DESIGNATED SCHOOL CHOICE

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DESIGNATED SCHOOL CHOICE

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ABSTRACT

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Turkish government changed the high-school placement system for several concerns in 2018. The government as a designer designates and orders schools to each student in terms of location. Then students reveal their preference list over these designated schools. The government desires students to be assigned to as possible as the closest schools. However, students' preference list is independent from the designation order. In this context, there is an incompatibility between the students' preferences and the concern of the designer. The thesis will solve this sort of incompatibility. Two-Stage-Generalized-Priority-Mechanism proposed in the thesis finds the set of all possible designer-optimal matchings. At the second-stage, TSGPM yields the best designer-optimal matching in terms of the students' preference list. At the last part of the thesis, strategic properties of the mechanism will be discussed.

ÖZET

ATANMIŞ OKUL SEÇİMİ

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2018 senesinde Türk hükümeti çeşitli sebeplerle lise yerleştirme sistemini değiştirmiştir. Bir dizayncı olarak hükümet, öğrencilerin büyük bir çoğunluğu için tam merkezi bir sistemde sıralamalara göre yerleştirilecek okulların yerine lokasyonlara göre öğrencilerin yerleştirileceği mahalle liseleri kurmayı hedeflemiştir. Bu sistemde, hükümet her bir öğrenciye lokasyonlarına göre okullar atar. Ardından öğrenciler kendilerine atanan okullar üzerine oluşturdukları tercih listesini sunarlar. Hükümet her bir öğrenciyi mümkün olan en yakın okula yerleştirmeyi arzular. Ancak öğrencilerin tercih listesi hükümetin isteğinden bağımsızdır. Böylece hükümetin isteği ve öğrencilerin tercihleri arasında bir uyumsuzluk oluşacaktır. Bu çalışma, bu uyumsuzluğu ortadan kaldıracaktır. Bu tezde önerilen İki-Etaplı-Genelleştirilmiş-Takaddüm-Mekanizması ilk etabında bütün dizayncı-optimal sonuçların kümesini bulur. Ardından ikinci etabında öğrencilerin tercih listelerine göre en-iyi dizayncı-optimal (kısıtlı etkin) eşleşmeyi verecektir. Bu tezin son kısmında ise, önerilen mekanizmanın stratejik analizi yapılacak ve olası manipülasyon durumları değerlendirilecektir.

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To Fırat Yılmaz Çakıroğlu

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1. INTRODUCTION

1.1 Matching Theory

Matching theory is a research area concerning matching, allocation and exchange of indivisible goods. Matching is one of the most recent important area of economics that focuses on who gets what. In a traditional way, economists consider the price mechanism as the determinant of who gets what. However, price mechanism in the allocation of indivisible goods such as school seats, jobs, kidneys and houses, is dysfunctional in the exchange of these sort of objects. The main reason behind that is not also indivisible goods, but also the fact that price mechanism is not legal or ethical in certain markets such as school choice, kidney exchange etc. Matching can involve two-sided markets such as men and women, students and schools, or firms and workers etc. Or matching can involve exchange of indivisible objects, such as dormitory rooms, transplant organs, courses, houses etc. Recently, matching theory and its application to market design have emerged as the most successful discipline of economic theory and applied mechanism design.

Polanyi (1944) points out that there are three types of economic relationship: *exchange, reciprocity and gift.* One of them might be dominant in a different conditions of the age and society. According to Polanyi, economic relationships were generally embedded in social relationships before Industrial Revolution. In other words, social relationships were determinant of economic conditions. Industrial Revolution brought along the mass production, expansion of market economy and then dominance of exchange. Mechanism design approach basically constructs an environment as the market and determine rules of exchange mechanism. As a special area of mechanism design, matching theory basically constructs exchange mechanisms without money, such as college admission, kidney exchange, etc. Hence, we can conclude that the main meaning of matching theory is to deepen Polanyi's exchange to the detriment of reciprocity and gift. Before the mechanism design approach to school choice and kidney exchange, students and patients had to accept the randomness of allocation or to need reciprocal relationships or gifts. Matching theory designs the environment and rules of exchange of such indivisible goods. The other name of mechanism design is market design. Actually, the intrinsic argument when using the term, market design, is the assertion about marketization of non-markets like school choice, transplant organs etc. We can interpret the recent great development in mechanism design studies and applications as a deepening of exchange relationship and continuation of what industrial revolution did.

School choice and such assignment problems are the principal application area of mechanism design. That is because school choice need to be designed the allocation and exchange of discrete resources in environments in which monetary transfers are not allowed. School choice is obviously an allocation problem because there are many students who desires to consume the schools and who compete with each other for desired schools, and there is a restricted number of schools and seats. Matching theory as a sub-discipline of microeconomic theory has several different axioms and properties about the allocation. A matching theorist tries to design a mechanism that yields the outcome desired by the designer for all possible combination of preferences, priorities etc. Some of these axioms may be competing with each other, in other words, there may be a trade-off between two different good properties. In such a case, a market designer tries to develop a mechanism that yields an outcome which satisfies one of these axioms, and then, satisfies the other axiom among the first. This is like a multi-object optimization problem in operation research literature. In the thesis, a special school choice problem in Turkey will be analyzed as a multi-object optimization problem by concerning the properties originated from matching theory literature.

1.2 The School Choice Problem in Turkey

Until 2018, students who graduated from secondary school had taken a fully-centralized exam scores and then they were assigned to high-schools in terms of their preferences and exam scores in Turkey. That assignment problem was a simple school choice problem. In that case, the government would satisfy stability, Pareto-efficiency and strategy-proofness by using school-proposing deferred acceptance algorithm proposed by Gale and Shapley (1962). However, a fully-centralized exam and then placement in terms of only exam scores have many intrinsic problem although it has an important advantage. What is the advantage of a fully-centralized measurement and placement system?

Countries like Turkey that have both regional[1] and income inequalities[2] have an important problem about education system: equality in opportunities. At first sight, a fully-centralized selection system eases the vertical mobility. A student coming from low socio-economic background may be successful in the centralized selection mechanism in an easier way than in a non-centralized system because students may not access the opportunities to prepare a portfolio for an application in the non-centralized system. Taş et al. (2016) show that the 9% of the success of students in Turkey is explained by socio-economic background although OECD average of the role of socio-economic background is 13%. Observe that the fact is despite relatively higher Gini coefficient in Turkey. To sum up, Turkey has eased vertical mobility by using centralized exam although Turkey has more economic inequality.

Obviously, there is a trade-off between vertical mobility and selection quality in terms of centralized selection system. Taş et al. (2016) show that fully-centralized system brings along more anxiety and unsuccessful scores in PISA tests although students spend lots of time out of school time to study for the exam.[3] Moreover, about 1.5 millions of students have entered this exam and so on, that is a multiple-choice exam. We can say that a multiple choice exam brings along a tendency to choose among options provided, not a tendency to create a new thing. On the other hand, a fully-centralized exam creates more apathy to extracurricular activities such as foreign language, music and sport etc.

The main negative impact of an assignment mechanism based on ranking is on quality distribution of schools. Since the government has assigned the most successful students to the most successful schools and the least successful students to the least successful schools, the gap between student quality of schools was an important issue. Before the government implemented the new system in 2018, it had carried out many educational policy such as obligatory rotation to all teachers and reduction in the number of types of high-schools. Moreover, the high-school education became compulsory in 2013. The government is trying to stabilize a compulsory high-school education. The new system should be thought with these policies. Moreover, this type of inequality in quality of schools is the main reason behind the average scores

[1]See Figure 4, Figure 5, Figure 8 of Taş et al. (2016)

[2]See the online dataset published by OECD; https://data.oecd.org/inequality/income-inequality.htm

[3]See Table 7 in afore mentioned report

in PISA. I think that main concern of the Turkish government is to stabilize high-school education for the reasons above.

Turkish government decided to change the high-school assignment system in 2018. In the new system, a certain set of successful schools are allowed for selecting through exam scores. Total capacity of these schools is about 20% of all students. These successful students are assigned in terms of the ranking coming from the centralized exam. The remaining students are assigned in terms of their location. The main concern of the Turkish government is to minimize transportation costs when using locations. The second one is to combine successful and unsuccessful students into the same school. Since the students from the same neighbourhood are assigned to the same schools, we can assume that their socio-economic background will be similar. It is an open question whether peer-effect in outcomes is negative or positive when we gather students who not only have the same socio-economic background, but also have the different abilities.

The government takes each student's location, then they draw three nested circle around the each student's location. Each circle for each student is designed in the manner that each circle contains one of each three different schools: Anatolian, Imam-Hatip (religious) and vocational high schools. The schools in the first circle are the closest schools to that student and so on. Hence, 9 different schools with 3 different types are designated to schools.

After the designation process, each student reveals his/her preference list over designated schools. However, one of the students may not desire to be assigned to the school which is in the first circle of that student. In other words, there may be an incompatibility between students' desire and the government's concern. The main problem in this thesis will be to solve this type of incompatibility.

That school choice problem in Turkey will be modelled in the thesis. Consider a school choice problem where the social planner desires to match students to schools in a controlled way. More specifically, each student is first desired to assign a certain subset of schools. If it is not possible, then to assign another subset of schools, and it goes like that. The government desires students to match students to as possible as the closest schools. However, students may not desire to be assigned to that school. How this incompatibility can be solved?

1.3 A Brief Outline of the Thesis

In the second chapter of the thesis, a related literature will be reviewed. That school choice problem in Turkey can be also interpreted as an important theoretical problem in school choice literature. Relationships between marriage market, housing market, college admission model and school choice problems will be introduced. Since I will use generalized priority mechanism to solve incompatibility between students' and the government's desires, I will introduce kidney exchange literature.

In the third chapter, I will introduce the school choice model that captures the real life school choice problem in Turkey in a more general way. The main focus will be on the designer-optimality as the definition of the government's concern and Pareto efficiency as the definition of students' desire. Proposition 1 shows that Pareto efficiency, non-wastefulness and fairness are separately incompatible with designer-optimality. The definition of constrained efficiency as the solution of the incompatibility mentioned above will be introduced.

In the fourth chapter, the mechanism design approach I proposed will be introduced. That mechanism is named as *Two-Staged Generalized Priority Mechanism*. Priority Mechanism introduced in Pairwise Kidney Exchange will be generalized in order to find a designer optimal matching. Theorem 1 shows that the first stage of TSGPM necessarily yields a designer-optimal matching. Theorem 2 shows that we can obtain the set of all designer-optimal matchings by applying the first stage of TSGPM for each permutation of priority ordering. After the first stage of TSGPM generates the set of all designer-optimal matchings, the second stage of TSGPM yields a constrained efficient matching by using again the same algorithm in the first stage. Theorem 3 shows that TSGPM yields a constrained efficient matching.

In the fifth chapter, vulnerability of TSGPM to possible manipulations will be analysed. The main question will be whether a student, as a strategic agent in the school choice problems, can be better-off to reveal his/her preference list untruthfully. Theorem 4 shows that if strategy space only includes revealed preference profiles, TSGPM is strategy-proof. However, parents can change their location in order to send their child to a better school. This is a real-life example in Turkey. Theorem 5 shows that TSGPM is open to manipulation by changing location. Or TSGPM is not strategy-proof if the strategy space includes location decision.

In the conclusion part of the thesis, whole work in the thesis will be briefly summarized. Then further possible studies will be discussed.

2. RELATED LITERATURE

Matching theory literature starts with the seminal paper written by Gale and Shapley (1962). They constructed a marriage market, as a two-sided market, in which men have strict preference list over women and women have strict preference list over men. They define stability and efficiency notions. An assignment of applicants to colleges is unstable if there are two applicants α and β who are assigned to colleges A and B, respectively, although β prefers A to B and A prefers β to α. As usual, the first question mark is about the existence of stable matching. If stable matching necessarily exists, the second question is how to find a stable matching. They introduce Deferred Acceptance Algorithm which guarantees the stable matching in the marriage market [1]. The second concern is Pareto efficiency, i.e. A matching is Pareto efficient if there exist no other matching such that every men or women is weakly better when one of them is strictly better-off. We know that stability implies Pareto-efficiency, stability is not necessarily implied by Pareto-efficiency. Matching theorists define men(women) optimality as the following: a matching is men(women) optimal stable matching if this matching is at least good as any other stable matching for all men(women). Moreover, Men(Women) Proposing Deferred Acceptance Algorithm yields not only stable assignment but also men(women) optimal assignment.[2] Moreover, Knuth (1976)'s Decomposition Theorem shows that men(women)-optimal stable matching is also women(men)-worst stable matching.

In college admission model, there exist two sides of agents referred to colleges and students. Each student desires to admit a college and has a preference list over colleges. Each college desires to satisfy a maximum number of students determined by their exogenously given capacity. They have preferences over individual students. If we divide each college with capacity q into the q separate pieces in which each

[1]See Theorem 1 in Gale and Shapley (1962)

[2]See Theorem 2 in afore paper

piece has a capacity of one, we can construct a relationship between marriage market and college admission model. Lemma 1 introduced by Roth and Sotomayor (1989) shows that a matching of a college admissions problem is stable if and only if the corresponding matching of its related marriage problem is stable. Hence, the student and college-proposing deferred acceptance algorithm yield stable matchings for each college admissions model by both Gale's First Theorem and Sotomayor's Lemma. We can replicate the optimality by the same way.

DAA satisfies stability and optimality. However, that mechanism is not perfect. Roth (1982)'s Impossibility Theorem shows that there exists no mechanism that is both stable and strategy-proof in the marriage market. Proposition 1 introduced by Alcalde and Barberà (1994) shows that there exists no mechanism that is Pareto-efficient, individually rational, and strategy-proof. Theorem 5 of Roth (1982) shows that student-optimal stable mechanism is strategy-proof in college admission problem.

After the marriage market and college admission model, matching literature goes into school choice studies. A school choice problem consists of a finite set of students, a finite set of schools, quota profiles for schools, strong preference profiles for students and a weak priority profiles for schools. In a school choice problem, the priorities of schools are exogenous. In other words, students are strategic agents but schools are objects to be consumed. So a school choice problem is a one-sided matching problem. It is the most important difference between the school choice problem and the college admission problem. If each school has a strong priority ordering, then it is obvious that a school choice problem is naturally isomorphic with college admission problem by letting each school's preference relation be its priority relation.

Balinski and Sönmez (1999) introduced university assignment problem in Turkey as a school choice problem. Firstly they show that the multi-category serial dictatorship algorithm that Turkish government used is equivalent to school-proposing deferred acceptance mechanism. That mechanism is fair, however, the mechanism has a number of serious problems, such as inefficiency, vulnerability to manipulation, and the potential of penalizing students for improved test scores. They propose Gale and Shapley's student proposing deferred acceptance mechanism to solve all these deficiencies and show that that mechanism is only the second best mechanism in that context.

The Boston mechanism as the mechanism used in Boston until 2005 assigns maximum number of students to their first choices based on their revealed preferences; next, maximum number of remaining students to their second choices; and so on. Abdulkadiroğlu and Sönmez (2003) showed that Boston mechanism is not necessarily stable. The second important problem of that widely used mechanism is its vulnerability to manipulation or lack of strategy-proofness. We can imagine that a parent with high-level socio-economic status has more information about the rules of assignment than a parent with low-level socio-economic status. Then the former has more chance to manipulate. One can say that strategy-proofness is important for equal opportunities. Hence, worries about commonly used Boston mechanism is a real-life problem and open to be designed by school choice mechanism. Gale-Shapley's Student-Proposing Deferred Acceptance Algorithm was implemented in Boston in 2006 and is in use. By Gale-Shapley's seminal paper, we know that the mechanism yields student-optimal stable matching and by Roth's paper it is strategy-proof. Moreover, Theorem 3 in Alcalde and Barberà (1994) shows that algorithm is the unique stable and strategy-proof mechanism in school choice problems.

For a school choice problem, there exists one more competing mechanism: Top Trading Cycles Mechanism. That mechanism was proposed by Shapley and Scarf (1974) for housing market although it is attributed to David Gale. Shapley and Scarf showed that the core is non-empty by using Top Trading Cycles Mechanism. What if agents would have weak preferences? Quint and Wako (2004) introduced Top Trading Segmentation Algorithm[3], which yields strict core allocation if it is non-empty in the case of weak domain of agents. The other, by Yılmaz (2009), presented a random mechanism satisfying individual rationality, ex-ante efficiency and no justified-envy. However, this mechanism proposed is not a strict core mechanism and is not strategy-proof. As a great progress in house allocation problem, Alcalde-Unzu and Molis (2011) proposed Top Trading Absorbing Mechanism. That mechanism allows weak Pareto-improvements through absorbing sets. They show that TTAS always selects a deterministic, individual rational, Pareto-efficient, in the core and strategy-proof allocation. In other words, that mechanism yields an allocation in the core if it is non-empty, but if the strict core is empty, that mechanism yields a Pareto-efficient mechanism on the contrary of Top Trading Segmentation.

Then, Abdulkadiroğlu and Sönmez (2003) introduces TTC into school choice problem. The intuition behind that mechanism is that it starts with students who have the highest priorities, and they trade the schools for which they have the highest priorities in case that a Pareto improvement is possible. Hence, they showed that The Top Trading Cycles Mechanism is Pareto-efficient. Roth (1982) showed that TTC is strategy-proof. Hence, Abdulkadiroğlu and Sönmez (2003) shows that The Top Trading Cycles Mechanism is strategy-proof. Let's remember DAA is also

[3]See Alcalde-Unzu and Molis (2011) to understand why that is not a mechanism

Pareto-efficient. The other remark is that these two competing mechanism cannot be ranked in terms of Pareto-domination.

Although given strict priorities in school choice problem, there exist no efficient and strategy-proof mechanism that Pareto-dominates DA, it has been empirically demonstrated that the efficiency loss of deferred acceptance algorithm can be significant in practice Abdulkadiroğlu, Pathak, and Roth (2009). That demonstration yields a trade-off between efficiency and strategy-proofness. What about school choice problems with weak priorities? Erdil and Ergin (2008) introduces the model in which students have strict preferences over schools, but schools have weak priorities over students. Student proposing deferred acceptance algorithm obviously does not work in the case of weak priorities because when schools get proposals, schools cannot choose the best due weak priorities. Erdil and Ergin (2008) showed that when the random tie-breaking is implemented to weak priorities and then DAA is applied, DAA may yield a stable matching that is Pareto-dominated by another stable matching. The question here is how to improve efficiency without violating stability. For that purpose, they define constrained efficiency notion as the following: "A matching is constrained efficient if that matching is not Pareto-dominated by another stable matching." To guarantee constrained efficient matching, i.e. all Pareto improvements by keeping stability, they introduces Stable Improvement Cycle Mechanism. They proves that that mechanism is constrained efficient and strategy-proof in the case weak priorities in school choice problems.

The problem modelled in that thesis is similar to constrained efficiency problem mentioned above. While stability is the main concern of Erdil and Ergin (2008), the main concern will be designer optimality. Both mechanism solves the constrained efficiency problem. The designation order structure is very similar to weak priority structure as you will see. Why the designation order structure is needed rather than weak priority structure? That is because I am defining an optimality notion over the designation ordering. While the paper tries to develop a mechanism that yields Pareto-undominated matching from the set of all stable matchings, the thesis chooses tries to develop a mechanism that yields Pareto-undominated matching from the set of all designer-optimal matchings. Since stability is a two-sided notion, finding a constrained efficient mechanism from the set of designer-optimal matchings requires a new mechanism approach. The TSGP mechanism uses the idea of priority mechanism in the following kidney exchange literature.

I will introduce a summary of two different kidney exchange mechanisms. A kidney exchange problem consists of a set of donor kidney-transplant patient pairs, a set of compatible kidneys for each patient, a strict preference relation over kidneys and waiting list. Roth, Sönmez, and Ünver (2004) introduces Top Trading Cycles and Chains Algorithm in order to yield an efficient matching for any chain rule and strategy-proof for some chain rules. Chains are important for kidneys from cadavers because TTCC provides a new patient giving his/her donor's kidney in return to take kidney from cadaver.

After the kidney exchange mechanism above was proposed, Roth, Sönmez, and Ünver (2005) introduces Pairwise Kidney Exchange problem. That problem arises because they argue that the experience of American surgeons suggest that preferences over kidneys can be indicated as a binary relation. In other words, transplants of compatible live kidneys have about equal survival probabilities, regardless of the closeness of tissue types between patient and donor. In this set-up, each patient is indifferent between all compatible kidneys and only pairwise exchanges are allowed due to incentive reasons. They define Pareto efficiency as maximality in this context. The main concern is to assign kidneys to the maximum number of patients. They show that the set of Pareto efficient matchings constitute a matroid structure. By using this property, they proved that the same number of patients will be matched at every Pareto-efficient matching.[4] They propose priority mechanism, known as *greedy mechanism* in the matroid literature. Priority mechanism necessarily yields not only priority respectful matching, but also Pareto-efficient (maximal) matching.

Lastly, Afacan, Bó, and Turhan (2018) introduces maximality and efficiency problem in a school choice set-up in which there are students with strict preferences and schools with strict priorities. They argue that maximality may be the most crucial concern of the designer. They prove that no fair mechanism is maximal and no strategy-proof mechanism is maximal. However, they introduces Efficient Assignment Mechanism that yields efficient and maximal matching. First step of EAM mechanism yields the set of all maximal matchings by using priority mechanism introduced by Roth, Sönmez, and Ünver (2005). This matching may not be efficient. Then the second step of EAM fixes the efficiency by improving welfare through improving cycles and chains introduced by Roth, Sönmez, and Ünver (2004). In other words, Afacan, Bó, and Turhan (2018) uses two different kidney exchange algorithms in order to guarantee maximality and efficiency in the school choice model. That thesis can be interpreted as the continuation of this paper. I will design a constrained efficient mechanism in a different, but similar environment from Erdil and Ergin (2008) by using priority mechanism introduced for kidney exchange problem.

[4] If exchange among more than two pairs was allowed, this property would not be satisfied. See Example 1 of Roth, Sönmez, and Ünver (2005)

3. MODEL

The school choice model in the thesis consists of the following elements:

- A finite set of students $I = \{i_1, ..., i_n\}$
- A finite set of schools $S = \{s_1, ..., s_m\}$
- A capacity vector $q = \langle q_{s_1}, ..., q_{s_m} \rangle$ where q_s is the number of available seats at school s
- A **designation order** structure $\tau_i(s)$ where $\tau_i : S \cup \emptyset \to \{1, ..., |S|, |S|+1\}$ is a function such that $s \in S$, $\tau_i(s)$ is the designation order of school s to student i. For instance, τ_i^{-1} is the first order designated schools of student i. For any $i \in I$ and $s \in S$, $\tau_i(s) < \tau_i(\emptyset) = |S|+1$. Let $\mathcal{T}$ be the set of all possible designation order structure for students.
- An exogenous priority function $f : \{1, 2, ..., n\} \to I$ be a bijection and $\mathcal{F}$ be the class of all such bijections. Note that $|\mathcal{F}| = n!$ and each of these bijections is called as the **priority ordering** of students. Notice that each of these bijection is independent from τ_i.
- A profile of strict preference of students $P = (P_i)_{i \in I}$, where P_i is the student i's preference relation over $S \cup \{\emptyset\}$ where $\emptyset$ denotes being unassigned. Let $\mathcal{P}$ denote the set of all possible preferences for students. A school s is **acceptable** to i if $sP_i\emptyset$, and unacceptable otherwise Let R_i denote *at-least-as-good-as* preference relation associated with P_i, that is: $sR_is' \Leftrightarrow sP_is'$ or $s = s'$.

Consider the tuple $\langle I, S, \tau, P, q, f \}$ as the market. A **matching** μ is an assignment of students to schools such that any student is assigned to at most one school. Formally, a matching is a function $\mu : I \to S \cup \{\emptyset\}$ where $\emptyset$ such that for each student i, $\mid \mu^{-1}(s) \mid \leq q_s$. A student i is **assigned** under μ if $\mu_i \neq \emptyset$. Let $\mid \mu_s \mid$ denote the number of students assigned to school s under μ and $\mid \mu \mid$ denote the total number of students assigned under μ. Let $\mathcal{M}$ be the set of all matchings.

A matching μ is **individually rational** if, for each student i, $\mu_i R_i \emptyset$. Matching μ is

non-wasteful if there is no student-school pair(i,s) such that $sP_i\mu_i$ and $|\mu_s| < q_s$. Matching μ is **fair** if there is no pair of students i; j such that $\mu_j P_i \mu_i$ and $i \succ_j j$. Matching is **stable** if it is individually rational, non-wasteful, and fair.[1]

A matching μ is **Pareto-superior** to μ' if, for each student i, $\mu_i R_i \mu_i'$, with this strictly holding for at least one student. In other words, matching μ **Pareto-dominates** matching μ'. A matching μ is **Pareto-efficiency** if no individually rational matching is Pareto-superior than μ.

A matching μ **respects designation orders more** than matching μ' if, for each student i, $\tau_i(\mu_i) \leq \tau_i(\mu_i')$, with this holding strictly for some student. A matching μ is **designer optimal** if no individually rational matching respects designations more than μ.

In other words, a matching μ is designer-optimal if there is no individually rational matching μ' such that for every student i, $\tau_i(\mu^{'}) \leq \tau_i(\mu)$ with this holding strictly for some student.

A **mechanism** ϕ is a systematic way that assigns a matching for each problem $\langle I, S, \tau, P, q, f\}$, that is, a function $\phi^f : \mathcal{P}^{|I|} \cup \mathcal{T}^{|I|} \to \mathcal{M}$. A mechanism simply selects a matching for every possible combination of preference profiles and designation order structure given a priority ordering f. A mechanism ϕ is efficient, designer-optimal, individually rational, stable if, for any problem $P \in \phi$, $\phi(P)$ is efficient, designer-optimal, individually rational, stable respectively.

Proposition 1 *Pareto efficiency, non-wastefulness and fairness are separately incompatible with designer-optimality.*

Proof. Suppose there are only one student and two schools in the market. Let $I = \{i\}$ and $S = \{a, b\}$ with $q = \langle 1, 1\rangle$. Let $\tau_i(a) = 1$ and $\tau_i(b) = 2$. Suppose $Pi : b, a, \emptyset$. In other words, the student prefers the outermost school than the closest school. However, the unique designer optimal matching is $\mu_i = a$. The unique designer optimal matching assigns school a to the student although student a prefers school b than school a and $\mid \mu_b \mid = 0$ Hence, non-wastefulness is incompatible with designer-optimality.

Suppose $I = \{i, j\}$ and $S = \{a, b\}$ with $q = \langle 1, 1\rangle$. Let student i be the top priority student and $\tau_i(b) = 1$ and $\tau_i(a) = 2$ and $\tau_j(a) = 1$. Then, the only designer-optimal matching assigns school b to student i. Hence, it would not be fair once student i's top choice is school a. Hence, fairness is incompatible with designer-optimality.

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[1] See Lemma 2 in Balinski and Sönmez (1999). Justified-envy is equivalent to the definition of fairness.

Suppose $I = \{i,j\}$ and $S = \{a,b\}$ with $q = \langle 1,1 \rangle$. Let $\tau_i(b) = 1$ and $\tau_i(a) = 2$ and $\tau_j(a) = 1$ and $\tau_j(b) = 2$. Then, the only designer-optimal matching assigns school b to student i and assigns school a to student j. However, matching such that assigns school b to student j and assigns school a to student i is Pareto-superior than the unique designer-optimal matching once student i's top choice a and student j's top choice b . Hence, Pareto-efficiency is incompatible with designer-optimality.

■

The main issue that will be interested in that thesis is on the incompatibility between Pareto-efficiency and designer-optimality. As you can see in the theorem above, there may be a conflict between the preference profiles of students and policies of the designer. How can we solve this conflict? That is a multi-object optimization problem. Let us prioritize the policy interest of the designer. How can we obtain the best solution for the students among the designer-optimal matchings? How can we guarantee the designer-optimal matching which is not Pareto-dominated by any other designer-optimal matching?

For this purpose, define an efficiency notion restricted by a certain set. Matching μ is **constrained efficient** if μ' Pareto-dominates μ, μ is respecting designer-optimality more than μ'. In other words, matching μ is constrained efficient if there exists no μ' such that μ' Pareto-dominates μ and is also designer-optimal.

The most substantial question is how to find a constrained efficient matching. A school choice problem with incompatibility between the preference profiles of students and the interest of the designer can be modelled in that way. The constrained efficient mechanism I proposed will be demonstrated in the next section.

4. MECHANISM DESIGNS

I am proposing Two-Stage Generalized Priority Mechanism (TSGPM) to solve the constrained efficiency problem. Firstly, a constrained efficient mechanism must guarantee the designer optimality because constrained efficiency is defined as the best matching among the designer-optimal ones. After that, that mechanism should find the designer-optimal matching that is not Pareto-dominated by any designer-optimal matching.

4.1 Priority Mechanism in Kidney Exchange

Roth, Sönmez, and Ünver (2005) introduces a kidney exchange model in which patients have binary preference profile over kidneys. A kidney may be compatible or incompatible to the patient, there is strict domination among two compatible kidneys. In that context, a Pareto efficient matching is a maximal matching. Roth, Sönmez, and Ünver (2005) proves that proposition through matroid structure.

A *matroid* is a pair (χ, Γ) such that such that χ is a set and Γ is a collection of subsets of χ (called the independent sets) such that

- if I is in Γ and $J \subset I$ then J is in Γ; and
- if I and J are in Γ and $\mid I \mid > \mid J \mid$ then there exists an $i \in I/J$ such that $J \cup \{i\}$ in in Γ.

They argue that all patients and the sets of simultaneously matchable patients as a collection of subsets of all patients constitutes a matroid structure.[1] Then, they show that the same number of patients will receive a transplant at every Pareto-efficient matching by using the second property of the matroid structure of matchable

[1] See Proposition 1 in the paper

patients. [2] Hence, maximality and Pareto-efficiency is equivalent in this context and the following priority mechanism yields a maximal matching by construction.

- Let ε^0 be the set of all possible matchings,
- In general for $k \leq n$, let $\varepsilon^k \subseteq \varepsilon^{k-1}$ be such that

$$\varepsilon^k = \begin{cases} \{\mu \in \varepsilon^{k-1} : \mu(k) \neq k\} & if \;\; \exists \mu \in \varepsilon^{k-1} \;\; s.t. \;\; \mu(k) \neq k \\ \varepsilon^{k-1} & otherwise \end{cases}$$

The patient with the first priority order chooses the set of matchings (ε^1) in which a compatible kidney is assigned to him/her from the set of all matchings(ε^0). Then, the second patient chooses the set of matchings(ε^2) in which a compatible kidney is assigned to him/her from the set of matchings (ε^1). The same procedure follows until the last patient and the set of maximal matchings (ε^n) is obtained.

4.2 Finding a Designer-Optimal Matching

The notion of efficiency in the kidney exchange problem with binary preferences over kidneys is the same with maximality, however, we cannot say the same thing for the designer-optimality in the designated school choice model. That is because there is no binary relation of designation orders. On the other hand, priority mechanism will be useful to find designer-optimal mechanism because of the weak ordering of designation orders.

Let us consider the following algorithm. Let first enumerate the students $I = \{i_1, i_2, .., i_n\}$ according to the exogenous priority function, f, and $d \in \{1, ..., D\}$ denotes the designation order of school s for student i, *i.e.* $\tau_i(s) = d$. Let ε^0 be the set of all feasible matchings. $\varepsilon^{k,d}$ be the subset of ε^{k-1} for the designation order d at step k. Let ε^k be the subset of ε^{k-1} at the end of Step k. Let ϕ^f be the set of remaining matchings at the last step n given priority function f. Consider the following generalized priority mechanism as the first stage of TSGPM:

[2] See Lemma 1. Lemma 1 brings along the fact that there is no trade-off between priority allocation and the number of kidneys assigned.

Step 1. Student i_1 chooses matchings in which s/he is assigned to the first designated schools (ε^1) from the set of all feasible matchings ε^0, i.e. $\varepsilon^1 = \{\mu \in \varepsilon^0 : \tau_{i_1}(\mu_{i_1}) = 1\}$.

Step 2. Student i_2 chooses matchings in which s/he is assigned to the lowest designated schools (ε^2) from ε^1. In a more detailed way, student i_2 chooses matchings in which s/he is assigned to the first designated schools ($\varepsilon^{2,1}$) from ε^1. If there is such a matching, $\varepsilon^{2,1} \neq \varepsilon^1$, then continue Step 3 with $\varepsilon^2 = \varepsilon^{2,1}$. If there is no such a matching, $\varepsilon^{2,1} = \varepsilon^1$, then student i_2 chooses matchings in which s/he is assigned to his/her second designated schools ($\varepsilon^{2,2}$) from ε^1. Follow the same process until $\varepsilon^{2,d} \neq \varepsilon^1$ is obtained. Then continue Step 3 with $\varepsilon^2 = \varepsilon^{2,d}$. i.e. ε^1,i.e. $\varepsilon^2 = \{\mu \in \varepsilon^1 : \tau_{i_2}(\mu_{i_2}) \leq \tau_{i_2}(\mu'_{i_2})\ \forall \mu' \in \varepsilon^1\}$.

Step k. Student i_k chooses matchings in which s/he is assigned to the first designated schools ($\varepsilon^{k,1}$) from ε^{k-1}. If there is such a matching, $\varepsilon^{k,1} \neq \varepsilon^{k-1}$, then continue Step $k+1$ with $\varepsilon^k = \varepsilon^{k,1}$. If there is no such a matching, $\varepsilon^{k,1} = \varepsilon^{k-1}$, then student i_k chooses matchings in which s/he is assigned to his/her second designated schools ($\varepsilon^{k,2}$) from ε^{k-1}. Follow the same process until $\varepsilon^{k,d} \neq \varepsilon^{k-1}$ is obtained. Then continue Step $k+1$ with $\varepsilon^k = \varepsilon^{k,d}$. i.e. ε^{k-1}, i.e. $\varepsilon^k = \{\mu \in \varepsilon^{k-1} : \tau_{i_k}(\mu_{i_k}) \leq \tau_{i_k}(\mu'_{i_k})\ \forall \mu' \in \varepsilon^{k-1}\}$

Follow the same rule until student i_n for the last designation order.

More formally,

- Let ε^0 be the set all possible matchings
- In general for $k \leq n$, let $\varepsilon^k \subseteq \varepsilon^{k-1}$ be such that

$$\varepsilon^k = \begin{cases} \{\mu \in \varepsilon^{k-1} : \tau_{i_k}(\mu_{i_k}) \leq \tau_{i_k}(\mu'_{i_k})\ \forall \mu' \in \varepsilon^{k-1}\} & if\ \ \exists \mu \in \varepsilon^{k-1}\ \ s.t.\ \mu_{i_k} \neq \emptyset \\ \varepsilon^{k-1} & otherwise \end{cases}$$

Theorem 1 *Given a priority ordering $f \in \mathcal{F}$ and designation order structure, the generalized priority mechanism, or the first stage of TSGPM, ϕ^f yields a designer-optimal matching, i.e. $\phi^f \in \xi\ \forall f \in \mathcal{F}$.*

Proof. Let $\mu \subseteq \phi^f$, i.e. the mechanism yields μ. Suppose that there exists a matching μ' such that μ' respects designations more than μ.

Then $\tau_i(\mu'_i) < \tau_i(\mu_i)$ for some $i \in I$ and $\tau_j(\mu'_j) \leq \tau_j(\mu_j)$ for all $j \in I$ by definition. We have two cases:

Case 1: Given priority ordering f, matching μ' is also produced by ϕ^f, i.e. $\mu' \in \phi^f$.

If Case 1 is true, then $\mu, \mu' \in \phi^f$. Since both $\mu, \mu' \in \phi^f$, both $\mu, \mu' \in \varepsilon^k$ for all Step k. Each student i_k chooses the best subset of ε^{k-1} by construction, then $\tau_i(\mu_i') = \tau_i(\mu_i)$ for each student $i_k \in I$. *Contradiction*

Hence, we have only Case 2.

Case 2: Given priority ordering f, matching μ' is not produced by ϕ^f, i.e. $\mu' \notin \phi^f$.

Since $\mu' \notin \phi^f$, μ' is necessarily eliminated at any step k by student i_k because of $\tau_i(\mu_i) < \tau_i(\mu_i')$ for student i_k. Hence, μ' does not respect designations more than μ. *Contradiction.*

Hence, given priority ordering f, each μ in ϕ^f is a designer optimal matching. ■

The first stage of TSGPM necessarily produces a subset of all designer-optimal matchings. Let ε be the set of all designer-optimal matchings. Theorem 1 proves $\phi^f \subseteq \varepsilon$ given priority function f. The next problem will be to find a constrained efficient matching. Before that, the next section will demonstrate how to obtain the set of all designer optimal matchings through the first stage of TSGPM.

4.3 The Set of All Designer-Optimal Matchings

The first stage of TSGPM can find a subset of all designer-optimal matchings by using a random priority ordering of students. However, given a random priority ordering, we obviously cannot argue that the outcome of first stage of TSGPM includes all designer-optimal matchings. We should compare designer optimal matchings with each other in order to understand whether there is a Pareto-domination or not. In order to find a constrained efficient matching, the students can order all possible designer-optimal matchings in terms of the preference profiles over designated schools. Hence, the intermediary stage of TSGPM that enables the transition from designer-optimal to constrained efficiency will be to obtain the set of all designer-optimal matchings.

Since the first stage of TSGPM above produces a designer-optimal matching for each priority ordering of students, we can obtain the set of all designer optimal matchings for each combination of possible $n!$ priority ordering if I can prove the following theorem:

Theorem 2 *For each designer-optimal matching, there exists at least one priority ordering of students which corresponds to that matching in the first-stage of TSGPM, i.e.* $\xi \subseteq \phi^{\mathcal{F}}$.

Proof. Let $\mu \in \varepsilon$ be a designer-optimal matching. We need to show $\exists\, f \in \mathcal{F}$ such that $\mu \subseteq \phi^f$.

Define again $\omega_d(\mu) = \{i \in I : \tau_i(\mu_i) = d\}$, i.e. ω_d is the set of students assigned to their dth designated school at μ.

The main idea for the proof will be to prioritize students according to their designation orders at μ. At the top of priority ordering, there are students assigned to one of their first designated schools at μ, student i whose $\tau_i(\mu_i) = 1$. The first problem is whether there necessarily exists a student assigned to his/her first designated school or not. Let prove the following Lemma.

Lemma 1 $\omega_1(\mu) \neq \emptyset$, *i.e. there exists at least one student who is assigned to his/her first-designated school at* μ.

Proof. Suppose $\omega_1(\mu) = \emptyset$, i.e. there exists no student who is assigned to his/her first designated school at μ. Then we can say that $\tau_i(\mu_i) > 1$ for all students $i \in I$.

Take one of the students say i_1. Say $\mu_{i_1} = s_1$ such that $\tau_{i_1}(s_1) > 1$. We know that there is a school s_2 such that $\tau_{i_1}(s_2) = 1$. Since μ is designer-optimal, there exists at least one student i_2 who is assigned to s_2 at μ. Otherwise, there would be another matching μ' where student i_1 is assigned to s_1 such that μ' respects designations more than μ.

Notice that there is a school s such that $\tau_{i_2}(s) = 1$. There are two cases: Firstly, $s \in S$ may be the school s_1. If that is the case, $\langle i_1, i_2 \rangle$ constitutes an improving cycle. Then μ would be not designer-optimal. Hence, there exists another school s_3 such that $\tau_{i_2}(s_3) = 1$. Let us continue the same argument for s_3. Since μ is designer-optimal, there exists at least one student i_3 who is assigned to s_3 at μ. Otherwise, there would be another matching μ' where student i_2 is assigned to s_3 such that μ' respects designations more than μ.

Continue the same process until the last student i_n . Obtain the following table from the process:

Remember the same arguments above for student i_n by induction. There is a school s such that $\tau_{i_n}(s) = 1$. Since μ is designer-optimal, there exists at least one student $i \in I$ who is assigned to s at μ. Because of the finiteness of the set of students I, s should be the same school in the set of school $s_t : \{t < n\}$. Notice that $\mu_{i_t} = s_t$ by construction.

- student $i_1 : 1 = \tau_{i_1}(s_2) < \tau_{i_1}(s_1)$
- student $i_2 : 1 = \tau_{i_2}(s_3) < \tau_{i_2}(s_2)$
- student $i_n : 1 = \tau_{i_n}(s_t) < \tau_{i_n}(s_n)$

Let me define an improving cycle. Each student points to one of his/her lower designated school. Each school points to the owner. A cycle is an ordered list of distinct schools and distinct students $\langle s_t, i_t, s_{t+1}, i_{t+1}, s_{t+2}, ..., i_k\rangle$ where s_1 points to i_1, i_1 points to s_2, . . . , s_k points to i_k, i_k points to s_1. If there exists an improving cycle under μ, μ is not a designer-optimal matching by definition.

Observe that schools students $\langle s_t, i_t, s_{t+1}, i_{t+1}, s_{t+2}, i_{t+2}, s_{t+3}, ..., i_n\rangle$ constitutes an improving cycle. Then μ is not designer-optimal. *Contradiction*

$\blacksquare$

Hence, we know that there exists at least one student at the first designated school at μ for each problem. Consider such students in $\omega_1(\mu)$. Let the bijective function f map $\omega_1(\mu)$ to $[1, |\omega_1(\mu)|]$. When the first stage of TSGPM is applied for only those students between $[1, |\omega_1(\mu)|]$. Then obtain $\varepsilon^{|\omega_1(\mu)|}$.

Observe $\mu \in \varepsilon^{|\omega_d(\mu)|}$ because $\tau_i(\mu_i) = 1 \, \forall i \in \omega_1(\mu)$. In other words, μ cannot be eliminated by students who is assigned to one of their first-designated schools at μ because there is no matching with lower designation order.

Who will be the next student? Obviously, we have to find the student who is not assigned to a lower designated school at matching $\mu' \in \varepsilon^{|\omega_1(\mu)|}$ than μ. If there did not exist such a student, then the student would eliminate μ whoever that student $i_{|\omega_1(\mu)|}$ is.

Define $S(\mu) = \{i \in I \backslash w_1(\mu) : \nexists \, \mu' \in \varepsilon^{|\omega_1(\mu)|} \, s.t. \, \tau_i(\mu'_i) < \tau_i(\mu_i)\}$.

Following lemma guarantees the existence of such a student.

Lemma 2 $S(\mu) \neq \emptyset$

Proof. Suppose $S(\mu) = \emptyset$. For each student except $\omega_1(\mu)$, we can find a matching which assigns that student to a closer school than at μ. More formally, $\forall \, j \in I \backslash w_1(\mu) : \nexists \, \mu\prime \in \varepsilon_{|\omega_1(\mu)|} such that \tau_j(\mu'_j) < \tau_j(\mu_j)\}$. If that is false, then we can find a student who will not eliminate matching μ.

Since $\mu\prime, \mu \in \varepsilon^{|\omega_1(\mu)|}$, then $\tau_i(\mu'_i) = \tau_i(\mu_i) = 1 \forall \, i \in w_\mu \forall \, \mu\prime \in \varepsilon^{|\omega_1(\mu)|}$.(1) In other words, $\varepsilon^{|\omega_1(\mu)|}$ includes only matchings that assign students between $[1, |\omega_1(\mu)|]$ to their first designated schools by construction.

Since μ is a designer optimal matchings, μ' should not respect designations more than μ.

Then $\exists\, k \in I$ such that $\tau_k(\mu_k) < \tau_k(\mu'_k)$ by definition.(2) Notice that student k is not in $\omega_1(\mu)$ by (1). Notice that student k is an element of $S(\mu)$ by using (1) and the definition of set $S(\mu)$. Observe $\mu'_j = \mu_k$.

Then we can repeat the same argument of student i for the student k. There exists a matching $\mu'' \in \varepsilon^{|\omega_1(\mu)|}$ such that $\tau_k(\mu''_k) < \tau_k(\mu_k)$.(3) Let us combine (2) and (3) and obtain the following for student k:

$$\exists \mu' \mu'' \in \varepsilon^{|\omega_1(\mu)|} \text{ such that } \tau_k(\mu''_k) < \tau_k(\mu_k) < \tau_k(\mu'_k).$$

Then there exists another student with the same result for a matching μ'''. Hence, we can continue that process until the last student n. See the following:

- student $j : \tau_j(\mu'_j) < \tau_j(\mu_j)$
- student $k : \tau_k(\mu''_k) < \tau_k(\mu_k) < \tau_k(\mu'_k)$

$\vdots$

-
- student $n : \tau_n(\mu^T_n) < \tau_n(\mu_n) < \tau_n(\mu^{T-1}_n)$

Follow the same argument in (2) for student n. There should be a student who is assigned to a lower designation at μ than at μ^T.

Remember that $\mu'_j = \mu_k, \mu''_k = \mu_{k+1}, ..., \mu^{T-1}_{n-1} = \mu_n$. Notice that μ^T_n should be equal to μ_j because of the finiteness of the set of students. However, these students and schools consist of an improving cycle. Since μ is designer optimal, existence of an improving cycle is a *contradiction* $\blacksquare$

Lemma 2 proves the existence of a student who will not eliminate μ from $\varepsilon^{|\omega_1(\mu)|}$. Let the bijective function f map $S(\mu)$ to $[|\omega_1(\mu)|, |S(\mu)|]$ randomly. Randomness will not violate the survival of μ because μ is not eliminated in a smaller set if it is not eliminated in a bigger set, $\varepsilon^{|\omega_1(\mu)|}$.

We can continue the same process for the other students in $I \backslash (\omega_1(\mu) \cup S(\mu))$ because the existence proof in Lemma 2 survives for the remaining students. By the finiteness of the set of students, we can get a complete bijective priority ordering including all students. Since any of them can eliminate matching μ at each step by construction, $\mu \in \phi^{\{}$.

■

Hence, we have proved that we can obtain all designer-optimal matchings if we apply the first-stage of TSGPM for all permutation of priority ordering, $n!$ times. Obviously, the number of applying the algorithm is weakly lower than $n!$ because the mechanism can produce more than one designer-optimal matchings with only one priority ordering. However, we can exactly obtain the set of all designer-optimal matchings by applying the first-stage of TSGPM for each permutation of the priority ordering.

The second-stage of TSGPM will be introduced in the next section. The designer obtains the set of all designer optimal matchings through the first-step and then intermediary step of TSGPM. We can say that the designer is indifferent between these matchings because there is no feasible matching that respects designations more than these matchings. Since we have the set of matchings that the designer is indifferent between, we can choose the best matching in terms of the preference profile of the students among designer-optimal matchings.

4.4 Finding a Constrained Efficient Matching

We can introduce the second part of the mechanism in order to guarantee constrained efficiency. We have the set of all designer-optimal matchings through the first-step of TSGPM and the intermediary stage. Let the set of all possible designer-optimal matchings §⟩ be $\varepsilon^0_{D.O.}$.

Students' strict preference list over schools are transformed to a weak preference list over matchings. If a student is assigned to a designated school at any matching, this matching is preferred to other matchings where that student is assigned to a less preferred school in terms of the preference list of the corresponding student. If a student is assigned to the same school at two different matchings, the student is indifferent between these two matchings. Matchings in which a student is unassigned is the least preferred matching for this student and that student is indifferent between these matchings. The revealed strict preference list for each student can be written as the weak preference list over matchings for each student.

For each student i,

- If $\mu_i \; P_i \; \mu'_i$ then $\mu \; P_i \; \mu\prime$.

- If $\mu_i = \mu'_i$ then $\mu \; R_i \; \mu\prime$.
- If $\mu_i = \emptyset$ then remove μ from the preference list of student i.

After that transformation from preferences over schools to preferences over matchings, the next step is to find a constrained efficient mechanism. Since we have the set of all designer-optimal matchings, $\varepsilon^0_{D.O.}$, constrained efficiency problem is reduced to find a Pareto-efficient matching from $\varepsilon^0_{D.O.}$. That is because constrained efficient matching is equivalent to Pareto-undominated matching among the set of designer-optimal matchings. Hence, we can use a Pareto-efficient mechanism in order to find a constrained efficient matching.

Let us consider the following mechanism as the second-stage of TSGPM. Let us again enumerate the students $I = \{i_1, i_2, .., i_n\}$. Say that is a random priority ordering $f \in \mathcal{F}$. The following mechanism has the same logic of the priority mechanism.

Step 1. The first student i_1 chooses his/her best subset of the set of all designer optimal matchings, i.e. $\varepsilon^1_{D.O.} \subseteq \varepsilon^0_{D.O.}$ where for each $\mu^1 \in \varepsilon^1_{D.O.}$, student i_1 has the preference list such that $\mu^1_{i_1} \; R_{i_1} \; \mu^0_{i_1}$ for all $\mu^0 \in \varepsilon^0_{D.O.}$.

Step 2. If $\varepsilon^0_{D.O.}$ includes only one matching, the process is terminated. If not, the second student i_2 chooses his/her best subset of the previous student's best subset, i.e. $\varepsilon^2_{D.O.} \subseteq \varepsilon^1_{D.O.}$ where for each $\mu^2 \in \varepsilon^2_{D.O.}$, student i_2 has the preference list such that $\mu^2_{i_2} \; R_{i_2} \; \mu^1_{i_2}$ for all $\mu^1 \in \varepsilon^1_{D.O.}$.

Follow the same rule until only one matching remains or the last student chooses his/her best.

The second-stage of TSGPM is a modified priority mechanism. More formally;

Consider $\varepsilon^0_{D.O.}$ (set of all designer-optimal matchings).

In general for $k \le n$, let $\varepsilon^k_{D.O.} \subseteq \varepsilon^{k-1}_{D.O.}$ be such that

$$\varepsilon^k_{D.O.} = \begin{cases} \{\mu \in \varepsilon^{k-1}_{D.O.} : \mu \in \varepsilon^{k-1}_{D.O.} \;\; s.t. \;\; \mu R_{i_k} \mu' \;\; \forall \mu' \in \varepsilon^{k-1}_{D.O.}\} & if \;\; \exists \mu \in \varepsilon^{k-1}_{D.O.} \;\; s.t. \;\; \mu_{i_k} \neq \emptyset \\ \varepsilon^{k-1}_{D.O.} & otherwise \end{cases}$$

Theorem 3 *Two-stage mechanism yields a constrained efficient matching.*

Proof. Let μ be the matching generated by TSGPM. Suppose μ is not constrained efficient. We know that a designer optimal matching μ' such that $\mu'_i \; R_i \; \mu_i$ for all $i \in I$, with this strictly holding for some student $i_k \in I$. Say student i_k has the

k^{th} priority. Student i_k chooses a subset $\varepsilon^k_{D.O.} \subseteq \varepsilon^{k-1}_{D.O.}$ and $\mu \in \varepsilon^k_{D.O.}$. Observe that $\mu' \notin \varepsilon^k_{D.O.}$, otherwise, student i_k would choose μ' since $\mu'_{i_k}\ P_i\ \mu_{i_k}$. Hence, μ' was eliminated at any step $m < k$ by the student i_m. That is why $\mu_{i_m}\ P_{i_m}\ \mu'_{i_m}$ by definition. *Contradiction.* ■

Hence, TSGP mechanism is a constrained efficient mechanism. The second-stage of TSGPM finds a designer optimal matching that is not Pareto-dominated by another designer-optimal matching. In other words, the mechanism satisfies the government's concern in the first-stage. Then the mechanism produces all matchings that satisfy the government's concern by using Theorem 2. Then the second-stage of TSGPM guarantees the best matching among the designer-optimals.

Notice that we could easily introduce the constrained maximality issue before the second-stage. A matching μ is constrained maximal if the matching contains the maximum number of assigned students among the designer-optimal matchings. Then we would find a constrained efficient matching by the same way if we define constrained efficiency as Pareto-undominated designer-optimal and maximal matching by another designer-optimal and constrained maximal matchings. The order of optimization would be designer-optimality, maximality and efficiency relatively.

What if the optimization order would be maximality, designer optimality and efficiency? Find maximal matchings, then find designer-optimal matchings among the maximal matchings and then find the constrained efficient matching among maximal and designer-optimal matching. How to find this type of constrained efficient matching is still an open question. If the primary concern of the designer was maximality, TSGPm would not respond to that concern.

5. STRATEGY PROOFNESS

In this chapter, the thesis will analyze the vulnerability of TSGPM to the possible profitable manipulation. Students may have two different ways to manipulate the mechanism. The first one is the conventional way to manipulate in the school choice literature: untruthful revealing of the preference list. In the first section, Theorem 6 shows that TSGPM does not permit a profitable manipulation if the strategic actions are defined over only preferences.

The second way to manipulate is to change his/her location by moving a closer area to a more desirable school. That situation is a real issue in Turkey because the parents may prefer to compensate the cost of moving another apartment in order to assign their child to a more desirable school. In the model, a student can change his/her designated schools and designation orders by moving another apartment. That is a realistic way to manipulate the mechanism. This sort of analysis brings along the link between preference list and designation orders because designation orders will change in terms of the preference lists.

5.1 Manipulation via Preference Profile

Theorem 4 *TSGPM is strategy-proof if the strategy space is limited by preference profile.*

Proof. Firstly, no strategic action has effect on $\varepsilon^{k-1}_{D.O.}$ as the choice set of student i by definition. In other words, whatever student i does, she is restricted to choose a subset of $\varepsilon^{k-1}_{D.O.}$. That is why students can only determine their own preference list, not designated schools and their designation ordering.

Suppose there exists a profitable deviation by revealing preference list untruthfully. Consider student i with k^{th} priority.

Let P_i be the real preference list of student i over schools and P'_i be the manipulative preference list of student i. Let $\mu \in \varepsilon^0_{D.O.}$ be the designer-optimal outcome of the truthful revealing (P_i) for student i in TSGPM and $\mu' \in \varepsilon^0_{D.O.}$ be the designer-optimal outcome of manipulation (P'_i) of student i in TSGPM. Suppose that $\mu'_i P_i \mu_i$.

In the second-stage of TSGPM, student i with priority k chooses $\varepsilon^k_{D.O.}$ from $\varepsilon^{k-1}_{D.O.}$. In order of the existence of possible profitable deviation, there has to exist $\mu' \in \varepsilon^{k-1}_{D.O.}/\varepsilon^k_{D.O.}$ such that $\mu'_i P_i \mu_i$. If μ' was in $\varepsilon^k_{D.O.}$, student i would be indifferent between μ and μ'. When the student i choose from $\varepsilon^{k-1}_{D.O.}$, the second-stage of TSGPM with P_i would choose μ' from $\varepsilon^{k-1}_{D.O.}$ because student i chooses the best subset and $\mu'_i P_i \mu_i$. In other words, $\mu' \in \varepsilon^k_{D.O.}$. *Contradiction*

That proof is valid for the case that unacceptable revealing is permitted. Since the mechanism assigns the best subset of student i, student i cannot obtain an extra gain by revealing some schools unacceptable. ■

5.2 Manipulation via Location

Another way to manipulate the mechanism is to change his/her location in order to get a more desirable school. This sort of manipulations may be a reason behind the inflated house market prices for houses around more desirable schools. That is an important issue for a school choice mechanism.

In the model, a student can not only change preference profile, but also his/her location to get a more desirable good. Desirability is based on the original preference list. It is known that students have no incentive to reveal their preferences untruthfully if the strategy space is restricted by the preferences by Theorem 6. However, the next proposition is not vulnerable to manipulations via location change.

The first-stage of TSGPM is the generalization of priority mechanism. Actually, we can think of weak ordering structure of designation orders as the weak domain of preferences in another set-up. Hence, the first-step of TSGPM can be think as a Pareto efficient mechanism that works in weak preference domain. The following result shows that the generalized priority mechanism, that is Pareto-efficient, is not strategy-proof.

Proposition 2 *If student i is permitted to change the location, TSGPM is not strategy-proof.*

Proof. The following example shows that there may be a situation in which a manipulation via location is profitable for a student.

Let be three students and three schools with one capacity, i.e. $I = \{i_1, i_2, i_3\}$, $S = \{a, b, c\}$ and $q = \langle 1, 1, 1 \rangle$.

Let the preference profile and designation order structure for each student be the following:

P_{s_1}	P_{s_2}	P_{s_3}
c	c	c
b	b	b
a	a	a

τ_{s_1}	τ_{s_2}	τ_{s_3}
a	a	b
b	b	c
c	c	a

- $\tau_{i_1}(a) = 1, \tau_{i_1}(b) = 2, \tau_{i_1}(c) = 3$
- $\tau_{i_2}(a) = 1, \tau_{i_2}(b) = 2, \tau_{i_2}(c) = 3$
- $\tau_{i_3}(b) = 1, \tau_{i_3}(c) = 2, \tau_{i_3}(a) = 3$

In that example, there is also one school for each designation order of each student. Apply the first-step of TSGPM (The Generalized Priority Mechanism).

Notice that we have $|I|! = 6$ different possible priority ordering f. In other words, $|\mathcal{F}| = 6|$. Let say $f_1 = \langle i_1, i_2, i_3 \rangle$, $f_2 = \langle i_1, i_3, i_2 \rangle$, $f_3 = \langle i_2, i_1, i_3 \rangle$, $f_4 = \langle i_2, i_3, i_1 \rangle$, $f_5 = \langle i_3, i_1, i_2 \rangle$, $f_6 = \langle i_3, i_2, i_1 \rangle$. Let us apply TSGPM for each of priority ordering above.

Fix the priority ordering f_1. At step 1, student i_1 chooses matchings (ε_1) that assigns him to first-designated school, a. Since each school has one capacity, ε_1 does not include matchings in which i_2 is assigned to school a. Then student i_2 chooses matchings(ε_2) where s/he is assigned to school b from ε_1. Hence, ε_2 includes matchings in which student i_1 is assigned to school a and student i_2 is assigned to school b. Since capacities are 1 for each school, the only matching that student i_3 can choose is the matching in which student i_3 is assigned to school c. Hence, ϕ^{f_1} yields a unique matching. Follow the same process for each priority ordering f and obtain the set of all designer-optimal matchings by using Theorem 2 and 3.

$$\phi^{f_1} = \left\{ \mu^1 = \begin{pmatrix} i_1 & i_2 & i_3 \\ a & b & c \end{pmatrix} \right\} \quad \phi^{f_2} = \left\{ \mu^2 = \begin{pmatrix} i_1 & i_2 & i_3 \\ a & c & b \end{pmatrix} \right\} \quad \phi^{f_3} = \left\{ \mu^3 = \begin{pmatrix} i_1 & i_2 & i_3 \\ b & a & c \end{pmatrix} \right\}$$

$$\phi^{f_4} = \left\{ \mu^4 = \begin{pmatrix} i_1 & i_2 & i_3 \\ c & a & b \end{pmatrix} \right\} \quad \phi^{f_5} = \left\{ \mu^2 = \begin{pmatrix} i_1 & i_2 & i_3 \\ a & c & b \end{pmatrix} \right\} \quad \phi^{f_6} = \left\{ \mu^4 = \begin{pmatrix} i_1 & i_2 & i_3 \\ c & a & b \end{pmatrix} \right\}$$

Then the set of all designer optimal matchings, ε has four different designer-optimal matchings, $\xi = \{\mu^1, \mu^2, \mu^3, \mu^4\}$. Suppose priority ordering f_3 is the priority ordering at the second-stage of TSGPM. Students will choose the best matchings according to preference list they revealed with respect to priority ordering f. Student i_2 chooses only matching μ_2 because school c is the best school for i_2 and μ_2 is the unique designer-optimal matching in which i_2 is assigned to school c. Then the algorithm is terminated and mechanism yields a unique matching, μ_2

Notice student i_1 is assigned to the least preferred school at matching μ_2. Remember that student i_1 cannot obtain more preferred school by changing preference profile according to Theorem 6. Suppose students are permitted to change their location. Student i_1 moves close to school c. Let τ'_{i_1} be new designation order function for student i_1.

- $\tau'_{i_1}(c) = 1, \tau'_{i_1}(b) = 2, \tau'_{i_1}(a) = 3$

Observe that there is a unique designer-optimal matching $\mu' = \begin{pmatrix} i_1 & i_2 & i_3 \\ c & a & b \end{pmatrix}$. For each priority ordering f, ϕ^f gives the matching μ'. Hence, whatever priority ordering of the second-stage of TSGPM is, μ' will be the unique constrained efficient matching produced by TSGPM. Observe that $\mu'_{i_1} P_{i_1} \mu^2_{i_1}$. Hence, student i_1 can obtain a more preferred school by changing his/her location even though s/he does not know the priority ordering of the second-stage of TSGPM and the others' preference profile. Moving close to school c is the dominant strategy for student i_1.

■

Proposition 2 suggests that TSGPM is not strategy-proof if the mechanism permits students to change their location. That is a real problem for the government as a designer. However, moving is an costly action for the agents. That cost may prevent the possible manipulation of the mechanism. That issue can be studied in further studies.

6. CONCLUSION

That special school choice problem in Turkey is about almost 1 millions students. Many other educational concern discussed in the introduction part requires this sort of mechanism design. The thesis tried to solve the incompatibility between students' preferences and the government's concern.

In the school choice literature, weak priorities is one of the most significant problems because satisfying a set of axioms become difficult when we have such a designation order structure used by the Turkish government as a different policy concern of weak priorities. Firstly, Two-Staged-Generalized-Priority-Mechanism solves designer optimality problem. I showed that the first-stage necessarily yields a designer optimal matching. Hence, TSGPM is a designer-optimal mechanism.

Then before the second stage, it is proven that for each designer-optimal matching there exists at least one priority ordering of students that the mechanism yields the corresponding matching. Hence, the set of all designer optimal matchings are guaranteed by applying the first-stage of the mechanism for $n!$ times.

The set of all designer-optimal matchings has all matchings that the government's concern about locational placement due to transportation cost, neighbourhood schools and peer effect etc. is satisfied. Before the second-stage, we have all such matchings. In the model, students have revealed their own preference profiles over designated schools. Then the second-stage of the mechanism chooses Pareto-undominated matching in the set of all designer optimal matchings. Hence, TSGPM is a constrained efficient mechanism.

After satisfying the government's concern, the mechanism proposed in the thesis finds the best matching among the matchings that satisfies the government's concern. That is the solution to incompatibility between preferences and the government's concern.

After that characterization of the mechanism, vulnerability to manipulation was discussed. According to findings in the fifth chapter, the mechanism is strategy-proof if we permit only changing preference profiles. In other words, no student can benefit from misreporting preference list or truth-telling is a weakly dominant strategy for all students under the mechanism ϕ. However, if we permit students changing their location, the mechanism ϕ can be manipulated by moving to a closer school. In other words, keeping location constant may be dominated by moving to a closer school for some student.

Notice that the thesis is not a complete work about both the theoretical problem of this sort of school choice problem and the special real-life problem in Turkey. Firstly, a percentage of efficiency gain provided by this constrained efficient mechanism should be determined by simulations. Further works will enrich the discussion of generalized priority mechanism.

Secondly, issue of changing location may be important for house prices. Nash equilibrium in a costly moving set-up would be important. The last thing is TSGPM requires to obtain the set of all designer-optimal matchings in order to get constrained efficient matching. However, that may be so costly way in terms of computer power. NP-hard analysis will enrich the discussion about that.

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