Symmetry, uniqueness and blow-up in some elliptic variational problems with critical exponent

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Contents

Abstract

This thesis contains some new results on several different nonlinear elliptic variational problems with critical exponent and the associated partial differential equations.

Paper [P1] gives a classification of positive solutions for the biharmonic Yamabe equation on $\mathbb{R}^n \setminus \{0\}$, $n \geq 5$, which possess a non-removable one-point singularity at the origin. It is shown that these solutions are up to dilation given by a negative power of $|x|$ times a one-parameter family of radial functions periodic in $\ln |x|$. This is in complete analogy to the well-known classification in the second-order case.

Paper [P2] extends this classification to include more general critical nonlinearities, e.g. of Hardy-Rellich type. A main achievement in this paper is the use of the moving planes method to prove radial symmetry for arbitrary positive singular weak solutions.

Paper [P3] classifies the solutions of a conformally invariant log-Sobolev equation on the sphere $\mathbb{S}^n$. It is shown that all positive finite-energy solutions are given by the conformal factors. This extends the known classification results concerning both minimizers of the associated functional and solutions to the corresponding equation with fractional exponent $s > 0$.

Paper [P4] is devoted to studying the asymptotics in the three-dimensional Brezis–Nirenberg problem as the lower-order perturbation tends to its critical value. Our setting includes non-constant critical potentials and non-constant perturbations. Through a careful expansion of the energy functional, we obtain the asymptotics of the energy to first subleading order in the perturbation parameter as well as precise blow-up asymptotics for a sequence of almost minimizers and a characterization of their possible concentration points.

In Paper [P5], the methods and results from [P4] are carried over to non-critical (with respect to the Brezis–Nirenberg problem) dimensions $N \geq 4$. This extends and sharpens previous results in the literature.

Zusammenfassung

Die vorliegende Arbeit enthält einige neue Resultate über eine Reihe nichtlinearer elliptischer Variationsprobleme und die zugehörigen partiellen Differentialgleichungen.

In dem Artikel [P1] werden positive Lösungen der biharmonischen Yamabe-Gleichung auf $\mathbb{R}^n \setminus \{0\}$, $n \geq 5$, klassifiziert, welche eine nichthebbare Punktsingularität im Ursprung besitzen. Die Lösungen sind bis auf Dilatation gegeben durch eine negative Potenz von $|x|$ mal eine einparametrige Familie radialer Funktionen, die periodisch in $\ln |x|$ sind. Dies ist vollständig analog zu der bekannten Klassifikation singulärer Lösungen im Fall zweiter Ordnung.

In dem Artikel [P2] wird diese Klassifikation auf eine allgemeine Klasse kritischer Nichtlinearitäten ausgedehnt. Eines der Hauptresultate dieser Arbeit ist der Beweis von Radialsymmetrie für beliebige positive singuläre schwache Lösungen mithilfe der Moving-Planes-Methode.

Der Artikel [P3] befasst sich mit der Klassifikation der Lösungen einer konform invarianten logarithmischen Sobolevgleichung auf der Sphäre $\mathbb{S}^n$. Es wird gezeigt, dass alle positiven Lösungen mit endlicher Energie durch die konformen Faktoren gegeben sind. Dies verbessert die bekannten Resultate über Minimierer des zugehörigen Funktionals bzw. über Lösungen der Gleichung für fraktionellen Exponenten $s > 0$.

Artikel [P4] befasst sich mit asymptotischer Analysis für das Brezis–Nirenberg-Problem, wenn der Term niederer Ordnung gegen seinen kritischen Wert konvergiert. Es werden auch nicht-konstante kritische Potentiale sowie nicht-konstante Störungen betrachtet. Mithilfe einer sorgfältigen Entwicklung des Energiefunktionals erhalten wir das asymptotische Verhalten der Energie zu erster nachführender Ordnung im Störungsparameter sowie die exakten Wachstumsraten und möglichen Konzentrationspunkte für eine Folge von (Fast-) Minimierern.

In Artikel [P5] werden die Methoden von [P4] im Fall nicht-kritischer (im Brezis–Nirenberg-Sinn) höherer Raumdimension $N \geq 4$ angewendet und analoge Resultate erzielt. Dies erweitert und verschärft frühere Resultate aus der Literatur.

Acknowledgements

In the very first place, I want to express my deep gratitude to my supervisor, Rupert Frank. I feel lucky and honored to have been able to spend close to four years working with him and learning from him in countless ways. I thank him for his unfailing helpfulness, his many advices which turned out useful, precise and adequate in just about any situation. His way of thinking about mathematics and his working rigor have truly been a source of inspiration to me over the past years for which I am very grateful.

I also thank all my other coauthors with whom I had the pleasure to collaborate during my thesis, namely Lukas Emmert, Hynek Kovařík and Hanli Tang, for their enthusiasm, their hard work and their good ideas.

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A sincere thanks goes to all the other members of my thesis committee: to Tobias Weth and Tianling Jin, who accepted the duty of external referees, and to Thomas Sørensen and Phan Thành Nam, who were willing to act as head examiner and substitute examiner, respectively. I thank all of them for their reliability and supportiveness at all stages of the final preparation of this thesis.

From when I started as a PhD student to the hand-in of this thesis, no less than 1295 days have passed. Spending most of those at the LMU math department was a pleasure thanks to all of you fellow mathematicians and friends, and the memories of Cabane evenings, self-made seminars, Schafkopf attempts, blackboard discussions, World Cup defeats and coffee sessions could fill many more pages than the few ones below. Whoever knows they are meant, thank you for your company and for your friendship!

My parents, Elisabeth and Robert König, have been through more with me than anybody else. I only begin to understand and appreciate the love and support they have been giving me until today, and certainly can never honor it enough. Danke, Mama und Papa, dass ihr mich den sein lasst, der ich bin, und mich ohne an euch selbst zu denken immer und überall unterstützt!

My last word of thanks I saved for Telse, who always brings me back to the non-mathematical ground, being the only person in the world who knows how to divide by zero. Thank you for inspiring, questioning and standing by me every day, in short: thank you for your love.

Eidesstattliche Versicherung

(Siehe Promotionsordnung vom 12.07.11, § 8, Abs. 2 Pkt. .5.)

Hiermit erkläre ich an Eidesstatt, dass die Dissertation von mir selbstständig, ohne unerlaubte Beihilfe angefertigt ist.

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Ort, Datum Unterschrift Doktorand/in

Formular 3.2

CHAPTER 1

Introduction

The purpose of this thesis is to give a unified presentation of the following publications[1] obtained in the course of my doctoral studies under the supervision of Prof. Rupert L. Frank.

[P1] *Classification of positive singular solutions to a nonlinear biharmonic equation with critical exponent* (with Rupert L. Frank), arXiv:1711.00776, Anal. PDE **12** (2019), no. 4, 1101–1113.

[P2] *Singular solutions to a semilinear biharmonic equation with a general critical nonlinearity* (with Rupert L. Frank), arXiv:1903.02385, Atti Accad. Naz. Lincei Rend. Lincei Mat. Appl. **30** (2019), no. 4, 817–846.

[P3] *Classification of solutions of an equation related to a conformal log Sobolev inequality* (with Rupert L. Frank, Hanli Tang), arXiv:2003.08135, submitted to Adv. Math.

[P4] *Energy asymptotics in the three-dimensional Brezis–Nirenberg problem* (with Rupert L. Frank, Hynek Kovařík), arXiv:1908.01331, submitted to Ann. Henri Poincaré C.

[P5] *Energy asymptotics in the higher-dimensional Brezis–Nirenberg problem* (with Rupert L. Frank, Hynek Kovařík), arXiv:1910.11036, Mathematics in Engineering, **2** (2020), no. 1, 119–140.

The structure of the rest of this thesis is as follows. The purpose of Chapter 1 is to introduce the mathematical background and to review the existing literature concerning the problems under study. More precisely, after introducing the basic objects as well as recalling the Sobolev and HLS inequalities, two main lines of research shall be discussed in more detail. The first, particularly relevant to Papers [P1]–[P3], is the classification of positive solutions to conformally invariant nonlinear elliptic equations, see Section 1.3. The second, presented in Section 2.4, is the blow-up asymptotics for Brezis–Nirenberg-type problems, which are studied in Papers [P4]–[P5].

[1] The publication of mine *Liquid Drop Model for nuclear matter in the dilute limit* (with L. Emmert, R. L. Frank), arXiv:1807.11904, accepted for publication in SIAM Journal on Mathematical Analysis, is not a part of this thesis for reasons of thematic coherence.

Chapter 2 is then devoted to presenting the results from Papers [P1]–[P5]. Building on the exposition in Section 1, we will introduce, state and comment on the main results of each paper and give a sketch of the main proof steps.

Finally, a short declaration on my contributions as a coauthor for each of the papers is given.

1.1. Fractional powers of the Laplacian and Sobolev spaces

The main topic of this thesis is to study the behavior of certain classes of solutions to some variational problems set on spaces of functions. The unifying structural feature that all of these problems share is that they are *elliptic*. While the notion of ellipticity can be defined for a general pseudodifferential operator [**78**], in all that follows we shall not go beyond considering the prototypical examples of elliptic differential operators. These are of course the *Laplace operator*, given in space dimension $n \geq 1$ by $\Delta := \sum_{i=1}^n \partial_{x_i}^2$, and its powers, which we shall introduce now. The following facts can mostly be found in the introductory article [**32**]. For the sake of brevity, we only discuss the operator acting on $\mathbb{R}^n$ rather than on domains and make no mention of the more general Sobolev spaces $W^{s,p}$ with $p \neq 2$. For a much more thorough treatment, see [**32**] or, e.g., the textbooks [**1, 61, 37**].

We start by defining, for $s > 0$, the *fractional Laplacian* $(-\Delta)^s$ acting on a function $u \in C_0^\infty(\mathbb{R}^n)$, say, by

$$(-\Delta)^s u := \mathcal{F}^{-1}(|\xi|^{2s} \mathcal{F}u(\xi)). \tag{1}$$

Here $\mathcal{F}$ is the Fourier transform defined below. This definition extends naturally to functions u lying in the space $H^{2s}(\mathbb{R}^n)$, with

$$H^s(\mathbb{R}^n) := \{u \in L^2(\mathbb{R}^n) \, : \, |\xi|^s \mathcal{F}u \in L^2(\mathbb{R}^n)\}.$$

The space $H^s(\mathbb{R}^n)$ is referred to as *Sobolev space (of order $s > 0$)*. Equipped with the scalar product $(u,v)_{H^s} := \int_{\mathbb{R}^n} (1+|\xi|^{2s}) \overline{\mathcal{F}u(\xi)} \mathcal{F}v(\xi) \, \mathrm{d}\xi$, it is a Hilbert space for every $s > 0$. Of some importance below will also be a variant of this space, namely the homogeneous Sobolev space $\dot{H}^s(\mathbb{R}^n)$ defined as the completion of $C_0^\infty(\mathbb{R}^n)$ with respect to the norm $\||\xi|^s \mathcal{F}u\|_{L^2(\mathbb{R}^n)}$.

For $s \in (0,1)$ and a sufficiently smooth function u, the fractional Laplacian can also be expressed as a singular integral by the formula

$$(-\Delta)^s u(x) := C_{n,s} P.V. \int_{\mathbb{R}^n} \frac{u(x) - u(y)}{|x-y|^{n+2s}} \, \mathrm{d}y$$

for some constant $C_{n,s} > 0$, where *P.V.* denotes the principal value integral. In the same parameter regime, as observed by Caffarelli and Silvestre in [**21**], $(-\Delta)^s$ can be viewed as the Dirichlet-to-Neumann operator of a degenerate elliptic extension problem on the upper half-space $\mathbb{R}^{n+1}_+$. For many more equivalent definitions $(-\Delta)^s$, see [**56**].

If $0 < s < n/2$, then the inverse operator $(-\Delta)^{-s}$ is given through convolution by

$$(-\Delta)^{-s} v(x) = c_{n,s} \int_{\mathbb{R}^n} |x-y|^{2s-n} v(y) \, \mathrm{d}y, \tag{2}$$

for a constant $c_{n,s} > 0$.

Notation. For $p \in [1, \infty)$, a measurable set $\Omega \subset \mathbb{R}^n$ and a measurable function $u : \Omega \to \mathbb{C}$, we shall denote by $\|u\|_{L^p(\Omega)} := \left(\int_\Omega |u|^p\right)^{1/p}$ its L^p norm. If the domain Ω is understood, we abbreviate $\|u\|_p := \|u\|_{L^p(\Omega)}$. The scalar product on $L^2(\Omega)$ is always denoted by $(\cdot, \cdot)$. For a function $u \in L^1(\mathbb{R}^n)$, we take its Fourier transform to be defined by

$$\mathcal{F}u(\xi) := \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{-i\xi \cdot x} u(x) \, \mathrm{d}x. \tag{3}$$

With this definition, $\mathcal{F}$ extends to a unitary map on $L^2(\mathbb{R}^n)$, see e.g. [**73**, Chapter IX].

For a differentiable function $\Phi : M \to N$ between Riemannian manifolds M and N and $x \in M$, we denote by $D\Phi(x)$ the differential of Φ at x. If $N = \mathbb{R}$, we write $D\Phi(x) = \nabla \Phi(x)$.

We denote by $\mathbb{S}^n := \{x \in \mathbb{R}^{n+1} : |x| = 1\} \subset \mathbb{R}^{n+1}$ the n-dimensional sphere and by $N := (0, ..., 0, 1)$ and $S := (0, ..., 0, -1)$ its north and south pole, respectively. We will always take $\mathbb{S}^n$ to be equipped with the surface measure induced by its embedding into $\mathbb{R}^{n+1}$.

Finally, we denote by $B_r(x_0) := \{x \in \mathbb{R}^n : |x - x_0| < r\}$ the open ball in $\mathbb{R}^n$ with radius $r > 0$ and center $x_0 \in \mathbb{R}^n$.

1.2. The Sobolev inequality and the HLS inequality

In this section we bring the calculus of variations into play by discussing the fundamental functional inequalities for the Laplacian and its powers, namely the classical inequalities of Sobolev and Hardy-Littlewood-Sobolev. The concepts and results introduced in relation with them will play an important role throughout the following.

For clarity, we start by discussing the historically significant case $s = 1$. The Sobolev quotient functional is given, for $u \in C_0^\infty(\mathbb{R}^n)$ say, by

$$\mathcal{S}_n[u] := \frac{\int_{\mathbb{R}^n} |\nabla u|^2 \, \mathrm{d}x}{\left(\int_{\mathbb{R}^n} u^{\frac{2n}{n-2}} \, \mathrm{d}x\right)^{\frac{n-2}{n}}} = \frac{\int_{\mathbb{R}^n} u(-\Delta u) \, \mathrm{d}x}{\|u\|_{\frac{2n}{n-2}}^2}. \tag{4}$$

The exponent $\frac{2n}{n-2}$ is chosen to make the functional invariant under the replacement $u(x) \to u(\lambda x)$. For this reason, the exponent $\frac{2n}{n-2}$ is said to be *critical.*

The *Sobolev inequality* states that

$$S_n := \inf \left\{ \mathcal{S}_n[u] \,:\, 0 \not\equiv u \in \dot{H}^1(\mathbb{R}^n) \right\} \tag{5}$$

is a strictly positive number. (A simple scaling argument using the family of functions $(u(\lambda\cdot))_{\lambda>0}$ shows that the infimum is zero if the exponent $\frac{2n}{n-2}$ in (4) is replaced by any other positive number.) This inequality, without finding the value of the sharp constant S_n, was first proved in the 1930s by Sobolev [**84**]. Twenty years later, Gagliardo [**41**] and Nirenberg [**66**] gave independently a more modern proof, still without the sharp constant. The value of the sharp constant was first given in unpublished work by Rodemich [**76**] and through a non-rigorous argument by Rosen [**77**], who noted that the functions

$$U_{x,\lambda}(y) := \left(\frac{\lambda}{1+\lambda^2|x-y|^2}\right)^{\frac{n-2}{2}} \qquad \text{for some} \quad \lambda > 0,\ x \in \mathbb{R}^n, \tag{6}$$

and their constant multiples realize the infimum in (5). Building on an early result of Bliss [**13**], a full proof involving the sharp constant S_n and the family of optimizers (6) was then given in independent work by Aubin [**7**] and Talenti [**88**],

We now turn to the case of general fractional exponents $s \in (0, n/2)$. The *fractional Sobolev inequality* asserts that $S_{n,s} := \inf\{\mathcal{S}_{n,s}[u] \,:\, 0 \not\equiv u \in H^s(\mathbb{R}^n)\}$ is a positive number, that is,

$$\|(-\Delta)^{s/2}u\|_2^2 \geq S_{n,s}\left(\int_{\mathbb{R}^n} u^{\frac{2n}{n-2s}}\,\mathrm{d}x\right)^{\frac{n-2s}{n}} \qquad \text{for all } u \in H^s(\mathbb{R}^n). \tag{7}$$

In the seminal work [**60**], Lieb determined the optimal constant $S_{n,s}$ for all $s \in (0, n/2)$ and proved that all optimizers for $\mathcal{S}_{n,s}$ must be of the form

$$u(y) = c\left(\frac{\lambda}{1+\lambda^2|x-y|^2}\right)^{\frac{n-2s}{2}} \qquad \text{for some} \quad c \neq 0,\ \lambda > 0,\ x \in \mathbb{R}^n. \tag{8}$$

In fact, Lieb proved this classification for optimizers of the (diagonal case of the) *Hardy-Littlewood-Sobolev inequality*, which says that

$$\iint_{\mathbb{R}^n\times\mathbb{R}^n} \frac{v(x)v(y)}{|x-y|^{n-2s}}\,\mathrm{d}x\,\mathrm{d}y \leq C_{n,s}^{\mathrm{HLS}}\|v\|_{\frac{2n}{n+2s}}^2 \qquad \text{for all } v \in L^{\frac{2n}{n+2s}}(\mathbb{R}^n). \tag{9}$$

Here we take $C_{n,s}^{\mathrm{HLS}}$ to be the sharp constant. The classification of optimizers for (9) yields the one for (7) by observing that the Sobolev and the HLS inequalities are *dual* to each other. This means that

$$\frac{1}{2}\|(-\Delta)^s u\|_2^2 = \sup_{v\in L^{\frac{2n}{n+2s}}(\mathbb{R}^n)}\left\{(u,v) - \frac{1}{2}c_{n,s}\iint_{\mathbb{R}^n\times\mathbb{R}^n}\frac{v(x)v(y)}{|x-y|^{n-2s}}\,\mathrm{d}x\,\mathrm{d}y\right\}, \tag{10}$$

i.e., up to a suitable normalization, the HLS double integral is the Legendre-Fenchel transform of $\|(-\Delta)^{s/2}u\|_2^2$. The relation (10) can easily be proved by using the inversion formula (2) and completing a square. From the duality relation (10) one can see that u optimizes the Sobolev inequality if and only if $v = u^{\frac{n+2s}{n-2s}}$ optimizes the HLS inequality and that moreover the optimal constants are related by $S_{n,s}^{-1} = c_{n,s}C_{n,s}^{\mathrm{HLS}}$. See [**61**, proof of Theorem 8.3] and the introductions of the papers [**23, 33**] for a more detailed explanation of duality.

A proof of the classification result (8) which works directly on the Sobolev inequality (7) instead of using duality, was later given in [**86**] for positive integers and in [**31**] for all fractional exponents $s \in (0, n/2)$. By now, many other proofs for (8) have been found, valid for some or all values of s and based on a wide variety of methods, e.g. mass transportation [**30**], the Brunn-Minkowski inequality [**14**], fast diffusion flows [**23**], inversion positivity [**39**] and second variation [**40**].

1.3. The Yamabe equation and the role of symmetries

We now turn to discussing the PDE arising from the Sobolev inequality (7), namely its Euler-Lagrange equation (up to normalization)

$$(-\Delta)^s u = u^{\frac{n+2s}{n-2s}} \qquad \text{on } \mathbb{R}^n. \tag{11}$$

Equation (11) is referred to as the *(fractional) Yamabe equation* in parts of the literature, due to its important role, for $s = 1$, in the Yamabe problem from differential geometry [**99, 93, 6, 79**].

By applying $(-\Delta)^{-s}$ to (11), we obtain the equation

$$u(x) = ((-\Delta)^{-s} u^{\frac{n+2s}{n-2s}})(x) = c_{n,s} \int_{\mathbb{R}^n} \frac{u^{\frac{n+2s}{n-2s}}(y)}{|x-y|^{n-2s}} \, \mathrm{d}y, \tag{12}$$

which is essentially equivalent to (11) (see [**27**, Section 4] for a detailed statement). By setting $u^{\frac{n+2s}{n-2s}} = v$, we recognize (12) in fact to be, up to normalization, the Euler-Lagrange equation of the HLS inequality (9).

Conformal invariance. The equations (11) and (12) possess a remarkable symmetry property, namely that of *conformal invariance*. Recall that a conformal map $\Phi : M \to N$ between two Riemannian manifolds M and N by definition preserves the angle under which any two given curves intersect. The group of conformal maps from $\mathbb{R}^n \cup \{\infty\}$ to itself is generated by the translations, rotations and dilations together with the inversion about the unit sphere. For more background on conformal maps, see [**61**, Section 4.4]. Any conformal map $\Phi : \mathbb{R}^n \setminus N \to \mathbb{R}^n \setminus \Phi(N)$, for some Lebesgue null-set N, induces an isometry from $L^{\frac{2n}{n-2s}}(\mathbb{R}^n)$ into itself by setting

$$u_\Phi(x) := \det D\Phi(x)^{\frac{n-2s}{2n}} u(\Phi(x)). \tag{13}$$

Conformal invariance now refers to the fact that u_Φ is a solution to equation (11), respectively (12), if and only if u is. This is easiest seen for (12). Indeed, the property

$$|\Phi(x) - \Phi(y)| = \det D\Phi(x)^{\frac{1}{2n}} |x-y| \det D\Phi(y)^{\frac{1}{2n}}$$

satisfied by any conformal map, implies via change of variables the transformation rule

$$((-\Delta)^{-s} u_\Phi^{\frac{n+2s}{n-2s}})(x) = \det D\Phi(x)^{\frac{n-2s}{2n}} (-\Delta)^{-s} u^{\frac{n+2s}{n-2s}}(\Phi(x)),$$

from which the conformal invariance of (12) easily follows.

By the same computation for the inverse stereographic projection $\mathcal{S} : \mathbb{R}^n \to \mathbb{S}^n \setminus \{S\}$, which is conformal, Lieb observed in [**60**] that the HLS inequality can be recast on the sphere $\mathbb{S}^n$, a reformulation that was crucial for his classification of optimizers. Indeed, given a function $u \in L^{\frac{2n}{n+2s}}(\mathbb{R}^n)$, similarly to (13) we define a function $v \in L^{\frac{2n}{n+2s}}(\mathbb{S}^n)$ by the relation

$$u(x) = \det D\mathcal{S}(x)^{\frac{n+2s}{2n}} v(\mathcal{S}(x)). \tag{14}$$

Then inequality (9) for u is equivalent to the inequality

$$\iint_{\mathbb{S}^n \times \mathbb{S}^n} \frac{v(\omega)\, v(\eta)}{|\omega - \eta|^{n-2s}} \,\mathrm{d}\omega\, \mathrm{d}\eta \le C^{\mathrm{HLS}}_{n,s} \|v\|^2_{L^{\frac{2n}{n+2s}}(\mathbb{S}^n)}, \tag{15}$$

for v with the same sharp constant $C^{\mathrm{HLS}}_{n,s}$, see e.g. [**61**, Theorem 4.5]. Moreover, via (14) Lieb's classification result for the extremizers can be reformulated by saying that v extremizes (15) if and only if

$$v(\omega) = c \left(\frac{\sqrt{1 - |\zeta|^2}}{1 - \zeta \cdot \omega} \right)^{(n+2s)/2} \tag{16}$$

for some $c \neq 0$ and $\zeta \in \mathbb{R}^{n+1}$ with $|\zeta| < 1$.

Classification of positive solutions and the method of moving planes. An interesting problem posed by Lieb in the above-mentioned work [**60**], whose by-now complete solution has sparked many interesting developments, is to classify the positive solutions $u \in L^{\frac{2n}{n-2s}}_{\mathrm{loc}}(\mathbb{R}^n)$ to (12). More precisely, the goal is to prove that they are all again of the form (8). In view of the Euler-Lagrange formalism, this clearly extends the classification of Sobolev optimizers discussed in the preceding section.

For the simplest case $s = 1$, this classification has been known since the 1980s, by work of Gidas, Ni and Nirenberg [**43, 44**] (with an additional decay hypothesis at infinity) and Caffarelli, Gidas and Spruck [**22**], see also [**67**]. Their main achievement was to prove that any positive solution is necessarily radially symmetric about some point in $\mathbb{R}^n$. Once this is known, equation (11) reduces to an ODE which can be solved explicitly. The radial symmetry is proved using the *method of moving planes*, which will be an important tool in what follows. This device goes back to an idea of Alexandrov [**2**] and was widely popularized by Serrin [**82**] and the above-mentioned works [**43, 44**]. Many more important references can be found in the review article [**16**] and in the discussion below.

Among the cases $s \neq 1$, the first classification result was given for the biharmonic case $s = 2$ by Lin [**62**] using a remarkable adaptation of the moving planes method, which was extended to cover all integers s less than $n/2$ by Wei and Xu [**97**]. See also [**92, 24, 98**] for related results. The classification result for the full fractional range was given for the integral equation (12) independently by Chen, Li and Ou [**27**] and Yanyan Li [**59**].

Inversion symmetry and moving spheres. We discuss now in some more depth the special role of the inversion symmetry since this will be a recurrent theme in Papers [P2] and

[P3]. In the context of the above-cited classification results, inversion symmetry has been made crucial use of in at least two different ways. Firstly, when applying the moving planes method, one can generate decay at infinity by passing to the *Kelvin transform* $u_{I_{\lambda,x_0}}$ of a solution u (in the notation of (13)) about a regular point x_0. Here, $I_{\lambda,x_0}(x) = \frac{\lambda^2(x-x_0)}{|x-x_0|^2} + x_0$ is the inversion about the sphere with center $x_0 \in \mathbb{R}^n$ and radius $\lambda > 0$. This strategy is carried out in [P2], see Section 2.2 below.

A more systematic and powerful way of exploiting inversion symmetry in the context of conformally invariant equations is the *method of moving spheres* set forth in [**68, 29, 58**]. Essentially, it replaces the reflection across a family of hyperplanes, central to the method of moving planes, by the inversion $(I_{\lambda,x_0})_{\lambda>0}$ about a family of spheres.

The crucial observation in the moving spheres method is that a function which is inversion symmetric with respect to spheres of every possible center $x_0 \in \mathbb{R}^n$ is *automatically* of the form (8). Surprising though this property may sound, it can be proved by a relatively simple 'calculus lemma', see [**58**, Lemma 2.5] for $s = 1$ and [**57**, Lemma 11.1] for general $s \in (0, n/2)$. This clarifies the special role of the functions (8) and allows for a more direct classification proof than the moving planes approach, which only yields the partial conclusion of radial symmetry up to translation. A somewhat more detailed description of the argument, applied in the setting of [P3], can be found in Section 2.3 below.

Among the above-cited literature, the work [**59**] uses the moving spheres approach, in an integral formulation. The alternative proof in [**27**] uses the Kelvin transform in connection with an integral version of the moving planes method, but also invokes inversion symmetry in a second step to conclude (8). The same spirit is already present in Lieb's classification argument for optimizers in [**60**], which also exploits the additional explicit conformal symmetries obtained by projecting to the sphere.

Classification of positive singular solutions. At this point it is natural to ask about solutions to (11) which violate the integrability assumption $u \in L^{\frac{2n}{n-2s}}_{\text{loc}}(\mathbb{R}^n)$. The simplest case to study is to assume a one-point singularity at the origin. That is, we ask which positive functions $u \in L^{\frac{2n}{n-2s}}_{\text{loc}}(\mathbb{R}^n \setminus \{0\})$ satisfy

$$(-\Delta)^s u = u^{\frac{n+2s}{n-2s}} \qquad \text{on} \quad \mathbb{R}^n \setminus \{0\}. \tag{17}$$

An explicit example of a solution to (17) which is not in the class (8) of full-space solutions is given by the pure-power function

$$u(x) = d_{n,s}|x|^{-\frac{n-2s}{2}}, \tag{18}$$

which for suitable $d_{n,s} > 0$ indeed solves (17).

For the easiest case $s = 1$, all positive singular solutions $u \in L^{\frac{2n}{n-2s}}_{\text{loc}}(\mathbb{R}^n \setminus \{0\})$ to the equation

$$-\Delta u = u^{\frac{n+2}{n-2}} \qquad \text{on } \mathbb{R}^n \setminus \{0\} \tag{19}$$

have been classified in the 1980s in the above-mentioned work of Caffarelli, Gidas and Spruck [**22**] and of Schoen [**81**]. The result is as follows. There exists $a_0 > 0$ and a family $(\varphi_a)_{(0,a_0]}$ of periodic functions $\varphi_a : \mathbb{R} \to \mathbb{R}_+$ such that if $u \in L^{\frac{2n}{n-2}}_{\text{loc}}(\mathbb{R}^n \setminus \{0\})$ is a positive solution to (17) with $u \notin L^{\frac{2n}{n-2s}}_{\text{loc}}(\mathbb{R}^n)$, then

$$u(x) = |x|^{-\frac{n-2}{2}} \varphi_a(\ln|x| + T), \qquad \text{for some} \quad a \in (0, a_0],\ T \in \mathbb{R}. \tag{20}$$

There are two key steps to the proof of this fact. Firstly, one proves that u is radial via the method of moving planes [**22**]. Then, through the logarithmic change of variables

$$u(x) = |x|^{-\frac{n-2}{2}} \varphi(\ln|x|) \tag{21}$$

equation (19) transforms to the ODE

$$-\varphi'' + \frac{(n-2)^2}{4}\varphi = \varphi^{\frac{n+2}{n-2}}. \tag{22}$$

Thanks to the autonomy of (22), the proof can be concluded by simple phase-plane analysis [**81**].

The classification (20) yields in turn a precise result on asymptotic radial symmetry for singular positive solutions u to (19) on the punctured ball $\{0 < |x| < 1\}$ which is in fact one of the main results in [**22**], namely that

$$u(x) = |x|^{-\frac{n-2}{2}} \varphi_a(x + T)(1 + o(1)) \qquad \text{as} \quad |x| \to 0.$$

This asymptotic expansion has been further refined in [**55**].

Secondly, the singular solutions (20) play an important role in the construction of constant scalar curvature metrics with prescribed isolated singularities by Mazzeo and Pacard [**65**], see also [**80**].

For singular solutions to (17) with general s, the situation is much more challenging. The best result to date is a two-sided growth bound $u \sim |x|^{-\frac{n-2s}{2}}$, which was proved in the innovative works [**20**] (for $s \in (0,1)$) and [**54**] (for $s \in (0, n/2)$) by using extension techniques and some blow-up analysis results from [**51, 52**].

In Paper [P1], we consider the biharmonic case $s = 2$. We prove a classification result for solutions to (17) with $s = 2$ which is fully analogous to (20). In fact, we show that, even though the equation is of fourth order, a suitable adaptation of 'phase-plane' analysis still works. Our methods and results from [P1] have since been used by other authors to study open questions related to biharmonic equations and Q-curvature [**72, 71, 3**]. See Section 2.1 for details.

1.4. Blow-up asymptotics in the Brezis–Nirenberg problem

In the following section we introduce the second main topic of this thesis, treated in Papers [P4] and [P5], which is the study of blow-up asymptotics for elliptic equations with critical exponent. In contrast to the above section, the problems are naturally set on bounded domains of $\mathbb{R}^N$. (We switch here to denoting the space dimension by capital $N \geq 3$, in accordance with the notation in [P4] and [P5] and the major part of the related literature.) Therefore, the role of symmetry and uniqueness will be less dominant in what follows. Nevertheless the reader will recognize the important role of the Sobolev inequality and its full-space optimizers. Moreover, the problems we study here only concern the second-order case $s = 1$, even though they are meaningful also for fractional exponents, see e.g. [**70, 35, 89, 83**].

The Brezis–Nirenberg problem. The interest in the questions we shall study grew out of the seminal paper [**17**] by Brezis and Nirenberg. There, the authors study under what conditions on the potential $a \in C(\overline{\Omega})$ the following nonlinear elliptic boundary value problem possesses a solution.

$$\begin{aligned} -\Delta u + au &= N(N-2)u^p && \text{in } \Omega, \\ u &> 0 && \text{in } \Omega, \\ u &= 0 && \text{on } \partial\Omega. \end{aligned} \tag{23}$$

Here, $\Omega \subset \mathbb{R}^N$ is a bounded open set, $N \geq 3$ and $p := \frac{N+2}{N-2}$ is the critical exponent with respect to the Sobolev inequality. Our choice of the normalization factor $N(N-2)$ is motivated by the fact that the functions $U_{x,\lambda}$ introduced in (6) satisfy $-\Delta U_{x,\lambda} = N(N-2)U_{x,\lambda}$ on $\mathbb{R}^N$.

By the variational approach, a solution to (23) may be found as an optimizer for the minimization problem

$$S(a) := \inf_{0 \not\equiv u \in H_0^1(\Omega)} \mathcal{S}_a[u], \tag{24}$$

where

$$\mathcal{S}_a[u] := \frac{\int_\Omega (|\nabla u|^2 + au^2)\, \mathrm{d}x}{\|u\|_{L^{p+1}(\Omega)}^2}. \tag{25}$$

Due to the lack of compactness in the embedding $H^1(\Omega) \hookrightarrow L^{p+1}(\Omega)$, the existence of a minimizer for (24) cannot be established by the direct method. To remedy this, Brezis and Nirenberg showed that a sufficient condition for the existence of a minimizer is the strict inequality [**17**, Lemma 1.2]

$$S(a) < S(0). \tag{26}$$

(This lemma is attributed to Lieb in [**17**].)

One of the main findings in [**17**] is that the validity of (26) depends rather strikingly on the space dimension. Indeed, for $N \geq 4$, (26) holds whenever $a(x) < 0$ for some $x \in \Omega$. Since the best Sobolev constant is never achieved on a bounded domain, one easily deduces from this that in fact the equivalence

$$S(a) \quad \text{is achieved} \quad \Leftrightarrow \quad S(a) < S(0) \tag{27}$$

holds. On the other hand, in dimension $N = 3$, there exists $\mu > 0$ such that if $\|a\|_\infty < \mu$, then $S_a = S_0$. We denote the largest possible of these numbers by $\mu_0 = \mu_0(\Omega)$ and shall refer to it as the *critical Brezis–Nirenberg constant* (of the domain $\Omega \subset \mathbb{R}^3$). The fact that (27) also holds if $N = 3$ was proved later by Druet [**34**].

The question of existence of a solution to (23) is even more subtle than the question for existence of a minimizer of (24). If the domain Ω is star-shaped, then (23) has no solutions for $a \geq 0$ by the Pohozaev identity. However, if Ω has non-trivial (in a suitable sense) topology, then (23) may have solutions regardless of the sign of a which do not arise as minimizers of (24) [**9**]. For a more detailed discussion, we refer to the excellent review article [**15**].

Blow-up asymptotics. The main question to be addressed here in connection with the Brezis–Nirenberg problem is the study of blow-up of low-energy solutions to (23). We start by discussing the case of constant potentials for simplicity. Thus we study solutions u_ϵ to

$$\begin{aligned} -\Delta u_\epsilon - (\mu_0 + \epsilon) u_\epsilon &= N(N-2) u_\epsilon^p && \text{in } \Omega, \\ u_\epsilon &> 0 && \text{in } \Omega, \\ u_\epsilon &= 0 && \text{on } \partial\Omega. \end{aligned} \tag{28}$$

If $N = 3$, we take here μ_0 to be the critical Brezis–Nirenberg constant. If $N \geq 4$, we take $\mu_0 = 0$. By the results from [**17**] and [**34**] presented above, (28) has a solution u_ϵ for all $\epsilon \to 0$ obtained via minimization of $S(-\mu_0 - \epsilon)$, while there is no such minimizer for $\epsilon = 0$. The natural question is thus whether the solutions u_ϵ blow up as $\epsilon \to 0$, in the sense that $\|u_\epsilon\|_{L^\infty(\Omega)} \to \infty$. If yes, can one describe in detail the blow-up profile, the blow-up speed and the points of concentration?

The study of this kind of questions has been initiated in [**4, 5**] and most of all by Brezis and Peletier in [**18**].

To describe the related results and conjectures we fix some notation first. For a function $a \in C(\overline{\Omega})$ such that $-\Delta + a$ is coercive as an operator on $L^2(\Omega)$, we let the (Dirichlet) *Green's function* G_a be the unique function on $\Omega \times \Omega$ that satisfies in the distributional sense

$$\begin{aligned} (-\Delta + a) G_a(x, \cdot) &= (N-2)|\mathbb{S}^{N-1}| \delta_x && \text{in } \Omega, \\ G_a(x, \cdot) &= 0 && \text{on } \partial\Omega. \end{aligned}$$

Moreover, we let $H_a(x,y) := \frac{1}{|x-y|^{N-2}} - G_a(x,y)$ be its *regular part.* Notice that $H_0(x,y)$ is harmonic, and hence smooth, in y. Through the resolvent formula one can deduce from this that $H_a(x,y)$ has a continuous extension to the diagonal $\{x = y\} \subset \Omega \times \Omega$. Thus we can define the *Robin function* $\phi_a(x) := H_a(x,x)$. The importance of the Robin function in connection with problems involving the critical Sobolev exponent has been first pointed out in [**79, 15, 8**].

In dimensions $N \geq 4$, the blow-up behavior can be considered completely understood by virtue of the following theorem.

Theorem 1 (Atkinson–Peletier, Brezis–Peletier, Han, Rey). *Let $N \geq 4$ and let (u_ϵ) be a family of solutions to* (28) *(with $\mu_0 = 0$). Suppose moreover that (u_ϵ) is a minimizing sequence for the Sobolev quotient. Then there is a constant c_N only depending on N and a point $x_0 \in \Omega$ such that as $\epsilon \to 0$, up to extraction of a subsequence,*

$$\lim_{\epsilon \to 0} \epsilon \|u_\epsilon\|_{L^\infty(\Omega)}^{\frac{2(N-4)}{N-2}} = c_N |\phi_0(x_0)| \qquad \text{if } N \geq 5$$

and

$$\lim_{\epsilon \to 0} \epsilon \log \|u_\epsilon\|_{L^\infty(\Omega)} = c_4 |\phi_0(x_0)| \qquad \text{if } N = 4.$$

Moreover, x_0 is a critical point for ϕ_0.

Notice carefully that in Theorem 1 the (u_ϵ) are not assumed to be minimizers of $S(-\epsilon)$. Indeed, in **[75]** Rey constructs a sequence of solutions to (28) satisfying the assumptions of Theorem 1 and concentrating about *any* critical point of ϕ_0. For ϵ small enough, however, these functions do *not* minimize $S(-\epsilon)$ unless x_0 is a global minimum of ϕ_0, which follows from the result in **[87]**. Still, a (much cruder) low-energy condition is imposed through the assumption that the (u_ϵ) are a minimizing sequence of the Sobolev quotient (i.e., of (4), with $\mathbb{R}^n$ replaced by Ω). Such an assumption is clearly necessary for a subsequence of the u_ϵ to have precisely one concentration point by a concentration-compactness argument, see **[85]**, **[63]**.

Theorem 1 has been proved in **[5]** when Ω is the ball and conjectured in **[18]** for general smooth open sets Ω. The conjecture has been proved subsequently in two independent works by Han **[46]** and Rey **[74]**.

The counterpart in dimension $N = 3$ is the following. We let $\mu_0 > 0$ be the critical Brezis-Nirenberg constant associated to the set $\Omega \subset \mathbb{R}^3$.

Conjecture 2 (à la Brezis–Peletier). *Let $N = 3$ and let (u_ϵ) be a family of solutions to* (23)*. Suppose moreover that (u_ϵ) is a minimizing sequence for the Sobolev quotient. Then there is a universal constant $c_3 > 0$ and $x_0 \in \Omega$ such that as $\epsilon \to 0$, up to extraction of a subsequence,*

$$\lim_{\epsilon \to 0} \epsilon \, \|u_\epsilon\|_{L^\infty(\Omega)}^2 = c_3 \left(\int_\Omega G_{-\mu_0}(x_0, y)^2 \, \mathrm{d}y \right)^{-1} . \tag{29}$$

Moreover, x_0 is a critical point of $\phi_{-\mu_0}$.

The statement of Conjecture 2 does not appear explicitly as a conjecture in **[18]**, but it is contained there in spirit. Indeed, its analogue for the related problem

$$\begin{aligned} -\Delta u &= u^{p-\epsilon} + \mu_0 u && \text{in } \Omega, \\ u &> 0 && \text{in } \Omega, \\ u &= 0 && \text{on } \partial\Omega, \end{aligned} \tag{30}$$

with critical constant $\mu_0 > 0$ and subcritical exponent $p - \epsilon$ (in the limit $\epsilon \to 0$) is stated in [**18**, Conjecture 3].

Both Conjecture 2 and [**18**, Conjecture 3] are so far only known when Ω is a ball; see [**18**, Theorems 2 and 3].

CHAPTER 2

Overview of the papers

2.1. Paper [P1]. Classification of positive singular solutions to a nonlinear biharmonic equation with critical exponent

In this paper, we study positive singular solutions to the critical biharmonic equation

$$\Delta^2 u = u^{\frac{n+4}{n-4}} \qquad \text{on} \qquad \mathbb{R}^n \setminus \{0\}, \tag{31}$$

for dimension $n \geq 5$. This is the special case $s = 2$ of the equation (17) introduced and discussed in Section 1.3.

Our main result is a complete classification of positive singular solutions to (31), which is fully analogous to (20) for the classical case $s = 1$. In view of the regularity theory from [**94**], it is no restriction to state it for classical solutions $u \in C^4(\mathbb{R}^n \setminus \{0\})$ only.

THEOREM 3. *Let* $a_0 = \left(\frac{n(n-4)}{4}\right)^{\frac{n-4}{4}}$. *There exists a family of functions* $(v_a)_{a \in [0,a_0]}$ *defined on* $\mathbb{R}$ *with* $\inf_{\mathbb{R}} v_a = a$ *and* $v_a(0) = \max_{\mathbb{R}} v$ *such that* $u \in C^4(\mathbb{R}^n \setminus \{0\})$ *is a positive solution to* (31) *if and only if it is of the form*

$$u(x) = |x|^{-\frac{n-4}{2}} v_a(\ln|x| + T), \tag{32}$$

for some $a \in [0, a_0]$ *and* $T \in \mathbb{R}$. *Moreover,* u *is strictly radial-decreasing if* $a > 0$.

If $a = a_0$, *then* $v_{a_0} \equiv a_0$.

If $a_0 > a > 0$, *then* v_a *is periodic with* $v_a(0) > a_0$ *and symmetric with respect to its local extrema.*

If $a = 0$, *then* $v_0 = c_n(\cosh t)^{-\frac{n-4}{2}}$, *for some explicit constant* c_n.

Let us give a few remarks concerning Theorem 3.

(1) This theorem answers affirmatively a recent conjecture on log-periodicity of singular solutions stated in [**45**]. Moreover, our result shows that the additional assumption made there on the positivity of the associated scalar curvature is unnecessary.

(2) By undoing the logarithmic change of variables $u(x) = |x|^{-\frac{n-4}{2}} v(\ln|x|)$, the two special solutions to (31) mentioned above are recovered as the endpoints of the

family v_a. Indeed, v_0 and its translates yield the family of smooth solutions from (8) (with suitable normalization constant c). On the other hand, v_{a_0} corresponds to the pure inverse power solution (18).

(3) Theorem 3 has since been used by several authors to extend some classical results from the case $s = 1$ to the biharmonic setting. Firstly, in the recent preprint [**72**] Ratzkin uses Theorem 3 to refine the asymptotic radial symmetry statement for local solutions of Jin and Xiong [**54**]. This constitutes the analogue of [**55**] for $s = 2$. Secondly, the same author uses in [**71**] the family of solutions (32) to prove a property of a fourth-order conformal invariant which corresponds to a result of Schoen [**81**] on the classical Yamabe invariant. Thirdly, in [**3**] Andrade and do Ò use a variant of our ODE analysis to extend the statement of Theorem 3 to fourth-order systems.

Besides these applications, we believe that Theorem 3 should be useful in the construction of constant Q-curvature metrics with isolated singularities, similarly to the works of Schoen [**80**] and Mazzeo and Pacard [**65**] for $s = 1$.

(4) The model power nonlinearity $g(u) = u^{\frac{n+4}{n-4}}$ on the right side of (31) can in fact be replaced by a large class of functions satisfying a suitable growth bound. Extending our analysis to these general nonlinearities is the main achievement of Paper [P2], described in Section 2.2 below.

Sketch of proof. We now explain the proof strategy for Theorem 3.

The first step is to observe that any solution u to (31) with a non-removable singularity at 0 must be radially symmetric. This was proved by Lin [**62**] using the method of moving planes.

By radial symmetry and the change of variables $u(x) = |x|^{-\frac{n-4}{2}} v(\ln |x|)$, the problem is thus equivalent to studying solutions $v \in C^4(\mathbb{R})$ to the ODE

$$v^{(4)} - Av'' + Bv = v^{\frac{n+4}{n-4}} \qquad \text{on } \mathbb{R}, \tag{33}$$

with coefficients

$$A = \frac{n(n-4)+8}{2} \quad \text{and} \quad B = \frac{n^2(n-4)^2}{16}. \tag{34}$$

A similar change of variables could be performed if the right side of (31) was given by any power nonlinearity u^p with $p > 1$. However, only the critical exponent $p = \frac{n+4}{n-4}$ leads to an autonomous ODE [**42**, Section 3], which will be essential for the following analysis.

The existence part of Theorem 3 is proved via a relatively straightforward shooting argument for the ODE (33). (Alternatively, in [**45**], existence of a family of positive periodic solutions parametrized by the period length rather than the minimal value is proved using a variational argument.)

We focus in what follows on the proof of the uniqueness statement of Theorem 3, which contains in fact the main innovation. The idea is to carry out a phase-plane analysis similar to the second-order case $s = 1$ for the solutions to the ODE (33). At first sight, it cannot be expected that a similar analysis can give the desired result, e.g. because of the additional degrees of freedom in the 'phase-plane' and because the maximum principle is in general unavailable. Indeed, autonomous fourth-order ODEs do exhibit in general a much more diverse behavior than second-order ones, and their solutions may include non-monotone kinks, multibump periodic solutions and chaotic solutions, see e.g. [**69**]. Overcoming these difficulties is the main achievement in [P1].

The key is to exploit the special structure of the ODE (33). Most importantly, for all $n \geq 5$, its coefficients A and B satisfy the structural condition

$$A > 0 \qquad \text{and} \qquad 4B < A^2.$$

This means that (33) can be written in the factorized form

$$\left(\frac{\mathrm{d}^2}{\mathrm{d}x^2} - \lambda\right)\left(\frac{\mathrm{d}^2}{\mathrm{d}x^2} - \mu\right) v = v^{\frac{n+4}{n-4}}, \tag{35}$$

with positive coefficients $\lambda > \mu > 0$. Using this second-order-type structure, one can prove the following comparison lemma, which is central to our argument.

LEMMA 4. *Let $v, w \in C^4(\mathbb{R})$ be nonnegative solutions to* (33) *such that*

$$\begin{aligned} v(0) &\geq w(0), \\ v'(0) &\geq w'(0), \\ v''(0) - \mu v(0) &\geq w''(0) - \mu w(0), \\ v'''(0) - \mu v'(0) &\geq w'''(0) - \mu w'(0). \end{aligned}$$

Then $v \equiv w$.

Lemma 4 goes back to [**19, 95**]. In these works, however, the strong additional a priori assumption of boundedness of v and w was imposed. By combining their proof with a result of Gazzola and Grunau [**42**], which asserts that nonnegative solutions can only blow up in finite time, we obtain Lemma 4 for arbitrary nonnegative solutions v, w. This simplified argument is only exposed in Lemma 13 of Paper [P2], but we chose to present it here for clarity. We point out that in the published version of Paper [P1], we do not use this last argument, but prove boundedness of v, w instead by an energy argument.

Lemma 4 immediately implies the following corollary.

COROLLARY 5. *Let $v, w \in C^4(\mathbb{R})$ be solutions to* (33) *and suppose that $v(t_0) = w(t_0)$ and $v'(t_0) = w'(t_0)$ for some $t_0 \in \mathbb{R}$. Then $v \equiv w$.*

Corollary 5 states the surprising fact that solutions to (33) which are defined on all of $\mathbb{R}$ are uniquely determined by only two instead of four initial values! In particular, if t_0 is a local extremum of v, by considering $w(t) = v(2t_0 - t)$, we obtain symmetry of v with respect to any local extremum t_0. By repeated application of this, any solution with at least two local extrema is periodic. Using these facts, the proof can be concluded by lifting the standard second-order phase-plane arguments.

2.2. Paper [P2]. Singular solutions to a semilinear biharmonic equation with a general critical nonlinearity

In this paper, we extend the techniques from Paper [P1] to prove a classification result analogous to (3) for positive singular solutions to the class of equations

$$\Delta^2 u = \frac{1}{|x|^{\frac{n+4}{2}}}\, g(|x|^{\frac{n-4}{2}} u) \qquad \text{in } \mathbb{R}^n \setminus \{0\} \tag{36}$$

for a general nonlinearity g satisfying the following natural assumptions.

$$\begin{cases} g \in C^1(\mathbb{R}_+),\ g > 0,\ \lim_{t\to 0} g(t) = 0, & \\ \frac{g(t)}{t} < g'(t) \le \frac{n+4}{n-4}\frac{g(t)}{t} & \text{for all } t > 0, \\ \beta := \lim_{t\to 0} g'(t) < \frac{n^2(n-4)^2}{16}, & \\ g(t) \ge ct^q & \text{for all } t \ge 1, \quad \text{for some } q > 1,\ c > 0. \end{cases} \tag{37}$$

Choosing $g(t) = t^{\frac{n+4}{n-4}}$ recovers the equation (31) treated in Paper [P1], but the conditions (37) also allow for the inclusion of a Hardy-Rellich term $|x|^{-4}u$ and a general subcritical power u^q (or a sum thereof), as in

$$\Delta^2 u = \beta |x|^{-4} u + |x|^{-\frac{n+4}{2} + q\frac{n-4}{2}}\, u^q \qquad \text{in } \mathbb{R}^n \setminus \{0\}, \tag{38}$$

with $0 < \beta < \frac{n^2(n-4)^2}{16}$ and $1 < q \le \frac{n+4}{n-4}$.

The way the nonlinearity in (36) is written becomes natural when passing to radial logarithmic coordinates. Indeed, if u is a radial solution to (36), then by the change of variables $u(x) = |x|^{-\frac{n-4}{2}} v(\ln |x|)$, (36) is equivalent to

$$v^{(4)} - Av'' + Bv = g(v) \qquad \text{in } \mathbb{R}, \tag{39}$$

with A and B as in (34).

The assumptions (37) ensure that the function $G(u) := \int_0^u g(t)\,\mathrm{d}t - B\frac{u^2}{2}$ has a unique global minimum on $(0, \infty)$, which we denote by $a_0 > 0$. Moreover, we denote by $\mu = \frac{1}{2}(A - \sqrt{A^2 - 4(B-\beta)})$ the smaller one of the two positive roots of the characteristic polynomial $\xi^4 - A\xi^2 + (B - \beta)$ of the linearization of (39). Then our classification result reads as follows.

Theorem 6. *There exists a family of functions $(v_a)_{a\in[0,a_0]}$ defined on $\mathbb{R}$ with $\inf_{\mathbb{R}} v_a = a$ and $v_a(0) = \max_{\mathbb{R}} v_a$ such that the following holds. The function $u \in C^4(\mathbb{R}^n \setminus \{0\})$ is a positive solution to* (31) *if and only if it is of the form*

$$u(x) = |x|^{-\frac{n-4}{2}} v_a(\ln |x| + T),$$

for some $a \in [0, a_0]$ *and* $T \in \mathbb{R}$. *Moreover,* u *is strictly radial-decreasing if* $a > 0$.

If $a = a_0$, *then* $v_{a_0} \equiv a_0$.

If $a_0 > a > 0$, *then* v_a *is periodic with* $\sup_{\mathbb{R}} v_a > a_0$ *and symmetric with respect to its local extrema.*

If $a = 0$, *then for any* $\epsilon > 0$ *there is* $C > 0$ *such that* $v_0(t) \leq Ce^{-(\sqrt{\mu}-\epsilon)|t|}$.

This classification result is thus identical to the one from Theorem 3, up to the fact that for general g there is no explicit expression for the homoclinic solution v_0 and we can only give exponential decay bounds.

While in the ODE part of the proof of Theorem 6, the only novelty compared to Paper [P1] is the derivation of the exponential decay estimates, a substantial amount of work is required to prove radial symmetry of any solution u to (36).

We use a variant of the method of moving planes which goes back at least to Terracini [**90**] and relies mostly on integral estimates. Unlike the existing symmetry proofs in the literature for higher-order equations [**62, 97**], where the method of moving planes is driven by pointwise estimates, our method can therefore be applied directly to weak solutions of the equation.

Let us give a somewhat more detailed sketch of the moving planes argument. The first step is to regularize the function at infinity via a (non-centered) Kelvin transformation. That is, we fix some $z \in \mathbb{R}^n \setminus \{0\}$, denote by $z^* = z/|z|^2$ its inversion about the unit sphere and consider

$$v(x) := u_z^*(x) := \frac{1}{|x|^{n-4}} u\left(\frac{x}{|x|^2} + z\right), \qquad x \in \mathbb{R}^n \setminus \{0, z^*\}.$$

This function satisfies the transformed equation

$$\Delta^2 v = k^{\frac{n+4}{2}} g(k^{-\frac{n-4}{2}} v) \qquad \text{in } \mathbb{R}^n \setminus \{0, -z^*\} \tag{40}$$

with

$$k(x) := k_z(x) := \frac{|z^*|}{|x||x+z^*|}, \qquad x \in \mathbb{R}^n \setminus \{0, -z^*\}.$$

We shall show that v is reflection-symmetric with respect to any hyperplane H passing through 0 and z. By letting $z \to 0$ along fixed directions, one recovers from this radial symmetry of u by an easy argument.

The main step is thus to carry out the method of moving planes for equation (40). Without loss, we assume that $z_1 = 0$ and $H = \{x_1 = 0\}$. For any number $\lambda < 0$, we let $\Sigma_\lambda = \{x_1 > \lambda\}$, $x^\lambda = (2\lambda - x_1, x_2, ..., x_n)$, $v_\lambda(x) = v(x^\lambda)$ and $k_\lambda(x) = k(x^\lambda)$.

We need to analyze the positivity of the difference function $w^{(\lambda)} := v - v_\lambda$ on the half-space Σ_λ. This function satisfies

$$\Delta^2 w^{(\lambda)} \geq V^{(\lambda)} w^{(\lambda)} \qquad \text{on } \Sigma_\lambda, \tag{41}$$

with

$$V^{(\lambda)}(x) = k_\lambda(x)^4 \, \frac{g(v(x)k_\lambda(x)^{-\frac{n-4}{2}}) - g(v_\lambda(x)k_\lambda(x)^{-\frac{n-4}{2}})}{v(x)k_\lambda(x)^{-\frac{n-4}{2}} - v_\lambda(x)k_\lambda(x)^{-\frac{n-4}{2}}}.$$

The following lemma gives a type of small-volume maximum principle which drives the moving planes method in our setting. It may be compared with Lemma 9 from [P3] below.

LEMMA 7 (Lemma 10 in [P2]). *There is a constant $\epsilon_0 > 0$, depending only on n, such that if $|\{w^{(\lambda)} < 0\}| > 0$, then $\int_{\{w^{(\lambda)}<0\}} \left(V^{(\lambda)}\right)^{\frac{n}{4}} \geq \epsilon_0$.*

To prove Lemma 7, the underlying simple idea is to multiply (41) by $w_-^{(\lambda)}$ and integrate over Σ_λ. Using integration by parts and Sobolev's and Hölder's inequalities, one deduces the inequality

$$\|w_-^{(\lambda)}\|^2_{\frac{2n}{n-4}} \lesssim \|V^{(\lambda)} 1_{\{w^{(\lambda)}<0\}}\|_{n/4} \|w_-^{(\lambda)}\|^2_{\frac{2n}{n-4}}, \tag{42}$$

which implies the conclusion of the lemma. This idea in its basic form is attributed to **[12]** in the review article **[16]**. The relevance of replacing pointwise estimates by integral ones was emphasized in Terracini's works **[90, 91]**; see also **[27]** for a version of this argument in the integral equation setting. Implementing the above idea and deriving (42) rigorously in our setting requires some careful estimates and approximations, however. This is because (36), (40) and (41) only hold in the weak sense and the class of test functions is restricted due to both the presence of singularities and because the equation is of fourth order. Details can be found in Section 2 of [P2].

2.3. Paper [P3]. Classification of solutions of an equation related to a conformal log Sobolev inequality

This paper, joint with Rupert Frank and Hanli Tang, is concerned with the endpoint case $s = 0$ of the classification results described in Section 1.3 above. To motivate our main theorem, let us first introduce the corresponding variational framework. The latter has a particularly nice expression on the sphere $\mathbb{S}^n$, where the following log-Sobolev inequality proved by Beckner **[10, 11]** holds.

$$\iint_{\mathbb{S}^n\times\mathbb{S}^n} \frac{|v(\omega)-v(\eta)|^2}{|\omega-\eta|^n}\, \mathrm{d}\omega\, \mathrm{d}\eta \geq C_n \int_{\mathbb{S}^n} |v(\omega)|^2 \ln \frac{|v(\omega)|^2|\mathbb{S}^n|}{\|v\|_2^2}\, \mathrm{d}\omega\,, \tag{43}$$

for all $v \in C^\infty(\mathbb{S}^n)$, say. Here, the constant

$$C_n := \frac{4}{n}\frac{\pi^{n/2}}{\Gamma(n/2)}\,, \tag{44}$$

is sharp, as can be seen by testing against the *conformal factors*

$$v(\omega) = c\left(\frac{\sqrt{1-|\zeta|^2}}{1-\zeta\cdot\omega}\right)^{n/2}, \tag{45}$$

where $c \neq 0$ and $\zeta \in \mathbb{R}^{n+1}$ with $|\zeta| < 1$. To recognize (43) as the endpoint case $s = 0$ of the fractional Sobolev inequality (7) respectively the HLS inequality (9), recall from Section 1.3 the equivalent formulation of (9) on $\mathbb{S}^n$ to be

$$\iint_{\mathbb{R}^n \times \mathbb{R}^n} \frac{v(x)v(y)}{|x-y|^{n-2s}} \, \mathrm{d}x \, \mathrm{d}y \leq C_{n,s}^{\mathrm{HLS}} \|v\|_{\frac{2n}{n+2s}}^2. \tag{46}$$

Then (43) is obtained from (46) through a differentiation argument in the limit $s \to 0$, which we explain briefly in the following. There are several equivalent ways to do this, e.g. using expansion in spherical harmonics, as done in [**10**]. We present here an alternative argument sketched in [**11**] which does not rely on spherical harmonics. Writing

$$\iint_{\mathbb{S}^n \times \mathbb{S}^n} \frac{|v(\omega) - v(\eta)|^2}{|\omega - \eta|^{n-2s}} \, \mathrm{d}\omega \, \mathrm{d}\eta = 2 \iint_{\mathbb{S}^n \times \mathbb{S}^n} \frac{v(\omega)(v(\omega) - v(\eta))}{|\omega - \eta|^{n-2s}} \, \mathrm{d}\omega \, \mathrm{d}\eta$$

we obtain by the HLS inequality (15) that

$$\iint_{\mathbb{S}^n \times \mathbb{S}^n} \frac{|v(\omega) - v(\eta)|^2}{|\omega - \eta|^{n-2s}} \, \mathrm{d}\omega \, \mathrm{d}\eta \geq 2 C_{n,s}^{\mathrm{HLS}} |\mathbb{S}^n|^{\frac{2s}{n}} \left(\|v\|_{L^2(\mathbb{S}^n)}^2 - |\mathbb{S}^n|^{-\frac{2s}{n}} \|v\|_{L^{\frac{2n}{n+2s}}(\Omega)}^2 \right). \tag{47}$$

Here we used the relation $\int_{\mathbb{S}^n} \frac{1}{|\omega - \eta|^{n-2s}} \, \mathrm{d}\eta = |\mathbb{S}^n|^{-\frac{2s}{n}} C_{n,s}^{\mathrm{HLS}}$ (for any $\omega \in \mathbb{S}^n$), which follows from evaluating (15) with the extremizing function $v \equiv 1$. Since the HLS constant is given by [**61**, Theorem 4.3]

$$C_{n,s}^{\mathrm{HLS}} = \pi^{\frac{n}{2} - s} \frac{\Gamma(s)}{\Gamma(\frac{n}{2} + s)} \left(\frac{\Gamma(n/2)}{\Gamma(n)} \right)^{-2s/n} \sim \frac{2\pi^{n/2}}{\Gamma(n/2)} \frac{1}{2s} + o(\frac{1}{2s}) \qquad \text{as } s \to 0,$$

the right side of (47) tends to

$$-\frac{4\pi^{n/2}}{\Gamma(n/2)} \frac{\mathrm{d}}{\mathrm{d}\alpha}\Big|_{\alpha=0} \left(|\mathbb{S}^n|^{-\frac{\alpha}{n}} \|v\|_{\frac{2n}{n+\alpha}}^2 \right) = \frac{4\pi^{n/2}}{n\Gamma(n/2)} \int_{\mathbb{S}^n} |v(\omega)|^2 \ln \frac{|v(\omega)|^2 |\mathbb{S}^n|}{\|v\|_2^2} \, \mathrm{d}\omega$$

in the limit $s \to 0$. This yields (43).

This passage to the limit does not preserve the characterization of optimizers (16), which for (43) was recovered in [**11**] by an additional argument. As discussed in Section 1.3 for the case $s > 0$, it is an interesting and non-trivial question to extend this classification to solutions of the Euler-Lagrange equation fulfilled by optimizers. This equation reads, in the weak sense and for functions u normalized so that $\|u\|_2^2 = |\mathbb{S}^n|$,

$$\mathcal{E}[\varphi, u] := \frac{1}{2} \iint_{\mathbb{S}^n \times \mathbb{S}^n} \frac{(\varphi(\omega) - \varphi(\eta))\,(u(\omega) - u(\eta))}{|\omega - \eta|^n} \, \mathrm{d}\omega \, \mathrm{d}\eta = C_n \int_{\mathbb{S}^n} \varphi(\omega)\, u(\omega) \ln u(\omega) \, \mathrm{d}\omega \tag{48}$$

for all $\varphi \in \mathcal{D}$. Here the finite-energy space $\mathcal{D}$ is given by

$$\mathcal{D} := \left\{ v \in L^2(\mathbb{S}^n) : \iint_{\mathbb{S}^n \times \mathbb{S}^n} \frac{|v(\omega) - v(\eta)|^2}{|\omega - \eta|^n} \, \mathrm{d}\omega \, \mathrm{d}\eta < \infty \right\}.$$

Our main result gives an answer to this question.

Theorem 8. *Let $0 \not\equiv u \in \mathcal{D}$ be a nonnegative weak solution of equation* (48). *Then*

$$u(\omega) = \left(\frac{\sqrt{1-|\zeta|^2}}{1-\zeta \cdot \omega} \right)^{n/2}$$

for some $\zeta \in \mathbb{R}^{n+1}$ with $|\zeta| < 1$.

That is, analogously to the case $s > 0$, the set of positive finite-energy solutions to the Euler-Lagrange equation (48) is precisely given by the conformal factors.

From the differentiation argument above, it is clear that the quadratic form $\mathcal{E}$, respectively the underlying operator

$$Hu(\omega) := P.V. \int_{\mathbb{S}^n} \frac{u(\omega) - u(\eta)}{|\omega - \eta|^n} \, \mathrm{d}\eta$$

is closely related to the logarithmic Laplacian $(-\Delta)^L$ with Fourier symbol $2 \ln |\xi|$ on $\mathbb{R}^n$, which was studied recently in [**25, 48**]. Analogously, this operator is given, at least formally, by differentation of $(-\Delta)^s u$ at $s = 0$.

We prove Theorem 8 by the same method that was already employed in [**59**] to solve the classification problem $s > 0$, namely the method of moving spheres discussed in Section 1.3.

Since, unlike for $s > 0$, (48) does not admit a simple reformulation as an integral equation, we use what has come to be called the *direct method* of moving spheres. This means that we apply the moving spheres argument directly to some non-local integro-differential equation, in our case (48), instead of its integral equation version. This approach has been pioneered in work of Jarohs and Weth [**49, 50**] and systemized by Chen, Li, Li and Zhang in [**26, 28**]. It emphasizes the role of small domain maximum principles for non-local operators as the main technical ingredient.

To complete the picture, we note here that in the integral equation setting, which is usually less challenging to deal with, maximum principles similar in spirit appear implicitly in earlier works like [**27, 59, 64**]. For non-local differential operators, versions of the strong maximum principle appear e.g. in the works [**38, 53**].

The adaptation of the direct method to the case $s = 0$ is not straightforward. Deriving the relevant maximum principles in the logarithmic setting is therefore one of the main accomplishments in this paper. The additional difficulties have two different, yet related, sources. Firstly, since the regularity proof from [**59**, Theorem 1.2] cannot be directly applied in our case, we cannot study the pointwise version of (48), but rather have to stick with the weak version. In particular, the proof of the needed strong maximum principle is made much more delicate by this restriction than e.g. the corresponding version, valid for $C^{1,1}$ functions, in [**28**, Theorem 2.2]. Secondly, it turns out that we cannot use bounds in the usual L^p norms to estimate the logarithmic nonlinearity. We need to substitute those by suitable inequalities of Orlicz type.

We now give a sketch of the proof of Theorem 8, which proceeds via the method of moving spheres, lifted from $\mathbb{R}^n$ to $\mathbb{S}^n$ via stereographic projection. That is, for every $\lambda > 0$ and $x_0 \in \mathbb{R}^n$, we let $I_{\lambda,x_0}(x) = \frac{\lambda^2(x-x_0)}{|x-x_0|^2} + x_0$ be the inversion about the sphere in $\mathbb{R}^n$ with radius λ and center x_0. Setting $\xi_0 = \mathcal{S}(x_0)$, we consider the conformal map $\Phi_{\lambda,\xi_0} : \mathbb{S}^n \setminus \{S, \xi_0\} \to \mathbb{S}^n \setminus \{S, \xi_0\}$ given by

$$\Phi_{\lambda,\xi_0} = \mathcal{S} \circ I_{\lambda,x_0} \circ \mathcal{S}^{-1}. \tag{49}$$

Given a positive solution u to (48), the method of moving spheres consists in comparing u to its reflection

$$u_{\lambda,\xi_0}(\omega) := \det D\Phi_{\lambda,\xi_0}(\omega)^{1/2} u(\Phi_{\lambda,\xi_0}(\omega)).$$

on the 'half-sphere' $\Sigma_{\lambda,\xi_0} := \mathcal{S}(B_\lambda(x_0))$. By conformal invariance of the equation, u_{λ,ξ_0} is a solution to (48) if and only if u is. The main step in the moving spheres method then consists in showing a strong symmetry property of the solution u, namely that

$$\text{for every} \quad \xi_0 \in \mathbb{S}^n \setminus \{S\}, \text{ there exists } \lambda = \lambda(\xi_0) \in (0,\infty) \text{ such that } u_{\lambda,\xi_0} \equiv u. \tag{50}$$

The proof of (50) proceeds through the analysis of the positivity of the difference function $w_{\lambda,\xi_0} := u_{\lambda,\xi_0} - u$. Notice that as a consequence of Φ_{λ,ξ_0} being an involution, w_{λ,ξ_0} is *antisymmetric*, meaning that it is the negative of its reflection with respect to Φ_{λ,ξ_0}.

The following maximum principle, valid for antisymmetric functions, plays an important role in the proof of (50).

Lemma 9 (Lemma 4 in [P3]). *Let $\lambda > 0$ and $\xi_0 \in \mathbb{S}^n \setminus \{S\}$, let $\Omega \subset \Sigma_{\lambda,\xi_0}$ be measurable and let $V : \Omega \to \mathbb{R}$ be a measurable function with*

$$\int_\Omega e^{2V_-/C_n} < |\mathbb{S}^n| . \tag{51}$$

If $w \in \mathcal{D}$ is antisymmetric with respect to Σ_{λ,ξ_0} and satisfies

$$\mathcal{E}[\varphi, w] + \int_\Omega \varphi V w \geq 0 \quad \text{for any } 0 \leq \varphi \in \mathcal{D} \text{ with } \varphi = 0 \text{ on } \Omega^c \tag{52}$$

and

$$w \geq 0 \quad \text{a.e. on } \Sigma_{\lambda,\xi_0} \setminus \Omega, \tag{53}$$

then $w \geq 0$ a.e. on Ω.

Assumption (51) can be thought of a small volume assumption on the domain Ω, which is standard for the maximum principles involved in the direct moving spheres method, as is antisymmetry. The new feature, which takes the logarithmic character of our problem into account, is the fact that we need to measure the smallness of the potential V_- in an integral norm involving the exponential rather than a p-th power.

Once property (50) is established, the proof can be concluded in a relatively standard manner. Indeed, (50) translates back to the usual moving spheres inversion symmetry condition on the functions $v(x) = \det D\mathcal{S}(x)^{1/2} u(\mathcal{S}(x))$ defined on $\mathbb{R}^n$. By using **[39,** Theorem 1.4], which generalizes [**58**, Lemma 2.5] to arbitrary finite measures, we conclude that v must be of the form

$$v(x) = c\left(\frac{2b}{b^2+|x-a|^2}\right)^{n/2}. \tag{54}$$

for some $a \in \mathbb{R}^n$, $b > 0$ and $c \geq 0$. Projecting back to the sphere, we find that u must be of the form (45), for some $c \geq 0$ and $\zeta \in \mathbb{R}^{n+1}$ with $|\zeta| < 1$. Finally, we use (48) to determine that $c = 1$. This completes the proof of Theorem 8.

2.4. Paper [P4]. Energy asymptotics in the three-dimensional Brezis–Nirenberg problem

This paper, joint with Rupert Frank and Hynek Kovařík, deals with the blow-up asymptotics for almost minimizers of a Brezis–Nirenberg-type functional in the critical dimension $N = 3$.

To motivate our results from the viewpoint of the Brezis–Peletier conjectures discussed in Section 1.4, for an open and bounded set $\Omega \subset \mathbb{R}^3$, consider the following slight generalization of (28).

$$\begin{aligned} -\Delta u_\epsilon + (a+\epsilon V)u_\epsilon &= 3u_\epsilon^5 \qquad \text{in } \Omega, \\ u_\epsilon &> 0 \qquad \text{in } \Omega, \\ u_\epsilon &= 0 \qquad \text{on } \partial\Omega. \end{aligned} \tag{55}$$

Here, we allow for $a \in C^1(\Omega) \cap C(\overline{\Omega})$ and $V \in C(\overline{\Omega})$ to be *non-constant* potentials without restrictions on their signs.

Our results from Paper [P4] are relevant for describing the subclass of *energy-minimizing* solutions to (55), that is, the (u_ϵ) arising as minima of the functional $\mathcal{S}_{a+\epsilon V}$ introduced in Section 1.4.

The following notion, introduced in [**47**], serves as a generalization of the Brezis–Nirenberg criticality discussed in Section 1.4, to a non-constant potential. We say that $a \in C(\overline{\Omega})$ is *critical* if $S(a) = S(0)$, but $S(\tilde{a}) < S(a)$ for every $\tilde{a} \in C(\overline{\Omega})$ with $\tilde{a} \leq a$ and $\tilde{a} \not\equiv a$. A noteworthy equivalent characterization proved by Druet [**34**] is that a is critical if and only if $\inf_\Omega \phi_a = 0$, where ϕ_a is the Robin function introduced in Section 1.4. In particular, the set

$$\mathcal{N}_a := \{x \in \Omega \,:\, \phi_a(x) = 0\}$$

is non-empty. Notice that if a is critical, then $-\Delta + a$ is coercive as a consequence of Hölder's inequality.

To state the main result of Paper [P4], we need to introduce some more notation. Recall that the family $(U_{x,\lambda})$ of normalized Sobolev optimizers has been defined in (6). We define

$PU_{x,\lambda} \in H^1_0(\Omega)$ to be the unique function satisfying

$$\Delta PU_{x,\lambda} = \Delta U_{x,\lambda} \quad \text{in } \Omega, \qquad PU_{x,\lambda} = 0 \quad \text{on } \partial\Omega \,. \tag{56}$$

Moreover, let

$$T_{x,\lambda} := \text{ span } \{PU_{x,\lambda}, \partial_\lambda PU_{x,\lambda}, \partial_{x_i} PU_{x,\lambda} \, (i = 1, 2, 3)\}$$

and let $T^\perp_{x,\lambda}$ be the orthogonal complement of $T_{x,\lambda}$ in $H^1_0(\Omega)$ with respect to the inner product $\int_\Omega \nabla u \cdot \nabla v \, \mathrm{d}y$. We denote by $\Pi_{x,\lambda}$ and $\Pi^\perp_{x,\lambda}$ the orthogonal projections in $H^1_0(\Omega)$ onto $T_{x,\lambda}$ and $T^\perp_{x,\lambda}$, respectively.

Finally, we set

$$Q_V(x) := \int_\Omega V(y) \, G_a(x,y)^2 \, \mathrm{d}y, \qquad x \in \Omega \,, \tag{57}$$

and

$$\mathcal{N}_a(V) := \{x \in \mathcal{N}_a : \; Q_V(x) < 0\} \,.$$

Here is the detailed statement of the main result of [P4]. We denote $S := S(0)$ in the following.

Theorem 10. *Let Ω be of class C^2 and assume that a is critical with $a(x) < 0$ for all $x \in \mathcal{N}_a$. Moreover, suppose that $\mathcal{N}_a(V) \neq \emptyset$. Let $(u_\epsilon) \subset H^1_0(\Omega)$ be a family of functions such that*

$$\lim_{\epsilon \to 0} \frac{\mathcal{S}_{a+\epsilon V}[u_\epsilon] - S(a+\epsilon V)}{S - S(a + \epsilon V)} = 0 \qquad \text{and} \qquad \int_\Omega u_\epsilon^6 \, \mathrm{d}x = \left(\frac{S}{3}\right)^{\frac{3}{2}} \,. \tag{58}$$

Then there are $(x_\epsilon) \subset \Omega$, $(\lambda_\epsilon) \subset (0, \infty)$ and $(\alpha_\epsilon) \subset \mathbb{R}$ such that

$$u_\epsilon = \alpha_\epsilon \left(PU_{x_\epsilon, \lambda_\epsilon} - \lambda_\epsilon^{-1/2} \, \Pi^\perp_{x_\epsilon,\lambda_\epsilon} (H_a(x_\epsilon, \cdot) - H_0(x_\epsilon, \cdot)) + r_\epsilon \right) \tag{59}$$

and, along a subsequence,

$$x_\epsilon \to x_0 \qquad \text{for some } x_0 \in \mathcal{N}_a(V) \text{ with } \; \frac{Q_V(x_0)^2}{|a(x_0)|} = \sup_{y \in \mathcal{N}_a(V)} \frac{Q_V(y)^2}{|a(y)|} \,,$$

$$\phi_a(x_\epsilon) = o(\epsilon) \,,$$

$$\lim_{\epsilon \to 0} \epsilon \, \lambda_\epsilon = 4\pi^2 \, \frac{|a(x_0)|}{|Q_V(x_0)|} \,,$$

$$\alpha_\epsilon = s + \mathcal{O}(\epsilon) \qquad \text{for some } s \in \{\pm 1\} \,.$$

Finally, $r_\epsilon \in T^\perp_{x_\epsilon,\lambda_\epsilon}$ and $\|\nabla r_\epsilon\|_2 = o(\epsilon)$.

The assumption that $a < 0$ on $\mathcal{N}_a$ is not severe, as for critical a one always has $a \leq 0$ on $\mathcal{N}_a$, see Corollary 2.2 of [P4]. Moreover, the denominator in (58) is always non-zero under the assumption that $\mathcal{N}_a(V) \neq \emptyset$, see Theorem 11 below.

As we already tried to make clear above, Theorem 10 should be seen as a variational version of the Brezis–Peletier-type conjecture stated in Conjecture 2. As a consequence of the variational approach, the result only holds for energy-minimizing solutions and the bounds are only obtained in H^1 instead of L^∞ norm. On the other hand, Theorem 10 also holds for almost minimizers (in the sense of (58)), which do not need to satisfy a corresponding equation.

As a byproduct of the proof of Theorem 10, we obtain the asymptotics of the energy $S(a+\epsilon V)$ as $\epsilon \to 0$.

THEOREM 11. (i) *Assume that $\mathcal{N}_a(V) \neq \emptyset$. Then $S(a+\epsilon V) < S$ for all $\epsilon > 0$ and*

$$\lim_{\epsilon \to 0+} \frac{S(a+\epsilon V) - S}{\epsilon^2} = -\left(\frac{3}{S}\right)^{\frac{1}{2}} \frac{1}{8\pi^2} \sup_{x \in \mathcal{N}_a(V)} \frac{Q_V(x)^2}{|a(x)|} . \tag{60}$$

(ii) *Assume that $\mathcal{N}_a(V) = \emptyset$. Then $S(a+\epsilon V) = S + o(\epsilon^2)$ as $\epsilon \to 0+$. If, in addition, $Q_V(x) > 0$ for all $x \in \mathcal{N}_a$, then $S(a+\epsilon V) = S$ for all sufficiently small $\epsilon > 0$.*

Theorem 11 further clarifies the role of the assumption $\mathcal{N}_a(V) \neq \emptyset$ made in Theorem 10. Indeed, we see that the condition $\mathcal{N}_a(V) \neq \emptyset$ is 'almost sharp' in the sense that unless $\min_{\mathcal{N}_a} Q_V = 0$, the asymptotics of $S(a+\epsilon V)$ as $\epsilon \to 0$ are trivial.

The first step in the proof of Theorems 10 and 11 is to prove a sharp upper bound on the minimal energy $S(a+\epsilon V)$. We test $\mathcal{S}_{a+\epsilon V}$ with the family of functions

$$\psi_{x,\lambda}(y) := PU_{x,\lambda}(y) - \lambda^{-1/2}(H_a(x,y) - H_0(x,y)) . \tag{61}$$

with parameters $x \in \Omega$ and $\lambda > 0$ to be determined. (The intuition behind this choice of test functions can be sketched non-rigorously as follows. Suppose that u_ϵ is a true minimizer. From (63) below, we should have $u_\epsilon^5 \lambda^{1/2} \sim \delta_{x_0}$, so that by (55) we get $u(x) \sim (-\Delta + a)^{-1} u_\epsilon^5 \sim \lambda^{-1/2} G_a(x_0, y)$. But now, as $\lambda \to \infty$,

$$PU_{x,\lambda} + \lambda^{1/2} H_0(x, \cdot) \sim U_{x,\lambda} \sim \frac{\lambda^{-1/2}}{|x-y|}$$

for any fixed $x \in \Omega$ (see [**75**, Proposition 1] for the first asymptotic equality). Thus as $x \to x_0$ and $\lambda \to \infty$, the $\psi_{x,\lambda} \in H_0^1(\Omega)$ should be a good approximation to the expected limiting profile $G_a(x_0, \cdot)$ of the minimizers u_ϵ.)

We compute

$$\begin{aligned}\mathcal{S}_{a+\epsilon V}[\psi_{x,\lambda}] = S &+ \left(\frac{S}{3}\right)^{-\frac{1}{2}} 4\pi\, \phi_a(x)\, \lambda^{-1} \\ &+ \left(\frac{S}{3}\right)^{-\frac{1}{2}} \left(\frac{\epsilon}{\lambda} Q_V(x) - 2\pi^2\, a(x)\, \lambda^{-2} - (15\pi^2 - 128)\, \phi_a(x)^2\, \lambda^{-2}\right) \\ &+ o(\lambda^{-2}) + o(\epsilon \lambda^{-1}) .\end{aligned} \tag{62}$$

In view of Druet's result, the subleading λ^{-1} is minimized by taking $x \in \mathcal{N}_a$. Optimizing the remaining terms first in $\lambda > 0$ and then in $x \in \mathcal{N}_a(V)$ yields the upper bound in the expansion (60).

Let us insert a brief comment on a subtlety in the choice of the test functions in (61). In view of the expansion (59), it may be surprising that one can get the sharp upper bound by testing with the $\psi_{x,\lambda}$. After all, the $\psi_{x,\lambda}$ differ from (59) by $\lambda^{-1/2}\,\Pi_{x,\lambda}(H_a(x,\cdot)-H_0(x,\cdot))$, which can be shown to be of order $\mathcal{O}(\lambda^{-1})$, but not smaller. However, due to cancellations in the quotient the contribution to $\mathcal{S}_{a+\epsilon V}$ caused by this additional term is only of the order $o(\lambda^{-2})$ and thus negligible for our purposes.

The strategy in the proof of the corresponding lower bound is to start with an arbitrary sequence (u_ϵ) of almost minimizers (normalized so that $\int_\Omega u_\epsilon^6\,\mathrm{d}x = \int_{\mathbb{R}^3} U_{0,1}^6\,\mathrm{d}x = (S/3)^{3/2}$) and to show that they must essentially coincide with the test function $\psi_{x,\lambda}$ used to derive the upper bound. The starting point for this is the asymptotic decomposition

$$u_\epsilon = \alpha_\epsilon\,(PU_{x_\epsilon,\lambda_\epsilon} + w_\epsilon)\,, \tag{63}$$

where up to a subsequence

$$\begin{aligned} \alpha_\epsilon &\to s \qquad \text{for some } s\in\{-1,+1\}\,,\\ x_\epsilon &\to x_0 \qquad \text{for some } x_0\in\overline{\Omega}\,,\\ \lambda_\epsilon \mathrm{dist}(x_\epsilon,\partial\Omega) &\to \infty\,,\\ \|\nabla w_\epsilon\|_2 &\to 0 \qquad \text{and} \qquad w_\epsilon \in T^\perp_{x_\epsilon,\lambda_\epsilon}\,. \end{aligned} \tag{64}$$

The asymptotics (63) and (64) are well-known for exact minimizers u_ϵ of $S(a+\epsilon V)$ and can be derived from results of Struwe [**85**] and Bahri-Coron [**9**], see e.g. [**75**, Proposition 2]. The proof extends to almost minimizers without major problems.

If we expand the functional $\mathcal{S}_{a+\epsilon V}[u_\epsilon]$ according to the decomposition (63), the remainder terms containing w will a priori be of order $o(1)$ as $\epsilon\to 0$, which is by far not precise enough. However, the key point is that we can improve the bound on w by using the orthogonality condition on w from (64), the sharp upper bound already proved in (62) and the coercivity inequality

$$\int_\Omega \left(|\nabla v|^2 + av^2 - 15\,U_{x,\lambda}^4 v^2\right)\mathrm{d}y \geq \rho\int_\Omega |\nabla v|^2\,\mathrm{d}y\,, \qquad \text{for all } v\in T^\perp_{x,\lambda}. \tag{65}$$

This is a variant of an inequality proved by Rey [**75**, Appendix D], which is stated in (70) below. Inequality (65) has been proved and applied in the present context by Esposito [**36**], who used it to give a simple alternative proof of Druet's result that $\inf_\Omega \phi_a = 0$ for critical a. As can be guessed from (62), this amounts to expanding $\mathcal{S}_{a+\epsilon V}[u_\epsilon]$ to the first subleading order λ^{-1}.

To derive the results stated in Theorems 10 and 11, however, it is apparent from (62) that we need to expand the energy to the second subleading order λ^{-2}. Hence the idea is to iterate the procedure of expanding and using coercivity to conclude a sharper error bound. This requires substantially more work and new techniques. In particular, we need to use the inequality (65) two more times before we can expand u_ϵ and $S(a+\epsilon V)$ to the needed precision. In this process, the *zero-modes*, i.e., the functions in $T_{x,\lambda}$, need to be specially taken into account, which can be seen as a reflection of the translation and dilation invariance of the Sobolev inequality.

2.5. Paper [P5]. Energy asymptotics in the Brezis–Nirenberg problem. The higher-dimensional case

Paper [P5], also joint with Rupert Frank and Hynek Kovařík, is a companion work to Paper [P4] and contains the corresponding asymptotics of the Brezis–Nirenberg quotient functional and its almost minimizers in space dimension $N \geq 4$.

As already mentioned in the discussion in Section 1.4, the Brezis-Nirenberg problem in case $N \geq 4$ presents fewer difficulties and subtleties than the critical case $N = 3$. Indeed, the Brezis–Nirenberg result from **[17]** implies that for $N \geq 4$ the only critical function is $a = 0$, and that $S(\epsilon V) < S(0)$ if and only if

$$\mathcal{N}(V) := \{x \in \Omega \,:\, V(x) < 0\}$$

is non-empty.

Here are the main results of Paper [P5]. We use the notation introduced in Section 2.4. Moreover, we denote $S_N := S(0)$ and $\phi(x) := \phi_0(x)$.

THEOREM 12. *Let $\Omega \subset \mathbb{R}^N$ be open and bounded of class C^2 and suppose that $\mathcal{N}(V) \neq \emptyset$. Let $(u_\epsilon) \subset H^1_0(\Omega)$ be a family of functions such that*

$$\lim_{\epsilon \to 0} \frac{\mathcal{S}_{\epsilon V}[u_\epsilon] - S(\epsilon V)}{S_N - S(\epsilon V)} = 0 \qquad \text{and} \qquad \int_\Omega |u_\epsilon|^{\frac{2N}{N-2}}\,\mathrm{d}x = \left(\frac{S_N}{N(N-2)}\right)^{\frac{N}{2}}. \tag{66}$$

Then there are $(x_\epsilon) \subset \Omega$, $(\lambda_\epsilon) \subset (0,\infty)$, $(\alpha_\epsilon) \subset \mathbb{R}$ and $(w_\epsilon) \subset H^1_0(\Omega)$ with $w_\epsilon \in T^\perp_{x_\epsilon,\lambda_\epsilon}$ such that

$$u_\epsilon = \alpha_\epsilon \left(PU_{x_\epsilon,\lambda_\epsilon} + w_\epsilon\right) \tag{67}$$

and, along a subsequence, $x_\epsilon \to x_0$ for some $x_0 \in \mathcal{N}(V)$. Moreover, there are constants a_N, b_N and D_N such that

$$\begin{cases} \phi(x_0)^{-\frac{2}{N-4}}\ |V(x_0)|^{\frac{N-2}{N-4}} = \sup_{x\in\mathcal{N}(V)} \left(\phi(x)^{-\frac{2}{N-4}}\ |V(x)|^{\frac{N-2}{N-4}}\right), & N \geq 5, \\ \phi(x_0)^{-1}|V(x_0)| = \sup_{x\in\mathcal{N}(V)} \left(\phi(x)^{-1}|V(x)|\right), & N = 4, \end{cases}$$

$$\begin{cases} \|\nabla w_\epsilon\|_2 = o(\epsilon^{\frac{N-2}{2N-8}}), & N \geq 5, \\ \|\nabla w_\epsilon\|_2 \leq \exp\Big(-\frac{2}{\epsilon}\left(1+o(1)\right)\sigma_4(\Omega,V)^{-1}\Big), & N = 4, \end{cases}$$

$$\begin{cases} \lim_{\epsilon\to 0} \epsilon\,\lambda_\epsilon^{N-4} = \frac{N\,(N-2)^2\,a_N\,\phi(x_0)}{2\,b_N\,|V(x_0)|}, & N \geq 5, \\ \lim_{\epsilon\to 0} \epsilon\,\ln\lambda_\epsilon = \frac{2\,\phi(x_0)}{|V(x_0)|}, & N = 4, \end{cases}$$

$$\begin{cases} \alpha_\epsilon = s\left(1 + D_N\epsilon^{\frac{N-2}{N-4}} + o(\epsilon^{\frac{N-2}{N-4}})\right), & N \geq 5, \\ \alpha_\epsilon = s\left(1 + \exp\Big(-\frac{4}{\epsilon}\left(1+o(1)\right)\left(\sup_{x\in\mathcal{N}(V)}\left(\phi(x)^{-1}|V(x)|\right)\right)^{-1}\Big)\right), & N = 4, \end{cases}$$

for some $s \in \{\pm 1\}$.

The corresponding energy asymptotics are the following.

THEOREM 13. *There exists a constant* C_N *such that as* $\epsilon \to 0+$,

$$S(\epsilon V) = S_N - C_N \sup_{x\in\mathcal{N}(V)} \left(\phi(x)^{-\frac{2}{N-4}}\, |V(x)|^{\frac{N-2}{N-4}}\right) \epsilon^{\frac{N-2}{N-4}} + o(\epsilon^{\frac{N-2}{N-4}}) \qquad \text{if } N \geq 5 \quad (68)$$

and

$$S(\epsilon V) = S_4 - \exp\Big(-\frac{4}{\epsilon}\,(1+o(1)) \left(\sup_{x\in\mathcal{N}(V)} \big(\phi(x)^{-1}|V(x)|\big)\right)^{-1} \Big) \qquad \text{if } N = 4. \quad (69)$$

The proof of Theorems 12 and 13 is along the same lines as those of the results from Paper [P4] sketched in Section 2.4. As already pointed out above, the case $N \geq 4$ is technically less involved. This is reflected in the proofs by the fact that testing with the simpler family of functions $PU_{x,\lambda}$ (without the correction term $-\lambda^{-1/2}(H_a(x,\cdot) - H_0(x,\cdot))$ appearing in (61)) already yields the sharp upper bound for the energy in (68) resp. (69). As a consequence, in the asymptotic decomposition (63) (which holds in $N \geq 4$ as well) we do not need to extract a subleading term of u_ϵ, so that only one application of the appropriate coercivity inequality [**75**, Appendix D]

$$\int_\Omega |\nabla v|^2 \,\mathrm{d}y - N(N+2) \int_\Omega U_{x,\lambda}^{p-1}\, v^2 \,\mathrm{d}y \;\geq\; \frac{4}{N+4} \int_\Omega |\nabla v|^2 \,\mathrm{d}y\,, \qquad \text{for } v \in T_{x,\lambda}^{\perp} \quad (70)$$

suffices.

We explained in Section 2.4 that Theorem 10 from Paper [P4] can be seen as an H^1 version of the three-dimensional critical Brezis–Peletier Conjecture 2 for energy-minimizing solutions. In the same way, Theorem 12 is related to the higher-dimensional Brezis–Peletier conjecture from Theorem 1 which was solved by Han [**46**] and Rey [**74**]. For the special case of constant V and exact minimizers (u_ϵ), Theorems 12 and 13 are essentially proved in a work by Takahashi [**87**] by combining variational ideas similar to ours with results from Han and Rey. We also mention the work [**96**] which has a result and proof similar to [**87**], but for the subcritical problem (30) (with $\mu_0 = 0$). Paper [P5], other than elucidating the methods and ideas from [P4], therefore has the merit of giving a new self-contained proof of Takahashi's results and extending them to non-constant V and to almost minimizers (u_ϵ).

To conclude the discussion, let us briefly review the similarities and discrepancies between the cases $N = 3$ and $N \geq 4$ reflected in the results of Papers [P4] and [P5]. On the one hand, given the fundamentally different behavior of dimension $N = 3$, one may find the structure of the asymptotics surprisingly similar in the two cases. Namely, any normalized minimizing sequence u_ϵ develops the asymptotic profile $PU_{x_\epsilon,\lambda_\epsilon}$, where the concentration scale λ_ϵ is given to leading order by an inverse power of ϵ times a coefficient determined by an auxiliary optimization problem in x which involves the Green's function of $-\Delta + a$ (with critical a) and the perturbation potential V. On the other hand, an important difference is that for $N \geq 4$ merely the local values $V(x)$ are relevant to said optimization problem, while in $N = 3$ the global behavior of V enters through the quantity $Q_V(x)$. Moreover, in $N = 3$, concentration points are a priori restricted to the set $\mathcal{N}_a$, while in $N \geq 4$ no such additional restriction occurs.

Statement on my contributions as a coauthor

Conformally to the requirements, I attempt to estimate in the following my contributions to the papers [P1]–[P5] as a coauthor. Such a listing is necessarily both subjective and inexact, especially in a field like mathematics where the quintessential 'research work' is highly non-tangible and division of labor only works at a rudimentary level. I nevertheless hope that the following description can be useful to assess my thesis accomplishments. What is common to all the papers below is that the first impetus to study the problem was given by my advisor, Rupert Frank.

[P1] *Classification of positive singular solutions to a nonlinear biharmonic equation with critical exponent* (with Rupert L. Frank), Anal. PDE **12** (2019), no. 4, 1101–1113.

and

[P2] *Singular solutions to a semilinear biharmonic equation with a general critical nonlinearity* (with Rupert L. Frank), Atti Accad. Naz. Lincei Rend. Lincei Mat. Appl. **30** (2019), no. 4, 817–846.

I did almost all of the research work, as well as the redaction of the main body of papers [P1] and [P2], autonomously, supported by regular discussions with Rupert Frank.

[P3] *Classification of solutions of an equation related to a conformal log Sobolev inequality* (with Rupert L. Frank, Hanli Tang), arXiv:2003.08135, submitted to Adv. Math.

Hanli Tang and I completed most of the research work autonomously. Rupert Frank supported us by regular discussions, contributed a number of valuable ideas and took part in the final redaction of the proofs.

[P4] *Energy asymptotics in the three-dimensional Brezis–Nirenberg problem* (with Rupert L. Frank, Hynek Kovařík), arXiv:1908.01331, submitted to Ann. Henri Poincaré C,

and

[P5] *Energy asymptotics in the higher-dimensional Brezis–Nirenberg problem* (with Rupert L. Frank, Hynek Kovařík), Mathematics in Engineering, **2** (2020), no. 1, 119–140:

Papers [P4] and [P5] originated in an unfinished project that my two coauthors began several years ago. After I was introduced to the problem by my advisor, I was able to contribute a new idea that made it possible to overcome the decisive remaining difficulty and complete the proof of the result. This corresponds essentially to Section 6 and Appendix A of [P4]. In [P5] I worked out Section 5 and Appendix A and was in charge of the redaction of the main body of the text.

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