Università Degli Studi Di Napoli "Federico II"

Dottorato di Ricerca in Scienze Matematiche ed Informatiche, XXXI Ciclo

Thesis submitted for the degree of Doctor of Philosophy

Isoperimetric Problems in Quantitative Form

Supervisor:
Chiar.mo Prof.
Nicola Fusco

Author:
Domenico Angelo
La Manna

CHAPTER 1

Introduction

In this thesis we collect some results on two problems of isoperimetric type obtained by the author and contained in [**46**],[**47**], [**48**]. The first result concerns the isoperimetric problem in the Gauss space in the class of sets E symmetric with respect to the origin, i.e. $E = -E$. We recall that the Gauss space is the space $\mathbb{R}^n$ equipped with the probability measure γ defined by

$$\gamma(E) = \frac{1}{(2\pi)^{\frac{n}{2}}} \int_E e^{-\frac{|x|^2}{2}} dx.$$

For a set of locally finite Euclidean perimeter, the Gaussian perimeter is defined by setting

$$P_\gamma(E) = \frac{1}{(2\pi)^{\frac{n}{2}}} \int_{\partial^* E} e^{-\frac{|x|^2}{2}} d\mathcal{H}^{n-1}$$

where $\partial^* E$ is the reduced boundary of E in the sense of De Giorgi. It is well known that the isoperimetric sets in Gauss space are the halfspaces

$$H^-_{\omega,s} = \{x \in \mathbb{R}^n : \quad x \cdot \omega \le s\}$$

where $\omega \in \mathbb{S}^{n-1}$ and $s \in \mathbb{R}$. More precisely, if E is a measurable subset of $\mathbb{R}^n$, then

$$P_\gamma(E) \ge P_\gamma(H^-_{\omega,s}) \tag{1.1}$$

where the halfspace $H^-_{\omega,s}$ is such that $\gamma(H^-_{\omega,s}) = \gamma(E)$. Moreover, in (1.1) equality holds if and only if $E = H^-_{\omega,s}$, for some $\omega \in \mathbb{S}^{n-1}$, up to a set of null measure. On the other hand, if one restricts the problem to the class of symmetric sets, the characterization of the isoperimetric sets is still open. As explained in details in the introduction to Chapter 4, it has been suggested by several authors ([**18, 40, 41, 59**]) that they could be, depending on the Gaussian volume of E, either the ball centered at the origin (or its complement) or a symmetric strip. In this thesis we prove that for $n \ge 2$ and $r \in (0, \sqrt{n+1})$, the ball centered at the origin is a local minimizer of the perimeter among symmetric sets of prescribed Gaussian volume. More precisely, one has the following quantitative inequality: *Given $r \in (0, \sqrt{n+1})$ there exist $\delta, \kappa > 0$ such that if $E = -E$, $\gamma(E) = \gamma(B_r)$ and $\gamma(E \Delta B_r) \le \delta$, then*

$$P_\gamma(E) - P_\gamma(B_r) \ge \kappa \gamma(E \Delta B_r).$$

Moreover, δ and κ are bounded away from zero if $r \in [a, b] \subset (0, \sqrt{n+1})$. This local minimality result is optimal since in the same chapter it is also proved that if $r > \sqrt{n+1}$ the ball B_r is not a local minimizer. Differently from what happens in higher dimension, if $n = 1$ we prove that there exists $r_0 > 0$ such that if $r > r_0$ then the unique minimizer of the Gaussian perimeter among symmetric sets of prescribed Gaussian volume is the interval centered at the origin, while if $r < r_0$ the unique minimizer is given by $C_s = (-\infty, s) \cup (s, \infty)$, with s such that $\gamma(B_r) = \gamma(C_s)$. Finally, if $r = r_0$, the interval $(-r_0, r_0)$ and its complement are the unique minimizers. The results obtained in Chapter 4 are proven using a technique based on the second variation, introduced for the first time in this context by Fuglede in [**34**], together with a selection principle introduced by Cicalese and Leonardi in [**23**] as modified by Acerbi, Fusco and Morini in [**1**]. The use of this strategy is based on the regularity of minimizers (and of quasi minimizers) of the Euclidean perimeter, see the introduction to Chapter 4 for a more detailed explanation. Note that in our case the above strategy is more complicated. An obvious difficulty comes from the constraint that the competing sets must be symmetric with respect to the origin. But the main source of problems is due to the presence of possible unbounded competitors of balls.

In the fifth chapter we study a nonlocal isoperimetric problem, namely the minimization of the functional

$$I(E) = P(E) + \int_E \int_E \frac{1}{|x-y|^{n-2}} dxdy - K \int_E \frac{1}{|x|^{n-2}} dx \tag{1.2}$$

among all sets of prescribed measure, where $P(E)$ denotes the (Euclidean) perimeter of E in the De Giorgi sense and K is a given nonnegative constant. As explained in the introduction to Chapter 5, this functional was proposed as a sharp interface version of the Thomas-Fermi-Dirac-Von Weiszacker model. From the energetic point of view, here the perimeter has the role of a cohesion force, in the sense that it tends to keep the particles close to each other, while the non local interaction would like to spread the set as much as possible and the Coulombic attraction tries to attract the particles to the charge fixed at the origin of the space.

This model is a variant of a simpler one where $K = 0$, which has been widely studied in literature and for which various authors ([**13, 43, 44, 55**]) proved the characterization of global minimizers and the non existence of the latters, depending on the volume of E. For the functional (1.2), Lu and Otto in [**49**] proved that there exists a critical mass m_c such that the constrained problem for $I(E)$ does not admit minimizers for $m \geq m_c$. In Chapter 5 of this thesis we study the local and global minimality of balls. In particular, we prove that there exists a critical radius r_0 such that if B_r is the ball centered at the origin with radius $r < r_0$, then B_r is a local minimizer of the costrained problem with respect to L^1 variations, while this local minimality property fails if $r > r_0$. Furthermore, we prove that there exists a radius r_1 such that if $r < r_1$, the ball B_r is the unique global minimizer for the functional. While the radius r_1 has an explicit expression, the value of r_0 is characterized as solution of an algebraic equation of degree n. Both r_0 and r_1 tend to infinity as $K \to \infty$ while the quotient $\frac{r_0}{r_1}$ is bounded from above, for every K, by a constant depending only on the dimension. The strategy to prove the local minimality is again a combination of arguments based on the second variation and a sort of selection principle. Nevertheless, the non local nature of the potential terms in (1.2) makes the use of this strategy much more delicate then in most cases treated in literature. As result of this strategy, also for the functional I the local minimality of B_r, for $r < r_0$, comes together with a stability estimate. More precisely, we prove that, given $r \in (0, r_0)$, there exist $\delta, C > 0$ such that if $E \subset \mathbb{R}^n$, $|E| = |B_r|$ and $|E \Delta B_r| \leq \delta$, then

$$I(E) \geq I(B_r) + C|E\Delta B_r|^2.$$

Two introductory chapters have been added before Chapters 4 and 5. In the first of them we start with some definitions and preliminary results which are used in the rest of the thesis. We also give a short account of the proof of the quantitative isoperimetric inequality of Fusco, Maggi and Pratelli ([**37**]), a result used in the sequel. In Chapter 3 we sketch the proof of the quantitative isoperimetric inequality in the Gauss space given by Barchiesi, Brancolini and Julin in [**5**], which, even if it is not directly used in the following, could help the reader to better understanding the results of Chapter 4.

CHAPTER 2

The quantitative isoperimetric inequality

We start this section giving some definitions and results that will be useful for the rest of the present thesis. First of all, we give the definition of a function of bounded variation and after that we define the class of sets of finite perimeter. In order to do that, we follow the notation of [**3**].

0.1. BV functions and sets of finite perimeter.

DEFINITION 2.1. Let $n \geq 1$, $\Omega \subset R^n$ an open set and $u \in L^1(\Omega)$. We will say that u is a function of bounded variation in Ω iff there exist a vector valued Radon measure $\mu = (\mu_1, \mu_2, \ldots, \mu_n)$ in Ω such that

$$\int_\Omega u \frac{\partial \phi}{\partial x_i} = -\int_\Omega \phi d\mu_i, \qquad \forall \phi \in C_c^\infty(\Omega). \tag{2.1}$$

The vector measure μ is the distributional derivative of u and we denote it as Du. The total variation of the measure Du is denoted, as usual, by $|Du|(\Omega)$ and the set of all functions of bounded variation is denoted by $\mathrm{BV}(\Omega)$.

Now we define the variation of u.

DEFINITION 2.2. Let $u \in L^1(\Omega)$. The variation of u in Ω is indicated as $V(u, \Omega)$ and is defined as follows

$$V(u, \Omega) = \sup \left\{ \int_\Omega u \mathrm{div} \phi dx : \quad \phi \in C_c^\infty(\Omega, \mathbb{R}^n), \, ||\phi||_\infty \leq 1 \right\}. \tag{2.2}$$

The following proposition gives us a characterization of the functions belonging to the space $\mathrm{BV}(\Omega)$.

PROPOSITION 2.3. *Let $u \in L^1(\Omega)$. Then $u \in BV(\Omega)$ if and only if $V(u, \Omega) < \infty$. In addition, $V(u, \Omega) = |Du|(\Omega)$ for all $u \in BV(\Omega)$ and $u \to V(y, \Omega)$ is lower semicontinuous with respect to the L^1 convergence.*

Now we state two important theorems about BV functions: the first one is the analogous of the celebrated Meyers and Serrin's theorem while the second is a compactness result.

THEOREM 2.4. *Let $u \in L^1(\Omega)$. Then $u \in BV(\Omega)$ if and only if there exists a sequence $\{u_h\}_{h \in \mathbb{N}} \subset C^\infty(\Omega)$ converging to u in L^1 and such that*

$$L : \lim_{h \to +\infty} \int_\Omega |Du_h| dx < \infty. \tag{2.3}$$

Moreover, the sequence can be chosen so that $L = |Du|(\Omega)$.

THEOREM 2.5. *Let Ω be a bounded open subset of $\mathbb{R}^n$ with Lipschitz boundary. Then every sequence $\{u_h\}_{h \in \mathbb{N}} \subset BV(\Omega)$ such that*

$$\sup \left\{ \int_A |u_h| dx + |Du_h|(\Omega) \right\}$$

admits a subsequence $u_{h(k)}$ converging in L^1 to a function $u \in BV(\Omega)$ and $Du_{h(k)}$ weakly converges to Du^* in the sense of measure.*

The last theorem we want to state about BV functions, is the so called Gagliardo Nirenberg inequality, which is strictly related to the isoperimetric inequality.

Theorem 2.6 (Gagliardo Nirenberg inequality). *For any function $u \in L^1_{loc}(\mathbb{R}^n)$ there exists a real number m such that*

$$||u - m||_{L^{\frac{n}{n-1}}(\mathbb{R}^n)} \leq c(n)V(u, \mathbb{R}^n) \tag{2.4}$$

If $u \in BV(\mathbb{R}^n)$, then $m = 0$ and hence $||u||_{L^{\frac{n}{n-1}}(\mathbb{R}^n)} \leq c(n)|Du|(\mathbb{R}^n)$

Now we are ready to give the definition of set of finite perimeter.

Definition 2.7. Let $n \geq 1$, $E \subset \mathbb{R}^n$ a Borel set and $\Omega \subset R^n$. E is said to be of finite perimeter in Ω if $\chi_E \in \mathrm{BV}(\Omega)$. The total variation of χ_E in Ω will be denoted as $P(E; \Omega)$, the perimeter of E in Ω.

Note that if E is a smooth set, by the divergence theorem

$$\int_E \mathrm{div}\phi dx = \int_{\partial E \cap \Omega} \phi \cdot \nu_E d\mathcal{H}^{n-1}, \qquad \forall \phi \in C^\infty_c(\Omega, \mathbb{R}^n). \tag{2.5}$$

Thus, taking the supremum over ϕ with $||\phi||_\infty \leq 1$, we get

$$P(E; \Omega) = \mathcal{H}^{n-1}(\partial E).$$

We give now the definition of reduced boundary of a set of finite perimeter.

Definition 2.8. Let E be a set of finite perimeter in Ω. The reduced boundary of E in Ω, denoted by $\partial^* E$, is defined as the collection of points in $\Omega \cap \mathrm{supp}|D\chi_E|$ where the limit

$$\nu_E(x) := -\lim_{\rho \to 0} \frac{D\chi_E(B_\rho(x))}{|D\chi_E(B_\rho(x))|}$$

exists and satisfies $|\nu_E(x)| = 1$. The function $\nu_E(x)$ is called the generalized inner normal to E.

Note that by the Besicovitch derivation theorem we have that ν_E exists $|D\chi_E|-$a.e. and by the Radon-Nikodym theorem $D\chi_E = \nu_E|D\chi_E|$ and thus for a set of finite perimeter E we can rewrite the Definition (2.5) as

$$\int_E \mathrm{div}\phi dx = -\int_{\partial^* E \cap \Omega} (\phi, \nu_E) d|D\chi_E| \quad \forall \phi \in C^1_c(\Omega, \mathbb{R}^n).$$

Theorem 2.9 (De Giorgi). *Let $E \subset \mathbb{R}^n$ be a set of finite perimeter, then the following hold:*

(1) *$\partial^* E$ is countably $(n-1)$-rectifiable, i.e., $\partial^* E = K_i \cap N_0$ where K_i are compact subsets of C^1 manifolds M_i of dimension $n-1$ and $\mathcal{H}^{n-1}(N_0) = 0$*
(2) $|D\chi_E| = \mathcal{H}^{n-1} \llcorner \partial E$
(3) *for $\mathcal{H}^{n-1}$-a.e. $x \in K_i$ the generalized exterior normal ν_E is orthogonal to the tangent hyperplane to the manifold M_i at x*
(4) *for all $x \in \partial^* E$*

$$\lim_{r \to 0} \frac{|E \cap B_r(x)|}{|B_r(x)|} = \frac{1}{2}$$

(5) *for all $x \in \partial^* E$*

$$\lim_{r \to 0} \frac{\mathcal{H}^{n-1}(\partial^* E \cap B_r(x))}{\mathcal{H}^{n-1}(B_r)} = 1$$

Property (4) of Theorem 2.9 suggests an alternative, equivalent, definition of the reduced boundary of E.

DEFINITION 2.10. Let $E \subset \mathbb{R}^n$ be a measurable set and $t \in [0,1]$. We set

$$E^{(t)} = \{x \in \mathbb{R}^n : \lim_{r\to 0} \frac{|E \cap B_r(x)|}{|B_r(x)|} = t\}$$

. Then, the measure theoretic boundary of E is defined as $\partial^M E := \mathbb{R}^n \setminus (E^{(0)} \cup E^{(1)})$

The following theorem tells us that, up to $\mathcal{H}^{n-1}$ negligible sets, $\partial^* E$ and $\partial^M E$ coincide.

THEOREM 2.11 (Federer). *Let E be a set of finite perimeter of $\mathbb{R}^n$. Then*

$$\partial^* E \subset E^{(\frac{1}{2})} \subset \partial^M E, \qquad \mathcal{H}^{n-1}(\mathbb{R}^n \setminus (E^{(1)} \cup E^{(0)} \cup \partial^* E)) = 0.$$

The introduction of sets of finite perimeter in the De Giorgi sense has been very useful to better understanding the isoperimetric inequality and the Plateau problem. We give now the definition of perimeter minimizer set.

DEFINITION 2.12. Let $n \geq 2$ and $\Omega \subset \mathbb{R}^n$ open. A set E is said to be a perimeter minimizer in Ω if

$$P(E;\Omega) \leq P(F;\Omega)$$

for all $F \subset \mathbb{R}^n$ such that $F\Delta E \subset\subset \Omega$.

DEFINITION 2.13. Given $\Lambda, r_0 \in \mathbb{R}$, a set E is a (Λ, r_0)-perimeter minimizer if

$$P(E; B_r(x)) \leq P(F; B_r(x)) + \Lambda |E\Delta F| \tag{2.6}$$

whenever $E\Delta F \subset B_r(x) \cap \Omega$ and $r < r_0$.

We state here the fundamental regularity theorem for (Λ, r_0)-perimeter minimizers. Before that we introduce the following convergence of measurable sets. We say that a sequence of measurable sets $E_h \subset \mathbb{R}^n$ converges in measure in Ω to E if $|(E_h \Delta E) \cap \Omega| \to 0$ as $h \to \infty$, or equivalently if $\chi_{E_h} \to \chi_E$ in $L^1(\Omega)$. The local convergence in measure is defined in the obvious way.

THEOREM 2.14 ($C^{1,\gamma}$-regularity). *If $n \geq 2$, $\Omega \subset \mathbb{R}^n$ is an open set and E is a (Λ, r_0)-perimeter minimizer in Ω with $\Lambda r_0 < 1$, then $\Omega \cap \partial^* E$ is a $C^{1,\gamma}$ hypersurface for every $\gamma < \frac{1}{2}$, it is relatively open in $\Omega \cap \partial E$ and $\mathcal{H}^s(\partial E \setminus \partial^* E) = 0$ for all $s < 8$. Moreover, if E_j is a sequence of equibounded (Λ, r_0)-perimeter minimizers converging in measure to a C^2 set E, then for j large each E_j is of class $C^{1,\gamma}$ and $E_j \to E$ in $C^{1,\gamma}$.*

REMARK 2.15. The estimate on the Hausdorff dimension of the singular set in Theorem 2.14 is sharp. Indeed, in dimension 8 the Simon's cone defined as

$$C = \{(x,y) \in \mathbb{R}^4 \times \mathbb{R}^4 : |x|^2 = |y|^2\}$$

is a minimal surface whose singular set has Hausdorff dimension equal to 0, see [**63**],[**12**] for the original proof and [**27**] for a new and short proof of the same result.

1. The isoperimetric inequality

We start this section stating the geometric form of the isoperimetric inequality.

THEOREM 2.16 (Isoperimetric inequality). *Let $n \geq 2$, $m > 0$ and r such that $|B_r| = m$. Then*

$$P(E) \geq P(B_r) \tag{2.7}$$

for all sets of finite perimeter E such that $|E| = m$. Furthermore, equality holds if and only if $E = B_r(x_0)$ for some $x_0 \in \mathbb{R}^n$.

REMARK 2.17. The isoperimetric inequality can be restated in an analitic way: for every set of finite perimeter and finite measure E the following inequality holds:

$$(n\omega_n)^{\frac{1}{n}}|E|^{\frac{n-1}{n}} \leq P(E).$$

If we apply the Gagliardo-Nirenberg inequality to the characteristic function of a set of finite perimeter we get

$$|E|^{\frac{n-1}{n}} \leq c(n)P(E).$$

This observation suggests us that the best constant in the Gagliardo Niremberg inequality is $c(n) = (n\omega_n)^{-\frac{1}{n}}$.

The isoperimetric inequality has been investigated for many years by very famous mathematicians and it has been proved in several ways. Around the beginning of the last century, Bonnesen in [**14**] proved that in the plane the isoperimetric inequality can be stated in a stable way. Later, Osserman in [**61**] continued the study of the stability of the isoperimetric inequality calling this kind of inequalities after Bonnesen. In dimension higher than two, the optimal Bonnesen type inequality has been proved by Fusco, Maggi and Pratelli in [**37**] and it is stated as follows.

THEOREM 2.18 (The sharp quantitative isoperimetric inequality). *Let $n \geq 1$. There exists a constant $\gamma(n)$ such that for every measurable set E of finite measure*

$$D(E) \geq \gamma(n)\alpha(E)^2, \tag{2.8}$$

where $D(E)$ and $\alpha(E)$ are scaling invariant quantities called, respectively, the isoperimetric deficit and Fraenkel asymmetry and are defined as

$$D(E) := \frac{P(E) - P(B_r)}{P(E)},\ \alpha(E) := \inf_{x_0 \in \mathbb{R}^n} \left\{ \frac{|E \Delta B_r(x_0)}{|E|} \right\}$$

where $|B_r| = |E|$.

After the paper [FMP], the study of the stability of isoperimetric inequalities has become an important branch of the modern analysis, see [F] for the main generalizations of the problem.

1.1. The quantitative isoperimetric inequality. In this section we give a sketch of the proof of the quantitative isoperimetric inequality done by Fusco, Maggi and Pratelli. To this end, we now introduce the Steiner symmetrization of a set. For $x \in \mathbb{R}^n$, we will use the notation $x = (x', y)$ with $x' \in \mathbb{R}^{n-k}$ and $y \in \mathbb{R}^k$.

Given a measurable set $E \subset \mathbb{R}^n$, we define the essential projection of E over the the first k coordinate hyperplanes, for all $1 \leq k \leq n-1$ as

$$\pi(E)^+ = \{x' \in \mathbb{R}^{n-k} : \mathcal{H}^k(\{y \in \mathbb{R}^n (x', y) \in E\}) > 0\}.$$

With $r_E(x')$ we denote the radius of the k dimensional ball having the same H^k measure of $E_{x'} = \{y \in \mathbb{R}^k : (x', y) \in E\}$. The Steiner symmetral of E of codimension k is defined as

$$E^S = \{(x', y) : x' \in \pi^+(E), |y| < r_E(x')\}. \tag{2.9}$$

Note that if E has finite perimeter then the same is true for E^s, see [**6**]. Observe also that by Fubini's theorem we have that $|E| = |E^s|$, while the perimeter decreases under Steiner symmetrization (see [**6**]). Let $v_E(x') = \mathcal{H}^k(E_{x'})$ and $p_E(x') = \mathcal{H}^{k-1}(\partial^*(E_{x'}))$.

THEOREM 2.19. *Let $E \subset R^k$ be a set of finite perimeter and let E^S its Steiner symmetral of codimension k. Then for any Borel set $U \subset \mathbb{R}^{n-k}$*

$$P(E^S; U \times \mathbb{R}^k) \leq P(E; U \times \mathbb{R}^k) \tag{2.10}$$

Moreover,

$$P(E^S;U) = \int_U \sqrt{p_{E^S}(x')^2 + |\nabla v_E(x')|^2}dx' + |D^s v_E|(U).$$

In particular, if $k = n-1$ we have

$$P(E^S;U) = \int_U \sqrt{(n-1)^2\omega_{n-1}^{\frac{2}{n-1}} v(t)^{\frac{2(n-2)}{n-1}} + v'(t)^2}dx' + |D^s v_E|(U), \tag{2.11}$$

Since the quantities involved in the quantitative isoperimetric inequality are scaling invariant, we prove it when $|E| = |B|$. The first thing to prove is that we can reduce ourselves to the case of equibounded sets.

LEMMA 2.20. *There exists two constants. l, C depending only on n such that given any set of finite perimeter E, with $|E| = |B|$, there exists a measurable set F such that $F \subset [-l, l]^n = Q_l$, $|F| = |B|$ and*

$$D(F) \leq CD(E), \qquad \alpha(E) \leq \alpha(F) + CD(E). \tag{2.12}$$

The proof of this lemma is technical and it is not the key point of the proof of the quantitative isoperimetric inequality. Thus we will skip it. Before going on, we state two useful lemmas: the first one is a qualitative result while the second lemma, although easy to prove, will be a very powerful tool for the rest of the proof.

LEMMA 2.21. *Let $l > 0$. For any $\varepsilon > 0$ there exist $\delta > 0$ such that if $E \subset Q_l$, $|E| = |B|$ and $D(E) \leq \delta$ then $\alpha(E) < \varepsilon$.*

The proof of this lemma is not hard. It may be obtained with a contradiction argument and using the compactness theorem for BV functions. Next, following the terminology introduced in [**37**], we say that $E \subset \mathbb{R}^n$ n is an n-symmetric set if it is symmetric with respect to the n coordinate hyperplanes.

LEMMA 2.22. *Let E be an $n-$symmetric set with $|E| = |B|$. Then*

$$\alpha(E) \leq \frac{|E\Delta B_r|}{|B_r|} \leq 3\alpha(E).$$

More over, if E is also convex

$$\alpha(E) = \frac{|E\Delta B_r|}{|B_r|}.$$

The next step is to show that we may reduce to a $n-$symmetric set. Given a hyperplane H, we consider the two half spaces H^+, H^- in which $\mathbb{R}^n$ is divided by H and denote by r_H the reflection about H. Let E be a measurable set divided by the hyperplane H in two parts of equal volume. Then consider the two halves in which E is divided by H: $E \cap H^+$ and $E \cap H^-$ and the sets obtained by adding to each half its symmetral with respect to H, i.e.

$$E^+ := (E \cap H^+) \cup r_H(E \cap H^+), \qquad E^- := (E \cup H^-) \cap r_H(E \cap H^-). \tag{2.13}$$

By construction, $|E^+| = |E^-| = E|$. Moreover, it is not hard to show that

$$P(E^+) + P(E^-) \leq 2P(E), \quad \text{hence} \quad D(E^\pm) \leq 2D(E), \tag{2.14}$$

with the first inequality possibly being strict. Thus, if for some universal constant $C(n)$ one has

$$\alpha(E) \leq C(n)\alpha(E^+) \quad \text{or} \quad \alpha(E) \leq C(n)\alpha(E^-)$$

iterating this estimate we would immediately get (2.15). Unfortunately, this does not hold. The next proposition provides the right strategy.

Proposition 2.23. *There exist δ and C, depending on n, such that if $E \subset Q_l$, $|E| = |B|$ and $D(E) \leq \delta$, given any two orthogonal hyperplanes H_1, H_2 dividing E in four parts of equal measure and the four sets $E\pm_1, E_2^{\pm}$ defined as (2.13) with H replaced by H_1 and H_2, respectively, we have that at least one of them, call it $\tilde{E}$ satisfies the estimate*

$$\alpha(E) \leq C_0 \alpha(\tilde{E}) \tag{2.15}$$

Proof. *Step 1.* Let $E_1^+, E_1^-, E_2^+, E_2^-$ be the four sets obtained by reflecting E around the two orthogonal hyperplanes H_1 , H_2 dividing E in parts of equal measure. Let $B_i^{\pm}$ the respective optimal balls for each of the sets $E_i^{\pm}$, for $i = 1, 2$ and observe that for each i

$$\begin{aligned}\min_{x \in \mathbb{R}^n} |E \Delta B(x)| &\leq |E \Delta B_i^+| = |(E \Delta B_i^+) \cap H_i^+| + |(E \Delta B_i^+) H_i^-| \\ &\leq |(E \Delta B_i^+) \cap H_i^+| + |(E \Delta B_i^-) H_i^-| + |(B_i^+ \Delta B_i^+) H_i^-| \\ &= \frac{1}{2}(|E \Delta B_i^+| + |E \Delta B_i^-| + |B_i^+ \Delta B_i^-|)\end{aligned} \tag{2.16}$$

where in the last equality we used the symmetry of $E_i^{\pm}$. We now claim that if we show that

$$|B_i^+ \Delta B_i^-| \leq 16(|B_i^+ \Delta E_i^+| + |B_i^- \Delta E_i^-|) \tag{2.17}$$

for $i = 1$ or $i = 2$ we are done. Indeed, if it holds for $i = 1$

$$\min_{x \in \mathbb{R}^n} |E \Delta B(x)| \leq 9(|(E \Delta B_i^+) \cap H_i^+| + |(E \Delta B_i^-) H_i^-|) \leq 27|B|(\alpha(E_1^+) + \alpha(E_1^-)),$$

thus proving (2.15) with $C_0 = 54$ and $\tilde{E}$ equal to E_1^+ or E_1^-.
Step 2. Assume by contradiction that

$$|B_i^+ \Delta B_i^-| > 16(|B_i^+ \Delta E_i^+| + |B_i^- \Delta E_i^-|) \tag{2.18}$$

for $i = 1, 2$ and introduce the following unions of half balls

$$S_1 = (B_i^+ \cap H_i^+) \cup (B_i^- \cap H_i^-).$$

Then, by (2.18) we get

$$|S_1 \Delta S_2| \leq |S_1 \Delta E| + |S_2 \Delta E| = \frac{1}{2} \sum_{i=1}^{i=2} |B_i^+ \Delta B_i^-| < \frac{1}{32} \sum_{i=1}^{i=2} |B_i^+ \Delta B_i^-|. \tag{2.19}$$

Using Lemma 2.21, given $\epsilon > 0$ we can chose δ_0 small enough such that E is ϵ-close to its optimal ball and the same happens for each $E_i^{\pm}$. This implies that the center of the four balls for the symmetral sets are close to each other, thus each region $H_1^{\pm} \cap H_2^{\pm}$ must contain almost one quarter of the balls $B_i^{\pm}$. Then, if δ_0 is small enough,

$$|(B_1^\sigma \Delta B_2^\tau) \cap (H_1^\sigma \cap H_2^\tau| > \frac{|B_1^\sigma \Delta B_2^\tau|}{8}$$

and then

$$|B_1^+ \cap B_1^-| \leq |B_1^+ \cap B_2^+| + |B_2^+ \cap B_1^-| < 16|S_1 \Delta S_2|$$

and in a similar way that $|B_2^+ \cap B_2^-| < 16|S_1 \Delta S_2|$. This inequality combined with (2.19) leads to a contradiction. □

Theorem 2.24. *There exist δ and C, depending only on n, such that if $E \subset Q_l$, $|E| = |B|, D(E) \leq \delta$, then there exists an n-symmetric set F such that $F \subset Q_{2l}$, $|F| = |E|$ and*

$$\alpha(E) \leq C\alpha(F),\ D(F) \leq 2^n D(E) \tag{2.20}$$

PROOF. Take $\delta_1 = \delta_0 2^{-(n-1)}$, where δ_0 is chosen as in Proposition 2.23. By applying this proposition $n-1$ times to different pairs of orthogonal directions and recalling (2.14) we find a set $\tilde{E}$, with $|\tilde{E}| = |B|$ such that

$$\alpha(E) \leq C_0^{n-1}\alpha(\tilde{E}), \;\; D(\tilde{E}) \leq 2^{n-1}D(E).$$

Moreover, by translating $\tilde{E}$ and relabeling the coordinate axes, if needed, we may assume without loss of generality that $\tilde{E}$ is symmetric about all the coordinate hyperplanes $\{x_1 = 0\}, \ldots, \{x_{n-1} = 0\}$. In order to get the last symmetry we take a hyperplane H orthogonal to e_n dividing $\tilde{E}$ into two parts of equal measure and consider the corresponding sets $\tilde{E}^+$,$\tilde{E}^-$. Again, by translating $\tilde{E}$ in the direction of e_n, if necessary, we may assume that $H = \{x_n = 0\}$. As before we have

$$D(\tilde{E}^{\pm}) \leq 2D(\tilde{E}) \leq 2^n D(E).$$

To control the asymmetry of $\tilde{E}^{\pm}$ observe that since E is symmetric with respect to the first $n-1$ coordinate hyperplanes, $\tilde{E}^{\pm}$ and E are both n-symmetric so we can use Lemma 2.22 to get

$$\alpha(\tilde{E}) \leq \frac{3}{2}[\alpha(\tilde{E}^+) + \alpha(\tilde{E}^-]$$

Thus, at least one of the sets $\tilde{E}^{\pm}$ has asymmetry index greater than $\frac{1}{3}\alpha(\tilde{E})$. Therefore, denoting by F this set, we have

$$D(F) \leq 2D(\tilde{E}) \leq 2^n D(E) \;\text{ and }\; \alpha(E) \leq C_0^{n-1}\alpha(\tilde{E}) \leq 3C_0^{n-1}\alpha(F).$$

Finally, the inclusion $F \subset Q_{2l}$ follows immediately from the construction performed in the proof of Proposition 2.23 and the one performed here. $\square$

From the results obtained in the previous section it is clear that in order to prove the quantitative isoperimetric inequality (2.8) we may assume without loss of generality that there exist $\delta_0 \in (0, 1)$ and $l > 0$ such that

$$|E| = |B|, \quad E \subset Q_l, \quad D(E) \leq \delta_0, \quad E \quad \text{is n-symmetric.} \tag{2.21}$$

In fact, since $\alpha(E) \leq 2$ it is clear that if $D(E) \geq \delta_0$ (2.8) follows immediately with $\gamma(n) = 4/\delta_0$. Thus, if δ_0 is sufficiently small Lemma 2.20 and Theorem 2.24 tell us that we may assume without loss of generality that E is contained in some cube of fixed size and that it is $n-$symmetric. Therefore, throughout this section we shall always assume that E satisfies the above assumptions.

The next step consists in reducing the general case to the case of an axially symmetric set, i.e., a set E having an axis of symmetry such that every non-empty cross section of E perpendicular to this axis is an $(n-1)$-ball. To this aim we recall that the Schwarz symmetral of a measurable set E with respect to the x_n axis is defined as

$$E^* = \{(x, t) \in R^{n-1} \times \mathbb{R} : t \in \mathbb{R}, |x| < r_E(t)\}$$

where $r_E(t)$ is the radius of the $(n-1)$-dimensional ball having the same measure of the section E_t , that is $\mathcal{H}^{n-1}(E_t) = \omega_{n-1}r_E(t)^{n-1}$.

PROPOSITION 2.25. *Let $E \in Q_l$ be an n symmetric set with $D(E) < \delta_0$ and $|E| = |B|$. If $n = 2$ or if $n \geq 3$ and the quantitative isoperimetric inequality* (2.8) *holds true in $\mathbb{R}^{n-1}$, then there exists a constant $C(n)$ such that*

$$|E\Delta E^*| \leq C(n)\sqrt{D(E)} \;\text{ and }\; D(E^*) \leq D(E) \tag{2.22}$$

We will only prove this proposition in the case $n \geq 3$, where the induction assumption is used, since in the case $n = 2$ the proof is similar and actually simpler. So this lemma is telling us that once we are reduced to consider equibounded n-symmetric sets, arguing by induction we can prove the quantitative isoperimetric inequality only studying axially symmetric sets.

Proof. The second inequality in (2.22) follows immediately from the fact that $|E^*| = |E|$ and $P(E^*) \leq P(E)$. In order to prove the first inequality, we start assuming that

$$\mathcal{H}^{n-1}(\{x = (x', y) \in \partial^* E : \nu_{x'}(x) = 0\}) = 0. \tag{2.23}$$

Using the expression of the perimeter in (2.11), the isoperimetric inequality and Holder inequality one can get

$$\begin{aligned} P(E) - P(B) > P(E) - P(E^*) &\geq \int_{\mathbb{R}} \left(\sqrt{p_E^2 + (v_E')^2} - \sqrt{p_{E^*}^2 + (v_E')^2} \right) dt \\ &= \int_{\mathbb{R}} \frac{p_E^2 - p_{E^*}^2}{\sqrt{p_E^2 + (v_E')^2} + \sqrt{p_{E^*}^2 + (v_E')^2}} dt \\ \geq \left(\int_{\mathbb{R}} \sqrt{p_E^2 - p_{E^*}^2}\, dt \right)^2 & \frac{1}{\int_{\mathbb{R}} \left(\sqrt{p_E^2 + (v_E')^2} + \sqrt{p_{E^*}^2 + (v_E')^2} \right) dt} \\ & \frac{1}{P(E) + P(E^*)} \left(\int_{\mathbb{R}} \sqrt{p_E^2 - p_{E^*}^2} \right)^2 dt. \end{aligned}$$

Therefore, since $D(E) < \delta_0 < 1$ we have $P(E^*) \leq P(E) \leq 2P(B)$. Thus, observing that $p_E \geq p_{E^*}$ we obtain

$$\begin{aligned} \sqrt{D(E)} \geq c \int_{\mathbb{R}} \sqrt{(p_E^2 - p_{E^*}^2)} dt &\geq c \int_{\mathbb{R}} \sqrt{p_E^*} \sqrt{p_E + p_{E^*}} \sqrt{(p_E - p_{E^*})/p_{E^*}} dt \\ &\geq \sqrt{2} c \int_{\mathbb{R}} \sqrt{(p_E - p_{E^*})/p_{E^*}} dt \end{aligned} \tag{2.24}$$

where c is a dimensional constant. Now observe that since $(E^*)_t$ is a $(n-1)$-dimensional ball of radius $r_E(t)$ with $\mathcal{H}^{n-1}$ measure equal to $\mathcal{H}^{n-1}(E_t)$, the ratio

$$\frac{(p_E - p_{E^*})}{p_{E^*}}$$

is the isoperimetric deficit in $\mathbb{R}^{n-1}$ of E_t. Since by assumption, the quantitative isoperimetric inequality holds true in $\mathbb{R}^{n-1}$, we have

$$\gamma(n-1) \frac{(p_E - p_{E^*})}{p_{E^*}} \geq \alpha_{n-1}(E_t)^2$$

where $\alpha_{n-1}(E_t)$ is the $n-1$ dimensional Fraenkel asymmetry of E_t. Since E_t is an $(n-1)$-symmetric set laying on an $(n-1)$-hyperplane, thanks to Lemma 2.22 we infer

$$\sqrt{\gamma(n-1)} \sqrt{\frac{(p_E(t) - p_{E^*}(t))}{p_{E^*}(t)}} \geq \alpha_{n-1}(E_t) \geq \frac{1}{3} \frac{\mathcal{H}^{n-1}((E^*)_t \Delta (E)_t)}{\mathcal{H}^{n-1}((E^*)_t)}.$$

Therefore, plugging the last inequality in (2.24) and using Fubini's theorem we have

$$\sqrt{D(E)} \geq c' \int_{\mathbb{R}} \frac{\mathcal{H}^{n-1}(E_t \Delta (E*)_t)}{r_E(t)} dt \geq \int_{-l}^{l} \frac{\mathcal{H}^{n-1}(E_t \Delta (E^*)_t)}{l} dt \geq \frac{c'}{l} |E \Delta E^*|.$$

where in the second last inequality we used that $E \subset Q_l$. This proves (2.22). To conclude the proof, suppose (2.23) does not hold. Then, we can approximate E with a sequence of sets E_h converging to E in measure obtained by rotating a little E so that so that (2.23) holds true for all E_h. The conclusion follows observing that the sequence E_h^* converges to E^* in measure and (2.22) holds for all the sets E_h. □

We need a further reduction before proving the inequality (2.8). The proof of this lemma can be found in [**35**].

LEMMA 2.26. *There exist two universal constants C and δ, depending only on the dimension n, such that for every open bounded C^∞ set E with $D(E) \leq \delta$ there exist a connected component F of E such that $|F| > |E|/2$,*

$$\alpha(E) \leq \alpha(F) + CD(E), \text{ and } D(F) \leq CD(E).$$

PROOF OF THEOREM 2.18. As observed before, it is enough to prove the Theorem when (2.21) holds for a sufficient small δ that will be specified during the proof. Moreover, we may assume without loss of generality that the set E is C^∞. Otherwise by a standard approximation procedure, see for instance the proof of [**3**, Th. 3.42], we may find a sequence of n-symmetric smooth open sets E_h converging in measure to E, $|E_h| = |E|$ for all h, $P(E_h) \to P(E)$, satisfying the assumptions (2.21) with l possibly replaced by $2l$. Then the quantitative isoperimetric inequality for E will follow from the same inequality for E_h. Finally, we can also assume that E is connected, otherwise choosing δ_0 possibly smaller we apply Lemma 2.26 and consider the open connected component F with $|F| > |E|/2$. Observing that F is $n-$symmetric as well, we may replace E with λF, where λ is chosen such that $|\lambda F| = |E|$. Consider the strips $S = \{x : |x_n| < \sqrt{2}/2\}$ and $S' = \{x : |x_1| < \sqrt{2}/2\}$. Since $B \subset S \cup S'$, one of the two strips, say S, must contain half of the measure of B/E. Therefore

$$|E\Delta B| < 4|(B \setminus E) \cap S|.$$

Assume $n = 2$ or $n = 3$ and that (2.8) holds true in dimension $n-1$. Then

$$|B|\alpha(E) \leq 3|E\Delta B| \leq 12|(B \setminus E) \cap S| \leq 12|E\Delta E^*| + |(B\Delta E^*) \cap S| \leq C(n)\sqrt{D(E)} + 12|B\Delta E^*|$$

where we used Lemma 2.24. Thus, the proof will be done if we prove

$$|B\Delta E^*| \leq C(n)\sqrt{D(E)}.$$

Set $v(t) := \mathcal{H}^{n-1}(E^* \cap \{x_n = t\}$ and $w(t) = \mathcal{H}^{n-1}(B \cap \{x_n = t\}$. By an approximation argument we may assume that $v(t)$ is a $W^{1,1}$ function (a priori, it is just BV), and thus absolutely continuous. Using that $D(E^*) < \delta_0$, by Lemma 2.21 $\alpha(E^*)$ is small too- Then, comparing $v(t)$ with $w(t)$ one can find that there exists $c_0(n)$ such that $v(t) \geq c_0$ for all $t \in [-\sqrt{2}, \sqrt{2}]$. The next step is based on a optimal transportation argument.

Let $\tau = \tau(t)$ be the monotone increasing function from $(-a, a)$ to $(-1, -1)$ such that

$$\int_{-\infty}^{t} v(s)ds = \int_{-\infty}^{\tau(t)} w(t)dt$$

Differentiating the above inequality

$$\tau'(t) = \frac{v(t)}{w(\tau(t))} \tag{2.25}$$

and then τ is in $W^{2,1}_{\text{loc}}(-a, a)$ (and then locally Lipschitz) since v is in $W^{1,1}(\mathbb{R})$. Thanks to the equation above, we have

$$\begin{aligned}
|(E\Delta E^*) \cap S| = \int_I |w(t) - v(t)|dt &= \int_I |w(t) - w(\tau)\tau'(t)|dt \\
&\leq \int_I |w(t) - w(\tau)|dt + \int_I |w(\tau) - w(\tau)\tau'(t)|dt \\
\leq C(n) \int_I [|t - \tau(t)| + ||1 - \tau'(t)|]dt] &\leq C(n) \int_I |1 - \tau'(t)|dt
\end{aligned}$$

where we used that, by symmetry, $\tau(0) = 0$ and then

$$t - \tau(t) = t - 0 - (\tau(t) - \tau(0) \leq \int_0^t (1 - \tau'(t))dt.$$

To complete the proof, the last thing to prove is

$$\int_I |1-\tau'(t)|dt < C(n)\sqrt{D(E^*}.$$

For that, let $T : (-a, a) \times \mathbb{R}^{n-1} \to \mathbb{R}$ be defined as follows:

$$T(x) = \sum_{i=1}^{n-1} \left(\frac{w(\tau(x_n))}{v(x_n)}\right)^{\frac{1}{n-1}} x_i e_i + \tau(x_n)e_n.$$

Note that T maps the level set $E\cap\{x_n = t\}$ to the level set $B\cap\{x_n = t\}$. For a.e $x \in (-a, a)\times\mathbb{R}^{n-1}$

$$\text{div}T(x) = \frac{n-1}{\tau'(x_n)^{1/(n-1)}} + \tau'(x_n) \geq n$$

Using that $\lim_{b\to a^-} v(\pm b) = 0$ and the divergence theorem we have

$$P(E^*) \geq \int_{\partial E^*} T\cdot\nu_{E^*}d\mathcal{H}^{n-1} = \lim_{b\to a^-}\int_{\partial E^*\cap|x_n|<b} T\cdot\nu_{E^*}d\mathcal{H}^{n-1} =$$
$$\lim_{b\to a^-}\int_{E^*\cap|x_n|<b} \text{div}T dx = \int_{E^*} \text{div}T dx.$$

Then using that $|E^*| = |B|$ and the above inequality

$$\begin{aligned}P(E^*) - P(B) &\geq \int_{E^*}(\text{div}T - n) = \int_{E^*}\left(\frac{n-1}{\tau'(x_n)^{1/(n-1)}} + \tau'(x_n) - n\right)dx\\
&= \int_{-a}^{a} v(t)\left(\frac{n-1}{\tau'(t)^{1/(n-1)}} + \tau'(x_n) - n\right)dt\\
&= \int_{-a}^{a}\frac{v(t)}{\tau'(t)^{1/(n-1)}}\left(n-1+\tau'(t)^{n/(n-1)} - n\tau'(t)^{1/(n-1)}\right)dt\\
&\geq \int_{-a}^{a}\frac{v(t)}{\tau'(t)^{1/(n-1)}}(1-\tau'(t)^{1/(n-1)})^2dt,\end{aligned}$$

where we used coarea formula and the fact that the function $t \to (n-1) + t^n - nt - (1-t)^2$ has a strict minimum for $t = 1$ in $[0, +\infty)$. Setting $\sigma(t) = \tau'(1)^{n/(n-1)}$, from the above chain of inequality we get

$$\int_{-a}^{a} v|1-\sigma|dt \leq \sqrt{\left(\int_{-a}^{a}\frac{v}{\sigma}(1-\sigma)^2dt\right)}\sqrt{\int_{-a}^{a} v\sigma dt} \leq C(n)D(E),$$

where we used that $v \leq (2l)^{n-1}$, $\tau(a) = \tau(-a) = 1$ and thus

$$\int_{-a}^{a} v\sigma dt \leq \left(\int_{-a}^{a}\sigma^{n-1}dt\right)^{\frac{1}{n-1}}\left(\int_{-a}^{a} v^{\frac{n.1}{n.2}}dt\right)^{\frac{n.2}{n.1}} \leq C(n)\left(\int_{-a}^{a}\tau'dt\right)^{\frac{1}{n-1}} = 2^{\frac{1}{n-1}}C(n).$$

To conclude, we observe that $\sup_{t\in I}\tau \leq \lambda(n) < 1$. Indeed,

$$|E^*\cap\{x_n > \sqrt{2}/2\}| \geq |B\cap\{x_n > \sqrt{2}/2\}| - 3|B|\alpha(E^*) > c(n) - 3|B|\alpha(E^*) > c_2(n),$$

provided that $\alpha(E^*)$ is small enough, and

$$|E^*\cap\{x_n > \sqrt{2}/2\}| = |B\cap\{x_n > \tau(\sqrt{2}/2)\}| \leq C(n)(1-\tau(\sqrt{2}/2))$$

and then $1-\tau(\sqrt{2}/2)C(n)$. Since τ is a strict increasing function we get the desired bound. Note that (2.25) yields $\sup_{t\in I}\tau'(t) \leq C(n)$ and then

$$\int_I |\tau'-1|dt \leq C\int_I v|\sigma-1| \leq C(n)\sqrt{D(E)}.$$

□

CHAPTER 3

Gauss space

1. The isoperimetric inequality in Gauss space

In this section we will recall the basic properties and definitions about Gauss space. For $n \geq 1$, we indicate by $\gamma(E)$ the Gaussian measure of the set E, i.e.

$$\gamma(E) = \frac{1}{(2\pi)^{\frac{n}{2}}} \int_E e^{-\frac{|x|^2}{2}} dx.$$

In analogy with the Euclidean case, the perimeter in Gauss space is defined via the divergence theorem as

$$P_\gamma(E) = \frac{1}{(2\pi)^{\frac{n}{2}}} \sup \left\{ \int_E (\mathrm{div}\varphi - \varphi \cdot x) e^{-\frac{|x|^2}{2}} dx : \varphi \in C_c^\infty(\Omega) \right\}.$$

Note that if E is a smooth set, using the divergence theorem, we easily get that

$$P_\gamma(E) = \frac{1}{(2\pi)^{\frac{n}{2}}} \int_{\partial E} e^{-\frac{|x|^2}{2}} d\mathcal{H}^{n-1}.$$

Before stating the isoperimetric inequality in this framework, let us introduce some notation. Given $\omega \in \mathbb{S}^{n-1}$ and $l \in \mathbb{R}$, we set

$$H_{\omega,l} = \{x \in \mathbb{R}^n : x \cdot \omega = l\}, \quad H^+_{\omega,l} = \{x \in \mathbb{R}^n : x \cdot \omega \geq l\}, \quad H^-_{\omega,l} = \mathbb{R}^n \setminus H^+_{\omega,l}.$$

We define the function $\Phi : \mathbb{R} \to (0,1)$ as

$$\Phi(s) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{s} e^{-\frac{t^2}{2}} dt$$

Then, given any $s \in \mathbb{R}$ and $\omega \in \mathbb{S}^{n-1}$

$$\gamma(H^-_{\omega,s}) = \Phi(s), \; P_\gamma(H^-_{\omega,s}) = e^{-\frac{s^2}{2}}$$

Then the Gaussian perimeter of an halfspace with Gaussian volume r is

$$I(r) := e^{-\frac{\Phi^{-1}(r)^2}{2}}.$$

The isoperimetric problem in Gauss space states that among all sets of prescribed Gaussian volume, the halfspace minimizes the Gaussian perimeter. This result has been proved by several authors (see [**8**], [**53**]) , while the proof that halfspaces are the unique minimizers has been first obtained in [**17**].

Theorem 3.1 (Gaussian isoperimetric inequality). *Let $a \in (0,1)$ and $E \subset \mathbb{R}^n$ a measurable set. Then*

$$P_\gamma(E) \geq I(\gamma(E)). \tag{3.1}$$

Moreover, the equality holds if and only if E is an halfspace.

As in the Euclidean case, a very natural question concerns the stability of the isoperimetric inequality (3.1). This problem has been studied by several authors, see [**53**], [**22**], [**5**]. Before stating the theorem, we introduce the analog of the Fraenkel asymmetry for the Gauss space, i.e.

$$\lambda(E) = \min_{\nu \in \mathbb{S}^{n-1}} \gamma(E \Delta H^+_{\nu,s})$$

where s is such that $\gamma(E) = \gamma(H^+_{\omega,s})$. A first non optimal stability result is the following.

THEOREM 3.2. *Let $n \geq 2$. For any $r \in (0,1)$ there exists a positive constant $C(n,r)$ such that if E is a measurable set with $\gamma(E) = r$, then*

$$P_\gamma(E) \geq I(r) + \frac{\lambda(E)^2}{C(n,r)}. \tag{3.2}$$

If $n = 1$, we have

$$P_\gamma(E) \geq I(r) + \frac{C(n,r)}{\lambda(E)} \sqrt{\log \frac{1}{\lambda(E)}}$$

The proof of the above result is due to Chianchi, Fusco, Maggi and Pratelli in [**22**] and is based on an accurate estimate of the Ehrhard symmetrization. Indeed, in the mentioned paper the authors first gave a proof of the classic isoperimetric inequality in Gauss space, and then, inspired by the proof of the quantitative isoperimetric inequality in the Euclidean space, they start with a reduction argument which allow them to reduce to the case of $(n-1)$-symmetric sets.

REMARK 3.3. If one computes the right hand side of (3.1), one discovers that it does not depend on the dimension n. This suggested that the constant in (3.2) should be independent on n. This is, indeed, true and it has been proved by Barchiesi, Brancolini and Julin in [**5**], using a functional involving the Gaussian perimeter and the Gaussian baricenter.

Here we present the main steps of the proof of the optimal inequality in Gauss space given in [**5**]. The key point of this proof is a smart choice of the test function in a suitable functional given by the Gaussian perimeter and the baricenter. The idea is to show that if one perturbs a set along the normal with initial velocity proportional to one the coordinates of the normal, this deformation reduces the Gaussian perimeter. This argument leads to a dimension reduction which allows the authors to reduce to the one dimensional case. Note that the difference between this reduction argument and the one provided by the Ehrard symmetrization in [**22**] is that in the latter paper at every dimension reduction a new constant, possibly depending on the dimension, appears. On the contrary, the use of the second variation in [**5**] allows the authors to reduce to the one dimensional case without creating any new constant.

Before starting the proof, we now fix some notation. Given a smooth manifold M embedded in $\mathbb{R}^n$, let X be a smooth vector field on it. Since we are assuming M to be smooth we can extend X in a neighborhood of M, say U, and with an abuse of notations we still denote the extension by X. Then we may calculate the differential of X in U and project it on the tangent plane to the manifold. We call this projection the tangential differential, which will be denoted as $D_\tau X$. More explicitly

$$D_\tau X = DX - (DX\nu_M) \otimes \nu_M$$

where $\otimes$ denotes the tensor product. From this, we define the tangential divergence of X as the trace of the tangent differential:

$$\mathrm{div}_\tau X = \mathrm{Trace} D_\tau = \mathrm{div} X - (DX\nu_M) \cdot \nu_M.$$

For a function $u \in C^\infty(M)$, we can define the tangential gradient similarly as

$$D_\tau u = Du - (Du \cdot \nu_M)\nu_M.$$

Note that given a basis $\{e_i\}_{1\le i\le n}$ of $\mathbb{R}^n$ we set

$$\delta_i u = D_\tau u\cdot e_i = D_{e_i}u - D_{\nu_M}u\nu_M\cdot e_i,$$

where $D_{\nu_M}u$ stands for $Du\cdot\nu_M$. In case that no ambiguity occurs, we will write ν and ν_i instead of ν_M and $\nu_M\cdot e_i$.

From this, we define the tangential laplacian, i.e. the Laplace Beltrami operator, as

$$\Delta_\tau u = \mathrm{div}_\tau(D_\tau u) = \sum_{i=1}^n \delta_i(\delta_i u).$$

Since M is smooth, the normal vector field ν_M is also smooth and we can define the mean curvature H_M at a point x as

$$H_M(x) = \mathrm{div}_\tau \nu_M(x),$$

while the sum of the squares of the principal curvatures is given by

$$|B_M|^2 = \mathrm{Trace}(D_\tau\nu_M, D_\tau\nu_M).$$

The tangential divergence theorem states that given a vector field $X\in C_c^\infty(M,\mathbb{R}^n)$

$$\int_M \mathrm{div}X d\mathcal{H}^{n-1} = \int_M H_M X\cdot\nu_M d\mathcal{H}^{n-1}.$$

When M is the boundary of a set $E\subset\mathbb{R}^n$, we will always use E as subscript instead of ∂E and we will drop the subscript if the dependence on the set is clear from the context.

1.1. Proof. In order to prove the sharp quantitative isoperimetric inequality in Gauss space, we introduce the functional

$$\mathcal{F}(E) = P_\gamma(E) + \sqrt{\pi/2}\varepsilon|b(E)|^2, \tag{3.3}$$

where

$$b(E) = \frac{1}{(2\pi)^{\frac{n}{2}}}\int_E xe^{-\frac{|x|^2}{2}}dx$$

is the baricenter of the set E with the Gaussian weight and ε a positive constant that will be chosen later. In this section we will study the problem

$$\min\{\mathcal{F}(E),\, E\subset\mathbb{R}^n,\, \gamma(E)=\Phi(s)\}, \tag{3.4}$$

where $s\le 0$ is given. It is important to note that the baricenter can be seen as an "antiperimeter", in the sense that its modulus is maximized when the set E is an halfspace. Indeed, let b_s be the modulus of the baricenter of the halfspace $H^-_{\omega,s}$ (it does not depend on ω) and let E be a set with the same Gaussian volume of $H^-_{\omega,s}$. Then if $\omega = -\frac{b(E)}{|b(E)|}$, we have

$$|b(E)| - b_s = -(b(E)+b_s\omega)\cdot\omega = -\int_E x\cdot\omega d\gamma + \int_{H^-_{\omega,s}} x\cdot\omega d\gamma$$

$$= -\int_{E\setminus H^-_{\omega,s}} x\cdot\omega d\gamma + \int_{H^-_{\omega,s}} x\cdot\omega d\gamma = -\int_{E\setminus H^-_{\omega,s}}(x\cdot\omega - s)d\gamma + \int_{H^-_{\omega,s}}(x\cdot\omega-s)d\gamma \le 0$$

because the last integrals are both negative. We now state the Gaussian isoperimetric inequality proven in [**5**].

Theorem 3.4. *There exists an absolute constant c such that for every $s\in\mathbb{R}$ and for every set $E\subset\mathbb{R}^n$ with $\gamma(E)=\Phi(s)$ the following estimate holds:*

$$\beta(E)\le c(1+s^2)D(E). \tag{3.5}$$

$$\beta(E) := \min_{\omega\in S^{n-1}} |b(E) - b(H^-_{\omega,s})|$$

The next proposition tells us that the quantitative isoperimetric inequality in Gauss space is true provided that we are able to prove that halfspaces with the correct mass are the only minimizers of $\mathcal{F}$.

PROPOSITION 3.5. *Let $E \subset \mathbb{R}^n$ be a set with $\gamma(E) = \Phi(s)$. Then $\beta(e) \geq \frac{e^{-\frac{s^2}{2}}}{4}\tilde{\alpha}^2$, where*

$$\tilde{\alpha}(E) := \begin{cases} 2\Phi(-|s|) & \text{if } b(E) = 0 \\ \gamma(E\Delta H^-_{\omega,s}) & \text{otherwise,} \end{cases} \tag{3.6}$$

with $\omega = -b(E)/|b(E)|$.

The asymmetry β has been introduced in [**29**] where the author proved a similar estimate but without control on the constant of the right hand side.

PROOF. Since $\tilde{\alpha}(E) = \tilde{\alpha}(\mathbb{R}^n \setminus E)$, it is enough to consider the case $s \leq 0$. Consider the function

$$f(s) = e^{-\frac{s^2}{2}} - \sqrt{\frac{2}{\pi}} \int_{-\infty}^{s} e^{-\frac{t^2}{2}}\, dt.$$

Differentiating the function $f(t)$, one easily get that f is a nonnegative function. Then $e^{-\frac{s^2}{2}} \geq 2\Phi(s)$ and thus if $b(E) = 0$,

$$\beta(E) = b_s = \frac{e^{-\frac{s^2}{2}}}{\sqrt{2\pi}} \geq \frac{e^{\frac{s^2}{2}}}{\sqrt{2\pi}}\tilde{\alpha}(E)^2.$$

Assume now $b(E) \neq 0$ and, up to a rotation that $b(E) = -|b(E)|e_n$ and call $H = H^-_{e_n,s}$. Let a_1 and a_2 be positive numbers such that

$$\gamma(E \setminus H) = \frac{1}{\sqrt{2\pi}} \int_{s-a_1}^{s} e^{-\frac{t^2}{2}}\, dt = \frac{1}{\sqrt{2\pi}} \int_{s}^{s+a_2} e^{-\frac{t^2}{2}}\, dt.$$

Set now $E^+ := E \setminus H$, $E^- := E \cap H$, $F^+ := \mathbb{R}^{n-1} \times [s, s + a_2)$, $F^- := \mathbb{R}^{n-1} \times (-\infty, s - a_1)$ and $F = F^+ \cup F^-$. By the definition of a_1 and a_2, $\gamma(F) = \Phi(s)$, $\gamma(F^+) = \gamma(E^+)$ and $\gamma(F^-) = \gamma(F^+)$. Hence, using that $\gamma(E^\pm) = \gamma(E^\pm)$ provides $\gamma(F^\pm \setminus E^\pm) = \gamma(E^\pm \setminus F^\pm)$. Moreover,

$$\begin{aligned} \beta(E) - \beta(F) &= \int_E x_n d\gamma - \int_F x_n d\gamma \\ = \int_{E^+\setminus F^+} (x_n - s - a_2) d\gamma &+ \int_{F^+\setminus E^+} (-x_n + s + a_2) d\gamma \\ + \int_{E^-\setminus F^-} (x_n - s + a_1) d\gamma &+ \int_{F^-\setminus E^-} (-x_n + s - a_1) d\gamma \geq 0 \end{aligned}$$

because all the integrands are positive. At this point, using that for a fixed $s \leq 0$

$$g(t) := \int_{s-t}^{s} (-x_n + s) e^{-\frac{x_n}{2}}\, dx_n - \frac{e^{\frac{s^2}{2}}}{2} \left(\int_{s-t}^{s} e^{-\frac{(x_n^2}{2}}\, dx_n \right)^2$$

is a nonnegative function in $(0, \infty)$, it is not hard to show that $\beta(F) > e^{\frac{s^2}{2}}\gamma(E \setminus H)^2$, which is of course equivalent to the thesis thanks to the above inequality and to the fact that $2\gamma(E \setminus H) = \gamma(E\Delta H)$. □

Instead of studying the functional $\mathcal{F}$, it is more convenient to study the functional

$$\mathcal{G}(E) = P_\gamma(E) + \sqrt{\pi/2}\epsilon|b(E)|^2 + \sqrt{\pi/2}\Lambda|\gamma(E) - \Phi(s)|.$$

The problem we want to solve is now to minimize $\mathcal{G}$ without a volume constraint. The existence of minimizers comes directly from a standard compactness argument and the lower semicontinuity of the Gaussian perimeter under the L^1_{loc} convergence. The next proposition ensures us that a minimizer of $\mathcal{F}$ enjoys good regularity properties and this allows us to calculate the Euler equation of $\mathcal{G}$.

PROPOSITION 3.6. *Let E be a minimizer of $\mathcal{G}$. Then the reduced boundary $\partial^* E$ is a relatively open, smooth hypersurface and satisfies the Euler equation*

$$H_E - x \cdot \nu + \varepsilon b \cdot x = \lambda$$

where $b = b(E)$ and $\nu = \nu_E$. Here λ is a suitable constant such that $|\lambda| \leq \Lambda$.

The regularity result stated in the above proposition follows from the fact that a minimizer of $\mathcal{G}$ is also a (Λ, r) minimizer of the perimeter. A proof of this fact for a similar functional will be given in the next chapter, where also the first and second variations will be calculated.

PROPOSITION 3.7. *Let E be a minimizer of $\mathcal{G}$. The quadratic form associated with the second variation satisfies*

$$J[\phi] := \int_{\partial^* E} \left(|D_\tau \phi|^2 - |B_E|^2 \phi^2 - \phi^2 + \varepsilon (b \cdot \nu) \phi^2 \right) d\mathcal{H}^{n-1}_\gamma + \frac{\varepsilon}{\sqrt{2\pi}} \left| \int_{\partial^* E} \phi x d\mathcal{H}^{n-1}_\gamma \right|^2 \geq 0$$

for all $\phi \in C^\infty_0(\partial^* E)$ such that

$$\int_{\partial^* E} \phi d\mathcal{H}^{n-1}_\gamma = 0.$$

Next, we introduce a space more flexible than $C^\infty_c(\partial^* E)$. To this aim, let us define the space $H^1_\gamma(\partial^* E)$ as the closure of $C^\infty_0(\partial^* E)$ with respect to the norm $||u||_{H^1_\gamma} = ||u||_{L^2_\gamma} + ||D_\tau u||_{L^2_\gamma}$. For a general manifold that space can be even empty, the reason comes from the fact that dealing with Gauss space one always expect to have non compact manifold as solution of isoperimetric problems, thus if the singular set of the manifold is not small enough we can not expect to approximate even constant functions. In our case, since for a minimizer E the dimension of the singular set is not greater than $n - 8$, we can conclude that H^1_γ contains a large and interesting set of functions.

PROPOSITION 3.8. *Let E be a minimizer of $\mathcal{F}$ and H^1_γ as above. If $u \in C^\infty(\partial^* E)$ is such that $||u||_{H^1_\gamma} < \infty$, then $u \in H^1_\gamma(\partial^* E)$.*

This proposition is proved by approximation: the main point is to prove the existence of a sequence of Lipschitz functions with compact support $\zeta_k : \partial^* E \to \mathbb{R}$ such that

$$\zeta_k \to 1, \;\; ||D_\tau \zeta_k||_{L^2_\gamma(\partial^* E)} \leq \frac{1}{k}. \tag{3.7}$$

Note that since we want to prove that the constant of the quantitative isoperimetric inequality in the Gauss space does not depend on the dimension n, it is important that the estimates above do not depend on n as well.

PROOF OF THEOREM 3.4. First of all, let us fix

$$\varepsilon = \frac{\sqrt{2\pi}}{6(1 + \Lambda^2)} \text{ and } \Lambda = \frac{e^{-\frac{s^2}{2}}}{\sqrt{\pi}\Phi(s)}. \tag{3.8}$$

Step 1. The first thing to prove is that for every $\omega \in \mathbb{S}^{n-1}$

$$\int_{\partial^* E} (x \cdot \omega)^2 d\mathcal{H}^{n-1}_\gamma \leq \frac{13}{3} (\Lambda^2 + 1) e^{-\frac{s^2}{2}}$$

Using H_s as competitor for $\mathcal{F}$, the minimality of E and the definition of ε, we get

$$P_\gamma(E) \leq \mathcal{F}(H_s) = P(H_s) + \varepsilon \sqrt{(\pi/2)} |b(H_s)|^2 \leq \frac{13}{12} e^{-\frac{s^2}{2}}$$

and then by the Gaussian isoperimetric inequality, if r is such that $\Phi(r) = \gamma(E)$, we have

$$|b| \le |b(H_r)| = \frac{1}{\sqrt{2\pi}} P(H_r) \le \frac{1}{2\pi} P_\gamma(E) \le \frac{13}{12\sqrt{2\pi}} e^{-\frac{s^2}{2}},$$

which yields

$$\varepsilon |b| \le \frac{1}{4} \tag{3.9}$$

from the definition of ε in (3.8). Since $\partial^* E$ is smooth, if we choose a Lipschitz vector field X multiply the Euler equation by $X \cdot \nu$ and integrate by parts we get

$$\int_{\partial^* E} (\mathrm{div}_\tau X - X \cdot x) d\mathcal{H}^{n-1} - \varepsilon \int_{\partial^* E} x_n X \cdot \nu d\mathcal{H}^{n-1} = \lambda \int_{\partial^* E} X \cdot \nu d\mathcal{H}^{n-1}.$$

Let ζ_k be the sequence of Lipschitz functions with compact support given by Proposition 3.8, $\omega \in \mathbb{S}^{n-1}$ and $X = \zeta_k^2 \omega x_\omega$ with $x_\omega = x \cdot \omega$ and plug this vector field in the above equation. Then, using Young inequality, we get

$$\begin{aligned} \int_{\partial^* E} (x_\omega^2 - (1 - (\nu \cdot \omega)^2) \zeta_k^2 d\mathcal{H}_\gamma^{n-1} - \frac{1}{8} \int_{\partial^* E} (x_n^2 + x_\omega^2) \zeta_k^2 d\mathcal{H}_\gamma^{n-1} \\ \le |\lambda| \int_{\partial^* E} |x_\omega| \zeta_k^2 d\mathcal{H}_\gamma^{n-1} + 2 \int_{\partial^* E} \zeta_k |D_\tau \zeta_j| d\mathcal{H}_\gamma^{n-1} \\ \le \lambda^2 P_\gamma(E) + \frac{1}{2} \int_{\partial^* E} x_\omega^2 \zeta_k^2 d\mathcal{H}_\gamma^{n-1} + 4 \int_{\partial^* E} |D_\tau \zeta_k|^2 d\mathcal{H}_\gamma^{n-1}. \end{aligned}$$

This yields

$$\frac{3}{8} \int_{\partial^* E} x_\omega^2 \zeta_k^2 d\mathcal{H}_\gamma^{n-1} \le \frac{1}{8} \int_{\partial^* E} x_n^2 \zeta_k^2 d\mathcal{H}_\gamma^{n-1} + (\lambda^2 + 1) P_\gamma(E) + 4 \int_{\partial^* E} |D_\tau \zeta_k| d\mathcal{H}_\gamma^{n-1}.$$

Since the right hand side does not depend on ω, we can take the maximum over $\omega \in \mathbb{S}^{n-1}$ and pass to the limit as $k \to \infty$ to get the thesis:

$$\frac{1}{8} \max_{\omega \in \mathbb{S}^{n-1}} \le \int_{\partial^* E} x_\omega^2 d\mathcal{H}_\gamma^{n-1} 4(\lambda^2 + 1) P_\gamma(E) \le \frac{13}{3} (\Lambda^2 + 1), e^{-\frac{s^2}{2}}$$

where we used that $|\lambda| \le \Lambda$.

Step 2. With Step 1 in our hands, we want now to get rid of the last addend of the second variation. The aim now is to get that for every $\phi \in H_\gamma^1$, with $\int_{\partial^* E} \phi d\mathcal{H}_\gamma^{n-1} = 0$,

$$\int_{\partial^* E} \left(|D_\tau \phi|^2 - |B_E|^2 \phi^2 - \frac{5}{8} \phi^2 - \varepsilon |b| \nu_n \phi^2 \right) d\mathcal{H}_\gamma^{n-1} \ge 0 \tag{3.10}$$

By the definition of H_γ^1 there exists a sequence $\phi_k \in C_0^\infty(\partial^* E)$ such that $\phi_k \to \phi$ in the H_γ^1 sense. Note that this implies that $a_k = \int_{\partial^* E} \phi_k d\mathcal{H}_\gamma^{n-1} \to 0$ and then up to replace ϕ_k with $\phi_k - a_k$, we may assume $\int_{\partial^* E} \phi_k d\mathcal{H}_\gamma^{n-1} = 0$. If ω_k is such that

$$\left| \int_{\partial^* E} \phi_k x d\mathcal{H}_\gamma^{n-1} \right| = \int_{\partial^* E} x \phi_k d\mathcal{H}_\gamma^{n-1} \cdot \omega_k = \int_{\partial^* E} x_{\omega_k} \phi_k d\mathcal{H}_\gamma^{n-1},$$

from step 2 and Holder inequality we get

$$\left| \int_{\partial^* E} \phi_k x d\mathcal{H}_\gamma^{n-1} \right|^2 \le \int_{\partial^* E} x_{\omega_k}^2 d\mathcal{H}_\gamma^{n-1} \int_{\partial^* E} \phi_k^2 d\mathcal{H}_\gamma^{n-1} \le \frac{13}{3} (\Lambda^2 + 1) e^{-\frac{s^2}{2}} \int_{\partial^* E} \phi_k d\mathcal{H}_\gamma^{n-1}.$$

Then using Proposition 3.7

$$\int_{\partial^* E} (|D_\tau \phi_k|^2 - |B_E|^2 - \phi_k^2 + \varepsilon (b \cdot \nu) \phi^2) d\mathcal{H}_\gamma^{n-1} \ge \frac{13\varepsilon}{3\sqrt{2\pi}} (\Lambda^2 + 1) e^{-\frac{s^2}{2}} \int_{\partial^* E} \phi_k d\mathcal{H}_\gamma^{n-1} \tag{3.11}$$

and then letting $k \to \infty$, this immediately yield (3.10) by the definition of ε. Note that from (3.10) we can infer that

$$\int_{\partial^* E} |B_E|^2 \phi^2 d\mathcal{H}_\gamma^{n-1} \le C \left(\int_{\partial^* E} \phi^2 d\mathcal{H}_\gamma^{n-1} + \int_{\partial^* E} |D_\tau \phi|^2 d\mathcal{H}_\gamma^{n-1} \right),$$

which together with Proposition 3.8, gives

$$\int_{\partial^* E} |B_E|^2 d\mathcal{H}_\gamma^{n-1} < \infty.$$

Step 3. In this step, using some well known geometric equations and Proposition 3.7 we will prove that halfspaces are the only minimizers of $\mathcal{F}$. First of all, note that since up to rotation we can assume that $b(E) = |b(E)e_n$, then

$$\int_{\partial^* E} \nu_j e^{-\frac{|x|^2}{2}} d\mathcal{H}^{n-1} = \int_E x_j e^{-\frac{|x|^2}{2}} = 0$$

for all $j \in \{1, 2, \dots, n-1\}$. Moreover,

$$\int_{\partial^* E} \nu_j d\mathcal{H}_\gamma^{n-1} \le \int_{\partial^* E} |B_E|^2 d\mathcal{H}_\gamma^{n-1} < \infty$$

and then $\nu_j \in H^1_\gamma(\partial^* E)$. Differentiate the Euler equation with respect to δ_j to get

$$\delta_j H - \delta_j(x \cdot \nu) + \varepsilon \delta_j (b \cdot x) = 0.$$

Using the well known equations, for $j \neq n$,

$$\Delta_\tau \nu_j = -|B_E|^2 \nu_j + \delta_j H, \quad \delta_j(x \cdot \nu) = D_\tau \nu_j \cdot x \quad \text{and} \quad \delta_j x_n = \nu_j \nu_n,$$

we arrive at

$$\Delta_\tau \nu_j - D_\tau \nu_j \cdot x = -|B_E|^2 \nu_j - \varepsilon |b| \nu_n \nu_j. \tag{3.12}$$

Take ζ_k as in Proposition 3.8, multiply (3.12) by $\zeta_k \nu_j$ and then use the divergence theorem to get

$$\begin{aligned}
\int_{\partial^* E} \zeta_k (|B_E|^2 + \varepsilon |b| \nu_n \nu_j^2) d\mathcal{H}_\gamma^{n-1} &= -\int_{\partial^* E} \zeta_k \nu_j (\Delta_\tau \nu_j - D_\tau \nu_j \cdot x)) d\mathcal{H}_\gamma^{n-1} \\
&\quad - \int_{\partial^* E} \zeta_k \nu_j \mathrm{div}_\tau (D_\tau \nu_j e^{-\frac{|x|^2}{2}}) d\mathcal{H}_\gamma^{n-1} = \\
= -\int_{\partial^* E} \mathrm{div}_\tau (\zeta_k \nu_j D_\tau \nu_j e^{-\frac{|x|^2}{2}} d\mathcal{H}_\gamma^{n-1} &+ \int_{\partial^* E} D_\tau (\zeta_k \nu_j \cdot D_\tau \nu_j) d\mathcal{H}_\gamma^{n-1} \\
&= \int_{\partial^* E} \nu_j D_\tau \zeta_k \cdot D_\tau \nu_j d\mathcal{H}_\gamma^{n-1} + \int_{\partial^* E} \zeta_k |D_\tau \nu_j|^2 d\mathcal{H}_\gamma^{n-1}.
\end{aligned}$$

Then from (3.7) we obtain

$$-\frac{5}{18} \int_{\partial^* E} \nu_j^2 d\mathcal{H}^{n-1} \ge 0,$$

which implies $\nu_j = 0$ $\mathcal{H}^{n-1}$-a.e., for $j \neq n$. Actually, using the De Giorgi structure theorem, one can prove that $\partial E = \partial^* E$ and then E is smooth.

To prove that E is *a* halfspace we are left to show that E is connected. Assume by contradiction that $\partial E = \Gamma_1 \cup \Gamma_2$ with Γ_1, Γ_2 disjoint, closed manifolds. Let $a_1 < 0 < a_2$ be real numbers such that the function ϕ

$$\phi(x) = \begin{cases} a_1, & x \in \Gamma_1 \\ a_2, & x \in \Gamma_2 \end{cases}$$

satisfies $\int_{\partial E} \phi d\mathcal{H}^{n-1}_\gamma = 0$. Using ϕ as test function in (3.10) we get

$$\int_{\partial^* E} \left(|B_E|^2 \phi^2 + \frac{5}{18}\phi^2 + \varepsilon |b| \nu_n \phi^2 \right) d\mathcal{H}^{n-1} \leq 0.$$

and this is clearly a contradiction since $\phi \neq 0$. Using that by step 1 $\varepsilon|b| \leq \frac{1}{4}$, we conclude

$$\int_{\partial^* E} \left(|B_E|^2 + \frac{1}{36} \right) \phi^2 d\mathcal{H}^{n-1}_\gamma \geq 0,$$

which is clearly false since $\phi \neq 0$. The last thing to prove is that E has the correct Gaussian volume. Since we already know that the minimizer E must be a halfspace, the last thing to check is that the function

$$f(t) = e^{-\frac{t^2}{2}} + \frac{\varepsilon}{2\sqrt{2\pi}} e^{-t^2} - \Lambda\sqrt{2\pi}|\Phi(t) - \Phi(s)|.$$

has a minimum for $t = s$, which can be checked by recalling the definition of ε and Λ in (3.8). $\square$

CHAPTER 4

Symmetric Gaussian Problem

Another interesting question in Gauss space is the following: *Among n-symmetric sets with prescribed Gaussian volume, which one is the isoperimetric figure ?* The above question is known in literature as *Symmetric Gaussian Problem* (SGP). The main difficulty coming from this problem is that all the classical methods known to be working for the Gaussian isoperimetric problem, seem to fail in this contest: for instance, the Ehrard symmetrization ([**28**], [**22**]) and the Ornstein-Uhlenbeck semigroup argument ([**4**]) do not look feasible. The first issue one meets is to understand which could be an appropriate conjecture. This problem is stated as an open problem in [**18**] and [**32**]. Since in [**15**] Barthe proved that if one replaces the standard Gaussian perimeter by a certain anisotropic perimeter, the solution of the isoperimetric problem among n-symmetric sets is the symmetric strip or its complement, it looks natural to think that the strip or its complement are the solutions to the SGP. Later on other authors ([**18, 59**]) suggested instead that the perimeter minimizer was the ball centered at the origin or its complement. This latter conjecture was disproved by Heilman in [**40**]. In the same paper he also shows that for very small masses the one dimensional balls centered at the origin maximize the *Gaussian noise stability*, a result that in particular implies the minimality of these balls among symmetric sets. On the other hand still Heilman, in the more recent paper [**41**], showed that at least for sets of Gaussian measure 1/2, or close to this value, the corresponding symmetric strip cannot be a minimizer. The results proved in [**41**] suggest instead that the minimizer could be a cylinder C whose boundary, after rotation, is given by $\partial C = r\mathbb{S}^k \times \mathbb{R}^{n-k}$ for some $r > 0$ and $0 \leq k \leq n$. In the following we will argue as in [**46**] to prove that the ball, in general, is not the correct candidate as solution of the minimum problem: the result we will provide is that if $r \geq \sqrt{n+1}$, then B_r cannot be the optimal shape for the Gaussian perimeter among symmetric sets with $\gamma(E) = \gamma(B_r)$. Moreover, we will prove that the ball is a local minimizer for the Gaussian perimeter. Our Theorem 4.1 is closely related to a well known conjecture, known as *Symmetric Gaussian Problem*, see [**40**].

THEOREM 4.1. *Let $n \geq 2$ and $\sigma \in (0, 1/2)$. There exist δ and κ such that if $r \in [\sigma, \sqrt{n+1}-\sigma]$, E is a set of locally finite perimeter with $E = -E$, $\gamma(E) = \gamma(B_r)$ and $\gamma(E\Delta B_r) < \delta$, then*

$$P_\gamma(E) - P_\gamma(B_r) \geq \kappa(n, \sigma)\gamma(E\Delta B_r)^2. \tag{4.1}$$

Let us now comment briefly on this result. First, observe that Theorem 4.1, beside stating the local minimality of balls B_r when $r \in (0, \sqrt{n+1})$, provides also a quantitative estimate of the isoperimetric gap $P_\gamma(E) - P_\gamma(B_r)$ in terms of the square of the measure of the symmetric difference between E and B_r. In this respect this inequality is close to the recent quantitative isoperimetric inequalities in Gaussian space proved in [**5**], [**37**], [**53**], [**54**],[**29**]. Note also that the constant κ in (2) is uniformly bounded from below when r is away from 0 and $\sqrt{n+1}$. In addition, Proposition 4.4 shows that the result above is sharp in the sense that if $r > \sqrt{n+1}$ then B_r is *never* a local minimizer for the perimeter. Also the power 2 is optimal, as it can be easily checked with an argument similar to the one used for the quantitative inequality in the Euclidean case, see [**35**, Section 4]. However, balls B_r are not in general global minimizers among symmetric sets with the same Gaussian measure, at least if r is sufficiently small, see Proposition 4.11.

Finally, in the 1-dimensional case we show that B_r is always a local minimizer of the perimeter among symmetric sets with the same Gaussian measure, see Section 4. Moreover, balls are the unique global minimizers for $r > r_0$, where r_0 is the unique positive number such that

$$\frac{1}{\sqrt{2\pi}}\int_{-r_0}^{r_0} e^{-\frac{t^2}{2}}\,dt = \frac{1}{2},$$

while $\mathbb{R}\setminus B_r$ is the unique global minimizer when $0 < r < r_0$.

We conclude this introduction with a few words on the proof of Theorem 4.1, which is achieved following the strategy introduced in this context by Cicalese and Leonardi in [**23**] and later on modified in [**1**]. More precisely, we first prove inequality (4.1) for nearly spherical sets, i.e., sets that are close in C^1 to a ball B_r with the same Gaussian volume and symmetric around the origin. Then we extend it to the general case with a contradiction argument based on the regularity theory for sets of minimal perimeter, see a more detailed account of this strategy in Section 3, before the proof of Theorem 4.1. Note that in our case the above strategy is more complicated. An obvious difficulty comes from the constraint that the competing sets must be symmetric with respect to the origin. However the main source of problems is represented by possible unbounded competitors of balls.

0.1. Nearly spherical sets. A set $E\subset\mathbb{R}^n$ is said to be *nearly spherical* if there exist a ball B_r and a Lipschitz function $u:\mathbb{S}^{n-1}\to(-1/2,1/2)$ such that

$$E=\{y=trx(1+u(x)):\,x\in\mathbb{S}^{n-1},\,0\le t<1\}. \tag{4.2}$$

In the following, given any function $u:\mathbb{S}^{n-1}\to\mathbb{R}$, we shall always assume that u is extended to $\mathbb{R}^n\setminus\{0\}$ by setting $u(x)=u\big(\frac{x}{|x|}\big)$.

It is easily checked that if E is defined as in (4.2) then its Gaussian measure and it Gaussian perimeter are given, respectively, by the two formulas below

$$\gamma(E)=\frac{r^n}{(2\pi)^{n/2}}\int_B(1+u(x))^n e^{-\frac{r^2|x|^2(1+u(x))^2}{2}}\,dx \tag{4.3}$$

$$P_\gamma(E)=\frac{r^{n-1}}{(2\pi)^{\frac{n-1}{2}}}\int_{\mathbb{S}^{n-1}}(1+u(x))^{n-1}\sqrt{1+\frac{|D_\tau u(x)|^2}{(1+u(x))^2}}\,e^{-\frac{r^2(1+u(x))^2}{2}}\,d\mathcal{H}^{n-1}, \tag{4.4}$$

where $D_\tau u$ stands for the tangential gradient of u on $\mathbb{S}^{n-1}$.

When E is a measurable set such that $\gamma(E)=\gamma(B_r)$ we shall often use the following notation

$$D_\gamma(E)=(2\pi)^{n/2}\big[P_\gamma(E)-P_\gamma(B_r)\big] \tag{4.5}$$

to denote its *Gaussian isoperimetric deficit* with respect to the ball B_r.

Next theorem states that if r is smaller than a critical radius depending on the dimension, the Gaussian isoperimetric deficit of a nearly spherical set symmetric with respect to the origin is strictly positive and the following Fuglede type estimate holds.

THEOREM 4.2. *Let $n\ge 2$ and $r\in(0,\sqrt{n+1})$. There exist $\varepsilon\in(0,1/2)$, depending on n and r, and κ_0, depending only on n, with the following property. If E is a nearly spherical set as in (4.2) with $\|u\|_{W^{1,\infty}(\mathbb{S}^{n-1})}<\varepsilon$, symmetric with respect to the origin and such that $\gamma(E)=\gamma(B_r)$, then*

$$P_\gamma(E)-P_\gamma(B_r)\ge\kappa_0 r^{n-1}(n+1-r^2)\|u\|^2_{W^{1,2}(\mathbb{S}^{n-1})}. \tag{4.6}$$

PROOF. **Step 1.** Fix $r\in(0,\sqrt{n+1})$. Using the expression of $P_\gamma(E)$ provided in (4.4) we may split

$$D_\gamma(E) = (2\pi)^{n/2}\big[P_\gamma(E) - P_\gamma(B_r)\big] = r^{n-1}\int_{\mathbb{S}^{n-1}}(1+u)^{n-1}e^{-\frac{r^2(1+u)^2}{2}}\left(\sqrt{1+\frac{|D_\tau u|^2}{(1+u)^2}}-1\right)d\mathcal{H}^{n-1}$$
$$+ r^{n-1}\int_{\mathbb{S}^{n-1}}\Big[(1+u)^{n-1}e^{-\frac{r^2(1+u)^2}{2}} - e^{-\frac{r^2}{2}}\Big]d\mathcal{H}^{n-1}$$
$$= r^{n-1}e^{-\frac{r^2}{2}}I_1 + r^{n-1}e^{-\frac{r^2}{2}}I_2. \tag{4.7}$$

Observe that $\sqrt{1+t} \geq 1 + \frac{t}{2} - \frac{t^2}{8}$ for all $t > 0$. Therefore, from the smallness assumption $\|u\|_{W^{1,\infty}(\mathbb{S}^{n-1})} < \varepsilon \leq \frac{1}{2}$, we get

$$I_1 = \int_{\mathbb{S}^{n-1}}(1+u)^{n-1}e^{-r^2(u+u^2/2)}\left(\sqrt{1+\frac{|D_\tau u|^2}{(1+u)^2}}-1\right)d\mathcal{H}^{n-1}$$
$$\geq \int_{\mathbb{S}^{n-1}}(1+u)^{n-1}e^{-r^2(u+u^2/2)}\Big(\frac{1}{2}\frac{|D_\tau u|^2}{(1+u)^2} - \frac{1}{8}\frac{|D_\tau u|^4}{(1+u)^4}\Big)d\mathcal{H}^{n-1}$$
$$\geq \Big(\frac{1}{2} - C\varepsilon\Big)\int_{\mathbb{S}^{n-1}}(1+u)^{n-1}e^{-r^2(u+u^2/2)}|D_\tau u|^2\,d\mathcal{H}^{n-1} \geq \Big(\frac{1}{2} - C\varepsilon\Big)\int_{\mathbb{S}^{n-1}}|D_\tau u|^2\,d\mathcal{H}^{n-1}, \tag{4.8}$$

for some constant C depending only on n, but not on r. Concerning the integral term I_2 we have, by Taylor expansion,

$$I_2 = \int_{\mathbb{S}^{n-1}}\big[(1+u)^{n-1}e^{-r^2(u+u^2/2)} - 1\big]\,d\mathcal{H}^{n-1}$$
$$= (n-1-r^2)\int_{\mathbb{S}^{n-1}}u\,d\mathcal{H}^{n-1} + \Big[\frac{(n-1)(n-2)}{2} - \Big(n-\frac{1}{2}\Big)r^2 + \frac{r^4}{2}\Big]\int_{\mathbb{S}^{n-1}}u^2\,d\mathcal{H}^{n-1} + R_1,$$

where the remainder term R_1 can be again estimated by $C\varepsilon\|u\|_2^2$, for some constant C depending only on n. Therefore, recalling the previous estimate (4.8) and the equality in (4.7) we have

$$r^{1-n}e^{\frac{r^2}{2}}D_\gamma(E) \geq \frac{1}{2}\int_{\mathbb{S}^{n-1}}|D_\tau u|^2\,d\mathcal{H}^{n-1} + (n-1-r^2)\int_{\mathbb{S}^{n-1}}u\,d\mathcal{H}^{n-1}$$
$$+ \Big[\frac{(n-1)(n-2)}{2} - \Big(n-\frac{1}{2}\Big)r^2 + \frac{r^4}{2}\Big]\int_{\mathbb{S}^{n-1}}u^2\,d\mathcal{H}^{n-1} - C\varepsilon\|u\|^2_{W^{1,2}(\mathbb{S}^{n-1})}. \tag{4.9}$$

To estimate the integral of u in the previous inequality we are going to use the assumption that the Gaussian measures of E and B_r are equal. This equality, using (4.3), can be written as

$$\int_0^1 t^{n-1}\,dt\int_{\mathbb{S}^{n-1}}\Big[(1+u)^n e^{-\frac{r^2t^2(1+u)^2}{2}} - e^{-\frac{r^2t^2}{2}}\Big]\,d\mathcal{H}^{n-1} = 0$$

Using again Taylor expansion, we then easily get

$$0 = \int_0^1 t^{n-1}e^{-\frac{r^2t^2}{2}}\,dt\int_{\mathbb{S}^{n-1}}\big[(1+u)^n e^{-r^2t^2(u+u^2/2)} - 1\big]\,d\mathcal{H}^{n-1}$$
$$= \int_0^1 t^{n-1}e^{-\frac{r^2t^2}{2}}\,dt\int_{\mathbb{S}^{n-1}}\Big[(n-r^2t^2)u + \Big(\frac{n(n-1)}{2} - \frac{(2n+1)r^2t^2}{2} + \frac{r^4t^4}{2}\Big)u^2\Big]\,d\mathcal{H}^{n-1} + R_2$$
$$= \int_{\mathbb{S}^{n-1}}\Big[(na_n - r^2b_n)u + \Big(\frac{n(n-1)a_n}{2} - \frac{(2n+1)r^2b_n}{2} + \frac{r^4c_n}{2}\Big)u^2\Big]\,d\mathcal{H}^{n-1} + R_2, \tag{4.10}$$

where we have set

$$a_n = \int_0^1 t^{n-1}e^{-\frac{r^2t^2}{2}}\,dt, \qquad b_n = \int_0^1 t^{n+1}e^{-\frac{r^2t^2}{2}}\,dt, \qquad c_n = \int_0^1 t^{n+3}e^{-\frac{r^2t^2}{2}}\,dt$$

and where the remainder term R_2 is estimated as before

$$|R_2| \le C\varepsilon \|u\|^2_{L^2(\mathbb{S}^{n-1})}. \tag{4.11}$$

A simple integration by parts gives that

$$b_n = \frac{na_n}{r^2} - \frac{e^{-\frac{r^2}{2}}}{r^2}, \qquad c_n = \frac{n(n+2)a_n}{r^4} - \frac{(n+2)e^{-\frac{r^2}{2}}}{r^4} - \frac{e^{-\frac{r^2}{2}}}{r^2}.$$

Thus, inserting the above values of b_n and c_n into (4.10) we immediately get that

$$\int_{\mathbb{S}^{n-1}} u\, d\mathcal{H}^{n-1} = -\frac{n-1-r^2}{2}\int_{\mathbb{S}^{n-1}} u^2\, d\mathcal{H}^{n-1} - e^{\frac{r^2}{2}} R_2. \tag{4.12}$$

Then, substituting in (4.9) the integral of u on $\mathbb{S}^{n-1}$ by the right hand side of the above equality, we obtain the following estimate

$$r^{1-n} e^{\frac{r^2}{2}} D_\gamma(E) \ge \frac{1}{2}\int_{\mathbb{S}^{n-1}} |D_\tau u|^2\, d\mathcal{H}^{n-1} - \frac{n-1+r^2}{2}\int_{\mathbb{S}^{n-1}} u^2\, d\mathcal{H}^{n-1} - C\varepsilon\|u\|^2_{W^{1,2}(\mathbb{S}^{n-1})}. \tag{4.13}$$

Step 2. For any integer $k \ge 0$, let us denote by $y_{k,i}$, $i = 1, \dots, G(n,k)$, the spherical harmonics of order k, i.e., the restrictions to $\mathbb{S}^{n-1}$ of the homogeneous harmonic polynomials of degree k, normalized so that $||y_{k,i}||_{L^2(\mathbb{S}^{n-1})} = 1$, for all $k \ge 0$ and $i \in \{1, \dots, G(n,k)\}$. The functions $y_{k,i}$ are eigenfunctions of the Laplace-Beltrami operator on $\mathbb{S}^{n-1}$ and for all k and i

$$-\Delta_{\mathbb{S}^{n-1}} y_{k,i} = k(k+n-2) y_{k,i}\,.$$

Therefore if we write

$$u = \sum_{k=0}^{\infty}\sum_{i=1}^{G(n,k)} a_{k,i} y_{k,i}, \quad \text{where} \quad a_{k,i} = \int_{\mathbb{S}^{n-1}} u y_{k,i} d\mathcal{H}^{n-1},$$

we have

$$||u||^2_{L^2(\mathbb{S}^{n-1})} = \sum_{k=0}^{\infty}\sum_{i=1}^{G(n,k)} a^2_{k,i}, \quad ||D_\tau u||^2_{L^2(\mathbb{S}^{n-1})} = \sum_{k=1}^{\infty} k(k+n-2)\sum_{i=1}^{G(n,k)} a^2_{k,i}\,. \tag{4.14}$$

Note that since E is symmetric with respect to the origin, we have that u is an even function, hence in the harmonic decomposition only the terms with k even will appear. In particular $a_{1,i} = 0$ for all $i = 1, \dots, n$. Note also that from (4.12) and (4.11) we have

$$|a_{0,1}| \le C\varepsilon \|u\|_{L^2(\mathbb{S}^{n-1})}. \tag{4.15}$$

Thus, from (4.13),(4.14) and (4.15) we have

$$\begin{aligned}
r^{1-n} e^{\frac{r^2}{2}} D_\gamma(E) &\ge \frac{1}{2}\sum_{k=2}^{\infty} k(k+n-2)\sum_{i=1}^{G(n,k)} a^2_{k,i} - \frac{n-1+r^2}{2}\sum_{k=2}^{\infty}\sum_{i=1}^{G(n,k)} a^2_{k,i} - C_0\varepsilon\|u\|^2_{W^{1,2}(\mathbb{S}^{n-1})} \\
&= \frac{n+1-r^2}{2}\sum_{i=1}^{G(n,2)} a^2_{2,i} + \frac{1}{2}\sum_{k=2}^{\infty}[k(k+n-2)-(n-1-r^2)]\sum_{i=1}^{G(n,k)} a^2_{k,i} - C_0\varepsilon\|u\|^2_{W^{1,2}(\mathbb{S}^{n-1})}, \\
&\ge c_0(n+1-r^2)\sum_{k=2}^{\infty} k(k+n-2)\sum_{i=1}^{G(n,k)} a^2_{k,i} - C_0\varepsilon\|u\|^2_{W^{1,2}(\mathbb{S}^{n-1})},
\end{aligned}$$

for some positive constants c_0, C_0 depending only on n. Using again (4.15) and the fact that $a_{1,i} = 0$ for $i = 1, \dots, n$, from the previous inequality we deduce that there exist two constants $c_1, C_1 > 0$ depending only on n such that

$$r^{1-n} e^{\frac{r^2}{2}} D_\gamma(E) \ge c_1(n+1-r^2)\|u\|^2_{W^{1,2}(\mathbb{S}^{n-1})} - C_1\varepsilon\|u\|^2_{W^{1,2}(\mathbb{S}^{n-1})}.$$

From this inequality (4.6) immediately follows provided that we choose

$$0 < \varepsilon \le \min\Big\{\frac{1}{2}, \frac{c_1(n+1-r^2)}{2C_1}\Big\}. \tag{4.16}$$

□

The following uniform estimate is a straightforward consequence of the previous theorem.

COROLLARY 4.3. *Let $n \ge 2$ and $r_0 \in (0, \sqrt{n+1})$. There exist $\varepsilon \in (0, 1/2)$, $\kappa_1 > 0$, depending only on n and r_0, such that if $r \in (0, r_0]$ and E is a nearly spherical set as in (4.2) with $\|u\|_{W^{1,\infty}(\mathbb{S}^{n-1})} < \varepsilon$, then*

$$P_\gamma(E) - P_\gamma(B_r) \ge \kappa_1 r^{-1-n} \gamma(E\Delta B_r)^2. \tag{4.17}$$

PROOF. Fix r_0 and a nearly spherical set E as in the statement. Then, arguing as in the proof of (4.10), we get

$$\begin{aligned} \gamma(E\Delta B_r) &= \frac{r^n}{(2\pi)^{\frac{n}{2}}} \int_B \Big|(1+u)^n e^{-\frac{r^2|x|^2(1+u)^2}{2}} - e^{-\frac{r^2|x|^2}{2}}\Big|\, dx \\ &= \frac{r^n}{(2\pi)^{\frac{n}{2}}} \int_0^1 t^{n-1} e^{-\frac{r^2t^2}{2}}\, dt \int_{\mathbb{S}^{n-1}} \Big|(1+u)^n e^{-r^2t^2(u+u^2/2)} - 1\Big|\, d\mathcal{H}^{n-1} \\ &\le C(n) r^n \int_{\mathbb{S}^{n-1}} |u|\, d\mathcal{H}^{n-1}, \end{aligned} \tag{4.18}$$

where in the last inequality we have used the assumption that $\|u\|_{W^{1,\infty}(\mathbb{S}^{n-1})} \le 1/2$. Then, choosing

$$\varepsilon = \min\Big\{\frac{1}{2}, \frac{c_1(n+1-r_0^2)}{2C_1}\Big\},$$

where c_1 and C_1 are as in (4.16), from (4.18) and (4.6) we get at once

$$\gamma(E\Delta B_r)^2 \le \frac{C(n) r^{n+1}}{n+1-r^2}\big[P_\gamma(E) - P_\gamma(B_r)\big],$$

for a suitable constant $C(n)$. Hence (4.17) follows. □

Consider the isoperimetric problem in the Gaussian space

$$\min\{P_\gamma(E) :\, \gamma(E) = m\} \tag{4.19}$$

for some fixed $m > 0$. The Euler-Lagrange equation associated to the minimum problem (4.19)

$$H_E - x \cdot \nu_E = \lambda \qquad \text{on } \partial E, \tag{4.20}$$

where $H_{\partial E}$ is the sum of the the principal curvatures of the boundary of E and λ is a suitable Lagrange multiplier. Observe that B_r is a solution of (4.20), hence a critical point of the isoperimetric problem (4.19) for all $r > 0$. Theorem 4.2 shows that if $0 < r < \sqrt{n+1}$ then B_r is also a local minimizer for the isoperimetric problem with respect to small variations, close to B_r in C^1 and symmetric with respect to the origin. In this respect the above theorem is sharp since if $r > \sqrt{n+1}$ then B_r is *never* a local minimizer for the Gaussian perimeter under the constraints $\gamma(E) = m$ and $E = -E$, as it can be shown by a simple second variation argument.

PROPOSITION 4.4. *Let $n \ge 2$, $r > \sqrt{n+1}$ and k a positive integer. For every $\varepsilon > 0$ there exists a function $u \in C^\infty(\mathbb{S}^{n-1})$, with $\|u\|_{C^k(\mathbb{S}^{n-1})} < \varepsilon$ such that the corresponding nearly spherical set*

$$E = \{y = trx(1 + u(x)) :\, x \in \mathbb{S}^{n-1},\, 0 \le t < 1\} \tag{4.21}$$

is symmetric with respect to the origin, $\gamma(E) = \gamma(B_r)$ and $P_\gamma(E) < P_\gamma(B_r)$.

Proof. Fix $r > \sqrt{n+1}$, a positive integer k and $\varepsilon > 0$. Given an even function $\varphi \in C^\infty(\mathbb{S}^{n-1})$ such that

$$\int_{\mathbb{S}^{n-1}} \varphi(x)\, d\mathcal{H}^{n-1} = 0, \tag{4.22}$$

let $X \in C_c^\infty(\mathbb{R}^n;\mathbb{R}^n)$ be a vector field such that $X(-x) = -X(x)$ and

$$X(x) = \frac{e^{\frac{|x|^2}{2}}}{|x|^n}\varphi\Big(\frac{x}{|x|}\Big)x \qquad \text{for } x \in B_{2r} \setminus \overline{B}_{\frac{r}{2}}. \tag{4.23}$$

Let Φ be the flow associated to X, that is the unique C^∞ map $\Phi : \mathbb{R}^n \times (-1,1) \to \mathbb{R}^n$ such that for all $x \in \mathbb{R}^n$ and $t \in (-1,1)$

$$\frac{\partial \Phi}{\partial t}(x,t) = X(\Phi(x,t)), \quad \Phi(x,0) = x. \tag{4.24}$$

Set $E_t = \Phi(\cdot,t)(B_r)$ for all $t \in (-1,1)$. Since $\Psi(x,t) = -\Phi(-x,t)$ is also a solution to (4.24), by uniqueness we have that $\Phi(-x,t) = -\Phi(x,t)$, hence each E_t is symmetric with respect to the origin. Moreover, there exists $\delta > 0$ such that for $|t| < \delta$ the set E_t is a nearly spherical set as in (4.21) and the corresponding function u satisfies $\|u\|_{C^k(\mathbb{S}^{n-1})} < \varepsilon$. We claim that we can choose δ so that $\gamma(E_t) = \gamma(B_r)$ for $|t| < \delta$. To see this, let us choose $\delta > 0$ so that $B_{\frac{r}{2}} \subset\subset E_t \subset\subset B_r$. Then, see [**51**, Prop. 17.8], for all $t \in (-\delta,\delta)$

$$\frac{d}{dt}\gamma(E_t) = \frac{1}{(2\pi)^{\frac{n}{2}}}\int_{\partial E_t} X \cdot \nu^{E_t} e^{-\frac{|x|^2}{2}}\, d\mathcal{H}^{n-1}.$$

The equality $\gamma(E_t) = \gamma(B_r)$ will follow by observing that the integral on the right hand side of the above formula vanishes for all $t \in (-\delta,\delta)$. Indeed, if $\varrho > r/2$ is such that $B_\varrho \subset\subset E_t$, from the divergence theorem, recalling (4.22) and (4.23), we have

$$\begin{aligned}\int_{\partial E_t} X \cdot \nu_{E_t} e^{-\frac{|x|^2}{2}}\, d\mathcal{H}^{n-1} &= \int_{\partial B_\varrho} X \cdot \frac{x}{|x|} e^{-\frac{|x|^2}{2}}\, d\mathcal{H}^{n-1} + \int_{E_t \setminus B_\varrho} \operatorname{div}\big(X e^{-\frac{|x|^2}{2}}\big)\, dx \\ &= \frac{1}{\varrho^{n-1}}\int_{\partial B_\varrho} \varphi\Big(\frac{x}{|x|}\Big)\, d\mathcal{H}^{n-1} + \int_{E_t \setminus B_\varrho} \operatorname{div}\Big(\frac{x}{|x|^n}\varphi\Big(\frac{x}{|x|}\Big)\Big)\, dx = 0.\end{aligned}$$

Set now $p(t) = (2\pi)^{\frac{n}{2}} P_\gamma(E_t)$ for $t \in (-\delta,\delta)$. Using the formula for the first variation of the perimeter, see [**51**, Th. 17.5], the divergence theorem on manifolds and (4.22), we have

$$\begin{aligned}p'(0) &= \int_{\partial B_r} (\operatorname{div}_\tau X - X \cdot x) e^{-\frac{|x|^2}{2}}\, d\mathcal{H}^{n-1} = \int_{\partial B_r} \Big(X \cdot \frac{x}{|x|} H_{B_r} - X \cdot x\Big) e^{-\frac{|x|^2}{2}}\, d\mathcal{H}^{n-1} \\ &= \int_{\partial B_r} \Big(\frac{n-1}{r^{n+1}} - \frac{1}{r^{n-2}}\Big)\varphi\Big(\frac{x}{|x|}\Big)\, d\mathcal{H}^{n-1} = 0\end{aligned}$$

Thus, in order to conclude the proof it will be enough to show that we may always choose φ satisfying (4.22) and such that $p''(0) < 0$.
To this aim, let us evaluate $p''(0)$. Note that the general formula for the second variation of the Gaussian perimeter is quite complicate, see for instance [**5**, Eq. (17)]. However in our case, since B_r satisfies the Euler-Lagrange equation (4.20), it simplifies a lot. Indeed, see [**5**, Prop. 3] we have

$$p''(0) = \int_{\partial B_r} \big[|D_\tau(X \cdot \nu_{B_r})|^2 - |B_{B_r}|^2 (X \cdot \nu_{\partial B_r})^2 - (X \cdot \nu_{B_r})^2\big] e^{-\frac{|x|^2}{2}}\, d\mathcal{H}^{n-1},$$

where $|B_{B_r}|^2$ is the sum of the squares of the principal curvatures of ∂B_r. Hence,

$$p''(0) = \frac{e^{\frac{r^2}{2}}}{r^{2n-2}} \int_{\partial B_r} \Big[\Big| D_\tau \Big(\varphi\Big(\frac{x}{|x|}\Big)\Big)\Big|^2 - \frac{n-1}{r^2}\varphi\Big(\frac{x}{|x|}\Big)^2 - \varphi\Big(\frac{x}{|x|}\Big)^2 \Big] \, d\mathcal{H}^{n-1}$$
$$= \frac{e^{\frac{r^2}{2}}}{r^{n-1}} \int_{\mathbb{S}^{n-1}} \Big(\frac{1}{r^2}|D_\tau \varphi(x)|^2 - \frac{n-1}{r^2}\varphi(x)^2 - \varphi(x)^2 \Big) \, d\mathcal{H}^{n-1}.$$

Then, choosing $\varphi = y_2$, where y_2 is any homogeneous harmonic polynomial of degree 2, normalized so that $\|y_2\|_{L^2(\mathbb{S}^{n-1})} = 1$, (4.22) is obviously satisfied and from the above formula we get

$$p''(0) = \frac{e^{\frac{r^2}{2}}}{r^{n-1}} \Big(\frac{2n}{r^2} - \frac{n-1}{r^2} - 1 \Big) = \frac{(n+1-r^2)e^{\frac{r^2}{2}}}{r^{n+1}} < 0,$$

thus concluding the proof. □

0.2. L^1-local minimality. In this section we show how to derive from Theorem 4.2 the L^1-local minimality of balls centered at the origin with sufficiently small radii. Our proof follows the strategy devised in [**1**] with a few difficulties due to the fact that in the Gauss space the presence of a density in the measure γ does not allow us to reduce the proof to the case of bounded sets as it happens in the Euclidean case.

We now introduce a functional that will be used in the proof of the L^1-local minimality of the ball. Given $r > 0$ for every set E of locally finite perimeter we define

$$J(E) = P_\gamma(E) + \Lambda_1 \gamma(E \Delta B_r) + \Lambda_2 |\gamma(E) - \gamma(B_r)|, \tag{4.25}$$

where $\Lambda_1, \Lambda_2 \geq 0$. Next lemma, which is the counterpart in our setting of [**1**, Lemma 4.1], shows that if Λ_1 is sufficiently large, then the unique minimizer of J among all sets of locally finite perimeter E is the ball B_r.

LEMMA 4.5. *Let $n \geq 2$. There exists $C_0(n) > 0$ such that if $r > 0$, $\Lambda_1 > C_0\big(r + \frac{1}{r}\big)$ and $\Lambda_2 \geq 0$, then B_r is the unique minimizer of J among all sets $E \subset \mathbb{R}^n$ of locally finite perimeter.*

PROOF. Let $\eta : \mathbb{R} \to [0,1]$ be a smooth function such that $\eta \equiv 0$ outside the interval $(1/3, 3)$, $\eta \equiv 1$ on the interval $(1/2, 2)$. Fix $r > 0$ and denote by X_r the vector field

$$X_r(x) = \eta\Big(\frac{|x|}{r}\Big)\frac{x}{|x|} \qquad \text{for all } x \in \mathbb{R}^n.$$

Then

$$J(E) - J(B_r) \geq \int_{\partial^* E} X_r \cdot \nu_E \, d\mathcal{H}^{n-1}_\gamma - \int_{\partial B_r} X_r \cdot \nu_{B_r} \, d\mathcal{H}^{n-1}_\gamma + \Lambda_1 \gamma(E \Delta B_r)$$
$$= \int_E (\operatorname{div} X_r - X_r \cdot \ x) \, d\gamma - \int_{B_r} (\operatorname{div} X_r - X_r \cdot \ x) \, d\gamma + \Lambda_1 \gamma(E \Delta B_r)$$
$$= \int_{E \Delta B_r} (\operatorname{div} X_r - X_r \cdot \ x) \, d\gamma + \Lambda_1 \gamma(E \Delta B_r).$$

Since by construction $||\operatorname{div} X_r - X_r \cdot x||_\infty \leq C_0\big(\frac{1}{r} + r\big)$ for some constant C_0 depending only on n, the result immediately follows. □

One difficulty in the proof of L^1-local minimality of the balls B_r for small radii is that the Gaussian measure is not scaling invariant. The following simple lemma is a helpful tool to deal with this issue.

LEMMA 4.6. *Let $n \geq 2$, $\sigma \in (0,1/2)$. For every $\varepsilon > 0$ there exists $\delta < \sigma/2$ depending only on ε, n, σ, such that for all $r \in \big[\sigma, \sqrt{n+1}\big]$ and $\tau \in (0,\delta)$*

$$P_\gamma(B_r) - P_\gamma(B_{r-\tau}) \leq \varepsilon P_\gamma(H^-_{e_n,s(r,\tau)}), \tag{4.26}$$

where the half space $H^-_{e_n,s(r,\tau)}$ is such that $\gamma(H^-_{e_n,s(r,\tau)}) = \gamma(B_r) - \gamma(B_{r-\tau})$.

PROOF. For $r \in \big[\sigma, \sqrt{n+1}\big]$, we set

$$f(r,\tau) = e^{-\frac{s(r,\tau)^2}{2}} \quad \text{for } 0 < \tau \leq r, \qquad f(r,0) = 0.$$

Then, for $0 < \tau \leq \sigma/2$ we have

$$\frac{P_\gamma(B_r) - P(_\gamma(B_{r-\tau})}{P_\gamma(H^-_{s(r,\tau)})} = \frac{n\omega_n}{(2\pi)^{\frac{n-1}{2}}} \frac{r^{n-1}e^{-\frac{r^2}{2}} - (r-\tau)^{n-1}e^{-\frac{(r-\tau)^2}{2}}}{f(r,\tau) - f(r,0)}.$$

Therefore, by the Cauchy's mean value theorem there exists $\vartheta \in (0,\tau)$, such that

$$\frac{P_\gamma(B_r) - P(_\gamma(B_{r-\tau})}{P_\gamma(H^-_{e_n,s(r,\tau)})} = \frac{n\omega_n}{(2\pi)^{\frac{n-1}{2}}} \frac{-(n-1)(r-\vartheta)^{n-2}e^{-\frac{(r-\vartheta)^2}{2}} + (r-\vartheta)^n e^{-\frac{(r-\vartheta)^2}{2}}}{-s(r,\vartheta)e^{-\frac{s(r,\vartheta)^2}{2}}\dfrac{\partial s}{\partial \tau}(r,\vartheta)}. \tag{4.27}$$

On the other hand, since by definition

$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{s(r,\tau)} e^{-\frac{t^2}{2}}\,dt = \frac{n\omega_n}{(2\pi)^{\frac{n}{2}}}\int_{r-\tau}^{r} t^{n-1}e^{-\frac{t^2}{2}}\,dt,$$

differentiating this equation with respect to τ we have that

$$\frac{\partial s}{\partial \tau}(r,\vartheta) = \frac{n\omega_n}{(2\pi)^{\frac{n-1}{2}}}(r-\vartheta)^{n-1}e^{\frac{s(r,\vartheta)^2 - (r-\vartheta)^2}{2}}.$$

Thus, inserting this value in (4.27) we have

$$\begin{aligned}\frac{P_\gamma(B_r) - P(_\gamma(B_{r-\tau})}{P_\gamma(H^-_{s(r,\tau)})} &= \frac{-(n-1) + (r-\vartheta)^2}{-s(r,\vartheta)(r-\vartheta)} \\ &\leq \frac{2}{-\Phi^{-1}\big(\gamma(B_r) - \gamma(B_{r-\vartheta})\big)(r-\vartheta)} \\ &\leq \frac{2}{-\Phi^{-1}(C(n)\vartheta)\sigma} \leq \frac{4}{-\Phi^{-1}(C(n)\tau)\sigma},\end{aligned}$$

for a suitable constant depending only on n. Then the conclusion follows since $\Phi^{-1}(\tau) \to -\infty$ as $\tau \to 0^+$. □

As in [**1**] the proof of Theorem 4.1 uses heavily the regularity theory for area minimizing sets. For the reader's convenience we recall the relevant definitions and the main results that we need in the sequel.

DEFINITION 4.7. Let $E \subset \mathbb{R}^n$ be a set of locally finite perimeter, $\omega, r_0 > 0$ and let Ω be an open subset of $\mathbb{R}^n$. We say that E is a (ω, r_0)-*quasiminimizer of the (Euclidean) perimeter* in Ω if for every ball $B_\varrho(x) \subset \Omega$ with $\varrho < r_0$ and any set F of locally finite perimeter such that $E\Delta F \subset\subset B_\varrho(x)$

$$P(E; B_\varrho(x)) \leq P(E; B_\varrho(x)) + \omega\varrho^n. \tag{4.28}$$

Note that this notion of minimality is slightly weaker than the $(\lambda, r_0)-$minimality defined in Definition 2.13 where on the right hand side of (4.28) the term $\omega\varrho^n$ is replaced by $\Lambda|E\Delta F|$, where Λ is a fixed positive constant. Nevertheless, the regularity theory for perimeter minimizers carries also to quasiminizers. In particular we have the following two results. For the first one we refer to [**67**, Th. 1.9], see also [**51**, Th. 21.14]. For a proof of Theorem 4.9 the reader may see [**23**, Prop. 2.2] or [**51**, Th. 26.6]. Note that when dealing with a set of finite perimeter E we always tacitly assume that E is a Borel set such that its topological boundary ∂E coincides with the support of the perimeter measure, i.e.,

$$\partial E = \{x \in \mathbb{R}^n : 0 < |E \cap B_r(x)| < \omega_n r^n \text{ for every } r > 0\},$$

see for instance [**51**, Prop. 12.19]. Observe that

$$E_h \text{ converge locally in measure to } E \quad \Longrightarrow \quad \gamma(E_h) \to \gamma(E) \text{ as } h \to \infty. \tag{4.29}$$

THEOREM 4.8. *Let E_h be a sequence of (ω, r_0)-quasiminimizers in Ω converging locally in measure to a set of locally finite perimeter E. Then the two following properties hold:*

(i) if $x_h \in \partial E_h \cap \Omega$ and $x_h \to x \in \Omega$, then $x \in \partial E$;

(ii) if $x \in \partial E \cap \Omega$, then there exists a sequence x_h such that $x_h \in \partial E_h \cap \Omega$ for all h and $x_h \to x$.

We will also need a regularity theorem stating that if F is a perimeter quasiminimizer, sufficiently close in L^1 to a smooth open set E, then F is indeed $C^{1,\alpha}$ close to E.

THEOREM 4.9. *Let E_h be a sequence of equibounded (ω, r_0)-quasiminimizers in $\mathbb{R}^n$, converging in $\mathbb{R}^n$ to a bounded open set E of class C^2. Then, for h large enough E_h is of class $C^{1,\frac{1}{2}}$ and*

$$\partial E_h = \{x + \psi_h(x)\nu_E(x) \;:\; x \in \partial E\}$$

with $\psi_h \to 0$ in $C^{1,\alpha}$ for all $\alpha \in (0, \frac{1}{2})$.

We are now ready to prove our main result. Roughly speaking it states that if B_r is a ball whose radius is below the critical value $\sqrt{n+1}$, then it is a local minimizer of the Gaussian perimeter among all sets with the same Gaussian measure and symmetric with respect to the origin. Moreover this local minimality property holds with a uniform quantitative estimate, provided r is away from 0 and from $\sqrt{n+1}$.

Before giving the proof of this theorem, let us briefly describe its strategy. We argue by contradiction assuming that there exists a sequence of symmetric sets E_h, with $\gamma(E_h) = \gamma(B_{r_h})$, such that $\varepsilon_h = \gamma(E_h \Delta B_{r_h}) \to 0$ as $h \to \infty$, for which the inequality (4.1) is violated. At this point, as first observed in this context by Cicalese and Leonardi in [**23**], one may replace the sets E_h with a new sequence F_h, still violating inequality (4.1), but converging in $C^{1,\alpha}$ to a ball $B_{\overline{r}}$. This leads to a contradiction with the local minimality property of $B_{\overline{r}}$, provided the constant κ is sufficiently small. The new sequence F_h is obtained by minimizing the functionals

$$J_h(F) = P_\gamma(F) + \Lambda_1|\gamma(F\Delta B_{r_h}) - \varepsilon_h| + \Lambda_2|\gamma(F) - \gamma(B_{r_h})|, \tag{4.30}$$

for suitable $\Lambda_1, \Lambda_2 > 0$, among all subsets of $\mathbb{R}^n$ symmetric with respect to the origin. The choice of this particular functional is inspired by a similar one first introduced in [**1**] and later on successfully modified in [**10**], [**11**], [**16**], [**36**]. To get the $C^{1,\alpha}$ convergence of the minimizers F_h we prove that they are also (ω, r_0)-quasiminimizers of the Euclidean perimeter in every ball B_R, a fact that in our case is not completely trivial, since each F_h minimizes the functional J_h only among sets which are symmetric with respect to the origin. At this point, if we knew that the sets F_h were equibounded, the $C^{1,\alpha}$ convergence would follow immediately from Theorem 4.9. However, there is no reason why this should be true and to overcome this difficulty we have to show that even if the F_h may be unbounded they all split into two regions, a bounded one which converge in $C^{1,\alpha}$ to the ball $B_{\overline{r}}$ and another one of small mass which disappears at infinity.

Proof of Theorem 4.1. Throughout this proof we are going to use the following notation. Given a measurable set E we denote by $r(E)$ the radius of the ball centered at the origin such that

$$\gamma(E) = \gamma(B_{r(E)}). \tag{4.31}$$

Step 1. We argue by contradiction assuming that there exists a sequence E_h of sets symmetric with respect to the origin, with $\gamma(E_h) = \gamma(B_{r_h})$, $r_h \in [\sigma, \sqrt{n+1} - \sigma]$, such that

$$\varepsilon_h = \gamma(E_h \Delta B_{r_h}) \to 0, \qquad P_\gamma(E_h) - P_\gamma(B_{r_h}) \leq \kappa \varepsilon_h^2, \tag{4.32}$$

for a suitable κ that will be fixed later in the proof. Let us fix

$$\Lambda_1 > C_0\Big(\sqrt{n+1} + \frac{1}{\sigma}\Big), \qquad \Lambda_2 \geq \max\{3\Lambda_1, \widetilde{C}\}, \tag{4.33}$$

where C_0 is the constant provided in Lemma 4.5 and $\widetilde{C}$ is a constant, depending only on n and σ, that will be fixed later.
For every h we consider the following minimum problem

$$\min\big\{J_h(F) :\, F = -F,\, F \text{ has locally finite perimeter}\big\}, \tag{4.34}$$

where J_h is the functional defined in (4.30).
The existence of a minimizer for the the problem in (4.34) is readily proved by observing that any minimizing sequence is compact with respect to the local convergence in $\mathbb{R}^n$ and recalling the lower semicontinuity of the perimeter and the continuity of the Gaussian measure, see (4.29), with respect to the local convergence in $\mathbb{R}^n$.
Let us now assume, without loss of generality, that $r_h \to \overline{r}$ for $h \to \infty$ and observe that the minimizers F_h converge locally in L^1 to $B_{\overline{r}}$. In fact, since by the minimality of F_h

$$P_\gamma(F_h) \leq P_\gamma(E_h) \leq C(n), \qquad \text{for all } h,$$

we have, see [**3**, Th. 3.39], that up to a (not relabelled) subsequence, they converge locally in $\mathbb{R}^n$ to some set of locally finite perimeter $\widetilde{F}$. We claim that $\widetilde{F} = B_{\overline{r}}$. To see this let us take a set of locally finite perimeter E, symmetric with respect to the origin. By the minimality of F_h we have that

$$P_\gamma(F_h) + \Lambda_1|\gamma(F_h \Delta B_{r_h}) - \varepsilon_h| + \Lambda_2|\gamma(F_h) - \gamma(B_{r_h})| \leq J_h(E).$$

Recalling (4.29), from the previous inequality we get immediately that

$$P_\gamma(\widetilde{F}) + \Lambda_1\gamma(\widetilde{F}\Delta B_{\overline{r}}) + \Lambda_2|\gamma(\widetilde{F}) - \gamma(B_{\overline{r}})| \leq P_\gamma(E) + \Lambda_1\gamma(E\Delta B_{\overline{r}}) + \Lambda_2|\gamma(E) - \gamma(B_{\overline{r}})|.$$

Hence, recalling the first inequality in (4.33), Lemma 4.5 yields $\widetilde{F} = B_{\overline{r}}$.
Note that for every B_R there exist $\omega > 0, r_0 \in (0,1)$ such that, for h large, the sets F_h are all (ω, r_0)-quasiminimizers in B_R. The proof of this latter property is given in Lemma 4.10 below.
Step 2 We claim that for h large

$$\gamma(F_h \Delta B_{r_h}) \geq \frac{\varepsilon_h}{4}. \tag{4.35}$$

To this end observe that by the minimality of F_h, the inequality in (4.32) and Lemma 4.5 again, we have

$$\begin{aligned} P_\gamma(F_h) + \Lambda_1|\gamma(F_h\Delta B_{r_h}) - \varepsilon_h| + \Lambda_2|\gamma(F_h) - \gamma(B_{r_h})| &\leq P_\gamma(E_h) \\ \leq P_\gamma(B_{r_h}) + \kappa\varepsilon_h^2 &\leq P_\gamma(F_h) + \Lambda_1\gamma(F_h\Delta B_{r_h}) + \kappa\varepsilon_h^2. \end{aligned} \tag{4.36}$$

If $\gamma(F_h\Delta B_{r_h}) \geq \varepsilon_h/2$, inequality (4.35) is trivially satisfied. Otherwise, if $\gamma(F_h\Delta B_{r_h}) \leq \varepsilon_h/2$, from (4.36) we deduce

$$\varepsilon_h - \gamma(F_h\Delta B_{r_h})) \leq \gamma(F_h\Delta B_{r_h}) + \frac{\kappa}{\Lambda_1}\varepsilon_h^2.$$

Hence, the claim (4.35) follows for h sufficiently large, since $\varepsilon_h \to 0$ by (4.32).
Since $\gamma(F_h)$ may be different from $\gamma(B_{r_h})$, it is convenient to consider the balls $B_{r(F_h)}$ defined as in (4.31). From (4.35) and (4.36), recalling the second inequality in (4.33), we have for h large

$$|\gamma(F_h) - \gamma(B_{r_h})| \leq \frac{\Lambda_1}{\Lambda_2}\gamma(F_h\Delta B_{r_h}) + \frac{\kappa}{\Lambda_2}\varepsilon_h^2 \leq \frac{\gamma(F_h\Delta B_{r_h})}{2}.$$

Thus, we may estimate, for h large

$$\begin{aligned}\gamma(F_h\Delta B_{r_h}) &\leq \gamma(F_h\Delta B_{r(F_h)}) + \gamma(B_{r(F_h)}\Delta B_{r_h})\\ &= \gamma(F_h\Delta B_{r(F_h)}) + |\gamma(F_h) - \gamma(B_{r_h})| \leq \gamma(F_h\Delta B_{r(F_h)}) + \frac{\gamma(F_h\Delta B_{r_h})}{2}.\end{aligned}$$

Therefore, recalling (4.35), we have

$$\gamma(F_h\Delta B_{r_h}) \leq 2\gamma(F_h\Delta B_{r(F_h)}), \text{ hence } \gamma(F_h\Delta B_{r(F_h)}) \geq \frac{\varepsilon_h}{8}. \tag{4.37}$$

From (4.36) and the second inequality in (4.37) we have with some easy computations

$$\begin{aligned}P_\gamma(F_h) + \Lambda_2|\gamma(F_h) - \gamma(B_{r_h})| &\leq P_\gamma(B_{r_h}) + 64\kappa\gamma(F_h\Delta B_{r(F_h)})^2\\ &= P_\gamma(B_{r(F_h)})) + [P_\gamma(B_{r_h}) - P_\gamma(B_{r(F_h)})] + 64\kappa\gamma(F_h\Delta B_{r(F_h)})^2\\ &\leq P_\gamma(B_{r(F_h)})) + C(n)|r_h - r(F_h)| + 64\kappa\gamma(F_h\Delta B_{r(F_h)})^2\\ &\leq P_\gamma(B_{r(F_h)})) + \widetilde{C}(n,\sigma)|\gamma(B_{r(F_h)}) - \gamma(B_{r_h})| + 64\kappa\gamma(F_h\Delta B_{r(F_h)})^2,\end{aligned}$$

for a suitable constant $\widetilde{C}$ depending only on n and σ. Therefore, recalling that $\Lambda_2 \geq \widetilde{C}$ by (4.32), we end up by proving that also the sets F_h satisfy a 'reverse' quantitative inequality as the one in (4.32), with a possibly bigger constant

$$P_\gamma(F_h) \leq P_\gamma(B_{r(F_h)})) + 64\kappa\gamma(F_h\Delta B_{r(F_h)})^2. \tag{4.38}$$

Note that if we knew that the F_h were equibounded we would have by Theorem 4.9 that they were converging in $C^{1,\alpha}$ to the ball $B_{\overline{r}}$ and, taking κ sufficiently small, from (4.38) we would get a contradiction to (4.17), thus concluding the proof. Instead, since it may happen that the sets F_h are unbounded or that they are not equibounded, we split them as follows

$$G_h = F_h \cap B_n, \qquad L_h = F_h \setminus B_n.$$

Clearly, the G_h converge in L^1 to $B_{\overline{r}}$, while $\gamma(L_h) \to 0$ as $h \to \infty$. Moreover, since

$$\begin{aligned}\gamma(F_h\Delta B_{r(F_h)}) &\leq \gamma(F_h\Delta G_h) + \gamma(G_h\Delta B_{r(G_h)}) + \gamma(B_{r(G_h)}\Delta B_{r(F_h)})\\ &= \gamma(G_h\Delta B_{r(G_h)}) + 2\gamma(L_h),\end{aligned}$$

from (4.38) we conclude that

$$P_\gamma(F_h) \leq P_\gamma(B_{r(F_h)})) + C_2\kappa\big[\gamma(G_h\Delta B_{r(G_h)})^2 + \gamma(L_h)^2\big], \tag{4.39}$$

for some universal constant C_2 not even depending on n.
Step 3. We claim now that for h large

$$F_h \cap (\overline{B}_{n+1} \setminus B_n) = \emptyset. \tag{4.40}$$

To prove this we argue by contradiction assuming that for infinitely many h the intersection $F_h \cap (\overline{B}_{n+1} \setminus B_n)$ is not empty. On the other hand, since $F_h \cap \overline{B}_n$ is converging in $\mathbb{R}^n$ to $B_{\overline{r}}$, we have that for h large also $(\overline{B}_{n+1} \setminus B_n) \setminus F_h$ is not empty. Therefore, there exists an increasing

sequence $h_k \to \infty$ such that for any k there exists $x_k \in \partial F_{h_k} \cap (\overline{B}_{n+1} \setminus B_n)$ (note that since the sets F_h are quasiminimizers of the perimeter in every ball B_R, they are of class C^1 and thus ∂F_h coincides with the topological boundary). Passing possibly to another, and not relabelled, subsequence we may assume that $x_k \to x$ for some $x \in \overline{B}_{n+1} \setminus B_n$. But this is impossible since by Theorem 4.8 the point x should belong to $\partial B_{\overline{r}}$.
Note that (4.40) yields in particular that

$$G_h \subset B_n, \qquad L_h \subset \mathbb{R}^n \setminus B_{n+1}. \tag{4.41}$$

As an immediate consequence of the above inclusions we have that the sets G_h are quasiminimizers of the Euclidean perimeter in $\mathbb{R}^n$.
Another consequence of (4.41) is that for h large $P_\gamma(F_h) = P_\gamma(G_h) + P_\gamma(L_h)$. Thus, from (4.39) we get that for h large

$$P_\gamma(G_h) \le P_\gamma(B_{r(F_h)}) - P_\gamma(L_h) + C_2\kappa\big[\gamma(G_h \Delta B_{r(G_h)})^2 + \gamma(L_h)^2\big].$$

Now, let $s_h \in \mathbb{R}$ be such that $\gamma(H^-_{s_h}) = \gamma(L_h)$. From the inequality above and the Gaussian isoperimetric inequality we have

$$P_\gamma(G_h) \le P_\gamma(B_{r(F_h)}) - P_\gamma(H^-_{e_n,s_h}) + C_2\kappa\big[\gamma(G_h \Delta B_{r(G_h)})^2 + \gamma(H_{e_n,s_h})^2\big].$$

In turn, using (4.26) with $\varepsilon = 1/2$ to estimate $P_\gamma(B_{r(F_h)})$, we have that for h sufficiently large

$$P_\gamma(G_h) \le P_\gamma(B_{r(G_h)}) - \frac{1}{2}P_\gamma(H^-_{e_n,s_h}) + C_2\kappa\big[\gamma(G_h \Delta B_{r(G_h)})^2 + \gamma(H^-_{e_n,s_h})^2\big]. \tag{4.42}$$

Finally, observe that

$$\lim_{s\to-\infty} \frac{\gamma(H^-_{e_n,s})}{P_\gamma(H^-_{e_n,s})} = 0.$$

Therefore, from (4.42) we may conclude that for h sufficiently large

$$P_\gamma(G_h) \le P_\gamma(B_{r(G_h)}) + C_2\kappa\gamma(G_h \Delta B_{r(G_h)})^2. \tag{4.43}$$

Now, since the sets G_h are converging to $B_{\overline{r}}$ in $\mathbb{R}^n$, by Lemma 4.9 they also converge in $C^{1,\alpha}$ to $B_{\overline{r}}$, i.e.

$$\partial G_h = \{x(1 + u_h(x)) : x \in \partial B_{\overline{r}}\}$$

where $u_h \to 0$ in $C^{1,\alpha}(\partial B_{\overline{r}})$. Thus, still denoting by u_h the 0-homogeneous extension of the above functions u_h, we conclude that

$$G_h = \Big\{y = tr(G_h)x\Big(1 + \frac{\overline{r}(1 + u_h(x)) - r(G_h)}{r(G_h)}\Big) : x \in \mathbb{S}^{n-1}, 0 \le t < 1\Big\},$$

where, since $r(G_h) \to \overline{r}$,

$$\frac{\overline{r}(1 + u_h(x)) - r(G_h)}{r(G_h)} \to 0 \qquad \text{in } C^{1,\alpha}(\mathbb{S}^{n1}).$$

Thus, by (4.17) we conclude that for h sufficiently large

$$P_\gamma(G_h) - P_\gamma(B_{r(G_h)}) \ge \kappa_1 r(G_h)^{-1-n}\gamma(G_h \Delta B_{r(G_h)})^2 \ge \frac{\kappa_1}{(n+1)^{\frac{n+1}{2}}}\gamma(G_h \Delta B_{r(G_h)})^2,$$

which contradicts (4.43) if we choose

$$\kappa < \frac{\kappa_1}{C_2(n+1)^{\frac{n+1}{2}}}.$$

Hence the conclusion follows by this contradiction. □

The arguments used in the proof of next lemma are similar to the ones used for the standard perimeter. However, in our case the proof is more involved due presence of the constraint $F=-F$ in the minimum problems (4.34).

Lemma 4.10. *Let $n\geq 2$ and $\sigma\in(0,1/2)$ as in Theorem 4.1. Moreover, let F_h be a sequence of minimizers of the problems (4.34), with F_h converging locally in $\mathbb{R}^n$ to a ball $B_{\overline{r}}$, with $\overline{r}\in[\sigma,\sqrt{n+1}-\sigma]$. There exists h_0 such that for every ball B_R, with $R\geq 1$, there exist $\omega, r_0>0$, such that F_h is a (ω,r_0)-quasiminimizer in B_R for all $h\geq h_0$.*

Proof. **Step 1.** Let us fix $R\geq 1$. We start proving that there exist $r_1,\vartheta>0$, depending on R, such that if $x\in\partial F_h\cap B_R$, $\varrho<r_1$, then

$$|F_h\cap B_\varrho(x)|\leq(\omega_n-\vartheta)\varrho^n. \tag{4.44}$$

To this end, let us observe that if $x\in\mathbb{R}^n$, $\varrho>0$ and G is a set of locally finite perimeter with $G=-G$, such that $F_h\Delta G\subset\subset B_\varrho(x)\cup B_\varrho(-x)$, then from the minimality inequality $J_h(F_h)\leq J_h(G)$ we get

$$P_\gamma(F_h;B_\varrho(x)\cup B_\varrho(-x))\leq P_\gamma(G;B_\varrho(x)\cup B_\varrho(-x))+(\Lambda_1+\Lambda_2)\gamma(F_h\Delta G).$$

From this inequality, setting

$$m(x,\varrho)=\frac{1}{(2\pi)^{\frac{n}{2}}}\min_{y\in B_\varrho(x)}e^{-\frac{|y|^2}{2}},\qquad M(x,\varrho)=\frac{1}{(2\pi)^{\frac{n}{2}}}\max_{y\in B_\varrho(x)}e^{-\frac{|y|^2}{2}},$$

we immediately get the following inequality for the Euclidean perimeter

$$m(x,\varrho)P(F_h;B_\varrho(x)\cup B_\varrho(-x))\leq M(x,\varrho)P(G;B_\varrho(x)\cup B_\varrho(-x))+(\Lambda_1+\Lambda_2)M(x,\varrho)|F_h\Delta G|.$$

Thus, dividing both sides of this inequality by $m(x,\varrho)$ and observing that if $0<\varrho<1$ we have $(M(x,\varrho)-m(x,\varrho))/m(x,\varrho)<C\varrho$, for some constant C depending on R, we get that

$$P(F_h;B_\varrho(x)\cup B_\varrho(-x))\leq(1+C\varrho)P(G;B_\varrho(x)\cup B_\varrho(-x))+C'|F_h\Delta G|. \tag{4.45}$$

Recalling that $F_h\Delta G\subset\subset B_\varrho(x)\cup B_\varrho(-x)$ from the standard isoperimetric inequality we get

$$\begin{aligned}|F_h\Delta G|&\leq|B_\varrho(x)\cup B_\varrho(-x)|^{\frac{1}{n}}|F_h\Delta G|^{\frac{n-1}{n}}\leq n\omega_n\varrho P(F_h\Delta G;B_\varrho(x)\cup B_\varrho(-x))\\&\leq n\omega_n\varrho[P(F_h;B_\varrho(x)\cup B_\varrho(-x))+P(G;B_\varrho(x)\cup B_\varrho(-x))],\end{aligned}$$

where the last inequality follows by using the precise expression of the reduced boundary of the symmetric difference of two sets of finite perimeter, see [**51**, Th. 16.3]. Inserting this inequality in (4.45) we conclude that there exists $\chi>1$ depending only on n, Λ_1 and Λ_2 and R such that for all $0<\varrho<1$

$$(1-\chi\varrho)P(F_h;B_\varrho(x)\cup B_\varrho(-x))\leq(1+\chi\varrho)P(G;B_\varrho(x)\cup B_\varrho(-x)). \tag{4.46}$$

Let us now fix $x\in\mathbb{R}^n$ and $0<\varrho<1/\chi$ and set $G=F_h\cup(B_{\varrho'}(x)\cup B_{\varrho'}(-x))$ for some $0<\varrho'<\varrho$. Note that G is an admissible comparison set since $G=-G$. With this choice of G, using again the precise expression of the reduced boundary of the difference between two sets of finite perimeter, see again [**51**, Th. 16.3], from (4.46) we easily obtain that

$$\begin{aligned}(1-\chi\varrho)P(F_h;B_\varrho(x)\cup B_\varrho(-x))\leq(1+\chi\varrho)\big[&\mathcal{H}^{n-1}\big(F_h^{(0)}\cap\partial(B_{\varrho'}(x)\cup B_{\varrho'}(-x))\big)\\&+P(F_h;(B_\varrho(x)\cup B_\varrho(-x))\setminus(\overline{B}_{\varrho'}(x)\cup\overline{B}_{\varrho'}(-x))),\end{aligned}$$

where $F_h^{(0)}$ denotes the sets of points in $\mathbb{R}^n$ where F_h has density 0. From this inequality, letting $\varrho'\to\varrho$, we deduce that if $0<\varrho<1/\chi$, then

$$P(F_h,B_\varrho(x))\leq\frac{2(1+\chi)}{1-\chi}\mathcal{H}^{n-1}\big(F_h^{(0)}\cap\partial B_\varrho(x)\big). \tag{4.47}$$

Let us now fix $x \in \partial F_h$. In this way, setting $m(\varrho) = |B_\varrho(x) \setminus F_h|$, we have that $m(\varrho) > 0$ for all $\varrho > 0$. Since $m'(\varrho) = \mathcal{H}^{n-1}\big(F_h^{(0)} \cap \partial B_\varrho(x)\big)$ for a.e. $\varrho > 0$, from (4.47) we get that for all $\varrho \in (0, 1/\chi)$

$$m(\varrho)^{\frac{n-1}{n}} \leq \frac{2\kappa_n(1+\chi)}{1-\chi} m'(\varrho),$$

where κ_n is the constant of the Euclidean relative isoperimetric in balls, see for instance [**3**, Eq. 3.43]. Integrating this inequality we then get that for all $0 < \varrho < 1/\chi$

$$|B_\varrho(x) \setminus F_h| \geq \vartheta \varrho^{n-1},$$

hence (4.44) follows.

Step 2. Let us now prove that there exists an integer h_0 such that

$$|B_{\frac{\sigma}{2}} \setminus F_h| = 0 \qquad \text{for all } h \geq h_0. \tag{4.48}$$

To prove this inclusion we argue by contradiction assuming that there exists a strictly increasing sequence h_k of integers such that $|B_{\frac{\sigma}{2}} \setminus F_{h_k}| > 0$ for all k. On the other hand, since the sets $F_{h_k} \cap B_{\frac{\sigma}{2}}$ are converging in $\mathbb{R}^n$ to $B_{\frac{\sigma}{2}}$, we have also that $|B_{\frac{\sigma}{2}} \cap F_{h_k}| > 0$ for all k sufficiently large. Thus, from the relative isoperimetric inequality on balls we have that for all k large $P(F_{h_k}; B_{\frac{\sigma}{2}}) > 0$, hence there exists a point $x_k \in \partial F_{h_k}$. Without loss of generality we may assume that the sequence x_{h_k} converges to a point $x \in \overline{B}_{\frac{\sigma}{2}}$. We now apply (4.44) with R=1 and $0 < \varrho < \min\{r_1, \sigma/2\}$. From the local convergence of F_h in $\mathbb{R}^n$ and we then have

$$|B_\varrho(x)| = \lim_k |F_{h_k} \cap B_\varrho(x_k)| \leq (\omega_n - \vartheta)\varrho^n.$$

From this contradiction (4.48) immediately follows.

Step 3. Let us now fix $R \geq 1$ and set $r_0 = \min\{\sigma/4, 1/\chi\}$, where χ is the constant in (4.47) (note that this constant depends on R but not on h). Let us consider a ball $B_\varrho(x) \subset B_R$, with $0 < \varrho < r_0$, and a set G of locally finite perimeter such that $F_h \Delta G \subset\subset B_\varrho(x)$, for a given $h \geq h_0$. Assume first that $|x| \geq \sigma/4$ and observe that in this case $B_\varrho(x) \cap B_\varrho(-x) = \emptyset$. Then, define

$$G' = [F \setminus (B_\varrho(x) \cup B_\varrho(-x))] \cup (G \cap B_\varrho(x)) \cup (-G \cap B_\varrho(-x)).$$

By construction, the set $-G' = G'$ and, inserting it in (4.45) we immediately get that

$$P(F_h; B_\varrho(x)) \leq (1 + C\varrho)P(G; B_\varrho(x)) + C'|F_h \Delta G|.$$

Adding $C\varrho$ to both sides of this inequality and recalling (4.47) we have

$$(1 + C\varrho)P(F_h; B_\varrho(x)) \leq (1 + C\varrho)P(G; B_\varrho(x)) + \frac{2C\varrho(1+\chi)}{1-\chi}\mathcal{H}^{n-1}\big(F_h^{(0)} \cap \partial B_\varrho(x)\big) + C'|F_h \Delta G|.$$

Dividing this inequality by $1 + C\varrho$ we immediately get that

$$P(F_h; B_\varrho(x)) \leq P(G; B_\varrho(x)) + \omega\varrho^n, \tag{4.49}$$

for a suitable ω depending only on $n, \chi, \Lambda_1, \Lambda_2$ and R.

If $|x| < \sigma/4$, recalling the inclusion (4.48), we have that $P(F_h; B_\varrho(x)) = 0$ and thus (4.49) holds trivially. This concludes the proof of the lemma. □

Now we want to show that B_r is not a global minimizer among symmetric sets with prescribed Gaussian measure, at least if r is small. To this end, we set $C_s = \mathbb{R}^n \setminus B_s$, and for every $r > 0$ we denote by $s(r)$ the unique number such that

$$\int_0^r t^{n-1} e^{-\frac{t^2}{2}}\, dt = \int_{s(r)}^\infty t^{n-1} e^{-\frac{t^2}{2}}\, dt \tag{4.50}$$

In other words, $s(r)$ is such that $\gamma(B_r) = \gamma(C_{s(r)})$

PROPOSITION 4.11. *For every $n > 2$ there exists $r_0 > 0$ such that*

$$P_\gamma(C_{s(r)}) < P_\gamma(B_r) \tag{4.51}$$

for every $r < r_0$.

PROOF. Differentiating (4.50) with respect to r, we have

$$r^{n-1}e^{-\frac{r^2}{2}} = -s^{n-1}(r)e^{-\frac{s^2(r)}{2}}s'(r) \tag{4.52}$$

In order to show that $P_\gamma(B_r) > P_\gamma(C_{s(r)})$ for r small enough, using (4.52) we evaluate the quotient as follows

$$\frac{P_\gamma(B_r)}{P_\gamma(C_{s(r)})} = \frac{r^{n-1}e^{-\frac{r^2}{2}}}{s^{n-1}(r)e^{-\frac{s^2(r)}{2}}} = -s'(r) \tag{4.53}$$

Since $\lim_{r\to 0^+} s(r) = +\infty$, $\lim_{r\to 0^+} s'(r) = -\infty$. Then there exists $r_0 > 0$ such that if $r < r_0$ we have $s'(r) < -1$. Therefore

$$\frac{P_\gamma(B_r)}{P_\gamma(C_{s(r)})} = -s'(r) > 1,$$

hence (4.51) follows.

□

0.3. The 1-dimensional case. In this section we shall briefly discuss the 1-dimensional case. Beside being much simpler, this case exhibits quite different features. Before stating the local minimality result we recall that in one dimension a set of locally finite perimeter is locally the union of a finitely many intervals.

PROPOSITION 4.12. *Let $n = 1$. For every $r > 0$, there exists $\delta = \delta(r)$ such that if $E \subset \mathbb{R}$ is a set of locally finite perimeter, $0 < \gamma(B_r \Delta E) < \delta$, $E = -E$ and $\gamma(E) = \gamma(B_r)$, then*

$$P_\gamma(E) > P_\gamma(B_r) \tag{4.54}$$

Moreover, there exists r_0 such that:

(a) *if $r > r_0$ then B_r is the unique global minimizer of the perimeter among all the sets E such that $E = -E$ and $\gamma(B_r) = \gamma(E)$.*
(b) *if $r < r_0$ then $C_s = (-\infty, -s) \cup (s, +\infty)$ is the unique global minimizer of the perimeter among all the sets E such that $E = -E$ and $\gamma(B_r) = \gamma(C_s) = \gamma(E)$.*
(c) *if $r = r_0$, then both B_{r_0} and C_{r_0} are global minimizers.*

PROOF. The proof is quite easy, and it is based on the minimality property of the halfline. Fix any $r > 0$ and E such that $\gamma(E) = \gamma(B_r)$. Since a set of locally finite perimeter is locally the union of a finite number of intervals, the generic symmetric set E will be of the type

$$E = \bigcup_{i=1}^{M}(-b_i, -a_i) \cup \bigcup_{i=1}^{M}(a_i, b_i) \cup (-a, a)$$

for some $0 \le a < a_1 < b_1 < \cdots < a_i < b_i < \dots$, with $M \in \mathbb{N} \cup \{\infty\}$ and $b_M \in (0, \infty]$.
Take $R > r$ such that $R \neq a_i, R \neq b_i,\ \forall i \in \mathbb{N}$ and such that $b_j < R < a_{j+1}$ for some $j \in \mathbb{N}$. Using the isoperimetric inequality it is easy to check that if H_{-s} is a halfline such that

$$\gamma(H_{-s}) = \frac{1}{2}\gamma(E \setminus B_R),$$

then

$$P_\gamma((E \cap B_R) \cup C_s) \le P_\gamma(E).$$

Therefore we may assume without loss of generality that

$$E = (-\infty, -b) \cup \bigcup_{i=1}^{k} (-b_i, -a_i) \cup \bigcup_{i=1}^{k} (a_i, b_i) \cup (-a, a) \cup (b, +\infty) \tag{4.55}$$

where $k = \max\{i \in \mathbb{N} : b_i < R\}$ and $\gamma(E) = \gamma(B_r)$.
Observe that if $0 < a < r$ then

$$\frac{\sqrt{2\pi}}{2} P_\gamma(B_r) = e^{-\frac{r^2}{2}} < e^{-\frac{a^2}{2}} \leq \frac{\sqrt{2\pi}}{2} P_\gamma(E)$$

and thus (4.54) follows. On the other hand, since $\gamma(E) = \gamma(B_r)$, $a = r$ if and only if $E = B_r$. Therefore we are left with the case $a = 0$. In this case we fix $\delta < \frac{\gamma(B_{\frac{r}{2}})}{2}$ and let $\gamma(B_r \Delta E) < \delta$. This last inequality implies that $a_1 < \frac{r}{2}$. In fact, if $a_1 > \frac{r}{2}$, we would have

$$\frac{\sqrt{2\pi}}{2} \gamma(B_{\frac{r}{2}}) = \int_0^{\frac{r}{2}} e^{-\frac{x^2}{2}} dx \leq \int_0^{a_1} e^{-\frac{x^2}{2}} < \frac{\sqrt{2\pi}}{2} \gamma(E \Delta B_r) < \frac{\sqrt{2\pi}}{4} \gamma(B_{\frac{r}{2}}).$$

This contradiction shows that $a_1 < \frac{r}{2}$, hence $P_\gamma(B_r) < P_\gamma(E)$.
Let us prove (a). Let $r_0 > 0$ be such that

$$\frac{1}{\sqrt{2\pi}} \int_{-r_0}^{r_0} e^{-x^2} dx = \frac{1}{2}.$$

Let $r > r_0$ and assume by contradiction that there exists a set E such that $E = -E$, $\gamma(E) = \gamma(B_r)$ and $P_\gamma(E) < P_\gamma(B_r)$. Arguing as before we may assume

$$E = (-\infty, -b) \cup \bigcup_{i=1}^{k} (-b_i, -a_i) \cup \bigcup_{i=1}^{k} (a_i, b_i) \cup (b, +\infty) \tag{4.56}$$

for some $a_1 > 0$. Let $s > 0$ be such that

$$\frac{1}{2} \gamma(E) = \frac{1}{\sqrt{2\pi}} \int_s^\infty e^{-x^2} dx$$

and consider $C_s = (-\infty, -s) \cup (s, \infty)$. Using the isoperimetric inequality and the fact that $a_1 > 0$ we have

$$P_\gamma(E) > P_\gamma(C_s).$$

Since $\gamma(C_s) = \gamma(B_r) > \frac{1}{2}$ we have that $s < r$ and thus

$$P_\gamma(B_r) < P_\gamma(H_s) < P_\gamma(E).$$

Assume now $r < r_0$. In this case $P_\gamma(C_s) < P_\gamma(B_r)$ since $r < s$. Hence B_r cannot be a global minimizer.
In case (b) the proof that C_s is a global minimizer among all the symmetric sets follows by the same argument as in (a).
Finally if $r = r_0$, $P_\gamma(B_{r_0}) = P_\gamma(C_{r_0})$ and both minimize the Gaussian perimeter among symmetric sets. □

We want to emphasize that this argument applies only when $n = 1$ because of the rigidity of the structure of the sets of locally finite perimeter and because in one dimension the measure of the perimeter of the ball is a strictly decreasing function of the radius r. On the other hand, for $n > 1$ the measure of the perimeter of the ball is increasing for $r < \sqrt{n-1}$ and decreasing for $r > \sqrt{n-1}$.

The previous minimality result holds indeed also in a quantitative form. The simple proof of this property uses an argument of [**22**].

PROPOSITION 4.13. *Let $n = 1$. For every $r > 0$ there exists $\delta(r) > 0$ such that for any $E \subset \mathbb{R}$, $E = -E$, $\gamma(E) = \gamma(B_r)$ and $\gamma(E\Delta B_r) \leq \delta(r)$ there exist a positive constant $C(r)$ such that*

$$P_\gamma(E) - P_\gamma(B_r) > C(r)\gamma(E\Delta B_r)\sqrt{\log\left(\frac{1}{\gamma(E\Delta B_r)}\right)} \tag{4.57}$$

PROOF. First, note that it is enough to prove the inequality in the case $P_\gamma(E)-P_\gamma(B_r) < \delta_0(r)$, for some positive δ_0 to be chosen later. Let $E \subset \mathbb{R}$ be a set of locally finite perimeter. As before, we may assume without loss of generality that E is of the form (4.56). Let $\delta(r)$ be as in Proposition 4.12. We have 2 cases: $a = 0$ and $r > a > 0$.
Let $a = 0$. Since $\gamma(E\Delta B_r) < \delta$, as before we have that $a_1 < \frac{r}{2}$ and then

$$P_\gamma(E) - P_\gamma(B_r) > e^{-\frac{r^2}{8}} - e^{-\frac{r^2}{2}} = f(r).$$

Then, we set $\delta_0(r) = f(r)$. With such a choice of $\delta_0(r)$ we are immediately reduced to the case $a > 0$.
Let $0 < a < r$. Since $\gamma(E\Delta B_r) < \delta$, arguing as in the proof of Proposition 4.12, we have that there exists $\varepsilon > 0$ such that $a > r - \varepsilon$. Let K_ε such that

$$\frac{2}{\sqrt{2\pi}}\int_{K_\epsilon}^{\infty} e^{-\frac{t^2}{2}}\,dt = \gamma(E) - \frac{1}{\sqrt{2\pi}}\int_{-a}^{a} e^{-\frac{t^2}{2}}\,dt$$

If we set $E' = (-a, a) \cup C_{K_\epsilon}$, we have $P_\gamma(E') \leq P_\gamma(E)$ and $\gamma(E'\Delta B_r) \geq \gamma(E\Delta B_r)$ and then it is enough to estimate the isoperimetric gap for E'. For that, we recall the two elementary inequalities proved in [**22**]

$$\int_s^\infty e^{-\frac{t^2}{2}}\,dt < \frac{e^{-\frac{s^2}{2}}}{s} \quad \text{for } s > 0 \tag{4.58}$$

and

$$\int_s^\infty e^{-\frac{t^2}{2}}\,dt > \frac{e^{-\frac{s^2}{2}}}{4s} \quad \text{for } s \geq 1. \tag{4.59}$$

Since $\gamma(E'\Delta B_r) = 2\gamma(C_{K_\varepsilon})$, from the definition of C_{K_ε} and the two inequalities above and the fact that $K_\epsilon \to \infty$ we deduce

$$\frac{e^{-\frac{K_\epsilon^2}{2}}}{4\sqrt{2\pi}K_\epsilon} < \gamma(E'\Delta B_r) < \frac{e^{-\frac{K_\epsilon^2}{2}}}{\sqrt{2\pi}K_\epsilon}, \tag{4.60}$$

while for the isoperimetric deficit

$$\frac{P_\gamma(E') - P_\gamma(B_r)}{2} = e^{-\frac{a^2}{2}} + e^{-\frac{K_\varepsilon^2}{2}} - e^{-\frac{r^2}{2}}. \tag{4.61}$$

Since

$$e^{-\frac{a^2}{2}} - e^{-\frac{r^2}{2}} = \int_a^r te^{-\frac{t^2}{2}}\,dt > a\int_a^r e^{-\frac{t^2}{2}}\,dt = a\sqrt{\frac{\pi}{2}}\gamma(B_r\Delta B_a) = \frac{a}{2}\sqrt{\frac{\pi}{2}}\gamma(E'\Delta B_r)$$

and thanks to (4.60),

$$\frac{P_\gamma(E') - P_\gamma(B_r)}{2} > \sqrt{2\pi}K_\epsilon\gamma(E'\Delta B_r)$$

From the first inequality in (4.60) we get

$$\gamma(E'\Delta B_r) \geq \frac{e^{-\frac{K_\epsilon^2}{2}}}{4K_\varepsilon} \geq e^{-K_\epsilon^2}$$

which can be read also as $K_\varepsilon \geq \sqrt{\frac{1}{\gamma(E'\Delta B_r)}}$ and that immediately gives the desired result.

□

CHAPTER 5

Liquid drop

Another isoperimetric problem which has attracted the interest of many researchers is the one related to the liquid drop problem. Namely, the aim is to minimize the following energy

$$I(E) = P(E) + \int_E \int_E \frac{1}{|x-y|^{n-2}} dxdy - K \int_E \frac{1}{|x|^{n-2}} \tag{5.1}$$

among all sets of fixed measure, where $E \subset \mathbb{R}^n$ and $K \geq 0$. The perimeter here has the role of a cohesion force, in the sense that it tends to keep the particles close to each other while the non local interaction would like to spread the set as much as possible and the Coulombic attraction tries to attract the particles to the charge fixed at the origin of the space.

1. The case K=0

One of the earliest scientist to investigate this problem was Gamow in 1928, see [**38**], when he proposed a model for a nucleus made by α-particles very similar to a water-drop held together by surface tension. This mathematical model had a great success because of his flexibility. In fact it has been used to model the mechanism of nuclear fission (see [**24**], [**25**], [**56**], [**62**]), the behavior of a variety of polymers when they are quickly cooled (see [**26**], [**39**], [**45**], [**52**], [**58**], [**60**], [**65**]) and many physical situations (see [**19**], [**21**], [**30**], [**57**]) and in particular to . To better understand the physics behind, let us spend few words about the last phenomena, for a deeper explanation see [**2**] and [**20**].

A diblock copolymer molecule is a linear chain consisting of two subchains (made of two different monomers, say A and B) joined covalently to each other. Below a critical temperature, even a weak repulsion between unlike monomers A and B induces a strong repulsion between the subchains, causing the subchains to segregate. However the chemical bond between them prevent a macroscopic segregation. Rather, in a system of many such macromolecules, the immiscibility of these monomers drives the system to form structures which minimize the surface area separating the unlike monomers and this tendency to separate the monomers into A and B-rich domains is counter balanced by the entropy cost associated with a chain. Because of this energetic competition, a phase separation on a mesoscopic scale with A and B-rich domains emerges. Roughly speaking, we are dealing with a system where two materials are bonded by a strong connection, but having big "sets" of the same material costs too much.

One of the main difficulty that one finds trying to minimize $I(E)$ is that the symmetrization techniques fail: while the Steiner symmetrization decreases the perimeter, since it is enclosing the particles it is reasonable to think that it increases the nonlocal interaction. In fact, a simple use of Hardy inequality gives that for any set E with $|E| = |B_r|$, E not equal to a ball,

$$V(E) < V(B_r).$$

This maximizing property of the ball makes the variational problem highly nontrivial. In [**44**], the authors studied the three dimensional problem

$$I_\alpha(m) = \inf_{E\subset\mathbb{R}^3}\{P(E) + \int_E\int_E \frac{1}{|x-y|^\alpha}dxdy, \quad |E| = m\} \tag{5.2}$$

where $\alpha \in (0,3)$. What they discovered is that for every $\alpha \in (0,2)$, there exists $m_0 = m_0(\alpha)$ such that if $m < m_0$ the ball of the proper radius is the only solution of the isoperimetric problem. Actually, they proved that the global minimimality of the ball for small masses holds up to dimension $n = 7$, if $\alpha \in (0, n-1)$. In the same paper, they also prove that for $\alpha \in (0,2)$ and $n = 3$ there exists a value $m_1 = m_1(\alpha)$ such that for $m > m_1$ the infimum $I_\alpha(m)$ is not achieved. The main tool to prove nonexistence when a nonlocal competing term appears is the use of density estimates in order to have a priori bounds for the energy of a minimizer. We would like to stress that to find an upper bound is not the problem. In fact, as Knupfer and Muratov pointed out, for every $\alpha \in (0,3)$ there exists a constant C such that $I_\alpha(m) \leq Cm$ whenever m is bigger than a fixed quantity. The main problem is to find a proper lower bound: the isoperimetric inequality in the analytic form immediately yield that $I_\alpha(m) > C_1 m^{\frac{2}{3}}$ and then

$$m^{\frac{2}{3}} \leq C_2 m$$

which does not give any information for m big enough. Thus, to find a better exponent for the lower bound one needs to find a fine estimate from below for the repulsion of the minimizer. The main tool for that is the use of uniform density estimates. The procedure essentially goes like follows: first of all one notes that a minimizer needs to be a connected set, then one shows a uniform density estimate that together with the connectedness gives an upper bound for the diameter of the set of the type $\text{diam}(E) < Cm$ and then one estimates the double integral present in the definition of the problem. Unfortunately, this procedure works only for $\alpha < 2$.

For $\alpha \geq 2$ nothing about the nonexistence is known yet. The reason why the problem becomes more difficult when α is close enough to the dimension of the ambient space comes from the fact that the repulsion term starts to be more alike a local interaction term, and it is not trivial even to understand if a thin and long "sausage-shaped" set has less or more energy than the union of many balls placed far away from each other. At the same time, independently from Knupfer and Muratov, Julin in [**42**] proved the global minimality of the ball in every dimension for small masses. The techniques are a little different but they are in the same spirit: while in [**44**] they use the quantitative isoperimetric inequality, Julin used an improved version of the quantitative isoperimetric inequality provided by himself in the same paper.

After the two papers of Knupfer and Muratov, several authors tried to generalize their result. In particular here we will briefly discuss about two papers which came out almost at the same time: the first one due to Bonacini and Cristoferi and the second one by Figalli, Fusco, Maggi, Millot and Morini. In [**13**], Bonacini and Cristoferi studied the same functional in the n dimensional case with $\alpha \in (0, n-1)$. They prove the existence of a threshold value $m_1^{\text{loc}}(\alpha)$ for which the ball of mass $m < m_1^{\text{loc}}(\alpha)$ is an isolated local minimizer in L^1 of the functional in (5.2). They also showed that the argument provided by Knupfer and Muratov to prove the nonexistence can be extended in the n dimensional case if one restricts himself to the case $\alpha \in (0,2)$. What seems to be clear, comparing the pioneer paper of Knupfer and Muratov and the paper by Bonacini and Cristoferi, is that what really plays an important role is not the dimension of the space but the exponent α. The techniques to approach either the problem of minimality of a given set, in this case the ball, or the issue of the nonexistence for large values of the mass are pretty much the same in any dimension as long the exponent α is fixed, say $\alpha = 1$, although the study of the physical case $\alpha = n-2$ appears to be very challenging. Furthermore, they provide the existence of a small $\bar{\alpha}$ such that if

$\alpha < \bar{\alpha}$, then $m_0(\alpha) = m_1^{\text{glob}}(\alpha)$. Heuristically, this last result comes from the fact that if one send α to 0, then the only surviving term is the perimeter which is minimized by the ball. To make this argument rigorous, they argue by contradiction and use the quantitative isoperimetric inequality. For a general $\alpha \in (0, n-1)$ they find the existence of a value $m_1^{\text{glob}}(\alpha)$ such that if $m < m_1^{\text{glob}}(\alpha)$, then the ball of mass m is again the only minimizer of the problem. We want also to underline that they proved that for $\alpha \in (0, 1)$, it holds $m_1^{\text{glob}}(\alpha) < m_1^{\text{loc}}(\alpha)$, differently from what was conjectured earlier. As mentioned, the proof of the local minimality contained in [**13**] does not apply to the case $\alpha \in (n-1, n)$, which contains nontrivial technical difficulties. At the same time, in [**31**], the authors wrote a paper where they not only were able to deal with the case $\alpha \in (n-1, n)$, but also to generalize the problem studying the case when the fractional perimeter is taken in account instead of the classical one. Namely, the authors studied the functional

$$I_{s,\alpha}(E) = P^s(E) + V_\alpha(E) = \int_E \int_E \frac{1}{|x-y|^\alpha} dxdy + \int_E \int_{E^c} \frac{1}{|x-y|^{n+s}} dxdy$$

with $E \subset \mathbb{R}^n$, $s \in (0, 1)$ and $\alpha \in (0, n)$. To deal with this functional, the paper contains a very deep calculation of the energy using the first and second variation. Using the selection principle, they show the existence of a threshold value $m_0(\alpha, s)$ such that for every $m < m_0$, the ball of mass m is a local minimizer of $I_{\alpha,s}$, while if $m > m_0$ the ball of Lebesgue measure m does not have this property. Moreover, again with the use of the quantitative isoperimetric inequality, they find a value $m_1(\alpha, s)$ such that, if $m < m_1$, the ball of mass m is the only global minimizer of $I_{\alpha,s}$.

2. Case $K > 0$

Here we study the minimizers under the volume constraint $|E| = m$ of the functional

$$I(E) = P(E) + V(E) - KR(E), \tag{5.3}$$

where

$$V(E) = \int_E \int_E \frac{1}{|x-y|^{n-2}} dxdy$$

is a Coulombic repulsive potential of the set with itself and

$$R(E) = \int_E \frac{1}{|x|^{n-2}} dx \tag{5.4}$$

is a repulsive term of the set with a point charge. Here $P(E)$ stands for the standard Euclidean perimeter in the De Giorgi sense and $K \geq 0$. In the three dimensional case this functional has been studied by Lu and Otto in [**49**]. In that paper they prove that if m is sufficiently large then the constrained minimum problem has no solutions. They also show that there exists a critical value m_c such that if $m < m_c$ the ball centered at the origin is the unique global minimizer. This result is obtained using a quantitative version of the isoperimetric inequality with a Coulombic term proved by Julin in [**42**] .

In this chapter we prove that there exists a critical radius $r_0 > 0$ such that if $r < r_0$ the ball B_r centered at the origin is a local minimizer of the constrained minimum problem and that this property fails when $r > r_0$. As in [**31**], we show also the global minimality of balls B_r when $r < r_1$, for some $0 < r_1 < r_0$. Note that both critical radii r_1 and r_0 tend to infinity as $K \to \infty$ and an argument provided in the last section, see Lemma 5.18, shows that the ratio r_0/r_1 stays bounded independently of K.

This section is organized as follows. We start fixing the notation and giving some preliminary results, then we provide a Fuglede type estimate for the functional I, see Theorem 5.2. Precisely, we prove that if $r < r_0$ the ball B_r is a local minimizer with respect to small C^1 variations.

Then, after calculating the second variation of I, we show that the radius r_0 provided by Theorem 5.2 is indeed optimal since balls of radius $r > r_0$ are not local minimizers. Note that while the formula of the second variation of V can be obtained by more or less standard arguments, see for instance [**31**], the calculations leading to the second variation of the attractive term turn out to be more delicate due to the presence of a singularity in the integrand in (5.4).

We conlcude this section passing from the local minimality of the ball B_r with respect to small C^1 variations implies the full local minimality result. As usual in this framework, we follow a strategy first devised by Cicalese and Leonardi in [**23**], see also [**1**], based on the regularity theory for quasi minimizers of the perimeter. However, in our setting this approach turns out to be more complicated. Indeed, the main difficulty comes from the fact that, differently from most cases studied in the literature, see [**23**], [**1**], [**36**], [**10**], [**31**], [**16**], our functional is not translation invariant. Overcoming this difficulty requires a delicate estimate of the behavior of the repulsive term R on sets which are C^1 close to a ball centered at the origin.

2.1. Notation and preliminary results. For any measurable set $E \subset \mathbb{R}^n$ we define the Coulombic potential $V(E)$ and the repulsive term $R(E)$ as follows

$$V(E) = \int_E \int_E \frac{1}{|x-y|^{n-2}}\, dxdy, \qquad R(E) = \int_E \frac{1}{|x|^{n-2}}\, dx.$$

We are interested in minimizing the nonlocal energy given by

$$I(E) = P(E) + V(E) - KR(E), \tag{5.5}$$

where K is a positive constant that will be fixed throughout the paper.

Note that in order to avoid trivial statements, *we shall assume throughout that the dimension of the ambient space* $\mathbb{R}^n$ *is greater than or equal to 3*, unless specified otherwise.

It is easily checked that if E is defined as in (4.2) then its measure and perimeter are given, respectively, by the following formulas

$$|E| = r^n \int_B (1+u(x))^n \, dx = \frac{r^n}{n} \int_{\mathbb{S}^{n-1}} (1+u(x))^n \, d\mathcal{H}^{n-1}, \tag{5.6}$$
$$P(E) = r^{n-1} \int_{\mathbb{S}^{n-1}} (1+u(x))^{n-1} \sqrt{1 + \frac{|D_\tau u(x)|^2}{(1+u(x))^2}} \, d\mathcal{H}^{n-1},$$

where $D_\tau u$ stands for the tangential gradient of u on $\mathbb{S}^{n-1}$.

Similarly, $V(E)$ and $R(E)$ can be also represented as

$$\begin{aligned} V(E) &= \int_E \int_E \frac{1}{|x-y|^{n-2}} \, dxdy = r^{n+2} \int_B \int_B \frac{(1+u(x))^n (1+u(x))^n}{|x(1+u(x)) - y(1+u(y))|^{n-2}} \, dxdy \\ &= r^{n+2} \int_{\mathbb{S}^{n-1}} d\mathcal{H}^{n-1}_x \int_{\mathbb{S}^{n-1}} d\mathcal{H}^{n-1}_y \int_0^{1+u(x)} d\rho \int_0^{1+u(y)} \frac{\rho^{n-1}\sigma^{n-1}}{\left(|\rho-\sigma|^2 + \rho\sigma|x-y|^2\right)^{\frac{n-2}{2}}} \, d\sigma \\ \frac{R(E)}{r^2} &= \int_B \frac{(1+u(x))^2}{|x|^{n-2}} \, dx = \frac{1}{2} \int_{\mathbb{S}^{n-1}} (1+u(x))^2 \, d\mathcal{H}^{n-1}. \end{aligned}$$

For any integer $k \geq 0$, let us denote by $y_{k,i}$, $i = 1, \ldots, G(n,k)$, the spherical harmonics of order k, i.e., the restrictions to $\mathbb{S}^{n-1}$ of the homogeneous harmonic polynomials of degree k, normalized so that $||y_{k,i}||_{L^2(\mathbb{S}^{n-1})} = 1$, for all $k \geq 0$ and $i \in \{1, \ldots, G(n,k)\}$. The functions $y_{k,i}$ are eigenfunctions of the Laplace-Beltrami operator on $\mathbb{S}^{n-1}$ and for all k and i

$$-\Delta_{\mathbb{S}^{n-1}} y_{k,i} = \lambda_k y_{k,i} \, .$$

where $\lambda_k = k(k+n-2)$. Moreover if $u \in L^2(\mathbb{S}^{n-1})$ we have

$$u = \sum_{k=0}^{\infty} \sum_{i=1}^{G(n,k)} a_{k,i} y_{k,i}, \quad \text{where} \quad a_{k,i} := \int_{\mathbb{S}^{n-1}} u(x) y_{k,i}(x) \, d\mathcal{H}^{n-1}.$$

Therefore, for a function $u \in H^1(\mathbb{S}^{n-1})$ we have that

$$\|u\|^2_{L^2(\mathbb{S}^{n-1})} = \sum_{k=0}^{\infty} \sum_{i=1}^{G(n,k)} a^2_{k,i}, \qquad \|D_\tau u\|^2_{L^2(\mathbb{S}^{n-1})} = \sum_{k=1}^{\infty} \sum_{i=1}^{G(n,k)} \lambda_k a^2_{k,i}. \tag{5.7}$$

If $s \in (-1,1)$ and $u \in L^2(\mathbb{S}^{n-1})$, we set

$$[u]^2_{s,\mathbb{S}^{n-1}} := \int_{\mathbb{S}^{n-1}} \int_{\mathbb{S}^{n-1}} \frac{|u(x)-u(y)|^2}{|x-y|^{n-1+2s}} \, d\mathcal{H}^{n-1}_x d\mathcal{H}^{n-1}_y.$$

Also these seminorms can be represented using the Fourier coefficients of u and suitable sequences of eigenvalues. In particular, see formulas (7.12) and (7.5) in [**31**], we have

$$[u]^2_{-\frac{1}{2},\mathbb{S}^{n-1}} = \int_{\mathbb{S}^{n-1}} \int_{\mathbb{S}^{n-1}} \frac{|u(x)-u(y)|^2}{|x-y|^{n-2}} \, d_x\mathcal{H}^{n-1} \, d\mathcal{H}^{n-1}_y = \sum_{k=1}^{\infty} \sum_{i=1}^{G(n,k)} \mu_k a^2_{i,k}, \tag{5.8}$$

where the eigenvalues μ_k are given, for an integer $k \geq 0$, by the following expressions

$$\mu_k := \frac{4\pi^{\frac{n}{2}}}{\Gamma(\frac{n-2}{2})} \left(\frac{\Gamma(\frac{n-2}{2})}{\Gamma(\frac{n}{2})} - \frac{\Gamma(k+\frac{n-2}{2})}{\Gamma(k+\frac{n}{2})} \right). \tag{5.9}$$

It is easily checked that the above sequence is bounded and strictly increasing. Moreover, see [**31**, Prop. 7.5],

$$\mu_1 = 2(n+2)\frac{V(B)}{P(B)}, \qquad \mu_2 = \frac{2n}{n+2}\mu_1. \tag{5.10}$$

Finally, we recall the following useful estimate, proved in [**31**, Appendix C],

$$\frac{\lambda_k - \lambda_1}{\mu_k - \mu_1} \geq \frac{\lambda_2 - \lambda_1}{\mu_2 - \mu_1} \qquad \forall k \geq 2. \tag{5.11}$$

Let us now define a function $\mathcal{P} : [0, \infty) \to \mathbb{R}$ setting for $r \geq 0$

$$\mathcal{P}(r) := \inf_{k \geq 2} \left\{ \frac{\lambda_k - \lambda_1}{\mu_k - \mu_1} r^{n-3} - r^n + \frac{K(n-2)}{\mu_k - \mu_1} \right\}. \tag{5.12}$$

Lemma 5.1. *The function $\mathcal{P}$ defined in* (5.12) *is continuous. Moreover there exists $r_0 > 0$ such that*

$$\mathcal{P}(r_0) = 0, \tag{5.13}$$

$\mathcal{P}(r) > 0$ for $0 < r < r_0$ and $\mathcal{P}(r) < 0$ for $r > r_0$.

Proof. Observe that from (5.10) and (5.9)

$$\frac{2(n-2)V(B)}{P(B)} \leq \mu_k - \mu_1 \leq \frac{2(n-2)P(B)}{n} \qquad \forall k \geq 2. \tag{5.14}$$

From this inequality we have that $(\lambda_k - \lambda_1)/(\mu_k - \mu_1) \to \infty$ as $k \to \infty$, hence for any interval $a > 0$ there exists $k_a \geq 2$ such that

$$\mathcal{P}(r) = \inf_{2 \leq k \leq k_a} \left\{ \frac{\lambda_k - \lambda_1}{\mu_k - \mu_1} r^{n-3} - r^n + \frac{K(n-2)}{\mu_k - \mu_1} \right\} \qquad \text{for all } r \in [0, a].$$

This proves that $\mathcal{P}$ is continuous. Observe also that from (5.10), (5.11) and the second inequality in (5.14)

$$\mathcal{P}(r) \geq \frac{(n+1)P(B)}{2(n-2)V(B)} r^{n-3} - r^n + \frac{Kn}{2P(B)},$$

hence $\mathcal{P} > 0$ in a right neighborhood of the origin. Note also that $\mathcal{P}(r) \to -\infty$ as $r \to +\infty$.

Let us now set for any integer $k \geq 2$ and any $r \geq 0$

$$P_k(r) := \frac{\lambda_k - \lambda_1}{\mu_k - \mu_1} r^{n-3} - r^n + \frac{K(n-2)}{\mu_k - \mu_1}. \tag{5.15}$$

It is easily checked that P_k has exactly one zero $r_k > 0$ and that $P_k(r) < 0$ for $r > r_k$. Therefore, denoting by $r_0 > 0$ the first zero of $\mathcal{P}$ and by $k_0 \geq 2$ an integer such that $\mathcal{P}(r_0) = P_{k_0}(r_0) = 0$, we have that $\mathcal{P}(r) \leq P_{k_0}(r) < 0$ for all $r > r_0$. Hence, the proof follows. □

2.2. Nearly spherical sets. In this section we prove the local minimality of balls B_r with $r < r_0$ with respect to small variations in C^1.

Theorem 5.2. *Let $\sigma \in (0, r_0/2)$, where r_0 is defined as in* (5.13)*. There exist two positive constants ε_0 and c_0, depending only on n and σ, with the following property. If E is a nearly spherical set as in* (4.2)*, with $|E| = B_r$ and barycenter at the origin, $r \in (\sigma, r_0 - \sigma)$ and $\|u\|_{W^{1,\infty}(\mathbb{S}^{n-1})} \leq \varepsilon_0$, then*

$$I(E) - I(B_r) \geq c_0 \|u\|^2_{L^2(\mathbb{S}^{n-1})}. \tag{5.16}$$

Proof. We are going to prove (5.16) by an argument similar to the one introduced by Fuglede in [**34**]. To this end, it is convenient to rephrase the assumption replacing E by the set

$$E_t : \{y = r(1 + tu(x)) : x \in B\},$$

with $u \in W^{1,\infty}(\mathbb{S}^{n-1})$, $\|u\|_{W^{1,\infty}(\mathbb{S}^{n-1})} \le 1/2$,

$$|E_t| = |B_r|, \qquad \int_{E_t} x\,dx = 0,$$

$t \in (0, 2\varepsilon_0)$, where the constant $\varepsilon_0 < 1/2$ will be determined at the end of the proof. Thus, our assertion (5.16) becomes

$$I(E_t) - I(B_r) \ge c_0 t^2 \|u\|^2_{L^2(\mathbb{S}^{n-1})}, \tag{5.17}$$

for a suitable constant $c_0 > 0$ depending only on n and σ. In order to prove this inequality we estimate the differences between the various quantities appearing in the definition (5.5) of I. We start by the perimeter term. In this case, see for instance the proof of Theorem 3.1 in [**34**], we have, provided ε_0 is sufficiently small,

$$\frac{P(E_t) - P(B_r)}{r^{n-1}} \ge \frac{t^2}{2}\left(\int_{\mathbb{S}^{n-1}} |D_\tau u|^2\,d\mathcal{H}^{n-1} - (n-1)\int_{\mathbb{S}^{n-1}} u^2\,d\mathcal{H}^{n-1}\right) - C(n)t^3\|u\|^2_{L^2}, \tag{5.18}$$

for some constant $C(n)$ depending only on n. The difference between the two potential terms is estimated in [**31**, (5.20)] as follows

$$\frac{V(E_t) - V(B_r)}{r^{n+2}} \ge \frac{t^2}{2}\Big(2(n+2)\frac{V(B)}{P(B)}\|u\|^2_{L^2} - [u]^2_{-\frac12}\Big) - C(n)t^3\big(\|u\|^2_{L^2} + [u]^2_{-\frac12}\big). \tag{5.19}$$

Let us now estimate the remaining difference.

$$\begin{aligned} \frac{R(E_t) - R(B_r)}{r^2} &= \frac12\int_{\mathbb{S}^{n-1}} \big((1 + tu(x))^2 - 1\big)\,d\mathcal{H}^{n-1} = \frac12\int_{\mathbb{S}^{n-1}} 2tu + t^2u^2\,d\mathcal{H}^{n-1} \\ &= t\int_{\mathbb{S}^{n-1}} u\,d\mathcal{H}^{n-1} + \frac{t^2}{2}\int_{\mathbb{S}^{n-1}} u^2\,d\mathcal{H}^{n-1}. \end{aligned} \tag{5.20}$$

Using now the assumption $|E_t| = |B_r|$, from (4.3), after expanding $(1 + tu)^n$ we obtain

$$n\int_{\mathbb{S}^{n-1}} tu\,d\mathcal{H}^{n-1} + \frac{n(n-1)}{2}\int_{\mathbb{S}^{n-1}} t^2u^2\,d\mathcal{H}^{n-1} + \sum_{k=3}^{n}\binom{n}{k}t^k\int_{\mathbb{S}^{n-1}} u^k d\mathcal{H}^{n-1} = 0. \tag{5.21}$$

Inserting in (5.20) the expression of the integral of u on $\mathbb{S}^{n-1}$ obtained from this identity, and recalling that $|u| < 1/2$ and $0 < t < 2\varepsilon_0 < 1$, we get

$$\frac{R(E_t) - R(B_r)}{r^2} \ge -\frac{n-2}{2}\int_{\mathbb{S}^{n-1}} t^2u^2 d\mathcal{H}^{n-1} - C(n)t^3\|u\|_{L^2}.$$

Collecting this inequality, (5.18) and (5.19), we have

$$\begin{aligned} I(E_t) - I(B) \ge{}& \frac{t^2r^{n-1}}{2}\big(\|D_\tau u\|^2_{L^2} - (n-1)\|u\|^2_{L^2}\big) + \frac{t^2r^{n+2}}{2}\Big(2(n+2)\frac{V(B)}{P(B)}\|u\|^2_{L^2} - [u]^2_{-\frac12}\Big) \\ &+ \frac{t^2r^2K(n-2)}{2}\|u\|^2_{L^2} - C(n)t^3\big(\|u\|^2_{L^2} + [u]^2_{-\frac12}\big). \end{aligned} \tag{5.22}$$

We now write all the norms in the previous inequality in terms of the Fourier coefficients $a_{k,i}$ of u. To this end, observe that from (5.21), using the fact that $|u| \le 1/2$ and $0 < t < 1$, we have in particular that

$$|a_0| \le C(n)t\|u\|^2_{L^2}. \tag{5.23}$$

From the condition that the barycenter of E_t is at the origin we have

$$\int_{\mathbb{S}^{n-1}} x(1+u(x))^n \, d\mathcal{H}^{n-1} = 0.$$

Therefore, arguing as in the proof of (5.21), we get

$$\sup_{i=1,\dots,n} |a_{1,i}| \le C(n)t\|u\|_{L^2}^2. \tag{5.24}$$

Finally, recalling that the eigenvalues μ_k are all bounded, from (5.8) we get that

$$[u]_{-\frac{1}{2}} \le C(n)\|u\|_{L^2}.$$

Using this inequality, recalling (5.7), (5.8) and that $\lambda_1 = n-1$ and $\mu_1 = 2(n+2)V(B)/P(B)$, see (5.10), from the estimate (5.22) we get

$$\begin{aligned} I(E_t) - I(B) &\ge \frac{t^2r^2}{2}\sum_{k=2}^{\infty}\sum_{i=1}^{G(k,n)} \big((\lambda_k-\lambda_1)r^{n-3} + (\mu_1-\mu_k)r^n + K(n-2)\big)\, a_{k,i}^2 - C(n)t^3\|u\|_{L^2}^2 \\ &\ge \frac{t^2r^2}{2}\sum_{k=2}^{\infty}\sum_{i=1}^{G(k,n)} (\mu_k-\mu_1)\Big(\frac{\lambda_k-\lambda_1}{\mu_k-\mu_1}r^{n-3} - r^n + \frac{K(n-2)}{\mu_k-\mu_1}\Big)a_{k,i}^2 - C(n)t^3\|u\|_{L^2}^2. \end{aligned}$$

From this estimate, using (5.23), (5.24) and recalling Lemma 5.1, we readily obtain

$$\begin{aligned} I(E_t) - I(B) &\ge \frac{(n-2)V(B)t^2r^2}{P(B)}\Big(\frac{(n+1)P(B)}{2(n-2)V(B)}r^{n-3} - r^n + \frac{Kn}{2P(B)}\Big)\sum_{k=2}^{\infty}\sum_{i=1}^{G(k,n)} a_{k,i}^2 - C(n)t^3\|u\|_2^2 \\ &\ge c(n,\sigma)t^2\sum_{k=2}^{\infty}\sum_{i=1}^{G(k,n)} a_{k,i}^2 - C(n)t^3\|u\|_{L^2}^2 \ge c(n,\sigma)t^2\|u\|_{L^2}^2 - C(n,\sigma)t^3\|u\|_{L^22}^2, \end{aligned}$$

for some suitable constants $c(n,\sigma), C(n,\sigma)$ depending only on n and σ.

From the inequality above, taking t, hence ε_0, sufficiently small we get (5.17). This proves the theorem. □

Observe that there exists a constant $C(n)$ depending only on n such that if E is a nearly spherical set as in (4.2) then

$$\frac{|E\Delta B_r|}{C(n)} \le \|u\|_{L^2(\mathbb{S}^{n-1})} \le C(n)|E\Delta B_r|.$$

In view of the above inequalities we may rewrite the previous theorem in the following equivalent way.

THEOREM 5.3. *Let $\sigma \in (0, r_0/2)$, where r_0 is as in* (5.13). *There exist two positive constants ε_0 and c_1, depending only on n and σ, such that if E is a nearly spherical set satisfying the assumptions of Theorem 4.2, then*

$$I(E) - I(B_r) > c_1|E\Delta B_r|^2. \tag{5.25}$$

2.3. Second variation. In this section we will calculate the second variation of the functional $I(E)$. The resulting formula will be used to show that a ball B_r with $r>0$ is never a local minimizer for the functional I with respect to L^1 variations.

First, we fix some notation. Given a vector field $X \in C_c^2(\mathbb{R}^n, \mathbb{R}^n)$, the *associated flow* is defined as the solution of the Cauchy problem

$$\begin{cases} \dfrac{\partial}{\partial t}\Phi(x,t) = X(\Phi(x,t)) \\ \Phi(x,0) = x. \end{cases} \tag{5.26}$$

In the following we shall always write Φ_t to denote the map $\Phi(\cdot, t)$. Note that for any given X there exists $\delta > 0$ such that for $t \in [-\delta, \delta]$, the map Φ_t is a diffeomorphism coinciding with the identity map outside a compact set.

If $E \subset \mathbb{R}^n$ is measurable, we set $E_t := \Phi_t(E)$. Denoting by $J\Phi_t$ the n-dimensional jacobian of $D\Phi_t$, the first and second derivatives $J\Phi_t$ are given by, see [**51**],

$$\frac{\partial}{\partial t} J\Phi_t\big|_{t=0} = \mathrm{div} X, \qquad \frac{\partial^2}{\partial t^2} J\Phi_t\big|_{t=0} = \mathrm{div}((\mathrm{div} X)X). \tag{5.27}$$

From this formulas we have in particular that if E is a sufficiently smooth open set then

$$\frac{d}{dt}|E_t| = \int_{\partial E_t} X \cdot \nu_{E_t}\, d\mathcal{H}^{n-1}, \qquad \frac{d^2}{dt^2}|E_t| = \int_{\partial E_t} (X \cdot \nu_{E_t}) \mathrm{div} X\, d\mathcal{H}^{n-1}.$$

If the flow is *volume preserving*, i.e., $|E_t| = |E|$ for all $t \in [-\delta, \delta]$, then in particular we have that for all $t \in [-\delta, \delta]$

$$\int_{\partial E_t} X \cdot \nu_{E_t}\, d\mathcal{H}^{n-1} = 0, \qquad \int_{\partial E_t} (X \cdot \nu_{E_t}) \mathrm{div} X\, d\mathcal{H}^{n-1} = 0. \tag{5.28}$$

Finally, given a sufficiently smooth bounded open E and a vector field X we recall that the *first variation of the perimeter of* E at X is defined by setting

$$\delta P(E)[X] := \frac{d}{dt} P(\Phi_t(E))\big|_{t=0},$$

where Φ_t is the flow associated with X. The *second variation of the perimeter of* E at X is defined by

$$\delta^2 P(E)[X] := \frac{d^2}{dt^2} P(\Phi_t(E))\big|_{t=0}.$$

The first and second variations of the functionals R, V and I are defined accordingly.

If E is a C^2 open set we denote by H_E its *scalar mean curvature* of ∂E, i.e., the sum of the principal curvatures of ∂E. We denote by B_E the *second fundamental form of* ∂E and recall that the square $|B_E|^2$ of its euclidean norm is equal to the sum of the squares of the principal curvatures of ∂E.

As we shall see below, the second variation of V involves some nonlocal variants of H_E and $|B_E|^2$. To this end, if E is a bounded open set of class C^2, we set for every $x \in \partial E$

$$H^*_E(x) := 2 \int_E \frac{1}{|x-y|^{n-2}}\, dy.$$

The quantity H^*_E plays the role of H_E, while the analogue of $|B_E|^2$ is defined by setting for $x \in \partial E$

$$C^2_E(x) := \int_{\partial E} \frac{|\nu_E(x) - \nu_E(y)|^2}{|x-y|^{n-2}}\, d\mathcal{H}^{n-1}_y. \tag{5.29}$$

We start by calculating the first and second variation of R.

LEMMA 5.4. *Let $E \subset \mathbb{R}^n$ be a bounded open set of class C^2. Assume that $X \in C^2_c(\mathbb{R}^n; \mathbb{R}^n)$. Then*

$$\delta R(E)[X] = \int_{\partial E} \frac{X \cdot \nu_E}{|x|^{n-2}}\, d\mathcal{H}^{n-1}. \tag{5.30}$$

Moreover, if $0 \notin \partial E$,

$$\delta^2 R(E)[X] = \int_{\partial E} \left(\frac{(X \cdot \nu_E)\mathrm{div} X}{|x|^{n-2}} - (n-2) \frac{(X \cdot \nu_E)(X \cdot x)}{|x|^n} \right) d\mathcal{H}^{n-1}. \tag{5.31}$$

Proof. Given $X \in C^2_c(\mathbb{R}^n;\mathbb{R}^n)$, let Φ_t be the associated flow defined as in (5.26). Let $\delta > 0$ be such that the map Φ_t is a diffeomorphism for all $t \in [-\delta,\delta]$. As above, we set $E_t = \Phi_t(E)$ and denote by ∂_t and ∂_{tt} the first and second partial derivatives with respect to t, respectively.

In order to prove the formulas (5.30) and (5.31) we regularize R by setting for $\varepsilon > 0$

$$R_\varepsilon(E) = \int_E \frac{1}{|x|^{n-2}+\varepsilon}\,dx.$$

Since $\Phi_{t+s}(x) = \Phi_s(\Phi_t(x))$, changing variable, we have

$$\begin{aligned}\frac{d}{dt}R_\varepsilon(E_t) &= \frac{d}{ds}R_\varepsilon(E_{t+s})|_{s=0} = \frac{d}{ds}\Big(\int_{E_t}\frac{J\Phi_s}{|\Phi_s|^{n-2}+\varepsilon}\,dx\Big)|_{s=0}\\ &= \Big(\int_{E_t}\frac{\partial_s J\Phi_s}{|\Phi_s|^{n-2}+\varepsilon}\,dx - (n-2)\int_{E_t}\frac{|\Phi_s|^{n-4}\Phi_s\cdot\partial_s\Phi_s}{(|\Phi_s|^{n-2}+\varepsilon)^2}J\Phi_s\,dx\Big)|_{s=0}.\end{aligned}$$

Therefore, recalling the first identity in (5.27), we have

$$\frac{d}{dt}R_\varepsilon(E_t) = \int_{E_t}\Big(\frac{\mathrm{div}X}{|x|^{n-2}+\varepsilon} - (n-2)\frac{(X\cdot x)|x|^{n-4}}{(|x|^{n-2}+\varepsilon)^2}\Big)\,dx = \int_{\partial E_t}\frac{X\cdot\nu_{E_t}}{|x|^{n-2}+\varepsilon}\,d\mathcal{H}^{n-1}.$$

From this formula it follows that the functions $R_\varepsilon(t)$ converge uniformly in $[-\delta,\delta]$, together with their first derivatives, as $\varepsilon \to 0$. Thus, (5.30) follows immediately letting $\varepsilon \to 0$.

Let us differentiate $R(E_t)$ once again. Arguing as before we have

$$\begin{aligned}\frac{d^2}{dt^2}R(E_t) = \frac{d^2}{ds^2}R_\varepsilon(E_{t+s})|_{s=0} &= \Big(\int_{E_t}\frac{\partial_{ss}J\Phi_s}{|\Phi_s|^{n-2}+\varepsilon} - 2(n-2)\frac{|\Phi_s|^{n-4}\Phi_s\cdot\partial_s\Phi_s}{(|\Phi_s|^n+\varepsilon)^2}\partial_s J\Phi_s\,dx\Big)|_{s=0}\\ &\quad + \Big(\int_{E_t}\frac{\partial^2}{\partial s^2}\Big(\frac{1}{|\Phi_s|^{n-2}+\varepsilon}\Big)J\Phi_s\,dx\Big)|_{s=0} = J_1(t)+J_2(t).\end{aligned}\tag{5.32}$$

Recalling the identities (5.27), we have

$$\begin{aligned}J_1(t) &= \int_{E_t}\frac{\mathrm{div}(X\mathrm{div}X)}{|x|^{n-2}+\varepsilon}\,dx - 2(n-2)\int_{E_t}\frac{|x|^{n-4}(X\cdot x)\mathrm{div}X}{(|x|^{n-2}+\varepsilon)^2}\,dx\\ &= \int_{E_t}\mathrm{div}\Big(\frac{X\mathrm{div}X}{|x|^{n-2}+\varepsilon}\Big)\,dx - (n-2)\int_{E_t}\frac{|x|^{n-4}(X\cdot x)\mathrm{div}X}{(|x|^{n-2}+\varepsilon)^2}\\ &= \int_{\partial E_t}\frac{(X\cdot\nu_{E_t})\mathrm{div}X}{|x|^{n-2}+\varepsilon}\,d\mathcal{H}^{n-1} - (n-2)\int_{E_t}\frac{|x|^{n-4}(X\cdot x)\mathrm{div}X}{(|x|^{n-2}+\varepsilon)^2},\end{aligned}\tag{5.33}$$

where the last equality follows from the divergence theorem. Differentiating twice $1/(|\Phi_t|^{n-2}+\varepsilon)$ with respect to t, we have, using again the divergence theorem,

$$\begin{aligned}\frac{J_2(t)}{n-2} &= -\int_{E_t}\frac{|x|^{n-4}|X|^2 + |x|^{n-4}\langle DXX, x\rangle + (n-4)|x|^{n-6}(X\cdot x)^2}{(|x|^{n-2}+\varepsilon)^2}\,dx\\ &\qquad\qquad - 2(n-2)\int_{E_t}\frac{(|x|^{n-4}(X\cdot x))^2}{(|x|^{n-2}+\varepsilon)^3}\,dx\\ &= -\int_{E_t}\mathrm{div}\Big(\frac{|x|^{n-4}X(X\cdot x)}{(|x|^{n-2}+\varepsilon)^2}\Big)\,dx + \int_{E_t}\frac{|x|^{n-4}\mathrm{div}X(X\cdot x)}{(|x|^{n-2}+\varepsilon)^2}\,dx\\ &= -\int_{\partial E_t}\frac{|x|^{n-4}(X\cdot\nu_{E_t})(X\cdot x)}{(|x|^{n-2}+\varepsilon)^2}\,d\mathcal{H}^{n-1} + \int_{E_t}\frac{|x|^{n-4}\mathrm{div}X(X\cdot x)}{(|x|^{n-2}+\varepsilon)^2}\,dx.\end{aligned}$$

Then, from this last equality, (5.32) and (5.33), we have

$$\frac{d^2}{dt^2}R(E_t) = \int_{\partial E_t}\frac{(X\cdot\nu_{E_t})\mathrm{div}X}{|x|^{n-2}+\varepsilon}\,d\mathcal{H}^{n-1} - (n-2)\int_{\partial E_t}\frac{|x|^{n-4}(X\cdot\nu_E)(X\cdot x)}{(|x|^{n-2}+\varepsilon)^2}\,d\mathcal{H}^{n-1}.$$

As before the validity of (5.31) follows by observing that since $0 \notin \partial E$ also the second derivatives of $R_\varepsilon(E_t)$ converge uniformly in a neighborhood of the origin as $\varepsilon \to 0$. $\square$

Let us now recall the first and second variation formulas for P and V. To this end, we shall denote by div_τ the tangential divergence and by X_τ the tangential component of the vector field X. For a proof of the next lemma we refer to [**31**, Sect. 6].

LEMMA 5.5. *Let $E \subset \mathbb{R}^n$ be a bounded open set of class C^2 and $X \in C^2_c(\mathbb{R}^n;\mathbb{R}^n)$. Then*

$$\delta P(E)[X] = \int_{\partial E} H_E(X\cdot\nu_E)\,d\mathcal{H}^{n-1}, \tag{5.34}$$

$$\delta^2 P(E)[X] = \int_{\partial E}\left(|D_\tau(X\cdot\nu_E)|^2 - |B_E|^2(X\cdot\nu_E)^2\right)\,d\mathcal{H}^{n-1} + \int_{\partial E} H_E(\mathrm{div}X(X\cdot\nu_E) - \mathrm{div}_\tau((X\cdot\nu_E)X_\tau).$$

Moreover,

$$\delta V(E)[X] = \int_{\partial E} H^*_{\partial E}(X\cdot\nu_E)\,d\mathcal{H}^{n-1}, \tag{5.35}$$

$$\delta^2 V(E)[X] = -\int_{\partial E}\int_{\partial E}\frac{|X\cdot\nu_E(x) - X\cdot\nu_E(y)|^2}{|x-y|^{n-2}}\,d\mathcal{H}^{n-1}_x\,d\mathcal{H}^{n-1}_y + \int_{\partial E} C^2_E(X\cdot\nu_E)^2 d\mathcal{H}^{n-1} + \int_{\partial E} H^*_E(\mathrm{div}X(X\cdot\nu_E) - \mathrm{div}_\tau((X\cdot\nu_E)X_\tau))d\mathcal{H}^{n-1}.$$

DEFINITION 5.6. We say that a set of locally finite perimeter $E \subset \mathbb{R}^n$ is a *constrained, strict L^1-local minimizer* for the functional I if there exists $\delta > 0$ such that whenever F is a set of locally finite perimeter such that $|F| = |E|$ and $0 < |E\Delta F| \le \delta$, then

$$I(F) > I(E).$$

Using (5.30), (5.34) and (5.35), it is easily checked that if E is a C^2, bounded constrained local minimizer for I, there exists $\lambda \in \mathbb{R}$ such that

$$H_E + H^*_E - \frac{K}{|x|^{n-2}} = \lambda \qquad \text{on } \partial E. \tag{5.36}$$

Conversely, any C^2 bounded open set satisfying (4.20) will be called *a constrained critical set* for the functional I. Note that any ball B_r centered at the origin trivially satisfies (4.20), hence it is a constrained critical set for I. Moreover, if $0 \notin \partial E$ and the flow associated with X is volume preserving, then, setting $\phi := X\cdot\nu_{B_r}$, we have, recalling (5.28),

$$\delta^2 I(B_r)[X] := \partial^2 I(B_r)[\phi] = \int_{\partial B_r}\left(|D_\tau\phi|^2 - \frac{n-1}{r^2}\phi^2\right)d\mathcal{H}^{n-1} + \frac{K(n-2)}{r^{n-1}}\int_{\partial B_r}\phi^2 d\mathcal{H}^{n-1} - \int_{\partial B_r}\int_{\partial B_r}\frac{|\phi(x)-\phi(y)|^2}{|x-y|^{n-2}}d\mathcal{H}^{n-1} + rC^2_B\int_{\partial B_r}\phi^2 d\mathcal{H}^{n-1}. \tag{5.37}$$

Given a function $\phi \in H^1(\partial B_r)$ with $\int_{\partial B_r}\phi = 0$, it is always possible to construct a sequence of vector fields $X_j \in C^\infty_c(\mathbb{R}^n;\mathbb{R}^n)$, such that div $X_j = 0$ in a ball B_R with $R > r$ and such that $X_j\cdot\nu_E \to \phi$ in $H^1(\partial B_r)$, see for instance [**1**, Cor. 3.4]. Since the flows associated with the vector fields X_j are all volume preserving, from this approximation result and (5.37) it follows immediately that if the ball B_r is a constrained local minimizer of I, then for any function $\phi \in H^1(\partial B_r)$ with $\int_{\partial B_r}\phi = 0$

$$\partial^2 I(B_r)[\phi] \ge 0. \tag{5.38}$$

Next result shows that for r sufficiently large the ball B_r is a never a constrained local minimizer for I.

THEOREM 5.7. *Let $r_0 > 0$ be as in (5.13). If $r > r_0$ the ball B_r is not a constrained local minimizer of I.*

PROOF. Fix $r > r_0$. From lemma 5.1 it follows that there exists $k \geq 2$ such that

$$\frac{\lambda_k - \lambda_1}{\mu_k - \mu_1} r^{n-3} - r^n + \frac{K(n-2)}{\mu_k - \mu_1} < 0. \tag{5.39}$$

For every $x \in \partial B_r$ set $\phi(x) := y_k(x/r)$, where y_k is the restriction to $\mathbb{S}^{n-1}$ of a homogeneous harmonic polynomial of degree k, normalized so that $||y_k||_{L^2(\mathbb{S}^{n-1})} = 1$. Recalling that $\|D_\tau y_k\|_{L^2(\mathbb{S}^{n-1})} = \lambda_k$, from (5.38) we have

$$\partial^2 I(B_r)[\phi] = (\lambda_k - \lambda_1) r^{n-3} + K(n-2) - \int_{\partial B_r} \int_{\partial B_r} \frac{|\phi(x) - \phi(y)|^2}{|x-y|^{n-2}} d\mathcal{H}^{n-1} + r^n C_B^2. \tag{5.40}$$

On the other hand from (5.8) we have

$$\int_{\partial B_r} \int_{\partial B_r} \frac{|\phi(x) - \phi(y)|^2}{|x-y|^{n-2}} d\mathcal{H}^{n-1} = \mu_k r^n \int_{\mathbb{S}^{n-1}} \phi^2 d\mathcal{H}^{n-1} = \mu_k r^n. \tag{5.41}$$

From the definition (5.29), using again (5.8) and recalling that the first order normalized spherical harmonic are the functions $x_i/\sqrt{\omega_n}$, we have

$$\begin{aligned} C_B^2 &= \frac{1}{n\omega_n} \int_{\mathbb{S}^{n-1}} \int_{\mathbb{S}^{n-1}} \frac{|x-y|^2}{|x-y|^{n-2}} \, d\mathcal{H}_x^{n-1} \, d\mathcal{H}_y^{n-1} = \frac{1}{n\omega_n} \sum_{i=1}^n \int_{\mathbb{S}^{n-1}} \int_{\mathbb{S}^{n-1}} \frac{|x_i - y_i|^2}{|x-y|^{n-2}} \, d\mathcal{H}_x^{n-1} \, d\mathcal{H}_y^{n-1} \\ &= \frac{1}{n\omega_n} \mu_1 \sum_{i=1}^n \left(\int_{\mathbb{S}^{n-1}} \frac{x_i^2}{\sqrt{\omega_n}} \, d\mathcal{H}^{n-1} \right)^2 = \mu_1. \end{aligned}$$

Therefore, from the equality above, (5.40), (5.41) and (5.39) we get that

$$\partial^2 I(B_r)[\phi] = (\lambda_k - \lambda_1) r^{n-3} + K(n-2) - (\mu_k - \mu_1) r^n < 0.$$

Hence, the result follows. □

2.4. L^1-local minimality. In this section we show the main result of the paper, i.e., the strict L^1-local minimality of balls centered at the origin with radius smaller than the radius r_0 defined in (5.13). This result will be proved using Theorem 5.2, following a strategy first introduced in this framework in [**23**] and later on improved in [**1**]. Our result goes as follows.

THEOREM 5.8. *Let $n \geq 3$, $\sigma \in (0, r_0/2)$, where r_0 is defined as in (5.13). There exist δ, γ, depending only on n, K, σ, such that if $E \subset \mathbb{R}^n$ is a measurable set such that $|E \Delta B_r| \leq \delta$ and $|E| = |B_r|$, then*

$$I(E) \geq I(B_r) + \gamma |E \Delta B_r|^2.$$

We start with a simple lemma on the potential energy V.

LEMMA 5.9. *Let $F, E \subset \mathbb{R}^n$ be measurable sets and $|F| < \infty$. Then*

$$V(F) - V(E) \leq \frac{n}{\omega_n} |F|^{\frac{2}{n}} |F \setminus E|. \tag{5.42}$$

PROOF. Denote by r the radius of a ball with the same measure of F. Note that for every measurable set G with $|G| = |B_r|$

$$\int_G \frac{1}{|x|^{n-2}} \, dx \leq \int_{B_r} \frac{1}{|x|^{n-2}} \, dx.$$

Thus we have

$$V(F) - V(E) \le 2\int_{F\setminus E} dx \int_F \frac{1}{|x-y|^{n-2}}\, dy = 2\int_{F\setminus E} dx \int_{x-F} \frac{1}{|z|^{n-2}}\, dz$$
$$\le 2|F\setminus E| \int_{B_r} \frac{1}{|x|^{n-2}}\, dx = |F\setminus E| n\omega_n r^2.$$

Hence, (5.42) follows. □

Let us now state another simple lemma which we be useful to treat the perimeter term and the attraction term in the energy. In all the remaining part of this section we shall always assume $n \ge 3$.

LEMMA 5.10. *Let $\sigma \in (0, r_0/2)$, where r_0 is defined as in* (5.13)*. There exists Λ_0, depending on n, K, σ, such that if $\Lambda \ge \Lambda_0$ and $r \in [\sigma, r_0]$, the ball B_r is the unique minimizer of the functional*

$$P(E) - KR(E) + \Lambda\big||E| - |B_r|\big|$$

among all sets of finite measure.

PROOF. Recall that for every set E of finite measure, we have

$$P(B_{r_E}) - KR(B_{r_E}) \le P(E) - KR(E),$$

where B_{r_E} is the ball with the same volume of E. Therefore, to prove the lemma it is enough to show that if Λ is sufficiently large, then the function

$$f(\varrho) := n\varrho^{n-1} - \frac{Kn\varrho^2}{2} + \Lambda\big|\varrho^n - r^n\big|$$

has a unique minimum in $[0, \infty)$ at $\varrho = r$. Indeed if $0 \le \varrho \le r$

$$f'(\varrho) = n(n-1)\varrho^{n-2} - Kn\varrho - \Lambda n\varrho^{n-1} \le 0,$$

provided $\Lambda \ge (n-1)/r_0$. Similarly, if $\varrho \ge r$

$$f'(\varrho) = n(n-1)\varrho^{n-2} - Kn\varrho + \Lambda n\varrho^{n-1} \ge 0,$$

provided $\Lambda \ge K/\sigma^{n-2}$. Then the conclusion follows from the two previous estimates choosing $\Lambda \ge \max\{(n-1)/r_0, K/\sigma^{n-2}\}$. □

LEMMA 5.11. *There exists $C_1 > 0$, depending only on n, such that, if $\eta \in (0,1)$ and $E \subset \mathbb{R}^n$ is a measurable set such that $|E \setminus B_r| < \eta$ for some $r > 0$, then we can find $r \le r_E \le r + C_1\eta^{\frac{1}{n}}$ such that*

$$P(E \cap B_{r_E}) \le P(E) - \frac{|E \setminus B_{r_E}|}{C_1\eta^{\frac{1}{n}}} \tag{5.43}$$

PROOF. For any $\varrho > 0$ we set $u(\varrho) := |E \setminus B_\varrho|$. By the area formula $u'(\varrho) = -\mathcal{H}^{n-1}(\partial B_\varrho \cap E)$ for $\mathcal{L}^1$-a.e. $\varrho > 0$. We set

$$C_1 := \frac{2n+1}{(n\omega_n)^{\frac{1}{n}}}. \tag{5.44}$$

If $u(r + C_1\eta^{\frac{1}{n}}) = 0$ then (5.43) trivially holds with $r_E = r + C_1\eta^{\frac{1}{n}}$.
If $u(r + C_1\eta^{\frac{1}{n}}) > 0$ we argue by contradiction assuming that for every $r \le \varrho \le r + C_1\eta^{\frac{1}{n}}$

$$-2u'(\varrho) - P(E \setminus B_\varrho) = P(E \cap B_\varrho) - P(E) > -\frac{u(\varrho)}{C_1\eta^{\frac{1}{n}}}.$$

Using the isoperimetric inequality we have that for all $\varrho \in \big(r, r + C_1\eta^{\frac{1}{n}}\big)$

$$-u'(\varrho) > \frac{1}{2}(n\omega_n)^{\frac{1}{n}}|E \setminus B_\varrho|^{\frac{n-1}{n}} - \frac{u(\varrho)}{2C_1\eta^{\frac{1}{n}}} = \frac{1}{2}(n\omega_n)^{\frac{1}{n}}u(\varrho)^{\frac{n-1}{n}} - \frac{u(\varrho)}{2C_1\eta^{\frac{1}{n}}}.$$

Since $u(\varrho) \le |E \setminus B_r| < \eta$, recalling the definition (5.44) of C_1, we get

$$-u'(\varrho) > \frac{n}{C_1}u(\varrho)^{\frac{n-1}{n}} \qquad \text{for all } r \le \varrho \le r + C_1\eta^{\frac{1}{n}}.$$

Integrating this inequality in $\big(r, r + C_1\eta^{\frac{1}{n}}\big)$ we obtain

$$u(r)^{\frac{1}{n}} - u(r + C_1\eta^{\frac{1}{n}})^{\frac{1}{n}} > \eta^{\frac{1}{n}},$$

which contradicts the assumption $\eta > |E \setminus B_r|$. Hence the result follows. □

The following lemma will be used in the proof of Theorem 5.8.

LEMMA 5.12. *Let $\sigma \in (0, r_0/2)$, where r_0 is defined as in (5.13), and let $\Lambda_1 \ge 2n\omega_n^{\frac{2-n}{n}}r_0^2$, $\Lambda_2 \ge \Lambda_0$, with Λ_0 as in Lemma 5.10. There exists $\varepsilon_0 > 0$ such that if $0 \le \varepsilon \le \varepsilon_0$ and $r \in [\sigma, r_0 - \sigma]$, then the minimum problem*

$$\min\big\{I(E) + \Lambda_1\big||E\Delta B_r| - \varepsilon\big| + \Lambda_2\big||E| - |B_r|\big| :\, |E| < \infty\big\}$$

as at least a solution $F \subset B_R$, with $R = r_0 + C_1$, where C_1 is the constant in Lemma 5.11.

PROOF. Given a set of finite perimeter and finite measure E, we define for $\varepsilon > 0$

$$J_\varepsilon(E) := I(E) + \Lambda_1\big||E\Delta B_r| - \varepsilon\big| + \Lambda_2\big||E| - |B_r|\big|.$$

Let E_h be a minimizing sequence for J_ε such that

$$J_\varepsilon(E_h) \le \inf J_\varepsilon + \frac{1}{h}.$$

From this inequality, recalling Lemmas 5.9 and 5.10, we have,

$$\begin{aligned} J_\varepsilon(E_h) &\le I(B_r) + \Lambda_1\varepsilon + \frac{1}{h} \le I(E_h) + V(B_r) - V(E_h) + \Lambda_1\varepsilon + \Lambda_2\big||E_h| - |B_r|\big| + \frac{1}{h} \\ &\le I(E_h) + n\omega_n r^2|B_r \setminus E_h| + \Lambda_1\varepsilon + \Lambda_2\big||E_h| - |B_r|\big| + \frac{1}{h}. \end{aligned}$$

Therefore, from this inequality, recalling that $\Lambda_1 \ge 2n\omega_n^{\frac{2-n}{n}}r_0^2$, we have

$$\Lambda_1\big||E_h\Delta B_r| - \varepsilon\big| \le \frac{\Lambda_1}{2}|B_r \setminus E_h| + \Lambda_1\varepsilon + \frac{1}{h},$$

hence

$$|E_h\Delta B_r| \le 4\varepsilon + \frac{2}{h\Lambda_1}.$$

Assume now that $\varepsilon \le \varepsilon_0 < 1/5$, with ε_0 to be chosen. Set $\eta := 5\varepsilon_0$. For h so large that $4\varepsilon + 2/(h\Lambda_1) < \eta$, denote by $r_h := r_{E_h} \in [r, r + C_1\eta^{\frac{1}{n}}]$ the radius provided by Lemma 5.11. Thus, recalling (5.43), we estimate for h large

$$\begin{aligned} J_\varepsilon(E_h \cap B_{r_h}) &\le \Big(P(E_h) - \frac{|E_h \setminus B_{r_h}|}{C_1\eta^{\frac{1}{n}}}\Big) + V(E_h) - KR(E_h) + KR(E_h \setminus B_{r_h}) + \Lambda_1\big||E_h\Delta B_r| - \varepsilon\big| \\ &\quad + \Lambda_1\big||(E_h \cap B_{r_h})\Delta B_r| - |E_h\Delta B_r|\big| + \Lambda_2\big||E_h| - |B_r|\big| + \Lambda_2|E_h \setminus B_{r_h}| \\ &\le J_\varepsilon(E_h) + \Big(\frac{K}{r_h^{n-2}} + \Lambda_1 + \Lambda_2 - \frac{1}{C_1\eta^{\frac{1}{n}}}\Big)|E_h \setminus B_{r_h}| \le J_\varepsilon(E_h), \end{aligned}$$

provided we choose η, hence ε_0, sufficiently small. Thus also $E_h \cap B_{r_h}$ is a minimizing sequence for J_ε. Since for h large the sets $E_h \cap B_{r_h} \subset B_R$ are equibounded and have equibounded perimeters, a standard argument shows that up to a not relabelled subsequence they converge in measure to a set $E \subset B_R$ who is an absolute minimizer for J_ε. □

In order to make the presentation clearer we split the proof of Theorem 5.8 in several lemmas. We will argue by contradiction.

Let r_0 be defined as in (5.13). Given $\sigma \in (0, r_0/2)$, we assume that there exists a sequence E_h of sets such that $|E_h| = |B_{r_h}|$, with $r_h \in [\sigma, r_0 - \sigma]$ and

$$\lim_h |E_h \Delta B_r| = 0, \qquad I(E_h) \le I(B_{r_h}) + C_0 |E_h \Delta B_{r_h}|^2 \quad \text{for all} \quad h \in \mathbb{N}, \tag{5.45}$$

for some $C_0 > 0$ to be fixed later.

The idea of the proof is to replace the sets E_h with a sequence of sets still satisfying (5.45), possibly with a larger constant, and converging in C^1 to a ball B_r with $0 < r < r_0$. This convergence will then contradict the quantitative estimate (5.25), provided C_0 is sufficiently small.

To this end we consider the functionals

$$J_{\varepsilon_h}(E) := I(E) + \Lambda_1 ||E \Delta B_{r_h}| - \varepsilon_h| + \Lambda_2 ||E| - |B_{r_h}||, \tag{5.46}$$

where $\varepsilon_h := |E_h \Delta B_{r_h}|$, and Λ_1, Λ_2 satisfy the assumptions of Lemma 5.12. Thanks to this lemma we may conclude that for h sufficiently large the functional J_{ε_h} has an absolute minimizer F_h contained in B_R, where R is the radius provided by the lemma.

Next lemma shows that the above minimizers F_h converge in measure to a ball.

LEMMA 5.13. *Let the sets F_h be defined as above. Then, up to a subsequence, they converge in measure to a ball B_r with $r \in [\sigma, r_0 - \sigma]$.*

PROOF. Recall that for h large the sets F_h are equibounded. Moreover, still assuming h sufficiently large,

$$J_{\varepsilon_h}(F_h) \le J_{\varepsilon_h}(B_{r_h}) = I(B_{r_h}) + \Lambda_1 \varepsilon_h \le C,$$

for some $C > 0$ independent of h. Thus, the F_h have equibounded perimeters. Therefore, up to a not relabeled subsequence, we may assume that they converge in measure to a set $F_\infty \subset B_R$ and that $r_h \to r \in [\sigma, r_0 - \sigma]$. It is easily checked that F_∞ is a minimizer of the functional

$$J(E) = I(E) + \Lambda_1 |E \Delta B_r| + \Lambda_2 ||E| - |B_r||.$$

Let us now show that $F_\infty = B_r$. To this end we estimate, using Lemmas 5.10 and 5.9 ,

$$\begin{aligned} J(F_\infty) &= P(F_\infty) + V(F_\infty) - KR(F_\infty) + \Lambda_1 |F_\infty \Delta B_r| + \Lambda_2 ||F_\infty| - |B_r|| \\ &\ge P(B_r) + V(F_\infty) - KR(B_r) + \Lambda_1 |F_\infty \Delta B_r| \\ &= J(B_r) + V(F_\infty) - V(B_r) + \Lambda_1 |F_\infty \Delta B_r| \\ &\ge J(B_r) - n\omega_n r_0^2 |B_r \setminus F_\infty| + \Lambda_1 |F_\infty \Delta B_r|. \end{aligned}$$

Then the conclusion follows by recalling that $\Lambda_1 \ge 2n\omega_n r_0^2$. □

The next lemma provides a density estimate for F_h. We give only a sketch of the proof since it follows quite closely a standard argument in the regularity theory of sets of finite perimeter.

LEMMA 5.14. *There exist $\varrho_0 > 0$ and $\vartheta_0 > 0$ such that if $F_h \subset B_R$ is a minimizer of J_{ε_h} and $0 < \varrho \le \varrho_0$ then for all $y \in \partial^* F_h$*

$$\frac{|F_h \cap B_\varrho(y)|}{|B_\varrho(y)|} \le 1 - \vartheta_0. \tag{5.47}$$

PROOF. From the minimality of F_h we have that $J_{\varepsilon_h}(F_h) \le J_{\varepsilon_h}(F_h \cup B_\varrho(y))$ for all $\varrho \in (0,1)$. From this inequality we get that for $\mathcal{L}^1$-a.e. $\varrho \in (0,1)$

$$\begin{aligned} P(F_h; B_\varrho(y)) &\le \mathcal{H}^{n-1}(\partial B_\varrho(y) \setminus F_h) + V(B_\varrho(y) \cup F_h) - V(F_h) + KR(F_h) - KR(B_\varrho(y) \cup F_h) \\ &\qquad + \Lambda_1 ||(B_\varrho(y) \cup F_h)\Delta B_{r_h}| - |F_h \Delta B_{r_h}|| + \Lambda_2 ||B_\varrho(y) \cup F_h| - |F_h|| \\ &\le \mathcal{H}^{n-1}(\partial B_\varrho(y) \setminus F_h) + C|B_\varrho(y) \setminus F_h|, \end{aligned}$$

where the constant C depends only on n, r_0, Λ_1 and Λ_2. Starting from this estimate, the conclusion then follows arguing exactly as in [**51**, Th. 16.14]. □

LEMMA 5.15. *Let $\sigma \in (0, r_0/2)$, Λ_1, Λ_2 and ε_h be as above and let $F_h \subset B_R$ be a minimizer of the functional J_{ε_h} defined in (5.46). There exist $\Lambda, \overline{r} > 0$ and a not relabelled subsequence F_h such that every F_h is a $(\Lambda, \overline{r})$-almost minimizer of the perimeter.*

PROOF. Observe that by Lemma 5.13 it follows that, passing possibly to a subsequence, we may assume that F_h converges in measure to a ball B_r with $r \in [\sigma, r_0 - \sigma]$. We set $\bar\varrho := \min\{\sigma/2, \varrho_0\}$, where ϱ_0 is the radius provided by Lemma 5.14. We claim that there exists h_0 such that

$$|B_{\bar\varrho} \setminus F_h| = 0 \qquad \text{for all } h \ge h_0. \tag{5.48}$$

Indeed, if the above claim were not true we could find a subsequence F_{h_k} such that $|B_{\bar\varrho} \setminus F_{h_k}| > 0$ for all k. Since F_h converges in measure to B_r and $r \ge 2\bar\varrho$, we may also assume that $|B_{\bar\varrho} \cap F_{h_k}| > 0$ for all k. Therefore, by the relative isoperimetric inequality we get that $P(F_{h_k}; B_{\bar\varrho}) > 0$. Hence, for all k there exists $y_k \in \partial^* F_{h_k} \cap B_{\bar\varrho}$. Passing possibly to another to a subsequence, we may assume that $y_k \to y \in \overline{B}_{\bar\varrho}$. By applying the estimate (5.47) we the get

$$|B_{\bar\varrho}(y)| = \lim_k |F_{h_k} \cap B_{\bar\varrho}(y_k)| \le (1-\vartheta_0) \lim_k |B_{\bar\varrho}(y_k)| = (1-\vartheta_0)|B_{\bar\varrho}(y)|.$$

This contradiction proves (5.48).

Let us now set $\overline{r} = \bar\varrho/3$. Let $E \subset \mathbb{R}^n$ be such that $E\Delta F_h \subset B_\varrho(y)$, with $\varrho < \overline{r}$ and $h \ge h_0$. If $|y| \le 2\overline{r}/3$, then, since $B_\varrho(y) \cap F_h = B_\varrho(y)$ by (5.48), we have $P(F_h; B_\varrho(y) = 0$, hence, trivially

$$P(F_h; B_\varrho(y)) \le P(E; B_\varrho(y)).$$

If instead $|y| > 2\overline{r}/3$, we are going to show that

$$P(F_h; B_\varrho(y)) \le P(E; B_\varrho(y)) + \Lambda |E\Delta F_h|,$$

for some $\Lambda > 0$ that will be chosen below. From the minimality of F_h we get, recalling (5.42),

$$\begin{aligned} P(F_h; B_\varrho(y)) &\le P(E; B_\varrho(y)) + V(E) - V(F_h) - KR(E) + KR(F_h) \\ &\qquad + \Lambda_1 ||F_h \Delta B_r| - |E_h \Delta B_r| + \Lambda_2 ||F_h|| - |E|| \\ &\le P(E; B_\varrho(y)) + n\omega_n^{\frac{n-2}{n}} |E|^{\frac{2}{n}} |E \setminus F_h| + K \int_{F_h \Delta E} \frac{1}{|x|^{n-2}} dx + (\Lambda_1 + \Lambda_2)|F_h \Delta E|. \end{aligned}$$

Since $|E| \le |F_h| + |B_{\overline{r}}| \le C(n, r_0)$ and $F_h \Delta E \subset \mathbb{R}^n \setminus B_{\overline{r}/3}$ from the above estimate we easily get that

$$P(F_h; B_\varrho(y)) \le P(E; B_\varrho(y)) + C(n, r_0, \Lambda_1, \Lambda_2)|F_h \Delta E| + C(n)\overline{r}^{2-n} K |F_h \Delta E|,$$

for some positive constants $C(n)$ and $C(n, r_0, \Lambda_1, \Lambda_2)$. From this inequality the conclusion immediately follows by taking Λ sufficiently large. □

We are ready now to prove Theorem 5.8.

Proof of Theorem 5.8. **Step 1.** Given $\sigma \in (0, r_0/2)$, we argue by contradiction, assuming that there exists a sequence of sets of finite perimeter E_h satisfying (5.45). Then, we take $\Lambda_1 \geq 2n\omega_n^{\frac{2-n}{n}} r_0^2$ and $\Lambda_2 \geq \max\{2\Lambda_0, 4\Lambda_1\}$ and consider a sequence F_h of minimizers of the functionals (5.46), where $\varepsilon_h = |E_h \Delta B_{r_h}|$. Thanks to Lemma 5.12 and Lemma 5.13 we may assume, passing possibly to a subsequence, that $F_h \subset B_R$ for all h and that they converge in measure to the ball B_r for some $r \in [\sigma, r_0 - \sigma]$. Then by Lemma 5.15 we may also assume that the F_h are all $(\Lambda, \overline{r})$-almost minimizers of the perimeter for some $\Lambda, \overline{r} > 0$. Therefore, Theorem 4.9 yields that the sequence F_h converges in $C^{1,\alpha}$ to B_r. In particular, denoting by $\widetilde{r}_h$ the radius of the ball such that $|F_h| = |B_{\widetilde{r}_h}|$, we may assume that $\widetilde{r}_h \in [\sigma/2, r_0 - \sigma/2]$ for all h and that there exists a sequence $\psi_h \in C^1(\mathbb{S}^{n-1})$ converging in C^1 to 0 such that for all h

$$F_h = \{y = \widetilde{r}_h x(1 + \psi_h(x)) : x \in B\}. \tag{5.49}$$

By the minimality of the F_h, recalling Lemma 5.10 and Lemma 5.9 we have

$$\begin{aligned} I(F_h) + \Lambda_1\big||F_h \Delta B_{r_h}| - \varepsilon_h\big| + \Lambda_2\big||F_h| - |B_{r_h}|\big| &\leq I(E_h) \leq I(B_{r_h}) + C_0|E_h \Delta B_r|^2 \\ &\leq I(F_h) + V(B_{r_h}) - V(F_h) + \Lambda_0\big||F_h| - |B_{r_h}|\big| + C_0\varepsilon_h^2 \\ &\leq I(F_h) + n\omega_n r_0^2 |B_{r_h} \setminus F_h| + \Lambda_0\big||F_h| - |B_{r_h}|\big| + C_0\varepsilon_h^2 \\ &\leq I(F_h) + \frac{\Lambda_1}{2}|B_{r_h} \Delta F_h| + \frac{\Lambda_2}{2}\big||F_h| - |B_{r_h}|\big| + C_0\varepsilon_h^2, \end{aligned} \tag{5.50}$$

where the last inequality follows from the choice of Λ_1 and Λ_2. From the above inequality we then get easily that

$$\varepsilon_h + \frac{\Lambda_2}{2\Lambda_1}\big||F_h| - |B_{r_h}|\big| \leq \frac{3}{2}|B_{r_h} \Delta F_h| + \frac{C_0}{\Lambda_1}\varepsilon_h^2.$$

Note that in particular we have that for h large $\varepsilon_h \leq 2|B_{r_h} \Delta F_h|$. Therefore, passing possible to another subsequence if needed, we may assume without loss of generality that for all h

$$\begin{aligned} \varepsilon_h + \frac{\Lambda_2}{2\Lambda_1}\big||F_h| - |B_{r_h}|\big| \leq 2|B_{r_h} \Delta F_h| &\leq 2|F_h \Delta B_{\widetilde{r}_h}| + 2\big||B_{\widetilde{r}_h}| - |B_{r_h}|\big| \\ &= 2\,|F_h \Delta B_{\widetilde{r}_h}| + 2\big||F_h| - |B_{r_h}|\big|. \end{aligned}$$

Recalling that we have chosen $\Lambda_2 \geq 4\Lambda_1$, from the above inequality we have that for all h

$$\varepsilon_h \leq 2|F_h \Delta B_{\widetilde{r}_h}|. \tag{5.51}$$

Thus, using the second inequality in (5.50) we have that for all h

$$\begin{aligned} I(F_h) + \Lambda_1\big||F_h \Delta B_{r_h}| - \varepsilon_h\big| + \Lambda_2\big||F_h| - |B_{r_h}|\big| &\leq I(B_{r_h}) + C_0\varepsilon_h^2 \\ &\leq I(B_{\widetilde{r}_h}) + C(n,\sigma)|\widetilde{r}_h - r_h| + C_0\varepsilon_h^2, \end{aligned}$$

for a positive constant $C(n, \sigma)$ independent of h. Note however that there exists another constant $c(n, \sigma)$ still depending only on n and σ, such that

$$c(n,\sigma)|\widetilde{r}_h - r_h| \leq \big||F_h| - |B_{r_h}|\big|.$$

Therefore, choosing $\Lambda_2 \geq C(n, \sigma)/c(n, \sigma)$, and Λ_1 accordingly, we have, recalling (5.51),

$$I(F_h) \leq I(B_{\widetilde{r}_h}) + 4C_0|F_h \Delta B_{\widetilde{r}_h}|^2. \tag{5.52}$$

Let us now denote by x_h the barycenter of F_h and observe that

$$|x_h| = \frac{1}{|B_{\widetilde{r}_h}|}\left|\int_{F_h} x\,dx\right| \leq \frac{1}{|B_{\widetilde{r}_h}|}\int_{F_h \Delta B_{\widetilde{r}_h}} |x|\,dx \to 0 \qquad \text{as } h \to \infty.$$

We set $G_h := F_h - x_h$. Since x_h is converging to 0, from (5.49) we deduce that there exists a sequence $\varphi_h \in C^1(\mathbb{S}^{n-1})$ converging in C^1 to 0 such that for all h

$$G_h = \{y = \widetilde{r}_h x(1+\varphi_h(x)) : x \in B\}. \tag{5.53}$$

Step 2. We now estimate $R(F_h) - R(G_h)$. To this end we use Lemma 5.4 (note that $0 \notin \partial G_h$), observing that $F_h = \Phi_1(G_h)$, where the flow is given by $\Phi_t(x) := x + tx_h$. Thus, recalling (5.30) and (5.31) we have

$$R(F_h) - R(G_h) = \int_{\partial G_h} \frac{x_h \cdot \nu_{G_h}}{|x|^{n-2}}\, d\mathcal{H}^{n-1} - \frac{n-2}{2}\int_{\partial G_{h,t_h}} \frac{(x_h\cdot \nu_{G_{h,t_h}})(x_h\cdot x)}{|x|^n}\, d\mathcal{H}^{n-1} + o(|x_h|^2), \tag{5.54}$$

where $G_{h,t_h} = G_h + t_h x_h$ for some $t_h \in (0,1)$. Observe now that

$$\begin{aligned}\Big| \int_{\partial G_{h,t_h}} &\frac{(x_h\cdot \nu_{G_{h,t_h}})(x_h\cdot x)}{|x|^n}\, d\mathcal{H}^{n-1} - \int_{\partial G_h} \frac{(x_h\cdot \nu_{G_h})(x_h\cdot x)}{|x|^n}\, d\mathcal{H}^{n-1}\Big| \\ &= \Big| \int_{\partial G_h} (x_h\cdot \nu_{G_h}(x))\Big(\frac{x_h\cdot(x+t_hx_h)}{|x+t_hx_h|^n} - \frac{x_h\cdot x}{|x|^n}\Big)\, d\mathcal{H}^{n-1}\Big| \le C|x_h|^3.\end{aligned}$$

Therefore, from (5.54) we have

$$R(F_h) - R(G_h) = \int_{\partial G_h} \frac{x_h \cdot \nu_{G_h}}{|x|^{n-2}}\, d\mathcal{H}^{n-1} - \frac{n-2}{2}\int_{\partial G_h} \frac{(x_h\cdot \nu_{G_h})(x_h\cdot x)}{|x|^n}\, d\mathcal{H}^{n-1} + o(|x_h|^2). \tag{5.55}$$

Recalling (5.53), we have that at the point $y = \widetilde{r}_h x(1+\varphi_h(x))$ with $x \in \mathbb{S}^{n-1}$,

$$\nu_{G_h}(z) = \frac{x(1+\varphi_h(x)) - D_\tau\varphi_h(x)}{\sqrt{(1+\varphi_h(x))^2 + |D_\tau\varphi_h(x)|^2}}.$$

Thus, denoting by div_τ the tangential divergence on the sphere and using the divergence theorem, we get

$$\begin{aligned}\int_{\partial G_h} \frac{x_h\cdot\nu_{G_h}}{|x|^{n-2}}\, d\mathcal{H}^{n-1} &= \widetilde{r}_h \int_{\mathbb{S}^{n-1}} x_h\cdot(x(1+\varphi_h(x)) - D_\tau\varphi_h(x))\, d\mathcal{H}^{n-1} \\ &= \widetilde{r}_h \int_{\mathbb{S}^{n-1}} (x_h\cdot x)\varphi_h\, d\mathcal{H}^{n-1} - \widetilde{r}_h\int_{\mathbb{S}^{n-1}} \mathrm{div}_\tau(x_h\varphi_h)\, d\mathcal{H}^{n-1} \\ &= -(n-2)\widetilde{r}_h \int_{\mathbb{S}^{n-1}} (x_h\cdot x)\varphi_h\, d\mathcal{H}^{n-1}.\end{aligned}$$

Since G_h has barycenter at the origin, arguing as in the proof of (5.24) we have that for h large

$$\Big|\int_{\partial G_h} \frac{x_h\cdot\nu_{G_h}}{|x|^{n-2}}\, d\mathcal{H}^{n-1}\Big| = (n-2)\widetilde{r}_h\Big|x_h\cdot \int_{\partial\mathbb{S}^{n-1}} x\varphi(x)\, d\mathcal{H}^{n-1}\Big| \le C(n)|x_h|\|\varphi_h\|^2_{L^2(\mathbb{S}^{n-1})}. \tag{5.56}$$

Let us now estimate the second integral on the right hand side of (5.55). To this end we estimate

$$\begin{aligned}\int_{\partial G_h} \frac{(x_h\cdot\nu_{G_h})(x_h\cdot x)}{|x|^n}\, d\mathcal{H}^{n-1} &= \int_{\mathbb{S}^{n-1}} \frac{\big(x_h\cdot(x(1+\varphi_h) - D_\tau\varphi_h)\big)(x_h\cdot x)}{1+\varphi_h(x)}\, d\mathcal{H}^{n-1} \\ &\ge \int_{\mathbb{S}^{n-1}} \big(x_h\cdot(x(1+\varphi_h) - D_\tau\varphi_h)\big)(x_h\cdot x)\, d\mathcal{H}^{n-1} - C(n)|x_h|^2\|\varphi_h\|^2_{H^1(\mathbb{S}^{n-1})} \\ &\ge \int_{\mathbb{S}^{n-1}} |x_h\cdot x|^2\, d\mathcal{H}^{n-1} - C(n)|x_h|^2\|\varphi_h\|_{H^1(\mathbb{S}^{n-1})} = \omega_n|x_h|^2 - C(n)|x_h|^2\|\varphi_h\|_{H^1(\mathbb{S}^{n-1})}.\end{aligned}$$

From this estimate and from (5.56) we finally obtain, recalling (5.55), that for h large

$$\begin{aligned}R(F_h) - R(G_h) &\le C(n)|x_h|\|\varphi_h\|^2_{L^2(\mathbb{S}^{n-1})} - \frac{n-2}{2}\omega_n|x_h|^2 + C(n)|x_h|^2\|\varphi_h\|_{H^1(\mathbb{S}^{n-1})} \\ &\le C(n)|x_h|\|\varphi_h\|^2_{L^2(\mathbb{S}^{n-1})} - \frac{n-2}{3}\omega_n|x_h|^2.\end{aligned}$$

Therefore, for h large, we have

$$I(G_h) = I(F_h) + KR(F_h) - KR(G_h) \le I(F_h) + C(n)K|x_h|\|\varphi_h\|^2_{L^2(\mathbb{S}^{n-1})} - \frac{K(n-2)}{3}\omega_n|x_h|^2.$$

Therefore, using (5.52), from the above inequality we have for h large, recalling that $x_h \to 0$,

$$\begin{aligned}
I(G_h) &\le I(B_{\widetilde{r}_h}) + 4C_0|F_h\Delta B_{\widetilde{r}_h}|^2 + C(n)K|x_h|\|\varphi_h\|^2_{L^2(\mathbb{S}^{n-1})} - \frac{K(n-2)}{3}\omega_n|x_h|^2 \\
&\le I(B_{\widetilde{r}_h}) + 8C_0(|G_h\Delta B_{\widetilde{r}_h}|^2 + |G_h\Delta F_h|^2) + C(n)K|x_h|\|\varphi_h\|^2_{L^2(\mathbb{S}^{n-1})} - \frac{K(n-2)}{3}\omega_n|x_h|^2 \\
&\le I(B_{\widetilde{r}_h}) + 9C_0|G_h\Delta B_{\widetilde{r}_h}|^2 + 8C_0|G_h\Delta F_h|^2 - \frac{K(n-2)}{3}\omega_n|x_h|^2 \\
&\le I(B_{\widetilde{r}_h}) + 9C_0|G_h\Delta B_{\widetilde{r}_h}|^2,
\end{aligned}$$

where the last inequality follows by observing that

$$8C_0|G_h\Delta F_h|^2 - \frac{K(n-2)}{3}\omega_n|x_h|^2 \le C(n)8C_0|x_h|^2 - \frac{K(n-2)}{3}\omega_n|x_h|^2 < 0,$$

provided C_0 is sufficiently small. In conclusion we have shown that for h large

$$I(G_h) \le I(B_{\widetilde{r}_h}) + 9C_0|G_h\Delta B_{\widetilde{r}_h}|^2$$

and this inequality contradicts (5.25) if we assume also $C_0 < c_1/9$. □

2.5. Global minimality. In this last section we prove the existence of a critical radius $r_1 \le r_0$ such that if $r < r_1$, the ball centered at the origin with radius r is the unique global minimizer of I among all sets of prescribed measure. We start with a simple lemma.

Lemma 5.16. *Let $n \ge 3$. There exists a constant $C(n) > 0$ such that if $E \subset \mathbb{R}^n$ is a Borel set with $|E| = |B_r|$ then*

$$\frac{R(B_r) - R(E)}{r^2} > C(n)\left(\frac{|E\Delta B_r|}{r^n}\right)^2. \tag{5.57}$$

Proof. Since both quantities in (5.57) are scaling invariant we may assume $r = 1$. Thus, let E be a Borel set with $|E| = |B|$ with $|E\Delta B| > 0$ and let us decompose it as $E = (E\cap B)\cup(E\setminus B)$. Let $0 < \varrho < 1 < r$ such that $|B_\varrho| = |B\setminus E|$, $|B_r\setminus B| = |E\setminus B|$ and set

$$E^* := B_\rho \cup (B_r \setminus B).$$

Clearly, we have that

$$|E\Delta B| = |E^*\Delta B|, \qquad R(E^*) > R(E).$$

At this point we can easily evaluate the left handside of (5.57)

$$R(B) - R(E) \ge R(B) - R(E^*) = \frac{n\omega_n}{2}(2 - (\varrho^2 + r^2)).$$

Since $r^n = 2 - \varrho^n$, from the inequality above we have

$$R(B) - R(E) \ge \frac{n\omega_n}{2}(2 - \varrho^2 - (2-\varrho^n)^{\frac{2}{n}}) := f(\varrho).$$

The conclusion then follows by observing that

$$\lim_{\varrho\to 1}\frac{f(\varrho)}{(1-\varrho)^2} = c(n) > 0.$$

□

Before stating the main result of this section, let us define

$$r_1 := \left(\frac{K}{2\omega_n}\right)^{\frac{1}{n}}. \tag{5.58}$$

THEOREM 5.17. *Let $n \geq 3$. If $r < r_1$, where r_1 is defined as in (5.58), the ball centered at the origin is the only gobal minimizer of I among all sets $E \subset \mathbb{R}^n$ with prescribed volume $|E| = |B_r|$. Moreover,*

$$I(E) - I(B_r) \geq c|E\Delta B_r|^2,$$

for some positive constant c depending only on n and r.

PROOF. We start by observing that

$$V(B_r) - V(E) = \int_{B_r}\int_{B_r} \frac{1}{|x-y|^{n-2}}dxdy - \int_E\int_E \frac{1}{|x-y|^{n-2}}dxdy \tag{5.59}$$

$$= 2\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{\chi_{B_r}(x)(\chi_{B_r}(y)-\chi_E(y))}{|x-y|^{n-2}}\,dxdy - \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{(\chi_{B_r}(y)-\chi_E(y))(\chi_{B_r}(x)-\chi_E(x))}{|x-y|^{n-2}}\,dxdy.$$

Set for all $x \in \mathbb{R}^n$

$$u(x) := \int_{\mathbb{R}^n} \frac{(\chi_{B_r}(y)-\chi_E(y))}{|x-y|^{n-2}}. \tag{5.60}$$

Then

$$-\Delta u = c_n(\chi_{B_r} - \chi_E),$$

for some constant $c(n) > 0$. Therefore, integrating by parts,

$$\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{(\chi_{B_r}(y)-\chi_E(y))(\chi_{B_r}(x)-\chi_E(x))}{|x-y|^{n-2}}dxdy = -\int_{\mathbb{R}^n} u(y)\Delta u(y)dy = c_n\int_{\mathbb{R}^n}|Du|^2\,dx. \tag{5.61}$$

Combining (5.59), (5.61) and the fact that u is superharmonic in B_r, we get

$$V(B_r) - V(E) \leq 2\int_{B_r} u(y)dy \leq 2|B_r|u(0) = 2|B_r|\,(R(B_r) - R(E))$$

Thus using the isoperimetric inequality, the above lemma and that $r < r_1$ we can conclude that

$$I(E) - I(B_r) \geq P(E) - P(B_r) + (K - 2|B_r|)\big(R(B_r) - R(E)\big) \geq c(n,r)(K-2|B_r|)|E\Delta B_r|^2.$$

□

From definitions (5.13) and (5.58) it is clear that both r_0 and r_1 tend to ∞ as $K \to \infty$. However the ratio r_0/r_1 stays bounded.

LEMMA 5.18. *Let $n \geq 3$. Then*

$$\limsup_{K\to+\infty} \frac{r_0}{r_1} \leq \Big(\frac{n\omega_n^2}{V(B)}\Big)^{\frac{1}{n}}.$$

PROOF. Let P_2 be defined as in (5.15) and denote by $r_2 \geq r_0$ the unique zero of P_2. From the second equation in (5.10), we have that

$$P_2(r) = a(n)r^{n-3} - r^n + \frac{K(n+2)}{\mu_1},$$

where, using the first equation in (5.10), $a_n = (n+1)P(B)/[2(n-2)V(B)]$. Recalling the first equation in (5.10), (5.14) and the definition (5.58) of r_1 we have at once that

$$\frac{K(n+2)}{\mu_1} = \gamma_n^n r_1^n, \qquad \text{where } \gamma_n = \Big(\frac{n\omega_n^2}{V(B)}\Big)^{\frac{1}{n}} > 1.$$

Fix now $\varepsilon > 0$. Then

$$P_2(\gamma_n(1+\varepsilon)r_1) = \gamma_n^n r_1^n\bigg(\frac{a_n(1+\varepsilon)^{n-3}}{r_1^3} - (1+\varepsilon)^n + 1\bigg) < 0,$$

provided r_1, hence K, is large enough. Therefore we may conclude that for K sufficiently large

$$r_0 \leq r_2 < \gamma_n(1+\varepsilon)r_1.$$

Hence, the result follows. □

3. Non existence of minimizers

In this section we prove the non existence of minimizers in dimension three. Here we briefly explain the reason why we expect that. If one considers each term of (5.1) separately, it is well known that the ball is an extremal for all of them: precisely it minimizes the perimeter and maximizes both $V(E)$ and $R(E)$ under volume constraint. Indeed it is the competition among these three terms that makes the problem mathematically challenging. Therefore, while it can be proved that for $m < K$ the ball is the unique minimizer of (5.1), (see [**50**], [**46**] and [**44**], [**42**] and [**49**] for related results), it is natural to expect that minimizers do not occur when m is large enough. To see this, assume that $|E| = 1$, consider the rescaled set λE and observe that

$$I_K(\lambda E) = \lambda^2 P(E) + \lambda^4 V(E) - K\lambda^2 R(E). \tag{5.62}$$

When λ is large enough, the leading term in (5.62) is $V(\lambda E) = \lambda^4 V(E)$. Therefore, in order to minimize the energy (5.62) it would be convenient to lower as much as possible the value of $V(E)$. However this is not feasible since the functional V does not admit minimizers. In [**50**], Lu and Otto proved the existence of a critical mass m_c such that if $m > m_c$ the constrained minimum problem for I_K has no minimizer, see also [**44**]. The advantage of the proof presented here is that the critical mass is explicitly calculated. The main theorem of this section is the following.

THEOREM 5.19. *If $m > 8 + 2K$ the problem*

$$\min\{I_K(E) : E \subset \mathbb{R}^3, \quad |E| = m\}$$

has no solutions.

Note that the above theorem gives also a lower bound for the critical threshold m_c. We define the quantity

$$\mathcal{I}_K[m] := \inf_{|E|=m} I_K(E)$$

Since the functional is not invariant under translation, we can not expect I_K to be subadditive. However the following weak form of subadditivity was proved in [**50**, Lemma 4].

LEMMA 5.20. *Let A, B real positive numbers. Then it holds*

$$\mathcal{I}_K[A+B] \le \mathcal{I}_K[A] + \mathcal{I}_0[B].$$

PROOF OF THEOREM 5.19. We use a strategy introduced in [**33**]. For $\omega \in \mathbb{S}^2$ and $l \in \mathbb{R}$ we set

$$H_{\omega,l} = \{x \in \mathbb{R}^3 : x\cdot\omega = l\}, \quad H^+_{\omega,l} = \{x \in \mathbb{R}^3 : x\cdot\omega \ge l\}, \quad H^-_{\omega,l} = \mathbb{R}^3 \setminus H^+_{\omega,l}.$$

Then, if $\Omega \subset \mathbb{R}^3$.

$$\Omega^+_{\omega,l} = \Omega \cap H^+_{\omega,l}, \quad \Omega^-_{\omega,l} = \Omega \cap H^-_{\omega,l}.$$

Given $m > 0$, let E be a minimizer of I_K under the constraint $|E| = m$. Using Lemma 5.20 and the minimizing property of E, we have

$$I_K(E) = \mathcal{I}_K[m] \le \mathcal{I}_K[|E^-_{\omega,l}|] + \mathcal{I}_0[|E^+_{\omega,l}|] \le I_K(|E^-_{\omega,l}|) + I_0(|E^+_{\omega,l}|). \tag{5.63}$$

The above inequality can be rewritten as

$$P(E) + V(E) - KR(E) \le P(E^-_{\omega,l}) + V(E^-_{\omega,l}) - KR(E^-_{\omega,l}) + P(E^+_{\omega,l}) + V(E^+_{\omega,l}). \tag{5.64}$$

Given $\omega \in \mathbb{S}^2$, for a.e. $l \in \mathbb{R}$ we have $P(E^-_{\omega,l}) = P(E; H^-_{\omega,l}) + \mathcal{H}^2(E \cap H_{\omega,l})$ and

$$V(E) = V(E^-_{\omega,l}) + V(E^+_{\omega,l}) + \int_{E^-_{\omega,l}}\int_{E^+_{\omega,l}} \frac{1}{|x-y|}dxdy.$$

Therefore, from (5.64) we obtain

$$\int_{E^-_{\omega,l}}\int_{E^+_{\omega,l}}\frac{1}{|x-y|}dxdy\le 2\mathcal{H}^2(E\cap H_{\omega,l})+K\int_{E^+_{\omega,l}}\frac{1}{|x|}dx.$$

Integrating this inequality with respect to l from 0 to ∞, we have

$$\int_0^\infty\int_{E^-_{\omega,l}}\int_{E^+_{\omega,l}}\frac{1}{|x-y|}dxdydl\le 2|E^+_{\omega,0}|+K\int_0^\infty\int_{E^+_{\omega,l}}\frac{1}{|x|}dx.$$

In order to estimate the last integral we observe, using the layer cake formula and Fubini's theorem, that

$$\int_{E^+_{\omega,0}}\frac{x\cdot\omega}{|x|}dx=\int_{E^+_{\omega,0}}\frac{1}{|x|}\int_0^\infty\chi_{(0,x\cdot\omega)}(t)dtdx=\int_0^\infty\int_{E^+_{\omega,0}}\frac{1}{|x|}\chi_{(t,\infty)}(x\cdot\omega)dxdt=\int_0^\infty\int_{E^+_{\omega,l}}\frac{1}{|x|}dxdl.$$

Thus, we have

$$\int_0^\infty\int_{E^-_{\omega,l}}\int_{E^+_{\omega,l}}\frac{1}{|x-y|}dxdydl\le 2|E^+|+K\int_{E^+_{\omega,0}}\frac{|x\cdot\omega|}{|x|}dx. \tag{5.65}$$

Interchanging the role of $E^-_{\omega,l}$ and $E^+_{\omega,l}$ in the (5.64), we have

$$I_K(E)=\mathcal{I}_K[m]\le\mathcal{I}_K[|E^+_{\omega,l}|]+\mathcal{I}_0[|E^-_{\omega,l}|]\le I_K(|E^+_{\omega,l}|)+I_0(|E^-_{\omega,l}|).$$

From which, arguing as in the proof of (5.65), we obtain

$$\int_{-\infty}^0\int_{E^-_{\omega,l}}\int_{E^+_{\omega,l}}\frac{1}{|x-y|}dxdydl\le 2\int_{-\infty}^0\mathcal{H}^2(E\cap H_{\omega,l})dl+K\int_{E^-_{\omega,0}}\frac{|x\cdot\omega|}{|x|}dx.$$

Summing this inequality with (5.65) we have

$$\int_{-\infty}^\infty\int_{E^-_{\omega,l}}\int_{E^+_{\omega,l}}\frac{1}{|x-y|}dxdydl\le 2|E|+K\int_E\frac{|x\cdot\omega|}{|x|}dx. \tag{5.66}$$

Using Fubini's theorem,

$$\int_{-\infty}^\infty\int_{E^-_{\omega,l}}\int_{E^+_{\omega,l}}\frac{1}{|x-y|}dxdydl=\int_E\int_E\int_{-\infty}^\infty\frac{\chi_{\{y\cdot\omega<l<x\cdot\omega\}}(y)}{|x-y|}dldydx=\int_E\int_E\frac{(\omega\cdot(x-y))_+}{|x-y|}dxdy.$$

Since for $a\in\mathbb{R}^3$

$$\int_{\mathbb{S}^2}|\omega\cdot a|d\omega=2\int_{\mathbb{S}^2}(\omega\cdot a)_+d\omega=2\pi|a|,$$

averaging over $\omega\in\mathbb{S}^2$ and using Fubini once again, we obtain

$$\frac{1}{4\pi}\int_E\int_E\int_{\mathbb{S}^2}\frac{(\omega\cdot(x-y))_+}{|x-y|}d\omega dxdy=\frac{1}{4}|E|^2,\qquad \frac{1}{4\pi}\int_E\int_{\mathbb{S}^2}\frac{|x\cdot\omega|}{|x|}d\omega dx=\frac{1}{2}|E|$$

and thus (5.66) yields

$$\frac{m^2}{4}\le\left(2+\frac{K}{2}\right)m.$$

From this inequality the result follows. □

Bibliography

[1] E. Acerbi, N. Fusco & M. Morini, *Minimality via second variation for a nonlocal isoperimetric problem.* Comm. Math. Phys. **322** (2013), 515-557.

[2] G. Alberti, R. Chorksi & F.Otto, *Uniform energy distribution for an isoperimetric problem with long-range interactions* Journal of the American Mathematical Society **22** (2009), 569-605

[3] L. Ambrosio, N. Fusco & D. Pallara, *Functions of Bounded Variation and Free Discontinuity Problems.* Oxford University Press, 2000.

[4] D. Bakry & M. Ledoux. *Lévy-Gromov isoperimetric inequality for an infinite dimensional diffusion generator*. Invent. Math. **123** (1995), 259–281.

[5] M. Barchiesi, A. Brancolini & V. Julin, *Sharp dimension free quantitative estimates for the Gaussian isoperimetric inequality.* Ann. Probab., **45** (2017), 668-697.

[6] M. Barchiesi, F. Cagnetti & N. Fusco, *Stability of the Steiner symmetrization of convex sets.* J. Eur. Math. Soc. **15** (2013), 1245,1278

[7] M. Barchiesi & V. Julin, *Symmetry of minimizers of a Gaussian isoperimetric problem* preprint (2018), http://cvgmt.sns.it/paper/3870/ .

[8] F. Barthe, *An isoperimetric result for the Gaussian measure and unconditional sets.* Bull. London Math. Soc. **33** (2001), 408–416.

[9] F. Bernstein, *Uber die isoperimetriche Eigenschaft des Kreises auf der Kugeloberflache und in der Ebene*, Math. Ann. **60** (1905), 117–136.

[10] V. Bögelein, F. Duzaar & N. Fusco, *A sharp quantitative isoperimetric inequality in higher codimension,* Rend. Lincei Mat. Appl. **26** (2015), 309-362.

[11] V. Bögelein, F.Duzaar & N. Fusco, *A quantitative isoperimetric inequality on the sphere,* Adv. Calc. Var. **10** (2017), 223-265

[12] E. Bombieri, E. De Giorgi & E. Giusti, *Minimal cones and the Bernstein problem.* Inventiones math., **7** (1969), 243–268

[13] M. Bonacini & R. Cristoferi *Local and global minimality results for a nonlocal isoperimetric problem on* $\mathbb{R}^n$ SIAM J. Math. Anal. **46** (2014) 2310-2349

[14] T. Bonnesen, *Uber die isoperimetrische Defizit ebener Figuren*, Math. Ann. **91** (1924),252–268.

[15] C. Borell, *The Brunn-Minkowski inequality in Gauss space.* Invent. Math. **30** (1975), 207-216.

[16] L. Brasco, G. De Philippis & B. Velichkov, *Faber-Krahn inequalities in sharp quantitative form,* Duke Math. J. **164** (2015), 1777-1831.

[17] E. A. Carlen & C. Kerce, *On the cases of equality in Bobkov's inequality and Gaussian rearrangement.* Calc. Var. Partial Differential Equations **13** (2001), 1-18.

[18] A. Chakrabarti & O. Regev, *An optimal lower bound on the communication complexity of gap-Hammingdistance.* STOC'11Proceedings of the 43rd ACM Symposium on Theory of Computing, **5160**, ACM, New York, (2011)

[19] L. Q. Chen & A. G. Khachaturyan, *Dynamics of simultaneous ordering and phase separation and effect of long-range Coulomb interactions.* Phys. Rev. Lett. **70** (1993), 1477–1480.

[20] R. Choksi, *Mathematical Aspects of Microphase Separation of Diblock Copolymers.* Surikaisekikenkyusko Kokyuroku, **1330** (2003) of Conference Proceedings, 10–17 .

[21] R. Choksi & M. A. Peletier, *Small volume fraction limit of the diblock copolymer problem: I. Sharp-interface functional*. SIAM J. Math. Anal., **42** (2010), 1334–1370.

[22] A. Cianchi, N. Fusco, F. Maggi, & A. Pratelli, *On the isoperimetric deficit in Gauss space.* Amer. J. Math. **133** (2011), 131-186.

[23] M. Cicalese & G.P. Leonardi, *A selection principle for the sharp quantitative isoperimetric inequality.* Arch. Ration. Mech. Anal. **206** (2012), 617-643.

[24] S. Cohen, F. Plasil & W. J. Swiatecki, *Equilibrium configurations of rotating charged or gravitating liquid masses with surface tension. II.* Annals of Physics, **82** (1974), 557–596.

[25] S. Cohen & W. J Swiatecki, *The deformation energy of a charged drop: IV. Evidence for a discontinuity in the conventional family of saddle point shapes.* Annals of Physics, **19** (1962), 67–164.

[26] P.G. de Gennes, *Effect of cross-links on a mixture of polymers.* J. de Physique – Lett.,**40** (2015), 69–72.

[27] G. De Philippis & E. Paolini, *A short proof of the minimality of Simons cone.* Rend. Sem. Mat. Univ. Padova **121** (2009) 233-241.

[28] A. Ehrhard, *Symetrisation dans l'espace de Gauss.* Math. Scand. **53** (1983), 281–301.

[29] R. Eldan, *A two-sides estimate for the Gaussian noise stability deficit.* Invent. Math. **201** (2015), 561-624.

[30] V. J. Emery & S. A. Kivelson, *Frustrated electronic phase-separation and high-temperature superconductors.* Physica C. **209** (1993), 597–621.

[31] A. Figalli, N. Fusco, F. Maggi, V. Millot & M. Morini, *Isoperimetry and stability properties of balls with respect to nonlocal energies.* Comm. Math. Phys. **336** (2015), 441-507.

[32] Y. Filmus, H. Hatami, S. Heilman, E. Mossel. R. O' Donnell, S. Sachdeva, A. Wan, & K. Wimmer, *Real Analysis in Computer Science: A collection of Open Problems*, available online (2014

[33] R.L. Frank, R. Killip & P.T. Nam, *Nonexistence of large nuclei in the liquid drop model.* Letters in Mathematical Physics, **106** (2016), 1033-1036.

[34] B. Fuglede, *Stability in the isoperimetric problem for convex or nearly spherical domains in* $\mathbb{R}^n$*.* Trans. Amer. Math. Soc. **314** (1989), 619-638.

[35] N. Fusco, *The quantitative isoperimetric inequality and related topics*,Bulletin of Mathematical Sciences **5** (2015), 517-607.

[36] N. Fusco & V. Julin, *A strong form of the quantitative isoperimetric inequality,* Calc. Var. Partial Differential Equations **50** (2014), 925-937.

[37] N. Fusco, F. Maggi & A. Pratelli, *The sharp quantitative isoperimetric inequality.* Ann. of Math. **168** 2008), 941-980.

[38] G. Gamow *G. Gamow. Mass defect curve and nuclear constitution.* Proceedings of the Royal Society of London. Series A, **126** (1930), 632–644.

[39] S. Glotzer, E. A. Di Marzio, & M. Muthukumar, *Reaction-controlled morphology of phase-separating mixtures. Phys. Rev. Lett.*, 74 (1995), 2034–2037.

[40] S. Heilman, *Low Correlation Noise Stability of Symmetric Sets.* arXiv:1511.00382v4, 2016.

[41] S. Heilman, *Symmetric convex sets with minimal Gaussian surface area* arXiv:1705.06643v1 (2017).

[42] V. Julin, *Isoperimetric problem with a Coulombic repulsive term.* Indiana Univ. Math. J. **63** (2014), 77–89.

[43] H. Knüpfer & C.B. Muratov, *On an isoperimetric problem with a competing nonlocal term I: The planar case.* Comm. Pure Appl. Math. **66** (2013), 1129-1162.

[44] H. Knüpfer & C.B. Muratov, *On an isoperimetric problem with a competing nonlocal term II: The general case.* Comm. Pure Appl. Math. **67** (2014), 1974–1994.

[45] V. F. Kovalenko & E. L. Nagaev, *Photoinduced magnetism.* Sov. Phys. Uspekhi, 29 (1986), 297–321.

[46] D.A. La Manna, *An Isoperimetric problem with a Coulombic repulsive and attractive term*, (2018) ESAIM:COCV, https://doi.org/10.1051/cocv/2018008

[47] D.A. La Manna, *Local minimality of the ball for the Gaussian perimeter*,(2017) Advances in calculus of Variations, https://doi.org/10.1515/acv-2017-0007.

[48] D.A. La Manna, *A short proof of a nonexistence result*, preprint (2018), http://cvgmt.sns.it/paper/3932/

[49] J. Lu & F. Otto, *An isoperimetric problem with Coulomb repulsion and attraction to a background nucleus.* Available at https://arxiv.org/pdf/1508.07172.pdf .

[50] J. Lu & F. Otto, *Nonexistence of minimizer for Thomas-Fermi-Dirac-von Weizsäcker model.* Comm. Pure Appl.Math. **67** (2014), 1605-1617.

[51] F. Maggi, *Sets of finite perimeter and geometric variational problems. An introduction to geometric measure theory.* Cambridge Studies in Advanced Mathematics **135**. Cambridge University Press, Cambridge, 2012.

[52] R.F. Mamin, *Domain structure of a new type near a photostimulated phase transitions: autosolitons.* JETP Lett., **60** (1994), 52–56.

[53] E. Mossel & J. Neeman, *Robust Dimension Free Isoperimetry in Gaussian Space.* Ann. Probab. **43** (2015), 971-991.

[54] E. Mossel, R. O'Donnell & K. Oleszkiewicz, *Noise stability of functions with low influences: invariance and optimality.* Ann. of Math. **171** (2010), 295-341.

[55] C.B. Muratov & A. Zaleski, *On an isoperimetric problem with a competing non-local term: quantitative results.* Ann. Global Anal. Geom. **47** (2015), 63–80.

[56] W.D. Myers & W.J. Swiatecki *Nuclear properties according to the Thomas-Fermi model.* Nuclear Physics A, **601** (1996), 141–167.

[57] E. L. Nagaev, *Phase separation in high-temperature superconductors and related magnetic systems*Phys. Uspekhi **38** (1995), 497–521

[58] I.A. Nyrkova, A.R. Khokhlov, & M. Doi, *Microdomain structures in polyelectrolyte systems: calculation of the phase diagrams by direct minimization of the free energy.* Macromolecules, **27** (1994), 4220–4230.

[59] R. O'Donnell, *Open problems in analysis of boolean functions.* arXiv:1204.6447v1 (2012).

[60] T. Ohta & K. Kawasaki, *Equilibrium morphology of block copolymer melts. Macromolecules.* **19** (2009), 2621–2632.

[61] R. Osserman, *Bonnesen-style isoperimetric inequalities*, Amer. Math. Monthly **86** (1979), 1–29.

[62] N.A. Pelekasis, J.A. Tsamopoulos, & G.D. Manolis, *Equilibrium shapes and stability of charged and conducting drops.* Phys. Fluids A: Fluid Dynamics, **2** (1990), 1328–1340.

[63] P. Simoes, *A class of minimal cones in* $\mathbb{R}^n$*,* $n \geq 8$*, that minimize area* PhD thesis, University of California, 1973.

[64] E. Spadaro, *Uniform energy and density distribution: diblock copolymers functional*, Interfaces and Free Boundaries, **11** (2009), 447-474

[65] F.H. Stillinger, *Variational model for micelle structure.* J. Chem. Phys., **78** (1983), 4654–4661.

[66] V.N. Sudakov, & B.S. Tsirel'son, *Extremal properties of half-spaces for spherically invariant measures* (Russian). Problems in the theory of probability distributions, II. Zap. Naučn. Sem. Leningrad. Otdel. Mat.Inst. Steklov. (LOMI) **41** (1974), 14-24.

[67] I. Tamanini, *Regularity results for almost minimal oriented hypersurfaces,* Quaderni del Dipartimento di Matematica dell'Università di Lecce, Lecce, 1984. See also http://cvgmt.sns.it/paper/1807/