Università degli Studi di Napoli "Federico II"

Corso di Dottorato in Economia

XXXI ciclo

Selection methods for subgame perfect Nash equilibrium in a continuous setting

Tutor
Prof.ssa Jacqueline Morgan
Coordinatore
Prof.ssa Maria Gabriella Graziano

Candidato
Francesco Caruso

Anno Accademico 2017-2018

A Miriam
Alla mia famiglia

Contents

Acknowledgements

At the end of the run, I would like to express my sincere gratitude to the key persons who supported me throughout the last three years.

Firstly, I wish to strongly thank my supervisor, Prof. Jacqueline Morgan. She introduced me to Game Theory during my graduate studies, she believed in me and gave me endless support through the PhD program. Her guidance, motivation and incitements have been allowing me to train and increase my research attitudes.

Besides my supervisor, I am really grateful to Dr. Maria Carmela Ceparano for being present in any situation I needed, for the infinite suggestions and for the patience she showed. Her contribution in our collaboration has been fundamental for achieving this result.

Moreover, I would like to thank all the professors and researchers of the Department of Economics and Statistics of the University of Naples Federico II for the stimulating discussions and helpful explanations, and in particular Prof. Maria Gabriella Graziano for her professional, accurate and careful work as PhD director.

A special mention goes to my PhD colleagues and my friends. The enjoyable moments I spent with them helped me to face this three-years run less hardly.

Finally, thanks go to my great family and to Miriam for the tremendous support and limitless endurance. I dedicate this thesis to them.

Chapter 1

Introduction

Together with the incessant increase of interest for Game Theory and for the Nash equilibrium solution concept, certainly the mostly used non-cooperative game-theoretic notion especially in economics and management science, the need of developing a theory which allows to explain how an equilibrium can be picked in a game has been arising. This is both because a game could admit more than one Nash equilibrium, so difficulties could occur in players when choosing their actions, and because of the concerns (connected to the latter issue) regarding the psychological and philosophical foundations of this concept.
In the foreword to the book of Harsanyi and Selten [53], which represents the first contribution in the development of a theory of Nash equilibrium selection in games, Aumann emphasizes that

> An equilibrium in a game is defined as an assignment to each player of a strategy that is optimal for him when the others use the strategies assigned to them. [...] In general, a given game may have several equilibria. Nash equilibrium makes sense only if each player knows which strategies the others are playing; if the equilibrium recommended by the theory is not unique, the players will not have this knowledge. Thus it is essential that for each game, the theory selects one unique Nash equilibrium from the set of all Nash equilibria, [53, p.xi].

Consequently, the following two key points are significant in the theory of equilibrium selection: on the one hand, the possibility to obtain an equilibrium selection by means of a constructive (in the sense of algorithmic) process; on the other hand, the motivations that would induce players to choose the actions leading to the designed selection, together with the interpretation of the method defined to achieve such a selection (still linked to the behavioural peculiarities of each player).

Many Nash equilibrium selection and refinement concepts have been proposed in Game Theory literature based on perturbations of the data of the game and/or motivated by possible distortions of players' rationality, both for normal-form games (see, for example, [148, 52, 118, 122, 60, 53, 127, 30] and also [143] and references therein) and, especially when players have finite sets of actions, for extensive-form games (see, for example, [135, 134, 142, 61, 117, 46]). Nevertheless, the issue of equilibrium selection in sequential games of perfect information where players have a continuum of actions has been less investigated (see [49, 26] for the imperfect information case).

Therefore, in this thesis we examine the issue of selection of the *subgame perfect Nash equilibrium* (SPNE for short, see [135, 134]), which represents the mostly known and broadly applied solution concept in sequential games, in the class of one-leader N-follower two-stage games (with $N \in \mathbb{N}$), namely $N+1$-person non-cooperative sequential games where players have a continuum of actions and the interactions among players are ruled as follows: one player acting in the first stage, the *leader*, chooses an action in his action set, then in the second stage the remaining N players, the *followers*, after having observed the choice made by the leader, reply by choosing simultaneously an action each one in his own action set. Our main purpose is to propose constructive methods in order to select an SPNE that both satisfy the desirable features illustrated above regarding the theory of equilibrium selection and provide SPNEs existence results.

In Chapter 2, after giving the definitions of one-leader N-follower two-stage game and subgame perfect Nash equilibrium, the problem of providing a manageable existence result for SPNEs (motivated by the fact that the best reply correspondence of the followers is, in general, not a lower semicontinuous set-valued map) in one-leader N-follower two-stage games is introduced. Firstly, we analyze the case where the followers' best reaction is assumed to be unique for any action chosen by the leader (Subsection 2.2.1): in this case the lower semicontinuity issue regarding the best reply correspondence can be overcome, and we show that finding the SPNEs of the game is equivalent to find the Stackelberg solutions of the associated Stackelberg problem (introduced by von Stackelberg in [145]). Then, we examine the general case where the followers' best reaction is not always unique (Subsection 2.2.2): since in this situation multiple SPNEs could come up we introduce also the arising issue of selection of SPNEs, so we describe some ways to select an SPNE which are motivated according to the beliefs that the leader has about how the followers choose actions in their best reaction sets. Such SPNE selections are obtained by exploiting the solutions of widely studied problems in Optimization Theory, in particular the strong Stackelberg, the weak Stackelberg and the intermediate Stackelberg problems associated to the game (see [66] and [95] for the definitions of such types of Stackelberg problems).

When the followers' best reaction is assumed to be unique the SPNEs selection issue could not occur, but a related arising issue concerns, evidently, the sufficient conditions ensuring the uniqueness of the followers' best reaction. Hence, in Chapter 3, we present a result regarding the existence of a unique Nash equilibrium in two-player normal-form games where the action sets are Hilbert spaces and which allows (provided that the players' best reply correspondences are single-valued) the two compositions of the best reply functions to be not a contraction mapping (being the existence of a unique Nash equilibrium provided the compositions of the best reply functions to be a contraction a well-known established result; see, for example, [69, Theorem 1]). Moreover, when we restrict to the class of weighted potential games introduced by Monderer and Shapley in [105], the (lack of) connections between Nash equilibria and maximizers of the potential function is proved, referring to Caruso, Ceparano and Morgan [24]. Finally, still having in mind to provide sufficient conditions guaranteeing the uniqueness of the followers' best reaction, the Rosen uniqueness result [132], which concerns the uniqueness of Nash equilibrium in normal-form games where the action sets are constrained subsets of Euclidean spaces, is reminded.

Coming back to the SPNE selection issue arising when the followers' best reaction is not always unique, the selection methods illustrated in Chapter 2, although behaviourally motivated, do not provide however a constructive procedure to achieve an SPNE and, furthermore, require the leader the demanding task of knowing the best reply correspondence, by definition of (strong, weak and intermediate) Stackelberg solution. Hence, we focus on designing constructive methods in order to select an SPNE with the following features:

(i) relieving the leader of learning the best reply correspondence;

(ii) allowing to overcome the difficulties deriving from the possible non-single-valuedness of the best reply correspondence.

These features will be satisfied by exploiting the Tikhonov regularization [140] and the proximal point algorithm [99, 131] (based on the Moreau-Yosida regularization [106]). Such techniques, widely used in Optimization, are presented in Chapter 4 together with applications to the selection of Nash equilibria in normal-form games (Subsections 4.1.1 and 4.2.2): they have the advantage of allowing to approximate a solution of a (optimization or Nash equilibrium) problem by constructing sequences of regularized problems having a unique solution, so step by step the approximating sequence is uniquely identified.

Therefore, in Chapter 5 two selection methods for SPNEs which fit goals (i) and (ii) are presented. In both methods an SPNE is achieved by constructing a sequence of one-leader N-follower two-stage games where the best reply correspondence is single-valued, by using the regularizations illustrated in Chapter 4. Firstly, we analyze the constructive method introduced by Morgan and Patrone in [109], where the Tikhonov regularization is exploited (Section 5.1). Then, referring to Caruso, Ceparano and Morgan [23], we show an SPNE selection method for one-leader one-follower two-stage games based on proximal point algorithm which satisfies even the following feature:

(iii) it is based on a learning approach which has a behavioral interpretation linked to the costs that players face when deviating from their current actions,

according to the interpretation of proximal point methods provided in [3] and illustrated in Subsection 4.2.1. Besides, we highlight that both methods embody an existence result for SPNEs which does not require the lower semicontinuity of the best reply correspondence. Finally, in Section 5.3 some further under investigation issues and directions for future research are discussed.

Chapter 2

Subgame perfect Nash equilibria in a continuous setting

The two main objects of the thesis are presented in this introductory chapter: the class of one-leader N-follower two-stage games (where $N \in \mathbb{N}$ can be 1 or more than 1) and the subgame perfect Nash equilibrium solution concept. After providing the mathematical formulation, the definitions and examples of applications of such objects, the difficulties of providing a reasonable existence result for subgame perfect Nash equilibria in one-leader N-follower two-stage games where players have a continuum of actions are discussed. Motivated by this drawback, we continue the investigation by analyzing two scenarios.

- When the followers' best reaction is assumed to be unique for any action chosen by the leader: the SPNEs can be fully characterized in terms of solutions of the associated Stackelberg problem and the difficulties above mentioned can be overcome.
- When the followers' best reaction is not always unique: aside from the existence of SPNEs, the arising issue of the selection of SPNEs is analyzed. Hence, some existing methods to select SPNEs (in one-leader N-follower two-stage games) are described, which rely on the exploitation of the solutions of broadly studied problems in literature: the strong Stackelberg, the weak Stackelberg and the intermediate Stackelberg problem.

Finally, the need to define a constructive method in order to select a subgame perfect Nash equilibrium that relieves the leader of knowing the sets of the followers' best reactions and that allows to overcome the difficulties due to the non-uniqueness of the followers' best reactions is pointed out, mentioning that in Chapter 5 two such constructive methods will be illustrated.

2.1 One-leader N-follower two-stage games

By *one-leader N-follower two-stage game* Γ_N we intend an $N+1$-person non-cooperative sequential game where

1. **in the first stage:** one player, henceforth called *leader*, chooses an action x in his action set X;

2. **in the second stage:** N players, henceforth called *followers*, observe the action x chosen by the leader in the first stage and then simultaneously choose actions $y_1, y_2, \dots, y_N$, where y_i is the action chosen by the i's follower in his action set Y_i, for any $i \in \{1, \dots, N\}$;

3. **payoffs:** after the two-stage interaction, leader receives $L(x, y_1, \dots, y_N)$ where $L\colon X \times Y_1 \times \cdots \times Y_N \to \mathbb{R}$ is the leader's payoff function, and the i's follower receives $F_i(x, y_1, \dots, y_N)$ where $F_i\colon X \times Y_1 \times \cdots \times Y_N \to \mathbb{R}$ is the i's follower payoff function, for any $i \in \{1, \dots, N\}$.

If $N \geq 2$ the followers acting in the second stage, after having observed the leader's action x, engage in the non-cooperative simultaneous-move game

$$G_x = \{I, (Y_i)_{i\in I}, (F_i(x, \cdot, \dots, \cdot))_{i\in I}\}, \tag{2.1}$$

where $I \coloneqq \{1, \dots, N\}$ is the set of the followers, and they react to the leader choosing a Nash equilibrium of G_x, i.e. a strategy profile $(y_1^*, \dots, y_N^*) \in Y \coloneqq Y_1 \times \cdots \times Y_N$ such that for any $i \in I$

$$F_i(x, y_1^*, \dots, y_{i-1}^*, y_i^*, y_{i+1}^*, \dots, y_N^*) \geq F_i(x, y_1^*, \dots, y_{i-1}^*, y_i, y_{i+1}^*, \dots, y_N^*) \tag{2.2}$$

for any $y_i \in Y_i$. As usual in game theory, for any $i \in I$ we set $Y_{-i} \coloneqq Y_1 \times \cdots \times Y_{i-1} \times Y_{i+1} \times \cdots \times Y_N$, so that a strategy profile $y \in Y$ can be also written as $(y_i, y_{-i}) \in Y_i \times Y_{-i}$. Hence, inequality (2.2) can be rewritten in the more compact way: $F_i(x, y_i^*, y_{-i}^*) \geq F_i(x, y_i, y_{-i}^*)$.
We denote by $\mathcal{N}\colon X \rightrightarrows Y$ the set-valued map that associates with each leader's action $x \in X$ the set of Nash equilibria of G_x, i.e.

$$\begin{aligned}\mathcal{N}(x) = \{y^* \in Y \text{ s.t. } & F_i(x, y_i^*, y_{-i}^*) \geq F_i(x, y_i, y_{-i}^*) \\ & \text{for any } y_i \in Y_i \text{ and } i \in I\},\end{aligned} \tag{2.3}$$

and we call it *Nash equilibrium correspondence of the followers*. We use the notation $(X, Y_1, \dots, Y_N, L, F_1, \dots, F_N)$ to refer to the one-leader N-follower two-stage game Γ_N in order to focus on the relevant features of the game.

If $N = 1$ the (unique) follower acting in the second stage, after having observed the leader's action x, faces the optimization problem

$$P_x\colon \max_{y \in Y} F(x, y), \tag{2.4}$$

where we set $Y = Y_1$ and $F = F_1$, and he reacts to the leader choosing a maximizer of the function $F(x, \cdot)$, i.e. an action $y^* \in Y$ such that

$$F(x, y^*) \geq F(x, y) \text{ for any } y \in Y. \tag{2.5}$$

We denote by $\mathcal{M}\colon X \rightrightarrows Y$ the set-valued map that associates with each leader's action $x \in X$ the set of follower's actions that solve P_x, i.e.

$$\begin{aligned} \mathcal{M}(x) &= \{y^* \in Y \text{ s.t. } F(x, y^*) \geq F(x, y) \text{ for any } y \in Y\} \\ &= \operatorname*{Arg\,max}_{y \in Y} F(x, y), \end{aligned} \tag{2.6}$$

and we call it *best reply correspondence of the follower*. We use the notation (X, Y, L, F) to refer to the one-leader one-follower two-stage game Γ_1, also denoted simply by Γ. Moreover it is worth to note that, if $I = \{1\}$ in (2.1) finding a Nash equilibrium of G_x is equivalent to solve the optimization problem P_x in (2.4), since inequality (2.2) is reduced to (2.5); consequently, even the Nash equilibrium correspondence $\mathcal{N}$ is reduced to the follower's best reply correspondence $\mathcal{M}$.

Let us provide now some relevant examples of one-leader N-follower two-stage games.

Example 2.1.1 (*Ultimatum game* [50]) One player, also called the *proposer*, has to split an amount of money $S > 0$ (he is endowed) with another player, called the *responder*. The responder, after observed the decision communicated by the proposer, can accept or reject the split. In the first case, the money is split according to the decision of the proposer; otherwise, both players receive nothing. The ultimatum game is a one-leader one-follower two-stage game where the proposer acts as the leader and the responder as the follower, the action sets are $X = [0, S]$ for the leader (who decides the sum of money $x \in [0, S]$ to keep for himself) and $Y = \{a, r\}$ for the follower (who decides whether to accept or reject the split communicated by the leader), and the payoff functions are

$$L(x, y) = \begin{cases} x, & \text{if } y = a \\ 0, & \text{if } y = r \end{cases} \quad \text{and} \quad F(x, y) = \begin{cases} S - x, & \text{if } y = a \\ 0, & \text{if } y = r, \end{cases}$$

for the leader and the follower, respectively.

Example 2.1.2 (*Stackelberg competition* [145]) There are two firms in a market which have to choose quantities. Firm 1, the leader, chooses a quantity $q_1 \geq 0$; then firm 2, the follower, observes the quantity q_1 and chooses a quantity $q_2 \geq 0$. Firm i aims to maximize his profit function depending on the inverse demand function $P(\cdot)$, the total quantity $Q = q_1 + q_2$ in the market and his cost function $C_i(\cdot)$. In such one-leader one-follower two-stage game, well-known as *Stackelberg game*, the action sets are $X = Y = [0, +\infty[$ and the payoff functions are given by the following profit functions:

$$L(q_1, q_2) = q_1 P(Q) - C_1(q_1) \quad \text{and} \quad F(q_1, q_2) = q_2 P(Q) - C_2(q_2),$$

where $P(\cdot)$ is a differentiable function with $P'(Q) < 0$ and $C_i(\cdot)$ is a twice differentiable function with $C_i'(q_i) > 0$ and $C_i''(q_i) \geq 0$ for any $i \in \{1, 2\}$.

Example 2.1.3 (*Leontief wage-employment model* [68]) It concerns the relationship between a monopoly union and N firms in an oligopoly. The union, that has exclusive control over wages, makes a single wage demand $w \geq 0$ for all the firms; then the firms, that have exclusive control over employment, observe w and simultaneously choose the employment levels (firm i chooses $e_i \geq 0$ for any $i \in \{1, \dots, N\}$). This situation can be modelled as a one-leader N-follower two-stage game where the union acts as the leader and the firms act as the followers, the action set of the leader is $X = [0, +\infty[$, the action set of follower i is $Y_i = [0, +\infty[$ for any $i \in \{1, \dots, N\}$, and the payoff functions for the leader and the follower i are, respectively,

$$L(w, e_1, \dots, e_N) = (w - w_a)E \quad \text{and} \quad F_i(w, e_1, \dots, e_N) = e_i P(E) - w e_i,$$

where $w_a \geq 0$ is the wage that union members can earn in alternative employment, $E = e_1 + \dots + e_N$ is the total employment of all the firms and $P(\cdot)$ is the inverse demand function (assumed to be differentiable with $P'(E) < 0$).

Example 2.1.4 (*Stackelberg competition with two followers* [136, 96]) Three firms choose quantities in a market with inverse demand function given by $P(\cdot)$. The firms operate as follows: firm 1 (the leader) chooses the quantity $q_1 \geq 0$; then firm 2 and firm 3 (the followers), after having observed q_1, act as in a Cournot duopoly and choose simultaneously quantities $q_2 \geq 0$ and $q_3 \geq 0$, respectively. In such one-leader two-follower two-stage game the action set of the leader is $X = [0, +\infty[$, the action sets of the followers are $Y_1 = Y_2 = [0, +\infty[$, and the payoff functions are given by the following profit functions

$$L(q_1, q_2, q_3) = q_1 P(Q) - C_1(q_1) \quad \text{and} \quad \begin{aligned} F_1(q_1, q_2, q_3) &= q_2 P(Q) - C_2(q_2), \\ F_2(q_1, q_2, q_3) &= q_3 P(Q) - C_3(q_3), \end{aligned}$$

where $Q = q_1 + q_2 + q_3$ is the total quantity produced by all the firms and $C_i(\cdot)$ is the cost function of firm i (assumed to be twice differentiable with $C_i'(q_i) > 0$ and $C_i''(q_i) \geq 0$), for any $i \in \{1, 2, 3\}$. The model can be easily extended to the case of $N > 2$ followers.

2.2 Subgame perfect Nash equilibrium

Before discussing the subgame perfect Nash equilibrium solution concept, it is worth to recall the notions of player's strategy and subgame (see, e.g., [45, 100]). In sequential games, a *strategy* for a player is a function that specifies which action the player chooses at any point in the game he could be required to make a decision (even if some of these points are not actually reached during the play of the game).
A *subgame* is any "subset" of the game satisfying the following properties:

1. it starts at a point where a player, who knows the history of the game up to that point, is required to make a decision;
2. it involves all the interactions among players starting from that point, until the end of the game;
3. if it contains a point where a player is involved in a simultaneous-move game with other players, it must contains entirely such a simultaneous-move game.

In a one-leader N-follower two-stage game $\Gamma_N = (X, Y_1, \ldots, Y_N, L, F_1, \ldots, F_N)$ the set of leader's strategies coincides with the set of actions X, the set of follower i's strategies is $Y_i^X = \{\varphi \colon X \to Y_i\}$ and G_x defined in (2.1) is a subgame of Γ_N for any $x \in X$ (the only subgame of Γ_N different from G_x is Γ_N itself). Analogously, in a one-leader one-follower two-stage game $\Gamma = (X, Y, L, F)$ the set of leader's strategies is X, the set follower's strategies is $Y^X = \{\varphi \colon X \to Y\}$ and the subgames are the degenerate games where the follower is the unique player and reacts to the a given choice $x \in X$ of the leader by maximizing the function $F(x, \cdot)$, that is, he solves the optimization problem P_x defined in (2.4).

Now we introduce the solution concept we are mostly interested in this work, the most known refinement of the Nash equilibrium widely used in sequential games: the *subgame perfect Nash equilibrium* concept ([135, 134]; see also, e.g., [54, 100]). The main idea on which this concept is founded is the *principle of sequential rationality*: Reinhard Selten suggested to choose those equilibria where the players' strategies represent a Nash equilibrium when restricted to each subgame of the original game, i.e., the players have to behave optimally from any point of the game onwards, so even with regard to off the equilibrium path actions. Therefore, more precisely, a strategy profile (i.e., a vector of strategies, one for each player) is a subgame perfect Nash equilibrium of a sequential game if the restriction of the strategy profile to any subgame constitutes a Nash equilibrium when that subgame is considered in isolation. The principle of sequential rationality is strongly connected to the *backward induction* procedure, the most common tool used to find subgame perfect Nash equilibria in sequential games of perfect information (as the one-leader one-follower two-stage games): firstly, it is determined the optimal behaviour at the end of the game and then, knowing this behaviour, the optimal behaviour in the earlier stages of the game is found (such a procedure, in a generalized version, is used also in sequential games with almost perfect information, as the one-leader N-follower two-stage games with $N \geq 2$).

Let $\Gamma_N = (X, Y_1, \ldots, Y_N, L, F_1, \ldots, F_N)$ be a one-leader N-follower two-stage game. The set of strategy profiles is $X \times Y_1^X \times \cdots \times Y_N^X$ and the definition of subgame perfect Nash equilibrium is characterized in the following way.

Definition 2.2.1 A strategy profile $(\bar{x}, \bar{\varphi}_1, \ldots, \bar{\varphi}_N) \in X \times Y_1^X \times \cdots \times Y_N^X$ is a subgame perfect Nash equilibrium (SPNE for short) of Γ_N if the following conditions are satisfied:

(SG1$_N$) for each choice x of the leader, the followers react choosing a Nash equilibrium of G_x, i.e. for any $x \in X$:

$$(\bar{\varphi}_1(x), \ldots, \bar{\varphi}_N(x)) \in \mathcal{N}(x);$$

(SG2$_N$) the leader maximizes his payoff function taking into account his hierarchical advantage, i.e.

$$\bar{x} \in \operatorname*{Arg\,max}_{x \in X} L(x, \bar{\varphi}_1(x), \ldots, \bar{\varphi}_N(x)).$$

It is worth to give the definition of SPNE even in the special case of a one-leader one-follower two-stage game $\Gamma = (X, Y, L, F)$.

Definition 2.2.2 A strategy profile $(\bar{x}, \bar{\varphi}) \in X \times Y^X$ is an SPNE of Γ if the following conditions are satisfied:

(SG1) for each choice x of the leader, the follower maximizes his payoff function, i.e. for any $x \in X$:

$$\bar{\varphi}(x) \in \mathcal{M}(x);$$

(SG2) the leader maximizes his payoff function taking into account his hierarchical advantage, i.e.

$$\bar{x} \in \operatorname*{Arg\,max}_{x \in X} L(x, \bar{\varphi}(x)).$$

Example 2.2.1 (*SPNE in Ultimatum game*) The game of Example 2.1.1 has one SPNE: the pair $(S, \bar{\varphi})$ where $\bar{\varphi}(x) = a$ for any $x \in [0, S]$. In this case the follower agrees with any split decided by the leader; the follower gains nothing, while the leader receives the total amount of money S.

Example 2.2.2 (*SPNE in Stackelberg competition*) Setting, for example, the inverse demand function $P(Q) = a - Q$ and the cost functions $C_1(q_1) = cq_1$ and $C_2(q_2) = cq_2$, the unique SPNE of the Stackelberg game in Example 2.1.2 is $(\bar{q}_1, \bar{\varphi})$ where

$$\bar{q}_1 = \frac{a-c}{2} \quad \text{and} \quad \bar{\varphi}(q_1) = \frac{a - q_1 - c}{2} \quad \text{for any } q_1 \geq 0.$$

Example 2.2.3 (*SPNE in Leontief wage-employment model*) The game in Example 2.1.3, set the inverse demand function $P(E) = a - E$, has one SPNE: the strategy profile $(\bar{w}, \bar{\varphi}_1, \dots, \bar{\varphi}_N)$, where

$$\bar{w} = \frac{a + w_a}{2} \quad \text{and} \quad \bar{\varphi}_i(w) = \frac{a - w}{N+1} \quad \text{for any } w \geq 0 \text{ and } i = \{1, \dots, N\}.$$

Example 2.2.4 (*SPNE in Stackelberg competition with two followers*) Setting $P(Q) = a - Q$ and $C_i(q_i) = cq_i$ for any $i \in \{1, 2, 3\}$, the unique SPNE of the game in Example 2.1.4 is $(\bar{q}_1, \bar{\varphi}_1, \bar{\varphi}_2)$ where

$$\bar{q}_1 = \frac{a-c}{2} \quad \text{and} \quad \bar{\varphi}_i(q_1) = \frac{a - q_1 - c}{3} \quad \text{for any } q_1 \geq 0 \text{ and } i = \{1, 2\}.$$

Let us show now a simple result on the existence of SPNEs for one-leader N-follower two-stage games where the players have a continuum of actions.

Proposition 2.2.3. *Let $\Gamma_N = (X, Y_1, \dots, Y_N, L, F_1, \dots, F_N)$ be a one-leader N-follower two-stage game, where $N \in \mathbb{N}$. Assume that*

(i) X is a compact subset of a Euclidean space;

(ii) Y_i is a subset of a Euclidean space, for any $i \in \{1, \dots, N\}$;

(iii) L is upper semicontinuous on $X \times Y_1 \times \cdots \times Y_N$;

(iv) the Nash equilibrium correspondence $\mathcal{N}\colon X \rightrightarrows Y_1 \times \cdots \times Y_N$

1. *is a lower semicontinuous set-valued map, that is, for any sequence* $(x_k)_k \subseteq X$ *converging to* $x \in X$ *and for any* $y \in \mathcal{N}(x)$*, there exists a sequence* $(y_k)_k \subseteq Y$ *such that* $(y_k)_k$ *converges to* y *and* $y_k \in \mathcal{N}(x_k)$ *for any* $k \in \mathbb{N}$*;*
2. *has nonempty closed convex values, i.e.,* $\mathcal{N}(x)$ *is nonempty, closed and convex for any* $x \in X$*.*

Then Γ_N *has at least one SPNE.*

Proof. Assumptions (i),(ii) and (iv) guarantee the existence of a continuous function $\bar{\varphi}\colon X \to Y_1 \times \cdots \times Y_N$ such that $\bar{\varphi}(x) = (\bar{\varphi}_1(x), \ldots, \bar{\varphi}_N(x)) \in \mathcal{N}(x)$ for any $x \in X$ (by Michael selection theorem, see [101]). In light of (i), (iii) and the continuity of $\bar{\varphi}$, there exists $\bar{x} \in \operatorname{Arg\,max}_{x \in X} L(x, \bar{\varphi}_1(x), \ldots, \bar{\varphi}_N(x))$. By Definition 2.2.1, the strategy profile $(\bar{x}, \bar{\varphi}_1(x), \ldots, \bar{\varphi}_N(x))$ is an SPNE of Γ_N. □

It is crucial for our analysis to highlight that in Proposition 2.2.3 a very demanding assumption is involved: the lower semicontinuity of the set-valued map $\mathcal{N}$. Such a requirement makes Proposition 2.2.3 a hardly manageable existence result (for SPNEs), both because it does not involve assumptions explicitly on the payoff functions and because the lower semicontinuity of the set-valued map $\mathcal{N}$ is, in general, not satisfied even if the payoff functions of the followers are bilinear, as proved in many folk examples in literature (see, for example, [11, Remark 4.3] and [94, Example 2.1] for the case of one follower, and [111], where the lower semicontinuity of the ϵ-*Nash equilibrium correspondence* is investigated, for the case of two followers). Let us illustrate below two examples (the first one involving one follower, and the second one involving two followers) that emphasize the arguments just mentioned, and that involve games having infinitely many SPNEs.

Example 2.2.5 Let $\Gamma = (X, Y, L, F)$ where the action sets are $X = Y = [-1, 1]$ and the payoff functions are defined on $[-1,1] \times [-1,1]$ by $L(x, y) = x + y$ and $F(x, y) = xy$. The follower best reply correspondence $\mathcal{M}$ is defined on $[-1, 1]$ by

$$\mathcal{M}(x) = \begin{cases} \{-1\}, & \text{if } x \in [-1, 0[\\ [-1, 1], & \text{if } x = 0 \\ \{1\}, & \text{if } x \in]0, 1]. \end{cases} \tag{2.7}$$

The set-valued map $\mathcal{M}$ is not lower semicontinuous on X. In fact, consider the sequence $(x_k)_k = (1/k)_k$ convergent to $x = 0$ and let $y = -1 \in \mathcal{M}(0)$.

Any sequence $(y_k)_k$ such that $y_k \in \mathcal{M}(1/k)$ for any $k \in \mathbb{N}$ is not convergent to -1 (it converges to 1); therefore, $\mathcal{M}$ is not lower semicontinuous in $x = 0$. Nevertheless, Γ has infinitely many SPNEs: denoted by $\bar{\varphi}^\alpha$ the function defined on $[-1, 1]$ by

$$\bar{\varphi}^\alpha(x) = \begin{cases} -1, & \text{if } x \in [-1, 0[\\ \alpha, & \text{if } x = 0 \\ 1, & \text{if } x \in]0, 1], \end{cases}$$

the pair $(1, \bar{\varphi}^\alpha)$ is an SPNE of Γ for any $\alpha \in [-1, 1]$.

Example 2.2.6 (Section 1 in [111]) Let $\Gamma = (X, Y_1, Y_2, L, F_1, F_2)$ where the action sets are $X = Y_1 = Y_2 = [0, 1]$ and the payoff functions are defined on $[0, 1] \times [0, 1] \times [0, 1]$ by $L(x, y_1, y_2) = -x + y_1 + y_2$ and $F_1(x, y_1, y_2) = F_2(x, y_1, y_2) = -xy_1y_2$. The Nash equilibrium correspondence $\mathcal{N}$ is defined on $[0, 1]$ by

$$\mathcal{N}(x) = \begin{cases} [0, 1] \times [0, 1], & \text{if } x = 0 \\ [0, 1] \times \{0\} \cup \{0\} \times [0, 1], & \text{if } x \in]0, 1]. \end{cases} \tag{2.8}$$

The set-valued map $\mathcal{N}$ is not lower semicontinuous on X. In fact, consider the sequence $(x_k)_k = (1/k)_k$ convergent to $x = 0$ and let $(y_1, y_2) = (1, 1) \in \mathcal{M}(0)$. By definition of $\mathcal{N}$, the point $(1, 1)$ is never the limit of a sequence $(y_{1,k}, y_{2,k})_k$ such that $(y_{1,k}, y_{2,k}) \in \mathcal{N}(1/k)$ for any $k \in \mathbb{N}$, therefore $\mathcal{N}$ is not lower semicontinuous in $x = 0$. However, Γ has infinitely many SPNEs: any pair $(0, \bar{\varphi})$ where $\bar{\varphi}\colon X \to Y_1 \times Y_2$ is a (not necessarily continuous) selection of $\mathcal{N}$ is an SPNE of Γ.

Furthermore we highlight that, since in Examples 2.2.5 and 2.2.6 (as well as in [11, Remark 4.3]) the payoff functions are bilinear, Proposition 2.2.3 in not useful to guarantee the existence of SPNEs even in one-leader N-follower two-stage games derived from the *mixed extension* (see, for example, [100, Chapter 5]) of games where the action sets are finite, which are broadly and steadily used in Game Theory literature.

Therefore, in the sequel of this chapter we deal with the issue of how overcoming the lower semicontinuity of the Nash equilibrium correspondence $\mathcal{N}$ in order to obtain existence of SPNEs, discussing moreover some related arising topics. We distinguish two circumstances:

1. when the Nash equilibrium correspondence is assumed to be single-valued (analyzed in Subsection 2.2.1);

2. when the Nash equilibrium correspondence is not necessarily single-valued (analyzed in Subsection 2.2.2).

In the first situation the main issue can be positively answered and, regarding the sufficient conditions ensuring the single-valuedness of the Nash equilibrium correspondence, in Chapter 3 uniqueness results for Nash equilibrium in normal-form games will be presented. In the second situation, aside from the existence of SPNEs, the consequent arising issue of the selection of a SPNE must be faced: in Chapter 5 two constructive methods for selecting an SPNE will be illustrated, by exploiting the tools analyzed in Chapter 4 (such methods guarantee even the existence of SPNEs regardless of the lower semicontinuity of $\mathcal{N}$).

2.2.1 Assuming uniqueness of the best reaction

Let $\Gamma_N = (X, Y_1, \dots, Y_N, L, F_1, \dots, F_N)$ be a one-leader N-follower two-stage game. When the followers' best reaction to any choice of the leader is unique, i.e. when the Nash equilibrium correspondence $\mathcal{N}$ is single-valued, the players engage in the so-called *classical Stackelberg game*. In this case the leader can fully anticipate the reactions of the followers and he takes them into account before choosing his action, and the followers behave answering to the the leader in the best (expected) way. Therefore, assumed that $\mathcal{N}$ is single-valued and $\mathcal{N}(x) = \{(n_1(x), \dots, n_N(x))\}$ for any $x \in X$, the leader is interested in solving

$$SP_N\colon \begin{cases} \max_{x \in X} L(x, n_1(x), \dots, n_N(x)) \\ \text{where } (n_1(x), \dots, n_N(x)) \in Y_1 \times \cdots \times Y_N \\ \text{is the unique Nash equilibrium of } G_x \text{ defined in (2.1).} \end{cases} \tag{2.9}$$

Analogous arguments hold if we deal with a one-leader one-follower two-stage game $\Gamma = (X, Y, L, F)$. Supposed that the follower's best reply correspondence $\mathcal{M}$ is single-valued and $\mathcal{M}(x) = \{m(x)\}$ for any $x \in X$, the leader faces the problem

$$SP\colon \begin{cases} \max_{x \in X} L(x, m(x)) \\ \text{where } m(x) \in Y \\ \text{is the unique solution of } P_x \text{ defined in (2.4).} \end{cases} \tag{2.10}$$

Problem SP is called *Stackelberg problem* or even *two-level optimization problem*. A wide literature is devoted to Stackelberg problems: we mention that existence results are given in [90, 107], approximation schemes and optimality conditions are analyzed in [137, 83, 81, 123], the dynamic version of such problem is studied in [27, 138]. A more detailed list of references can be found in [11] and [12].

Regarding to problem SP_N, which represents a generalization of the traditional Stackelberg model introduced in [145], existence and approximation results can be found in [126] for the case of two followers. In the sequel we refer to both SP_N and SP as Stackelberg problem.

Remark 2.2.4 In the games of Examples 2.1.3 and 2.1.4 the Nash equilibrium correspondence is single-valued, so the leader acts as in Stackelberg problem SP_N. Similarly, the follower's best reply correspondence of the game in Example 2.1.2 is single-valued, hence the leader acts as in Stackelberg problem SP. Instead, the follower's best reply correspondence in Example 2.1.1 is not single-valued at $x = S$ since the follower, if the leader chooses to keep all the money for himself, is indifferent between accepting or rejecting the split proposal (he receives nothing in any case). Moreover, the follower's best reply correspondence in Example 2.2.5 and the Nash equilibrium correspondence in Example 2.2.6 are not single-valued, as displayed in (2.7) and (2.8), respectively.

Before discussing existence results for SPNEs when $\mathcal{N}$ is assumed to be single-valued and addressing the issues pointed out in the previous section, it is worth to give now the definition of the solution concepts associated with the Stackelberg problems and to show two results concerning the existence of solutions for SP (whose proof, involving Propositions 4.1 and 5.1 in [107], is presented for sake of completeness) and SP_N (that generalizes the result of Proposizione 2.1 in [126] regarding the case of $N = 2$ followers), which will be used to derive sufficient conditions also for the existence of SPNEs.

Definition 2.2.5 A leader's action $\bar{x} \in X$ is said to be a *Stackelberg solution* of SP if

$$\bar{x} \in \operatorname*{Arg\,max}_{x \in X} L(x, m(x)),$$

that is, if $\bar{x}$ is a solution of the Stackelberg problem SP. An action profile $(\bar{x}, m(\bar{x})) \in X \times Y$ where $\bar{x}$ is a Stackelberg solution of SP is called *Stackelberg equilibrium* of SP.

Definition 2.2.6 A leader's action $\bar{x} \in X$ is said to be a *Stackelberg solution* of SP_N if

$$\bar{x} \in \operatorname*{Arg\,max}_{x \in X} L(x, n_1(x), \dots, n_N(x)),$$

that is, if $\bar{x}$ is a solution of the Stackelberg problem SP_N. An action profile $(\bar{x}, n_1(\bar{x}), \dots, n_N(\bar{x})) \in X \times Y_1 \times \dots \times Y_N$ where $\bar{x}$ is a Stackelberg solution of SP_N is called *Stackelberg equilibrium* of SP_N.

Proposition 2.2.7 (Propositions 4.1 and 5.1 in [107]). *Let SP be the Stackelberg problem defined in (2.10). Assume that*

(i) X and Y are compact subsets of two Euclidean spaces;

(ii) L and F are upper semicontinuous on $X \times Y$;

(iii) for any $(x, y) \in X \times Y$ and any sequence $(x_k)_k \subseteq X$ converging to x, there exists a sequence $(\tilde{y}_k)_k \subseteq Y$ such that

$$\liminf_{k \to +\infty} F(x_k, \tilde{y}_k) \geq F(x, y).$$

Then, there exists at least one Stackelberg solution of SP.

Proof. Let $(x_k)_k \subseteq X$ be a sequence converging to $x \in X$ and consider the following real-valued functions defined on Y by

$$\begin{aligned} g_k(y) &= -F(x_k, y), \text{ for any } k \in \mathbb{N} \\ g(y) &= -F(x, y). \end{aligned} \tag{2.11}$$

The upper semicontinuity of F and assumption (iii) guarantee that the sequence of functions $(g_k)_k$ *converges variationally* to the function g (see [152, 153] for the definition of variational convergence). Therefore, in light of [153, Theorem 1], we get

$$\operatorname{Limsup}_{k \to +\infty} \operatorname*{Arg\,min}_{y \in Y} g_k(y) \subseteq \operatorname*{Arg\,min}_{y \in Y} g(y), \tag{2.12}$$

where the "Limsup" in the left hand side of (2.12) stands for the Painlevé-Kuratowski upper limit of sets (see [63]). By the definition of g_k and g in (2.11), and the definition of m in (2.10), from (2.12) we derive

$$\operatorname{Limsup}_{k \to +\infty} \{m(x_k)\} \subseteq \{m(x)\},$$

that implies, in light of the compactness of Y,

$$\lim_{k \to +\infty} m(x_k) = m(x). \tag{2.13}$$

Since (2.13) holds for an arbitrary sequence $(x_k)_k \subseteq X$ converging to $x \in X$, the follower's best reply function m is continuous on X. The upper semicontinuity of L, the continuity of m and the compactness of X ensure that $\operatorname{Arg\,max}_{x \in X} L(x, m(x)) \neq \emptyset$. Therefore, the Stackelberg problem SP has at least one Stackelberg solution. □

Proposition 2.2.8 (Proposizione 2.1 in [126]). *Let SP_N be the Stackelberg problem defined in (2.9). Assume that* *(i)* X *is a compact subset of a Euclidean space;*

(ii) Y_i *is a compact subset of a Euclidean space, for any* $i \in \{1, \dots, N\}$*;*

(iii) L *is upper semicontinuous on* $X \times Y_1 \times \cdots \times Y_N$*;*

(iv) F_i *is upper semicontinuous on* $X \times Y_1 \times \cdots \times Y_N$*, for any* $i \in \{1, \dots, N\}$*;*

(v) for any $i \in \{1, \dots, N\}$ *the following holds:*
for any $(x, y_i, y_{-i}) \in X \times Y_i \times Y_{-i}$*, any sequence* $(x_k)_k \subseteq X$ *converging to* x *and any sequence* $(y_{-i,k})_k \subseteq Y_{-i}$ *converging to* y_{-i}*, there exists a sequence* $(\tilde{y}_{i,k})_k \subseteq Y_i$ *such that*

$$\liminf_{k \to +\infty} F_i(x_k, \tilde{y}_{i,k}, y_{-i,k}) \geq F_i(x, y_i, y_{-i}).$$

Then, there exists at least one Stackelberg solution of SP_N*.*

Proof. Given, for the sake of completeness, in the general case of N followers. Let $(x_k)_k \subseteq X$ be a sequence converging to $x \in X$ and consider a subsequence $(n_1(x_{k_j}), \dots, n_N(x_{k_j}))_j \subseteq Y_1 \times \cdots \times Y_N$ of $(n_1(x_k), \dots, n_N(x_k))_k$ converging to $(\bar{y}_1, \dots \bar{y}_N) \in Y_1 \times \cdots \times Y_N$, whose existence in guaranteed by assumption (ii). Let us show that $(\bar{y}_1, \dots \bar{y}_N) = (n_1(x), \dots, n_N(x))$. Fixed $i \in \{1, \dots, N\}$, in light of assumption (iv) and the definition of n_i in (2.9), we have

$$\begin{aligned} F_i(x, \bar{y}_i, \bar{y}_{-i}) \geq & \limsup_{j \to +\infty} F_i(x_{k_j}, n_i(x_{k_j}), n_{-i}(x_{k_j})) \\ = & \limsup_{j \to +\infty} \sup_{y_i \in Y_i} F_i(x_{k_j}, y_i, n_{-i}(x_{k_j})). \end{aligned} \tag{2.14}$$

Let $z_i \in Y_i$. By assumption (v) there exists a sequence $(\tilde{y}_{i,j})_j \subseteq Y_i$ such that

$$\begin{aligned} \limsup_{j \to +\infty} \sup_{y_i \in Y_i} F_i(x_{k_j}, y_i, n_{-i}(x_{k_j})) \geq & \limsup_{j \to +\infty} F_i(x_{k_j}, \tilde{y}_{i,j}, n_{-i}(x_{k_j})) \\ \geq & F_i(x, z_i, \bar{y}_{-i}). \end{aligned} \tag{2.15}$$

Inequality (2.15) implies

$$\limsup_{j \to +\infty} \sup_{y_i \in Y_i} F_i(x_{k_j}, y_i, n_{-i}(x_{k_j})) \geq \sup_{z_i \in Y_i} F_i(x, z_i, \bar{y}_{-i}). \tag{2.16}$$

By (2.14) and (2.16) it follows that

$$F_i(x, \bar{y}_i, \bar{y}_{-i}) \geq \sup_{z_i \in Y_i} F_i(x, z_i, \bar{y}_{-i});$$

then, $\bar{y}_i = n_i(x)$ and since i was fixed at the beginning of the proof, $\bar{y}_i = n_i(x)$ for any $i \in \{1, \dots, N\}$. Hence, the limit of any convergent subsequence of $(n_1(x_k), \dots, n_N(x_k))_k$ is $(n_1(x), \dots, n_N(x))$. Il light of assumption (ii), even the entire sequence $(n_1(x_k), \dots, n_N(x_k))_k$ converges to $(n_1(x), \dots, n_N(x))$, and this holds for any sequence $(x_k)_k \subseteq X$ converging to $x \in X$, that is, n_i is a continuous function for any $i \in \{1, \dots, N\}$. Assumptions (i) and (iii), and the continuity of n_i guarantee that $\operatorname{Arg\,max}_{x \in X} L(x, n_1(x), \dots, n_N(x)) \neq \emptyset$. Therefore, the Stackelberg problem SP_N has at least one Stackelberg solution. □

Coming back to the issue of existence of SPNEs, the set of SPNEs of a one-leader N-follower two-stage game $\Gamma_N = (X, Y_1, \dots, Y_N, L, F_1, \dots, F_N)$ where the followers' best reaction to any choice of the leader is unique can be fully characterized in terms of Stackelberg solutions (and Stackelberg equilibria) of the associated Stackelberg problem SP_N defined in (2.9). Consequently, the same assumptions used to show the existence of Stackelberg solutions ensure also the existence of SPNEs. In fact, just by exploiting the Definitions 2.2.1 and 2.2.6 and Proposition 2.2.8, the following result can be immediately proved.

Corollary 2.2.9. *Let $\Gamma_N = (X, Y_1, \dots, Y_N, L, F_1, \dots, F_N)$ be a one-leader N-follower two-stage game, where $N \in \mathbb{N}$, and assume that the Nash equilibrium correspondence $\mathcal{N}$ is single-valued. Then the following equivalence holds:*

$$(\bar{x}, \bar{\varphi}_1, \dots, \bar{\varphi}_N) \in X \times Y_1^X \times \dots \times Y_N^X \text{ is an SPNE of } \Gamma_N$$

$$\Updownarrow$$

$$\bar{x} \text{ is a Stackelberg solution of } SP_N \text{ and}$$

$$\{(\bar{\varphi}_1(x), \dots, \bar{\varphi}_N(x))\} = \mathcal{N}(x) \text{ for any } x \in X.$$

Moreover, if all the assumptions of Proposition 2.2.8 are satisfied, then Γ_N has at least one SPNE.

Finally, we emphasize that Corollary 2.2.9 involves explicit assumptions on the payoff functions which guarantee the lower semicontinuity of the Nash equilibrium correspondence $\mathcal{N}$ (assumed to be single-valued): in fact, set $\mathcal{N}(x) = \{(n_1(x), \dots, n_N(x))\}$ for any $x \in X$ with $n_i \colon X \to Y_i$ for any $i \in \{1, \dots, N\}$, the continuity of the functions $n_1, \dots n_N$ comes from assumptions (ii), (iv) and (v) of Proposition 2.2.8 (as shown in the first part of the proof of Proposition 2.2.8). It is immediate to note that the continuity of n_i for any $i \in \{1, \dots, N\}$ is equivalent to the lower semicontinuity of the Nash equilibrium (single-valued) correspondence $\mathcal{N}$. Therefore, the issue of the lower semicontinuity of $\mathcal{N}$ is positively addressed when $\mathcal{N}$ is assumed to be single-valued.

2.2.2 When the best reaction is not always unique

It may happen in many situations that the followers, after having observed the action chosen by the leader in the first stage, can be indifferent between two or more alternatives, i.e. that the Nash equilibrium correspondence $\mathcal{N}$ is not single-valued. In this case the leader cannot predict the behaviour of the followers, differently from the situation analyzed in Subsection 2.2.1, and multiple SPNEs could arise. Hence, the issues that we intend to address now regard not only the existence of SPNEs, but even the way to select an SPNE.
Before introducing a new selection method for SPNEs (see Chapter 5), we remind here the existing concepts of selection for SPNEs, some of which are based on the solutions of problems widely investigated in literature. For traditional reasons, in this subsection we focus primarily on the case of $N = 1$ follower, pointing out that analogous discussion, observations and results hold even in the case of $N > 1$ followers.

Let us start by defining two well-known problems (and corresponding solution concepts) associated to a one-leader one-follower two-stage game $\Gamma = (X, Y, L, F)$, whose solutions induce selections of SPNEs of Γ. When the follower's best reply correspondence $\mathcal{M}$ (defined in (2.6)) is not single-valued, two extreme behaviors of the leader could arise regarding his beliefs about how the follower chooses inside his own set of optimal actions in response to each action chosen by the leader. In the first case, the leader is optimistic and believes that the follower chooses the best action for the leader; whereas in the second one, the leader is pessimistic and believes that the follower could choose the worst action for the leader. These behaviors lead to the following broadly studied problems, originally named *generalized Stackelberg* problems (see [66]):

- *strong Stackelberg* problem, also called *optimistic Stackelberg* or *optimistic bilevel optimization* problem (see, for example, [18, 51, 144, 79, 75, 32, 146, 33, 28] and references therein)

$$s\text{-}SP\colon \begin{cases} \max_{x\in X} \sup_{y\in\mathcal{M}(x)} L(x,y) \\ \text{where } \mathcal{M}(x) \text{ is defined in (2.6),} \end{cases} \tag{2.17}$$

- *weak Stackelberg* problem, also called *pessimistic Stackelberg* or *pessimistic bilevel optimization* problem (see, for example, [107, 86, 80, 89, 1, 150, 34, 78], and references therein)

$$w\text{-}SP\colon \begin{cases} \max_{x\in X} \inf_{y\in\mathcal{M}(x)} L(x,y) \\ \text{where } \mathcal{M}(x) \text{ is defined in (2.6).} \end{cases} \tag{2.18}$$

Definition 2.2.10 A leader's action $\bar{x} \in X$ is said to be *strong Stackelberg solution* (or *optimistic solution*) if

$$\bar{x} \in \operatorname*{Arg\,max}_{x \in X} \sup_{y \in \mathcal{M}(x)} L(x, y),$$

that is, if $\bar{x}$ is a solution of s-SP.
An action profile $(\bar{x}, \bar{y}) \in X \times Y$ is said to be *strong Stackelberg equilibrium* (or *optimistic equilibrium*) if

$$\bar{x} \in \operatorname*{Arg\,max}_{x \in X} \sup_{y \in \mathcal{M}(x)} L(x, y) \quad \text{and} \quad \bar{y} \in \operatorname*{Arg\,max}_{y \in \mathcal{M}(\bar{x})} L(\bar{x}, y).$$

Definition 2.2.11 A leader's action $\bar{x} \in X$ is said to be *weak Stackelberg solution* (or *pessimistic solution*) if

$$\bar{x} \in \operatorname*{Arg\,max}_{x \in X} \inf_{y \in \mathcal{M}(x)} L(x, y),$$

that is, if $\bar{x}$ is a solution of w-SP.
An action profile $(\bar{x}, \bar{y}) \in X \times Y$ is said to be *weak Stackelberg equilibrium* (or *pessimistic equilibrium*) if

$$\bar{x} \in \operatorname*{Arg\,max}_{x \in X} \inf_{y \in \mathcal{M}(x)} L(x, y) \quad \text{and} \quad \bar{y} \in \mathcal{M}(\bar{x}).$$

Regarding the existence of strong Stackelberg and weak Stackelberg solutions, the following two results show how the strong Stackelberg problem is more tractable than the weak Stackelberg problem, in the sense that s-SP is guaranteed to have solutions under mild compactness and continuity assumptions, instead for w-SP the lower semicontinuity of the follower's best reply correspondence is assumed. The proofs are presented for the sake of completeness and to make easier to understand the connections with the SPNEs of Γ.

Proposition 2.2.12 (Proposition 3.1 and Theorem 3.1 in [74]). *Let s-SP be the strong Stackelberg problem defined in* (2.17)*. Assume that*

(i) X and Y are compact subsets of two Euclidean spaces;

(ii) L and F are upper semicontinuous on $X \times Y$;

(iii) for any $(x, y) \in X \times Y$ and any sequence $(x_k)_k \subseteq X$ converging to x, there exists a sequence $(\tilde{y}_k)_k \subseteq Y$ such that

$$\liminf_{k \to +\infty} F(x_k, \tilde{y}_k) \geq F(x, y).$$

Then, s-SP has at least one strong Stackelberg solution.

Proof. Firstly, note that $\mathcal{M}(x) \neq \emptyset$ for any $x \in X$, since Y is compact and F is upper semicontinuous. Let $(x_k)_k \subseteq X$ be a sequence converging to $x \in X$. Arguing as in Proposition 2.2.7, the upper semicontinuity of F and assumption (iii) guarantee that inclusion (2.12) holds, where functions g_k and g are defined in (2.11). Therefore, by (2.11) and the definition of $\mathcal{M}$

$$\text{Limsup}_{k\to+\infty}\mathcal{M}(x_k) \subseteq \mathcal{M}(x), \tag{2.19}$$

i.e, the set-valued map $\mathcal{M}$ is *closed at* x. Since (2.19) holds for an arbitrary sequence $(x_k)_k \subseteq X$ converging to $x \in X$, the follower's best reply correspondence $\mathcal{M}$ is *closed*. Let $\omega \colon X \to \mathbb{R} \cup \{+\infty\}$ be the function defined by

$$w(\xi) \coloneqq \sup_{y\in\mathcal{M}(\xi)} L(\xi, y).$$

By definition of w, for any sequence $(\epsilon_k)_k \subseteq]0, +\infty[$ decreasing to 0, there exists a sequence $(y_k)_k \subseteq Y$ such that

$$L(x_k, y_k) \geq w(x_k) - \epsilon_k,$$

which implies, since $\lim_{k\to+\infty} \epsilon_k = 0$ and using the properties of limit superior,

$$\limsup_{k\to+\infty} L(x_k, y_k) \geq \limsup_{k\to+\infty} w(x_k).$$

Still by limit superior properties, there exists a subsequence of natural numbers $(k_j)_j$ such that

$$\lim_{j\to+\infty} L(x_{k_j}, y_{k_j}) = \limsup_{k\to+\infty} L(x_k, y_k) \geq \limsup_{k\to+\infty} w(x_k). \tag{2.20}$$

Since Y is compact and $(y_{k_j})_j \subseteq Y$, there exists a subsequence $(y_{k_{j_l}})_l \subseteq (y_{k_j})_j$ converging to $y_0 \in Y$. Moreover, since $\mathcal{M}$ is closed at x and $y_{k_{j_l}} \in \mathcal{M}(x_{k_{j_l}})$ for any $l \in \mathbb{N}$ where $(x_{k_{j_l}})_l$ is convergent to x, necessarily $y_0 \in \mathcal{M}(x)$. Hence, in light of the upper semicontinuity of L and (2.20), we get

$$\begin{aligned} w(x) \geq L(x, y_0) &\geq \limsup_{l\to+\infty} L(x_{k_{j_l}}, y_{k_{j_l}}) \\ &= \lim_{j\to+\infty} L(x_{k_j}, y_{k_j}) \geq \limsup_{k\to+\infty} w(x_k). \end{aligned}$$

Therefore, w is upper semicontinuous. This, and the compactness of X, guarantee that $\text{Arg}\max_{x\in X} w(x) \neq \emptyset$. So, *s-SP* has at least one strong Stackelberg solution. □

Remark 2.2.13 As shown in the first part of the proof of Proposition 2.2.12, the assumptions on the follower's payoff function F guarantee inclusion (2.19), i.e. that the follower's best reply correspondence $\mathcal{M}$ is closed at $x \in X$. Such a result is proved by exploiting variational convergence properties implied by assumptions (ii)-(iii) of Proposition 2.2.12 (regarding function F). We remind that we could obtain the closedness of $\mathcal{M}$ even by using the notion of *pseudo-continuous* function introduced by Morgan and Scalzo in [112] (see also [113]). More precisely, the pseudocontinuity of F guarantees the closedness of the follower's best reply correspondence $\mathcal{M}$ ([113, Theorem 3.1]) and such a notion of pseudocontinuity, embedding an economic interpretation connected to the continuity of the preference relations that players are endowed with (see [113, Proposition 2.2]), is not related to variational convergence properties (see [112, Examples 3.1 and 3.2]). Analogous arguments also apply in all the (preceding and subsequent) results involving the closedness of the follower's best reply correspondence or of the Nash equilibrium correspondence (in the case of $N \geq 2$ followers).

Proposition 2.2.14 (Proposition 4.1 in [107])**.** *Let w-SP be the weak Stackelberg problem defined in* (2.18)*. Assume that*

(i) X and Y are compact subsets of two Euclidean spaces;

(ii) L and F are upper semicontinuous on $X \times Y$;

(iii) the follower's best reply correspondence $\mathcal{M}$ is lower semicontinuous on X.

Then, w-SP has at least one weak Stackelberg solution.

Proof. Firstly, note that $\mathcal{M}(x) \neq \emptyset$ for any $x \in X$, since Y is compact and F is upper semicontinuous. Let $(x_k)_k \subseteq X$ be a sequence converging to $x \in X$ and let $y \in \mathcal{M}(x)$. By assumption (iii), there exists a sequence $(y_k)_k \subseteq Y$ such that $(y_k)_k$ converges to y and $y_k \in \mathcal{M}(x_k)$ for any $k \in \mathbb{N}$. Consider the function $v\colon X \to \mathbb{R} \cup \{-\infty\}$ defined by

$$v(\xi) \coloneqq \inf_{y \in \mathcal{M}(\xi)} L(\xi, y).$$

Since $y_k \in \mathcal{M}(x_k)$ for any $k \in \mathbb{N}$, by definition of v and the upper semicontinuity of L, we get

$$\limsup_{k \to +\infty} v(x_k) \leq \limsup_{k \to +\infty} L(x_k, y_k) \leq L(x, y). \tag{2.21}$$

Since y is arbitrarily chosen in $\mathcal{M}(x)$, by (2.21) it follows that

$$\limsup_{k \to +\infty} v(x_k) \leq v(x),$$

so, v is upper semicontinuous. This, and the compactness of X, ensure that $\operatorname{Arg\,max}_{x \in X} v(x) \neq \emptyset$. Therefore, *w-SP* has at least one weak Stackelberg solution. □

Remark 2.2.15 We note that Propositions 2.2.12 and 2.2.14 could be shown even by exploiting directly results on the semicontinuity properties of the marginal functions, proved in [79, 76]. Moreover, Propositions 2.2.12 and 2.2.14 evidently provide existence results even for strong Stackelberg equilibria and weak Stackelberg equilibria, respectively.

Remark 2.2.16 Regarding the existence of solutions of w-SP, we stress that assumptions of Proposition 2.2.14 involve the lower semicontinuity of the follower's best reply correspondence $\mathcal{M}$, so the key drawbacks already highlighted after Proposition 2.2.3 occur also in this framework. In fact, as well as the continuity (and even the bilinearity) of the follower's payoff function does not ensure, in general, the lower semicontinuity of the follower's best reply correspondence (see Example 2.2.5), the compactness of the action sets and continuity of payoff functions do not guarantee, in general, the existence of weak Stackelberg solutions, as illustrated in many examples in literature (see, for example [13], [90] and [11, Remark 4.1]) and in the following one.

Example 2.2.7 Let $X = Y = [-1, 1]$, $L(x, y) = x + y$ and $F(x, y) = -xy$. The follower's best reply correspondence $\mathcal{M}$ is is defined on $[-1, 1]$ by

$$\mathcal{M}(x) = \begin{cases} \{1\}, & \text{if } x \in [-1, 0[\\ [-1, 1], & \text{if } x = 0 \\ \{-1\}, & \text{if } x \in]0, 1]. \end{cases} \tag{2.22}$$

Since for any $x \in [-1, 1]$

$$\min_{y \in \mathcal{M}(x)} L(x, y) = \begin{cases} x + 1, & \text{if } x \in [-1, 0[\\ x - 1, & \text{if } x \in [0, 1], \end{cases}$$

then

$$\operatorname*{Arg\,max}_{x \in [-1,1]} \min_{y \in \mathcal{M}(x)} L(x, y) = \emptyset.$$

Hence, the weak Stackelberg solution (as well as the weak Stackelberg equilibrium) does not exist. Let us note that the follower's best reply correspondence $\mathcal{M}$ in (2.22) is not lower semicontinuous on $[-1, 1]$. As regards to the strong Stackelberg problem, the existence of a strong Stackelberg solution is guaranteed by Proposition 2.2.12. Let us compute such a solution. Since for any $x \in [-1, 1]$

$$\max_{y \in \mathcal{M}(x)} L(x, y) = \begin{cases} x + 1, & \text{if } x \in [-1, 0] \\ x - 1, & \text{if } x \in]0, 1], \end{cases},$$

then

$$\operatorname{Arg}\max_{x\in[-1,1]}\max_{y\in\mathcal{M}(x)} L(x,y)=\{0\}.$$

Therefore, the strong Stackelberg solution is $\bar{x}=0$, and the strong Stackelberg equilibrium is the action profile $(0,1)$ as $\{1\}=\operatorname{Arg}\max_{y\in\mathcal{M}(0)} L(0,y)$.

We emphasize that due to the impossibility of overcoming the lower semicontinuity of the follower's best reply correspondence and the consequent difficulty of ensuring the existence of weak Stackelberg solutions, approximate solutions have been investigated and existence and approximation results have been obtained for the *ϵ-regularized weak Stackelberg problem* under mild compactness and continuity assumptions (see, for example, [104, 82, 86, 84, 87, 94]).

As regards to the issue of selection of SPNEs, let us show how the strong Stackelberg solutions and the weak Stackelberg solutions induce SPNEs in a one-leader one-follower two-stage game. Let *s-SP* and *w-SP* be the strong Stackelberg problem and the weak Stackelberg problem associated to the one-leader one-follower two-stage game $\Gamma=(X,Y,L,F)$, defined in (2.17) and (2.18), respectively. Starting from a strong or a weak Stackelberg solution one could derive a selection of SPNE of Γ motivated according to the two different behaviours of the leader (that is, the optimistic and pessimistic beliefs of the leader discussed at the beginning of this subsection). More precisely, by Definitions 2.2.2, 2.2.10 and 2.2.11 it follows that

(i) if the leader's action $\bar{x}\in X$ is a strong Stackelberg solution, then the strategy profile $(\bar{x},\bar{\varphi})\in X\times Y^X$ is an SPNE of Γ when $\bar{\varphi}(x)\in\operatorname{Arg}\max_{y\in\mathcal{M}(x)} L(x,y)$ for any $x\in X$;

(ii) if the leader's action $\bar{x}\in X$ is a weak Stackelberg solution, then the strategy profile $(\bar{x},\bar{\varphi})\in X\times Y^X$ is an SPNE of Γ when $\bar{\varphi}(x)\in\operatorname{Arg}\min_{y\in\mathcal{M}(x)} L(x,y)$ for any $x\in X$.

By definition, strong Stackelberg solutions and weak Stackelberg solutions involve $\sup_{y\in\mathcal{M}(x)} L(x,y)$ and $\inf_{y\in\mathcal{M}(x)} L(x,y)$ respectively. In light of (i) and (ii) stated above, in order to ensure the existence of an SPNE selection induced by strong and weak Stackelberg solutions, the equalities $\sup_{y\in\mathcal{M}(x)} L(x,y)=\max_{y\in\mathcal{M}(x)} L(x,y)$ and $\inf_{y\in\mathcal{M}(x)} L(x,y)=\min_{y\in\mathcal{M}(x)} L(x,y)$ must hold for any $x\in X$, respectively. In the the following two results we show that such a requirement does not entail any additional assumption for the SPNEs generated by strong Stackelberg solutions (with respect to assumptions of Proposition 2.2.12

ensuring the existence of such solutions), whereas additional continuity conditions are assumed for the SPNEs generated by weak Stackelberg solutions (with respect to assumptions of Proposition 2.2.14 guaranteeing the existence of such solutions) and we emphasize that, in spite of this, the requirement of lower semicontinuity of the follower's best reply correspondence cannot be overcome.

Proposition 2.2.17. *Let $\Gamma = (X, Y, L, F)$ be a one-leader one-follower two-stage game and s-SP be the associated strong Stackelberg problem defined in (2.17). If all the assumptions of Proposition 2.2.12 are satisfied, then Γ has at least one SPNE induced by a strong Stackelberg solution, i.e. the pair $(\bar{x}_s, \bar{\varphi}_s) \in X \times Y^X$ where*

$$\bar{x}_s \text{ is a solution of } s\text{-}SP \quad \text{and} \quad \bar{\varphi}_s(x) \in \operatorname*{Arg\,max}_{y \in \mathcal{M}(x)} L(x, y) \text{ for any } x \in X,$$

is well-defined.

Proof. Firstly, note that the existence of a strong Stackelberg solution $\bar{x}_s \in X$ is guaranteed by Proposition 2.2.12. Reviewing the beginning of the proof of Proposition 2.2.12, inclusion (2.19) showed that the follower's best reply correspondence $\mathcal{M}$ is a closed set-valued map. Hence, $\mathcal{M}(x)$ is closed for any $x \in X$ and, since Y is compact, $\mathcal{M}(x)$ is compact for any $x \in X$. Then, in light of the upper semicontinuity of L, the equality

$$\sup_{y \in \mathcal{M}(x)} L(x, y) = \max_{y \in \mathcal{M}(x)} L(x, y)$$

holds for any $x \in X$ and result is proved. □

Proposition 2.2.18. *Let $\Gamma = (X, Y, L, F)$ be a one-leader one-follower two-stage game and w-SP be the associated weak Stackelberg problem defined in (2.18). Assume that all the hypotheses of Proposition 2.2.14 are satisfied and that*

(i) $L(x, \cdot)$ is lower semicontinuous on Y, for any $x \in X$;

(ii) for any $(x, y) \in X \times Y$ and any sequence $(x_k)_k \subseteq X$ converging to x, there exists a sequence $(\tilde{y}_k)_k \subseteq Y$ such that

$$\liminf_{k \to +\infty} F(x_k, \tilde{y}_k) \geq F(x, y).$$

Then, Γ has at least one SPNE induced by a weak Stackelberg solution, i.e. the pair $(\bar{x}_w, \bar{\varphi}_w) \in X \times Y^X$ where

$$\bar{x}_w \text{ is a solution of } w\text{-}SP \quad \text{and} \quad \bar{\varphi}_w(x) \in \operatorname*{Arg\,min}_{y \in \mathcal{M}(x)} L(x, y) \text{ for any } x \in X,$$

is well-defined.

Proof. Firstly, note that the existence of a weak Stackelberg solution $\bar{x}_w \in X$ is ensured by Proposition 2.2.14. Arguing as at the beginning of Proposition 2.2.12, the upper semicontinuity of F and assumption (ii) guarantee that the follower's best reply correspondence $\mathcal{M}$ is a closed set-valued map. Hence, $\mathcal{M}(x)$ is closed for any $x \in X$ and, since Y is compact, $\mathcal{M}(x)$ is compact for any $x \in X$. Then, by assumption (i), the equality

$$\inf_{y \in \mathcal{M}(x)} L(x,y) = \min_{y \in \mathcal{M}(x)} L(x,y)$$

holds for any $x \in X$ and result is proved. □

Let us compute the selections of SPNEs induced by strong and weak Stackelberg solutions in the games of Examples 2.2.5 and 2.2.7

Example 2.2.8 Let Γ be the game defined in Example 2.2.5. The follower's best reply correspondence is given in (2.7). Since

$$\max_{y \in \mathcal{M}(x)} L(x,y) = \begin{cases} x-1, & \text{if } x \in [-1,0[\\ x+1, & \text{if } x \in [0,1], \end{cases} \quad \text{and} \quad \operatorname{Arg\,max}_{x \in [-1,1]} \max_{y \in \mathcal{M}(x)} L(x,y) = \{1\},$$

the s-SP associated to Γ has a unique strong Stackelberg solution $\bar{x}_s = 1$, which induces, according to Proposition 2.2.17, the following SPNE of Γ: the strategy profile $(1, \bar{\varphi}_s)$ where $\bar{\varphi}_s$ is defined on $[-1,1]$ by

$$\bar{\varphi}_s(x) = \begin{cases} -1, & \text{if } x \in [-1,0[\\ 1, & \text{if } x \in [0,1], \end{cases}$$

as $\{\bar{\varphi}_s(x)\} = \operatorname{Arg\,max}_{y \in \mathcal{M}(x)} L(x,y)$ for any $x \in [-1,1]$.
The w-SP associated to Γ, has a unique weak Stackelberg solution $\bar{x}_w = 1$, since

$$\min_{y \in \mathcal{M}(x)} L(x,y) = \begin{cases} x-1, & \text{if } x \in [-1,0] \\ x+1, & \text{if } x \in]0,1], \end{cases} \quad \text{and} \quad \operatorname{Arg\,max}_{x \in [-1,1]} \min_{y \in \mathcal{M}(x)} L(x,y) = \{1\},$$

which induces, though Γ does not satisfy the assumptions of Proposition 2.2.18, the following SPNE of Γ: the strategy profile $(1, \bar{\varphi}_w)$ where $\bar{\varphi}_w$ is defined on $[-1,1]$ by

$$\bar{\varphi}_w(x) = \begin{cases} -1, & \text{if } x \in [-1,0] \\ 1, & \text{if } x \in]0,1], \end{cases}$$

since $\{\bar{\varphi}_w(x)\} = \operatorname{Arg\,min}_{y \in \mathcal{M}(x)} L(x,y)$ for any $x \in [-1,1]$. As pointed out in Example 2.2.5, we remind that Γ has infinitely many SPNEs.

Example 2.2.9 Let Γ be the game defined in Example 2.2.7. The unique strong Stackelberg solution of the s-SP associated to Γ is $\bar{x}_s = 0$ (see Example 2.2.7) and the SPNE induced in Γ, according to Proposition 2.2.17, is the strategy profile $(0, \bar{\varphi}_s)$ where $\bar{\varphi}_s$ is defined on $[-1, 1]$ by

$$\bar{\varphi}_s(x) = \begin{cases} 1, & \text{if } x \in [-1, 0] \\ -1, & \text{if } x \in]0, 1], \end{cases}$$

since

$$\operatorname*{Arg\,max}_{y \in \mathcal{M}(x)} L(x, y) = \begin{cases} \{1\}, & \text{if } x \in [-1, 0] \\ \{-1\}, & \text{if } x \in]0, 1], \end{cases}$$

where $\mathcal{M}$ is defined in (2.22). As computed in Example 2.2.7, the weak Stackelberg solution of w-SP does not exist, hence it is not possible to select an SPNE of Γ by exploiting the solutions of w-SP (note that Γ does not satisfy the assumptions of Proposition 2.2.18). Moreover, the strategy profile $(0, \bar{\varphi}_s)$ is the unique SPNE of Γ.

We remind that the SPNEs selections induced by the strong Stackelberg solution and the weak Stackelberg solution reflect just two (extreme) beliefs of the leader about which actions the follower chooses inside his own set of optimal reactions. Extending this analysis, the intermediate situation where the leader has some information on the follower's choice in his set of optimal actions and, consequently, attributes a probability measure on $\mathcal{M}(x)$ for any $x \in X$ has been considered by Mallozzi and Morgan in [95] (see also [96]). Therefore, arguing in the same spirit of the strong and weak Stackelberg circumstances, the solutions of the *intermediate Stackelberg problem* represent another way to provide a selection of SPNE in one-leader one-follower two-stage games.

Let us mention the case of $N \geq 2$ followers. For a one-leader N-follower two-stage game $\Gamma_N = (X, Y_1, \dots, Y_N, L, F_1, \dots, F_N)$, Stackelberg problems analogous to s-SP and w-SP can be defined:

- *strong hierarchical Nash equilibrium* problem, also called *optimistic bilevel problem with Nash equilibrium constraints* (see, for example, [151, 91, 79, 75, 74, 32, 124, 97, 71, 73] and references therein)

$$s\text{-}SP_N\colon \begin{cases} \max_{x \in X} \sup_{(y_1, \dots, y_N) \in \mathcal{N}(x)} L(x, y_1, \dots, y_N) \\ \text{where } \mathcal{N}(x) \text{ is defined in (2.3)}, \end{cases}$$

- *weak hierarchical Nash equilibrium* problem, also called *pessimistic bilevel problem with Nash equilibrium constraints* (see, for example, [95, 110, 97, 72, 77, 70] and references therein)

$$w\text{-}SP_N : \begin{cases} \max_{x \in X} \inf_{(y_1,\dots,y_N)\in\mathcal{N}(x)} L(x, y_1, \dots, y_N) \\ \text{where } \mathcal{N}(x) \text{ is defined in (2.3).} \end{cases}$$

It is immediate to check that if $N = 1$, then $s\text{-}SP_N$ and $w\text{-}SP_N$ are reduced to $s\text{-}SP$ and $w\text{-}SP$, respectively. As regards to the existence of solutions, analogously to what happens for $s\text{-}SP$ and $w\text{-}SP$, we have that $s\text{-}SP_N$ is more tractable than $w\text{-}SP_N$ (in the sense that $w\text{-}SP_N$ may fail to have a solution since, in general, the lower semicontinuity of the Nash equilibrium correspondence $\mathcal{N}$ does not hold). About the existence and selection of SPNEs of Γ_N, in an analogous way to how from strong and weak Stackelberg solutions of $s\text{-}SP$ and $w\text{-}SP$ can be derived SPNE selections of Γ, one can prove that the solutions of $s\text{-}SP_N$ and $w\text{-}SP_N$ induce selections of SPNEs in Γ_N motivated according to the optimistic or pessimistic beliefs of the leader regarding what the followers choose inside the set of parametric Nash equilibria $\mathcal{N}(x)$ for any leader's action $x \in X$.

Let Γ_N be a one-leader N-follower two-stage game, where $N \in \mathbb{N}$. Regarding the issue of selection of SPNEs, aside from the need of overcoming the lower semicontinuity requirement for the Nash equilibrium correspondence $\mathcal{N}$ (faced successfully in the case of the SPNEs induced by strong Stackelberg solutions), we are interested in SPNEs selection results that achieve also the following two goals:

1. relieving the leader of knowing the Nash equilibrium correspondence;

2. providing algorithmic procedures to approach such SPNEs, especially concerning the strategies chosen by the followers in the SPNEs, that allow to overcome the difficulties due to the non-single-valuedness of the Nash equilibrium correspondence.

We note that the ways to obtain SPNEs described up to now do not satisfy the features mentioned above. Instead, Morgan and Patrone in [109] presented a constructive method in order to select an SPNE in the cases of $N = 1$ and $N = 2$ followers that positively answers to the goals (i) and (ii). Such a method, exploiting the Tikhonov regularization, will be shown in detail at the beginning of Chapter 5. Furthermore, in Chapter 5 we present a learning approach to select an SPNE in one-leader one-follower two-stage games which fits (i) and (ii) and has a behavioral interpretation linked to the costs that players face when they deviate from their current actions, based on Caruso, Ceparano and Morgan [23].

Chapter 3

Uniqueness of Nash equilibrium in normal-form games

In the first part of Section 2.2 we investigated the issue of existence of SPNEs when the followers' best reaction is assumed to be unique and we mentioned that a related arising issue concerns, evidently, the sufficient conditions ensuring the uniqueness of the followers' best reaction (i.e., the single-valuedness of the Nash equilibrium correspondence). Hence, in this chapter we aim to show a result regarding the existence of a unique Nash equilibrium in two-player normal-form games where the action sets are (unconstrained) Hilbert spaces: when the best reply correspondences of the players are single-valued, we present a theorem ensuring the existence of a unique Nash equilibrium that allows the two compositions of the best reply functions to be not a contraction mapping. Moreover, referring to Caruso, Ceparano and Morgan [24], we illustrate the implications of such a result when we restrict to the widely investigated class of weighted potential games, proving the (lack of) connections between Nash equilibria and maximizers of the potential function, and providing results and examples in both finite and infinite-dimensional setting. Finally we recall the uniqueness result shown by Rosen in his seminal paper [132], where the action sets of the players are constrained subsets of finite-dimensional spaces.

Let us consider a normal-form game $\Omega = \{I, (A_i)_{i\in I}, (U_i)_{i\in I}\}$ where $I = \{1, \dots, N\}$ is the set of players, A_i is the set of actions of player $i \in I$ and $U_i \colon A_1 \times \cdots \times A_N \to \mathbb{R}$ is the payoff function of player $i \in I$.

We denote by $A \coloneqq A_1 \times \cdots \times A_N$ the set of action profiles and, for any $i \in I$ we denote by $-i$ the subset of players that does not contain i, i.e. $\{-i\} = I \setminus \{i\}$. So, an action profile $a = (a_1, \dots, a_N) \in A$ can be equivalently rewritten as $(a_i, a_{-i}) \in A_i \times A_{-i}$. We denote by B_i the *best reply correspondence of player* i, that is B_i is the set-valued map defined on A_{-i} by

$$B_i(a_{-i}) \coloneqq \operatorname*{Arg\,max}_{a_i \in A_i} U_i(a_i, a_{-i}) \subseteq A_i, \tag{3.1}$$

i.e., $B_i(a_{-i}) = \{a_i \in A_i \colon U_i(a_i, a_{-i}) \geq U_i(a'_i, a_{-i}), \text{ for any } a'_i \in A_i\}$. When B_i is (nonempty) single-valued, the function b_i such that $\{b_i(a_{-i})\} \coloneqq B_i(a_{-i})$ is well-defined and called *best reply function of player* i.

Let us recall the definition of Nash equilibrium.

Definition 3.0.1 (Nash [119, 120]) An action profile $a^* = (a_1^*, \dots, a_N^*) \in A$ is said to be a *Nash equilibrium* of Ω if, for any $i \in I$

$$U_i(a_i^*, a_{-i}^*) \geq U_i(a_i, a_{-i}^*) \quad \text{for any } a_i \in A_i.$$

In the following remark, an useful characterization of Nash equilibria through the best reply correspondences is provided.

Remark 3.0.2 An action profile $a^* \in A$ is a Nash equilibrium of Ω if and only if a^* is a fixed point of the set-valued map $B \colon A \rightrightarrows A$ defined by

$$B(a_1, \dots, a_N) \coloneqq B_1(a_{-1}) \times \cdots \times B_N(a_{-N}), \tag{3.2}$$

that is, if and only if $a_i^* \in B_i(a_{-i}^*)$ for any $i \in I$.

Before illustrating the results on the uniqueness of Nash equilibrium we outlined at the beginning of the chapter, let us remind preliminarily two well-known results about the existence of a unique solution of maximization problems. The first one concerns with the maximization of functions defined on constrained subsets of finite-dimensional spaces; the second one regards functions defined on (unconstrained) Hilbert spaces. Let us consider the following maximization problem:

$$P \colon \max_{a \in A} U(a),$$

where A is a set and U is a real-valued function defined on A. A point $a^* \in A$ is said to be a solution of P if $U(a^*) = \max_{a \in A} U(a)$.

Proposition 3.0.3. *Assume that*

(i) A is a compact and convex subset of a Euclidean space;

(ii) U is lower semicontinuous on A;

(iii) U is strictly concave on A, i.e., for any $a', a'' \in A$ with $a' \neq a''$ and any $t \in]0, 1[$

$$U(ta' + (1-t)a'') > tU(a') + (1-t)U(a'').$$

Then, the problem P has a unique solution.

The result proved in Proposition 3.0.3 holds also for extended real-valued functions defined over subsets of topological spaces (see, for example, [7, Section 2.1] and [29, Chapter 1]).

Remark 3.0.4 If A is not compact, the continuity and the strict concavity of U do not even guarantee the existence of a solution. For example, let $A = \mathbb{R}$ and $U(a) = -e^a$ for any $a \in \mathbb{R}$. Assumptions of Proposition 3.0.3 are satisfied, except the compactness of A, and U has not a maximizer over $\mathbb{R}$.

Therefore, in order to ensure the existence of a unique solution of P when A is an unconstrained set, the strict concavity of U is not sufficient: a stronger concavity assumption is required.

Proposition 3.0.5. *Assume that*

(i) A is a Hilbert space with norm $\|\cdot\|_A$;

(ii) U is lower semicontinuous on A;

(iii) U is strongly concave on A, i.e., there exists $m > 0$ such that for any $a', a'' \in A$ and any $t \in [0, 1]$

$$U(ta' + (1-t)a'') \geq tU(a') + (1-t)U(a'') + mt(1-t)\|a' - a''\|.$$

Then, the problem P has a unique solution.

We mention that the result in Proposition 3.0.5 can be extended to the case of extended real-valued supercoercive functions (see, for example, [14, Corollaries 11.15 and 11.16]).

3.1 Uniqueness result in games with Hilbert spaces action sets

In this section we consider a game $\Omega = \{I, (A_i)_{i\in I}, (U_i)_{i\in I}\}$ where $I = \{1, 2\}$ and the actions set of player i, namely A_i, is assumed to be a real Hilbert space, equipped with inner product $(\cdot,\cdot)_{A_i}$ and associated norm $\|\cdot\|_{A_i}$, for any $i \in I$.

The set of actions profiles $A = A_1 \times A_2$ is understood as a Hilbert space with inner product $(\cdot,\cdot)_A$ defined by

$$(a, a')_A \coloneqq \sum_{i=1}^{N} (a_i, a'_i)_{A_i}$$

for any $a, a' \in A$ and associated norm $\|\cdot\|_A$. Firstly, we remind some usual notations in Functional Analysis, following for example [92, 37, 10], and we state some properties of the best reply correspondences of the two players in Ω.

Let S and T be normed vector spaces equipped with the norms $\|\cdot\|_S$ and $\|\cdot\|_T$ respectively, and let $\mathcal{L}(S,T)$ be the normed vector space of all continuous linear operators from S to T, with the usual norm $\|\Lambda\|_{\mathcal{L}(S,T)} \coloneqq \sup\{\|\Lambda(s)\|_T \colon \|s\|_S = 1\}$. In particular, when $T = \mathbb{R}$, the space of all continuous linear operators from S to $\mathbb{R}$ is denoted by S^*, that is $S^* = \mathcal{L}(S,\mathbb{R})$, and the duality operation between S^* and S is denoted by $\langle\cdot,\cdot\rangle_{S^*\times S}$. When $f\colon S \to T$ is a twice differentiable function on S, we denote by $Df\colon S \to \mathcal{L}(S,T)$ and $D^2 f\colon S \to \mathcal{L}(S,\mathcal{L}(S,T))$, respectively, the *Fréchet derivative of* f and the *second Fréchet derivative of* f, and by $Df(s) \in \mathcal{L}(S,T)$ and $D^2 f(s) \in \mathcal{L}(S,\mathcal{L}(S,T))$ we mean, respectively, the *derivative of* f *at* $s \in S$ and the *second derivative of* f *at* $s \in S$. If $S = S_1 \times \cdots \times S_n$ and $i, j \in \{1,\ldots,n\}$, we denote by $D_{s_i} f\colon S \to \mathcal{L}(S_i,T)$ the *partial derivative of* f *with respect to* s_i, and by $D_{s_j}(D_{s_i} f)\colon S \to \mathcal{L}(S_j,\mathcal{L}(S_i,T))$ and $D^2_{s_i} f\colon S \to \mathcal{L}(S_i,\mathcal{L}(S_i,T))$ we denote, respectively, the *second partial derivative of* f *with respect to* s_i *and* s_j and the *second partial derivative of* f *with respect to* s_i (obviously, $D_{s_i}(D_{s_i} f) = D^2_{s_i} f$ for any $i \in \{1,\ldots,n\}$). In light of what above, if g is a twice differentiable function from $A = A_1 \times A_2$ to $\mathbb{R}$ then $D_{a_j} g(a) \in A_j^*$ and $D_{a_k}(D_{a_j} g)(a) \in \mathcal{L}(A_k, A_j^*)$, for any $j, k \in I$ and $a \in A$. Finally, let $\mathcal{GL}(S,T) \subseteq \mathcal{L}(S,T)$ be the set of all bijective continuous linear operators from S to T with continuous (and linear) inverse. If $f \in \mathcal{GL}(S,T)$, we denote by $f^{-1}\colon T \to S$ the inverse operator of f, where $f^{-1} \in \mathcal{L}(T,S)$.

Let $i \in I$. In the following we use:

$(\mathcal{C}_i)$ $\begin{cases} U_i \text{ is strongly concave in the } i\text{-th argument, i.e. the function } U_i(\cdot, a_{-i}) \\ \text{is strongly concave on } A_i, \text{ for any } a_{-i} \in A_{-i}. \end{cases}$

Condition $(\mathcal{C}_i)$ guarantees that the best reply correspondence B_i defined in (3.1) is single-valued (in light of Proposition 3.0.5), so it is well-defined the best reply function $b_i\colon A_{-i} \to A_i$ such that $\{b_i(a_{-i})\} \coloneqq B_i(a_{-i})$ for any

$a_{-i} \in A_{-i}$. Nevertheless, B_i may be single-valued even if U_i is not strongly concave in the i-th argument (this is the case, for example, of U_i defined on $\mathbb{R}^2$ by $U_i(a_1, a_2) = -a_1^4 - a_2^4$). Clearly a function could be strongly concave in any argument and, at the same time, not concave on A (take, for example, U_i defined on $\mathbb{R}^2$ by $U_i(a_1, a_2) = -a_1^2 - a_2^2 - 5a_1^2 a_2^2$).
Moreover, we use:

$$(\mathcal{D}_i) \begin{cases} U_i \text{ is a twice continuously differentiable function on } A \text{ and } D^2_{a_i} U_i(a) \in \\ \mathcal{GL}(A_i, A_i^*) \text{ for any } a \in A, \text{ i.e. there exists the inverse operator} \\ [D^2_{a_i} U_i(a)]^{-1} \in \mathcal{L}(A_i^*, A_i) \text{ for any } a \in A. \end{cases}$$

Proposition 3.1.1. *Let $i \in I$ and assume $(\mathcal{C}_i)$ and $(\mathcal{D}_i)$. Then the best reply function b_i is continuously differentiable on A_{-i}. Moreover, if $\lambda_i \in [0, +\infty[$, where*

$$\lambda_i \coloneqq \sup_{a \in A} \|[D^2_{a_i} U_i(a)]^{-1} \circ D_{a_{-i}}(D_{a_i} U_i)(a)\|_{\mathcal{L}(A_{-i}, A_i)}$$

then b_i is Lipschitz continuous with Lipschitz constant no greater than λ_i.

Proof. The arguments are similar to those used, for example, in [108], where algorithms were constructed in order to approach a saddle point of zero-sum games.
Firstly, note that the function b_i is well-defined by $(\mathcal{C}_i)$. Let $a_{-i} \in A_{-i}$. In light of the differentiability of U_i on A and by definition of best reply function, the pair $(b_i(a_{-i}), a_{-i})$ satisfies the equation:

$$D_{a_i} U_i(b_i(a_{-i}), a_{-i}) = 0.$$

Hence, by the Implicit Function Theorem, b_i is continuously differentiable on A_{-i}. Furthermore, $Db_i(a_{-i}) \in \mathcal{L}(A_{-i}, A_i)$ and

$$Db_i(a_{-i}) = -[D^2_{a_i} U_i(b_i(a_{-i}), a_{-i})]^{-1} \circ [D_{a_{-i}}(D_{a_i} U_i)(b_i(a_{-i}), a_{-i})]. \tag{3.3}$$

Thus, $\sup_{a_{-i} \in A_{-i}} \|Db_i(a_{-i})\|_{\mathcal{L}(A_{-i}, A_i)} \leq \lambda_i$. By the Mean Value Inequality

$$\begin{aligned} \|b_i(a'_{-i}) - b_i(a''_{-i})\|_{A_i} &\leq \sup_{t \in [0,1]} \|Db_i(ta'_{-i} + (1-t)a''_{-i})\|_{\mathcal{L}(A_{-i}, A_i)} \|a'_{-i} - a''_{-i}\|_{A_{-i}} \\ &\leq \lambda_i \|a'_{-i} - a''_{-i}\|_{A_{-i}} \end{aligned}$$

for any a'_{-i}, $a''_{-i} \in A_{-i}$. Therefore, if $\lambda_i \in [0, +\infty[$, then b_i is Lipschitz continuous with Lipschitz constant no greater that λ_i. □

Let $i \in I$ and define the function $\beta_i : A_i \to A_i$ by

$$\beta_i(a_i) \coloneqq (b_i \circ b_{-i})(a_i) = b_i(b_{-i}(a_i)), \quad \text{for any } a_i \in A_i. \tag{3.4}$$

Remark 3.1.2 An action $\bar{a}_i \in A_i$ is a fixed point of β_i on A_i if and only if $(\bar{a}_i, b_{-i}(\bar{a}_i))$ is a Nash equilibrium of Ω.

We conclude the analysis of the best replies properties with the following result on the derivative of β_i.

Proposition 3.1.3. *Assume $(\mathcal{C}_i)$ and $(\mathcal{D}_i)$ for any $i \in I$. Then, for any $i \in I$, the function $\beta_i = b_i \circ b_{-i}$ is continuously differentiable on A_i and, for any $a_i \in A_i$, $D\beta_i(a_i) \in \mathcal{L}(A_i, A_i)$ is defined by*

$$\begin{aligned} D\beta_i(a_i) =& [D^2_{a_i} U_i(\beta_i(a_i), b_{-i}(a_i))]^{-1} \circ [D_{a_{-i}}(D_{a_i} U_i)(\beta_i(a_i), b_{-i}(a_i))] \\ & \circ [D^2_{a_{-i}} U_{-i}(a_i, b_{-i}(a_i))]^{-1} \circ [D_{a_i}(D_{a_{-i}} U_{-i})(a_i, b_{-i}(a_i))]. \end{aligned} \tag{3.5}$$

Moreover, if $\lambda_1, \lambda_2 \in [0, +\infty[$, then β_i is Lipschitz continuous with Lipschitz constant no greater than λ, where

$$\lambda := \lambda_1 \cdot \lambda_2. \tag{3.6}$$

Proof. By the chain rule, β_i is continuously differentiable on A_i, $D\beta_i(a_i) = Db_i(b_{-i}(a_i)) \circ Db_{-i}(a_i)$ for any $a_i \in A_i$, and equality in (3.5) follows by (3.3). Furthermore,

$$\sup_{a_i \in A_i} \|D\beta_i(a_i)\|_{\mathcal{L}(A_i, A_i)} \leq \lambda_1 \cdot \lambda_2 = \lambda,$$

which implies, as a consequence of the Mean Value Inequality, that

$$\|\beta_i(a'_i) - \beta_i(a''_i)\|_{A_i} \leq \lambda \|a'_i - a''_i\|_{A_i}, \text{ for any } a'_i, a''_i \in A_i. \tag{3.7}$$

Hence, if $\lambda_1, \lambda_2 \in [0, +\infty[$, then β_i is Lipschitz continuous with Lipschitz constant no greater that λ. □

Remark 3.1.4 Propositions 3.1.1 and 3.1.3 hold even replacing assumption $(\mathcal{C}_i)$ with the single-valuedness of the best reply correspondence B_i (which is weaker than $(\mathcal{C}_i)$ as highlighted after the definition of assumption $(\mathcal{C}_i)$, but it does not entail explicit conditions on the payoff funtions).

Let λ be defined as in (3.6) and β_i be defined as in (3.4), for any $i \in I$. We discuss the existence and uniqueness of Nash equilibrium by distinguishing two cases: firstly we analyze the situation where $\lambda < 1$ illustrating a known uniqueness result, then we prove a uniqueness theorem in the case $\lambda \geq 1$ by means of an additional assumption.

When $\lambda < 1$, β_i is a contraction for any $i \in I$. Hence, by using the Contraction Mapping Theorem, a well-known Nash equilibrium uniqueness result is obtained (proved also by Li and Başar in [69, Theorem 1]).

Theorem 3.1.5. *Assume* $(\mathcal{C}_i)$ *and* $(\mathcal{D}_i)$ *for any* $i \in I$, *and* $\lambda \in [0, 1[$. *Then* Ω *has a unique Nash equilibrium.*

Proof. Since $\lambda < 1$ and in light of (3.7), the function β_i is a contraction for any $i \in I$. Fixed $i \in I$, by the Contraction Mapping Theorem (see, for example, [9, Theorem 7 p. 244]) there exists a unique fixed point of β_i. Hence, the existence of a unique Nash equilibrium of Ω follows by Remark 3.1.2. □

Remark 3.1.6 When $A_1 = A_2 = \mathbb{R}$, the *strict diagonal dominance* condition (used in [47] in the case where the actions sets are $A_1 = A_2 = [0, +\infty[$) is equivalent to require $|D_{a_i}(D_{a_{-i}}U_i)(a)/D^2_{a_i}U_i(a)| < 1$ for any $a \in \mathbb{R}^2$ and $i \in I$, which implies $\lambda \leq 1$.

When $\lambda \geq 1$, β_i could be not a contraction. Nevertheless, the existence of one and only one Nash equilibrium of the game Ω will be guaranteed adding the following hypothesis:

$(\mathcal{G})$ There exist $i_0 \in I$ and $\gamma_{i_0} \in]1, +\infty[$ such that, for any $\psi \in A_{i_0}$, a'_{i_0}, $a''_{i_0} \in A_{i_0}$ and $a_{-i_0} \in A_{-i_0}$, we have:

$$(G_{i_0}(a'_{i_0}, a''_{i_0}, a_{-i_0})\psi, \psi)_{A_{i_0}} \geq \gamma_{i_0} \|\psi\|^2_{A_{i_0}};$$

where $G_i(a'_i, a''_i, a_{-i}) : A_i \to A_i$ is the operator defined as:

$$\begin{aligned} G_i(a'_i, a''_i, a_{-i}) :=& [D^2_{a_i}U_i(a'_i, a_{-i})]^{-1} \circ D_{a_{-i}}(D_{a_i}U_i)(a'_i, a_{-i}) \\ & \circ [D^2_{a_{-i}}U_{-i}(a''_i, a_{-i})]^{-1} \circ D_{a_i}(D_{a_{-i}}U_{-i})(a''_i, a_{-i}), \end{aligned}$$

for any a'_i, $a''_i \in A_i$, $a_{-i} \in A_{-i}$ and $i \in I$.

Remark 3.1.7 When $A_1 = A_2 = \mathbb{R}$, then for any $i \in I$ the derivatives $D^2_{a_i}U_i$ and $D_{a_{-i}}(D_{a_i}U_i)$ can be identified with the usual derivatives of real-valued functions defined on $\mathbb{R}^2$, $\left[D^2_{a_i}U_i(a)\right]^{-1}$ exists provided that $D^2_{a_i}U_i(a) \neq 0$, and $\left[D^2_{a_i}U_i(a)\right]^{-1} = 1/D^2_{a_i}U_i(a)$. Then $(\mathcal{G})$ holds if there exist $i \in I$ and $\gamma_i > 1$ such that

$$G_i(a'_i, a''_i, a_{-i}) = \frac{D_{a_{-i}}(D_{a_i}U_i)(a'_i, a_{-i})\, D_{a_i}(D_{a_{-i}}U_{-i})(a''_i, a_{-i})}{D^2_{a_i}U_i(a'_i, a_{-i})\, D^2_{a_{-i}}U_{-i}(a''_i, a_{-i})} \geq \gamma_i,$$

for any a'_i, $a''_i \in A_i$ and $a_{-i} \in A_{-i}$.

Let us note that γ_{i_0} in hypothesis $(\mathcal{G})$, and λ are related. Indeed:

Lemma 3.1.8. *If* $(\mathcal{G})$ *holds, then* $\lambda \geq \gamma_{i_0}$.

Proof. Let $\psi \in A_{i_0} \setminus \{0\}$, a'_{i_0}, $a''_{i_0} \in A_{i_0}$ and $a_{-i_0} \in A_{-i_0}$. Hypothesis $(\mathcal{G})$ ensures that

$$(G_{i_0}(a'_{i_0}, a''_{i_0}, a_{-i_0})\psi, \psi)_{A_{i_0}} \geq \gamma_{i_0} \|\psi\|^2_{A_{i_0}}. \tag{3.8}$$

In light of the Cauchy-Schwarz inequality and the definition of operator norm

$$(G_{i_0}(a'_{i_0}, a''_{i_0}, a_{-i_0})\psi, \psi)_{A_{i_0}} \leq \|G_{i_0}(a'_{i_0}, a''_{i_0}, a_{-i_0})\|_{\mathcal{L}(A_{i_0}, A_{i_0})} \|\psi\|^2_{A_{i_0}}. \tag{3.9}$$

Hence, by (3.8)-(3.9) and the definition of λ

$$\gamma_{i_0} \|\psi\|^2_{A_{i_0}} \leq \|G_{i_0}(a'_{i_0}, a''_{i_0}, a_{-i_0})\|_{\mathcal{L}(A_{i_0}, A_{i_0})} \|\psi\|^2_{A_{i_0}} \leq \lambda \|\psi\|^2_{A_{i_0}}.$$

Therefore, $\gamma_{i_0} \leq \lambda$. □

Remark 3.1.9 Let us emphasize that if $(\mathcal{G})$ holds, then $\lambda > 1$.

Now, we introduce the function $g_i^\delta : A_i \to A_i$ defined by

$$g_i^\delta(a_i) := \delta a_i - (\delta - 1)\beta_i(a_i), \tag{3.10}$$

where $\delta \in \mathbb{R}$ and $i \in I$. When $\delta > 1$ we call such a function δ*-inverse convex combinator* since in this case a_i is a convex combination of $g_i^\delta(a_i)$ and $\beta_i(a_i)$, for any $a_i \in A_i$: this justifies the use of term "inverse".

Lemma 3.1.10. *Let $\delta \neq 1$. A point $\bar{a}_i$ is a fixed point of g_i^δ on A_i if and only if $\bar{a}_i$ is a fixed point of β_i on A_i.*

Proof. By definition, $\bar{a}_i \in A_i$ is a fixed point of g_i^δ on A_i if and only if $g_i^\delta(\bar{a}_i) = \bar{a}_i$, i.e., $\delta\bar{a}_i - (\delta - 1)\beta_i(\bar{a}_i) = \bar{a}_i$ which is equivalent to $\beta_i(\bar{a}_i) = \bar{a}_i$ being $\delta \neq 1$. □

Now we prove our main result, which ensures the uniqueness of Nash equilibrium for the class of games which satisfy hypoteses $(\mathcal{C}_i)$, $(\mathcal{D}_i)$ and $(\mathcal{G})$.

Theorem 3.1.11. *Assume $(\mathcal{C}_i)$ and $(\mathcal{D}_i)$ for any $i \in I$, $(\mathcal{G})$ and $\lambda \in]1, +\infty[$. Then Ω has one and only one Nash equilibrium.*

Proof. Let $\lambda \in]1, +\infty[$ and let $i_0 \in I$ and $\gamma_{i_0} > 1$ be such that $(\mathcal{G})$ holds. Let $g_{i_0}^\delta$ be the δ-inverse convex combinator where

$$\delta = \frac{\lambda^2 - \gamma_{i_0}}{\lambda^2 - 2\gamma_{i_0} + 1}. \tag{3.11}$$

Note that $\lambda^2 - 2\gamma_{i_0} + 1 > 0$ by Lemma 3.1.8 since $\lambda > 1$ and that $\delta > 1$ since $\gamma_{i_0} > 1$. By Remark 3.1.2 and Lemma 3.1.10, Ω has a unique Nash equilibrium if and only if $g_{i_0}^\delta$ has a unique fixed point on A_{i_0}. Let a'_{i_0}, $a''_{i_0} \in A_{i_0}$. Then

$$\begin{aligned}\|g_{i_0}^\delta(a'_{i_0}) - g_{i_0}^\delta(a''_{i_0})\|^2_{A_{i_0}} &= \|\delta[a'_{i_0} - a''_{i_0}] - (\delta - 1)[\beta_{i_0}(a'_{i_0}) - \beta_{i_0}(a''_{i_0})]\|^2_{A_{i_0}} \\ &= \delta^2 \|a'_{i_0} - a''_{i_0}\|^2_{A_{i_0}} + (\delta - 1)^2 \|\beta_{i_0}(a'_{i_0}) - \beta_{i_0}(a''_{i_0})\|^2_{A_{i_0}} \\ &\quad - 2\delta(\delta - 1)(\beta_{i_0}(a'_{i_0}) - \beta_{i_0}(a''_{i_0}), a'_{i_0} - a''_{i_0})_{A_{i_0}}.\end{aligned} \tag{3.12}$$

By applying the Mean Value Theorem for real-valued functions to the function φ defined by

$$\varphi(\theta) := (\beta_{i_0}(\theta a'_{i_0} + (1-\theta)a''_{i_0}), a'_{i_0} - a''_{i_0})_{A_{i_0}}, \quad \text{for any } \theta \in [0,1],$$

there exists $t \in]0,1[$ such that

$$\begin{aligned}(\beta_{i_0}(a'_{i_0}) - &\beta_{i_0}(a''_{i_0}), a'_{i_0} - a''_{i_0})_{A_{i_0}} \\ &= (D\beta_{i_0}(ta'_{i_0} + (1-t)a''_{i_0})(a'_{i_0} - a''_{i_0}), a'_{i_0} - a''_{i_0})_{A_{i_0}}.\end{aligned} \tag{3.13}$$

Note that $D\beta_{i_0}(a_{i_0}) = G(\beta_{i_0}(a_{i_0}), a_{i_0}, b_{-i_0}(a_{i_0}))$ by (3.5). Hence, hypothesis $(\mathcal{G})$ and condition (3.13) imply that

$$(\beta_{i_0}(a'_{i_0}) - \beta_{i_0}(a''_{i_0}), a'_{i_0} - a''_{i_0})_{A_{i_0}} \geq \gamma_{i_0} \|a'_{i_0} - a''_{i_0}\|^2_{A_{i_0}}, \tag{3.14}$$

that is β_{i_0} is strongly monotone with constant γ_{i_0}. Thus, in light of (3.12), (3.14) and (3.7) we have

$$\|g^\delta_{i_0}(a'_{i_0}) - g^\delta_{i_0}(a''_{i_0})\|^2_{A_{i_0}} \leq \left[\delta^2 + (\delta-1)^2\lambda^2 - 2\delta(\delta-1)\gamma_{i_0}\right] \|a'_{i_0} - a''_{i_0}\|^2_{A_{i_0}}.$$

Observe that $(0 \leq) \left[\delta^2 + (\delta-1)^2\lambda^2 - 2\delta(\delta-1)\gamma_{i_0}\right] < 1$ or, equivalently, that

$$\delta + 1 + (\delta - 1)\lambda^2 - 2\delta\gamma_{i_0} < 0. \tag{3.15}$$

Indeed, factoring out δ in inequality (3.15), we get

$$\delta(\lambda^2 - 2\gamma_{i_0} + 1) < \lambda^2 - 1,$$

that is satisfied since $\gamma_{i_0} > 1$.
Thus, $g^\delta_{i_0}$ defined in (3.10) is a contraction when δ is given by (3.11) and therefore Ω has one and only one Nash equilibrium. □

Remark 3.1.12 Existence of one and only one Nash equilibrium could be still obtained if we substitute hypotheses $(\mathcal{G})$ and $(\mathcal{C}_i)$ with the less restrictive assumptions:

$(\mathcal{G}')$ There exist $i_0 \in I$ and $\gamma_{i_0} \in]1, +\infty[$ such that β_{i_0} is a strongly monotone operator with coefficient γ_{i_0}, that is for any $a'_{i_0}, a''_{i_0} \in A_{i_0}$, we have:

$$(\beta_{i_0}(a'_{i_0}) - \beta_{i_0}(a''_{i_0}), a'_{i_0} - a''_{i_0})_{A_{i_0}} \geq \gamma_{i_0} \|a'_{i_0} - a''_{i_0}\|^2_{A_{i_0}}.$$

and

$(\mathcal{C}_i')$ $\left\{ \text{The best reply correspondence } B_i \text{ is single-valued.} \right.$

Indeed, as shown in the proof of Theorem 3.1.11, hypothesis $(\mathcal{G})$ implies inequality (3.14) and, thus, assumption $(\mathcal{G}')$. Furthermore, as emphasized after the definition of assumption $(\mathcal{C}_i)$, the strong concavity of U_i in the i-th argument ensures the single-valuedness of the best reply correspondence B_i.
We stated Theorem 3.1.11 requiring hypothesis $(\mathcal{G})$ and $(\mathcal{C}_i)$ because we are interested in finding explicit conditions on the payoff functions.

Remark 3.1.13 In Theorem 3.1.11 hypothesis $(\mathcal{G})$ cannot be dropped, as shown in the following example.

Example 3.1.1 Let $A_1 = A_2 = \mathbb{R}$, and $U_1(a_1, a_2) = -e^{a_1^2} + 3a_1a_2$ and $U_2(a_1, a_2) = -a_2^2/2 + 3a_1a_2$. Since $D^2_{a_1}U_1(a) = -2e^{a_1^2}(1 + 2a_1^2)$, $D^2_{a_2}U_2(a) = -1$ and $D_{a_2}(D_{a_1}U_1)(a) = D_{a_1}(D_{a_2}U_2)(a) = 3$, for any $a \in \mathbb{R}^2$, then

$$\lambda = 3 \cdot \sup_{a_1 \in \mathbb{R}} \frac{3}{|-2e^{a_1^2}(1 + 2a_1^2)|} = \frac{9}{2}$$

and $G_1(1, a_1'', a_2) = G_2(a_2', a_2'', 1) = 9/(6e) < 1$ for any a_1'', a_2, a_2', $a_2'' \in \mathbb{R}$. Hence, for any $i \in I$ there does not exist $\gamma_i > 1$ such that $(\mathcal{G})$ holds. Such a game has the following three Nash equilibria: $(-k, -3k)$, $(0, 0)$, $(k, 3k)$, with $k = \sqrt{\ln 9 - \ln 2}$.

Remark 3.1.14 The following example illustrates a game which satisfies the assumptions of Theorem 3.1.11 and where best reply functions and the Nash equilibria could not be computed explicitly. However, by applying Theorem 3.1.11, one can conclude that the game has one and only one Nash equilibrium.

Example 3.1.2 Let Ω be the game where $A_1 = A_2 = \mathbb{R}$ and the payoff functions are defined by

$$U_1(a_1, a_2) = -a_1^2 - \cos a_1 \sin a_2 - 5a_1a_2,$$
$$U_2(a_1, a_2) = \frac{1}{1 + a_2^2} - 4a_2^2 + a_2 - 12a_1a_2.$$

The function U_1 is strongly concave in a_1 since $D^2_{a_1}U_1(a) = -2 + \cos a_1 \sin a_2 \leq -1$ for any $(a_1, a_2) \in \mathbb{R}^2$, and the function U_2 is strongly concave in a_2 since $D^2_{a_2}U_2(a) = [(6a_2^2 - 2)/(1 + a_2^2)^3] - 8 \leq -15/2$ for any $(a_1, a_2) \in \mathbb{R}^2$. So $(\mathcal{C}_i)$ and $(\mathcal{D}_i)$ hold, for any $i \in I$. Moreover

$$\frac{4}{3} \leq \lambda_1 = \sup_{a \in \mathbb{R}^2} \left| \frac{D_{a_2}(D_{a_1}U_1)(a)}{D^2_{a_1}U_1(a)} \right| = \sup_{a \in \mathbb{R}^2} \frac{5 - \sin a_1 \cos a_2}{2 - \cos a_1 \sin a_2} \leq 6,$$
$$\lambda_2 = \sup_{a \in \mathbb{R}^2} \left| \frac{D_{a_1}(D_{a_2}U_2)(a)}{D^2_{a_2}U_2(a)} \right| = \sup_{a \in \mathbb{R}^2} \frac{6(a_2^2 + 1)^3}{4a_2^6 + 12a_2^4 + 9a_2^2 + 5} = \frac{8}{5}.$$

Therefore $\lambda > 1$, as $\lambda = \lambda_1\lambda_2 \in [32/15, 48/5]$. Furthermore

$$\begin{aligned} G_1(a_1', a_1'', a_2) &= \frac{(5 - \sin a_1' \cos a_2)[6(a_2^2+1)^3]}{(2 - \cos a_1' \sin a_2)(4a_2^6 + 12a_2^4 + 9a_2^2 + 5)} \\ &\geq \inf_{a\in\mathbb{R}^2} \frac{5 - \sin a_1 \cos a_2}{2 - \cos a_1 \sin a_2} \cdot \inf_{a\in\mathbb{R}^2} \frac{6(a_2^2+1)^3}{4a_2^6 + 12a_2^4 + 9a_2^2 + 5} \\ &\geq \frac{4}{3} \cdot \frac{6}{5} = \frac{8}{5} > 1, \end{aligned}$$

for any $a_1', a_1'' \in \mathbb{R}$ and $a_2 \in \mathbb{R}$. Hence, $(\mathcal{G})$ is satisfied by taking $i_0 = 1$ and $\gamma_{i_0} = 8/5$, so Ω has a unique Nash equilibrium in light of Theorem 3.1.11.

3.1.1 Weighted potential games case

Now, following the work of Caruso, Ceparano and Morgan [24], we focus on the class of two-player *weighted potential games* (introduced by Monderer and Shapley in [105], see also [39, 147, 17, 98], and the survey [93] and references therein) and we illustrate some consequences and applications of the results previously shown. Preliminarily, let us recall the definition and the characterizations of weighted potential games.
The game $\Omega = \{I, (A_i)_{i\in I}, (U_i)_{i\in I}\}$ where $I = \{1, 2\}$ is said to be a *weighted potential game* (see [105, Section 2]) if there exist a vector $w = (w_1, w_2) \in \mathbb{R}^2_{++} := \{(w_1, w_2) \in \mathbb{R}^2 : w_1 > 0, w_2 > 0\}$ and a real-valued function P defined on A such that

$$U_i(a_i, a_{-i}) - U_i(a_i', a_{-i}) = w_i(P(a_i, a_{-i}) - P(a_i', a_{-i})),$$

for any $a_i, a_i' \in A_i$ and $a_{-i} \in A_{-i}$, for any $i \in I$. The function P is called *weighted potential* (or w-potential for short) of Ω. When $w_1 = w_2 = 1$, the game Ω becomes a *potential game* and P is a *potential*.

A useful characterization of weighted potential games is recalled in the next proposition.

Proposition 3.1.15 (Theorem 2.1 in [39])**.** *Ω is a weighted potential game with w-potential P and weights $(w_i)_{i\in I}$ if and only if for any $i \in I$*

$$U_i(a) = w_i P(a) + h_i(a_{-i}) \quad \text{for any } a \in A, \tag{3.16}$$

where $h_i : A_{-i} \to \mathbb{R}$.

From Proposition 3.1.15 we derive the following result.

Proposition 3.1.16 (Proposition 2 in [24])**.** *Ω is a weighted potential game if and only if there exists $c : A \to \mathbb{R}$ such that for any $i \in I$*

$$U_i(a) = f_i(a_i) + g_i(a_{-i}) + w_i c(a), \quad \text{for any } a \in A, \tag{3.17}$$

where $f_i : A_i \to \mathbb{R}$ *and* $g_i : A_{-i} \to \mathbb{R}$.

Proof. If Ω is a weighted potential game, then the equality (3.17) follows immediately from Proposition 3.1.15.
Vice versa, let $i \in I$ and let $c : A \to \mathbb{R}$, $f_i : A_i \to \mathbb{R}$ and $g_i : A_{-i} \to \mathbb{R}$ such that (3.17) holds. Then, $U_i(a)$ can be expressed as in equation (3.16) with

$$P(a) = c(a) + \frac{f_1(a_1)}{w_1} + \frac{f_2(a_2)}{w_2}, \quad \text{for any } a \in A$$
$$h_i(a_{-i}) = g_i(a_{-i}) - \frac{w_i}{w_{-i}} f_{-i}(a_{-i}), \quad \text{for any } a_{-i} \in A_{-i} \text{ and } i \in I.$$

Thus, Ω is a weighted potential game by Proposition 3.1.15. □

The following result is a direct consequence of Proposition 3.1.15 and gives a necessary condition for a game to be a weighted potential game when the payoff functions are twice-continuously differentiable.

Corollary 3.1.17. *If Ω is a weighted potential game and the payoff functions are twice continuously differentiable then, for any $i \in I$ there exists $\alpha_i > 0$ such that*

$$D_{a_{-i}}(D_{a_i} U_i) = \alpha_i D_{a_{-i}}(D_{a_i} U_{-i}).$$

As regards to the Nash equilibria of weighted potential games, firstly it is worth to note that the set of Nash equilibria of a weighted potential game with w-potential P coincides with the set of Nash equilibria of a game in which all the payoff functions of the players are replaced by the w-potential P (see [105, Lemma 2.1]). Moreover, literature results about the existence and the uniqueness of Nash equilibria in weighted potential games exploit the property that any maximum point of the w-potential P is a Nash equilibrium of the game, but the converse is not true in general: it may exist a Nash equilibrium of the weighted potential game that is not a maximum point of P. Nevertheless, if we assume that A is a convex set and P is bounded and concave on A and continuously differentiable on the interior of A, then any Nash equilibrium of the potential game is also a maximum point of P (see [121, Corollary of Theorem 1]). If, in addition, P is strictly concave and attains a maximum, then the weighted potential game has one and only one Nash equilibrium which coincides with the maximum point of P. The latter result is implied by Rosen uniqueness result [132, Theorem 2]: in fact, if w-potential P is strictly concave then the *diagonal strict concavity* condition holds (see also Theorem 3.2.2 and Remark 3.2.3 in the next section). However, the strict concavity of P is a very strong assumption and, incidentally, it is not sufficient to ensure by itself the existence of a maximum point of P if the actions sets are not compact.

However, by using Theorem 3.1.11, we present a result concerning the existence of a unique Nash equilibrium in weighted potential games without assuming neither the existence of a maximum point of the w-potential P nor the strict concavity of P.
Let $\Omega = \{I, (A_i)_{i\in I}, (U_i)_{i\in I}\}$ be a weighted potential game with w-potential P. In light of Proposition 3.1.15, in a weighted potential games setting, conditions $(\mathcal{C}_i)$ and $(\mathcal{D}_i)$ for any $i \in I$, and condition $(\mathcal{G})$ stated before are equivalent to the following ones, respectively

$(\mathcal{C})$ $\begin{cases}$ P is strongly concave in the i-th argument for any $i \in I$, i.e. the function $P(\cdot, a_{-i})$ is strongly concave in A_i, for any $a_{-i} \in A_{-i}$ and the function $P(a_i, \cdot)$ is strongly concave in A_{-i}, for any $a_i \in A_i$;

$(\mathcal{D})$ $\begin{cases}$ P is a twice continuously differentiable function on A and $D^2_{a_i}P(a) \in \mathcal{GL}(A_i, A_i^*)$ for any $a \in A$ and $i \in I$, i.e. there exists the inverse operator $[D^2_{a_i}P(a)]^{-1} \in \mathcal{L}(A_i^*, A_i)$ for any $a \in A$ and $i \in I$;

and

$(\mathcal{H})$ $\begin{cases}$ There exist $i_0 \in I$ and $\gamma_{i_0} \in]1, +\infty[$ such that, for any $\psi \in A_{i_0}$, a'_{i_0}, $a''_{i_0} \in A_{i_0}$ and $a_{-i_0} \in A_{-i_0}$, we have:

$$(H_{i_0}(a'_{i_0}, a''_{i_0}, a_{-i_0})\psi, \psi)_{A_{i_0}} \geq \gamma_{i_0} \|\psi\|^2_{A_{i_0}};$$

where $H_i(a'_i, a''_i, a_{-i}) : A_i \to A_i$ is the operator defined as:

$$\begin{aligned} H_i(a'_i, a''_i, a_{-i}) :=& [D^2_{a_i}P(a'_i, a_{-i})]^{-1} \circ D_{a_{-i}}(D_{a_i}P)(a'_i, a_{-i}) \\ & \circ [D^2_{a_{-i}}P(a''_i, a_{-i})]^{-1} \circ D_{a_i}(D_{a_{-i}}P)(a''_i, a_{-i}), \end{aligned}$$

for any a'_i, $a''_i \in A_i$, $a_{-i} \in A_{-i}$ and $i \in I$.

From Theorem 3.1.11 it follows immediately:

Corollary 3.1.18 (Theorem 1 in [24]). *Let Ω be a weighted potential game with w-potential P and assume $(\mathcal{C})$, $(\mathcal{D})$, $(\mathcal{H})$ and $\lambda \in]1, +\infty[$. Then Ω has one and only one Nash equilibrium.*

The next proposition explores how the hypotheses of Corollary 3.1.18 are related to strict concavity and to the existence of maximum points of the w-potential, when $A_1 = A_2 = \mathbb{R}$.

Proposition 3.1.19 (Proposition 6 in [24]). *Under the assumptions of Corollary 3.1.18 with $A_1 = A_2 = \mathbb{R}$, P does not admit a maximum point on $\mathbb{R}^2$*

(and, therefore, the unique Nash equilibrium of Ω is not a maximum point of the w-potential P) and P is not strictly concave.

Proof. First, let $i_0 \in I$ and $\gamma_{i_0} > 1$ be such that $(\mathcal{H})$ holds and let $a = (a_{i_0}, a_{-i_0}) \in \mathbb{R}^2$. Choosing $a'_{i_0} = a''_{i_0} = a_{i_0}$, by Remark 3.1.7:

$$H_{i_0}(a_{i_0}, a_{i_0}, x_{-i_0}) = \frac{[D_{a_{-i_0}}(D_{a_{i_0}} P)(a_{i_0}, a_{-i_0})]^2}{D^2_{a_{i_0}} P(a_{i_0}, a_{-i_0}) D^2_{a_{-i_0}} P(a_{i_0}, a_{-i_0})} \geq \gamma_{i_0} > 1,$$

that is

$$D^2_{a_{i_0}} P(a_{i_0}, a_{-i_0}) D^2_{a_{-i_0}} P(a_{i_0}, a_{-i_0}) - [D_{a_{-i_0}}(D_{a_{i_0}} P)(a_{i_0}, a_{-i_0})]^2 < 0,$$

i.e., the Hessian matrix of P is indefinite at a. As a is arbitrary, we have that P does not attain a maximum in $\mathbb{R}^2$ and P is not strictly concave on $\mathbb{R}^2$. □

Remark 3.1.20 The following example illustrates a game which satisfies the assumptions of Corollary 3.1.18 and where best reply functions and the Nash equilibria could not be computed explicitly. However, by applying Corollary 3.1.18 and Proposition 3.1.19, one can conclude that the game has one and only one Nash equilibrium and that such a Nash equilibrium is not a maximum point of the w-potential, respectively.

Example 3.1.3 Let Ω be a weighted potential game with $A_1 = A_2 = \mathbb{R}$ and P defined for any $a \in \mathbb{R}^2$ by:

$$P(a_1, a_2) = \frac{1}{1 + a_1^2} + \frac{1}{1 + a_2^2} - 4a_1^2 + a_1 - 4a_2^2 + a_2 - 12a_1a_2.$$

Since $D^2_{a_i} P(a) \leq -15/2 < 0$ for any $a \in A$ and $i \in I$, then P is strongly concave in any argument, so $(\mathcal{C})$ and $(\mathcal{D})$ are satisfied. Moreover, for $i \in I$:

$$\lambda_i = \sup_{a_i \in \mathbb{R}} \frac{6(a_i^2 + 1)^3}{4a_i^6 + 12a_i^4 + 9a_i^2 + 5} = \frac{8}{5};$$

so $\lambda = 64/25 > 1$. Finally, for any $i \in I$ and a'_i, a''_i, $a_{-i} \in \mathbb{R}$:

$$H_i(a'_i, a''_i, a_{-i}) = \frac{36({a'_i}^2 + 1)^3(a_{-i}^2 + 1)^3}{(4{a'_i}^6 + 12{a'_i}^4 + 9{a'_i}^2 + 5)(4a_{-i}^6 + 12a_{-i}^4 + 9a_{-i}^2 + 5)}.$$

Since $\inf_{(a'_i, a''_i, a_{-i}) \in \mathbb{R}^3} H_i(a'_i, a''_i, a_{-i}) = H_i(0, a''_i, 0) = 36/25$, assumption $(\mathcal{H})$ is satisfied with $\gamma_i = 36/25 > 1$.

Now, we present an application of Corollary 3.1.18 when Ω is a *weighted potential game with bilinear common interaction*, i.e. when P is defined on $A_1 \times A_2$ by

$$P(a_1, a_2) = f_1(a_1) + f_2(a_2) + m(a_2, a_1), \tag{3.18}$$

where $f_1 : A_1 \to \mathbb{R}$ and $f_2 : A_2 \to \mathbb{R}$ are twice continuously differentiable and strongly concave operators, and $m : A_2 \times A_1 \to \mathbb{R}$ is a bilinear continuous operator which defines the linear continuous operator $M \in \mathcal{L}(A_2, A_1^*)$ such that

$$m(a_2, a_1) = \langle Ma_2, a_1\rangle_{A_1^* \times A_1}, \quad \text{for any } a_1 \in A_1 \text{ and } a_2 \in A_2. \tag{3.19}$$

Hence, P is twice continuously differentiable on A and

$$\begin{aligned} D_{a_1}P(a) &= Df_1(a_1) + Ma_2, \quad D^2_{a_1}P(a) = D^2f_1(a_1), \quad D_{a_2}(D_{a_1}P)(a) = M \\ D_{a_2}P(a) &= Df_2(a_2) + M^t a_1, \quad D^2_{a_2}P(a) = D^2f_2(a_2), \quad D_{a_1}(D_{a_2}P)(a) = M^t, \end{aligned}$$

where $M^t := M^*J$ and M^* is the adjoint of M and J is the natural embedding of A_1 into A_1^{**} (see, e.g., VI.2.1 and II.3.18 in [37]). Therefore, the best reply correspondences are single-valued since P is strongly concave in any argument. If $D^2f_i(a_i)$ is invertible for any $a_i \in A_i$ and $i \in I$, then

$$\begin{aligned} \lambda_1 &= \sup_{a_1 \in A_1} \|[D^2f_1(a_1)]^{-1} \circ M\|_{\mathcal{L}(A_2,A_1)}, \\ \lambda_2 &= \sup_{a_2 \in A_2} \|[D^2f_2(a_2)]^{-1} \circ M^t\|_{\mathcal{L}(A_1,A_2)}, \end{aligned} \tag{3.20}$$

and hypothesis $(\mathcal{H})$ holds when there exists $\gamma > 1$ such that

$$\begin{aligned} (H_1(a_1', a_1'', a_2)\psi, \psi)_{A_1} &= (\{[D^2f_1(a_1')]^{-1} \circ M \circ [D^2f_2(a_2)]^{-1} \circ M^t\}\psi, \psi)_{A_1} \\ &\geq \gamma\|\psi\|^2_{A_1}, \quad \text{for any } a_1',\, a_1'' \in A_1,\, a_2 \in A_2, \text{ and } \psi \in A_1, \end{aligned} \tag{3.21}$$

or

$$\begin{aligned} (H_2(a_2', a_2'', a_1)\psi, \psi)_{A_2} &= (\{[D^2f_2(a_2')]^{-1} \circ M^t \circ [D^2f_1(a_1)]^{-1} \circ M\}\psi, \psi)_{A_2} \\ &\geq \gamma\|\psi\|^2_{A_2}, \quad \text{for any } a_2',\, a_2'' \in A_2,\, a_1 \in A_1, \text{ and } \psi \in A_2. \end{aligned} \tag{3.22}$$

Let us analyze two special classes of weighted potential games with bilinear common interaction in which the assumptions in Corollary 3.1.18 become easier to prove.

- *Quadratic case.* Assume that, for any $i \in I$, f_i in (3.18) is defined by

$$f_i(a_i) = -k_i(a_i, a_i) + L_i(a_i) + c_i, \quad \text{for any } a_i \in A_i, \tag{3.23}$$

where $k_i : A_i \times A_i \to \mathbb{R}$ is a bilinear continuous operator, $L_i : A_i \to \mathbb{R}$ is a linear continuous operator and $c_i \in \mathbb{R}$. Furthermore, for any $i \in I$, assume that k_i is symmetric and that there exists $\alpha_i \in \mathbb{R}_{++}$ such that

$$k_i(a_i, a_i) \geq \alpha_i\|a_i\|^2_{A_i}, \quad \text{for any } a_i \in A_i.$$

and let $K_i \in \mathcal{L}(A_i, A_i^*)$ be the operator such that

$$k_i(a_i', a_i'') = \langle K_i a_i', a_i'' \rangle_{A_i^* \times A_i}, \quad \text{for any } a_i',\, a_i'' \in A_i. \tag{3.24}$$

When the actions sets of both players coincide and the operators k_1, k_2 and m are linear functions of the inner product of the Hilbert space, uniqueness of Nash equilibrium is proved in the following proposition.

Proposition 3.1.21 (Proposition 7 in [24]). *Let $A_1 = A_2 = \mathbb{H}$ be a real Hilbert space and let P be defined as in (3.18) where f_1, f_2 satisfy (3.23). Let*

$$k_i(a_i', a_i'') = \alpha_i \cdot (a_i', a_i'')_{\mathbb{H}}, \quad \textit{for any } a_i',\, a_i'' \in \mathbb{H} \textit{ and } i \in I, \tag{3.25}$$

$$m(a_2, a_1) = \rho \cdot (a_2, a_1)_{\mathbb{H}}, \quad \textit{for any } a_1,\, a_2 \in \mathbb{H}, \tag{3.26}$$

where α_1, $\alpha_2 \in \mathbb{R}_{++}$ and $\rho \in \mathbb{R}$.
Assume that $\frac{\rho^2}{\alpha_1 \alpha_2} \neq 4$. Then, Ω has one and only one Nash equilibrium.

Proof. First note that $(\mathcal{C})$ is satisfied and that, in light of Lax-Milgram Theorem (see, e.g., [65, Theorem 2.1]), the operators K_1 and K_2 are invertible, so even $(\mathcal{D})$ is satisfied. Moreover, (3.20) implies $\lambda_1 = \frac{1}{2}\|K_1^{-1} \circ M\|_{\mathcal{L}(\mathbb{H},\mathbb{H})} < +\infty$, $\lambda_2 = \frac{1}{2}\|K_2^{-1} \circ M^t\|_{\mathcal{L}(\mathbb{H},\mathbb{H})} < +\infty$ and (3.21)-(3.22) are equivalent to

$$([K_1^{-1} \circ M \circ K_2^{-1} \circ M^t]\psi, \psi)_{\mathbb{H}} \geq 4\gamma \|\psi\|_{\mathbb{H}}^2, \quad \text{for any } \psi \in \mathbb{H};$$
$$([K_2^{-1} \circ M^t \circ K_1^{-1} \circ B]\psi, \psi)_{\mathbb{H}} \geq 4\gamma \|\psi\|_{\mathbb{H}}^2, \quad \text{for any } \psi \in \mathbb{H}.$$

Let $\psi \in \mathbb{H}$. Then $M^t\psi \in \mathbb{H}^*$; moreover in light of the definitions of M^* and J, and by (3.19) and (3.26):

$$\begin{aligned}\langle M^t\psi, a_2 \rangle_{\mathbb{H}^* \times \mathbb{H}} &= \langle M^* J\psi, a_2 \rangle_{\mathbb{H}^* \times \mathbb{H}} = \langle J\psi, M a_2 \rangle_{\mathbb{H}^{**} \times \mathbb{H}^*} \\ &= \langle M a_2, \psi \rangle_{\mathbb{H}^* \times \mathbb{H}} = \rho \cdot (a_2, \psi)_{\mathbb{H}}, \quad \text{for any } a_2 \in \mathbb{H}.\end{aligned} \tag{3.27}$$

Consider the operator $K_2^{-1} \in \mathcal{L}(\mathbb{H}^*, \mathbb{H})$. Then, $K_2^{-1}(M^t\psi)$ is the unique $a_2 \in \mathbb{H}$ such that $K_2 a_2 = M^t\psi$, that is $\langle K_2 a_2, y \rangle_{\mathbb{H}^* \times \mathbb{H}} = \langle M^t\psi, y \rangle_{\mathbb{H}^* \times \mathbb{H}}$ for any $y \in \mathbb{H}$. In light of (3.24), (3.25) and (3.27)

$$\alpha_2 \cdot (a_2, y)_{\mathbb{H}} = \rho \cdot (y, \psi)_{\mathbb{H}}, \quad \text{for any } y \in \mathbb{H};$$

so $K_2^{-1}(M^t\psi) = a_2 = \frac{\rho}{\alpha_2}\psi$.
Moreover, the operator $M(K_2^{-1}(M^t\psi)) \in \mathbb{H}^*$ is defined on $\mathbb{H}$ by:

$$\langle M(K_2^{-1}(M^t\psi)), x \rangle_{\mathbb{H}^* \times \mathbb{H}} = \frac{\rho^2}{\alpha_2} \cdot (\psi, x)_{\mathbb{H}}, \quad \text{for any } x \in \mathbb{H}. \tag{3.28}$$

Finally, consider the operator $K_1^{-1} \in \mathcal{L}(\mathbb{H}^*, \mathbb{H})$. Then, $K_1^{-1}(M(K_2^{-1}(M^t\psi)))$ is the unique $a_1 \in H$ such that $K_1 a_1 = M(K_2^{-1}(M^t\psi))$, that is $\langle K_1 a_1, x \rangle_{\mathbb{H}^* \times \mathbb{H}} =$

$\langle M(K_2^{-1}(M^t\psi)), x\rangle_{\mathbb{H}^*\times\mathbb{H}}$ for any $x \in H$. Therefore, by (3.28)

$$\alpha_1 \cdot (a_1, x)_{\mathbb{H}} = \left(\frac{\rho^2}{\alpha_2}\psi, x\right)_{\mathbb{H}}, \quad \text{for any } x \in \mathbb{H};$$

so

$$K_1^{-1}(M(K_2^{-1}(M^t\psi))) = a_1 = \frac{\rho^2}{\alpha_1\alpha_2}\psi. \tag{3.29}$$

If $\frac{\rho^2}{\alpha_1\alpha_2} < 4$, then

$$\sup_{a\in A}\|D\beta_1(a)\|_{\mathcal{L}(\mathbb{H},\mathbb{H})} = \frac{1}{4}\|K_1^{-1}\circ M \circ K_2^{-1}\circ M^t\|_{\mathcal{L}(\mathbb{H},\mathbb{H})} = \frac{\rho^2}{4\alpha_1\alpha_2} = \lambda < 1$$

by (3.29), recalling that $D\beta_1(x_1) = H_1(\beta_1(a_1), a_1, b_2(a_1)) = \frac{1}{4}[K_1^{-1}\circ M\circ K_2^{-1}\circ M^t]$. Therefore, β_1 is a contraction and Ω has one and only one Nash equilibrium by Theorem 3.1.5.
If $\frac{\rho^2}{\alpha_1\alpha_2} > 4$, then by (3.29)

$$([K_1^{-1}\circ M\circ K_2^{-1}\circ M^t]\psi, \psi)_{\mathbb{H}} = \frac{\rho^2}{\alpha_1\alpha_2}\|\psi\|^2_{\mathbb{H}}, \quad \text{for any } \psi \in \mathbb{H},$$

so $(\mathcal{H})$ holds since (3.21) is satisfied. Then, Ω has one and only one equilibrium by Corollary 3.1.18. □

Proposition 3.1.21 allows to prove the existence of a unique *open-loop Nash equilibrium* of the following *differential game* (for definitions see, e.g., [11, 36, 54]).

Example 3.1.4 Consider a two-player differential game with state equation given by

$$\dot{x}(t) = u_1(t) + u_2(t) - nx(t), \qquad x(0) = x_0, \tag{3.30}$$

where $t \in [0,T]$, $T \in]0,+\infty[$, x is continuously differentiable on $[0,T]$, u_1, $u_2 \in \mathbb{U} := L^2([0,T])$, $n \in \mathbb{R}_{++}$ and $x_0 \in \mathbb{R}_{++}$.

Player i, $i \in I$, has an instantaneous profit at time t equal to

$$\pi_i(x(t), u_1(t), u_2(t)) = x(t) - \alpha_i[u_i(t)]^2 + \rho u_1(t)u_2(t),$$

where $\alpha_i > 0$ and $\rho \in \mathbb{R}$. So, player i's objective functional is

$$J_i(x, u_1, u_2) = \int_0^T e^{-rt}\pi_i(x(t), u_1(t), u_2(t))\,dt, \tag{3.31}$$

where $r \geq 0$ is the common discount rate. The differential game (3.30)-(3.31) is similar to Example 7.1 in [36] which describes a situation where two individuals invest in a public stock of knowledge (see also Section 9.5 in [36]). We mention that in [44] the class of *potential differential games* is introduced: it is defined as the class of differential games to which it is possible to associate an optimal control problem whose solutions are open-loop Nash equilibria for the original game. The differential game (3.30)-(3.31) does not belong to such a class.

The solution to the first-order differential equation (3.30) is

$$x(t) = x_0 e^{-nt} + e^{-nt} \int_0^t [u_1(s) + u_2(s)] e^{ns}\, ds. \tag{3.32}$$

Denote by F_i the real-valued function defined on $\mathbb{U} \times \mathbb{U}$ obtained by substituting (3.32) in (3.31), that is

$$\begin{aligned} F_i(u_1, u_2) &:= \int_0^T e^{-rt} \left[x_0 e^{-nt} + e^{-nt} \int_0^t [u_1(s) + u_2(s)] e^{ns}\, ds \right] dt \\ &\quad - \int_0^T e^{-rt} \{\alpha_i [u_i(t)]^2 - \rho u_1(t) u_2(t)\}\, dt. \end{aligned}$$

The game $\Omega = \{I, \mathbb{U}, \mathbb{U}, F_1, F_2\}$ is a potential game with potential

$$P(u_1, u_2) = F_1(u_1, u_2) - \int_0^T e^{-rt} \alpha_2 [u_2(t)]^2\, dt.$$

Such a potential game belongs to the class of weighted potential games with bilinear common interaction, since P belongs to the class of functions considered in (3.18) and (3.23), where:

$$k_i(u_i', u_i'') = \alpha_i \int_0^T e^{-rt} u_i'(t) u_i''(t)\, dt \quad \text{for any } u_i',\, u_i'' \in \mathbb{U} \text{ and } i \in I; \tag{3.33}$$

$$m(u_2, u_1) = \rho \int_0^T e^{-rt} u_1(t) u_2(t)\, dt \quad \text{for any } u_1,\, u_2 \in \mathbb{U}; \tag{3.34}$$

$$L_i(u_i) = \int_0^T e^{-(r+n)t} \left[\int_0^t e^{ns} u_i(s)\, ds \right] dt \quad \text{for any } u_i \in \mathbb{U};$$

$$c = \int_0^T x_0 e^{-(r+n)t}\, dt.$$

We highlight that the operators k_i and m in (3.33)-(3.34) are of the same type of (3.25)-(3.26) where $\mathbb{H} = \mathbb{U}$ and $\mathbb{U}$ is endowed with the inner product defined by

$$(u_1, u_2)_{\mathbb{U}} := \int_0^T e^{-rt} u_1(t) u_2(t)\, dt, \quad \text{for any } u_1,\, u_2 \in \mathbb{U}. \tag{3.35}$$

Note that $\mathbb{U} = L^2([0, T])$ with the inner product defined in (3.35) is a Hilbert space. Hence, arguing as in Proposition 3.1.21 we can conclude that the differential game defined by (3.30)-(3.31) has one and only one open-loop Nash equilibrium if $\frac{\rho^2}{\alpha_1 \alpha_2} \neq 4$.

- *Real case.* Assume that $A_1 = A_2 = \mathbb{R}$. Then, the operator m in (3.18) can be written as

$$m(a_2, a_1) = \rho a_1 a_2, \quad \text{for any } a_1,\, a_2 \in \mathbb{R}, \tag{3.36}$$

where $\rho \in \mathbb{R}$. As emphasized in Remark 3.1.7, we identify the first and second order partial derivatives of P with the usual partial derivatives of real-valued functions defined on $\mathbb{R}^2$.

Proposition 3.1.22 (Proposition 8 in [24]). *Let P be defined as in (3.18) and let $S_i := -\inf_{a_i \in \mathbb{R}} D^2 f_i(a_i)$ and $s_i := -\sup_{a_i \in \mathbb{R}} D^2 f_i(a_i)$, for any $i \in I$. Assume*

(i) $s_1 > 0$ and $s_2 > 0$;

(ii) $\frac{\rho^2}{S_1 S_2} > 1$

where ρ is defined in (3.36). Then, Ω has one and only one Nash equilibrium.

Proof. Let $i \in I$. By (i), $D^2_{a_i} P(a) \leq \sup_{x_i \in \mathbb{R}} D^2 f_i(a_i) = -s_i < 0$; hence P is strongly concave in any argument, so $(\mathcal{C})$ and $(\mathcal{D})$ are satisfied. Since

$$\lambda_i = \sup_{a \in \mathbb{R}^2} \left| \frac{D_{a_{-i}}(D_{a_i} P)(a_i, a_{-i})}{D^2_{a_i} P(a_i, a_{-i})} \right| = \frac{|\rho|}{\inf_{a \in \mathbb{R}^2} |D^2_{a_i} P(a_i, a_{-i})|} = \frac{|\rho|}{s_i},$$

then $\lambda = \lambda_1 \lambda_2 = \frac{\rho^2}{s_1 s_2} \geq \frac{\rho^2}{S_1 S_2} > 1$ in light of (ii). Hence $\lambda \in]1, +\infty[$. Moreover, by (ii), for any a_1', a_1'', a_2', $a_2'' \in \mathbb{R}$:

$$H_1(a_1', a_1'', a_2') = H_2(a_2', a_2'', a_1') = \frac{\rho^2}{D^2 f_1(a_1') D^2 f_2(a_2')} \geq \frac{\rho^2}{S_1 S_2} > 1.$$

Hence, $(\mathcal{H})$ holds. Then, in light of Corollary 3.1.18, Ω has one and only one Nash equilibrium. □

Remark 3.1.23 The game in Example 3.1.3 satisfies the assumptions of Proposition 3.1.22 since the w-potential P fits (3.18), $0 < 15/2 = s_i < S_i = 10$ for any $i \in I$ and $\rho^2/(S_1 S_2) = 36/25 > 1$.

We conclude by mentioning that in Caruso, Ceparano and Morgan [22] an adjustment process-based algorithm has been defined to approximate the unique Nash equilibrium of games satisfying the assumptions described in this section. In particular, we highlight that the unique Nash equilibrium of potential games satisfying $(\mathcal{C})$, $(\mathcal{D})$ and $(\mathcal{H})$ (as the game in Example 3.1.3) cannot be approximated through the usual methods based on the potential function (which exploit the property that any maximizer of the potential function is a Nash equilibrium of the potential game), since such equilibrium is not a maximizer of the w-potential (in light of Proposition 3.1.19). See, for example, [38, 133] and reference therein for further discussion regarding methods based on the potential function.

3.2 Rosen uniqueness result

Throughout this section we assume that, in the game $\Omega = \{I, (A_i)_{i\in I}, (U_i)_{i\in I}\}$ with $I = \{1, \ldots, N\}$, player i's action set A_i is a subset of the Euclidean space $\mathbb{R}^{m_i}$ for any $i \in I$, with $m_i \in \mathbb{N}$. Consequently, the set of action profiles A is an orthogonal subset of $\mathbb{R}^m$, with $m := m_1 + \cdots + m_N$. Inequalities between vectors have to be understood as inequalities between components (for example: $a_i \geq a'_i$ with $a_i, a'_i \in A_i \subseteq \mathbb{R}^{m_i}$ means $a_{ij} \geq a'_{ij}$ for any $j \in \{1, \ldots, m_i\}$).
Before dealing with uniqueness issues, we recall the following well-known Nash equilibrium existence theorem.

Theorem 3.2.1 (Nash [119, 120], Debreu [31], Glicksberg [48], Fan [40]). *Assume that, for any $i \in I$,*

(E1) A_i is nonempty convex and compact,

(E2) F_i is continuous on A,

(E3) $F_i(\cdot, a_{-i})$ is quasi-concave on A_i, for any $a_{-i} \in A_{-i}$.

Then a Nash equilibrium of Ω exists.

Sketch of the proof. The proof is based on the Kakutani fixed point theorem (see Theorem 1 and Corollary in [57]). In fact, assumptions (E1)-(E2) guarantee that the best reply correspondence of each player is nonempty and closed, whereas assumption (E3) implies that the best reply correspondences are convex-valued. Hence the set-valued map B defined in (3.2) satisfies the hypotheses of the Kakutani theorem and the existence of a Nash equilibrium follows from Remark 3.0.2.

Rosen in [132, Theorems 1 and 2] established a fundamental uniqueness result when A_i is a constrained set defined by the solutions of a finite number of inequalities:

$$A_i = \{a_i \in \mathbb{R}^{m_i} \mid h_i(a_i) \geq 0\},$$

where $h_i := (h_{i1}, \ldots, h_{ik_i})$ and $h_{ij} \colon \mathbb{R}^{m_i} \to \mathbb{R}$ for any $j \in \{1, \ldots, k_i\}$, with $k_i \in \mathbb{N}$. Hence, the set of action profiles A is a constrained orthogonal subset of $\mathbb{R}^m$.

Theorem 3.2.2 (Theorems 1 and 2 in [132]). *Assume that*

(R0) h_{ij} is continuously differentiable on $\mathbb{R}^{m_i}$ for any $i \in I$ and $j \in \{1, \ldots, k_i\}$, and there exists $\bar{a} \in A$ such that $h_{ij}(\bar{a}_i) > 0$ for any $i \in I$ and $j \in \{1, \ldots, k_i\}$ for which h_{ij} is a nonlinear function.

and that, for any $i \in I$,

(R1) h_{ij} *is concave on* $\mathbb{R}^{m_i}$*, for any* $j \in \{1, \dots, k_i\}$*,*

(R2) A_i *is nonempty and bounded,*

(R3) U_i *is continuous on* A*,*

(R4) $U_i(\cdot, a_{-i})$ *is continuously differentiable on* A_i*, for any* $a_{-i} \in A_{-i}$*,*

(R5) there exists $r \in]0, +\infty[^N$ *such that the function* $g \colon A \times [0, +\infty[^N \to \mathbb{R}^m$ *defined by*

$$g(a, r) \coloneqq -(r_1 \nabla_1 U_1(a), \dots, r_N \nabla_N U_N(a)) \tag{3.37}$$

is strictly monotone on A*, i.e., for any* $a', a'' \in A$ *with* $a' \neq a''$

$$(a' - a'', g(a', r) - g(a'', r))_m > 0,$$

where $\nabla_i U_i(a)$ *stands for the gradient of* U_i *with respect to* $a_i \in \mathbb{R}^{m_i}$ *at* $a \in A$*, and* $(\cdot, \cdot)_m$ *denotes the Euclidean scalar product in* $\mathbb{R}^m$*.*

Then, Ω *has a unique Nash equilibrium.*

Sketch of the proof. Equilibrium existence comes from assumptions (R1)-(R5) and Theorem 3.2.1, since (R1)-(R2) guarantee that A_i is a nonempty convex and compact set, and (R4)-(R5) ensure the concavity of $U_i(\cdot, a_{-i})$. Uniqueness is obtained by contradiction exploiting the strict monotonicity of the function defined in (3.37) and the differential form of the necessary and sufficient Kuhn-Tucker conditions for a constrained maximum (see [62]), that holds in light of assumption (R0).

Remark 3.2.3 The uniqueness result of Rosen [132, Theorem 2] is not stated in terms of strict monotonicity of the function g defined in (3.37), but it is assumed

(R5′) there exists $r \in]0, +\infty[^N$ such that the function $\sigma \colon A \times [0, +\infty[^N \to \mathbb{R}$ defined by

$$\sigma(a, r) \coloneqq \sum_{i=1}^{N} r_i U_i(a)$$

is *diagonally strictly concave* on A, i.e., for any $a', a'' \in A$ with $a' \neq a''$

$$(a' - a'', k(a'', r))_m + (a'' - a', k(a', r))_m > 0,$$

where k is the *pseudogradient* of σ, that is, $k \colon A \times [0, +\infty[^N \to \mathbb{R}$ is the function defined by $k(a, r) \coloneqq (r_1 \nabla_1 U_1(a), \dots, r_N \nabla_N U_N(a))$.

It is immediate to check that the diagonal strict concavity of function σ is equivalent to the strict monotonicity of function g defined in (3.37), since $k = -g$. Hence, assumptions (R5) and (R5′) are equivalent.

Furthermore, Theorem 6 in [132] provides a sufficient condition for the strict monotonicity of g in assumption (R5).

Proposition 3.2.4 (Theorem 6 in [132])**.** *Assume that U_i is twice continuously differentiable on A, for any $i \in I$. Let $J_g(a, r) \in \mathbb{R}^{m\times m}$ be the Jacobian matrix of the function g defined in (3.37) at the point $(a, r) \in A\times[0, +\infty[^N$ with respect to the variable a for fixed r, that is*

$$J_g(a, r) := \left[\frac{\partial g}{\partial a_1}(a, r)\ldots\frac{\partial g}{\partial a_m}(a, r)\right],$$

and let $J_g(a, r)^T$ be the transposed matrix of $J_g(a, r)$. If there exists $r \in]0, +\infty[^N$ such that the symmetric matrix $[J_g(a, r) + J_g(a, r)^T]$ is negative definite for any $a \in A$, then $g(\cdot, r)$ is strictly monotone on A.

The compactness of the actions sets is crucial for the existence (and also for the uniqueness) of Nash equilibria, as evident in all the results of this section. Karamardian in [59, Theorem 5.1] showed that this condition can be relaxed: he provided an existence and uniqueness result (by using the same assumptions on the payoff functions of Theorem 3.2.2), in the case where $A_i = [0, +\infty[^{m_i}$ for any $i \in I$. However, when $A_i = \mathbb{R}^{m_i}$ for any $i \in I$, the assumptions on the payoff functions in Theorem 3.2.2 do not even guarantee the existence of Nash equilibria, as illustrated in the following example.

Example 3.2.1 Let $\Omega = \{I, (A_i)_{i\in I}, (U_i)_{i\in I}\}$ where $I = \{1, 2\}$, $A_1 = A_2 = \mathbb{R}$ and $U_1(a_1, a_2) = U_2(a_1, a_2) = -e^{a_1} - e^{a_2}$. Assumptions (R3)-(R4) in Theorem 3.2.2 are satisfied. Moreover, $g(a, r) = (r_1e^{a_1}, r_2e^{a_2})$ for any $a = (a_1, a_2) \in \mathbb{R}^2$ and $r = (r_1, r_2) \geq (0, 0)$, and g is strictly monotone on $\mathbb{R}^2$ for any $r_1, r_2 > 0$ since the Jacobian matrix of g, namely

$$J_g(a, r) = \begin{pmatrix} r_1e^{a_1} & 0 \\ 0 & r_2e^{a_2} \end{pmatrix}$$

is positive defined on $\mathbb{R}^2$, for any $r \in \mathbb{R}^2_{++}$ (see, e.g., [58, Theorem 3.1]). Hence, even (R5) holds. But the best reply correspondences of both players are empty-valued, so Ω has not Nash equilibria.

Finally, we mention that Carlson in [20] extended the Rosen uniqueness result [132, Theorem 2] to a setting where the actions set of each player is a constrained subset of a separable Hilbert space.

Chapter 4

On the Tikhonov and Moreau-Yosida regularization

While in the previous chapters we discussed theoretical aspects related to the existence and the uniqueness of some relevant solution concepts in Game Theory, now we deal with an argument closely connected to such theoretical aspects: the construction of the solutions. In particular, in this chapter we present two methods for the approximation of solutions of optimization problems: Tikhonov regularization and Moreau-Yosida regularization. We highlight that such methods have two important advantage, especially from a numerical point of view:

- they allow to construct sequences of regularized problems having a unique solution, so step by step the approximating sequence is uniquely identified.
- in the regularized problems the objective function is modified by adding terms that make the solutions of the the regularized problems easier to be found, hence the regularized problems are better-behaved than the original optimization problem.

After showing the definitions, the interpretations and the convergence properties of Tikhonov and Moreau-Yosida regularization (in optimization setting), we analyze the applications of such methods to the selection of Nash equilibria in normal-form games: having in mind to define a constructive selection method for SPNEs in one-leader N-followers two-stage games that satisfies the features highlighted at the end of Chapter 2, the key feature we pursue (analogously to the optimization framework) is to construct a sequence of regularized normal-form games where the Nash equilibrium is unique and to give sufficient conditions in order to guarantee the convergence of the related sequence of Nash equilibria to a Nash equilibrium of the original game.

4.1 Tikhonov regularization

Let A be a subset of a Euclidean space $\mathbb{A}$ with norm $\|\cdot\|_{\mathbb{A}}$ and U be a real-valued function defined on A. We deal with the following maximization problem:

$$P\colon \max_{a\in A} U(a).$$

A point $a^* \in A$ is said to be a solution of P if a^* is a maximizer of U over A, i.e. if $U(a^*) = \max_{a\in A} U(a)$, and we denote by M the set of maximizers of U, that is

$$M := \{a^* \in A \text{ such that } U(a^*) \geq U(a), \text{ for any } a \in A\}. \tag{4.1}$$

Let us consider the sequence of maximization problems $(P_k)_k$, introduced by Tikhonov in [140] (see also [141]), defined as follows:

$$P_k\colon \max_{a\in A} U(a) - \frac{1}{2\lambda_k}\|a\|^2_{\mathbb{A}},$$

where $k \in \mathbb{N}$ and $\lambda_k > 0$. Problem P_k is called *Tikhonov regularized problem of parameter* λ_k and the function $U_k\colon A \to \mathbb{R}$ defined by $U_k(a) = U(a) - \frac{1}{2\lambda_k}\|u\|^2_{\mathbb{A}}$ is called *Tikhonov regularization of* U *of parameter* λ_k.
Before showing the well-known relationship between the solutions of P_k and the solutions of P, whose proof is given for the sake of completeness, we recall a key result that will be used in the sequel.

Lemma 4.1.1. *Assume that X is a closed convex subset of a Euclidean space $\mathbb{X}$. Then, there exists a unique point $\hat{x} \in X$ such that*

$$\|\hat{x}\|_{\mathbb{X}} = \min_{x\in X}\|x\|_{\mathbb{X}},$$

called minimum norm element of X.

Remark 4.1.2 A more general result regarding the existence and uniqueness of the *projection of best approximation* of $\mathbb{A}$ onto A, when A is a closed convex subset of a Hilbert space, holds. See, for example, [8, Theorem 2.3].

Theorem 4.1.3. *Assume that*

(i) A is compact and convex;

(ii) U is upper semicontinuous and concave;

(iii) $(\lambda_k)_k \subseteq]0, +\infty[$ and $\lim_{k\to+\infty} \lambda_k = +\infty$.

Then, problem P_k has a unique solution $\bar{a}_k \in A$, for any $k \in \mathbb{N}$, and the sequence $(\bar{a}_k)_k$ is convergent to the minimum norm element of the set M, defined in (4.1).

Proof. Firstly, let us note that M is non-empty, closed and convex by assumptions (i)-(ii) and that, in light of Lemma 4.1.1, there exists a unique minimum norm element of M, which we denote by $\hat{a}$.
For any $k \in \mathbb{N}$, the function $U(\cdot) - \frac{1}{2\lambda_k}\|\cdot\|_{\mathbb{A}}$ is upper semicontinuous and strictly concave on A, being sum of the upper semicontinuous and concave function U and the continuous and strictly concave function $-\frac{1}{2\lambda_k}\|\cdot\|_{\mathbb{A}}$ (since $\lambda_k>0$). Hence, in light of assumption (i) and Proposition 3.0.3, problem P_k has a unique solution $\bar{a}_k \in A$.
Let $(\bar{a}_{k_j})_j \subseteq A$ be a subsequence of $(\bar{a}_k)_k$ converging to $\bar{a} \in A$, whose existence is guaranteed by the compactness of A, and let $a \in A$. Being $\bar{a}_{k_j}$ the solution of P_{k_j}, we have

$$U(\bar{a}_{k_j}) - \frac{1}{2\lambda_{k_j}}\|\bar{a}_{k_j}\|^2_{\mathbb{A}} \geq U(a) - \frac{1}{2\lambda_{k_j}}\|a\|^2_{\mathbb{A}},$$

which implies, in light of the upper semicontinuity of U, assumption (iii) and the compactness of A,

$$U(\bar{a}) \geq U(a).$$

Hence, since a is arbitrarily chosen in A, then $\bar{a} \in M$. Moreover, by definition of $\hat{a}$,

$$\|\bar{a}\|_{\mathbb{A}} \geq \|\hat{a}\|_{\mathbb{A}}. \tag{4.2}$$

Let us prove the opposite inequality. Since $\bar{a}_{k_j}$ is the solution of P_{k_j} and $\hat{a} \in M$, we have

$$\begin{aligned} U(\bar{a}_{k_j}) - \frac{1}{2\lambda_{k_j}}\|\bar{a}_{k_j}\|^2_{\mathbb{A}} &\geq U(\hat{a}) - \frac{1}{2\lambda_{k_j}}\|\hat{a}\|^2_{\mathbb{A}} \\ &\geq U(\bar{a}_{k_j}) - \frac{1}{2\lambda_{k_j}}\|\hat{a}\|^2_{\mathbb{A}}, \end{aligned}$$

which implies

$$\|\bar{a}_{k_j}\|_{\mathbb{A}} \leq \|\hat{a}\|_{\mathbb{A}}.$$

Taking the limit as j goes to infinity, by the continuity of $\|\cdot\|_{\mathbb{A}}$, we get

$$\|\bar{a}\|_{\mathbb{A}} \leq \|\hat{a}\|_{\mathbb{A}},$$

that, together with inequality (4.2) proves $\|\bar{a}\|_{\mathbb{A}} = \|\hat{a}\|_{\mathbb{A}}$. Hence, in light of the uniqueness of the minimum norm element, necessarily $\bar{a} = \hat{a}$. So, any convergent subsequence of $(\bar{a}_k)_k$ converges to $\hat{a}$. Therefore, by the compactness of A, the whole sequence $(\bar{a}_k)_k$ is convergent to $\hat{a}$. □

4.1.1 Applications to the selection of Nash equilibria in normal-form games

Using the same notation of Section 3.2, we consider a normal-form game $\Omega = \{I, (A_i)_{i\in I}, (U_i)_{i\in I}\}$ where $I = \{1, \dots, N\}$ is the set of players, A_i is the set of actions of player $i \in I$ and $U_i \colon A \to \mathbb{R}$ is the payoff function of player $i \in I$ defined on the set of action profiles $A \coloneqq A_1 \times \cdots \times A_N$. Moreover, we assume that A_i is a subset of the Euclidean space $\mathbb{R}^{m_i}$ (endowed with the Euclidean scalar product $(\cdot,\cdot)_{m_i}$ and associated norm $\|\cdot\|_{m_i}$) for any $i \in I$, with $m_i \in \mathbb{N}$. Hence, the set of action profiles A is an orthogonal subset of $\mathbb{R}^m$ (endowed with the Euclidean scalar product $(\cdot,\cdot)_m$ and associated norm $\|\cdot\|_m$), with $m \coloneqq m_1 + \cdots + m_N$. Finally, let us denote by $E(\Omega)$ the set of all Nash equilibria of Ω.

Following the approach used in [109, Section 4], we construct a sequence of regularized normal-form games $(\Omega_k)_k$ by means of Tikhonov regularization: for any $k \in \mathbb{N}$, consider

$$\Omega_k = \{I, (A_i)_{i\in I}, (U_i^k)_{i\in I}\},$$

where $U_i^k \colon A \to \mathbb{R}$ is the Tikhonov regularization of U_i with respect to a_i of parameter λ_k, that is the function defined on A by

$$U_i^k(a_i, a_{-i}) = U_i(a_i, a_{-i}) - \frac{1}{2\lambda_k}\|a_i\|_{m_i}^2,$$

with $\lambda_k > 0$.

Through the sequence of Tikhonov regularized games described above, it is possible to select a Nash equilibrium of Ω, as shown in the next result (which is proved by using the same arguments of [109, Theorem 4.1], where a parametric Nash equilibrium problem with two players deriving from a one-leader two-follower two-stage game is involved).

Theorem 4.1.4 (Theorem 4.1 in [109])**.** *Assume that, for any $i \in I$*

(i) A_i is compact and convex;

(ii) U_i is upper semicontinuous on A and continuously differentiable with respect to a_i on A;

(iii) $U_i(\cdot, a_{-i})$ is concave on A_i, for any $a_{-i} \in A_{-i}$;

(iv) for any $(a_i, a_{-i}) \in A_i \times A_{-i}$ and any sequence $(a_{-i,k})_k \subseteq A_{-i}$ converging to a_{-i}, there exists a sequence $(\tilde{a}_{i,k})_k \subseteq A_i$ converging to a_i such that

$$\liminf_{k\to+\infty} U_i(\tilde{a}_{i,k}, a_{-i,k}) \geq U_i(a_i, a_{-i});$$

and that

(v) for any $(a', a'') \in A \times A$ *the following inequality is satisfied*

$$\sum_{i=1}^{N} (\nabla_i U_i(a') - \nabla_i U_i(a''), a'_i - a''_i)_{m_i} \leq 0.$$

(vi) $(\lambda_k)_k \subseteq]0, +\infty[$ *and* $\lim_{k\to+\infty} \lambda_k = +\infty$.

Then, Ω_k *has a unique Nash equilibrium* $\bar{a}_k \in A$*, for any* $k \in \mathbb{N}$*, and the sequence* $(\bar{a}_k)_k$ *is convergent to the minimum norm element of* $E(\Omega)$*.*

Proof. Given, for the sake of completeness, in the general case of N players. Firstly, note that assumptions (i)-(iv) guarantee the existence of at least one Nash equilibrium of Ω and Ω_k for any $k \in \mathbb{N}$ (in light of [74, Theorem 2.1]). Let $k \in \mathbb{N}$ and $W_k \colon A \to \mathbb{R}^m$ be the function defined on A by

$$W_k(a) = \left(\nabla_1 U_1^k(a), \ldots, \nabla_N U_N^k(a)\right).$$

Then, by assumption (v) and since $\lambda_k > 0$

$$\begin{aligned}
&(W_k(a''), a' - a'')_m + (W_k(a'), a'' - a')_m \\
&\qquad = \sum_{i=1}^{N} (\nabla_i U_i(a''), a'_i - a''_i)_{m_i} - \frac{1}{\lambda_k}(a''_i, a'_i - a''_i)_{m_i} \\
&\qquad\quad + \sum_{i=1}^{N} (\nabla_i U_i(a'), a''_i - a'_i)_{m_i} - \frac{1}{\lambda_k}(a'_i, a''_i - a'_i)_{m_i} \\
&\qquad = \sum_{i=1}^{N} (\nabla_i U_i(a'') - \nabla_i U_i(a'), a'_i - a''_i)_{m_i} + \frac{1}{\lambda_k}\|a'_i - a''_i\|^2_{m_i} \\
&\qquad > 0,
\end{aligned}$$

for any $a', a'' \in A$ with $a' \neq a''$. Therefore, for the game Ω_k the diagonal strict concavity condition of Rosen (see Remark 3.2.3) is satisfied and, in light of Rosen uniqueness result (see Theorem 3.2.2), Ω_k has a unique Nash equilibrium $\bar{a}_k \in A$.

Moreover, $\bar{a}_k$ is the unique solution of the variational inequality associated to the operator $F_k \colon A \to A$ defined in the following way: $F_k(a')$ is the unique element of A such that

$$(F_k(a'), a'')_m = \sum_{i=1}^{N} (\nabla_i U_i(a'_i), a''_i)_{m_i} - \frac{1}{\lambda_k}(a'_i, a''_i)_{m_i}.$$

In light of [114, Theorem C], the sequence $(\bar{a}_k)_k$ converges to the solution $\hat{a}$ of the variational inequality:

$$(\hat{a}, a - \hat{a})_m \geq 0 \quad \text{for any } a \in E(\Omega),$$

that is, $\hat{a}$ is the minimum norm element of $E(\Omega)$. □

Remark 4.1.5 Assumption (v) is equivalent to requiring that the function $J\colon A \to \mathbb{R}^m$ defined by

$$J(a) \coloneqq -(\nabla_1 U_1(a), \dots, \nabla_N U_N(a))$$

is monotone on A, and it is weaker than the diagonal strict concavity condition of Rosen, stated in Remark 3.2.3 or, equivalently, than requiring the strict monotonicity of J, which is used to prove the Rosen uniqueness result (see assumption (R5) in Theorem 3.2.2).

4.2 Moreau-Yosida regularization and proximal point methods

Let $U\colon \mathbb{A} \to \mathbb{R} \cup \{-\infty\}$ be an extended real-valued function defined on a Euclidean space $\mathbb{A}$ with norm $\|\cdot\|_{\mathbb{A}}$, and let $\lambda > 0$. The *Moreau-Yosida regularization of U of parameter λ* (also called *Moreau envelope* or *proximal approximation*) is the function $M_{\lambda U}\colon \mathbb{A} \to [-\infty, +\infty]$ defined by

$$M_{\lambda U}(a) \coloneqq \sup_{x \in \mathbb{A}} U(x) - \frac{1}{2\lambda}\|x - a\|_{\mathbb{A}}^2,$$

introduced by Moreau in [106] (see also [130, 149], and [2, 7] and references therein) for convex lower semicontinuous functions defined on Hilbert spaces. The Moreau-Yosida regularization satisfies various good properties, as shown in the following result (whose proof can be found, for example, in [14, Sections 12.1, 12.2 and 12.4]).

Proposition 4.2.1. *Assume that U is upper semicontinuous and concave on $\mathbb{A}$ with $U \neq -\infty$, and let $\lambda > 0$. Then*

(i) $\sup_{a \in \mathbb{A}} M_{\lambda U}(a) = \sup_{a \in \mathbb{A}} U(a)$*;*

(ii) the net $(M_{\nu U}(a))_{\nu > 0}$ is increasing, for any $a \in \mathbb{A}$;

(iii) $\lim_{\nu \to +\infty} M_{\nu U}(a) = \sup_{x \in \mathbb{A}} U(x)$*, for any $a \in \mathbb{A}$;*

(iv) $\lim_{\nu \to 0} M_{\nu U}(a) = U(a)$*, for any $a \in \mathbb{A}$;*

(v) $M_{\lambda U}$ is real-valued, differentiable and concave.

Remark 4.2.2 We highlight that, in addition to the properties stated in Proposition 4.2.1, the Moreau-Yosida regularization has implications in terms of epiconvergence or $\Gamma-$convergence (see [2, 29]). More precisely, the hypoconvergence (or Γ^+-convergence) properties of a sequence of functions $(U_k)_k$ can be restated equivalently in terms of pointwise convergence of the sequence of the associated Moreau-Yosida regurarizations $(M_{\lambda U_k})_k$. For further details, see [2, Theorem 2.65 and Corollary 2.67] and [29, Theorem 9.16].

Hence, the Moreau-Yosida regularization $M_{\lambda U}$ is essentially a smoothed form of U: it is a real-valued and differentiable function, even when U is not; moreover $M_{\lambda U}$ and U have the same set of maximizers and they attain the same maximum. So, the problems of maximizing U and $M_{\lambda U}$ are equivalent, and the latter is always a smooth optimization problem (being aware that $M_{\lambda U}$ could be difficult to evaluate).

Now, let us introduce another important tool strongly connected with the Moreau-Yosida regularization. When U is upper semicontinuous and concave on $\mathbb{A}$, the function

$$x \in \mathbb{A} \mapsto U(x) - \frac{1}{2\lambda}\|x-a\|_{\mathbb{A}}^2 \in \mathbb{R}$$

is upper semicontinuous and strongly concave on $\mathbb{A}$ for any $a \in \mathbb{A}$ and $\lambda > 0$, then, in light of Proposition 3.0.3, it has a unique maximizer on $\mathbb{A}$. Hence, it is well-defined the *proximal operator of U of parameter* λ, that is the function $Prox_{\lambda U}\colon \mathbb{A} \to \mathbb{A}$ defined by

$$\{Prox_{\lambda U}(a)\} := \operatorname*{Arg\,max}_{x\in\mathbb{A}} U(x) - \frac{1}{2\lambda}\|x-a\|_{\mathbb{A}}^2. \tag{4.3}$$

The term "proximal" is justified by the fact that, in the particular case where $-U$ is the indicator function of a closed convex set $A \subseteq \mathbb{A}$, i.e.,

$$U(x) = \begin{cases} 0, & \text{if } x \in A \\ -\infty, & \text{if } x \notin A, \end{cases}$$

then the operator $Prox_{\lambda U}$ is reduced to the projection onto A, that is

$$\{Prox_{\lambda U}(a)\} = \operatorname*{Arg\,min}_{x\in A}\|x-a\|_{\mathbb{A}}^2,$$

for any $a \in \mathbb{A}$. Hence, the proximal operator can be viewed as a generalized projection (for a more detailed discussion regarding the interpretation of the proximal operators see [125, Chapter 3]).
The proximal operator $Prox_{\lambda U}$ and the Moreau-Yosida regularization $M_{\lambda U}$ display many connections, some of which are presented in the next result (for the proofs see, for example, [125, Chapters 2 and 3] and [14, Section 12.4]).

Proposition 4.2.3. *Assume that U is upper semicontinuous and concave on $\mathbb{A}$ with $U \neq -\infty$, and let $\lambda > 0$. Then*

(i) for any $a \in \mathbb{A}$,

$$M_{\lambda U}(a) = U(Prox_{\lambda U}(a)) - \frac{1}{2\lambda}\|a - Prox_{\lambda U}(a)\|_{\mathbb{A}}^2;$$

(ii) denoted by Id the identity operator on $\mathbb{A}$, we have

$$\nabla M_{\lambda U}(a) = -\frac{1}{\lambda}(Id - Prox_{\lambda,-U})(a), \quad \textit{for any } a \in \mathbb{A}.$$

Remark 4.2.4 We emphasize that even $Prox_{\lambda U}$ and U share important direct relationships. In fact, denoted by $Fix(Prox_{\lambda U})$ the set of fixed points of $Prox_{\lambda U}$, i.e. $Fix(Prox_{\lambda U}) = \{a \in \mathbb{A} \text{ s.t. } Prox_{\lambda U}(a) = a\}$, it is immediate to prove

$$Fix(Prox_{\lambda U}) = \operatorname*{Arg\,max}_{x \in \mathbb{A}} U(x);$$

and, moreover

$$Prox_{\lambda U}(a) = (Id + \lambda \partial(-U))^{-1}(a) \quad \text{for any } a \in \mathbb{A},$$

where $\partial(-U)$ denotes the *subdifferential* of the function $-U$ (see [128] and also [8] for definition and properties of the subdifferential correspondence; for further discussion concerning the connections with the proximal operator see [129, 19, 131]).

In optimization literature there exist various algorithms relying on the use of proximal operators (and the Moreau-Yosida regularization, consequently). Such algorithms belong to the class of the so-called *proximal point methods*, and they have been developed with the aim of approximating the solutions of convex optimization problems (for a review of the proximal point methods, see the surveys [67, 56, 125] and the references therein). Now we describe the *proximal point algorithm* introduced by Martinet [99] and Rockafellar in [131], which represents the first and most natural exploitation of the proximal operator and its related properties.
Let us remind that $U \colon \mathbb{A} \to \mathbb{R} \cup \{-\infty\}$ is an extended real-valued function defined on a Euclidean space $\mathbb{A}$ with norm $\|\cdot\|_{\mathbb{A}}$, and let $(\lambda_k)_{k \in \mathbb{N} \cup \{0\}}$ be a sequence of positive real numbers. Fixed $\bar{a}_0 \in \mathbb{A}$, we define recursively the sequence

$(\bar{a}_k)_k \subseteq \mathbb{A}$ in the following way:

$$\begin{aligned} \{\bar{a}_1\} &= \operatorname*{Arg\,max}_{x\in\mathbb{A}} U(x) - \frac{1}{2\lambda_0}\|x-\bar{a}_0\|^2_{\mathbb{A}}; \\ &\vdots \\ \{\bar{a}_k\} &= \operatorname*{Arg\,max}_{x\in\mathbb{A}} U(x) - \frac{1}{2\lambda_{k-1}}\|x-\bar{a}_{k-1}\|^2_{\mathbb{A}}; \\ &\vdots \end{aligned} \tag{4.4}$$

It is immediate to note that $\bar{a}_k = Prox_{\lambda_{k-1}U}(\bar{a}_{k-1})$ for any $k \in \mathbb{N}$, by definition of proximal operator. The convergence of the proximal point algorithm is stated in the next result.

Theorem 4.2.5 (Theorem 4 in [131]). *Assume that*

(i) U *is upper semicontinuous and concave with* $U \neq -\infty$*;*

(ii) $\operatorname{Arg\,max}_{a\in\mathbb{A}} U(a) \neq \emptyset$*;*

(iii) $(\lambda_k)_{k\in\mathbb{N}\cup\{0\}} \subseteq]0,+\infty[$ *and* $\sum_{k=0}^{+\infty} \lambda_k = +\infty$*.*

Let $\bar{a}_0 \in \mathbb{A}$*. Then, the sequence* $(\bar{a}_k)_k$ *generated by the proximal point algorithm defined in* (4.4) *is convergent to a maximizer of* U*, and* $\lim_{k\to+\infty} U(\bar{a}_k) = \max_{a\in\mathbb{A}} U(a)$*.*

Remark 4.2.6 In the papers of Martinet [99] and Rockafellar [131] the convergence of the proximal point algorithm is guaranteed allowing $\mathbb{A}$ to be a real Hilbert space, and in this case the sequence $(\bar{a}_k)_k$ is weakly convergent to a maximizer of U (see [131, Theorem 4]). However, the strong convergence can be guaranteed by adding some extra concavity assumptions which imply the existence of a unique maximizer of U (see, for example, [14, Theorem 27.1]).

It is worth to note that if U is a real-valued function defined on a subset A of $\mathbb{A}$, a proximal point algorithm analogous to the one in (4.4) can be defined (just by replacing $\mathbb{A}$ with A below the "Arg max") in light of the following straightforward result.

Lemma 4.2.7. *Let* U *be a real-valued function defined on* $A \subseteq \mathbb{A}$ *and* $\bar{U}$ *be the extended real-valued function defined on* $\mathbb{A}$ *by*

$$\bar{U}(a) = \begin{cases} U(a), & \text{if } a \in A \\ -\infty, & \text{if } a \notin A. \end{cases}$$

If the function U *is upper semicontinuous and concave on* A*, then* *(i) the function $\bar{U}$ is upper semicontinuous and concave on $\mathbb{A}$;*

(ii) $\operatorname{Arg\,max}_{x \in A} U(x) = \operatorname{Arg\,max}_{x \in \mathbb{A}} \bar{U}(x)$*;*

(iii) $\operatorname{Arg\,max}_{x \in A} U(x) - \frac{1}{2\lambda}\|x - a\|^2_{\mathbb{A}} = \operatorname{Arg\,max}_{x \in \mathbb{A}} \bar{U}(x) - \frac{1}{2\lambda}\|x - a\|^2_{\mathbb{A}}$*, for any $\lambda > 0$ and $a \in \mathbb{A}$.*

Next result follows immediately from Theorem 4.2.5 and Lemma 4.2.7.

Corollary 4.2.8. *Let U be a real-valued function defined on $A \subseteq \mathbb{A}$. Assume that*

(i) A is compact and convex;

(ii) U is upper semicontinuous and concave;

(iii) $(\lambda_k)_{k \in \mathbb{N} \cup \{0\}} \subseteq]0, +\infty[$ and $\sum_{k=0}^{+\infty} \lambda_k = +\infty$.

Let $\bar{a}_0 \in A$. Then, the sequence $(\bar{a}_k)_k$ generated by the proximal point algorithm

$$\begin{aligned} \{\bar{a}_1\} &= \operatorname*{Arg\,max}_{x \in A} U(x) - \frac{1}{2\lambda_0}\|x - \bar{a}_0\|^2_{\mathbb{A}}; \\ &\vdots \\ \{\bar{a}_k\} &= \operatorname*{Arg\,max}_{x \in A} U(x) - \frac{1}{2\lambda_{k-1}}\|x - \bar{a}_{k-1}\|^2_{\mathbb{A}}; \\ &\vdots \end{aligned} \tag{4.5}$$

*which is obtained just by replacing $\mathbb{A}$ with A below the "*Arg max*" in (4.4), is convergent to a maximizer of U on A, and $\lim_{k \to +\infty} U(\bar{a}_k) = \max_{a \in A} U(a)$.*

We have restated the proximal point algorithm and its convergence properties for real-valued functions defined on a compact and convex subset of $\mathbb{A}$ in order to make a clear comparison between the proximal point algorithm in (4.5) and the Tikhonov regularization presented in the previous section.

Firstly, we emphasize that both methods allow to switch from a maximization problem, whose solution is not guaranteed to be unique, to a sequence of regularized problems where the solution is unique at each step. However, the sequences defined by the two method are deeply different: the sequence generated by the proximal point algorithm is recursively defined, so each element of the sequence depends on the preceding one; whereas the sequence of Tikhonov regularized problems $(P_k)_k$ is not defined by recursion.

Regarding the convergence of the two methods, the only difference between the assumptions of Corollary 4.2.8 and Theorem 4.1.3 involves the sequence $(\lambda_k)_k$: the convergence of the proximal point algorithm requires a less restrictive assumption on $(\lambda_k)_k$ (which is allowed, for example, to be a constant sequence, as in [99]) than the corresponding one used in the method based on Tikhonov regularization. This has relevant implications from a numerical point of view: in fact, if $\lim_{k\to+\infty} \lambda_k = +\infty$, the regularizing effect in the Tikhonov regularized problems disappears and the problems P_k will be not easier to solve than the original problem P; instead, requiring just $\sum_{k=0}^{+\infty} \lambda_k = +\infty$ does not imply that the regularizing effect vanishes, so proximal point methods allow to manage better-behaved problems and, moreover, to achieve numerical improvements (see [125, Chapter 4] for further discussion).

Nevertheless, while the limit point of the sequence generated by the solutions of the Tikhonov regularized problems is uniquely determined (being the minimum norm element of the set of maximizer of U), proximal point methods do not provide a similar characterization. However, the limit point of the sequence generated by proximal point algorithms obviously depends on the choice of the initial point $\bar{a}_0$, but just in some very particular circumstances such a limit point can be explicitly characterized in dependence on $\bar{a}_0$ (see [14, Section 5.1]).

4.2.1 Behavioural interpretation: costs to move

After discussing the mathematical derivation and some numerical aspects of proximal methods (included the comparison with Tikhonov regularization), we present now a behavioural interpretation of the proximal operators connected to the costs that agents encounter when deviating from their current actions. The leading idea is that, in real life, changing an action or improving the quality of actions has a cost: as explained by Attouch and Soubeyran in [6] where they aim to model such costs in decision making processes,

> if the agent moves, he must pay physical, physiological, psychological, and cognitive costs of moving from a to some better state a', [6, p.12].

Let $C(a, a')$ be the cost to move from $a \in \mathbb{A}$ to $a' \in \mathbb{A}$, where $\mathbb{A}$ is understood as the set of the alternatives of the agent. This cost can be, in general, decomposed in the following way:

$$C(a, a') = e(a, a')k(a, a'),$$

where $e(a, a')$ is the per unit of distance cost to move (which depends on the alternatives a and a') and $k(a, a')$ is an index of dissimilarity between a and a' (which is related to the distance between the alternatives). In particular, the proximal operator defined in (4.3) involves a special kind of costs to move, called *low local costs to move*:

$$C(a, a') = \underline{\mathrm{e}} \|a - a'\|_{\mathbb{A}}^2,$$

where the per unit of distance cost to move is fixed for all the alternatives, and the index of dissimilarity is the square of the distance between the alternatives embedding the idea that small changes induce very small costs (see [4, 5] for a detailed analysis on the typology of costs to move).

Focusing on the proximal point algorithm defined in (4.4), at the generic step k of the algorithm, the agent chooses his action $\bar{a}_k$ taking into account his previous action $\bar{a}_{k-1}$. In making such a choice, he finds an action that compromises between maximizing U and being near to $\bar{a}_{k-1}$. The latter purpose is motivated according to an *anchoring effect*:

> agents have a (local) vision of their environment which depends on their current actions. Each action is anchored to the preceding one, which means that the perception the agents have of the quality of their subsequent actions depends on the current ones. In economics and management, one may think of actions as routines, ways of doing, while costs to change reflect the difficulty of quitting a routine or entering another one or reacting quickly, [3, p.1066].

Such an anchoring effect is formulated by subtracting a quadratic cost to move that reflects the difficulty of changing the previous action. The coefficient $1/\lambda_{k-1}$ is the per unit of distance cost to move of the agent and represents the trade-off parameter between maximizing U and minimizing the distance from $\bar{a}_{k-1}$. Since the same arguments apply for the preceding steps until going up to the first step of the algorithm, it follows that $\bar{a}_k$ as well as the limit of the sequence $(\bar{a}_k)_k$ embeds the agent's willingness of being near to $\bar{a}_0$.

4.2.2 Applications to the selection of Nash equilibria in normal-form games

Similarly to the analysis made in Subsection 4.1.1, now we examine how the Moreau-Yosida regularization (and, in particular, the proximal point methods) can be used to select a Nash equilibrium in normal-form games. Firstly, we consider the case where the players choose their actions in (unconstrained) Hilbert spaces, reviewing the results of Flåm and Greco [43] and Attouch, Redont and Soubeyran [3]; then, we give some insights in the situation where players' actions are chosen in constrained sets.

Let $\Omega = \{I, (\mathbb{A}_i)_{i\in I}, (U_i)_{i\in I}\}$ be a normal-form game where $I = \{1, \ldots, N\}$ is the set of players, $\mathbb{A}_i$ is the set of actions of player $i \in I$, and $U_i \colon \mathbb{A} \to \mathbb{R} \cup \{-\infty\}$ is the payoff function of player $i \in I$ defined on the set of action profiles $\mathbb{A} \coloneqq \mathbb{A}_1 \times \cdots \times \mathbb{A}_N$. We assume that $\mathbb{A}_i$ is a real Hilbert space, endowed with the inner product $(\cdot,\cdot)_{\mathbb{A}_i}$ and associated norm $\|\cdot\|_{\mathbb{A}_i}$; hence $\mathbb{A}$ is a real Hilbert space endowed with the natural inner product $(\cdot,\cdot)_{\mathbb{A}}$ defined by $(a, a')_{\mathbb{A}} = \sum_{i=1}^{N}(a_i, a'_i)_{\mathbb{A}_i}$ for any $a, a' \in \mathbb{A}$, and associated norm $\|\cdot\|_{\mathbb{A}}$. Let us denote by $E(\Omega)$ the set of all Nash equilibria of Ω and by $E_\epsilon(\Omega)$ the set of all $\epsilon-$*Nash equilibria* of Ω, the latter defined by

$$E_\epsilon(\Omega) = \left\{ \hat{a} \in \mathbb{A} \text{ s.t. } \inf_{\bar{a}\in E(\Omega)} \|\hat{a} - \bar{a}\|_{\mathbb{A}} \le \epsilon \right\},$$

where $\epsilon \ge 0$. Clearly, we have $E_\epsilon(\Omega) \supseteq E(\Omega)$ for any $\epsilon \ge 0$ and $E_0(\Omega) = E(\Omega)$. Following the approach used in [43, Section 2], fixed an initial action profile $\bar{a}_0 = (\bar{a}_{0,1}, \ldots, \bar{a}_{0,N}) \in \mathbb{A}$ and two sequences $(\lambda_k)_{k\in\mathbb{N}\cup\{0\}} \subseteq]0, +\infty[$ and $(\epsilon_k)_{k\in\mathbb{N}} \subseteq [0, +\infty[$, we construct recursively a sequence of regularized normal-form games $(\Omega_k)_k$ and a sequence of ϵ_k-Nash equilibria $(\bar{a}_k)_k$ by means of the proximal operator in the following way:

$$\begin{aligned}
&(S_1) \quad \begin{cases} \Omega_1 = \{I, (\mathbb{A}_i)_{i\in I}, (U_i^1)_{i\in I}\}, \\ \bar{a}_1 = (\bar{a}_{1,1}, \ldots, \bar{a}_{1,N}) \in E_{\epsilon_1}(\Omega_1), \end{cases} \\
&\qquad\quad \vdots \\
&(S_k) \quad \begin{cases} \Omega_k = \{I, (\mathbb{A}_i)_{i\in I}, (U_i^k)_{i\in I}\}, \\ \bar{a}_k = (\bar{a}_{k,1}, \ldots, \bar{a}_{k,N}) \in E_{\epsilon_k}(\Omega_k), \end{cases} \\
&\qquad\quad \vdots
\end{aligned} \tag{4.6}$$

where $U_i^k \colon \mathbb{A} \to \mathbb{R} \cup \{-\infty\}$ is defined on $\mathbb{A}$ by

$$U_i^k(a_i, a_{-i}) = U_i(a_i, a_{-i}) - \frac{1}{2\lambda_{k-1}} \|a_i - \bar{a}_{k-1,i}\|^2_{\mathbb{A}_i}, \tag{4.7}$$

for any $i \in I$ and $k \in \mathbb{N}$. It is worth to note that the player i's payoff function in Ω_k, namely U_i^k in (4.7), is defined by subtracting to the player i's payoff function of Ω a proximal term depending on the equilibrium action chosen by player i in Ω_{k-1}; consequently, the ϵ_k-Nash equilibrium $\bar{a}_k$ reached at step (S_k) is an updating of the $\epsilon_{k-1}-$Nash equilibrium $\bar{a}_{k-1}$ obtained in the preceding step. Such an updating issue does not appear in the sequence of equilibria defined by means of Tikhonov regularization, analyzed in Subsection 4.1.1.

Before showing the convergence result, we recall the notion of monotonicity for set-valued maps (which extends the analogous notion used for functions, see [102, 106] or, for example, [14, Chapter 20]). A set-valued map $K\colon \mathbb{A} \rightrightarrows \mathbb{A}$ is said to be *monotone* if

$$(a - a', b - b')_{\mathbb{A}} \geq 0 \quad \text{for any } (a, b), (a', b') \in gra(K),$$

where $gra(K) = \{(a, b) \in \mathbb{A} \times \mathbb{A} \text{ s.t. } b \in K(a)\}$. If K is a monotone set-valued map, K is said to be *maximal monotone* if there exists no monotone set-valued map $H\colon \mathbb{A} \rightrightarrows \mathbb{A}$ such that $gra(H) \supset gra(K)$.

Theorem 4.2.9 (Theorem 2.1 in [43])**.** *Assume that*

(i) $U_i(\cdot, a_{-i})$ *is upper semicontinuous and concave on* $\mathbb{A}_i$*, for any* $a_{-i} \in \mathbb{A}_{-i}$ *and any* $i \in I$*;*

(ii) the set-valued map $K\colon \mathbb{A} \rightrightarrows \mathbb{A}$ *defined on* $\mathbb{A}$ *by*

$$K(a) = \partial_1(-U_1)(a) \times \cdots \times \partial_N(-U_N)(a),$$

where $\partial_i(-U_i)$ *is the subdifferential of* $-U_i$ *with respect to* a_i*, is maximal monotone on* $\mathbb{A}$*;*

(iii) $\sum_{k=0}^{+\infty} \lambda_k = +\infty$*;*

(iv) $\sum_{k=1}^{+\infty} \epsilon_k < +\infty$*.*

Let $\bar{a}_0 \in \mathbb{A}$*. Then the sequences* $(\Omega_k)_k$ *and* $(\bar{a}_k)_k$ *constructed in* (4.6) *are well-defined and*

- *if* $E(\Omega) = \emptyset$*, then* $\lim_{k\to+\infty} \|\bar{a}_k\|_{\mathbb{A}} = +\infty$*;*
- *if* $E(\Omega) \neq \emptyset$*, then* $(\bar{a}_k)_k$ *is weakly convergent to a Nash equilibrium of* Ω*.*

Sketch of the proof. In this setting, the Nash equilibria of Ω are characterized in terms of zeros of the correspondence K, that is $E(\Omega) = \{\bar{a} \in \mathbb{A} \text{ s.t. } 0 \in K(\bar{a})\}$, since K is nothing but the subdifferential of the vector-valued function $a \in \mathbb{A} \mapsto -(U_1(a), \ldots, U_N(a)) \in \mathbb{A}^N$. Analogously, $\bar{a}_k \in E(\Omega_k) \Leftrightarrow 0 \in \partial_i(-U_i)(\bar{a}_k) - (\bar{a}_{k,i} - \bar{a}_{k-1,i})/2\lambda_{k-1}$ for any $i \in I$, which is equivalent to $\bar{a}_k \in (Id + \lambda_{k-1}K)^{-1}(\bar{a}_{k-1})$. Finally, in light of Remarks 4.2.4 and 4.2.6, it is sufficient to apply Theorem 4.2.5 on the convergence of the proximal point algorithm.

When $\epsilon_k = 0$ for any $k \in \mathbb{N}$ and the action sets are Euclidean spaces, by adding some differentiability assumptions on the payoff functions, from Theorem 4.2.9 we can derive the following selection result for Nash equilibria.

Proposition 4.2.10. *Let $\mathbb{A}_i = \mathbb{R}^{m_i}$ for any $i \in I$ and $\mathbb{A} = \mathbb{R}^m$, with $m = m_1 + \cdots + m_N$. Assume that, for any $i \in I$*

(i) U_i is real-valued and continuously differentiable with respect to a_i on $\mathbb{R}^{m_i}$;

(ii) $U_i(\cdot, a_{-i})$ is concave on $\mathbb{R}^{m_i}$, for any $a_{-i} \in \prod_{j \neq i} \mathbb{R}^{m_j}$;

and that

(iii) the function $J \colon \mathbb{R}^m \to \mathbb{R}^m$ defined by

$$J(a) \coloneqq -(\nabla_1 U_1(a), \ldots, \nabla_N U_N(a))$$

is monotone on $\mathbb{R}^m$;

(iv) $E(\Omega) \neq \emptyset$;

(v) $\sum_{k=0}^{+\infty} \lambda_k = +\infty$.

Let $\bar{a}_0 \in \mathbb{R}^m$. Then the sequence of games $(\Omega_k)_k$ and the sequence of action profiles $(\bar{a}_k)_k$ constructed in the following way:

$$\begin{cases} \Omega_1 = \{I, (\mathbb{R}^{m_i})_{i \in I}, (U_i^1)_{i \in I}\}, \\ \{\bar{a}_1\} = E(\Omega_1), \end{cases}$$
$$\vdots \tag{4.8}$$
$$\begin{cases} \Omega_k = \{I, (\mathbb{R}^{m_i})_{i \in I}, (U_i^k)_{i \in I}\}, \\ \{\bar{a}_k\} = E(\Omega_k), \end{cases}$$
$$\vdots$$

where $U_i^k \colon \mathbb{R}^m \to \mathbb{R}$ is defined on $\mathbb{R}^m$ by

$$U_i^k(a_i, a_{-i}) = U_i(a_i, a_{-i}) - \frac{1}{2\lambda_{k-1}} \|a_i - \bar{a}_{k-1,i}\|_{m_i}^2, \tag{4.9}$$

for any $i \in I$ and $k \in \mathbb{N}$, are well-defined and the sequence $(\bar{a}_k)_k$ is convergent to a Nash equilibrium of Ω.

Proof. Firstly, we show by induction on k that each game of the sequence $(\Omega_k)_k$ has a unique Nash equilibrium. Let $k = 1$. By assumptions (i)-(iii) and the definition of U_i^1 in (4.9), the function $J_1 \colon \mathbb{R}^m \to \mathbb{R}^m$ defined by $J_1(a) = -(\nabla_1 U_1^1(a), \ldots, \nabla_N U_N^1(a))$ for any $a \in \mathbb{R}^m$ is continuous and *strongly monotone* on $\mathbb{R}^m$, i.e. there exists $c > 0$ such that

$$(a' - a'', J_1(a') - J_1(a''))_m \geq c\|a' - a''\|_m^2,$$

for any $a', a'' \in \mathbb{R}^m$. In fact, fixed $b', b'' \in \mathbb{R}^m$, then

$$\begin{aligned}(b'-b'', J_1(b')-J_1(b''))_m &= (b'-b'', J(b')-J(b''))_m \\ &\quad + \sum_{i=1}^{N}\left(b'_i - b''_i, \frac{1}{\lambda_0}\left[(b'_i-\bar{a}_{0,i})-(b''_i-\bar{a}_{0,i})\right]\right)_{m_i} \\ &= (b'-b'', J(b')-J(b''))_m + \frac{1}{\lambda_0}\sum_{i=1}^{N}\|b'_i-b''_i\|^2_{m_i} \\ &\geq \frac{1}{\lambda_0}\|b'-b''\|^2_m.\end{aligned}$$

Therefore, once observed that $E(\Omega_1) = \{\bar{a} \in \mathbb{R}^m \text{ s.t. } J_1(\bar{a}) = 0\}$, the game Ω_1 has a unique Nash equilibrium in light of, for example, [14, Example 22.9]. Hence, the base case is proved. Assume that the result holds for $k > 1$, so Ω_k has a unique Nash equilibrium $\bar{a}_k$. By assumptions (i)-(iii), the definition of U_i^{k+1} in (4.9) and exploiting the same arguments used above, the function $J_{k+1}\colon \mathbb{R}^m \to \mathbb{R}^m$ defined by $J_{k+1}(a) = -(\nabla_1 U_1^{k+1}(a), \ldots, \nabla_N U_N^{k+1}(a))$ for any $a \in \mathbb{R}^m$ is continuous and strongly monotone on $\mathbb{R}^m$. Therefore, Ω_{k+1} has a unique Nash equilibrium by [14, Example 22.9], once observed that $E(\Omega_{k+1}) = \{\bar{a} \in \mathbb{R}^m \text{ s.t. } J_{k+1}(\bar{a}) = 0\}$. Hence, the inductive step is proved and the sequences $(\Omega_k)_k$ and $(\bar{a}_k)_k$ described in (4.8) are well-defined.
Now, note that assumptions (i)-(ii) imply assumption (i) of Theorem 4.2.9 and that the correspondence K defined in the statement of Theorem 4.2.9 is single-valued and $K(a) = \{J(a)\}$ for any $a \in \mathbb{R}^m$. In light of (i) and (iii), the function J is continuous and monotone, so J is maximal monotone on $\mathbb{R}^m$ (see, for example, [14, Corollary 20.25]). Hence even assumption (ii) in Theorem 4.2.9 is satisfied. Finally, by assumption (iv) and applying Theorem 4.2.9 with $\epsilon_k = 0$ for any $k \in \mathbb{N}$, it follows that the sequence $(\bar{a}_k)_k$ converges to a Nash equilibrium of Ω. □

Remark 4.2.11 Assumption (iv) in Proposition 4.2.10 can be dropped by requiring, for example, that J is strongly monotone on $\mathbb{R}^m$. Nevertheless, in this case, Ω has a unique Nash equilibrium and the issue of selection does not arise.

Let us compare the assumptions and the theses of Theorem 4.1.4 and Proposition 4.2.10. Firstly, both results fit our main purpose: they allow to define a selection of Nash equilibrium in normal-form games via the construction of a sequence of regularized games where the Nash equilibrium is unique. Moreover, such a sequence of games (as well as the selection) is obtained by exploiting two different optimization techniques (the Tikhonov regularization in Theorem 4.1.4 and the Moreau-Yosida regularization in Proposition 4.2.10) whose main features and differences have been described after Corollary 4.2.8 and in Subsection 4.2.1. Finally, we highlight that in Theorem 4.1.4 players choose their actions in constrained sets and the assumptions ensure the existence of al least one Nash equilibrium of the original game Ω, whereas in Proposition 4.2.10, where the action sets of the players are unconstrained Euclidean spaces, the non-emptiness of the set of Nash equilibria of Ω must be explicitly assumed.

Let us analyze another result which provides a selection of Nash equilibria for a particular class of normal-form games by means of proximal point algorithm, which is due to Attouch, Redont and Soubeyran [3]. Suppose that Ω is a two-player game, so $I = \{1, 2\}$, that $\mathbb{A}_1$ and $\mathbb{A}_2$ coincide with the same Hilbert space $\mathbb{H}$, and that the payoff functions U_1 and U_2 of the two players are defined on $\mathbb{A} = \mathbb{H} \times \mathbb{H}$ in the following way:

$$\begin{aligned} U_1(a_1, a_2) &= u_1(a_1) - \frac{\alpha}{2}\|a_1 - a_2\|^2_{\mathbb{H}}, \\ U_2(a_1, a_2) &= u_2(a_2) - \frac{\beta}{2}\|a_1 - a_2\|^2_{\mathbb{H}}, \end{aligned} \tag{4.10}$$

where $u_i \colon \mathbb{H} \to \mathbb{R} \cup \{-\infty\}$ for any $i \in \{1, 2\}$ and $\alpha, \beta > 0$. We note that, in light of (4.10) and Proposition 3.1.16, the game Ω is weighted potential game. In [3, Section 4] an *alternating* proximal point algorithm is used to construct a sequence of action profiles $(\bar{a}_{k,1}, \bar{a}_{k,2})_k \subseteq \mathbb{H} \times \mathbb{H}$ as follows: fixed $(\bar{a}_{0,1}, \bar{a}_{0,2}) \in \mathbb{H} \times \mathbb{H}$, let

$$(AlP) \begin{cases} \{\bar{a}_{k,1}\} = \operatorname{Arg\,max}_{a_1 \in \mathbb{H}} u_1(a_1) - \frac{\alpha}{2}\|a_1 - \bar{a}_{k-1,2}\|^2_{\mathbb{H}} - \frac{\mu_{k-1}}{2}\|a_1 - \bar{a}_{k-1,1}\|^2_{\mathbb{H}} \\ \{\bar{a}_{k,2}\} = \operatorname{Arg\,max}_{a_2 \in \mathbb{H}} u_2(a_2) - \frac{\beta}{2}\|a_2 - \bar{a}_{k,1}\|^2_{\mathbb{H}} - \frac{\nu_{k-1}}{2}\|a_2 - \bar{a}_{k-1,2}\|^2_{\mathbb{H}}, \end{cases}$$

for any $k \in \mathbb{N}$, where $(\mu_k)_{k \in \mathbb{N} \cup \{0\}}$ and $(\nu_k)_{k \in \mathbb{N} \cup \{0\}}$ are two sequences of positive real numbers. It is worth to note that by (4.10) and the definition of proximal operator given in (4.3), the pair $(\bar{a}_{k,1}, \bar{a}_{k,2})$ in (AlP) can be rewritten in the more compact way:

$$\begin{aligned} \bar{a}_{k,1} &= Prox_{\frac{1}{\mu_{k-1}} U_1(\cdot, \bar{a}_{k-1,2})}(\bar{a}_{k-1,1}), \\ \bar{a}_{k,2} &= Prox_{\frac{1}{\nu_{k-1}} U_2(\bar{a}_{k,1}, \cdot)}(\bar{a}_{k-1,2}). \end{aligned}$$

In the next result the convergence of the sequence $(\bar{a}_{k,1}, \bar{a}_{k,2})_k$ is stated.

Theorem 4.2.12 (Theorem 4.1 in [3]). *Assume that*

(i) $u_i \colon \mathbb{H} \to \mathbb{R} \cup \{-\infty\}$ *is upper semicontinuous and concave on* $\mathbb{H}$, $u_i \neq -\infty$ *and* $\sup_{a_i \in \mathbb{H}} u_i(a_i) < +\infty$ *for any* $i \in \{1, 2\}$*;*

(ii) $E(\Omega) \neq \emptyset$*;*

(iii) $\lim_{k\to+\infty} \mu_k = \mu > 0$ *and* $\lim_{k\to+\infty} \nu_k = \nu > 0$.

Let $(\bar{a}_{0,1}, \bar{a}_{0,2}) \in \mathbb{H} \times \mathbb{H}$. *Then, the sequence* $(\bar{a}_{k,1}, \bar{a}_{k,2})_k \subseteq \mathbb{H} \times \mathbb{H}$ *generated by* (AlP) *is well-defined and is weakly convergent to a Nash equilibrium of* Ω.

Remark 4.2.13 When $\mathbb{H} = \mathbb{R}$, assumption (ii) in Theorem 4.2.12 can be dropped by requiring, for example, that u_i is differentiable and strongly concave on $\mathbb{R}$ for any $i \in \{1, 2\}$. Nevertheless, in this case, Ω has a unique Nash equilibrium in light of Theorem 3.1.5 and the issue of selection does not arise.

We highlight that, although Theorem 4.2.12 shows the convergence of an algorithm allowing to select a Nash equilibrium by using a (alternating) proximal point algorithm, the sequence of action profiles generated by (AlP), notwithstanding uniquely determined, is not defined as a sequence of Nash equilibria of some regularized game. Hence, the selection of a Nash equilibrium proved in Theorem 4.2.12 is not achieved via a sequence of Nash equilibria of regularized games, as it happens in Proposition 4.2.10 and in Theorem 4.2.9. Moreover, we note that even in Theorem 4.2.12, as well as in Proposition 4.2.10, the non-emptiness of the set of Nash equilibria of Ω is explicitly assumed, since players are allowed to choose their actions in an unconstrained set.

Motivated by the fact that assumptions in the selection result of Theorem 4.2.9 do not guaranteed a priori the existence of at least one Nash equilibrium as well as in Proposition 4.2.10 the non-emptiness of the set of Nash equilibria must be explicitly assumed, now we focus on the situation where player i's action set is a constrained subset A_i of the Euclidean space $\mathbb{R}^{m_i}$ and the payoff function U_i of player i is a real-valued function defined on $A = A_1 \times \cdots \times A_N \subseteq \mathbb{R}^m$ (with $m = m_1 + \cdots + m_N$) for any $i \in I$. Our purpose is to give some insights concerning the possibility of extending the results of Proposition 4.2.10 to the "constrained" game $\Omega = \{I, (A_i)_{i\in I}, (U_i)_{i\in I}\}$: we aim to provide a selection result for Nash equilibria in games having constrained action sets by means of a sequence of proximal-regularized games with a unique Nash equilibrium.
This transition from the unconstrained to the constrained case has been dealt and successfully solved in the optimization framework at the beginning of this section (see Corollary 4.2.8) by using Lemma 4.2.7. Hence, let us investigate whether an analogous result to Lemma 4.2.7 holds also in a game-theoretical framework. Consider the following assumptions on Ω.

Suppose that, for any $i \in I$

(i) A_i *is compact and convex;*

(ii) U_i *is upper semicontinuous on* A*;*

(iii) for any $(a_i, a_{-i}) \in A_i \times A_{-i}$ *and any sequence* $(a_{-i,k})_k \subseteq A_{-i}$ *converging to* a_{-i}*, there exists a sequence* $(\tilde{a}_{i,k})_k \subseteq A_i$ *converging to* a_i *such that*

$$\liminf_{k\to+\infty} U_i(\tilde{a}_{i,k}, a_{-i,k}) \geq U_i(a_i, a_{-i});$$

(iv) $U_i(\cdot, a_{-i})$ *is concave on* A_i*, for any* $a_{-i} \in A_{-i}$*;*

(v) U_i *is continuously differentiable with respect to* a_i *on* A*;*

and, furthermore, that

(vi) the function $J\colon A \to \mathbb{R}^m$ *defined by*

$$J(a) := -(\nabla_1 U_1(a), \ldots, \nabla_N U_N(a))$$

is monotone on A*.*

Denoted with $\bar{\Omega}$ the game obtained from Ω by "extending" the payoff functions of Ω, that is the game $\bar{\Omega} = \{I, (A_i)_{i\in I}, (\bar{U}_i)_{i\in I}\}$ where $\bar{U}_i\colon \mathbb{R}^m \to \mathbb{R} \cup \{-\infty\}$ is defined on $\mathbb{R}^m$ by

$$\bar{U}_i(a) = \begin{cases} U_i(a), & \text{if } a \in A \\ -\infty, & \text{if } a \notin A, \end{cases}$$

for any $i \in I$, let us discuss if assumptions (i)-(vi) on Ω imply the following statements related to $\bar{\Omega}$:

(a) the set-valued map $\bar{J}\colon \mathbb{R}^m \rightrightarrows \mathbb{R}^m$ *defined on* $\mathbb{A}$ *by*

$$\bar{J}(a) = \partial_1(-\bar{U}_1)(a) \times \cdots \times \partial_N(-\bar{U}_N)(a),$$

where $\partial_i(-\bar{U}_i)$ *is the subdifferential of* $-\bar{U}_i$ *with respect to* a_i*, is maximal monotone on* $\mathbb{R}^m$*;*

(b) $E(\Omega) = E(\bar{\Omega})$*.*

Firstly, note that assumptions (i)-(iv) guarantee the existence of at least one Nash equilibrium of Ω, in light of [74, Theorem 2.1]. Moreover, by assumption (v) and the definition of J, we have

$$\bar{J}(a) = \begin{cases} \{J(a)\}, & \text{if } a \in A \\ \emptyset, & \text{if } a \notin A. \end{cases}$$

Hence, the continuity and monotonicity of J on A provided by assumptions (v)-(vi) ensure the maximal monotonicity of $\bar{J}$ on $\mathbb{R}^m$, so (a) is satisfied. If even (b) held, then it would be possible to restate Theorem 4.2.9 and Proposition 4.2.10 for games where the players action sets are compact and convex subsets of Euclidean spaces and without requiring explicitly the non-emptiness of the set of Nash equilibria of Ω. Unfortunately, while the inclusion $E(\Omega) \subseteq E(\bar{\Omega})$ obviously holds, in general $E(\bar{\Omega}) \nsubseteq E(\Omega)$ as illustrated in the following simple example.

Example 4.2.1 Let Ω be the game where $I = \{1, 2\}$, $A_1 = A_2 = [0, 1]$ and $U_1(a_1, a_2) = U_2(a_1, a_2) = c \in \mathbb{R}$ for any $(a_1, a_2) \in [0, 1]^2$. Obviously $E(\Omega) = [0, 1]^2$, while $E(\bar{\Omega}) = E(\Omega) \cup]1, +\infty[^2 \cup] - \infty, 0[^2$. Therefore $E(\Omega) \neq E(\bar{\Omega})$.

However, assumptions (i)-(vi) stated before are sufficient to construct a sequence of regularized game (by means of proximal point algorithm) having a unique Nash equilibrium, as illustrated in the following result. At the moment, the convergence issue of the sequence of Nash equilibria obtained in such a way (as well as the related Nash equilibrium selection issue) is under investigation.

Proposition 4.2.14. *Assume that $\Omega = \{I, (A_i)_{i\in I}, (U_i)_{i\in I}\}$ satisfies hypotheses (i)-(v) for any $i \in I$ and hypothesis (vi) at page 69. Let $(\lambda_k)_{k\in\mathbb{N}\cup\{0\}} \subseteq]0, +\infty[$ and $\bar{a}_0 \in A$. Then the sequence of games $(\Omega_k)_k$ and the sequence of action profiles $(\bar{a}_k)_k$ constructed in the following way:*

$$\begin{cases} \Omega_1 = \{I, (A_i)_{i\in I}, (U_i^1)_{i\in I}\}, \\ \{\bar{a}_1\} = E(\Omega_1), \end{cases}$$

$$\vdots$$

$$\begin{cases} \Omega_k = \{I, (A_i)_{i\in I}, (U_i^k)_{i\in I}\}, \\ \{\bar{a}_k\} = E(\Omega_k), \end{cases}$$

$$\vdots$$

where $U_i^k \colon A \to \mathbb{R}$ is defined on A by

$$U_i^k(a_i, a_{-i}) = U_i(a_i, a_{-i}) - \frac{1}{2\lambda_{k-1}} \|a_i - \bar{a}_{k-1,i}\|_{m_i}^2,$$

for any $i \in I$ and $k \in \mathbb{N}$, are well-defined. In particular Ω_k has a unique Nash equilibrium for any $k \in \mathbb{N}$.

Proof. The non-emptiness of the set of Nash equilibria of Ω is guaranteed by assumptions (i)-(iv), in light of [74, Theorem 2.1]. We show by induction on

k that the sequence $(\bar{a}_k)$ is well-defined. Let $k = 1$ and $J_1 \colon A \to \mathbb{R}^m$ be the function defined on A by

$$J_1(a) = -\left(\nabla_1 U_1^1(a), \dots, \nabla_N U_N^1(a)\right).$$

Then, by definition of J and U_i^1, assumption (vi), and since $\lambda_0 > 0$ we have

$$\begin{aligned}(a' - a'', J_1(a') - J_1(a''))_m &= (a' - a'', J(a') - J(a''))_m \\ &\quad + \sum_{i=1}^{N} \left(a_i' - a_i'', \frac{1}{\lambda_0} \left[(a_i' - \bar{a}_{0,i}) - (a_i'' - \bar{a}_{0,i}) \right] \right)_{m_i} \\ &= (a' - a'', J(a') - J(a''))_m + \frac{1}{\lambda_0} \sum_{i=1}^{N} \|a_i' - a_i''\|_{m_i}^2 \\ &> 0,\end{aligned}$$

for any $a', a'' \in A$ with $a' \neq a''$. Therefore, the function J_1 is strictly monotone on A, so in light of Rosen uniqueness result (Theorem 3.2.2), the game Ω_1 has a unique Nash equilibrium $\bar{a}_1 \in A$ (the existence of such equilibrium holds in light of [74, Theorem 2.1]). Hence, the base case is proved.

Assume that the result holds for $k > 1$, so Ω_k has a unique Nash equilibrium $\bar{a}_k$. Let $J_{k+1} \colon A \to \mathbb{R}^m$ be the function defined on A by

$$J_{k+1}(a) = -\left(\nabla_1 U_1^{k+1}(a), \dots, \nabla_N U_N^{k+1}(a)\right).$$

Then, by definition of J and U_i^{k+1}, assumption (vi), and since $\lambda_k > 0$ we get

$$\begin{aligned}&(a' - a'', J_{k+1}(a') - J_{k+1}(a''))_m \\ &= (a' - a'', J(a') - J(a''))_m + \sum_{i=1}^{N} \left(a_i' - a_i'', \frac{1}{\lambda_k} \left[(a_i' - \bar{a}_{k,i}) - (a_i'' - \bar{a}_{k,i}) \right] \right)_{m_i} \\ &= (a' - a'', J(a') - J(a''))_m + \frac{1}{\lambda_k} \sum_{i=1}^{N} \|a_i' - a_i''\|_{m_i}^2 > 0,\end{aligned}$$

for any $a', a'' \in A$ with $a' \neq a''$. Hence, the function J_{k+1} is strictly monotone on A and in light of Rosen uniqueness result, the game Ω_{k+1} has a unique Nash equilibrium $\bar{a}_{k+1} \in A$ (the existence holds in light of [74, Theorem 2.1]). Therefore the inductive step is proved and the proof is complete. □

We conclude this chapter by mentioning that the idea of using regularizations or perturbations of the data of the game has been broadly exploited in Game Theory especially to define Nash equilibrium refinement concepts in normal-form games (see, for example, [148, 52, 118, 122, 60, 30]). Furthermore, recently the Moreau-Yosida regularization has been used to define a new Nash equilibrium refinement for normal-form games when there is uncertainty related to players' actions (see [15]). Finally, we highlight that algorithms involving proximal point methods for Nash equilibria (and, more generally, for equilibrium problems) in a constrained setting have been widely investigated (see, for example, [42, 115, 116, 55, 64] and references therein). However, we highlight that in the references just mentioned and differently from the analysis presented in this chapter, the assumptions involve *Nikaido-Isoda*-type functions and not directly the players' payoff functions.

Chapter 5

Selection of subgame perfect Nash equilibria

Let us face now the issue of selecting a subgame perfect Nash equilibrium (SPNE) in one-leader N-follower two-stage games (where $N \in \mathbb{N}$) by exploiting the two regularization methods examined in Chapter 4.

In Chapter 2 we showed that, in one-leader one-follower two-stage games, the strong Stackelberg equilibria and the weak Stackelberg equilibria can provide particular selections of SPNEs (analogous arguments hold in the case of $N \geq 2$ followers). Nevertheless, as emphasized at the end of Section 2.2, the SPNEs derived from strong or weak Stackelberg equilibria require the leader of knowing the follower's best reply correspondence and, moreover, they are not achievable as limit point of any algorithmic procedure (as well as for the SPNEs of one-leader N-followers two-stage games deriving from the solutions of the corresponding strong or weak hierarchical Nash equilibrium problems).

In this chapter we introduce two constructive methods in order to select an SPNE with the following features:

(i) they relieve the leader of learning the Nash equilibrium correspondence;

(ii) they allow to overcome the difficulties deriving from the possible non-single-valuedness of the Nash equilibrium correspondence.

Such features are obtained by using the Tikhonov regularization and the proximal point algorithm. In particular, regarding the selection of SPNEs by means of the proximal point algorithm, an additional feature is achieved:

(iii) the constructive method is based on a learning approach which has a behavioral interpretation linked to the costs that players face when they deviate from their current actions.

Firstly, we analyze the constructive method introduced by Morgan and Patrone in [109], where the Tikhonov regularization is exploited for selecting an SPNE in one-leader N-follower two-stage games with $N = 1, 2$ (that fits features (i)-(ii)). Then, referring to Caruso, Ceparano and Morgan [23], we show an SPNE selection method for one-leader one-follower two-stage games based on proximal point algorithm which satisfies even feature (iii), according to the interpretation presented in Subsection 4.2.1.

5.1 Selection of SPNE via Tikhonov regularization

Let us describe in this section the SPNE selection method proposed by Morgan and Patrone in [109], starting from the analysis of case of one follower. Using the same notation of Chapter 2 let $\Gamma = (X, Y, L, F)$ be a one-leader one-follower two-stage game, where X and Y are the action sets of the leader and the follower, respectively, $L\colon X \times Y \to \mathbb{R} \cup \{-\infty\}$ and $F\colon X \times Y \to \mathbb{R} \cup \{-\infty\}$ are the payoff functions of the leader and the follower, respectively, and $\mathcal{M}\colon X \rightrightarrows Y$ denotes the follower's best reply correspondence defined in (2.6). Assume that X is a subset of a Euclidean space $\mathbb{X}$ and that Y is a subset of a Euclidean space $\mathbb{Y}$ equipped with the inner product $(\cdot,\cdot)_{\mathbb{Y}}$ and associated norm $\|\cdot\|_{\mathbb{Y}}$.
As illustrated in Section 2.1, the follower, after having observed the leader's action $x \in X$, faces the optimization problem P_x in (2.4). In [109, Section 3] a regularized optimization problem for the follower is defined by using Tikhonov regularization:

$$P_{x,k}\colon \max_{y \in Y} F(x, y) - \frac{1}{2\lambda_k}\|y\|_{\mathbb{Y}}^2, \tag{5.1}$$

where $k \in \mathbb{N}$ and $\lambda_k > 0$; that is, $P_{x,k}$ is the Tikhonov regularized problem of parameter λ_k related to the function $F(x,\cdot)$, according to the notation used in Section 4.1. Regarding the connections between the solutions of $P_{x,k}$ and P_x, from Theorem 4.1.3 it follows immediately the next result.

Proposition 5.1.1. *Assume that:*

(i) Y *is compact and convex;*

(ii) $F(x,\cdot)$ *is upper semicontinuous and concave on* Y*, for any* $x \in X$*;*

(iii) $(\lambda_k)_k \subseteq]0, +\infty[$ *and* $\lim_{k\to+\infty} \lambda_k = +\infty$*.*

Then, problem $P_{x,k}$ *has a unique solution* $\bar{m}_k(x) \in Y$*, for any* $x \in X$ *and any* $k \in \mathbb{N}$*, and the sequence* $(\bar{m}_k(x))_k$ *is convergent to the (unique) minimum norm element* $\hat{m}(x)$ *of the set* $\mathcal{M}(x)$*, defined in* (2.6)*, for any* $x \in X$*.*

Therefore, under the assumptions of Proposition 5.1.1, it is well-defined the following Stackelberg problem:

$$SP_k\colon \begin{cases} \max_{x\in X} L(x, \bar{m}_k(x)) \\ \text{where } \bar{m}_k(x) \in Y \\ \text{is the unique solution of } P_{x,k} \text{ defined in (5.1).} \end{cases}$$

In the next result, we illustrate the constructive method proposed in [109] to select an SPNE of Γ.

Theorem 5.1.2 (Theorem 3.1 in [109])**.** *Let* $\Gamma = (X, Y, L, F)$ *be a one-leader one-follower two-stage game. Assume that*

(i) X *is compact;*

(ii) Y *is compact and convex;*

(iii) L *is upper semicontinuous on* $X \times Y$*;*

(iv) $L(x, \cdot)$ *is lower semicontinuous on* Y*, for any* $x \in X$*;*

(v) F *is upper semicontinuous on* $X \times Y$*;*

(vi) for any $(x, y) \in X \times Y$ *and any sequence* $(x_k)_k \subseteq X$ *converging to* x*, there exists a sequence* $(\tilde{y}_k)_k \subseteq Y$ *converging to* y *such that*

$$\liminf_{k\to+\infty} F(x_k, \tilde{y}_k) \geq F(x, y);$$

(vii) $F(x, \cdot)$ *is concave on* Y*, for any* $x \in X$*;*

(viii) $(\lambda_k)_k \subseteq]0, +\infty[$ *and* $\lim_{k\to+\infty} \lambda_k = +\infty$*.*

Let $(\bar{x}_k, \bar{y}_k)_k \subseteq X \times Y$ *be the sequence of actions profiles where*

$$\bar{x}_k \textit{ is a solution of } SP_k \quad \textit{and} \quad \bar{y}_k = \bar{m}_k(\bar{x}_k) \textit{ is the solution of } P_{\bar{x}_k,k}.$$

If the sequence $(\bar{x}_k, \bar{y}_k)_k$ *is convergent to* $(\bar{x}, \bar{y}) \in X \times Y$*, then the strategy profile* $(\bar{x}, \bar{m}) \in X \times Y^X$ *where*

$$\bar{m}(x) = \begin{cases} \bar{y}, & \textit{if } x = \bar{x} \\ \hat{m}(x), & \textit{if } x \in X \setminus \{\bar{x}\}, \end{cases}$$

is an SPNE of Γ*.*

Sketch of the proof. Firstly, assumptions (i)-(iii), (v)-(viii) guarantee that SP_k has at least one solution for any $k \in \mathbb{N}$, in light of Proposition 2.2.7. Hence, the sequence $(\bar{x}_k, \bar{y}_k)_k$ is well-defined. Moreover, assumptions (ii), (v)-(vi) ensure that the follower's best reply correspondence $\mathcal{M}$ is closed at $\bar{x}$ (see, for example, the first part of the proof of Proposition 2.2.12), so $\bar{y} = \bar{m}(\bar{x}) \in \mathcal{M}(\bar{x})$ and property (SG1) in Definition 2.2.2 is satisfied (since $\hat{m}(x) \in \mathcal{M}(x)$ for any $x \neq \bar{x}$ by definition of $\hat{m}(x)$). Finally, assumptions (iii)-(iv) and Proposition 5.1.1 prove (SG2) in Definition 2.2.2.

Let us illustrate the features of the constructive method provided in Theorem 5.1.2 to select an SPNE in one-leader one-follower two-stage games by using the Tikhonov regularization.

- Denoted with $F_k\colon X \times Y \to \mathbb{R} \cup \{-\infty\}$ the function defined on $X \times Y$ by

$$F_k(x, y) = F(x, y) - \frac{1}{2\lambda_k} \|y\|_{\mathbb{Y}}^2,$$

and with $\Gamma_k = (X, Y, L, F_k)$ the one-leader one-follower two-stage game obtained from Γ by replacing the follower payoff function F with F_k, in light of Proposition 5.1.1 the follower's best reply correspondence in Γ_k is single-valued, so Γ_k is a classical Stackelberg game whose follower's best reply function is $\bar{m}_k\colon X \to Y$. Hence, Γ is approximated via a sequence of one-leader one-follower two-stage games $(\Gamma_k)_k$ where the follower's best reply correspondences are single-valued and the SPNE of Γ selected according to Theorem 5.1.2 is generated by the limit of a sequence of Stackelberg equilibria $(\bar{x}_k, \bar{y}_k)_k \subseteq X \times Y$, where $(\bar{x}_k, \bar{y}_k)$ is a Stackelberg equilibrium of the Stackelberg problem SP_k (associated to Γ_k). Therefore, the selection method illustrated above allows to overcome the difficulties deriving from the possible non-single-valuedness of the follower's best reply correspondence $\mathcal{M}$ and, by construction, the leader is not demanded to know the follower's best reply correspondence, so such a method fits the goals (i) and (ii) presented in the introduction of this chapter.

- Theorem 5.1.2 guarantees the existence of SPNEs in one-leader one-follower two-stage games regardless of the lower semicontinuity of the follower's best reply correspondence $\mathcal{M}$, as the following example shows.

Example 5.1.1 Let Γ be the game defined in Example 2.2.5. All the assumptions of Theorem 5.1.2 are evidently satisfied since L and F are continuous and F is a bilinear function. Choosing $\lambda_k = k$ for any $k \in \mathbb{N}$, the sequence $(\bar{x}_k, \bar{y}_k)_k$ obtained by applying the method described in Theorem 5.1.2 is convergent to $(1,1)$, in particular, $(\bar{x}_k, \bar{y}_k) = (1,1)$ for any $k \in \mathbb{N}$. Hence, the SPNE selected according to the method is $(1, \bar{m}(\cdot))$, where $\bar{m}\colon [-1,1] \to [-1,1]$ is the function defined on $[-1,1]$ by

$$\bar{m}(x) = \begin{cases} -1, & \text{if } x \in [-1,0[\\ 0, & \text{if } x = 0 \\ 1, & \text{if } x \in]0,1]. \end{cases}$$

Furthermore, as proved in Example 2.2.5, the follower's best reply correspondence $\mathcal{M}$, given in (2.7), is not lower semicontinuous on $[-1,1]$.

Still regarding the existence of SPNEs, we note that even if the sequence $(\bar{x}_k, \bar{y}_k)_k$ is not convergent in $X \times Y$, the compactness of X and Y ensure the existence of a convergent subsequence $(\bar{x}_{k_j}, \bar{y}_{k_j})_j$ whereby an SPNE can be generated in the same way as in Theorem 5.1.2.

- The SPNE constructed via the method presented in Theorem 5.1.2 is, in general, different from the SPNE selections that could be induced by the strong and the weak Stackelberg solutions, respectively, of the strong and the weak Stackelberg problems associated to Γ (see Propositions 2.2.17 and 2.2.18). To investigate this lack of connections, it is sufficient to compare the limit $(\bar{x}, \bar{y})$ of the sequence $(\bar{x}_k, \bar{y}_k)_k$ with the strong and the weak Stackelberg equilibria associated to Γ, as illustrated in the following example.

Example 5.1.2 (Example 3.4 in [109]) Let $\Gamma = (X, Y, L, F)$ where the action sets are $X = [-2,2]$ and $Y = [-1,1]$, and the payoff functions are defined on $[-2,2] \times [-1,1]$ by $L(x,y) = x + y$ and

$$F(x,y) = \begin{cases} -(x+7/4)y, & \text{if } x \in [-2,-7/4[\\ 0, & \text{if } x \in [-7/4, 7/4[\\ -(x-7/4)y, & \text{if } x \in [7/4, 2]. \end{cases}$$

The sequence $(\bar{x}_k, \bar{y}_k)_k$ defined in Theorem 5.1.2 converges to $(7/4, 0)$, which is different both from the strong Stackelberg equilibrium $(7/4, 1)$ and from the weak Stackelberg equilibrium $(2, -1)$ of the strong and weak Stackelberg problems associated to Γ.

Now let $\Gamma_N = (X, Y_1, \ldots, Y_N, L, F_1, \ldots, F_N)$ be a one-leader N-follower two-stage game with $N \geq 2$ and, referring to [109, Section 4], let us illustrate a constructive selection method for SPNE defined by using the Tikhonov regularization. We remind that X and Y_i are the action sets of the leader

and the follower i, respectively, $L\colon X \times Y_1 \times \cdots \times Y_N \to \mathbb{R} \cup \{-\infty\}$ and $F_i\colon X \times Y_1 \times \cdots \times Y_N \to \mathbb{R} \cup \{-\infty\}$ are the payoff functions of the leader and the follower i, respectively, and $\mathcal{N}\colon X \rightrightarrows Y_1 \times \cdots \times Y_N$ denotes the Nash equilibrium correspondence defined in (2.3). Analogously to the case of one follower, we assume that X is a subset of a Euclidean space $\mathbb{X}$ and that Y_i is a subset of a Euclidean space $\mathbb{Y}_i$ equipped with the inner product $(\cdot,\cdot)_{\mathbb{Y}_i}$ and associated norm $\|\cdot\|_{\mathbb{Y}_i}$ for any $i \in I = \{1,\dots,N\}$.
By definition of one-leader N-follower two-stage game, we recall that the followers, after having observed the leader's action $x \in X$, engage in the simultaneous-move game G_x defined in (2.1). Let us consider the following regularized normal-form game

$$G_{x,k} = \{I, (Y_i)_{i\in I}, (F_i^k(x,\cdot))_{i\in I}\}, \tag{5.2}$$

where $F_i^k\colon X \times Y_1 \times \cdots \times Y_N \to \mathbb{R} \cup \{-\infty\}$ is the Tikhonov regularization of F_i with respect to y_i of parameter λ_k, that is the function defined on $X \times Y_1 \times \cdots \times Y_N$ by

$$F_i^k(x, y_i, y_{-i}) = F_i(x, y_i, y_{-i}) - \frac{1}{2\lambda_k}\|y_i\|^2_{\mathbb{Y}_i},$$

with $k \in \mathbb{N}$ and $\lambda_k > 0$, according to the notation used in Subsection 4.1.1. By applying the same arguments of the proof of Theorem 4.1.4, it follows immediately the next result concerning the connections between the set of Nash equilibria of $G_{x,k}$ and G_x (that provides a generalization to N followers of [109, Theorem 4.1], where two followers are considered).

Proposition 5.1.3 (Theorem 4.1 in [109]). *Assume that, for any $i \in I$*

(i) Y_i is compact and convex;

(ii) $F_i(x,\cdot)$ is upper semicontinuous on $Y_1 \times \cdots \times Y_N$ and continuously differentiable with respect to y_i on $Y_1 \times \cdots \times Y_N$, for any $x \in X$;

(iii) $F_i(x,\cdot,y_{-i})$ is concave on Y_i, for any $y_{-i} \in Y_{-i}$ and any $x \in X$;

(iv) for any $(x, y_i, y_{-i}) \in X \times Y_i \times Y_{-i}$ and any sequence $(y_{-i,k})_k \subseteq Y_{-i}$ converging to y_{-i}, there exists a sequence $(\tilde{y}_{i,k})_k \subseteq Y_i$ converging to y_i such that

$$\liminf_{k\to+\infty} F_i(x, \tilde{y}_{i,k}, y_{-i,k}) \geq F_i(x, y_i, y_{-i});$$

and that

(v) for any $x \in X$ and any $(y', y'') \in (Y_1 \times \cdots \times Y_N)^2$ the following inequality is satisfied

$$\sum_{i=1}^{N} (\nabla_i F_i(x,y') - \nabla_i F_i(x,y''), y'_i - y''_i)_{\mathbb{Y}_i} \leq 0.$$

(vi) $(\lambda_k)_k \subseteq]0, +\infty[$ *and* $\lim_{k\to+\infty} \lambda_k = +\infty$.

Then, $G_{x,k}$ *has a unique Nash equilibrium* $\bar{n}_k(x) = (\bar{n}_{k,1}(x), \ldots, \bar{n}_{k,N}(x)) \in Y_1 \times \cdots \times Y_N$*, for any* $x \in X$ *and any* $k \in \mathbb{N}$*. Furthermore, equipped* $\mathbb{Y}_1 \times \cdots \times \mathbb{Y}_N$ *with the norm defined by*

$$\|(y_1, \ldots, y_N)\|^2_{\mathbb{Y}_1 \times \cdots \times \mathbb{Y}_N} = \sum_{i=1}^{N} \|y_i\|^2_{\mathbb{Y}_i},$$

for any $(y_1, \ldots, y_N) \in Y_1 \times \cdots \times Y_N$*, the sequence* $(\bar{n}_k(x))_k$ *is convergent to the (unique) minimum norm element* $\hat{n}(x)$ *of the set* $\mathcal{N}(x)$*, defined in* (2.3)*, for any* $x \in X$*.*

Hence, under the assumptions of Proposition 5.1.3, it is well-defined the following Stackelberg problem:

$$SP_{N,k}\colon \begin{cases} \max_{x\in X} L(x, \bar{n}_{k,1}(x), \ldots, \bar{n}_{k,N}(x)) \\ \text{where } \bar{n}_k = (\bar{n}_{k,1}(x), \ldots, \bar{n}_{k,N}(x)) \in Y_1 \times \cdots \times Y_N \\ \text{is the unique Nash equilibrium of } G_{x,k} \text{ defined in (5.2)}, \end{cases}$$

which plays a key role in the construction of an SPNE selection in Γ_N, as described in the next result (that represents a generalization to N followers of [109, Theorem 4.2], where two followers are considered).

Theorem 5.1.4 (Theorem 4.2 in [109])**.** *Let* Γ_N *be a one-leader* N*-follower two-stage game, with* $N \geq 2$*. Assume that, for any* $i \in I$

(i) Y_i *is compact and convex;*

(ii) F_i *is upper semicontinuous on* $X \times Y_1 \times \cdots \times Y_N$ *and* $F_i(x, \cdot)$ *is continuously differentiable with respect to* y_i *on* $Y_1 \times \cdots \times Y_N$*, for any* $x \in X$*;*

(iii) $F_i(x, \cdot, y_{-i})$ *is concave on* Y_i*, for any* $y_{-i} \in Y_{-i}$ *and any* $x \in X$*;*

(iv) for any $(x, y_i, y_{-i}) \in X \times Y_i \times Y_{-i}$ *and any sequence* $(x_k, y_{-i,k})_k \subseteq X \times Y_{-i}$ *converging to* (x, y_{-i})*, there exists a sequence* $(\tilde{y}_{i,k})_k \subseteq Y_i$ *converging to* y_i *such that*

$$\liminf_{k\to+\infty} F_i(x_k, \tilde{y}_{i,k}, y_{-i,k}) \geq F_i(x, y_i, y_{-i});$$

and that

(v) X *is compact;*

(vi) L *is upper semicontinuous on* $X \times Y_1 \times \cdots \times Y_N$*;*

(vii) $L(x, \cdot)$ *is lower semicontinuous on* $Y_1 \times \cdots \times Y_N$*, for any* $x \in X$*;*

(viii) *for any* $x \in X$ *and any* $(y', y'') \in (Y_1 \times \cdots \times Y_N)^2$ *the following inequality is satisfied*

$$\sum_{i=1}^{N} (\nabla_i F_i(x, y') - \nabla_i F_i(x, y''), y'_i - y''_i)_{\mathbb{Y}_i} \leq 0;$$

(ix) $(\lambda_k)_k \subseteq]0, +\infty[$ *and* $\lim_{k \to +\infty} \lambda_k = +\infty$.

Let $(\bar{x}_k, \bar{y}_k)_k \subseteq X \times Y_1 \times \cdots \times Y_N$ *be the sequence of actions profiles where*

$\bar{x}_k$ *is a solution of* $SP_{N,k}$ *and* $\bar{y}_k = \bar{n}_k(\bar{x}_k)$ *is the Nash equilibrium of* $G_{\bar{x}_k,k}$.

If the sequence $(\bar{x}_k, \bar{y}_k)_k$ *is convergent to* $(\bar{x}, \bar{y}) \in X \times Y_1 \times \cdots \times Y_N$*, then the strategy profile* $(\bar{x}, \bar{n}) \in X \times (Y_1 \times \cdots \times Y_N)^X$ *where*

$$\bar{n}(x) = \begin{cases} \bar{y}, & \text{if } x = \bar{x} \\ \hat{n}(x), & \text{if } x \in X \setminus \{\bar{x}\}, \end{cases}$$

is an SPNE of Γ_N.

Sketch of the proof. Firstly, assumptions (i)-(vi), (viii)-(ix) guarantee that $SP_{N,k}$ has at least one solution for any $k \in \mathbb{N}$, in light of Proposition 2.2.8. Hence, the sequence $(\bar{x}_k, \bar{y}_k)_k$ is well-defined. Moreover, assumptions (ii), (iv) ensure that the Nash equilibrium correspondence $\mathcal{N}$ is closed at $\bar{x}$ (see [74, Theorem 3.1]). The thesis follows by arguing as in Theorem 5.1.2 and, finally, by using assumptions (vi)-(vii) and Proposition 5.1.3.

We emphasize that the SPNE selection method for one-leader N-follower two-stage games (with $N \geq 2$) presented in Theorem 5.1.4 displays analogous features to one illustrated in Theorem 5.1.2. In fact, it allows to construct a sequence of one-leader N-follower two-stage games where the Nash equilibrium correspondences are single-valued, via the Tikhonov regularization, and to approach an SPNE by using the limit of the sequence of Stackelberg equilibria related the Stackelberg problems $(SP_{N,k})_k$ (which are associated to the sequence of "regularized" one-leader N-follower two-stage games constructed). Moreover, even in this case, the leader is not required to know the Nash equilibrium correspondence $\mathcal{N}$, hence the selection method defined in Theorem 5.1.4 satisfies both goals (i) and (ii) illustrated in the introduction of this chapter. Finally, analogously to the case of one follower, Theorem 5.1.4 provides also an existence result for SPNEs in one-leader N-follower two-stage games that does not require the Nash equilibrium correspondence $\mathcal{N}$ to be lower semicontinuous, and the SPNE achieved via Theorem 5.1.4 is not connected, in general, with the

SPNEs obtainable from the solutions of the strong or the weak hierarchical Nash equilibrium problem associated to Γ_N (defined at the end of Subsection 2.2.2).

We conclude by reminding that regularization methods has been already used to tackle the problem of non-single-valuedness of the follower's best reply correspondence both in strong Stackelberg problems (see [35] where the regularization of Molodtsov [103] is involved) and in weak Stackelberg problems (see [85], [88] and [89] where the regularizations of Solohovic [139], Tikhonov and Molodtsov are exploited, respectively). Furthermore, we mention that an approximation scheme involving sequences of simple functions has been used in [21] to show the existence of ϵ-SPNEs in games of perfect information in continuous setting.

5.2 Selection of SPNE via Moreau-Yosida regularization in one-leader one-follower two-stage games

Throughout this section we consider a one-leader one-follower two-stage game $\Gamma = (X, Y, L, F)$, where X and L are the set of actions and the payoff function of the leader, respectively, and Y and F are the set of actions and the payoff function of the follower, respectively, with L and F real-valued functions defined on $X \times Y$. Referring to Caruso, Ceparano and Morgan [23], we introduce a constructive method in order to select an SPNE of Γ by using a learning approach based on the proximal point algorithm (linked to the Moreau-Yosida regularization, analyzed in Section 4.2) with the following features:

(i) it relieves the leader of learning the follower's best reply correspondence $\mathcal{M}\colon X \rightrightarrows Y$;

(ii) it allows to overcome the difficulties deriving from the possible non-single-valuedness of $\mathcal{M}$;

(iii) it has a behavioral interpretation linked to the costs that players face when deviating from their current actions,

according to the goals highlighted in the introduction of this chapter.

5.2.1 Constructive procedure and interpretation

Before presenting a constructive procedure to select an SPNE which satisfies the features stated above, we define a class of games for which such an SPNE is achievable through the just mentioned procedure.

Definition 5.2.1 A one-leader one-follower two-stage game $\Gamma = (X, Y, L, F)$ belongs to the family $\mathcal{G}$ if the following assumptions are satisfied:

($\mathcal{A}1$) X is a compact subset of the Euclidean space $\mathbb{X}$, with norm $\|\cdot\|_{\mathbb{X}}$;

($\mathcal{A}2$) Y is a compact and convex subset of the Euclidean space $\mathbb{Y}$, with norm $\|\cdot\|_{\mathbb{Y}}$;

($\mathcal{L}1$) L is upper semicontinuous on $X \times Y$;

($\mathcal{L}2$) $L(x, \cdot)$ is lower semicontinuous on Y, for any $x \in X$;

($\mathcal{F}1$) F is upper semicontinuous on $X \times Y$;

($\mathcal{F}2$) for any $(x, y) \in X \times Y$ and for any sequence $(x_k)_k \subseteq X$ converging to x there exists a sequence $(\tilde{y}_k)_k \subseteq Y$ converging to y such that

$$\liminf_{k \to +\infty} F(x_k, \tilde{y}_k) \geq F(x, y);$$

($\mathcal{F}3$) $F(x, \cdot)$ is concave on Y, for any $x \in X$.

Remark 5.2.2 Assumptions ($\mathcal{F}1$)-($\mathcal{F}2$) have implications in term of epiconvergence or Γ-convergence (see, for example, [2, 29]). Indeed, let $x \in X$ and let $(x_k)_k \subseteq X$ be a sequence converging to x and consider the following real-valued functions defined on Y by

$$W_k(y) = F(x_k, y), \text{ for any } k \in \mathbb{N},$$
$$W(y) = F(x, y).$$

Then the sequence of functions $(W_k)_k$ Γ^+-converges to W (that is, $(-W_k)_k$ epiconverges to $-W$).

In the following remarks some properties of the family $\mathcal{G}$ are stated. The main computations for the first remark are provided in Subsection 5.2.4, while the proofs related to the second remark can be obtained by using Γ-convergence results (see, for example, Propositions 6.21 and 6.16 in [29]).

Remark 5.2.3 Requiring ($\mathcal{F}1$)-($\mathcal{F}3$) is weaker than requiring the continuity of F. Indeed, the function F defined on $X \times Y$, where $X = [1,2]$ and $Y = \overline{B}_{((1,0),1)}$ (i.e. Y is the closed ball in $\mathbb{R}^2$ centered in $(1,0)$ with radius 1), by

$$F(x,(y_1,y_2)) = \begin{cases} -\frac{y_2^2}{2y_1}x, & \text{if } (y_1,y_2) \neq (0,0) \\ 0, & \text{if } (y_1,y_2) = (0,0) \end{cases}$$

satisfies ($\mathcal{F}1$)-($\mathcal{F}3$) but $F(x,\cdot)$ is not lower semicontinuous at $(0,0)$, for any $x \in [1,2]$. The main computations for this result are provided in Subsection 5.2.4.

Remark 5.2.4 Assume $(X,Y,U,V) \in \mathcal{G}$ and $(X,Y,\hat{U},\hat{V}) \in \mathcal{G}$.

(i) The game $(X,Y,hU,kV) \in \mathcal{G}$ for any $h,k \geq 0$.

(ii) If $\hat{V}$ is continuous, then the game $(X,Y,U+\hat{U},V+\hat{V}) \in \mathcal{G}$.

(iii) If Ψ and Φ are real-valued functions defined on $\mathbb{R}$ with Ψ continuous and Φ increasing and concave, then the game $(X,Y,\Psi \circ U, \Phi \circ V) \in \mathcal{G}$.

The method we use to select an SPNE of Γ relies on the Costs to Move Procedure ($\mathcal{CM}$) illustrated below, which allows to construct recursively a sequence of perturbed games $(\Gamma_n)_n$ and a sequence of strategy profiles $(\bar{x}_n, \varphi_n)_n \subseteq X \times Y^X$, by means of the proximal point algorithm.

Procedure **($\mathcal{CM}$)**

Fix an initial point $(\bar{x}_0,\bar{y}_0) \in X \times Y$ and define for any $n \in \mathbb{N}$

$$(\mathcal{S}_n) \quad \begin{cases} \Gamma_n = (X,Y,L_n,F_n) \\ \{\varphi_n(x)\} = \operatorname{Arg\,max}_{y \in Y} F_n(x,y), \text{ for any } x \in X \\ \bar{x}_n \in \operatorname{Arg\,max}_{x \in X} L_n(x,\varphi_n(x)) \end{cases}$$

where for any $(x,y) \in X \times Y$

$$F_n(x,y) := F(x,y) - \frac{1}{2\gamma_{n-1}}\|y-\varphi_{n-1}(x)\|_{\mathbb{Y}}^2$$
$$L_n(x,y) := L(x,y) - \frac{1}{2\beta_{n-1}}\|x-\bar{x}_{n-1}\|_{\mathbb{X}}^2,$$

with $(\gamma_n)_{n \in \mathbb{N}\cup\{0\}} \subseteq]0,+\infty[$ and $\lim_{n\to+\infty}\gamma_n = +\infty$,
$(\beta_n)_{n \in \mathbb{N}\cup\{0\}} \subseteq]0,+\infty[$ and $\lim_{n\to+\infty}\beta_n = +\infty$,
and $\varphi_0(x) := \bar{y}_0$ for any $x \in X$.

Procedure ($\mathcal{CM}$) is well-defined when $F_n(x,\cdot)$ has a unique maximizer on Y, for any $x \in X$ and for any $n \in \mathbb{N}$, and when $L_n(\cdot,\varphi_n(\cdot))$ admits a maximizer on X, for any $n \in \mathbb{N}$. For the class of games introduced in Definition 5.2.1 such properties are satisfied, as it is proved in the next proposition.

Proposition 5.2.5 (Proposition 1 in [23]). *Assume that $\Gamma \in \mathcal{G}$. Then, Procedure ($\mathcal{CM}$) is well-defined and φ_n is a continuous function on X, for any $n \in \mathbb{N}$.*

Proof. We prove the result by induction on n. Let $n = 1$. By Remark 5.2.4(ii), $\Gamma_1 \in \mathcal{G}$. Moreover $F_1(x, \cdot)$ is strictly concave for any $x \in X$, therefore $\varphi_1(x)$ is well-defined and the follower's best reply correspondence in Γ_1 is single-valued. Since $\Gamma_1 \in \mathcal{G}$, in particular

(a_1) F_1 is upper semicontinuous on $X \times Y$,

(b_1) for any $(x, y) \in X \times Y$ and for any sequence $(x_k)_k \subseteq X$ converging to x, there exists a sequence $(\tilde{y}_k)_k \subseteq Y$ converging to y such that

$$\liminf_{k \to +\infty} F_1(x_k, \tilde{y}_k) \geq F_1(x, y).$$

Conditions (a_1) and (b_1) are sufficient to guarantee that $\lim_{k \to +\infty} \varphi_1(x_k) = \varphi_1(x)$ for any sequence $(x_k)_k$ converging to x, i.e. that φ_1 is continuous (see, for example, [107, Proposition 5.1]). This fact and the upper semicontinuity of L_1 ensure that $\bar{x}_1$ is well-defined. Hence, the base case is proved.
Assume that the result holds for $n > 1$, so the strategy profile $(\bar{x}_n, \varphi_n)$ is well-defined and φ_n is a continuous function. In light of Remark 5.2.4(ii), $\Gamma_{n+1} \in \mathcal{G}$ since φ_n is continuous. Furthermore $F_{n+1}(x, \cdot)$ is strictly concave for any $x \in X$, so $\varphi_{n+1}(x)$ is well-defined and φ_{n+1} is the follower's best reply function in Γ_{n+1}. As $\Gamma_{n+1} \in \mathcal{G}$, then

(a_{n+1}) F_{n+1} is upper semicontinuous on $X \times Y$,

(b_{n+1}) for any $(x, y) \in X \times Y$ and for any sequence $(x_k)_k \subseteq X$ converging to x, there exists a sequence $(\tilde{y}_k)_k \subseteq Y$ converging to y such that

$$\liminf_{k \to +\infty} F_{n+1}(x_k, \tilde{y}_k) \geq F_{n+1}(x, y).$$

By (a_{n+1}) and (b_{n+1}) it follows that φ_{n+1} is continuous (again in light of, for example, [107, Proposition 5.1]). Hence $\bar{x}_{n+1}$ is well-defined, since L_{n+1} is upper semicontinuous. So the inductive step is proved and the proof is complete. □

Remark 5.2.6 Note that assumption ($\mathcal{L}2$) in the definition of the family $\mathcal{G}$ (i.e., the lower semicontinuity of $L(x, \cdot)$ for any $x \in X$) is unnecessary in the proof of Proposition 5.2.5. We assumed $\Gamma \in \mathcal{G}$ in the proposition only for simplicity of exposition.

In the proof of Proposition 5.2.5 we showed that the follower's best reply correspondence in Γ_n is single-valued, i.e., Γ_n is a classical Stackelberg game. Moreover, the follower's best reply function φ_n in Γ_n is continuous and the strategy profile $(\bar{x}_n, \varphi_n)$ is an SPNE of Γ_n, for any $n \in \mathbb{N}$. Hence, Procedure $(\mathcal{CM})$ allows to define a perturbation of the game Γ consisting of the sequence of one-leader one-follower two-stage games $(\Gamma_n)_n$ where the follower's best reply correspondence is single-valued, and to construct a sequence of SPNEs related to such a perturbation. We point out that the payoff functions of both players in Γ_n are obtained by subtracting to the payoff functions of Γ a quadratic term depending on the SPNE reached in Γ_{n-1}. Consequently $(\bar{x}_n, \varphi_n)$, SPNE of Γ_n, is an update of $(\bar{x}_{n-1}, \varphi_{n-1})$, SPNE of Γ_{n-1}, and it is constructed by using a parametric proximal point algorithm, since

$$\varphi_n(x) = Prox_{\gamma_{n-1}F(x,\cdot)}(\varphi_{n-1}(x)),$$

for any $x \in X$ and any $n \in \mathbb{N}$, according to the definition of proximal point operator illustrated in Section 4.2.

Regarding the behavioural interpretation of Procedure $(\mathcal{CM})$, issues related to the costs that players face when deviate from their current actions occur, similarly to what illustrated in Subsection 4.2.1. In fact, at the generic step $(\mathcal{S}_n)$ of the procedure, the follower chooses his strategy φ_n taking into account his previous strategy φ_{n-1}. In making such a choice, he finds an action that compromises between maximizing $F(x,\cdot)$ and being near to $\varphi_{n-1}(x)$, for any $x \in X$. The latter purpose is motivated according to the *anchoring effect* explained by Attouch, Redont and Soubeyran in [3, p.1066] (and quoted in Subsection 4.2.1). Such an anchoring effect is formulated by subtracting a quadratic slight cost to move that reflects the difficulty of changing the previous action. The coefficient γ_{n-1} is linked to the per unit of distance cost to move of the follower and it is related to the trade-off parameter between maximizing $F(x,\cdot)$ and minimizing the distance from $\varphi_{n-1}(x)$. Since the same arguments apply for the preceding steps until going up to step $(\mathcal{S}_1)$, it follows that $\varphi_n(x)$ as well as the limit of $\varphi_n(x)$ embeds the willingness of being near to $\bar{y}_0$. Analogous observations hold also for the leader, who chooses an action having in mind to be near to his previous choices, and therefore even with the purpose of being near to $\bar{x}_0$.

In the next proposition, we prove that the limit of the sequence $(\varphi_n)_n$ is a selection of the follower's best reply correspondence. The pointwise convergence of $(\varphi_n)_n$ is obtained by adapting to a parametric optimization context the classical results about the convergence of the proximal point algorithm. Before showing the result, we state the following lemma.

Lemma 5.2.7. *Let G be a real-valued function defined on $X \times Y$ and $\bar{G}$ be the extended real-valued function defined on $X \times \mathbb{Y}$ by*

$$\bar{G}(x,y) = \begin{cases} G(x,y), & \text{if } y \in Y \\ -\infty, & \text{if } y \notin Y. \end{cases}$$

Let $x \in X$. If the function $G(x,\cdot)$ is upper semicontinuous and concave on Y, then

(i) the function $\bar{G}(x,\cdot)$ is upper semicontinuous and concave on $\mathbb{Y}$;

(ii) $\operatorname{Arg\,max}_{y \in Y} G(x,y) = \operatorname{Arg\,max}_{y \in \mathbb{Y}} \bar{G}(x,y)$*;*

(iii) $\operatorname{Arg\,max}_{y \in Y} G(x,y) - \frac{1}{2\lambda}\|y - v\|_{\mathbb{Y}}^2 = \operatorname{Arg\,max}_{y \in \mathbb{Y}} \bar{G}(x,y) - \frac{1}{2\lambda}\|y - v\|_{\mathbb{Y}}^2$*, for any $\lambda > 0$ and $v \in \mathbb{Y}$.*

(iv) $\varphi^*(x) \in \operatorname{Arg\,max}_{y \in Y} G(x,y) \iff \{\varphi^*(x)\} = \operatorname{Arg\,max}_{y \in Y} G(x,y) - \frac{1}{2\lambda}\|y - \varphi^*(x)\|_{\mathbb{Y}}^2$*, for any $\lambda > 0$.*

Proof. Claims (i)-(iii) follow immediately from Lemma 4.2.7, the proof of claim (iv) comes from Remark 4.2.4, taking into account claims (i)-(iii). □

Proposition 5.2.8 (Proposition 2 in [23])**.** *Assume that ($\mathcal{A}2$), ($\mathcal{F}1$) and ($\mathcal{F}3$) hold. Then the sequence $(\varphi_n)_n$ pointwise converges to a function $\varphi \in Y^X$ and $\varphi(x) \in \mathcal{M}(x)$ for any $x \in X$, where $\mathcal{M}(x) = \operatorname{Arg\,max}_{y \in Y} F(x,y)$.*

Proof. Let $x \in X$. By assumptions ($\mathcal{F}1$) and ($\mathcal{F}3$) and Lemma 5.2.7(i), the function $\bar{F}(x,\cdot)$, where $\bar{F}$ is defined on $X \times \mathbb{Y}$ by

$$\bar{F}(x,y) = \begin{cases} F(x,y), & \text{if } y \in Y \\ -\infty, & \text{if } y \notin Y, \end{cases}$$

is upper semicontinuous and concave and is not identically $-\infty$. Moreover, in light of Lemma 5.2.7(ii), the compactness of Y and assumption ($\mathcal{F}1$),

$$\operatorname*{Arg\,max}_{y \in \mathbb{Y}} \bar{F}(x,y) \neq \emptyset.$$

Given the above, and since $\lim_{n \to +\infty} \gamma_n = +\infty$ with $(\gamma_n)_{n \in \mathbb{N} \cup \{0\}} \subseteq]0, +\infty[$, the function $\bar{F}(x,\cdot)$ satisfies the hypotheses for the convergence of proximal point algorithm stated in Theorem 4.2.5. Then, the sequence $(z_n)_n$ defined by

$$\{z_n\} = \operatorname*{Arg\,max}_{y \in \mathbb{Y}} \bar{F}(x,y) - \frac{1}{2\gamma_{n-1}}\|y - z_{n-1}\|_{\mathbb{Y}}^2 \quad \text{for any } n \in \mathbb{N},$$

where $z_0 := \bar{y}_0$, converges to a point in $\operatorname{Arg\,max}_{y \in \mathbb{Y}} \bar{F}(x,y)$. Since the unique maximizer of $\bar{F}(x,\cdot) - \frac{1}{2\gamma_{n-1}}\|\cdot - \varphi_{n-1}(x)\|_{\mathbb{Y}}^2$ over $\mathbb{Y}$ coincides with the unique maximizer of $F(x,\cdot) - \frac{1}{2\gamma_{n-1}}\|\cdot - \varphi_{n-1}(x)\|^2_{\mathbb{Y}}$ over Y in light of Lemma 5.2.7(iii), then $z_n = \varphi_n(x)$ for any $n \in \mathbb{N}$. Furthermore, since the set of maximizers of $\bar{F}(x,\cdot)$ over $\mathbb{Y}$ coincides with the set of maximizers of $\bar{F}(x,\cdot)$ over Y in light of Lemma 5.2.7(ii), sequence $(\varphi_n(x))_n$ converges to a maximizer of $F(x,\cdot)$ over Y. Hence, the function φ that associates with each $x \in X$ the point $\varphi(x) \coloneqq \lim_{n\to+\infty} \varphi_n(x) \in Y$ is well-defined and $\varphi(x) \in \mathcal{M}(x)$ for any $x \in X$. □

5.2.2 SPNE selection result

The next theorem provides an existence result of an SPNE selection achievable via Procedure $(\mathcal{CM})$ for $\Gamma = (X, Y, L, F) \in \mathcal{G}$. Recall that $(\bar{x}_n, \varphi_n)_n$ is the sequence of strategy profiles generated by Procedure $(\mathcal{CM})$, which is well-defined in light of Proposition 5.2.5.

Theorem 5.2.9 (Theorem 1 in [23]). *Assume that $\Gamma \in \mathcal{G}$ and that the sequence of action profiles $(\bar{x}_n, \varphi_n(\bar{x}_n))_n \subseteq X \times Y$ converges to $(\bar{x}, \bar{y}) \in X \times Y$. Then the strategy profile $(\bar{x}, \bar{\varphi}) \in X \times Y^X$, where*

$$\bar{\varphi}(x) \coloneqq \begin{cases} \bar{y}, & \text{if } x = \bar{x} \\ \lim_{n\to+\infty} \varphi_n(x), & \text{if } x \neq \bar{x}, \end{cases}$$

is an SPNE of Γ.

Proof. We start to prove property (SG1) of Definition 2.2.2. Let $x \in X$ and $\varphi(x) = \lim_{n\to+\infty} \varphi_n(x)$, as defined in Proposition 5.2.8. If $x \neq \bar{x}$, Proposition 5.2.8 ensures that $\bar{\varphi}(x) = \varphi(x) \in \mathcal{M}(x)$. If $x = \bar{x}$, pick $y \in Y$. By assumption $(\mathcal{F}2)$, there exists a sequence $(\tilde{y}_n)_n$ converging to y such that

$$\liminf_{n\to+\infty} F(\bar{x}_n, \tilde{y}_n) \geq F(\bar{x}, y). \tag{5.3}$$

By $(\mathcal{F}1)$ we have:

$$\begin{aligned} F(\bar{x}, \bar{y}) &\geq \limsup_{n\to+\infty} F(\bar{x}_n, \varphi_n(\bar{x}_n)) \\ &= \limsup_{n\to+\infty} \left[F(\bar{x}_n, \varphi_n(\bar{x}_n)) - \frac{1}{2\gamma_{n-1}} \|\varphi_n(\bar{x}_n) - \varphi_{n-1}(\bar{x}_n)\|^2_{\mathbb{Y}} \right] \\ &= \limsup_{n\to+\infty} F_n(\bar{x}_n, \varphi_n(\bar{x}_n)), \end{aligned} \tag{5.4}$$

where the first equality holds since the second addend in the lim sup converges to 0 being $(\gamma_n)_{n\in\mathbb{N}\cup\{0\}}$ a divergent sequence of positive real numbers and Y a compact set, and the second equality comes from the definition of F_n in

Procedure $(\mathcal{CM})$. By the definition of $\varphi_n(\bar{x}_n)$ we get

$$\begin{aligned}\limsup_{n\to+\infty} F_n(\bar{x}_n,\varphi_n(\bar{x}_n)) &\geq \limsup_{n\to+\infty} F_n(\bar{x}_n,\tilde{y}_n)\\ &= \limsup_{n\to+\infty}\left[F(\bar{x}_n,\tilde{y}_n)-\frac{1}{2\gamma_{n-1}}\|\tilde{y}_n-\varphi_{n-1}(\bar{x}_n)\|_{\mathbb{Y}}^2\right].\end{aligned} \tag{5.5}$$

Recalling the properties of $(\gamma_n)_{n\in\mathbb{N}\cup\{0\}}$ and the compactness of Y, by (5.3)-(5.5) we have

$$\begin{aligned}F(\bar{x},\bar{y}) &\geq \limsup_{n\to+\infty}\left[F(\bar{x}_n,\tilde{y}_n)-\frac{1}{2\gamma_{n-1}}\|\tilde{y}_n-\varphi_{n-1}(\bar{x}_n)\|_{\mathbb{Y}}^2\right]\\ &= \limsup_{n\to+\infty} F(\bar{x}_n,\tilde{y}_n) \geq \liminf_{n\to+\infty} F(\bar{x}_n,\tilde{y}_n) \geq F(\bar{x},y).\end{aligned}$$

Hence, $\bar{y}\in\mathcal{M}(\bar{x})$ and (SG1) is satisfied.

In order to prove condition (SG2) in Definition 2.2.2, we have to show that $L(\bar{x},\bar{y})\geq L(x,\bar{\varphi}(x))$ for any $x\in X$. So, let $x\in X\setminus\{\bar{x}\}$. In light of $(\mathcal{L}1)$ we get

$$\begin{aligned}L(\bar{x},\bar{y}) &\geq \limsup_{n\to+\infty} L(\bar{x}_n,\varphi_n(\bar{x}_n))\\ &= \limsup_{n\to+\infty}\left[L(\bar{x}_n,\varphi_n(\bar{x}_n))-\frac{1}{2\beta_{n-1}}\|\bar{x}_n-\bar{x}_{n-1}\|_{\mathbb{X}}^2\right]\\ &\geq \limsup_{n\to+\infty}\left[L(x,\varphi_n(x))-\frac{1}{2\beta_{n-1}}\|x-\bar{x}_{n-1}\|_{\mathbb{X}}^2\right]\\ &\geq \liminf_{n\to+\infty}\left[L(x,\varphi_n(x))-\frac{1}{2\beta_{n-1}}\|x-\bar{x}_{n-1}\|_{\mathbb{X}}^2\right]\\ &= \liminf_{n\to+\infty} L(x,\varphi_n(x)) \geq L(x,\varphi(x))\end{aligned}$$

where the first (resp. second) equality holds since the second addend in the lim sup (resp. lim inf) converges to 0 being $(\beta_n)_{n\in\mathbb{N}\cup\{0\}}$ a divergent sequence of positive real numbers and X a compact set, the second inequality comes from the definition of $\bar{x}_n$ in Procedure $(\mathcal{CM})$, and the last inequality follows by $(\mathcal{L}2)$. As $x\in X\setminus\{\bar{x}\}$, then $L(x,\varphi(x))=L(x,\bar{\varphi}(x))$ and, therefore, $L(\bar{x},\bar{y})\geq L(x,\bar{\varphi}(x))$. Hence (SG2) holds, and the proof is complete. □

Remark 5.2.10 A selection result for SPNEs analogous to Theorem 5.2.9 can be obtained if the leader's payoff function is not modified in Procedure $(\mathcal{CM})$ by subtracting the quadratic proximal term, that is if the costs to move only concern the follower stage (i.e., $L_n=L$, for any $n\in\mathbb{N}$).

We emphasize that the SPNE selection method illustrated in Theorem 5.2.9, based on Procedure $(\mathcal{CM})$, applies (by construction) also in situations where the follower's best reply correspondence $\mathcal{M}$ is not known. Hence, the learning approach to select an SPNE shown in Theorem 5.2.9 relieves the leader of knowing the follower's best reply correspondence and so, in light of what discussed

in Subsection 5.2.1 regarding the features and the interpretation of Procedure $(\mathcal{CM})$, the SPNE selection method defined in this section fulfills all the properties described at the beginning of Section 5.2.
In the following remarks, issues concerning the dependence on the initial point $(\bar{x}_0, \bar{y}_0)$ of the SPNE selected, the pointwise limit of $(\varphi_n)_n$, and the lower semicontinuity of $\mathcal{M}$ are discussed, and related examples are provided.

Remark 5.2.11 The SPNE selected according to Theorem 5.2.9 is affected, in general, by the choice of the initial point $(\bar{x}_0, \bar{y}_0)$ in Procedure $(\mathcal{CM})$: in fact, such an SPNE reflects both the leader's willingness of being near to $\bar{x}_0$ and the follower's willingness of being near to $\bar{y}_0$, as discussed in the interpretation of the procedure in Subsection 5.2.1. The next example, whose main computations are provided in Subsection 5.2.4, emphasizes this dependence especially from the follower's perspective, whereas in Examples 5.2.2 and 5.2.3 these insights are more evident also from the leader's point of view.

Example 5.2.1 Let $\Gamma = (X, Y, L, F)$ be the one-leader one-follower two-stage game where $X = Y = [-1, 1]$ and

$$L(x, y) = x, \qquad F(x, y) = -xy.$$

The follower's best reply correspondence $\mathcal{M}$ is defined on $[-1, 1]$ by

$$\mathcal{M}(x) = \begin{cases} \{1\}, & \text{if } x \in [-1, 0[\\ [-1, 1], & \text{if } x = 0 \\ \{-1\}, & \text{if } x \in]0, 1]. \end{cases} \tag{5.6}$$

Let $(\bar{x}_0, \bar{y}_0) \in [-1, 1] \times [-1, 1]$ be the initial point of the procedure and let $\beta_n = \gamma_n = 2^n$ for any $n \in \mathbb{N} \cup \{0\}$. Then Procedure $(\mathcal{CM})$ generates the following sequence $(\bar{x}_n, \varphi_n)_n$ of strategy profiles:

$$\bar{x}_n = \begin{cases} \min\{1 + \bar{x}_0, 1\}, & \text{if } n = 1 \\ 1, & \text{if } n \geq 2, \end{cases} \qquad \varphi_n(x) = \begin{cases} 1, & \text{if } x \in \left[-1, \frac{\bar{y}_0 - 1}{a_n}\right[\\ \bar{y}_0 - a_n x, & \text{if } x \in \left[\frac{\bar{y}_0 - 1}{a_n}, \frac{\bar{y}_0 + 1}{a_n}\right] \\ -1, & \text{if } x \in \left]\frac{\bar{y}_0 + 1}{a_n}, 1\right], \end{cases} \tag{5.7}$$

where the sequence $(a_n)_n$ is recursively defined by

$$\begin{cases} a_1 = 1 \\ a_{n+1} = a_n + 2^n \quad \text{for any } n \geq 1. \end{cases}$$

The SPNE of Γ selected according to Theorem 5.2.9 is $(\bar{x}, \bar{\varphi})$, where

$$\bar{x} = 1, \qquad \bar{\varphi}(x) = \begin{cases} 1, & \text{if } x \in [-1, 0[\\ \bar{y}_0, & \text{if } x = 0 \\ -1, & \text{if } x \in]0, 1]. \end{cases} \tag{5.8}$$

Let us note that all the SPNEs of Γ are obtained when varying $\bar{y}_0 \in [-1, 1]$ in (5.8). Hence $\bar{\varphi}$ is, among all the follower's strategies being part of an SPNE, the follower's strategy such that $\bar{\varphi}(x)$ minimizes the distance from the follower's initial point $\bar{y}_0$, for any $x \in [-1, 1]$. Therefore the SPNE constructed by our method is the nearest SPNE to the initial point $(\bar{x}_0, \bar{y}_0)$ in the sense illustrated in Subsection 5.2.1 about the interpretation of the procedure.

Remark 5.2.12 The follower's strategy $\bar{\varphi}$ in the SPNE selected according to Theorem 5.2.9 differs from the pointwise limit φ of sequence $(\varphi_n)_n$ at most in one point. In fact if the two limits

$$\lim_{n \to +\infty} \varphi_n(\bar{x}_n) \quad \text{and} \quad \lim_{n \to +\infty} \varphi_n(\bar{x}), \tag{5.9}$$

where $\bar{x} = \lim_{n \to +\infty} \bar{x}_n$, coincide, then $\bar{\varphi}(x) = \varphi(x)$ for any $x \in X$ and the strategy profile $(\bar{x}, \varphi)$ is an SPNE of Γ in light of Theorem 5.2.9. Instead, if the two limits in (5.9) do not coincide, then $\bar{\varphi}(\bar{x}) \neq \varphi(\bar{x})$ and the strategy profile $(\bar{x}, \varphi)$ could be not an SPNE of Γ, hence we need the follower's strategy $\bar{\varphi}$ as in statement of Theorem 5.2.9 in order to get an SPNE. The following two examples illustrate the two cases described above: in the first one the two limits in (5.9) are equal, whereas, in the second one the two limits in (5.9) are different. The main computations of both examples are provided in Subsection 5.2.4. We mention that the two limits in (5.9) coincide if, for example, the sequence $(\varphi_n)_n$ continuously converges to φ (see, for example, [29, Chapter 4] for the definition and properties of continuous convergence).

Example 5.2.2 Let $\Gamma = (X, Y, L, F)$ be the one-leader one-follower two-stage game where $X = Y = [-1, 1]$ and

$$L(x, y) = y, \qquad F(x, y) = -xy.$$

The follower's best reply correspondence $\mathcal{M}$ is defined on $[-1, 1]$ by

$$\mathcal{M}(x) = \begin{cases} \{1\}, & \text{if } x \in [-1, 0[\\ [-1, 1], & \text{if } x = 0 \\ \{-1\}, & \text{if } x \in]0, 1]. \end{cases} \tag{5.10}$$

Let $(\bar{x}_0, \bar{y}_0) = (1, 1)$ be the initial point of the procedure and let $\beta_n = \gamma_n = 2^n$ for any $n \in \mathbb{N} \cup \{0\}$. Then Procedure $(\mathcal{CM})$ generates the following sequence $(\bar{x}_n, \varphi_n)_n$ of strategy profiles:

$$\bar{x}_n = 0, \qquad \varphi_n(x) = \begin{cases} 1, & \text{if } x \in [-1, 0[\\ 1 - a_n x, & \text{if } x \in [0, 2/a_n] \\ -1, & \text{if } x \in]2/a_n, 1], \end{cases} \tag{5.11}$$

where the sequence $(a_n)_n$ is recursively defined by

$$\begin{cases} a_1 = 1 \\ a_{n+1} = a_n + 2^n & \text{for any } n \geq 1. \end{cases}$$

Hence, the SPNE of Γ selected according to Theorem 5.2.9 is $(\bar{x}, \bar{\varphi})$, where

$$\bar{x} = 0, \qquad \bar{\varphi}(x) = \begin{cases} 1, & \text{if } x \in [-1, 0] \\ -1, & \text{if } x \in]0, 1]. \end{cases}$$

In this case, $\bar{\varphi}$ coincides with the pointwise limit of $(\varphi_n)_n$ since $\lim_n \varphi_n(\bar{x}_n) = 1 = \lim_n \varphi_n(\lim_n \bar{x}_n)$.
Let us note that Γ has infinitely many SPNEs. In fact, denoted with $\hat{\varphi}^\alpha$ the function defined on $[-1, 1]$ by

$$\hat{\varphi}^\alpha(x) := \begin{cases} 1, & \text{if } x \in [-1, 0[\\ \alpha, & \text{if } x = 0 \\ -1, & \text{if } x \in]0, 1], \end{cases}$$

the set of SPNEs of Γ is $\{(\hat{x}, \hat{\varphi}^\alpha) \mid \hat{x} \in [-1, 0[, \, \alpha \in [-1, 1]\} \cup \{(0, \hat{\varphi}^1)\}$, only one of which is obtained via our method. Hence, the selection method defined by means of Procedure $(\mathcal{CM})$ is effective.
Moreover, $\bar{x} = 0$ is the nearest leader's action to $\bar{x}_0 = 1$ among all the actions takeable by the leader in an SPNE, and analogously $\bar{\varphi}$ is, among all the follower's strategies being part of an SPNE, the follower's strategy such that $\bar{\varphi}(x)$ minimizes the distance from the follower's initial point $\bar{y}_0$, for any $x \in [-1, 1]$. So the insights illustrated in Remark 5.2.11 fit this case.

Example 5.2.3 Let $\Gamma = (X, Y, L, F)$ be the one-leader one-follower two-stage game where $X = [1/2, 2]$, $Y = [-1, 1]$ and

$$L(x, y) = -x - y, \qquad F(x, y) = \begin{cases} 0, & \text{if } x \in [1/2, 1] \\ (1 - x)y, & \text{if } x \in]1, 2]. \end{cases}$$

The follower's best reply correspondence $\mathcal{M}$ is given by

$$\mathcal{M}(x) = \begin{cases} [-1,1], & \text{if } x \in [1/2,1] \\ \{-1\}, & \text{if } x \in]1,2]. \end{cases} \tag{5.12}$$

Let $(\bar{x}_0, \bar{y}_0) = (1,1)$ and $\beta_n = \gamma_n = n+1$ for any $n \in \mathbb{N} \cup \{0\}$. Then Procedure $(\mathcal{CM})$ generates the following sequence $(\bar{x}_n, \varphi_n)_n$ of strategy profiles:

$$\bar{x}_n = \begin{cases} 1/2, & \text{if } n = 1 \\ 1 + 2/a_n, & \text{if } n \geq 2, \end{cases} \qquad \varphi_n(x) = \begin{cases} 1, & \text{if } x \in [1/2,1] \\ a_n + 1 - a_n x, & \text{if } x \in]1, 1+2/a_n] \\ -1, & \text{if } x \in]1+2/a_n, 2], \end{cases} \tag{5.13}$$

where the sequence $(a_n)_n$ is recursively defined by

$$\begin{cases} a_1 = 1 \\ a_{n+1} = a_n + n + 1 & \text{for any } n \geq 1. \end{cases}$$

Hence, the SPNE of Γ selected according to Theorem 5.2.9 is $(\bar{x}, \bar{\varphi})$, where

$$\bar{x} = 1, \qquad \bar{\varphi}(x) = \begin{cases} 1, & \text{if } x \in [1/2, 1[\\ -1, & \text{if } x \in [1,2]. \end{cases} \tag{5.14}$$

As mentioned in Remark 5.2.12, in this case

$$\lim_n \varphi_n(\bar{x}_n) = -1 \neq 1 = \lim_n \varphi_n(\lim_n \bar{x}_n)$$

and, furthermore, the strategy profile $(1, \varphi)$, where φ is the pointwise limit of $(\varphi_n)_n$, is not an SPNE of Γ since $\operatorname{Arg\,max}_{x \in [1/2,2]} L(x, \varphi(x)) = \emptyset$.
Finally, it is worth to note that even in this example the SPNE obtained is the nearest SPNE to the initial point $(\bar{x}_0, \bar{y}_0) = (1,1)$, in the sense described in Remark 5.2.11.

Remark 5.2.13 If the sequence $(\bar{x}_n, \varphi_n(\bar{x}_n))_n$ involved in the statement of Theorem 5.2.9 does not converge, the thesis of Theorem 5.2.9 still holds if $(\bar{x}, \bar{y})$ is replaced with the limit of a convergent subsequence $(\bar{x}_{n_k}, \varphi_{n_k}(\bar{x}_{n_k}))_k \subseteq (\bar{x}_n, \varphi_n(\bar{x}_n))_n$, whose existence is guaranteed by the compactness of X and Y. Therefore, assumption $\Gamma \in \mathcal{G}$ ensures the existence of SPNEs in one-leader one-follower two-stage games regardless of the lower semicontinuity of the follower's best reply correspondence. Indeed, in the examples above, the follower's best reply correspondences in (5.6), (5.10) and (5.12) are not lower semicontinuous set-valued maps.

The definition of $(\varphi_n)_n$ in Procedure $(\mathcal{CM})$ is based on a parametric proximal point method. Since proximal point methods require that an initial point has

to be fixed (as illustrated in Section 4.2), we have taken in Procedure $(\mathcal{CM})$ the constant function $\varphi_0 \in Y^X$ defined by $\varphi_0(x) = \bar{y}_0$ as the follower's initial point. However, Procedure $(\mathcal{CM})$ could be also defined choosing any continuous function $\varphi_0 \in Y^X$ as follower's initial point and all the results of Subsections 5.2.1 and 5.2.2 would be still valid (in particular, Propositions 5.2.5, 5.2.8 and Theorem 5.2.9).
The next two propositions state some further properties of our constructive method when in Procedure $(\mathcal{CM})$ the initial constant function defined by $\bar{y}_0$ is replaced with a continuous function $\varphi_0 \in Y^X$. For the sake of simplicity, we continue to refer to $(\varphi_n)_n$ as the sequence generated by this modified procedure.

Proposition 5.2.14 (Proposition 3 in [23]). *Let $\Gamma \in \mathcal{G}$ and let the follower's initial point $\varphi_0 \in Y^X$ be a continuous function. Assume that $\varphi_0(x) \in \mathcal{M}(x)$ for any $x \in X$. Then $\varphi_n = \varphi_0$ for any $n \in \mathbb{N}$. Moreover, φ_0 is the strategy chosen by the follower in the SPNE selected according to Theorem 5.2.9.*

Proof. We prove the first part of the result by induction. Firstly, note that the function F satisfies the assumptions of Lemma 5.2.7 as $\Gamma \in \mathcal{G}$.
Let $n = 1$. Since $\varphi_0(x) \in \mathcal{M}(x)$ for any $x \in X$, in light of Lemma 5.2.7(iv) and the definition of φ_1, we have

$$\{\varphi_0(x)\} = \operatorname*{Arg\,max}_{y \in Y} F(x,y) - \frac{1}{2\gamma_0}\|y - \varphi_0(x)\|^2_{\mathbb{Y}} = \{\varphi_1(x)\}, \text{ for any } x \in X,$$

so, the base case is satisfied. Let $n > 1$ and suppose that $\varphi_n = \varphi_0$. Then $\varphi_n(x) \in \mathcal{M}(x)$ for any $x \in X$ and, by Lemma 5.2.7(iv) and definition of φ_{n+1}, we get

$$\{\varphi_n(x)\} = \operatorname*{Arg\,max}_{y \in Y} F(x,y) - \frac{1}{2\gamma_n}\|y - \varphi_n(x)\|^2_{\mathbb{Y}} = \{\varphi_{n+1}(x)\}, \text{ for any } x \in X,$$

thus, the inductive step is proved. Hence, $\varphi_n = \varphi_0$ for any $n \in \mathbb{N}$ and the first part of the proof is complete.
Since $\varphi_n = \varphi_0$ for any $n \in \mathbb{N}$ and φ_0 is continuous, then, for any sequence $(x_n)_n \subseteq X$ converging to $x \in X$, the sequence $\varphi_n(x_n)$ converges to $\varphi_0(x)$. So, φ_0 is the follower's strategy in the SPNE selected according to Theorem 5.2.9. □

Proposition 5.2.15 (Proposition 4 in [23]). *Let $\Gamma \in \mathcal{G}$ and let the follower's initial point $\varphi_0 \in Y^X$ be a continuous function. Assume that there exists $\nu \in \mathbb{N}$ such that $\varphi_\nu = \varphi_{\nu-1}$. Then $\varphi_\nu(x) \in \mathcal{M}(x)$ for any $x \in X$ and $\varphi_n = \varphi_\nu$ for any $n > \nu$. Moreover, φ_ν is the strategy chosen by the follower in the SPNE selected according to Theorem 5.2.9.*

Proof. By the definition of φ_ν and since $\varphi_\nu = \varphi_{\nu-1}$, we have

$$\begin{aligned}\{\varphi_\nu(x)\} &= \operatorname*{Arg\,max}_{y\in Y} F(x,y) - \frac{1}{2\gamma_{\nu-1}}\|y-\varphi_{\nu-1}(x)\|^2_{\mathbb{Y}}\\ &= \operatorname*{Arg\,max}_{y\in Y} F(x,y) - \frac{1}{2\gamma_{\nu-1}}\|y-\varphi_{\nu}(x)\|^2_{\mathbb{Y}}, \text{ for any } x\in X.\end{aligned}$$

Then, in light of Lemma 5.2.7(iv) we get $\varphi_\nu(x) \in \mathcal{M}(x)$ for any $x \in X$.
Consider the new constructive procedure whose follower's initial point is the continuous function φ_ν and with $(\gamma_{\nu+n})_{n\in\mathbb{N}\cup\{0\}}$ instead of $(\gamma_n)_{n\in\mathbb{N}\cup\{0\}}$ (such a procedure is nothing but the original procedure taken away the first $\nu-1$ steps). Applying Proposition 5.2.14 we have $\varphi_n = \varphi_\nu$ for any $n > \nu$. Given the above and by the continuity of φ_ν, arguing as in the last part of the proof of Proposition 5.2.14, it follows that φ_ν is the strategy chosen by the follower in the SPNE selected according to Theorem 5.2.9. □

5.2.3 Connections with other methods to select SPNEs

In this section, firstly we analyze the relation between the SPNE selection method based on Moreau-Yosida regularization (illustrated in Subsections 5.2.1 and 5.2.2) and the method based on Tikhonov regularization proposed in [109] (displayed in Section 5.1), then we compare the SPNE selection achievable via Theorem 5.2.9 with the SPNEs obtainable through the strong Stackelberg solutions and the weak Stackelberg solutions of the strong and the weak Stackelberg problems, respectively, associated to Γ.
Before addressing such issues, we discuss whether the results in [3], where an alternating proximal algorithm with costs to move is introduced in normal-form games as illustrated in Subsection 4.2.2, can be used in a one-leader one-follower two-stage games framework to select SPNEs. We recall that the method proposed in [3] (avoiding the restrictions on the payoff functions given in (4.10)), fixed an initial point $(\hat{x}_0, \hat{y}_0) \in X \times Y$, generates a sequence $(\hat{x}_n, \hat{y}_n)_n \subseteq X \times Y$ defined by

$$\begin{cases} \hat{y}_n \in \operatorname{Arg\,max}_{y\in Y} F(\hat{x}_{n-1}, y) - \frac{1}{2\gamma_{n-1}}\|y-\hat{y}_{n-1}\|^2, \\ \hat{x}_n \in \operatorname{Arg\,max}_{x\in X} L(x, \hat{y}_n) - \frac{1}{2\beta_{n-1}}\|x-\hat{x}_{n-1}\|^2, \end{cases} \tag{5.15}$$

for any $n \in \mathbb{N}$. Remind that in Procedure $(\mathcal{CM})$ at each step it is defined a strategy profile $(\bar{x}_n, \varphi_n) \in X \times Y^X$ made of one leader's action and one follower's strategy. Differently, the algorithm schematized in (5.15) constructs at each step an action profile $(\hat{x}_n, \hat{y}_n) \in X \times Y$ composed by one leader's action and one follower's action and, moreover, the limit $(\hat{x}, \hat{y})$ is not, in general, connected to an SPNE of the one-leader one-follower two-stage game $\Gamma = (X, Y, L, F)$, as highlighted in the following example.

Example 5.2.4 Let $\Gamma = (X, Y, L, F)$ where $X = Y = [-1, 1]$ and

$$L(x, y) = x + y, \qquad F(x, y) = -xy.$$

The follower's best reply correspondence $\mathcal{M}$ is defined on $[-1, 1]$ by

$$\mathcal{M}(x) = \begin{cases} \{1\}, & \text{if } x \in [-1, 0[\\ [-1, 1], & \text{if } x = 0 \\ \{-1\}, & \text{if } x \in]0, 1]. \end{cases}$$

The game Γ has a unique SPNE, namely $(\bar{x}, \bar{\varphi})$, where

$$\bar{x} = 0, \qquad \varphi(x) = \begin{cases} 1, & \text{if } x \in [-1, 0] \\ -1, & \text{if } x \in]0, 1]. \end{cases}$$

Let $(\hat{x}_0, \hat{y}_0) = (0, 0)$ and $\beta_n = \gamma_n = 2^n$ for any $n \in \mathbb{N} \cup \{0\}$. Then the sequence defined in (5.15) converges to $(\hat{x}, \hat{y}) = (1, -1)$, which is not related to the SPNE of Γ (being $x = 1$ not chosen by the leader in the SPNE).

As regards to the connections with the SPNE selection method introduced in [109], we note that the way in which an SPNE is constructed via the Tikhonov regularization-based method described in Section 5.1 does not involve any task of learning step by step. Indeed, problem $P_{x,k}$ in (5.1) is not recursively defined and therefore, at a given step k, neither the follower's strategy $\bar{m}_k$ is an updating of his previous strategy $\bar{m}_{k-1}$ nor $\bar{x}_n$ is an updating of $\bar{x}_{n-1}$ (see Theorem 5.1.2). Hence, the anchoring effects arising in Procedure ($\mathcal{CM}$), and explained in Subsection 5.2.1, do not appear in this framework, as well as other kinds of behavioral motivation. As a matter of fact, in general, the learning method based on Procedure ($\mathcal{CM}$) and the selection method in [109] do not generate the same SPNE, as shown in the next example.

Example 5.2.5 Let Γ be the game defined in Example 5.2.3. The SPNE selected by using the approach in [109] is $(1, \bar{\varphi})$, where

$$\bar{\varphi}(x) = \begin{cases} 0, & \text{if } x \in [1/2, 1[\\ -1, & \text{if } x \in [1, 2]; \end{cases}$$

that does not coincide with the SPNE found out in (5.14).

Let us investigate the connections with the SPNEs induced by the strong and the weak Stackelberg solutions associated to Γ. Firstly, we remind that the computation of both strong and weak Stackelberg solutions, and related SPNEs, would require the leader to know the best reply correspondence of the follower, by definition. Instead, an SPNE selected via the method based on the proximal point algorithm described in Procedure ($\mathcal{CM}$), as well as the one selected via the method based on Tikhonov regularization (illustrated in Section 5.1), relieves the leader of knowing the follower's best reply correspondence. Moreover, we note that the SPNE selection obtained via Procedure ($\mathcal{CM}$) does not coincide, in general, with the SPNEs induced by the strong and the weak Stackelberg solutions of the strong and the weak Stackelberg problems associated to Γ (not surprisingly since, in general, the motivations underlying all these typologies of selections are completely different from each other). To show this fact, analogously to what pointed out in Section 5.1, it is sufficient to check if the limit $(\bar{x}, \bar{y})$ of the sequence of action profiles $(\bar{x}_n, \varphi_n(\bar{x}_n))_n$ obtained through Procedure ($\mathcal{CM}$) is a strong or a weak Stackelberg equilibrium. This lack of connection is exhibited in the following example.

Example 5.2.6 Let Γ be the game defined in Example 5.2.3. The follower's best reply correspondence $\mathcal{M}$ is given in (5.12). Since for any $x \in [1/2, 2]$

$$\max_{y \in \mathcal{M}(x)} L(x,y) = -x+1, \qquad \min_{y \in \mathcal{M}(x)} L(x,y) = \begin{cases} -x-1, & \text{if } x \in [1/2, 1] \\ -x+1, & \text{if } x \in]1,2], \end{cases}$$

then

$$\operatorname*{Arg\,max}_{x \in [1/2,2]} \max_{y \in \mathcal{M}(x)} L(x,y) = \{1/2\}, \qquad \operatorname*{Arg\,max}_{x \in [1/2,2]} \min_{y \in \mathcal{M}(x)} L(x,y) = \emptyset.$$

Hence, the strong Stackelberg equilibrium is the action profile $(1/2, -1)$ as $\{-1\} = \operatorname{Arg\,max}_{y \in \mathcal{M}(1/2)} L(1/2, y)$. Instead, the weak Stackelberg equilibrium does not exist.
Procedure ($\mathcal{CM}$) generates the sequence $(\bar{x}_n, \varphi_n)_n$ defined in (5.13). The sequence of action profiles $(\bar{x}_n, \varphi_n(\bar{x}_n))_{n \geq 2} = (1 + 2/a_n, -1)_{n \geq 2}$ converges to $(1, -1)$, which is neither a strong nor a weak Stackelberg equilibrium.

We conclude by mentioning that very recently Flåm in [41], motivated by the idea that imperfections in people capacity to choose, foresee or know must be taken into account when defining equilibrium solution concepts, illustrated some observations concerning the fact that change usually entails cost and gave some results on the existence and how approaching equilibria *modulo cost of change* both in normal-form games and in one-leader one-follower two-stage games.

5.2.4 Main computations of remarks and examples

In this subsection the main computation of Remark 5.2.3, Examples 5.2.1, 5.2.2 and 5.2.3 are provided.

Main computations of Remark 5.2.3

Firstly, we show that the function F defined in Remark 5.2.3 satisfies $(\mathcal{F}_1)$-$(\mathcal{F}_3)$.

(i) Proof of $(\mathcal{F}_1)$:

We need to show the upper semicontinuity of F only at $(x,(0,0))$, as F is continuous for any $(x,(y_1,y_2)) \in X \times (Y \setminus \{(0,0)\})$. Let $x \in X$ and let $(x_k,(y_{1,k},y_{2,k}))_k \subseteq X \times Y$ be a sequence converging to $(x,(0,0))$. Since $F(x,(y_1,y_2)) \leq 0$ for any $(x,(y_1,y_2)) \in X \times Y$ and $F(x,(0,0)) = 0$, then

$$\limsup_{k\to+\infty} F(x_k,(y_{1,k},y_{2,k})) \leq F(x,(0,0)).$$

Therefore $(\mathcal{F}_1)$ holds.

(ii) Proof of $(\mathcal{F}_2)$:

We need to show $(\mathcal{F}_2)$ only at $(x,(0,0))$, since F is continuous for any $(x,(y_1,y_2)) \in X \times (Y \setminus \{(0,0)\})$. Let $x \in X$ and let $(x_k)_k \subseteq X$ be a sequence converging to x. Define $(\tilde{y}_{1,k},\tilde{y}_{2,k}) \coloneqq (1/k,0) \in Y$ for any $k \in \mathbb{N}$. As $F(x_k,(\tilde{y}_{1,k},\tilde{y}_{2,k})) = 0$ for any $k \in \mathbb{N}$ and $F(x,(0,0)) = 0$, then

$$\liminf_{k\to+\infty} F(x_k,(\tilde{y}_{1,k},\tilde{y}_{2,k})) = F(x,(0,0)).$$

Therefore $(\mathcal{F}_2)$ holds.

(iii) Proof of $(\mathcal{F}_3)$:

Let $x \in X$. In order to prove the concavity of $F(x,(\cdot,\cdot))$ on $Y \setminus \{(0,0)\}$, we consider the twice-continuously differentiable function $g\colon]0,+\infty[\times\mathbb{R} \to \mathbb{R}$ defined by

$$g(y_1,y_2) \coloneqq -\frac{y_2^2}{2y_1}x.$$

The Hessian matrix of g at (y_1,y_2) is

$$Hg(y_1,y_2) = \begin{pmatrix} -\frac{y_2^2}{y_1^3}x & \frac{y_2}{y_1^2}x \\ \frac{y_2}{y_1^2}x & -\frac{1}{y_1}x \end{pmatrix}.$$

Since $Hg(y_1,y_2)$ is negative semi-definite for any $(y_1,y_2) \in]0,+\infty[\times\mathbb{R}$ (being $x \in [1,2]$), then g is concave on $]0,+\infty[\times\mathbb{R}$. Therefore, $F(x,(\cdot,\cdot))$ is concave on $Y \setminus \{(0,0)\}$, as $F(x,(y_1,y_2)) = g(y_1,y_2)$ for any $(y_1,y_2) \in Y \setminus \{(0,0)\}$. The concavity of $F(x,(\cdot,\cdot))$ on Y follows by the equality

$$F(x,t(0,0)+(1-t)(y_1,y_2)) = tF(x,(0,0)) + (1-t)F(x,(y_1,y_2)),$$

that holds for any $t \in [0,1]$ and $(y_1, y_2) \in Y$. Hence $(\mathcal{F}_3)$ is satisfied.

Let $x \in X$. We show that the function $F(x,(\cdot,\cdot))$ is not lower semicontinuous at $(0,0)$. In fact, let $(\bar{y}_{1,k}, \bar{y}_{2,k}) := (1/k, 1/\sqrt{k}) \in Y$ for any $k \in \mathbb{N}$. Since $F(x,(\bar{y}_{1,k}, \bar{y}_{2,k})) = -x/2 \in [-1/2, -1]$ and $F(x,(0,0)) = 0$, then

$$\liminf_{k \to +\infty} F(x,(\bar{y}_{1,k}, \bar{y}_{2,k})) \ngeq F(x,(0,0)).$$

Main computations of Example 5.2.1
Firstly, note that $\Gamma \in \mathcal{G}$. We prove (5.7) by induction on n. Let $n = 1$, then

$$\{\varphi_1(x)\} = \operatorname*{Arg\,max}_{y \in Y} F_1(x,y) = \operatorname*{Arg\,max}_{y \in [-1,1]} -xy - \frac{(y-\bar{y}_0)^2}{2} = \begin{cases} 1, & \text{if } x \in [-1, \bar{y}_0 - 1[\\ \bar{y}_0 - x, & \text{if } x \in [\bar{y}_0 - 1, \bar{y}_0 + 1] \\ -1, & \text{if } x \in]\bar{y}_0 + 1, 1]. \end{cases}$$

and

$$\{\bar{x}_1\} = \operatorname*{Arg\,max}_{x \in X} L_1(x, \varphi_1(x)) = \operatorname*{Arg\,max}_{x \in [-1,1]} x - \frac{(x-\bar{x}_0)^2}{2} = \begin{cases} 1 + \bar{x}_0, & \text{if } \bar{x}_0 \in [-1, 0] \\ 1, & \text{if } \bar{x}_0 \in]0, 1]. \end{cases}$$

As $a_1 = 1$, the base case is fulfilled. Assume that (5.7) holds for $n > 1$. So

$$F_{n+1}(x,y) = \begin{cases} P_1(x,y) = -\frac{y^2}{2^{n+1}} - \left(x - \frac{1}{2^n}\right) y - \frac{1}{2^{n+1}}, & \text{if } x \in \left[-1, \frac{\bar{y}_0 - 1}{a_n}\right[\\ P_2(x,y) = -\frac{y^2}{2^{n+1}} - \left(x + \frac{a_n x - \bar{y}_0}{2^n}\right) y - \frac{(\bar{y}_0 - a_n x)^2}{2^{n+1}}, & \text{if } x \in \left[\frac{\bar{y}_0 - 1}{a_n}, \frac{\bar{y}_0 + 1}{a_n}\right] \\ P_3(x,y) = -\frac{y^2}{2^{n+1}} - \left(x + \frac{1}{2^n}\right) y - \frac{1}{2^{n+1}}, & \text{if } x \in \left]\frac{\bar{y}_0 + 1}{a_n}, 1\right], \end{cases}$$

and

$$L_{n+1}(x,y) = -\frac{x^2}{2^{n+1}} + \left(1 + \frac{1}{2^n}\right) x - \frac{1}{2^{n+1}}.$$

If $x \in [-1, (\bar{y}_0 - 1)/a_n[$, then the unique maximizer of $P_1(x,\cdot)$ on $Y = [-1,1]$ is 1 since the abscissa of the vertex of the parabola $\mathcal{P}_1 := \{(y,z) \in R^2 \mid z = P_1(x,y)\}$ is $1 - 2^n x > 1$. If $x \in [(\bar{y}_0 - 1)/a_n, (\bar{y}_0 - 1)/(2^n + a_n)[$, then the unique maximizer of $P_2(x,\cdot)$ on $Y = [-1,1]$ is 1 since the abscissa of the vertex of the parabola $\mathcal{P}_2 := \{(y,z) \in R^2 \mid z = P_2(x,y)\}$ is $\bar{y}_0 - (2^n + a_n)x > 1$. If $x \in [(\bar{y}_0 - 1)/(2^n + a_n), (\bar{y}_0 + 1)/(2^n + a_n)]$, then the unique maximizer of $P_2(x,\cdot)$ on $Y = [-1,1]$ is $\bar{y}_0 - (2^n + a_n)x$ since the abscissa of the vertex of the parabola $\mathcal{P}_2$ is $\bar{y}_0 - (2^n + a_n)x \in [-1,1]$. If $x \in](\bar{y}_0 + 1)/(2^n + a_n), (\bar{y}_0 + 1)/a_n]$, then the unique maximizer of $P_2(x,\cdot)$ on $Y = [-1,1]$ is -1 since the abscissa of the vertex of the parabola $\mathcal{P}_2$ is $\bar{y}_0 - (2^n + a_n)x < -1$. If $x \in [(\bar{y}_0 + 1)/a_n, 1[$, then the unique maximizer of $P_3(x,\cdot)$ on $Y = [-1,1]$ is -1 since the abscissa of the vertex of the parabola $\mathcal{P}_3 := \{(y,z) \in R^2 \mid z = P_3(x,y)\}$ is $-(2^n x + 1) < -1$. Given the above, since $2^n + a_n = a_{n+1}$,

$$\{\varphi_{n+1}(x)\} = \operatorname*{Arg\,max}_{y \in Y} F_{n+1}(x,y) = \begin{cases} 1, & \text{if } x \in \left[-1, \frac{\bar{y}_0 - 1}{a_{n+1}}\right[\\ \bar{y}_0 - a_{n+1}x, & \text{if } x \in \left[\frac{\bar{y}_0 - 1}{a_{n+1}}, \frac{\bar{y}_0 + 1}{a_{n+1}}\right] \\ -1, & \text{if } x \in \left]\frac{\bar{y}_0 + 1}{a_{n+1}}, 1\right], \end{cases} \quad (5.16)$$

for any $x \in [-1,1]$. Since $L_{n+1}(x, \varphi_{n+1}(x)) = L_{n+1}(x,y)$ for any $(x,y) \in X \times Y$ and the abscissa of the vertex of the parabola $\mathcal{T} = \{(x,z) \in \mathbb{R}^2 \mid z = L_{n+1}(x, \varphi_{n+1}(x))\}$ is $2^n + 1 > 1$, then

$$\{\bar{x}_{n+1}\} = \operatorname*{Arg\,max}_{x \in [-1,1]} L_{n+1}(x, \varphi_{n+1}(x)) = \{1\}. \tag{5.17}$$

Equalities (5.16)-(5.17) prove the inductive step, so (5.7) holds.
As $\lim_{n \to +\infty} a_n = +\infty$, we get

$$\bar{x} = \lim_{n \to +\infty} \bar{x}_n = 1, \qquad \varphi(x) = \lim_{n \to +\infty} \varphi_n(x) = \begin{cases} 1, & \text{if } x \in [-1,0[\\ \bar{y}_0, & \text{if } x = 0 \\ -1, & \text{if } x \in]0,1]. \end{cases}$$

Since $\lim_{n \to +\infty} \varphi_n(\bar{x}_n) = 1$, then the SPNE selected according to Theorem 5.2.9 is $(1, \bar{\varphi}) = (1, \varphi)$.

Main computations of Example 5.2.2
Firstly, note that $\Gamma \in \mathcal{G}$. We prove (5.11) by induction on n. Let $n = 1$, then

$$\{\varphi_1(x)\} = \operatorname*{Arg\,max}_{y \in Y} F_1(x,y) = \operatorname*{Arg\,max}_{y \in [-1,1]} -xy - \frac{(y-1)^2}{2} = \begin{cases} 1, & \text{if } x \in [-1,0[\\ 1-x, & \text{if } x \in [0,1]. \end{cases}$$

and

$$\{\bar{x}_1\} = \operatorname*{Arg\,max}_{x \in X} L_1(x, \varphi_1(x)) \quad \text{where } L_1(x, \varphi_1(x)) = \begin{cases} -\frac{x^2-2x-1}{2} & \text{if } x \in [-1,0[\\ -\frac{x^2-1}{2}, & \text{if } x \in [0,1], \end{cases}$$

that is $\bar{x}_1 = 0$. As $a_1 = 1$, the base case is fulfilled. Assume that (5.11) holds for $n > 1$. So

$$F_{n+1}(x,y) = \begin{cases} P_1(x,y) = -\frac{y^2}{2^{n+1}} - \left(x - \frac{1}{2^n}\right) y - \frac{1}{2^{n+1}}, & \text{if } x \in [-1,0[\\ P_2(x,y) = -\frac{y^2}{2^{n+1}} - \left(x + \frac{a_n x - 1}{2^n}\right) y - \frac{(1-a_n x)^2}{2^{n+1}}, & \text{if } x \in \left[0, \frac{2}{a_n}\right] \\ P_3(x,y) = -\frac{y^2}{2^{n+1}} - \left(x + \frac{1}{2^n}\right) y - \frac{1}{2^{n+1}}, & \text{if } x \in \left]\frac{2}{a_n}, 1\right], \end{cases}$$

and

$$L_{n+1}(x,y) = -\frac{x^2}{2^{n+1}} + y. \tag{5.18}$$

If $x \in [-1,0[$, then the unique maximizer of $P_1(x,\cdot)$ on $Y = [-1,1]$ is 1 since the abscissa of the vertex of the parabola $\mathcal{P}_1 := \{(y,z) \in \mathbb{R}^2 \mid z = P_1(x,y)\}$ is $1 - 2^n x > 1$. If $x \in [0, 2/(2^n + a_n)]$, then the unique maximizer of $P_2(x,\cdot)$ on $Y = [-1,1]$ is $1-(2^n+a_n)x$ since the abscissa of the vertex of the parabola $\mathcal{P}_2 := \{(y,z) \in \mathbb{R}^2 \mid z = P_2(x,y)\}$ is $1-(2^n+a_n)x \in [-1,1]$. If $x \in]2/(2^n+a_n), 2/a_n]$,

then the unique maximizer of $P_2(x,\cdot)$ on $Y=[-1,1]$ is -1 since the abscissa of the vertex of the parabola $\mathcal{P}_2$ is $1-(2^n+a_n)x<-1$. If $x\in]2/a_n,1]$, then the unique maximizer of $P_3(x,\cdot)$ on $Y=[-1,1]$ is -1 since the abscissa of the vertex of the parabola $\mathcal{P}_3 := \{(y,z)\in\mathbb{R}^2 \mid z=P_3(x,y)\}$ is $-(2^nx+1)<-1$.
Given the above, since $2^n+a_n=a_{n+1}$,

$$\{\varphi_{n+1}(x)\} = \operatorname*{Arg\,max}_{y\in Y} F_{n+1}(x,y) = \begin{cases} 1, & \text{if } x\in[-1,0[\\ 1-a_{n+1}x, & \text{if } x\in\left[0,\frac{2}{a_{n+1}}\right] \\ -1, & \text{if } x\in\left]\frac{2}{a_{n+1}},1\right], \end{cases} \tag{5.19}$$

for any $x\in[-1,1]$. Evaluating the function L_{n+1} given in (5.18) at $(x,\varphi_{n+1}(x))$, we get

$$L_{n+1}(x,\varphi_{n+1}(x)) = \begin{cases} T_1(x) = -\frac{x^2}{2^{n+1}}+1, & \text{if } x\in[-1,0[\\ T_2(x) = -\frac{x^2}{2^{n+1}}-a_{n+1}x+1, & \text{if } x\in\left[0,\frac{2}{a_{n+1}}\right] \\ T_3(x) = -\frac{x^2}{2^{n+1}}-1, & \text{if } x\in\left]\frac{2}{a_{n+1}},1\right], \end{cases}$$

Since

(i) the abscissa of the vertexes of the parabolas $\mathcal{T}_1=\{(x,z)\in\mathbb{R}^2 \mid z=T_1(x)\}$ and $\mathcal{T}_3=\{(x,z)\in\mathbb{R}^2 \mid z=T_3(x)\}$ is 0;

(ii) the abscissa of the vertex of the parabola $\mathcal{T}_2=\{(x,z)\in\mathbb{R}^2 \mid z=T_2(x)\}$ is $-2^na_{n+1}<0$;

(iii) $L_{n+1}(\cdot,\varphi_{n+1}(\cdot))$ is continuous on $[-1,1]$,

then

$$\{\bar{x}_{n+1}\} = \operatorname*{Arg\,max}_{x\in[-1,1]} L_{n+1}(x,\varphi_{n+1}(x)) = \{0\}. \tag{5.20}$$

Equalities (5.19)-(5.20) prove the inductive step, so (5.11) holds.
As $\lim_{n\to+\infty} a_n=+\infty$, we get

$$\bar{x} = \lim_{n\to+\infty}\bar{x}_n = 0, \qquad \varphi(x) = \lim_{n\to+\infty}\varphi_n(x) = \begin{cases} 1, & \text{if } x\in[-1,0] \\ -1, & \text{if } x\in]0,1]. \end{cases}$$

Since $\lim_{n\to+\infty}\varphi_n(\bar{x}_n)=1$, then the SPNE selected according to Theorem 5.2.9 is $(0,\bar{\varphi})=(0,\varphi)$.

Main computations of Example 5.2.3
Firstly, note that $\Gamma\in\mathcal{G}$. We prove (5.13) by induction on n. Let $n=1$, then

$$\{\varphi_1(x)\} = \operatorname*{Arg\,max}_{y\in Y} F_1(x,y) \quad \text{where } F_1(x,y) = \begin{cases} -\frac{(y-1)^2}{2} & \text{if } x\in\left[\frac{1}{2},1\right] \\ -\frac{y^2}{2}+(2-x)y-\frac{1}{2}, & \text{if } x\in]1,2], \end{cases}$$

that is

$$\varphi_1(x) = \begin{cases} 1 & \text{if } x \in \left[\frac{1}{2}, 1\right] \\ 2 - x, & \text{if } x \in]1, 2] \end{cases}$$

Moreover

$$\{\bar{x}_1\} = \operatorname*{Arg\,max}_{x \in X} L_1(x, \varphi_1(x)) \quad \text{where } L_1(x, \varphi_1(x)) = \begin{cases} -\frac{x^2+3}{2} & \text{if } x \in \left[\frac{1}{2}, 1\right] \\ -\frac{x^2-2x+5}{2}, & \text{if } x \in]1, 2], \end{cases}$$

that is $\bar{x}_1 = \frac{1}{2}$. As $a_1 = 1$, the base case is fulfilled. Assume that (5.13) holds for $n > 1$. So

$$F_{n+1}(x, y) = \begin{cases} P_1(x, y), & \text{if } x \in \left[\frac{1}{2}, 1\right] \\ P_2(x, y), & \text{if } x \in \left]1, 1 + \frac{2}{a_n}\right] \\ P_3(x, y), & \text{if } x \in \left]1 + \frac{2}{a_n}, 2\right], \end{cases}$$

where

$$P_1(x, y) = -\frac{(y-1)^2}{2(n+1)},$$
$$P_2(x, y) = -\frac{y^2}{2(n+1)} + \left(1 - x + \frac{a_n + 1 - a_n x}{n+1}\right) y - \frac{(a_n + 1 - a_n x)^2}{2(n+1)},$$
$$P_3(x, y) = -\frac{y^2}{2(n+1)} + \left(1 - x - \frac{1}{n+1}\right) y - \frac{1}{2(n+1)},$$

and

$$L_{n+1}(x, y) = -\frac{x^2}{2(n+1)} - \left(1 - \frac{a_n + 2}{(n+1)a_n}\right) x - \frac{(a_n+2)^2}{2(n+1)a_n^2} - y. \tag{5.21}$$

If $x \in \left[\frac{1}{2}, 1\right]$, then the unique maximizer of $P_1(x, \cdot)$ on $Y = [-1, 1]$ is 1 since the abscissa of the vertex of the parabola $\mathcal{P}_1 := \{(y, z) \in \mathbb{R}^2 \mid z = P_1(x, y)\}$ is 1. If $x \in \left]1, 1 + \frac{2}{a_n+n+1}\right]$, then the unique maximizer of $P_2(x, \cdot)$ on $Y = [-1, 1]$ is $a_n + n + 2 - (n + 1 + a_n)x$ since the abscissa of the vertex of the parabola $\mathcal{P}_2 := \{(y, z) \in \mathbb{R}^2 \mid z = P_2(x, y)\}$ is $a_n + n + 2 - (n + 1 + a_n)x \in [-1, 1]$. If $x \in \left]1 + \frac{2}{a_n+n+1}, 1 + \frac{2}{a_n}\right]$, then the unique maximizer of $P_2(x, \cdot)$ on $Y = [-1, 1]$ is -1 since the abscissa of the vertex of the parabola $\mathcal{P}_2$ is $a_n + n + 2 - (n + 1 + a_n)x < -1$. If $x \in \left]1 + \frac{2}{a_n}, 2\right]$, then the unique maximizer of $P_3(x, \cdot)$ on $Y = [-1, 1]$ is -1 since the abscissa of the vertex of the parabola $\mathcal{P}_3 := \{(y, z) \in \mathbb{R}^2 \mid z = P_3(x, y)\}$ is $n - (n + 1)x < -1$.
Given the above, since $n + 1 + a_n = a_{n+1}$,

$$\{\varphi_{n+1}(x)\} = \operatorname*{Arg\,max}_{y \in Y} F_{n+1}(x, y) = \begin{cases} 1, & \text{if } x \in \left[\frac{1}{2}, 1\right] \\ a_{n+1} + 1 - (a_{n+1})x, & \text{if } x \in \left]1, 1 + \frac{2}{a_{n+1}}\right] \\ -1, & \text{if } x \in \left]1 + \frac{2}{a_{n+1}}, 2\right], \end{cases} \tag{5.22}$$

for any $x \in \left[\frac{1}{2}, 2\right]$. Evaluating the function L_{n+1} given in (5.21) at $(x, \varphi_{n+1}(x))$, we get

$$L_{n+1}(x, \varphi_{n+1}(x)) = \begin{cases} T_1(x), & \text{if } x \in \left[\frac{1}{2}, 1\right] \\ T_2(x), & \text{if } x \in \left]1, 1 + \frac{2}{a_{n+1}}\right] \\ T_3(x), & \text{if } x \in \left]1 + \frac{2}{a_{n+1}}, 2\right], \end{cases}$$

where

$$\begin{aligned} T_1(x) &= -\frac{x^2}{2(n+1)} - \left(1 - \frac{a_n + 2}{(n+1)a_n}\right) x - \frac{(a_n+2)^2}{2(n+1)a_n^2} - 1, \\ T_2(x) &= -\frac{x^2}{2(n+1)} - \left(1 - a_{n+1} - \frac{a_n + 2}{(n+1)a_n}\right) x - \frac{(a_n+2)^2}{2(n+1)a_n^2} - a_{n+1} - 1, \\ T_3(x) &= -\frac{x^2}{2(n+1)} - \left(1 - \frac{a_n + 2}{(n+1)a_n}\right) x - \frac{(a_n+2)^2}{2(n+1)a_n^2} + 1. \end{aligned}$$

Since

(i) the abscissa of the vertexes of the parabolas $\mathcal{T}_1 = \{(x,z) \in \mathbb{R}^2 \mid z = T_1(x)\}$ and $\mathcal{T}_3 = \{(x,z) \in \mathbb{R}^2 \mid z = T_3(x)\}$ is $\frac{2}{a_n} - n < \frac{1}{2}$;

(ii) the abscissa of the vertex of the parabola $\mathcal{T}_2 = \{(x,z) \in \mathbb{R}^2 \mid z = T_2(x)\}$ is $(n+1)a_{n+1} - n + \frac{2}{a_n} > 1 + \frac{2}{a_n} > 1 + \frac{2}{a_{n+1}}$;

(iii) $T_1\left(\frac{1}{2}\right) < T_3(2)$;

(iv) $L_{n+1}(\cdot, \varphi_{n+1}(\cdot))$ is continuous on $\left[\frac{1}{2}, 2\right]$,

then

$$\{\bar{x}_{n+1}\} = \operatorname*{Arg\,max}_{x \in [-1,1]} L_{n+1}(x, \varphi_{n+1}(x)) = \left\{1 + \frac{2}{a_{n+1}}\right\}. \tag{5.23}$$

Equalities (5.22)-(5.23) prove the inductive step, so (5.13) holds.
As $\lim_{n \to +\infty} a_n = +\infty$, we get

$$\bar{x} = \lim_{n \to +\infty} \bar{x}_n = 1, \qquad \varphi(x) = \lim_{n \to +\infty} \varphi_n(x) = \begin{cases} 1, & \text{if } x \in \left[\frac{1}{2}, 1\right] \\ -1, & \text{if } x \in]1, 2]. \end{cases}$$

Since $\lim_{n \to +\infty} \varphi_n(\bar{x}_n) = -1$, then the SPNE selected according to Theorem 5.2.9 is $(0, \bar{\varphi})$, where

$$\bar{\varphi}(x) = \begin{cases} 1, & \text{if } x \in \left[\frac{1}{2}, 1\right[\\ -1, & \text{if } x \in [1, 2]. \end{cases}$$

5.3 Some further discussion

In Section 5.2 we presented a method to select an SPNE in one-leader one-follower two-stage game by using a learning approach based on costs to move that relies on proximal point algorithm (related to Moreau-Yosida regularization). Our primary purpose is to apply such a method in economics and management science frameworks, analyzing how costs to move affect the players' decisions and, consequently, the SPNEs chosen in the several important subjects of study in economics related to two-stage games in a continuous setting (for example, principal-agent models, incentive design problems, mechanism design problems, ...).

The analysis for one-leader N-follower two-stage games (with $N \geq 2$) is presently under investigation. In this case, the non-single-valuedness of the Nash equilibrium correspondence $\mathcal{N}$, defined in (2.3), will be possibly overcome by exploiting Proposition 4.2.14 (concerning the selection of Nash equilibria via Moreau-Yosida regularization in normal-form games where the players have constrained action sets) or, in general, by applying a learning method based on proximal point algorithm and known results about uniqueness of Nash equilibria as [132] (which is used in Proposition 4.2.14), [25], or [24] (illustrated in Section 3.1).

Moreover, we emphasize that the selection method based on Procedure $(\mathcal{CM})$, as well as the method based on Tikhonov regularization, could be implemented in any finite game in mixed strategies and for any game where the players have a continuum of actions and the functions φ_n, defined in Subsection 5.2.1, can be analytically determined for any $n \in \mathbb{N}$. However, we aim also to design an algorithm, based on the constructive selection methods presented in this chapter, in order to approximate an SPNE in one-leader one-follower two-stage games.

Finally, we mention that another direction for future research is to adapt Procedure $(\mathcal{CM})$ to *semivectorial bilevel optimal control problems* (see [16]), that are differential games with hierarchical play where one leader in the first stage faces a scalar optimal control problem and more followers in the second stage solve a cooperative differential game. In fact, our learning approach relying on proximal point algorithm could be useful to construct SPNEs when the followers' Pareto control paths are not unique by requiring only convexity assumptions, whereas in [16] the non-single-valuedness of the followers' best reply correspondence is overcome in the optimistic and the pessimistic situations associated with the problem by means of some strict convexity assumptions.

Bibliography

[1] A. Aboussoror and P. Loridan. "Existence of solutions to two-level optimization problems with nonunique lower-level solutions". In: *Journal of Mathematical Analysis and Applications* 254 (2001), pp. 348–357. DOI: https://doi.org/10.1006/jmaa.2000.7001.

[2] H. Attouch. *Variational convergence for functions and operators*. Vol. 1. Pitman Advanced Publishing Program, 1984.

[3] H. Attouch, P. Redont, and A. Soubeyran. "A new class of alternating proximal minimization algorithms with costs-to-move". In: *SIAM Journal on Optimization* 18 (2007), pp. 1061–1081. DOI: https://doi.org/10.1137/060657248.

[4] H. Attouch and A. Soubeyran. "Inertia and reactivity in decision making as cognitive variational inequalities". In: *Journal of Convex Analysis* 13 (2006), p. 207.

[5] H. Attouch and A. Soubeyran. "Local search proximal algorithms as decision dynamics with costs to move". In: *Set-Valued and Variational Analysis* 19 (2011), pp. 157–177. DOI: https://doi.org/10.1007/s11228-010-0139-7.

[6] H. Attouch and A. Soubeyran. "Worthwhile-to-move behaviors as temporary satisficing without too many sacrificing processes". Preprint. 2009. URL: https://arxiv.org/pdf/0905.1238.pdf.

[7] J.-P. Aubin. *Mathematical methods of game and economic theory*. Mineola, New York: Dover, 2007.

[8] J.-P. Aubin. *Optima and equilibria: an introduction to nonlinear analysis*. Vol. 140. Springer Science & Business Media, 2013.

[9] J.-P. Aubin and I. Ekeland. *Applied nonlinear analysis*. Dover Publications, 2006.

[10] A. Avez. *Calcul Différentiel*. Paris: Masson, 1983.

[11] T. Başar and G. Olsder. *Dynamic Noncooperative Game Theory*. Society for Industrial and Applied Mathematics, 1998.

[12] A. Bagchi. *Stackelberg differential games in economic models*. Vol. 64. Springer-Verlag Berlin Heidelberg, 1984.

[13] J. F. Bard and J. E. Falk. "An explicit solution to the multi-level programming problem". In: *Computers & Operations Research* 9 (1982), pp. 77–100. DOI: https://doi.org/10.1016/0305-0548(82)90007-7.

[14] H.H. Bauschke and P.L. Combettes. *Convex analysis and monotone operator theory in Hilbert spaces*. New York: Springer Science & Business Media, 2011.

[15] P. Bich. "Prudent equilibria and strategic uncertainty in discontinuous games". Preprint. 2016. URL: https://halshs.archives-ouvertes.fr/halshs-01337293/file/prudent-equilibria-preprint-24-juin.pdf.

[16] H. Bonnel and J. Morgan. "Semivectorial bilevel convex optimal control problems: existence results". In: *SIAM Journal on Control and Optimization* 50 (2012), pp. 3224–3241. DOI: 10.1137/100795450.

[17] R. Brânzei, L. Mallozzi, and S. Tijs. "Supermodular games and potential games". In: *Journal of Mathematical Economics* 39 (2003), pp. 39–49. DOI: https://doi.org/10.1016/S0304-4068(02)00082-4.

[18] M. Breton, A. Alj, and A. Haurie. "Sequential Stackelberg equilibria in two-person games". In: *Journal of Optimization Theory and Applications* 59 (1988), pp. 71–97. DOI: 10.1007/BF00939867.

[19] H. Brezis. *Opérateurs maximaux monotones et semi-groupes de contractions dans les espaces de Hilbert*. Vol. 5. Elsevier, 1973.

[20] D.A. Carlson. "The existence and uniqueness of equilibria in convex games with strategies in Hilbert spaces". In: *Advances in Dynamic Games and Applications*. Ed. by E. Altman and O. Pourtallier. Boston, MA: Springer, 2001, pp. 79–97. DOI: https://doi.org/10.1007/978-1-4612-0155-7_6.

[21] G. Carmona. "On games of perfect information: equilibria, ϵ-equilibria and approximation by simple games". In: *International Game Theory Review* 07 (2005), pp. 491–499. DOI: 10.1142/S0219198905000661.

[22] F. Caruso, M. C. Ceparano, and J. Morgan. "An adjustment process-based algorithm with error bounds for approximating a Nash equilibrium". CSEF Working Papers. 2018. URL: https://ideas.repec.org/p/sef/csefwp/502.html.

[23] F. Caruso, M. C. Ceparano, and J. Morgan. "Subgame perfect Nash equilibrium: a learning approach via costs to move". In: *Dynamic Games and Applications* (2018). DOI: 10.1007/s13235-018-0277-3.

[24] F. Caruso, M.C. Ceparano, and J. Morgan. "Uniqueness of Nash equilibrium in continuous two-player weighted potential games". In: *Journal of Mathematical Analysis and Applications* 459 (2018), pp. 1208–1221. DOI: https://doi.org/10.1016/j.jmaa.2017.11.031.

[25] M. C. Ceparano and Quartieri F. "Nash equilibrium uniqueness in nice games with isotone best replies". In: *Journal of Mathematical Economics* 70 (2017), pp. 154–165. DOI: https://doi.org/10.1016/j.jmateco.2017.02.011.

[26] M. C. Ceparano and J. Morgan. "Equilibrium selection in multi-leader-follower games with vertical information". In: *TOP* (2017), pp. 526–543. DOI: 10.1007/s11750-017-0444-5.

[27] C. Chen and J. Cruz. "Stackelberg solution for two-person games with biased information patterns". In: *IEEE Transactions on Automatic Control* 17 (1972), pp. 791–798. DOI: 10.1109/TAC.1972.1100179.

[28] B. Colson, P. Marcotte, and G. Savard. "An overview of bilevel optimization". In: *Annals of Operations Research* 153 (2007), pp. 235–256. DOI: https://doi.org/10.1007/s10479-007-0176-2.

[29] G. Dal Maso. *An introduction to* Γ*-convergence*. Basel: Birkhäuser, 1993.

[30] G. De Marco and J. Morgan. "Slightly Altruistic Equilibria". In: *Journal of Optimization Theory and Applications* 137 (2008), pp. 347–362. DOI: 10.1007/s10957-008-9353-y.

[31] G. Debreu. "A social equilibrium existence theorem". In: *Proceedings of the National Academy of Sciences* 38 (1952), pp. 886–893. DOI: https://doi.org/10.1073/pnas.38.10.886.

[32] S. Dempe. *Foundations of bilevel programming*. Springer US, 2002.

[33] S. Dempe, J. Dutta, and B. S. Mordukhovich. "New necessary optimality conditions in optimistic bilevel programming". In: *Optimization* 56 (2007), pp. 577–604. DOI: 10.1080/02331930701617551.

[34] S. Dempe, B.S. Mordukhovich, and A.B. Zemkoho. "Necessary optimality conditions in pessimistic bilevel programming". In: *Optimization* 63.4 (2014), pp. 505–533. DOI: https://doi.org/10.1080/02331934.2012.696641.

[35] S. Dempe and H. Schmidt. "On an algorithm solving two-level programming problems with nonunique lower level solutions". In: *Computational Optimization and Applications* 6 (1996), pp. 227–249. DOI: 10.1007/BF00247793.

[36] E. Dockner, S. Jørgensen, N. Van Long, and G. Sorger. *Differential games in economics and management science*. New York: Cambridge University Press, 2000.

[37] N. Dunford, J.T. Schwartz, W.G Bade, and R.G. Bartle. *Linear operators*. Wiley-interscience New York, 1971.

[38] F. Facchinei, V. Piccialli, and M. Sciandrone. "Decomposition algorithms for generalized potential games". In: *Computational Optimization and Applications* 50 (2011), pp. 237–262. DOI: 10.1007/s10589-010-9331-9.

[39] G. Facchini, F. van Megen, P. Borm, and S. Tijs. "Congestion models and weighted Bayesian potential games". In: *Theory and Decision* 42 (1997), pp. 193–206. DOI: https://doi.org/10.1023/A:1004991825894.

[40] K. Fan. "Fixed-point and minimax theorems in locally convex topological linear spaces". In: *Proceedings of the National Academy of Sciences* 38 (1952), pp. 121–126. DOI: https://doi.org/10.1073/pnas.38.2.121.

[41] S.D. Flåm. "On games and cost of change". Preprints. 2018. URL: https://www.preprints.org/manuscript/201805.0025/v1.

[42] S.D. Flåm and A.S. Antipin. "Equilibrium programming using proximal-like algorithms". In: *Mathematical Programming* 78 (1996), pp. 29–41. DOI: https://doi.org/10.1007/BF02614504.

[43] S.D. Flåm and G.H. Greco. "Non-cooperative games; methods of subgradient projection and proximal point". In: *Advances in Optimization: Proceedings of the 6th French-German Colloquium on Optimization Held at Lambrecht, FRG, June 2–8, 1991*. Ed. by W. Oettli and D. Pallaschke. Berlin, Heidelberg: Springer, 1992, pp. 406–419.

[44] A. Fonseca-Morales and O. Hernández-Lerma. "Potential differential games". In: *Dynamic Games and Applications* 8 (2018), pp. 254–279. DOI: https://doi.org/10.1007/s13235-017-0218-6.

[45] D. Fudenberg and J. Tirole. *Game Theory*. Cambridge, MA: MIT Press, 1991.

[46] D. Fudenberg and J. Tirole. "Perfect Bayesian equilibrium and sequential equilibrium". In: *Journal of Economic Theory* 53 (1991), pp. 236–260. DOI: https://doi.org/10.1016/0022-0531(91)90155-W.

[47] D. Gabay and H. Moulin. "On the uniqueness and stability of Nash equilibrium in noncooperative games". In: *Applied Stochastic Control in Econometrics and Management Science*. Ed. by A. Bensoussan, P.R. Kleindorfer, and C.S. Tapiero. Amsterdam: North-Holland, 1980, pp. 271–293.

[48] I.L. Glicksberg. "A further generalization of the Kakutani fixed theorem, with application to Nash equilibrium points". In: *Proceedings of the American Mathematical Society* 3 (1952), pp. 170–174. DOI: https://doi.org/10.1090/S0002-9939-1952-0046638-5.

[49] J. González-Díaz and M. A. Meléndez-Jiménez. "On the notion of perfect Bayesian equilibrium". In: *TOP* 22 (2014), pp. 128–143. DOI: 10.1007/s11750-011-0239-z.

[50] W. Güth, R. Schmittberger, and B. Schwarze. "An experimental analysis of ultimatum bargaining". In: *Journal of Economic Behavior & Organization* 3 (1982), pp. 367–388. DOI: https://doi.org/10.1016/0167-2681(82)90011-7.

[51] P. T. Harker and J.-S. Pang. "Existence of optimal solutions to mathematical programs with equilibrium constraints". In: *Operations Research Letters* 7 (1988), pp. 61–64. DOI: https://doi.org/10.1016/0167-6377(88)90066-1.

[52] J. Harsanyi. "Oddness of the number of equilibrium points: a new proof". In: *International Journal of Game Theory* 2 (1973), 235âĂŞ–250.

[53] J.C. Harsanyi and R. Selten. *A general theory of equilibrium selection in games*. The MIT Press, 1988.

[54] A. Haurie, J.B. Krawczyk, and G. Zaccour. *Games and dynamic games*. World Scientific Publishing Company, 2012.

[55] A. N. Iusem and W. Sosa. "On the proximal point method for equilibrium problems in Hilbert spaces". In: *Optimization* 59 (2010), pp. 1259–1274. DOI: 10.1080/02331931003603133.

[56] A.N. Iusem. “Augmented Lagrangian methods and proximal point methods for convex optimization”. In: *Investigación Operativa* 8 (1999), pp. 11–49.

[57] S. Kakutani. “A generalization of Brouwer’s fixed point theorem”. In: *Duke mathematical journal* 8 (1941), pp. 457–459.

[58] S. Karamardian. “The nonlinear complementarity problem with applications, Part 1”. In: *Journal of Optimization Theory and Applications* 4 (1969), pp. 87–98. DOI: https://doi.org/10.1007/BF00927414.

[59] S. Karamardian. “The nonlinear complementarity problem with applications, Part 2”. In: *Journal of Optimization Theory and Applications* 4 (1969), pp. 167–181. DOI: https://doi.org/10.1007/BF00930577.

[60] M. Kojima, A. Okada, and S. Shindoh. “Strongly stable equilibrium points of n-person noncooperative games”. In: *Mathematics of Operations Research* 10 (1985), pp. 650–663.

[61] D. Kreps and R. Wilson. “Sequential equilibria”. In: *Econometrica* 50 (1982), pp. 863–894.

[62] H.W. Kuhn and A.W. Tucker. “Nonlinear Programming”. In: *Proceedings of the Second Berkeley Symposium on Mathematical Statistics and Probability*. Berkeley, CA: University of California Press, 1951, pp. 481–492.

[63] K. Kuratowski. *Topology*. Academic Press, 1966.

[64] N. Langenberg. *Convergence analysis of proximal-like methods for variational inequalities and fixed point problems: applications to the Nash equilibrium problem*. Logos Verlag Berlin, 2011.

[65] P.D. Lax and A.N. Milgram. “Parabolic equations”. In: *Contributions to the Theory of Partial Differential Equations*. Ed. by L. Bers, S. Bochner, and F. John. Vol. 33. Annals of Mathematics Studies. Princeton University Press, 1955, pp. 167–190. DOI: 10.1515/9781400882182-010.

[66] G. Leitmann. “On generalized Stackelberg strategies”. In: *Journal of Optimization Theory and Applications* 26 (1978), pp. 637–643. DOI: 10.1007/BF00933155.

[67] B. Lemaire. “The proximal algorithm”. In: *International series of numerical mathematics* 87 (1989), pp. 73–87.

[68] W. Leontief. “The pure theory of the guaranteed annual wage contract”. In: *Journal of Political Economy* 54 (1946), pp. 76–79. DOI: https://doi.org/10.1086/256311.

[69] S. Li and T. Başar. "Distributed algorithms for the computation of noncooperative equilibria". In: *Automatica* 23 (1987), pp. 523–533. DOI: https://doi.org/10.1016/0005-1098(87)90081-1.

[70] M. B. Lignola and J. Morgan. "A method to bypass the lack of solutions in MinSup problems under quasi-equilibrium constraints". In: *Optimization Letters* 10 (2016), pp. 833–846. DOI: https://doi.org/10.1007/s11590-015-0956-6.

[71] M. B. Lignola and J. Morgan. "Approximate values for mathematical programs with variational inequality constraints". In: *Computational Optimization and Applications* 53 (2012), pp. 485–503. DOI: https://doi.org/10.1007/s10589-012-9470-2.

[72] M. B. Lignola and J. Morgan. "Approximating security values of min-sup problems with quasi-variational inequality constraints". In: *Pacific Journal of Optimization* 10 (2014), pp. 749–765.

[73] M. B. Lignola and J. Morgan. "Asymptotic behavior of semi-quasivariational optimistic bilevel problems in Banach spaces". In: *Journal of Mathematical Analysis and Applications* 424 (2015), pp. 1–20. DOI: https://doi.org/10.1016/j.jmaa.2014.10.059.

[74] M. B. Lignola and J. Morgan. "Existence for optimization problems with equilibrium constraints in reflexive Banach spaces". In: *Optimization in Economics, Finance and Industry*. Milano: Datanova, 2002, pp. 15–35.

[75] M. B. Lignola and J. Morgan. "Existence of solutions to bilevel variational problems in Banach spaces". In: *Equilibrium Problems: Nonsmooth Optimization and Variational Inequality Models*. Ed. by F. Giannessi, A. Maugeri, and P. M. Pardalos. Boston, MA: Springer US, 2001, pp. 161–174. DOI: https://doi.org/10.1007/0-306-48026-3_10.

[76] M. B. Lignola and J. Morgan. "Semi-continuities of marginal functions in a sequential setting". In: *Optimization* 24 (1992), pp. 241–252. DOI: 10.1080/02331939208843793.

[77] M. B. Lignola and J. Morgan. "Viscosity solutions for bilevel problems with Nash equilibrium constraints". In: *Far East Journal of Applied Mathematics* 88 (2014), pp. 15–34.

[78] M.B. Lignola and J. Morgan. "Inner regularizations and viscosity solutions for pessimistic bilevel optimization problems". In: *Journal of Optimization Theory and Applications* 173 (2017), pp. 183–202. DOI: 10.1007/s10957-017-1085-4.

[79] M.B. Lignola and J. Morgan. "Stability of regularized bilevel programming problems". In: *Journal of Optimization Theory and Applications* 93 (1997), pp. 575–596. DOI: 10.1023/A:1022695113803.

[80] M.B. Lignola and J. Morgan. "Topological existence and stability for stackelberg problems". In: *Journal of Optimization Theory and Applications* 84 (1995), pp. 145–169. DOI: https://doi.org/10.1007/BF02191740.

[81] P. Loridan and J. Morgan. "A theoretical approximation scheme for Stackelberg problems". In: *Journal of Optimization Theory and Applications* 61 (1989), pp. 95–110. DOI: 10.1007/BF00940846.

[82] P. Loridan and J. Morgan. "Approximate solutions for two-level optimization problems". In: *Trends in Mathematical Optimization: 4th French-German Conference on Optimization*. Ed. by K.-H. Hoffmann, J. Zowe, J.-B. Hiriart-Urruty, and C. Lemarechal. Basel: Birkhäuser Basel, 1988, pp. 181–196. DOI: 10.1007/978-3-0348-9297-1_13.

[83] P. Loridan and J. Morgan. "Approximation results for two-level optimization problem and application to penalty methods". Università di Napoli. Dipartimento di Matematica e Applicazioni "R. Caccioppoli". 1985.

[84] P. Loridan and J. Morgan. "ϵ-Regularized two-level optimization problems: Approximation and existence results". In: *Optimization*. Ed. by S. Dolecki. Berlin, Heidelberg: Springer Berlin Heidelberg, 1989, pp. 99–113.

[85] P. Loridan and J. Morgan. "Least-norm regularization for weak two-level optimization problems". In: *Optimization, Optimal Control and Partial Differential Equations*. Ed. by V. Barbu, D. Tiba, and J. F. Bonnans. Basel: Birkhäuser Basel, 1992, pp. 307–318. DOI: 10.1007/978-3-0348-8625-3_28.

[86] P. Loridan and J. Morgan. "New results on approximate solution in two-level optimization". In: *Optimization* 20 (1989), pp. 819–836. DOI: https://doi.org/10.1080/02331938908843503.

[87] P. Loridan and J. Morgan. "On strict ϵ-solutions for a two-level optimization problem". In: *Papers of the 19th Annual Meeting / Vorträge der 19. Jahrestagung*. Ed. by W. Bühler, G. Feichtinger, R. F. Hartl, F. J. Radermacher, and P. Stähly. Berlin, Heidelberg: Springer Berlin Heidelberg, 1992, pp. 165–172.

[88] P. Loridan and J. Morgan. "Regularizations for two-level optimization problems". In: *Advances in Optimization*. Ed. by W. Oettli and D. Pallaschke. Berlin, Heidelberg: Springer Berlin Heidelberg, 1992, pp. 239–255.

[89] P. Loridan and J. Morgan. "Weak via strong Stackelberg problem: new results". In: *Journal of Global Optimization* (1996), pp. 263–287. DOI: https://doi.org/10.1007/BF00121269.

[90] R. Lucchetti, F. Mignanego, and G. Pieri. "Existence theorems of equilibrium points in Stackelberg". In: *Optimization* 18 (1987), pp. 857–866. DOI: https://doi.org/10.1080/02331938708843300.

[91] Z.-Q. Luo, J.-S. Pang, and D. Ralph. *Mathematical programs with equilibrium constraints*. Cambridge: Cambridge University Press, 1996.

[92] L.A. Lusternik and V.J. Sobolev. *Elements of functional analysis*. Delhi, India: Hindustan Publishing Corp., 1974.

[93] L. Mallozzi. "An application of optimization theory to the study of equilibria for games: a survey". In: *Central European Journal of Operations Research* 21 (2013), pp. 523–539. DOI: 10.1007/s10100-012-0245-8.

[94] L. Mallozzi and J. Morgan. "ϵ-Mixed strategies for static continuous-kernel Stackelberg games". In: *Journal of Optimization Theory and Applications* 78 (1993), pp. 303–316. DOI: 10.1007/BF00939672.

[95] L. Mallozzi and J. Morgan. "Hierarchical systems with weighted reaction set". In: *Nonlinear Optimization and Applications*. Ed. by G. Di Pillo and F. Giannessi. Boston, MA: Springer US, 1996, pp. 271–282. DOI: 10.1007/978-1-4899-0289-4_19.

[96] L. Mallozzi and J. Morgan. "Oligopolistic markets with leadership and demand functions possibly discontinuous". In: *Journal of Optimization Theory and Applications* 125 (2005), pp. 393–407. DOI: 10.1007/s10957-004-1856-6.

[97] L. Mallozzi and J. Morgan. "On approximate mixed Nash equilibria and average marginal functions for two-stage three-players games". In: *Optimization with Multivalued Mappings: Theory, Applications, and Algorithms*. Ed. by S. Dempe and V. Kalashnikov. Boston, MA: Springer US, 2006, pp. 97–107. DOI: 10.1007/0-387-34221-4_5.

[98] M Margiocco and L. Pusillo. "Potential games and well-posedness properties". In: *Optimization* 57 (2008), pp. 571–579. DOI: https://doi.org/10.1080/02331930601124997.

[99] B. Martinet. “Brève communication. Régularisation d’inéquations variationnelles par approximations successives”. In: *ESAIM: Mathematical Modelling and Numerical Analysis - Modélisation Mathématique et Analyse Numérique* 4 (1970), pp. 154–158.

[100] M. Maschler, E. Solan, and S. Zamir. *Game Theory*. Cambridge: Cambridge University Press, 2013.

[101] E. Michael. “Continuous selections. I”. In: *Annals of Mathematics* 63 (1956), pp. 361–382.

[102] G. J. Minty. “Monotone (nonlinear) operators in Hilbert space”. In: *Duke Mathematical Journal* 29 (1962), pp. 341–346. DOI: doi:10.1215/S0012-7094-62-02933-2.

[103] D.A. Molodtsov. “The solution of a class of non-antagonistic games”. In: *USSR Computational Mathematics and Mathematical Physics* 16 (1976), pp. 67–72. DOI: https://doi.org/10.1016/0041-5553(76)90042-2.

[104] D.A. Molodtsov and V.V. Fedorov. “Approximation of two-person games with information exchange”. In: *USSR Computational Mathematics and Mathematical Physics* 13 (1973), pp. 123–142. ISSN: 0041-5553. DOI: 10.1016/0041-5553(73)90010-4.

[105] D. Monderer and L.S. Shapley. “Potential games”. In: *Games and Economic Behavior* 14 (1996), pp. 124–143. DOI: 10.1006/game.1996.0044.

[106] J.J. Moreau. “Proximité et dualité dans un espace hilbertien”. In: *Bulletin de la Société mathématique de France* 93 (1965), pp. 273–299.

[107] J. Morgan. “Constrained well-posed two-level optimization problems”. In: *Nonsmooth Optimization and Related Topics*. Ed. by F.H. Clarke, V.F. Dem’yanov, and F. Giannessi. Boston, MA: Springer US, 1989, pp. 307–325. DOI: https://doi.org/10.1007/978-1-4757-6019-4_18.

[108] J. Morgan. “Methode directe de recherche du point de selle d’une fonctlonnelle convexe-concave et application aux problémes variationnels elliptiques avec deux controles antagonistes”. In: *International Journal of Computer Mathematics* 4 (1974), pp. 143–175. DOI: https://doi.org/10.1080/00207167408803086.

[109] J. Morgan and F. Patrone. “Stackelberg problems: subgame perfect equilibria via Tikhonov regularization”. In: *Advances in Dynamic Games: Applications to Economics, Management Science, Engineering, and Environmental Management*. Ed. by A. Haurie, S. Muto, L. A. Petrosjan,

and T. E. S. Raghavan. Boston, MA: Birkhäuser Boston, 2006, pp. 209–221. DOI: 10.1007/0-8176-4501-2_12.

[110] J. Morgan and R. Raucci. "Continuity properties of ϵ-solutions for generalized parametric saddle point problems and application to hierarchical games". In: *Journal of Mathematical Analysis and Applications* 211 (1997), pp. 30–48. DOI: https://doi.org/10.1006/jmaa.1997.5362.

[111] J. Morgan and R. Raucci. "Lower semicontinuity for approximate social Nash equilibria". In: *International Journal of Game Theory* 31 (2003), pp. 499–509. DOI: 10.1007/s001820300134.

[112] J. Morgan and V. Scalzo. "Pseudocontinuity in optimization and nonzero-sum games". In: *Journal of Optimization Theory and Applications* 120 (2004), pp. 181–197. DOI: 10.1023/B:JOTA.0000012738.90889.5b.

[113] J. Morgan and V. Scalzo. "Pseudocontinuous functions and existence of Nash equilibria". In: *Journal of Mathematical Economics* 43 (2007), pp. 174–183. DOI: https://doi.org/10.1016/j.jmateco.2006.10.004.

[114] U. Mosco. "Convergence of convex sets and of solutions of variational inequalities". In: *Advances in Mathematics* 3 (1969), pp. 510–585. DOI: https://doi.org/10.1016/0001-8708(69)90009-7.

[115] A. Moudafi. "Proximal point algorithm extended to equilibrium problems". In: *Journal of Natural Geometry* 15 (1999), pp. 91–100.

[116] A. Moudafi and M. Théra. "Proximal and Dynamical Approaches to Equilibrium Problems". In: *Ill-posed Variational Problems and Regularization Techniques*. Ed. by M. Théra and R. Tichatschke. Berlin, Heidelberg: Springer Berlin Heidelberg, 1999, pp. 187–201. DOI: https://doi.org/10.1007/978-3-642-45780-7_12.

[117] R. Myerson. *Game theory: analysis of conflict*. Cambridge, MA: Harvard University Press, 1991.

[118] R. Myerson. "Refinements of Nash equilibrium concept". In: *International Journal of Game Theory* 7 (1978), pp. 73–80. DOI: 10.1007/BF01753236.

[119] J.F. Nash. "Equilibrium points in n-person games". In: *Proceedings of the National Academy of Sciences* 36 (1950), pp. 48–49. DOI: 10.1073/pnas.36.1.48.

[120] J.F. Nash. "Non-Cooperative Games". In: *Annals of Mathematics* 54 (1951), pp. 286–295. DOI: 10.2307/1969529.

[121] A. Neyman. “Correlated equilibrium and potential games”. In: *International Journal of Game Theory* 26 (1997), pp. 223–227. DOI: https://doi.org/10.1007/BF01295851.

[122] A. Okada. “On stability of perfect equilibrium points”. In: *International Journal of Game Theory* 10 (1981), pp. 67–73.

[123] J. V. Outrata. “Necessary optimality conditions for Stackelberg problems”. In: *Journal of Optimization Theory and Applications* 76 (1993), pp. 305–320. DOI: 10.1007/BF00939610.

[124] J.-S. Pang and M. Fukushima. “Quasi-variational inequalities, generalized Nash equilibria, and multi-leader-follower games”. In: *Computational Management Science* 2 (2005), pp. 21–56. DOI: 10.1007/s10287-004-0010-0.

[125] N. Parikh and S. Boyd. “Proximal algorithms”. In: *Foundations and Trends® in Optimization* 1 (2014), pp. 127–239. DOI: http://dx.doi.org/10.1561/2400000003.

[126] R. Raucci. “Problemi gerarchici di punti di Nash: esistenza e approssimazione nel caso di unicita al livello inferiore”. In: *Rendiconti del Circolo Matematico di Palermo* 47 (1998), pp. 113–128.

[127] K. Ritzberger. “The theory of normal form games from the differentiable viewpoint”. In: *International Journal of Game Theory* 23 (1994), pp. 207–236. DOI: 10.1007/BF01247316.

[128] R. T. Rockafellar. *Convex analysis*. Princeton university press, 1970.

[129] R. T. Rockafellar. “On the maximal monotonicity of subdifferential mappings”. In: *Pacific Journal of Mathematics* 33 (1970), pp. 209–216.

[130] R.T. Rockafellar. “Characterization of the subdifferentials of convex functions”. In: *Pacific Journal of Mathematics* 17 (1966), pp. 497–510.

[131] R.T. Rockafellar. “Monotone operators and the proximal point algorithm”. In: *SIAM Journal on Control and Optimization* 14 (1976), pp. 877–898. DOI: https://doi.org/10.1137/0314056.

[132] J.B. Rosen. “Existence and uniqueness of equilibrium points for concave N-person games”. In: *Econometrica* 33 (1965), pp. 520–534. DOI: 0012-9682(196507)33:3<520:EAUOEP>2.0.CO;2-2.

[133] S. Sagratella. “Algorithms for generalized potential games with mixed-integer variables”. In: *Computational Optimization and Applications* 68 (2017), pp. 689–717. DOI: 10.1007/s10589-017-9927-4.

[134] R. Selten. "Reexamination of the perfectness concept for equilibrium points in extensive games". In: *International Journal of Game Theory* 4 (1975), pp. 25–55. DOI: https://doi.org/10.1007/BF01766400.

[135] R. Selten. "Spieltheoretische behandlung eines oligopolmodells mit nachfrageträgheit". In: *Zeitschrift für die gesamte Staatswissenschaft* (1965), pp. 301–324, 667–689.

[136] H. D. Sherali, A. L. Soyster, and F. H. Murphy. "Stackelberg-Nash-Cournot equilibria: characterizations and computations". In: *Operations Research* 31 (1983), pp. 253–276. DOI: 10.1287/opre.31.2.253.

[137] K. Shimizu and E. Aiyoshi. "A new computational method for Stackelberg and min-max problems by use of a penalty method". In: *IEEE Transactions on Automatic Control* 26 (1981), pp. 460–466. DOI: 10.1109/TAC.1981.1102607.

[138] M. Simaan and J. B. Cruz. "On the Stackelberg strategy in nonzero-sum games". In: *Journal of Optimization Theory and Applications* 11 (1973), pp. 533–555. DOI: 10.1007/BF00935665.

[139] V.F. Solohovic. "Unstable extremal problems and geometric properties of Banach spaces". In: *Soviet Mathematics Doklady* 11 (1970), pp. 1470–1472.

[140] A.N. Tikhonov. "Solution of incorrectly formulated problems and the regularization method". In: *Soviet Mathematics Doklady* 4 (1963), pp. 1035–1038.

[141] A.N. Tikhonov, A. Goncharsky, V.V. Stepanov, and A. G. Yagola. *Numerical Methods for the Solution of Ill-Posed Problems*. Vol. 328. Springer Netherlands, 1995.

[142] E. van Damme. "History-dependent equilibrium points in dynamic games". In: ed. by O. Moeschlin and D. Pallaschke. Amsterdam: North-Holland, 1981, pp. 27–38.

[143] E. van Damme. *Refinements of the Nash equilibrium concept*. Springer-Verlag Berlin Heidelberg, 1983.

[144] L.N. Vicente and P.H. Calamai. "Bilevel and multilevel programming: A bibliography review". In: *Journal of Global Optimization* 5 (1994), pp. 291–306. DOI: https://doi.org/10.1007/BF01096458.

[145] H. von Stackelberg. *The theory of the market economy*. Oxford: Oxford University Press, 1952.

[146] B. von Stengel and S. Zamir. *Leadership with commitment to mixed strategies*. Tech. rep. Centre for Discrete and Applicable Mathematics, London School of Economics, 2004.

[147] M. Voorneveld and H. Norde. “A Characterization of Ordinal Potential Games”. In: *Games and Economic Behavior* 19 (1997), pp. 235–242. DOI: https://doi.org/10.1006/game.1997.0554.

[148] W. Wen-Tsun and J. Jia-He. “Essential equilibrium points of n-person non-cooperative games”. In: *Scientia Sinica* 11 (1962), pp. 1307–1322.

[149] R.J.B. Wets. “Convergence of convex functions, variational inequalities and convex optimization problems”. In: *Variational inequalities and complementarity problems*. 1980, pp. 375–403.

[150] W. Wiesemann, A. Tsoukalas, P.-M. Kleniati, and B. Rustem. “Pessimistic bilevel optimization”. In: *SIAM Journal on Optimization* 23 (2013), pp. 353–380. DOI: https://doi.org/10.1137/120864015.

[151] J. J. Ye and D. L. Zhu. “Optimality conditions for bilevel programming problems”. In: *Optimization* 33 (1995), pp. 9–27. DOI: https://doi.org/10.1080/02331939508844060.

[152] T. Zolezzi. “On convergence of minima”. In: *Bollettino dell’Unione Matematica Italiana* 8 (1973), pp. 246–257.

[153] T. Zolezzi. “On stability analysis in mathematical programming”. In: *Sensitivity, Stability and Parametric Analysis*. Ed. by A. V. Fiacco. Berlin, Heidelberg: Springer Berlin Heidelberg, 1984, pp. 227–242. DOI: 10.1007/BFb0121220.