j, g(x, y), t) = f_j(y_j, H(X + y_j + Y_{-j}), t)$$

for any $j \in \mathcal{N}$ and for $t \in \mathcal{T}$.

Remark 3.1. If $\xi : \Omega \to \mathbb{R}$ is a discrete random variable i.e., $\xi(\Omega)$ is finite or countable, $\xi(\Omega) = \{t_1, \ldots, t_h, \ldots\}$, the leaders' payoff functions are defined by

$$\mathbb{E}[l_i(x_i, H(X + Y(H(X, \xi(\omega)))), \xi(\omega)) = \sum_h l_i(x_i, H(x_i + X_{-i} + Y(H(x_i + X_{-i}, t_h))), t_h)p(t_h) \tag{3.6}$$

for $i = 1, \ldots, M$, where $p(t_h) = \mathcal{P}(\xi(\omega) = t_h)$ i.e the probability that the realization of the random variable is t_h, for any $h \in \mathbb{N}$.

Remark 3.2. In the deterministic case, when $M = N = 1$, the model corresponds to the classical Stackelberg Leader–Follower problem (see [100]). The case $M = 1$ and $N \geq 1$ has been introduced in the oligopolistic market context in [94], [95] and studied from a computational point of view in [52], where it has been called MPEC, and it has been applied in other several contexts, for example in transportation in [68].

3.2 The Regular Case

In order to prove the existence and uniqueness of the follower Nash equilibrium and the existence of a multi-leader multi-follower equilibrium with aggregate uncertainty, we will use the following assumptions, already presented in literature (see [2], [19]).

Assumption 3.1. The aggregative game

$$\langle \mathcal{M} + \mathcal{N}, (U_i)_{i=1}^M, (V_j)_{j=1}^N, (l_i)_{i=1}^M, (f_j)_{j=1}^N, g, t \rangle$$

is an aggregative nice game for every $t \in \mathcal{T}$ i.e.,

- the aggregator g is twice continuously differentiable;
- every strategy set U_i and V_j for $i = 1, \dots, M$ and $j = 1, \dots, N$ is compact and convex;
- the payoff functions $l_i(x_i, g(x, y), t)$ and $f_j(y_j, g(x, y), t)$ are twice continuously differentiable and pseudo-concave in the player's own strategy for all $i = 1, \dots, M$ and for all $j = 1, \dots, N$;
- $D_{x_i} l_i(x_i, g(x, y), t) = 0$ whenever $x_i \in \partial U_i$ and $(v - x_i) D_{x_i} l_i(x_i, g(x, y), t) \leq 0$ $\forall v \in U_i$ and $D_{y_j} f_j(y_j, g(x, y), t) = 0$ whenever $y_j \in \partial V_j$ and $(u - y_j) D_{y_j} f_j(y_j, g(x, y), t) \leq 0$ $\forall u \in V_j$ for all $i = 1, \dots, M$ and for all $j = 1, \dots, N$ (that means that the first-order conditions hold whenever a best response is on the boundary).

Remark 3.3. By using these assumptions on l_i, for all $i = 1, \dots, M$, and the theorem of differentiation under the integral sign, it also follows that the expected payoff functions $\mathbb{E}[l_i(x_i, H(X + Y(H(X, \xi(\omega)))), \xi(\omega))]$ satisfy the Assumption 3.1 for any $i = 1, \dots, M$.

Now let us give another assumption on the followers' side.

Let us introduce for any $t \in \mathcal{T}$ and $j = 1, \dots, N$ the marginal profit that we can denote by

$$\pi_j(y_j, H(X+Y), t) := D_1 f_j(y_j, H(X+Y), t) + D_2 f_j(y_j, H(X+Y), t)\frac{\partial H}{\partial y_j}(X+Y),$$

where $D_1 f_j = \frac{\partial f_j}{\partial y_j}$ and $D_2 f_j = \frac{\partial f_j}{\partial H}$.

Assumption 3.2. If $(y_j, H(X+Y))$ satisfies $y_j < H(X+Y)$ and the marginal profit $\pi_j(y_j, H(X+Y), t) = 0$, then

(i) $\frac{\partial \pi_j}{\partial y_j} < 0$,

where $\frac{\partial \pi_j}{\partial y_j} = D_{11} f_j(y_j, H(X+Y), t) + 2D_{12} f_j(y_j, H(X+Y), t)\frac{\partial H}{\partial y_j}(X+Y) + D_{22} f_j(y_j, H(X+Y), t)\frac{\partial H}{\partial y_j}(X+Y) + D_2 f_j(y_j, H(X+Y), t)\frac{\partial^2 H}{\partial y_j^2}(X+Y)$;

(ii) $y_j \frac{\partial \pi_j}{\partial y_j} + H(X+Y)\frac{\partial \pi_j}{\partial H} < 0$.

Note that (i) corresponds to the law of diminishing marginal utility, while (ii) is assumed in order to obtain that the share functions are strictly decreasing and so to ensure the uniqueness of the followers' Nash equilibrium (see [19]).

Example 3.1. (see [24]). An oligopolistic situation with $M + N$ firms that supply an homogeneous product non-cooperatively is given. M leader firms announce their quantities in $U_1, \dots U_M$ and the rest of the N firms react by choosing a Cournot–Nash equilibrium in $V_1, \dots, V_N$. We consider:

- U_i and V_j for all $i = 1, \dots, M$ and for all $j = 1, \dots, N$ are compact subsets of $\mathbb{R}^+$ [1];
- the aggregator is the sum of the strategies i.e., $g(x, y) = X + Y$;
- given an exogenous random variable $\xi(\omega)$, which represents the market size based on a state space Ω describing the preferential fluctuations for the product under consideration, the payoff function for every leader i is

$$\mathbb{E}[l_i(x_i, X + Y(X, \xi(\omega)), \xi(\omega))] =$$

[1] U_i and V_j are subsets of $\mathbb{R}^+$ with capacity limits.

$$\int_{\mathcal{T}} x_i p(x_i + X_{-i} + Y(x_i + X_{-i}, t), t)\rho(t)dt - C_i(x_i),$$

where p is the inverse demand function that depends on the aggregate quantity and the random variable $\xi(\omega)$ and C_i is the ith leader's cost function, with p and C_i for all $i = 1, \dots, M$ being twice continuously differentiable functions;

- the payoff function for every follower j is

$$f_j(y_j, X + Y, t) = y_j p(X + y_j + Y_{-j}, t) - c_j(y_j),$$

where c_j is the jth follower's cost function, that is twice continuously differentiable, and t is the realization of the random variable.

In [24] this framework is used to model competition in the Telecommunication Industry, proposing a computational approach for finding equilibria for this market.

Remark 3.4. Note that Assumptions 3.1 and 3.2 hold for Example 3.1.

Fixing $x = (x_1, \dots, x_M)$, for any $t \in \mathcal{T}$, we consider the reduced aggregative game $\langle \mathcal{N}, (V_j, f_j)_{j=1}^N, g, t\rangle$.

Theorem 3.1. ***(Existence of an MLMFA Equilibrium)*** *Under Assumptions 3.1 and 3.2, the following hold:*

(i) there exists an equilibrium $(y_1(H(X,t)), \dots, y_N(H(X,t))) \in V$ i.e., this N-tuple satisfies 3.4 with $X = \sum_{i=1}^M x_i$;

(ii) denoted by $Q(x,t) = g(x, y_1(H(X,t)), \dots, y_N(H(X,t)))$, which is called an equilibrium aggregate given t and given x, there exist the smallest and largest equilibrium aggregates with respect to t, while x is fixed, denoted by $Q_*(x,t)$ and $Q^*(x,t)$, respectively;

(iii) $Q_* : U \times \mathcal{T} \to \mathbb{R}$ *is a lower semicontinuous function* $\forall x$ *and* $Q^* : U \times \mathcal{T} \to \mathbb{R}$ *is an upper semicontinuous function* $\forall x$*;*

(iv) the equilibrium $(y_1(H(X,t)), \dots, y_N(H(X,t)))$ *is unique.*

Proof. The result (i) is obtained easily, applying Kakutani's fixed point theorem. In fact, the best-reply correspondences will be upper hemicontinuous and have convex values, since by Assumption 3.1 f_j $\forall j = 1, \dots, N$ are quasi-concave functions (because pseudo-concavity implies quasi-concavity).

Points (ii) and (iii) follow straightforwardly from Theorem 2.10.

Point (iv) follows from [19] because of Assumption 3.2. □

Remark 3.5. By points (ii) and (iv), it follows that $Q_*(x,t) = Q^*(x,t) = Q(x,t)$ and so, by point (iii), we can conclude that the function $Q : U \times \mathcal{T} \to \mathbb{R}$ is a continuous function.

Theorem 3.2. *Under Assumptions 3.1 and 3.2, there exists an MLMFA equilibrium.*

Proof. The existence follows from Theorem 3.1, Remark 3.3 and Theorem 2.10. □

Example 3.2. (Global Emission Game)

We assume that there are four countries, indexed by h, $h \in \{1, \dots, 4\}$, which production and consumption generates emissions $e_h \geq 0$ of a global pollutant as an output. The pollutant is global in the sense that we assume pollution as a public bad and that individual emissions impose negative externalities on all other countries. We assume that, among these countries, two countries, namely countries i, with $i \in \{1, 2\}$, have commitment power and thus they act as leaders in a two-stage game, with countries j, with $j \in \{3, 4\}$, acting as followers. Assume that a shock hits the payoff functions. This shock could be, for example, caused by a disaster event that generates an exogenous pollutant and we represent it by a random variable ξ.

Thus let us consider the game

$$\left\langle 4, (U_i, l_i)_{i=1}^2, (V_j, f_j)_{j=3}^4, g, (\Omega, \Sigma, \mathcal{P}), \xi \right\rangle,$$

where

- fixing $e_{max} > 0$, $U_i = [0, e_{max}]$ for $i = 1, 2$, $V_j = [0, e_{max}]$ for $j = 3, 4$;
- denoting by $e = (e_1, e_2, e_3, e_4)$ the vector of strategies, $g(e) = \left(\sum_{h=1}^4 e_h \right)^2$;
- $(\Omega, \Sigma, \mathcal{P})$ is a probability state space, where any $\omega \in \Omega$ represents some state of the world;
- $\xi : \Omega \to \mathbb{R}$ is a continuous random variable on the probability state space $(\Omega, \Sigma, \mathcal{P})$ with uniform distribution $\rho(t) = \frac{1}{T}$ $\forall t \in [0, T]$ $(T > 0)$.

Let us consider α,$\beta > 0$. Then,

- for any $t \in [0, T]$, the payoff functions for the followers are

$$f_j(e_j, g(e), t) = \alpha e_j - \frac{e_j^2}{2} - \frac{\beta}{2}(e_1 + e_2 + e_3 + e_4 + t)^2$$

  for $j = 3, 4$;
- the payoff functions for the leaders are

$$\mathbb{E}[l_i(e_i, g(e), \xi(\omega))] = \alpha e_i - \frac{e_i^2}{2} - \frac{\beta}{2} \int_0^T \left[(e_1 + e_2 + e_3 + e_4 + t)^2 \right] \frac{1}{T} dt$$

  for $i = 1, 2$.

Fixing (e_1, e_2) and t, the followers choose the unique symmetric Nash equilibrium

$$e_3(e_1, e_2) = e_4(e_1, e_2) = \frac{\alpha - \beta(e_1 + e_2 + t)}{1 + 2\beta}$$

with $\alpha \geq \beta(2e_{max} + T)$. Thus, the leaders maximize, with respect their own strategy, the following payoff function:

$$\mathbb{E}[l_i(e_i, g(e), \xi(\omega))] = \alpha e_i - \frac{e_i^2}{2} - \frac{\beta}{2} \int_0^T \left[\left(\frac{e_1 + e_2 + 2\alpha + t}{1 + 2\beta} \right)^2 \right] \frac{1}{T} dt,$$

and the MLMFA equilibrium is

$$(e_1^*, e_2^*, e_3^*, e_4^*) =$$

$$= \Big(\frac{2\alpha(1+2\beta)^2 - \beta(4\alpha+T)}{2[(1+2\beta)^2+2\beta]}, \frac{2\alpha(1+2\beta)^2 - \beta(4\alpha+T)}{2[(1+2\beta)^2+2\beta]},$$
$$\frac{(1+2\beta)^2(\alpha-\beta t-2\alpha\beta)+2\beta(\alpha-\beta t+2\alpha\beta)+\beta^2 T}{[(1+2\beta)^2+2\beta](1+2\beta)},$$
$$\frac{(1+2\beta)^2(\alpha-\beta t-2\alpha\beta)+2\beta(\alpha-\beta t+2\alpha\beta)+\beta^2 T}{[(1+2\beta)^2+2\beta](1+2\beta)}\Big),$$

$\forall t \in [0,T]$, $\forall \beta > 0$ and $\forall \alpha \geq \beta(2e_{max}+T)$.

This model corresponds to a global emission game in the context of an IEA (International Environmental Agreement) under the Stackelberg assumption (see [1], [28]), where leaders are signatory countries and followers are non-signatory countries, in a non-cooperative strategic game.

Note that Assumptions 3.1 and 3.2 are satisfied for this game and Theorems 3.1 and 3.2 hold.

Remark 3.6. It is possible to use the classical algorithm called Sample Average Approximations method (SAA) (see, for example, [44]) to give a computational evaluation of the MLMFA equilibrium. It is based on the use of a sample of $\xi(\omega)$ rather than the distribution of the random variable $\xi(\omega)$. .

Let $\xi^1, \dots, \xi^k$ be an independently and identically distributed (i.i.d.) random sample of k realizations of the random variable $\xi(\omega)$. We approximate the ith leader's decision problem by the following SAA problem:

$$max_{x_i \in U_i} \phi_i^k(x_i, X_{-i}) := \frac{1}{k}\sum_{h=1}^{k} \bar{l}_i(x_1, \dots, x_i, \dots, x_M, \xi^h),$$

where, for simplicity, we denote $\bar{l}_i(x_1, \dots, x_i, \dots, x_M, \xi^h) = l_i(x_i, H(x_i + X_{-i} + Y(H(X, \xi^h))), \xi^h)$.

If $(x_1^k, \ldots, x_M^k)$ satisfies $\phi_i^k(x_i^k, X_{-i}^k) = max_{x_i \in U_i} \phi_i^k(x_i, X_{-i}^k)$ for $i = 1, \ldots, M$, then $(x_1^k, \ldots, x_M^k)$ is called a *multi-leader multi-follower equilibrium with aggregate uncertainty of the SAA problem* (MLMFA-SAA equilibrium).

If we introduce the function

$$L(x, z, \xi^h) := \sum_{i=1}^{M} \bar{l}_i(x_1, \ldots, z_i, \ldots, x_M, \xi^h)$$

and the function

$$\phi_k(x, z) := \frac{1}{k} \sum_{h=1}^{k} L(x, z, \xi^h),$$

then $x^k = (x_1^k, \ldots, x_M^k)$ is an MLMFA-SAA equilibrium if and only if

$$\phi_k(x^k, x^k) = max_{z \in U} \phi_k(x^k, z).$$

Note that, if we consider $L(x, z, \xi(\omega)) := \sum_{i=1}^{M} \bar{l}_i(x_1, \ldots, z_i, \ldots, x_M, \xi(\omega))$ and $\phi(x, z) := \mathbb{E}(L(x, z, \xi(\omega))$, it is straightforward to see that the vector $x^* = (x_1^*, \ldots, x_M^*)$ is an MLMFA equilibrium if and only if

$$\phi(x^*, x^*) = max_{z \in U} \phi(x^*, z).$$

Example 3.3. (Computational Evaluation) In the Global Emission Game we have that

$$\bar{l}_1(z_1, x_2, \xi(\omega)) = \alpha z_1 - \frac{z_1^2}{2} - \frac{\beta}{2}\Big(\frac{z_1 + x_2 + 2\alpha + \xi(\omega)}{1 + 2\beta}\Big)^2$$

and

$$\bar{l}_2(x_1, z_2, \xi(\omega)) = \alpha z_2 - \frac{z_2^2}{2} - \frac{\beta}{2}\Big(\frac{x_1 + z_2 + 2\alpha + \xi(\omega)}{1 + 2\beta}\Big)^2$$

and so

$$L(x, z, \xi(\omega) = \alpha(z_1 + z_2) - \frac{(z_1^2 + z_2^2)}{2} - \frac{\beta}{2}\Big(\frac{z_1 + x_2 + 2\alpha + \xi(\omega)}{1 + 2\beta}\Big)^2 -$$

$$-\frac{\beta}{2}\Big(\frac{x_1 + z_2 + 2\alpha + \xi(\omega)}{1 + 2\beta}\Big)^2.$$

With the method proposed above, for a fixed k, we can compute an MLMFA-SAA equilibrium, maximizing over $z \in U$ the function $\phi_k(x_k, z)$:

$$x^k = (x_1^k, x_2^k) = \Big(\frac{\alpha(1+2\beta)^2}{(1+2\beta)^2 + 2\beta} - \frac{2\alpha\beta}{(1+2\beta)^2 + 2\beta} - \frac{\beta \sum_{h=1}^{k} \xi^h}{k[(1+2\beta)^2 + 2\beta]}, \\ \frac{\alpha(1+2\beta)^2}{(1+2\beta)^2 + 2\beta} - \frac{2\alpha\beta}{(1+2\beta)^2 + 2\beta} - \frac{\beta \sum_{h=1}^{k} \xi^h}{k[(1+2\beta)^2 + 2\beta]} \Big).$$

In order to investigate the convergence of a sequence of MLMFA-SAA equilibria for $k \to +\infty$, let us note that $L(x, z, \xi(\omega))$ is a Lipschitz continuous function. Thus, in this case, we can easily obtain that $\phi_k(x, z)$ converges to $\phi(x, z)$ uniformly and, with a probability of one, the sequence $\{x_k\}$ converges to the unique MLMFA equilibrium x^*.

3.3 A More General Case in an Optimistic View

In the previous section, we have considered that the payoff functions are twice continuously differentiable; in this section, we want to avoid this assumption, and, in order to obtain results on the existence of an MLMFA equilibrium in this more general framework, we need an assumption taken from [2].

Assumption 3.3. The aggregative game

$$\big\langle \mathcal{M} + \mathcal{N}, (U_i)_{i=1}^M, (V_j)_{j=1}^N, (l_i)_{i=1}^M, (f_j)_{j=1}^N, g, t \big\rangle$$

is an aggregative game with strategic substitutes for any $t \in \mathcal{T}$, i.e.,

- every strategy set U_i and V_j for $i = 1, \ldots, M$ and $j = 1, \ldots, N$ is a lattice;
- for all $i = 1, \ldots, M$ and for all $j = 1, \ldots, N$ the payoff functions $l_i(x_i, g(x, y), t)$ and $f_j(y_j, g(x, y), t)$ are supermodular in the player's own strategy and exhibit decreasing differences in x_i and X_{-i} and in y_j and Y_{-j}, respectively, for any $t \in \mathcal{T}$.

For any $x = (x_1, \dots, x_M)$ and for any $t \in \mathcal{T}$, we consider the reduced aggregative game $\langle \mathcal{N}, (V_j, f_j)_{j=1}^N, g, t \rangle$.

Theorem 3.3. *Under Assumption 3.3, the following hold:*

(i) there exists an equilibrium $(y_1(H(X,t)), \dots, y_N(H(X,t))) \in V$ *i.e., this* N*-tuple satisfies 3.4;*

(ii) there exist the smallest and largest equilibrium aggregates denoted by $Q_*(x,t)$ *and* $Q^*(x,t)$*, respectively;*

(iii) $Q_* : U \times \mathcal{T} \to \mathbb{R}$ *is a lower semicontinuous function and* $Q^* : U \times \mathcal{T} \to \mathbb{R}$ *is an upper semicontinuous function.*

Proof. This result is an immediate consequence of Theorem 2.11. □

This theorem gives us the existence of a Nash equilibrium among followers but not the uniqueness of it. Thus, in principle, there are multiple equilibria denoted by $NE(X,t)$. We can consider a selection of the correspondence $(X,t) \rightrightarrows NE(X,t)$, namely, a function

$$\lambda : (X,t) \to (y_1^\lambda(H(X,t)), \dots, y_N^\lambda(H(X,t))),$$

in order to choose a profile in the set of the possible Nash equilibria of the followers.

We suppose that $l_i(x_i, g(x,y), t) = l_i(x_i, H(X+Y), t)$ is increasing in the second variable i.e., the aggregator g. Since H is a strictly increasing function, $l_i(x_i, H(X+Y), t)$ is increasing in the aggregate of strategies. By Theorem 3.3, the equilibria aggregates are ordered from the smallest one to the largest one and we assume that the leaders adopt the max-selection (in line with [65]) i.e.,

$$(y_1^{max}(H(X,t)), \dots, y_N^{max}(H(X,t))),$$

such that $\sum_{j=1}^{N} y_j^{max}(H(X,t)) = Y^*(x,t)$, and they take into account the following functions:

$$l_i(x_i, g(x, Y^*(x,t)), t) = l_i(x_i, H(X + Y^*(x,t)), t),$$

and they solve a Nash equilibrium problem.

Remark 3.7. By integral's monotonicity and by Assumption 3.3, the function $\mathbb{E}[l_i(x_i, H(X + Y^*(x,t)), t)]$ is supermodular in the player's own strategy and exhibits decreasing differences in x_i and X_{-i}, for all $i = 1, \dots, M$ and for any $t \in \mathcal{T}$.

Remark 3.8. In the case of multiple followers' responses, the max-selection corresponds (for $M = 1$) to the so-called strong Stackelberg–Nash solution or optimistic Stackelberg–Nash solution (see [11], [46], [48], [81]).

Theorem 3.4. *If Assumption 3.3 holds and if $l_i(x_i, g(x,y), t)$ is an increasing function in the aggregator, then there exists an MLMFA equilibrium.*

Proof. Since Assumption 3.3 holds, then Theorem 3.3 holds, and, since the function $l_i(x_i, g(x,y), t)$ is increasing in the aggregator, using the max-selection, we can consider the reduced aggregative game $\langle \mathcal{M}, (U_i, l_i)_{i=1}^{M}, g, t \rangle$, where $l_i = l_i(x_i, H(X + Y^*(x,t)), t)$ $\forall i = 1, \dots, M$, which is an aggregative game with strategic substitutes. Considering the functions $\mathbb{E}[l_i(x_i, H(X + Y^*(x,t)), t)]$, for any $i = 1, \dots, M$, by Remark 3.7 and using Theorem 2.11, the result is proved. □

Example 3.4. (Teamwork Project)

Three agents must each complete a task and they form a team. Each agent's task is critical to the success of the team's project, i.e. to the agents' common project, in the sense that if everyone is successful in his individual project, the team's project succeeds, otherwise it fails. Let us denote by s_h, with $h \in \{1, 2, 3\}$ the probability that agent h succeeds in his own task. We assume that agent 1 has commitment power and thus he acts as leader in a two-stage game, with agents 2 and 3 as followers.

Let us suppose that the payoff functions of both leader and followers depend on own strategies and the probability of joint success i.e. $s_1 s_2 s_3$. Moreover, we assume that a shock hits the payoff function in a way that, if the joint probability is sufficiently high, i.e. if the team is in himself close to succeed, the payoff of each player is weakly affected by the external shock, while, if the joint probability is sufficiently low, i.e. if the team is close to fail, the payoff of each player is strongly negatively affected by the external shock. Let us represent this shock by a random variable ξ. Thus let us consider the game

$$\left\langle 3, (U_1, l_1), (V_j, f_j)_{j=2}^3, g, (\Omega, \Sigma, \mathcal{P}), \xi \right\rangle,$$

where

- $U_1 = [0,1]$, $V_j = [0,1]$ for $j = 2,3$;
- denoted by $s = (s_1, s_2, s_3)$ the vector of strategies, $g(s) = \prod_{h=1}^3 s_h$;
- $(\Omega, \Sigma, \mathcal{P})$ is a probability state space, where any $\omega \in \Omega$ represents some state of the world;
- $\xi : \Omega \to \mathbb{R}$ is a continuous random variable on the probability state space $(\Omega, \Sigma, \mathcal{P})$ with uniform distribution $\rho(t) = 1$ $\forall t \in [0,1]$.

Then,

- for any $t \in [0,1]$, the payoff functions for the followers are

  $$f_j(s_j, g(s), t) = (s_1 s_2 s_3)^{1+t}$$

  for $j = 2,3$;
- the payoff function for the leader is

  $$\mathbb{E}[l_1(s_1, g(s), \xi(\omega))] = \mathbb{E}\left[(s_1 s_2 s_3)^{1+\xi(\omega)} - \left(s_1 + \frac{1}{4}\right)^4\right]$$

$$= \int_0^1 (s_1 s_2 s_3)^{1+t} dt - \left(s_1 + \frac{1}{4}\right)^4.$$

Fixing s_1 and t, the followers' Nash equilibria are

$$NE(s_1, t) = \begin{cases} \{(0,0),(1,1)\} & if s_1 \neq 0, \\ [0,1]^2 & if s_1 = 0. \end{cases}$$

By using the max-selection, the leader considers $(s_2^*, s_3^*) = (1,1)$ and so he maximizes, with respect to his own strategy, the following payoff function:

$$\mathbb{E}\left[s_1^{1+\xi(\omega)} - \left(s_1 + \frac{1}{4}\right)^4\right] = \int_0^1 s_1^{1+t} dt - \left(s_1 + \frac{1}{4}\right)^4 =$$
$$\frac{s_1(s_1 - 1)}{log s_1} - \left(s_1 + \frac{1}{4}\right)^4.$$

It can be proved that there exists $s_M \in [0,1]$ where this function has a positive maximum.

This model corresponds to the Teamwork project with multiple task (see [26], [41]).

Note that Assumption 3.3 is satisfied in this game, and, since l_1 is an increasing function in the aggregator $g(s) = \prod_{h=1}^{3} s_h$, Theorems 3.3 and 3.4 hold.

Chapter 4

Common Pool Resources Games and Social Purpose Games as classes of Aggregative Games

In literature, many common games in industrial organization, public economics and macroeconomics are aggregative games: among them, we mention Cournot and Bertrand games, patent races, models of contests of fighting and model with aggregate demand externalities.

Moreover, also a lot of environmental games present an aggregative structure like pollution games or water resource games, for which the aggregation function usually is the summation of strategies. Thus, in this chapter, we examine two particular environmental games, viewed and studied as aggregative games, namely an investment game in Common-Pool Resources and, after introducing a particular class of aggregative games called Social Purpose Games, a withdrawal game of a water resource.

In Section 4.1 we describe an investment decision making situation for a CPR using an aggregative normal form game. For this game we investigate the existence question of Nash equilibrium solution, that describes situations where agents act in a non-cooperative way. Two types of results are given, with or without convexity-like assumptions. Moreover, in the special case of quadratic return functions, the game is also studied under uncertainty, i.e. when the possibility of a natural disaster with a given probability is considered in the model.

In Section 4.1.1 we describe our model, introduce two different kinds of equilibrium, the non-cooperative and the cooperative ones, and show the existence of such equilibria. Then, we apply these concepts in the context of Environmental Economics where the return functions are quadratic and in which, for example, players could be countries that chose the level of investment into green policies, in order to be more environmentally friendly (see [90] and the references therein). In Section 4.1.2 we introduce a threshold investment and we study the resulting game with aggregative uncertainty, computing and comparing the non-cooperative and the cooperative equilibria. For more details see [57].

In Section 4.2, following the literature of additively separable aggregative games and in line with the asymmetry considered in [70], we introduce a class of non-cooperative games, called Social Purpose Games, for which the payoff of each player depends separately on his own strategy and on a function of the strategy profile, the aggregation function, which is the same for all players, weighted by an individual benefit parameter which enlightens the asymmetry, between agents, towards the social part of the benefit. The two parts of the payoff function represent respectively the individual and the social benefits.

In Section 4.2.1 we introduce the class of Social Purpose Games for which we show that they have a potential and we prove existence results for the Nash equilibrium and the Social Optimum. In Section 4.2.2 we show an existence result of a coalition leadership equilibrium and we prove a stability result, depending on the weights affecting the aggregation function. An application to a water resource game is illustrated ([12], [47], [83]). For more details see [33].

4.1 Common-Pool Resources: an Equilibrium Analysis

4.1.1 Investment in a CPR

In order to describe an investment decision making situation for a Common-Pool Resource let us consider the following normal form game $\Gamma = \left\langle \mathcal{N}, (U_i)_{i=1}^N, (f_i)_{i=1}^N \right\rangle$ where:

- $\mathcal{N} = \{1, \ldots, N\}$, with N the fixed number of players that can access to the CPR;
- for any $i \in \mathcal{N}$, $U_i = [0, e]$ is the strategy set for player i, where e is the initial endowment that each player can invest in the CPR or in an outside activity;
- denoted by $U = \prod_{i=1}^N U_i$, $f_i : U \to \mathbb{R}$ is the payoff function of player i for any $i \in \mathcal{N}$.

In line with [31] and [32], let us denote by $\omega > 0$ the marginal payoff of the outside activity and by $x_i \in [0, e]$ the quantity invested by player i in the CPR. In order to explicitly write the payoff function for each player i, let us introduce a twice continuously differentiable function $G : [0, Ne] \to \mathbb{R}$ that depends on the aggregate invested quantity in the CPR, i.e. $\sum_{i=1}^N x_i$, and that is a concave function such that $G(0) = 0$ and $G'(0) > \omega$. This function represents the aggregate return to the investment in the CPR and so the assumption $G'(0) > \omega$ is nothing but an incentive constraint since it means that initially the marginal return to the investment in the CPR is greater than the marginal payoff that everyone achieves if he invests in an outside activity (see [32]).
Note that, for any $i \in \mathcal{N}$, if player i decides to invest part of his endowment in the CPR he will obtain a certain payoff $\omega(e - x_i)$ plus the return of investment

in the CPR, $G(\sum_{i=1}^N x_i)$, multiplied by the share that is for him, i.e. $x_i/\sum_{i=1}^N x_i$, while if he decides to not invest in the CPR he will obtain a payoff ωe. So, let $x = (x_1, \ldots, x_N)$ be a vector of players' investment, then for any $i \in \mathcal{N}$, the payoff of player i is given by

$$f_i(x_1, \ldots, x_N) = \omega(e - x_i) + \frac{x_i}{\sum_{i=1}^N x_i} G(\sum_{i=1}^N x_i). \tag{4.1}$$

Since $G(0) = 0$ and $G'(0) > 0$, mathematically 0 is a zero of order one for the function G and so $G(\sum_{i=1}^N x_i) = \sum_{i=1}^N x_i H(\sum_{i=1}^N x_i)$ with $H : [0, Ne] \to \mathbb{R}$ such that $H(0) \neq 0$ and H is twice continuously differentiable[1]. So, for any $i \in \mathcal{N}$, we can rewrite the payoff function of player i, in terms of average aggregate return $H(\sum_{i=1}^N x_i)$, in the following way:

$$f_i(x_1, \ldots, x_N) = \omega(e - x_i) + x_i H(\sum_{i=1}^N x_i). \tag{4.2}$$

Let us explicitely note that, in the context of CPR, the function $G(\sum_{i=1}^N x_i)$ is the production function and $H(\sum_{i=1}^N x_i)$ is the average production function (see [32]). In the context of Environmental Economics, if the considered strategies are emissions, the function $G(\sum_{i=1}^N x_i)$ represents the damage cost function and $H(\sum_{i=1}^N x_i)$ its average (see [13], [42]).

Since G is a concave function we can easily obtain, by using Lagrange's theorem, that

$$G'(\sum_{i=1}^N x_i) < \frac{G(\sum_{i=1}^N x_i)}{\sum_{i=1}^N x_i} \tag{4.3}$$

that means that the output elasticity is less then zero.

Using inequality 4.3, one can check that $H'(\sum_{i=1}^N x_i) < 0$ so H is a strictly decreasing function.

[1] This is a technical construction that allows us to have more manageable payoff functions, in order to prove equilibria existence results.

In addition, we assume that the following inequality holds:

$$H'(\sum_{i=1}^{N} x_i) + \sum_{i=1}^{N} x_i H''(\sum_{i=1}^{N} x_i) < 0. \tag{4.4}$$

Inequality 4.4 is satisfied if H is a linear function, if H is a differentiable concave function or if it is a differentiable convex function with additional assumptions (for example $H(t) = (t-k)^2$, with $k > 2Ne$).

Remark 4.1. Note that inequality 4.4 cannot be derived by concavity of function G. In fact, concavity of G implies that $2H'(\sum_{i=1}^{N} x_i)+\sum_{i=1}^{N} x_i H''(\sum_{i=1}^{N} x_i) \leq 0$ and, as already noted, that $H'(\sum_{i=1}^{N} x_i) < 0$. These two implications together do not imply 4.4.

Remark 4.2. Inequality 4.4 has been already used in the context of an oligopolistic market analysis by Okuguchi (see [82]) and by Sheraly-Soyster-Murphy (see [94]) for computing Stackelberg-Nash-Cournot equilibria, supposing that $p'(Q) + Qp''(Q) < 0$. Namely, in both cases the function H is nothing but the inverse demand function $p(Q)$ i.e. the price at which consumers will demand and purchase a quantity Q.

Let us give an interpretation of assumption 4.4. Let us consider one player, namely player i, that has to face the average production function $H(\cdot)$. This player invests quantity x_i while the other players' aggregate investment is $\sum_{j\neq i} x_j$. Player i's revenue is $x_i H(x_i + \sum_{j\neq i} x_j)$ and so his marginal revenue is $H(x_i + \sum_{j\neq i} x_j) + x_i H'(x_i + \sum_{j\neq i} x_j)$. The rate of change of his marginal revenue with an increase in the other players' aggregate investment is

$$H'(x_i + \sum_{j\neq i} x_j) + x_i H''(x_i + \sum_{j\neq i} x_j), \tag{4.5}$$

by computing the derivative of the marginal revenue $H(x_i+\sum_{j\neq i} x_j)+x_i H'(x_i+\sum_{j\neq i} x_j)$ with respect to $\sum_{j\neq i} x_j$.

Suppose that $\sum_{j\neq i} x_j > 0$ and $H''(x_i + \sum_{j\neq i} x_j) \leq 0$, then

$$H'(x_i + \sum_{j\neq i} x_j) + x_i H''(x_i + \sum_{j\neq i} x_j) < 0$$

since $H'(x_i + \sum_{j\neq i} x_j) < 0$.
Suppose that $\sum_{j\neq i} x_j > 0$ and $H''(x_i + \sum_{j\neq i} x_j) > 0$, then

$$H'(x_i + \sum_{j\neq i} x_j) + x_i H''(x_i + \sum_{j\neq i} x_j) \leq$$

$$H'(x_i + \sum_{j\neq i} x_j) + (x_i + \sum_{j\neq i} x_j) H''(x_i + \sum_{j\neq i} x_j) < 0$$

since inequality 4.4 holds.
Finally, if we suppose that $\sum_{j\neq i} x_j = 0$, it's straightforward to obtain the same result.
So 4.4 implies that, for any level of investment x_i chosen by player i, his marginal revenue is decreasing when the aggregate investment made by all other players is increasing.
Note that, since $G'(0) = H(0) > \omega$ and $H(\cdot)$ is a strictly decreasing function, if the aggregate investment made by all the players significantly increases then the return due to the investment in the CPR becomes negative. Thus the players would be discouraged to move forward with their investments in CPR.

The N players can act non-cooperatively, looking for the so called *CPR equilibrium*, or cooperatively, looking for the so called *fully cooperative CPR equilibrium* ([17]).
The non-cooperative approach as well as the cooperative one are taken into account in CPR, Public Goods, Oligopolies, R&D models (see [14], [22], [29], [32]) and, in case of equilibrium uniqueness in both the approaches, the two kinds of equilibria are compared.

Definition 4.1. A *fully cooperative CPR equilibrium*, $CPRE^c$, is an N-tuple

$(x_1^c, \dots, x_N^c)$ such that, for each player $i \in \mathcal{N}$, x_i^c solves the following problem:

$$x_i^c \in argmax_{x_i \in U_i} \sum_{j=1}^{N} f_j(x_1^c, \dots, x_{i-1}^c, x_i, x_{i+1}^c, \dots, x_N^c).$$

Remark 4.3. Note that $\sum_{i=1}^N f_i(x_1, \dots, x_N) = \omega e - \omega \sum_{i=1}^N x_i + G(\sum_{i=1}^N x_i)$ is a continuous function on $U = [0, e]^N$ so, applying Weirstrass theorem, there exists a symmetric fully cooperative CPR equilibrium.

If the agents involved in this CPR management behave in a non-cooperative way, they solve a Nash equilibrium problem and we call the resulting Nash equilibrium *CPR equilibrium* (CPRE).

The following result guarantees the existence of CPR equilibria in a differentiable payoffs framework.

Theorem 4.1. ***(Existence)*** *Let H be a twice continuously differentiable function such that $H(0) \neq 0$ and $H'(\sum_{i=1}^N x_i) < 0$. Suppose that inequality 4.4 holds. Then there exists a symmetric CPRE.*

In order to prove this result, let us first give the following lemma:

Lemma 4.1. *Let $H(\cdot)$ be the twice differentiable function considered before and let us assume that inequality 4.4 holds. Then, for each fixed $X_{-i} = \sum_{j \neq i} x_j \geq 0$, the function $K(x_i) = x_i H(x_i + X_{-i})$ is a strictly concave function of x_i over $x_i \geq 0$.*

Proof. First of all let us remind that, since G is a concave function, $H(\cdot)$ is a strictly decreasing function.

Let us show that

$$K''(x_i) = 2H'(x_i + X_{-i}) + x_i H''(x_i + X_{-i}) < 0$$

for each $X_{-i} \geq 0$.

Let us suppose $H''(x_i + X_{-i}) \leq 0$, then $K''(x_i) < 0$ since $H'(x_i + X_{-i}) < 0$.

Conversely, let us suppose $H''(x_i + X_{-i}) > 0$. So

$$K''(x_i) \leq 2H'(x_i + X_{-i}) + (x_i + X_{-i})H''(x_i + X_{-i}) <$$

$$H'(x_i + X_{-i}) + (x_i + X_{-i})H''(x_i + X_{-i})$$

and, since inequality 4.4 holds, we have proved the result. □

Proof. *(Theorem 4.1)* Since Lemma 4.1 holds, then, for any $i \in \mathcal{N}$, $f_i(x)$ is a strictly concave function in x_i. So $f_i(x)$ is a quasi-concave function in x_i and since, for any $i \in \mathcal{N}$, $f_i(x)$ is continuous and the strategy set U_i is non-empty closed and compact, then a symmetric CPR equilibrium exists (see [75]) . □

Remark 4.4. (Non differentiable case). It may happen that the H function is kinked in some level (or levels) of aggregation, due to a different increasing rate of the aggregate return, for example:

$$H(t) = \begin{cases} a - t & \text{if } t \leq \bar{X} \\ a - \frac{(t+\bar{X})}{2} & \text{if } t > \bar{X} \end{cases}$$

In this case, in order to obtain an existence result, it is possible to prove that the game has a potential structure. In fact, in the case in which the considered function H is continuous, we can define a function $P : U \to \mathbb{R}$ such that

$$P(x_1, \ldots, x_N) := x_1 \ldots x_N \cdot (H(\sum_{i=1}^{N} x_i) - \omega)$$

and we can easily prove that it is a potential function for the game Γ. Thus, denoting by $CPR(\Gamma)$ the set of all possible CPR equilibria of Γ, we have that $max_{x \in U} P \subseteq CPR(\Gamma)$ and so, there exists at least one CPR equilibrium.

Remark 4.5. The natural question that may raise is about the economic interpretation of this potential function that is what the investors in the CPR are trying to jointly maximize. As Monderer and Shapley point out in the context of Cournot games (see [73]), we do not have an answer to this question.

However, in this case the only thing we care about is that the mere existence of a potential function helps the players to reach an equilibrium in a easier way.

We have then the following theorem.

Theorem 4.2. ***(Existence)*** *If H is a continuous function, then there exists a CPRE.*

Let us explicitly remark that the differentiability assumption is a condition useful in several computational procedures ([17]), so sometimes the first existence result has to be considered by using differentiable payoff functions.

4.1.2 Quadratic return under uncertainty

In literature there are several papers dealing with cooperative as well as non-cooperative approach to game theoretical models involving Environmental Economics (see, for example[1], [7], [28], [69] and the references therein). A huge quantity of environmental problems, such as climate change, loss of biodiversity, ozone depletion, the widespread dispersal of persistent pollutants and many others, involves the commons (for example forests, energy, industries, water and so on). In numerous situations the considered payoff functions are quadratic functions. Then, in this section we deal with a quadratic return payoff function case.

As done in [32], let us consider the function

$$G(t) = at - bt^2$$

with $a, b > 0$ and $G'(0) = a > \omega$.

In this case, $H(t) = a - bt$ and for all $i \in \mathcal{N}$, the payoff function becomes

$$f_i(x_1, \ldots, x_N) = \omega(e - x_i) + x_i[a - b(x_1 + \cdots + x_N)]$$

that represents the welfare of player i that comprises benefits from investment, deriving from production and consumption of goods, and damages caused by the aggregate investment.

Following a non-cooperative approach, we can easily show that the CPR equilibrium is an N-tuple $(x_1^*, \ldots, x_N^*)$ with

$$x_i^* = min\Big\{\frac{a-\omega}{b(N+1)}, e\Big\}$$

for each player $i \in \mathcal{N}$.

Instead, following a cooperative approach, the fully cooperative CPR equilibrium is an N-tuple $(x_1^c, \ldots, x_N^c)$ with

$$x_i^c = min\Big\{\frac{a-\omega}{2bN}, e\Big\}$$

for each player $i \in \mathcal{N}$.

Comparing the two kinds of equilibrium, we can check in a straightforward way that for each player $i \in \mathcal{N}$,

$$x_i^c \leq x_i^*$$

and so, cooperating, each player can invest less in the CPR.

In the case of Environmental Economics, sometimes it may happen a disaster event that implies a loss in the payoff of any agent. The disaster can have natural causes (earthquakes, floods, ...) or it may be due to human harm. In both cases investments in the management of resources are very useful. Suppose that a loss is considered in the payoff if the investment is lower than a given upper bound. More precisely, we suppose that there exists a threshold investment, denoted here by $\overline{X}$, that is a random variable since it depends on the probability of a disaster involving the CPR. In line with [7], we suppose that if the aggregate investment is sufficiently large, the payoff functions do not change with respect to the case without uncertainty, otherwise, if the aggregate investment is relative low, every player suffers a loss.

In order to explicitly write the payoff functions, let us fix $\overline{X} \in (0, Ne)$, the critical level, and a constant $L \geq 0$, that represents the loss value. Thus, in the case where $G(t) = at - bt^2$, the payoff functions are

$$\hat{f}_i(x_1, \dots, x_N) = \begin{cases} \omega(e - x_i) + x_i[a - b(\sum_{i=1}^N x_i)] & if \quad \sum_{i=1}^N x_i \geq \overline{X} \\ \omega(e - x_i) + x_i[a - b(\sum_{i=1}^N x_i)] - L & if \quad \sum_{i=1}^N x_i < \overline{X} \end{cases}$$

As in Section 4.1.1, the N players can act either non-cooperatively, looking for a so called *CPR equilibrium under uncertainty*, or cooperatively, looking for a so called *fully cooperative CPR equilibrium under uncertainty.*

The uncertainty is about the threshold $\overline{X}$: in particular, let us assume that the threshold investment is distributed uniformly, i.e. with probability distribution function

$$f(X) = \frac{1}{Ne}$$

with $X = \sum_{i=1}^N x_i \in [0, Ne]$ and so the corresponding cumulative distribution function is

$$F(X) = \mathcal{P}(\overline{X} \leq X) = \frac{X}{Ne}$$

with $X \in [0, Ne]$.

If players use a non-cooperative approach, each of them will maximize the expectation of his own payoff function that is

$$\mathbb{E}(\hat{f}_i) = \omega(e - x_i) + x_i(a - bX) - L(1 - F(X)) =$$

$$= \omega(e - x_i) + x_i(a - bX) - L\Big(1 - \frac{X}{Ne}\Big)$$

and we can easily show that the CPR equilibrium under uncertainty is a N-tuple $(x_1^*, \dots, x_N^*)$ with

$$x_i^* = min\Big\{\frac{a - \omega}{b(N+1)} + \frac{L}{beN(N+1)}, e\Big\}$$

$\forall i \in \mathcal{N}$.

If players decide to cooperate they will maximize the expectation of the joint payoff function that is

$$\mathbb{E}(\hat{f}^c) = N\omega e - \omega X + Xa - bX^2 - LN(1 - F(X)) =$$
$$= N\omega e - \omega X + Xa - bX^2 - LN\Big(1 - \frac{X}{Ne}\Big).$$

In this other frame, we can easily show that the fully cooperative CPR equilibrium (in this case each payoff is strictly concave in its decision variable) under uncertainty is a N-tuple $(x_1^c, \ldots, x_N^c)$ with

$$x_i^c = min\Big\{\frac{a-\omega}{2bN} + \frac{L}{2bNe}, e\Big\}$$

$\forall i \in \mathcal{N}$.

We note that the CPR and the fully cooperative CPR equilibria are identical in the case in which $N = 1$. When $N \geq 2$, if we suppose that $L < (a - \omega)e$, we can easily show that

$$x_i^c \leq x_i^*$$

and so, only with a minor disaster, if players decide to cooperate, they can invest less in the CPR.

If agents have additional informations on the variability of $\overline{X}$, the threshold investment, then one can also consider different probability distributions, leading to different results.

Summing up, given $\Gamma = \langle \mathcal{N}, (U_i)_{i=1}^N, (f_i)_{i=1}^N \rangle$ with N players, $U_i = [0, e]$ strategy set of player i, for any $i \in \mathcal{N}$, and $f_i(x) = \omega(e - x_i) + x_i\Big(a - b\sum_{i=1}^N x_i\Big)$, $a, b > 0$, payoff function of player i, for any $i \in \mathcal{N}$,

- in the deterministic case, given the CPR equilibrium $(x_1^*, \ldots, x_N^*)$ and the fully cooperative CPR equilibrium $(x_1^c, \ldots, x_N^c)$, $x_i^c \leq x_i^*$ for any $i \in \mathcal{N}$;

- in the stochastic case, in which a threshold random variable investment is considered, given $(x_1^*, \ldots, x_N^*)$ CPR equilibrium under uncertainty and

$(x_1^c, \ldots, x_N^c)$ fully cooperative CPR equilibrium under uncertainty, $x_i^c \leq x_i^*$ for any $i \in \mathcal{N}$ if $L < (a - \omega)e$.

4.2 Social Purpose Games

4.2.1 Class and properties

Let us consider the following normal form game $\Gamma = \left\langle \mathcal{N}, (U_i)_{i=1}^N, (f_i)_{i=1}^N \right\rangle$ where:

- $\mathcal{N} = \{1, \ldots, N\}$, with N the fixed number of players involved in the game;
- for any $i \in \mathcal{N}$, $U_i = [0, \overline{Q}]$ where $\overline{Q} > 0$;
- denoted by $U = \prod_{i=1}^N U_i = [0, \overline{Q}]^N$, $f_i : U \to \mathbb{R}$ is the payoff function of player i for any $i \in \mathcal{N}$.

In order to explicitely write the payoff function for each player $i \in \mathcal{N}$, let us consider $\alpha = (\alpha_1, \ldots, \alpha_N) \in \mathbb{R}^N_{++}$ a given vector of positive weights such that $0 < \alpha_1 \leq \cdots \leq \alpha_N$, where each α_i represents an individual benefit parameter of player i. Moreover, let us consider a function $H : \mathbb{R} \to \mathbb{R}$ common for each player and functions $h_i, g_i : [0, \overline{Q}] \to \mathbb{R}$ for each $i = 1, \ldots, N$. Denoting by $x = (x_1, \ldots, x_N)$ a strategy profile, the player i's payoff function is

$$f_i(x) = \alpha_i H\Big(\sum_{i=1}^N h_i(x_i) \Big) - g_i(x_i),$$

that depends separately on player i's own strategy, throught the term $g_i(x_i)$ that represents the individual benefit of player i, and on the common term $H\Big(\sum_{i=1}^N h_i(x_i) \Big)$ that represents the social benefit, since it depends on the aggregation of strategies, weighted by the individual benefit parameter, α_i, that measures the importance that player i gives to the social part of the payoff with respect to the individual part, given by $g_i(\cdot)$, and thus allows for asymmetry of public benefit across agents. Recalling Definition 2.13, the game Γ is an additively separable aggregative game with $l_i(g(x)) = \alpha_i H\Big(\sum_{i=1}^N h_i(x_i)\Big)$ and $m_i(x_i) = -g_i(x_i)$. The game Γ is called *Social Purpose Game*.

Remark 4.6. Recall that, given the strategy set $U_i = [0, \overline{Q}]$ for each player $i \in \mathcal{N}$, in Abatement Emission Games the payoff function of player i is given by

$$f_i(q_i, Q) = \frac{\overline{b}}{N}\Big(\overline{a}Q - \frac{1}{2}Q^2\Big) - \frac{\overline{c}}{2}q_i^2$$

with, calling e_i the emission level of player i, $q_i = \overline{Q} - e_i$ is the abatement quantity of player i, $Q = \sum_{i=1}^N q_i$ and $\overline{a}, \overline{b}, \overline{c} > 0$ (see [25]) and that in Public Good Provision Games the payoff function of player i is given by

$$f_i(q_i, Q) = G_i(Q) + P_i(q_i)$$

with $G_i : \mathbb{R} \to \mathbb{R}$ and $P_i : U_i \to \mathbb{R}$ with, given $m, p > 0$, $P_i(q_i) = m - pq_i$ i.e. the payoff function of each player depends separately on the quantity of private good that he consumes in a linear way but also on all the gifts to the public good made by all individuals (see [9]). Note that the class of Abatement Emission Games and the class of Public Good Provision Games differ only in the cost term which is quadratic for Abatement Emission Games and linear for Public Good Provision Games.

Thus Abatement Emission Games and Public Good Provision Games are included in the class of Social Purpose Games if $\alpha_i = 1$ for any $i \in \mathcal{N}$. In fact in Abatement Emission Games $H\Big(\sum_{i=1}^N q_i\Big) = \frac{\overline{b}}{N}\Big(\overline{a}Q - \frac{1}{2}Q^2\Big)$ (with h_i identity functions for any $i \in \mathcal{N}$) and $g_i(q_i) = \frac{\overline{c}}{2}q_i^2$, while in Public Good Provision Games $H\Big(\sum_{i=1}^N q_i\Big) = G_i(Q)$ (with h_i identity functions for any $i \in \mathcal{N}$) and $g_i(q_i) = m - pq_i$.

In the following proposition we show that each Social Purpose Game is a weighted potential game with weights α_i for any $i \in \mathcal{N}$:

Proposition 4.1. *A Social Purpose game is a weighted potential game and the* α*-potential is given by* $P(x) = H\Big(\sum_{i=1}^N h_i(x_i)\Big) - \sum_{i=1}^N \frac{1}{\alpha_i} g_i(x_i)$.

Proof. Let us prove the result checking that the definition of weighted potential game holds true. For every $x_i, y_i \in U_i$ and $x_{-i} = (x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_N) \in U_i^{N-1}$ we have that:
$f_i(y_i, x_{-i}) - f_i(x_i, x_{-i}) = \alpha_i H(h_i(y_i) + \sum_{j\neq i} h_j(x_j)) - g_i(y_i) - \alpha_i H(h_i(x_i) + \sum_{j\neq i} h_j(x_j)) + g_i(x_i) = \alpha_i\Big(H(h_i(y_i) + \sum_{j\neq i} h_j(x_j)) - \frac{1}{\alpha_i} g_i(y_i) - H(h_i(x_i) + \sum_{j\neq i} h_j(x_j)) + \frac{1}{\alpha_i} g_i(x_i)\Big) = \alpha_i\Big(H(h_i(y_i) + \sum_{j\neq i} h_j(x_j)) - \frac{1}{\alpha_i} g_i(y_i) - \sum_{j\neq i} \frac{1}{\alpha_j} g_j(x_j) - H(h_i(x_i) + \sum_{j\neq i} h_j(x_j)) + \frac{1}{\alpha_i} g_i(x_i) + \sum_{j\neq i} \frac{1}{\alpha_j} g_j(x_j)\Big) = \alpha_i(P(y_i, x_{-i}) - P(x_i, x_{-i}))$.
□

Using tools from potential games theory, let us prove the existence of a Nash equilibrium for this class of games.

Proposition 4.2. *If* H *is an upper semicontinuous, increasing and concave function,* h_i *is a continuous and concave function for any* $i \in \mathcal{N}$*,* g_i *is a lower semicontinuous and convex function for any* $i \in \mathcal{N}$*, then there exists a Nash equilibrium* $x^{NE} = (x_1^{NE}, \dots, x_N^{NE})$ *of* Γ*. If moreover* H*,* h_i*, for any* $i \in \mathcal{N}$*, and* g_i*, for any* $i \in \mathcal{N}$*, are continuously differentiable and either* H *is strictly concave or* g_i *are strictly convex for all* $i \in \mathcal{N}$*, then there exists a unique Nash equilibrium.*

Proof. Since H is increasing and concave and each h_i is concave, $H\Big(\sum_{i=1}^N h_i(x_i)\Big)$ is concave. Moreover $-g_i$ is concave, thus the potential function P is concave. Since H is upper semicontinuous, h_i continuous and g_i lower semicontinuous, the potential function P is upper semicontinuous. Then there exists a maximum of P and thus, since for a weighted potential game $argmax P \subseteq NE$, where NE is the set of Nash equilibria, there exists at least one Nash equilibrium.
If moreover H, h_i and g_i are continuously differentiable then P is continuously

differentiable and if either H is strictly concave or g_i are strictly convex for all $i \in \mathcal{N}$ then P is strictly concave thus there exists a unique Nash equilibrium. □

Remark 4.7. In the case in which the involved functions are continuously differentiable the Nash equilibrium is solution of

$$\nabla P(x_1, \ldots, x_N) =$$

$$= \Big(H'\Big(\sum_{i=1}^{N} h_i(x_i)\Big)h_1'(x_1) - \frac{g_1'(x_1)}{\alpha_1}, \ldots, H'\Big(\sum_{i=1}^{n} h_i(x_i)\Big)h_N'(x_N) - \frac{g_N'(x_N)}{\alpha_N}\Big) = 0.$$

Remark 4.8. Note that $\frac{g_i'(x_i^{NE})}{\alpha_i h_i'(x_i^{NE})}$ is equal for all $i \in \mathcal{N}$. In the case in which h_i is the identity function for any $i \in \mathcal{N}$, this invariant is given by $\frac{g_i'(x_i^{NE})}{\alpha_i}$.

Proposition 4.3. *In the assumptions of Proposition 4.2, there exists a Social Optimum* $x^{SO} = (x_1^{SO}, \ldots, x_N^{SO})$ *of* Γ. *If moreover either* H *is a strictly concave function or or* g_i *are strictly convex for all* $i \in \mathcal{N}$, *the Social Optimum is unique.*

Proof. In our assumptions, the social profit function

$$S(x_1, \ldots, x_N) := \sum_{i=1}^{N} \Pi_i = H(\sum_{i=1}^{N} h_i(x_i)) - \frac{\sum_{i=1}^{N} g_i(x_i)}{\overline{\alpha}}$$

where $\overline{\alpha} = \sum_{i=1}^{N} \alpha_i$, is continuous and concave over $[0, Q]^N$ thus, applying Weirstrass theorem, there exists a social optimum. If either H is strictly concave or g_i are strictly convex for all $i \in \mathcal{N}$, the social profit is strictly concave, thus the social optimum is unique. □

Remark 4.9. If the involved function are continuously differentiable, the social optimum solves

$$\nabla S(x_1, \ldots, x_N) =$$

$$= \Big(H'\Big(\sum_{i=1}^{N} h_i(x_i)\Big)h_1'(x_1) - \frac{g_1'(x_1)}{\overline{\alpha}}, \ldots, H'\Big(\sum_{i=1}^{N} h_i(x_i)\Big)h_N'(x_N) - \frac{g_N'(x_N)}{\overline{\alpha}}\Big) = 0.$$

Remark 4.10. For any $\underline{x} \in U$, $P(\underline{x}) \leq S(\underline{x})$.

Let us firstly consider the case in which h_i are the identity functions. We can compare the aggregate Nash equilibrium strategies, i.e. $\sum_{i=1}^N x_i^{NE}$, and the aggregate Social optimum strategies, i.e. $\sum_{i=1}^N x_i^{SO}$, as follows:

Proposition 4.4. *Let functions H and g_i, for any $i \in \mathcal{N}$, be continuously differentiable and let functions h_i, for any $i \in \mathcal{N}$, be the identity functions. If the function H' is decreasing in $\sum_{i=1}^N x_i$ and the functions g_i' are positive and increasing in x_i for any $i \in \mathcal{N}$, then $\sum_{i=1}^N x_i^{NE} \leq \sum_{i=1}^N x_i^{SO}$.*

Proof. Let us suppose by contradiction that $\sum_{i=1}^N x_i^{NE} > \sum_{i=1}^N x_i^{SO}$. Since H' is decreasing in $\sum_{i=1}^N x_i$, then $H'\Big(\sum_{i=1}^N x_i^{NE}\Big) \leq H'\Big(\sum_{i=1}^N x_i^{SO}\Big)$. Since Remarks 4.7 and 4.9 hold true, then, for some $i \in \mathcal{N}$, $\frac{g_i'(x_i^{NE})}{\alpha_i} \leq \frac{g_i'(x_i^{SO})}{\overline{\alpha}}$.
Since $\overline{\alpha} = \sum_{i=1}^N \alpha_i > 0$ and g_i' is positive, then $\frac{g_i'(x_i^{NE})}{\overline{\alpha}} < \frac{g_i'(x_i^{NE})}{\alpha_i} \leq \frac{g_i'(x_i^{SO})}{\overline{\alpha}}$. Thus, since g_i' is increasing in x_i, we obtain that $x_i^{NE} \leq x_i^{SO}$. Similarly, we can show that $x_j^{NE} \leq x_j^{SO}$ for any $j \in \mathcal{N} - \{i\}$, thus $\sum_{i=1}^N x_i^{NE} \leq \sum_{i=1}^N x_i^{SO}$. □

Remark 4.11. We can easily generalise the previous result to the case in which functions h_i for any $i \in \mathcal{N}$ are not the identity functions, assuming that $\frac{g_i'(\cdot)}{h_i'(\cdot)}$ are increasing functions in x_i for any $i \in \mathcal{N}$.
This assumption simply means that, an increase of player i's strategy turns to produce more impact on the individual function, i.e. g_i, rather than on the function through which player i's strategy is measured by the society, i.e. h_i.

As soon as we suppose that functions g_i for any $i \in \mathcal{N}$ are concave and function H is convex, we lose the property ensured by Proposition 4.4:

Example 4.1. Consider $\Gamma = \langle 2, [0,1], \{f_i\}_{i=1}^2 \rangle$ where $f_i(x_1, x_2) = \frac{(x_1+x_2)^2}{2} - 4\Big(x_i - \frac{x_i^2}{2}\Big)$ for any $i \in \{1,2\}$. It results that $(x_1^{NE}, x_2^{NE}) = (\frac{2}{3}, \frac{2}{3})$ and $(x_1^{SO}, x_2^{SO}) = (\frac{1}{2}, \frac{1}{2})$. Thus $x_1^{NE} + x_2^{NE} = \frac{4}{3} > x_1^{SO} + x_2^{SO} = 1$.

We want now to compare each component of the Nash equilibrium with the corresponding component of the Social Optimum.

Proposition 4.5. *Let us suppose that assumptions of Proposition 4.4 with g_i' strictly increasing in x_i hold and that $\frac{g_1'(\cdot)}{\alpha_1} \leq \cdots \leq \frac{g_N'(\cdot)}{\alpha_N}$, then $x_i^{NE} \leq x_i^{SO}$ for any $i \in \mathcal{N}$.*

Proof. Let us suppose by contradiction that it exists $j \in \mathcal{N}$ such that $x_j^{NE} > x_j^{SO}$ and $x_h^{NE} \leq x_h^{SO}$ for any $h \in \mathcal{N} - \{j\}$. Since g_j' is strictly increasing , then $\frac{g_j'(x_j^{NE})}{\alpha_j} > \frac{g_j'(x_j^{SO})}{\alpha_j}$. Two cases are possible:

- Take $h > j$.

  Since Remarks 4.7 and 4.9 hold true and $\alpha_h \geq \alpha_j$, it follows that

$$\frac{g_h'(x_h^{NE})}{\alpha_h} > \frac{g_h'(x_h^{SO})}{\alpha_j} \geq \frac{g_h'(x_h^{SO})}{\alpha_h}.$$

  Since g_h' is strictly increasing, then $x_h^{NE} > x_h^{SO}$.

- Take $h < j$.

  Since Remark 4.7 holds true and since the invariant is increasing in i, it follows that

$$\frac{g_h'(x_h^{NE})}{\alpha_h} = \frac{g_j'(x_j^{NE})}{\alpha_j} > \frac{g_j'(x_j^{SO})}{\alpha_j} \geq \frac{g_h'(x_h^{SO})}{\alpha_h}.$$

  Since g_h' is strictly increasing, then $x_h^{NE} > x_h^{SO}$.

□

Remark 4.12. If $\frac{g_1'(\cdot)}{\alpha_1 h_1'(\cdot)} \leq \cdots \leq \frac{g_N'(\cdot)}{\alpha_N h_N'(\cdot)}$, the previous result can be generalized to the case in which functions h_i, for any $i \in \mathcal{N}$, are not the identity functions.

Example 4.2. (A water resource game) We consider a game with $N \geqslant 3$ players. The player set is as usual given by $\mathcal{N} = \{1, \ldots, N\}$. Each player has a strategy set given by the non-negative real line $\mathbb{R}_+$ and typical strategic selection for player $i \in \mathcal{N}$ is denoted by $x_i \geqslant 0$ that represents the amount of water that player i decides to not withdraw from a water basin, in order to preserve the water resource.

Given a strategic tuple $x = (x_1, \ldots, x_N) \in \mathbb{R}_+^N$, the payoff function of player $i \in \mathcal{N}$ is defined by

$$f_i(x) = \alpha_i \sum_{j \in \mathcal{N}} x_j - x_i^2 \tag{4.6}$$

depending on the total remaining quantity in the basin, weighted by $\alpha_i > 0$ that is a benefit parameter, and on the cost that each player i incurs for not withdrawing it. We assume throughout, without loss of generality, that the players are ordered in their payoff parameter with $0 < \alpha_1 \leqslant \alpha_2 \leqslant \cdots \leqslant \alpha_N$.

Moreover, throughout the following analysis we define

$$A = \sum_{j \in \mathcal{N}} \alpha_j > 0 \tag{4.7}$$

Every player $i \in \mathcal{N}$ is supposed to select a best response to the strategies selected by all other players in the game. Hence, every player $i \in \mathcal{N}$ maximizes his payoff $f_i(x)$, where x_{-i} is assumed to be given.

First order conditions are determined as

$$\frac{\partial f_i}{\partial x_i} = \alpha_i - 2x_i \equiv 0 \tag{4.8}$$

Therefore, the Nash equilibrium is determined as

$$x_i^{NE} = \tfrac{1}{2}\alpha_i \quad \text{and} \quad f_i^{NE} = f_i\left(x_1^{NE}, \ldots, x_N^{NE}\right) = \tfrac{\alpha_i}{4}\left(2A - \alpha_i\right) > 0. \tag{4.9}$$

Collectively all players optimize the total payoff given by $f = \sum_{i \in \mathcal{N}} f_i = A \sum_{i \in \mathcal{N}} x_i - \sum_{i \in \mathcal{N}} x_i^2$ over all strategies. This results in first order conditions with for every $i \in \mathcal{N}$:

$$\frac{\partial f}{\partial x_i} = A - 2x_i \equiv 0 \tag{4.10}$$

Thus, the social optimum is determined for every $i \in \mathcal{N}$ as

$$x_i^{SO} = \tfrac{1}{2}A \quad \text{and} \quad f_i^{SO} = f_i\left(x_1^{SO}, \ldots, x_N^{SO}\right) = \tfrac{A}{4}\left(2\alpha_i n - A\right). \tag{4.11}$$

We remark that

$$f_i^{SO} > 0 \quad \text{iff} \quad \alpha_i > \frac{A}{2N}$$

meaning that, if the weight of player i is sufficiently large, he gives high value to the social part and thus his profit is positive joining the grand coalition, i.e. the coalition made by all the players.

4.2.2 Endogenous emergence of collaboration in partial cooperative games

Given a N-person normal Social Purpose Game $\Gamma = \langle \mathcal{N}, (U_i)_{i=1}^N, (f_i)_{i=1}^N \rangle$, let us fix the level of cooperation $k \in \{0, 1, \ldots, N\}$. Thus let us suppose that k players cooperate and let $C = \{N-k+1, \ldots, N\}$ be the set of cooperators and $\overline{C} = \{1, \ldots, N\}$ be the set of non-cooperators. Following the literature of partial cooperation (see [15], [16], [64]), given the definition of partial cooperative leadership equilibrium recalled in Section 2.7, let us prove an existence result for the partial cooperative game $\overline{\Gamma} = \langle C, \overline{C}, (U_i)_{i\in C}, (U_j)_{j\in\overline{C}}, (f_i)_{i\in C}, (f_j)_{j\in\overline{C}} \rangle$, following the results obtained in [15].

Proposition 4.6. *Let $\overline{\Gamma} = \langle C, \overline{C}, (U_i)_{i\in C}, (U_j)_{j\in\overline{C}}, (f_i)_{i\in C}, (f_j)_{j\in\overline{C}} \rangle$ be a partial cooperative Social Purpose game where, fixed the level of cooperation $k \in \{0, 1, \ldots, N\}$, $C = \{N-k+1, \ldots, N\}$ is the set of cooperators and $\overline{C} = \{1, \ldots, N\}$ is the set of non-cooperators. Assume that H is concave increasing and continuous on $\mathbb{R}$, $h_1, \ldots, h_N$ are identity functions, $g_1, \ldots, g_N$ are convex and continuous on $[0, \overline{Q}]$. Then there exists a partial cooperative leadership equilibrium of $\overline{\Gamma}$.*

Before proving this proposition, let us prove that the following hold:

Lemma 4.2. *Suppose that assumptions of Proposition 4.6 hold true. Then the correspondence $\mathbf{E} : \prod_{i=N-k+1}^N U_i \to 2^{\prod_{j=1}^{N-k} U_j}$ that maps x^C into $\mathbf{E}(x^C) = NE_{x^C} \subset \prod_{j=1}^{N-k} U_j$ is non-empty, compact valued and upper hemicontinuous.*

Proof. In our assumptions, Proposition 4.2 holds, thus for any $x^C \in \prod_{i=N-k+1}^{N} U_i$, $NE_{x^C} \neq \emptyset$ i.e. **E** is non-empty.

Let $\left(x_p^C\right)_{p\in\mathbb{N}}$ a sequence such that $x_p^C \to x^C$. Let us take $x_p^{\overline{C}} \in NE_{x_p^C}$ such that $x_p^{\overline{C}} \to x^{\overline{C}}$. Since $x_p^{\overline{C}} \in NE_{x_p^C}$ we have

$$\alpha_j H\Big(\sum_{j=1}^{N-k} x_{i_p}^{\overline{C}} + \sum_{i=N-k+1}^{N} x_{i_p}^{C}\Big) - g_j(x_{j_p}^{\overline{C}}) \geq$$

$$\geq \alpha_j H\Big(x_j^{'\overline{C}} + \sum_{l\neq j} x_{l_p}^{\overline{C}} + \sum_{i=N-k+1}^{N} x_{i_p}^{C}\Big) - g_j(x_j^{'\overline{C}})$$

for any $j \in \overline{C}$ and for any $x_j^{'\overline{C}} \in U_j$.

If $p \to +\infty$ by continuity we obtain

$$\alpha_j H\Big(\sum_{j=1}^{N-k} x_j^{\overline{C}} + \sum_{i=N-k+1}^{N} x_i^{C}\Big) - g_j(x_j^{\overline{C}}) \geq$$

$$\geq \alpha_j H\Big(x_j^{'\overline{C}} + \sum_{l\neq j} x_l^{\overline{C}} + \sum_{i=N-k+1}^{N} x_i^{C}\Big) - g_j(x_j^{'\overline{C}})$$

for any $j \in \overline{C}$ and for any $x_j^{'\overline{C}} \in U_j$. Thus $x^{\overline{C}} \in NE_{x^C}$ i.e. **E** is closed valued. Thus NE_{x^C} is a closed set for any x^C and it is compact since $NE_{x^C} \subset \prod_{j=1}^{N-k} U_j$ and $\prod_{j=1}^{N-k} U_j$ is compact.

Every closed correspondence with compact codomain is upper hemicontinuous.

□

Proof. (Proposition 4.6) Each payoff function f_i for $i \in C$ and f_j for $j \in \overline{C}$ is continuous and quasi-concave thus, since Lemma 4.2 holds true, applying Theorem 2.12 there exists at least one PCE. □

Remark 4.13. These results can be extended to the general case in which functions h_i for any $i \in \mathcal{N}$ are not the identity functions, assuming that they are continuous and concave.

Remark 4.14. Denoting by $x^{PC} = (x_1^{PC}, \dots, x_N^{PC})$ the partial cooperative leadership equilibrium and giving C the cooperative coalition, in the same assumptions of Proposition 4.4, we obtain that $\sum_{i \in C} x_i^{NE} \leq \sum_{i \in C} x_i^{PC}$. Moreover, in the same assumptions of Proposition 4.5, we obtain that $x_i^{NE} \leq x_i^{PC}$ for any $i \in \mathcal{N}$.

Given the existence of a partial cooperative leadership equilibrium, the next objective is to address the following research question: is there a way to endogeneously determine the number of cooperators in a partial cooperative framework? In order to give an answer to this question, we are going to use the stability notion. More precisely, to establish the number of players of a stable coalition, we refer the notions of internal and external stability (see [23]). The basic idea is that a coalition is stable if none inside has an incentive to defect and none outside has an incentive to join in.

Thus, given $h \in \mathcal{N}$ a generic player of the game Γ, let us denote by $f_h(C)$ his payoff function if he joins the coalition of cooperators and by $f_h(NC)$ his payoff function if he does not join the coalition. Let us recall the following:

Definition 4.2. ([23]) A coalition C of k players is

- *internal stable* iff $f_i(C) \geq f_i(NC)$ for any $i \in C$
- *external stable* iff $f_j(NC) \geq f_j(C)$ for any $j \in N \setminus C$

Proposition 4.7. *The configuration with $\overline{C} = \{1, \dots, N-k\}$ and $C = \{N - k+1, \dots, N\}$ is stable if and only if*

$$\alpha_{N-k+1} > \frac{g_{N-k+1}(x_{N-k+1}^{PC}) - g_{N-k+1}(x_{N-k+1}^{NE})}{H\left(\overline{x} + x_{N-k+1}^{PC}\right) - H\left(\overline{x} + x_{N-k+1}^{NE}\right)}$$

and

$$\alpha_{N-k} < \frac{g_{N-k}(x_{N-k}^{PC}) - g_{N-k}(x_{N-k}^{NE})}{H\left(\overline{x} + x_{N-k}^{PC}\right) - H\left(\overline{x} + x_{N-k}^{NE}\right)},$$

where $\overline{x} = \sum_{i \in C} x_i^{PC} + \sum_{j \in \overline{C}} x_j^{NE}$.

Remark 4.15. Note that, avoiding the case in which the first cooperator and the last non-cooperator are indifferent between being cooperator or non-cooperator, i.e. the case in which in Partial Cooperation and in Nash equilibria they pick the same strategy, the right hand-side of both disequalities are positive. In fact H is an increasing function and also g_i is increasing for any $i \in \mathcal{N}$ since g_i' is positive and Remark 4.14 holds.
We do not consider the case in which $x_i^{NE} = x_i^{PC}$.

Proof. To check the **internal stability** of the coalition of cooperators C we let $h = N - k + 1$ be the marginal player of C as introduced above. Then we can compare the payoffs of this player if he cooperates with C or acts as a non-cooperator. Indeed, if he cooperates he receives payoff

$$f_h(C) = \alpha_h H\Big(\sum_{i\in C} x_i^{PC} + x_h^{PC} + \sum_{j\in\overline{C}} x_j^{NE}\Big) - g_h(x_h^{PC})$$

and if he does not cooperate with C he receives

$$f_h(NC) = \alpha_h H\Big(\sum_{i\in C} x_i^{PC} + x_h^{NE} + \sum_{j\in\overline{C}} x_j^{NE}\Big) - g_h(x_h^{NE}).$$

Internal stability requires now that $f_h(C) > f_h(NC)$. This is equivalent to

$$\alpha_h H\Big(\sum_{i\in C} x_i^{PC} + x_h^{PC} + \sum_{j\in\overline{C}} x_j^{NE}\Big) - g_h(x_h^{PC}) >$$

$$> \alpha_h H\Big(\sum_{i\in C} x_i^{PC} + x_h^{NE} + \sum_{j\in\overline{C}} x_j^{NE}\Big) - g_h(x_h^{NE})$$

which is equivalent to the first condition in the assertion.

For the **external stability** of the configuration as indicated, let $h = N - k$ be the marginal non-cooperator. Again we can compare the payoffs of player h if he cooperates with C or acts as a non-cooperator. Indeed, if he cooperates he receives payoff

$$f_h(C) = \alpha_h H\Big(\sum_{i\in C} x_i^{PC} + x_h^{PC} + \sum_{j\in\overline{C}} x_j^{NE}\Big) - g_h(x_h^{PC})$$

and if he does not cooperate with C he receives

$$f_h(NC) = \alpha_h H\Big(\sum_{i\in C} x_i^{PC} + x_h^{NE} + \sum_{j\in\overline{C}} x_j^{NE}\Big) - g_h(x_h^{NE}).$$

External stability now requires that $f_h(NC) > f_h(C)$, which is equivalent to

$$\alpha_h H\Big(\sum_{i\in C} x_i^{PC} + x_h^{NE} + \sum_{j\in\overline{C}} x_j^{NE}\Big) - g_h(x_h^{NE}) >$$
$$> \alpha_h H\Big(\sum_{i\in C} x_i^{PC} + x_h^{PC} + \sum_{j\in\overline{C}} x_j^{NE}\Big) - g_h(x_h^{PC})$$

which is equivalent to the second condition of the assertion. □

Partial cooperation in water resource game

Let us now show an application of these results to the game considered in the Example 4.2. Let $C = \{N-k+1, \dots, N\} \subset \mathcal{N}$ be the coalition of $2 \leqslant k \leqslant N-1$ cooperators with the highest preference for the generated benefits in this game. The non-cooperators are now the players in $\overline{C} = \{1, \dots, N-k\}$ with the lowest preference for the collectively generated benefit.

Every non-cooperator $j \in \overline{C}$ now selects a best response to all other players' strategies. Hence,

$$x_j^{PC} = \tfrac{1}{2}\alpha_j \quad \text{for every } j \in \overline{C}.$$

The cooperators in C determine their strategies collectively to maximise their collective payoff $f_C = \sum_{i\in C} f_i$. This results into

$$x_i^{PC} = \tfrac{1}{2}A_C = \tfrac{1}{2}\sum_{i=N-k+1}^{N} \alpha_i \quad \text{for every } i \in C.$$

Hence, in the partial cooperative equilibrium we have that

$$\sum_{h\in\mathcal{N}} x_h^{PC} = \tfrac{1}{2}A_{\mathcal{N}\setminus C} + \tfrac{k}{2}A_C \tag{4.12}$$

$$f_j^{PC} = \alpha_j\left(\tfrac{1}{2}A_{\mathcal{N}\setminus C} + \tfrac{k}{2}A_C\right) - \tfrac{1}{4}\alpha_j^2 \qquad \text{for every } j \in \mathcal{N}\setminus C \tag{4.13}$$

$$f_i^{PC} = \alpha_i\left(\tfrac{1}{2}A_{\mathcal{N}\setminus C} + \tfrac{k}{2}A_C\right) - \tfrac{1}{4}A_C^2 \qquad \text{for every } i \in C \tag{4.14}$$

In the following proposition we endogenously determine the coalition of cooperators:

Proposition 4.8. *The configuration with* $\mathcal{N} \setminus C = \{1, \ldots, N-k\}$ *and* $C = \{N-k+1, \ldots, N\}$ *is stable if and only if*

$$\alpha_{N-k+1} > \frac{A_C}{1+\sqrt{2(k-1)}} \tag{4.15}$$

as well as

$$\alpha_{N-k} < \frac{A_C}{\sqrt{2k}}. \tag{4.16}$$

Proof. Let $h = N-k+1$ be the marginal player of C as introduced above. If he cooperates he receives payoff

$$f_h(C) = \alpha_h \left(\tfrac{1}{2}A_{\mathcal{N}\setminus C} + \tfrac{k}{2}A_C\right) - \tfrac{1}{4}A_C^2$$

and if he does not cooperate with C he receives

$$f_h(NC) = \alpha_h \left(\tfrac{1}{2}A_{\mathcal{N}\setminus C} + \tfrac{1}{2}\alpha_h + \tfrac{k-1}{2}A_{C\setminus h}\right) - \tfrac{1}{4}\alpha_h^2.$$

Internal stability requires now that $f_h(C) > f_h(NC)$. This is equivalent to

$$\tfrac{\alpha_h}{2}kA_c - \tfrac{1}{4}A_C^2 > \alpha_h \left(\tfrac{\alpha_h}{2} + \tfrac{k-1}{2}(A-\alpha_h)\right) - \tfrac{1}{4}\alpha_h^2$$

or

$$A_C^2 - \alpha_h^2 < 2\alpha_h A_C + 2(k-2)\alpha_h^2$$

or

$$A_C^2 - 2\alpha_h A_C + \alpha_h^2 = (A_C - \alpha_h)^2 < 2(k-1)\alpha_h^2$$

This is equivalent to

$$A_C - \alpha_h < \sqrt{2(k-1)}\alpha_h$$

which is equivalent to the first condition in the assertion.

Let $h = N - k$ be the marginal non-cooperator. If he cooperates he receives payoff

$$f_h(C) = \alpha_h \left(\tfrac{1}{2}(A_{\mathcal{N}\setminus C} - \alpha_h) + \tfrac{k+1}{2}(A_C + \alpha_h)\right) - \tfrac{1}{4}\left(A_C + \alpha_h\right)^2$$

and if he does not cooperate with C he receives

$$f_h(NC) = f_h^{PC} = \alpha_h \left(\tfrac{1}{2}A_{\mathcal{N}\setminus C} + \tfrac{k}{2}A_C\right) - \tfrac{1}{4}\alpha_h^2.$$

External stability now requires that $f_h(NC) > f_h(C)$, which is equivalent to

$$\tfrac{k}{2}\alpha_h A_C - \tfrac{1}{4}\alpha_h^2 > \alpha_h \left(\tfrac{k}{2}\alpha_h + \tfrac{k+1}{2}A_C\right) - \tfrac{1}{4}\left(A_C + \alpha_h\right)^2$$

or

$$A_C^2 + 2\alpha_h A_C > \alpha_h \left[\tfrac{k+1}{2}A_C + \tfrac{k}{2}\alpha_h - \tfrac{k}{2}A_C\right] = \alpha_h \left(\tfrac{1}{2}A_C + \tfrac{k}{2}\alpha_h\right)$$

which is equivalent to $A_C^2 > 2k\alpha_h^2$. This implies the external stability stated in the assertion of the proposition. □

Next we consider an example that illustrates the main conclusion of the proposition.

Example 4.3. Consider a game with 7 players, i.e., $\mathcal{N} = \{1, 2, 3, 4, 5, 6, 7\}$. Using the notation introduced above the next table summarises the findings and various equilibria.

In this example we order the players such that $\alpha_1 \geqslant \alpha_2 \geqslant \cdots \geqslant \alpha_7$ and we introduce

$$A_k = \sum_{i=1}^{k} \alpha_i.$$

By applying the test formulated in the proposition we identify that $C = \{1, 2\}$ is the unique stable coalition of cooperators for the given parameter values in the table. The partial cooperative equilibrium reported in the table below is founded on this selection:

i	1^*	2^*	3	4	5	6	7	Sum
α_i	8	7	7	7	7	5	1	$A = 42$
A_k	8	15	22	29	36	41	42	
$\frac{A_k}{\sqrt{2k}}$	5.66	7.5	8.98	10.25	11.38	11.84	11.23	
$x_i^{NE} = \frac{\alpha_i}{2}$	4	3.5	3.5	3.5	3.5	2.5	0.5	21
f_i^{NE}	152	134.75	134.75	134.75	134.75	98.75	20.75	
$x_i^{SO} = \frac{A}{2}$	21	21	21	21	21	21	21	147
f_i^{SO}	735	588	588	588	588	294	-294	
x_i^{PC}	7.5	7.5	3.5	3.5	3.5	2.5	0.5	28.5
f_i^{PC}	171.75	143.25	187.25	187.25	187.25	136.25	28.25	

The conclusions of the proposition are confirmed. In particular, the unique stable coalition of cooperators is determined to be $C = \{1, 2\}$. Indeed, the payoff of both players in C are improvements compared to the Nash equilibrium payoffs: $f_1^{PC} = 171.75 > f_1^{NE} = 152$ and $f_2^{PC} = 143.25 > f_2^{NE} = 134.75$.
On the other hand, adding player 3 to C would result in an optimal cooperative strategy of $x_1 = x_2 = x_3 = \frac{1}{2}A_3 = 11$ and a partial cooperative equilibrium payoff of $f_1 = 223$ and $f_2 = f_3 = 180 < f_3^{PC} = 187.25$.

Chapter 5

Conclusions

In this thesis some aspects of the class of Aggregative Games have been investigated, firstly developing the theory of multi-leader multi-follower aggregate equilibrium under uncertainty, secondly doing an equilibrium analysis for a Common-Pool Resource aggregative game and finally presenting the class of Social Purpose Games, with an application to a withdrawal water resource game.

Some possible further directions of research are the following:

- in the context of CPR games, since the main problem arising is the overuse control of the CPR, we suppose that there is a social planner who, knowing that the total investment in the social optimum profile is higher than the total investment in the non-cooperative equilibrium profile (see Section 4.1), proposes a taxation scheme in order to lead the investors' decision toward the social optimum, by looking for a map that maximizes the social welfare. Starting from this applicative example, we will introduce the concept of Mean Inverse Stackelberg strategy and we will explain a new methodology, based on Calculus of Variations, that allows to solve an Inverse Stackelberg model which is in general a problem very difficult to solve, cause it involves functions' composition (see [84],

[85]). This results are going to be presented in [55];

- in a dynamic context, we consider a two-player pollution game that represents the interaction between two countries, namely developed and developing ones. We allows for adaptation measures that consist in the set of actions that prevent or decrease the adverse effects of accumulated pollution. The development of each country is measured by the level of capital that it owns. The developed country invests in capital and in adaptation while the developing one invests firstly in capital and then, after having reached a certain threshold level of development, also in adaptation. We will characterize and compare the cooperative and non-cooperative solutions, supposing that the information structure is feedback, that is adaptation and investment in capital are state-dependent, with the state being defined as a four-dimensional vector (t, P, K_1, K_2) with t time, P pollution stock and K_1, K_2 respectively developing country's and developed country's capital stock. This results are going to be presented in [71];

- the results presented in Section 4.2 for the class of Social Purpose Games, that is a particular class of Aggregative Games, could be generalized to the case of Fully Aggregative Games (see [20]), namely aggregative additively separable games for which the aggregative term in the payoff function of each player is weighted by a function of the corresponding player's strategy. Among others, examples of Fully Aggregative Games are given by contests and fighting game, Cournot game and pollution control game.

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