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PARAMETER ESTIMATION AND PREDICTION OF FUTURE FAILURES IN THE LOG-LOGISTIC DISTRIBUTIONS BASED ON HYBRID-CENSORED DATA

BY

WASSIM R. ABOU GHAIDA

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COMMITTEE PAGE

The members of the Committee approve the Thesis of Wassim R. Abou Ghaida defended on 12/05/2020.

Ayman Bakleezi
Thesis/Dissertation Supervisor

Faiz Elfaki
Committee Member

Reza Pakyari
Committee Member

Approved:

Dr. Ibrahim Mohamed Alkaabi, Dean, College of College of Arts and Sciences

ABSTRACT

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Title: Parameter Estimation and Prediction of Future Failures in the Log-Logistic Distributions Based on Hybrid-Censored Data

Supervisor of Thesis: Ayman, S, Bakleezi.

The main purpose of this thesis is to study the prediction of future observations of a Log-Logistic distribution from Hybrid Censored Samples. We will study parameter point estimation, interval estimation, different point predictors will be formed such as Maximum Likelihood Predictor (MLP), Best Unbiased Predictor (BUP), and Conditional Median Predictor (CMP). Different Prediction intervals will be constructed such as Intervals based on Pivotal quantities, and High-Density Intervals (HDI). A simulation study will be run using the R software to investigate and compare the performance of all point predictors and prediction intervals. It is observed that the (BUP) is the best point predictor and the (HDI) is the best prediction interval.

Key words: Hybrid Censoring Scheme, Log-Logistic distribution, Maximum Likelihood Predictor (MLP), Best Unbiased Predictor (BUP), Conditional Median Predictor (CMP), Prediction Intervals (PI), High Density Intervals (HDI), Maximum Likelihood Estimation (MLE).

DEDICATION

I dedicate this thesis to my wife who gave me an ultimate support and love,

my parents who believed in me and

are always proud of me.

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Firstly, I would like to show my gratitude to Professor Ayman Bakleezi for his continuous help and support; whom I would never forget his encouraging words “I expect the best from you”. Also, I would like to thank the committee members Professor Reza Pakyari and Professor Faiz Elfaki for their comments which helped me be on the right track. Moreover, I am thankful to my wife, mother, father, brother and sister for their encouragement and support. Finally, I would like to thank my colleagues and friends especially Mohammad Mollazehi.

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CHAPTER 1: INTRODUCTION

This chapter will present a background about the Log-Logistic distribution, different censoring schemes, the specific problem and objectives of the study in addition to the literature review that provides some previous studies that have been done relevant to our study.

1.1 Application and background (The Log-Logistic Distribution)

In probability and Statistics, Statisticians use many distributions for life testing and reliability studies. The distributions could be used in different fields such as economy, medical fields, physical and industrial fields and so many other fields. The most commonly used distributions are Weibull distribution, Log-Normal distribution, Generalized Gamma distributions, Logistic distributions, Burr XII, Generalized Exponential distributions, etc.

Log-Logistic distribution is one of the parametric distributions that could be used as a life testing distribution since it belongs to the Scale-Shape family and because of the simplicity of its CDF and survival function, as both of them could be written in enclosed form. It is one of the right skewed and heavy tails functions, and it could be used as an alternative to lognormal distribution.

Back in the 80s, Bennett (1983) has considered the Log- Logistic distribution as a model for cancer survival data, as the Hazard rate or mortality can be used for the analysis of a group of patients under study. The Log-Logistic distribution can be used further in real life prediction of events. Chung (2010) has used the Log-Logistic accelerated failure time to predict accident duration based on recorded accident data set.

Mathematically speaking, simple transformations can be done to well-known distributions to obtain Log-Logistic distribution. Shoukri, Mian, and Tracy (1988) considered the standard distribution with probability $g(z) = e^z.(1+e^z)^{-2}$ with the transformation $z = \beta \cdot \ln(\frac{T}{\alpha})$ to obtain the probability density function of T as

$$f(t) = \begin{cases} \frac{\left(\frac{\beta}{\alpha}\right)\cdot\left(\frac{t}{\alpha}\right)^{\beta-1}}{\left(1+\left(\frac{t}{\alpha}\right)^{\beta}\right)^2} & \text{where } t \geq 0, \lambda, \beta \geq 0. \\ 0 & otherwise \end{cases}$$

According to Al-Shomrani, Shawky, Arif, and Aslam (2016), the Log-Logistic distribution T is a continuous probability distribution with probability density function:

$$f(t) = \begin{cases} \frac{\frac{\beta}{\lambda}(\frac{t}{\lambda})^{\beta-1}}{(1+(\frac{t}{\lambda})^{\beta})^2} & \text{where } t \geq 0, \lambda, \beta > 0\,. \\ 0 & otherwise \end{cases}$$

where λ is the scale parameter and β is the shape parameter. Without loss of generality we let $\frac{1}{\lambda} = \alpha$ we get

$$f(t) = \begin{cases} \frac{\alpha\beta(\alpha t)^{\beta-1}}{(1+(\alpha t)^{\beta})^2} & \text{where } t \geq 0, \alpha, \beta > 0\,. \\ 0 & otherwise \end{cases} \quad (1)$$

The Moment Generating Function mgf of T according to Casella and Berger (2002) is $M_T(\rho) = \int_{-\infty}^{\infty} e^{\rho t} f(t)dt = \int_0^{\infty} e^{\rho t} f(t)dt = \int_0^{\infty} e^{\rho t} \frac{\alpha\beta(\alpha t)^{\beta-1}}{(1+(\alpha t)^{\beta})^2} dt$

According to Ekawati, Warsono, and Kurniasari (2015) $M_T(\rho)$ can be written as

$$M_T(\rho) = \sum_{n=0}^{\infty} \frac{p^n}{\alpha^n \cdot n!} \cdot B\left(\frac{\beta+n}{\beta}, \frac{\beta-n}{\beta}\right) \quad (2)$$

B is the type-II beta function.

The mean of the distribution is defined as $E(T) = \frac{1}{\alpha} \cdot B\left(\frac{\beta+1}{\beta}, \frac{\beta-1}{\beta}\right)$ (3)

In general, the r^{th} Expectation $E(T^r) = \frac{1}{\alpha^r} \cdot B\left(\frac{\beta+r}{\beta}, \frac{\beta-r}{\beta}\right)$ (4)

The variance $V(T) = E(T^2) - (E(T))^2 = \frac{1}{\alpha^2} \cdot B\left(\frac{\beta+2}{\beta}, \frac{\beta-2}{\beta}\right) - \left(\frac{1}{\alpha} \cdot B\left(\frac{\beta+1}{\beta}, \frac{\beta-1}{\beta}\right)\right)^2$ (5)

The Probability Distribution function is $F(t) = P(T \leq t) = \frac{(\alpha t)^\beta}{1+(\alpha t)^\beta}$ (6)

and its Survival function is: $S(t) = 1 - F(t) = 1 - \frac{(\alpha t)^\beta}{1+(\alpha t)^\beta} = \frac{1}{1+(\alpha t)^\beta}$ (7)

The Hazard function according to (Mitra, 2013) can be considered as the instantaneous rate of failure which could be given $h(t) = \frac{f(t)}{S(t)} = \frac{\alpha\beta(\alpha t)^{\beta-1}}{1+(\alpha t)^\beta}$ (8)

$$h'(t) = \frac{\alpha^2\beta[(\beta-1)(\alpha t)^{\beta-2} - (\alpha t)^{2\beta-2}]}{[1+(\alpha t)^\beta]^2} \quad (9)$$

For $0 < \beta \leq 1$, the Hazard function is decreasing for all values of $t > 0$.

For $\beta > 1$, the Hazard function is decreasing for $t < \frac{1}{\alpha} e^{\frac{\ln(\beta-1)}{\beta}}$ and increasing for $t > \frac{1}{\alpha} e^{\frac{\ln(\beta-1)}{\beta}}$, and its peak is at $t = \frac{1}{\alpha} e^{\frac{\ln(\beta-1)}{\beta}}$

The median M is defined such that $\int_0^M \frac{\alpha\beta(\alpha t)^{\beta-1}}{(1+(\alpha t)^\beta)^2} dt = \frac{1}{2}$ which will be reduced to $M = \frac{1}{\alpha}$

Referring to Al-Shomrani et al. (2016) The quantile function:

$$x_q = \frac{1}{\alpha}(q^{-1} - 1)^{\frac{-1}{\beta}}; 0 < q < 1 \quad (10)$$

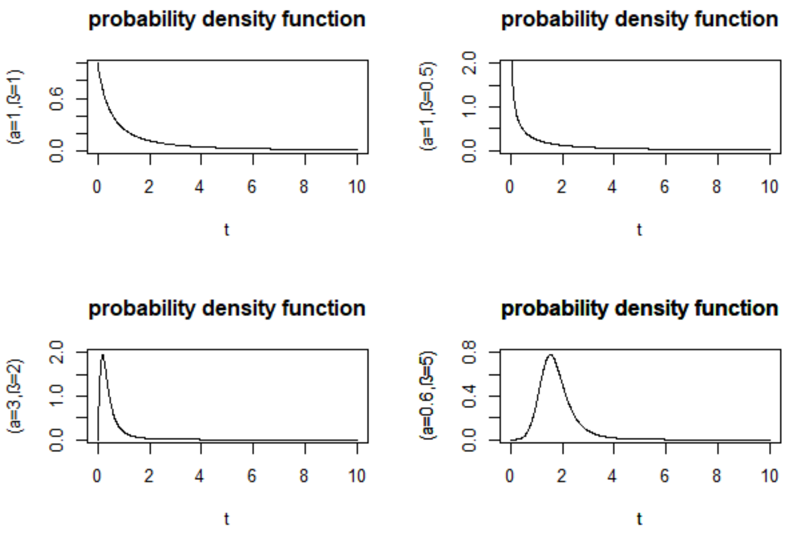

Figure 1: Graph of Probability Density Function of Log-Logistic Distribution

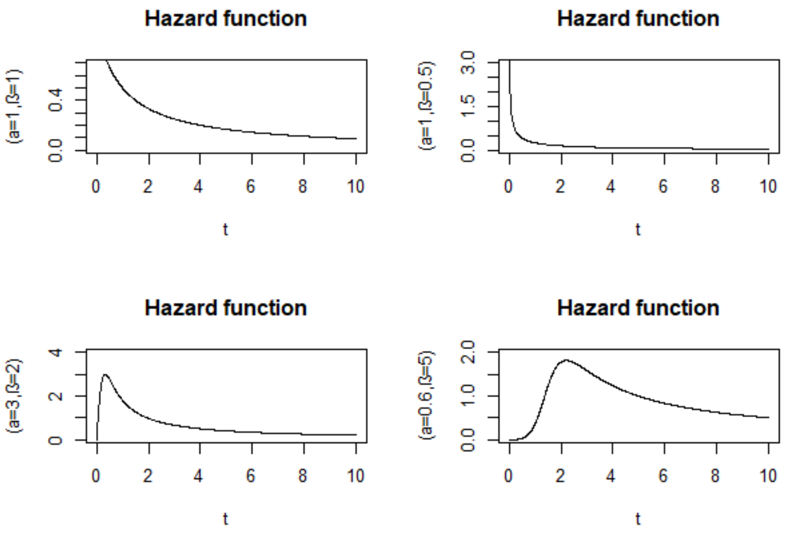

Figure 2: Graph of Hazard Function of Log-Logistic Distribution

1.2 Types of Censoring Schemes

1.2.1 Life Time Data

The term Lifetime refers to the time needed or covered until a certain event happens, and since the occurrence of such an event is random then Lifetime could be considered as a random variable T.

Two types of analysis that study the lifetimes of units arise. The first type is called reliability analysis which studies the lifetime of units such as electrical devises or machines, where the lifetime is considered as the time span of the failure of such unit since it is under operation or the time span of such a unit to stop working under the same working conditions.

The second type is called Survival analysis which studies the life of biological units (humans, animals...) and the lifetime in this case is considered the time elapsed until the death of such unit or the occurrence of a certain disease.

1.2.2 Censored Data

Censored data can be interpreted as the incomplete data when a researcher is looking for failure times of certain number of units under study. As a matter of fact, not all failure times or lifetimes can be observed by the researcher and hence a censored data is obtained.

Statisticians use censoring schemes for its importance in saving time and cost of performing experiments, making experiments on a limited number of units, and drawing inferences.

1.2.3 Type I-Censored Data

In Type I-Censored Data the time T where the experiment is terminated is previously determined by the researcher and hence it's a fixed number where the number of failures m cannot be determined until the end of the experiment and hence it is considered as a random variable.

1.2.4 Type II-Censored Data

In Type II-Censored Data the number of failures m is predetermined and the Time T where the experiment is terminated is not fixed where the experiment terminates when the number of failures m is reached and hence time T is a random variable.

1.2.5 Hybrid Censored Data

A mixture of the two types of censoring schemes is the hybrid censoring scheme. Epstein (1954) introduced this type of censoring scheme (type-I hybrid censoring) when the life distribution of electron tube data was assumed to be exponential.

This censoring scheme can be briefly explained as follows:

- when n units are placed into life testing, the researcher decides in advance to terminate the experiment when a pre-determined number r of failed units is reached or when the predetermined time T of terminating the experiment has reached.
- if X_r is the time at which the r^{th} failure occurs (X_r is a random variable) then the experiment will be terminated at time $T^* = \min(X_r, T)$ and hereby the experiment will not last longer than T or X_r. It is clear that type-I censoring scheme and type-II censoring schemes are a special case of hybrid censoring. That is type-I can be obtained from Hybrid censoring when r is set to be n. And type-II can be obtained when $T \to \infty$. If the researcher wants to guarantee a certain number of failures then the experiment could be terminated at $T^* = \max(X_r, T)$ and this censoring scheme could be referred to (type-II hybrid censoring).

1.2.6 Left Censored Data

When n units are that put into life-testing, and r out of n units have failed before a certain time T and the time of failure of such units is unknown. That is all what we know about the r units is that $0 \leq t_i \leq T$, $i = 1 \dots r$ where t_i is the time of failure of the i^{th} unit. In this case the r units are called left censored units.

1.3 Examples related to censored data and models

Censored Data might arise from real life situations as mentioned before, for example it is almost impossible to follow the failure of all electric bulbs and hence the researcher withdraws some of the units that are considered censored units, another example on censored data is when some patients are being monitored until the death or appearance of another disease, and hence the patients that are no longer being monitored are considered censored data. Regardless to the Censoring schemes we have earlier discussed, censored data might be left censored, right censored, left truncated, and double truncated and so on...

We will refer here to (Mitra, 2013) to illustrate some examples about left, double truncation, and right censoring.

Electronic devices and items are set into experiment of life testing before being released into the market. Suppose that the manufacturer sets a testing period of pre-specified time say $T = 100$ hours. Of course, some of the items will fail to work during this period of time and these items will not be set into market. Their lifetimes are unknown to the customers and these items are considered as left truncated data. Each item that is sold by the manufacturer is right censored as its actual life time is $t_i > T = 100$.

Before setting an example about double truncation, we will briefly relate this phenomenon to a variable having values included between two bounds: lower and upper bounds.

A good example about double truncation is sound waves. Sound waves can be any positive value but humans can only hear sounds whose sound waves frequencies f_i are included between 20 Hz and 20000 Hz that is: $20 \leq f_i \leq 20000$. The lower and upper bounds are 20 and 20000 respectively and the truncation points are on the left of 20 Hz and right of 20000 Hz.

1.4 Literature Review

In this section we will focus on reviewing previous studies and researches done by different statisticians whose work is related to life testing distributions. Our interest is basically to shade the light on different approaches taken to estimate the parameters of the different distributions. Moreover, we will basically focus on papers that discuss the prediction of future failures of censored data under adaptive or progressive censoring scheme.

Many statisticians have fitted many parametric distributions for life testing. The distributions that could be used for this purpose are: lognormal, Weibull, gamma, generalized gamma, exponential, logistic…

Sewailem and Baklizi (2019) have discussed the estimation of the parameters of Log-Logistic distribution using classical and Bayesian approach under adaptive progressive type-II censoring. Due to the non-linearity of the likelihood equations, the (NR) method was used to solve the MLEs. Asymptotic confidence intervals and approximate confidence intervals for the parameters, were obtained. Intervals based on (ML) estimation along with their corresponding coverage probability and credible intervals were obtained. Finally, the Bayesian approach was recommended as it has a better performance than that of the classical approach.

Dube, Pradhan, and Kundu (2011) have tried to estimate the parameters of a lognormal distribution based on Hybrid Censored data. Due to the non-linearity of the MLEs, they used the (EM) algorithm to calculate the MLEs. The (EM) algorithm can be summarized as follows: the complete data set W is the union of the observed data set X and the unobserved data set or censored set U. With the presence of missing data, the authors could find the asymptotic variance covariance matrix of the MLEs by inverting the observed information matrix ($I_X(\hat{\theta})$) where $\hat{\theta}$ is the vector of parameter estimates. Another approach has been used to approximate the parameters is the Approximate Maximum Likelihood Estimation (AMLE) which can be applied by using Taylor Expansion theorem around a certain point up to some order say first order.

Finally, the authors have compared the performance of both MLEs and the AMLEs through a simulation study and noticed that the MLEs and the AMLEs are the same with a slight difference in variance covariance matrix.

Hyun, Lee, and Yearout (2016)have analysed type-I and type-II Censored data where the lifetimes follow Log-Logistic distribution. The authors considered the MLE of the parameters and the asymptotic confidence intervals. The Log-Likelihood function and the Log-Likelihood equations were formed. It was observed that the Likelihood equations could not be obtained in closed form and then Newton Raphson was used to solve the non-linear likelihood equations. Moreover, in order to construct the asymptotic confidence intervals, the authors approximated the Fisher information matrix by the observed fisher information matrix at the MLEs. Finally, a simulation study have been done by generating type-I and type-II hybrid censored samples from Log-Logistic distribution. The average failure percent, the mean of the MLEs and the standard error were recorded in addition to the coverage probability of the confidence intervals based on 1000 replications.

Valiollahi, Asgharzadeh, and Kundu (2017) have studied the prediction of future observations based on Hybrid censored samples from generalized exponential distribution with parameters α and λ. The authors used the likelihood prediction approach to obtain the prediction likelihood function. Using the Markovian property, the conditional density of Y was obtained for $y \geq T_0$ where T_0 is the time where the experiment was terminated. The maximum likelihood predictor of y has been obtained as a function of α and λ that were replaced by their predictive maximum likelihood estimators that were estimated numerically. Another predictor has been discussed using the conditional prediction approach; the best unbiased predictor (BUP) as $\int_{T_0}^{\infty} y.f(y|x,\alpha,\lambda)\,dy$. Additional predictor; the conditional median predictor (CMP) defined as $P\left(Y \leq \hat{Y}/X = x\right) = P\left(Y \geq \hat{Y}/X = x\right)$. The predictors were obtained as a function of α and λ that were replaced by their predictive maximum likelihood estimators. Moreover, prediction intervals based on pivotal quantity, (HCD), and Bayesian prediction intervals were obtained. Finally, Simulation study based on 10000 replications was run to compare the performance of the different estimators. They noticed that the Mean Square Prediction Error Estimates increased as s increased for all prediction methods. (Here $s = 1, s = 2,\ s = 3$ represent the first, second, third failure and so on..). It was also noticed that Mean Square Prediction Error Estimates of the different predictors used under type –II were smaller than those under type-I.

The (BUP) has slightly less Mean Square Prediction Error than MLP and CLP. Also, the average length of coverage percentages increased as s increased. And in conclusion, the authors recommended to use the (BUP) for prediction of future observations and the (HCD) interval as it works better than the pivotal quantity interval (for $s > 1$).

In a similar fashion, Asgharzadeh, Valiollahi, and Kundu (2015) have studied the prediction of future observations based on type-I hybrid censored samples for two parameter Weibull random variable. The three classical point predictors (MLP, BUP, & CMP) were obtained in addition to Bayesian point predictor. First of all, the MLP was obtained as a function of the predictive maximum likelihood estimates of the parameters (PMLE) which were obtained numerically. The BUP and the CMP were obtained each as a function of the parameters that were replaced by their MLEs. Moreover, a Bayesian prediction approach was used and prediction intervals (PIs) based on hybrid censored data were obtained. A real data analysis was performed to check the validity of the model. Of the three used point predictors, the best predictor in terms of Bias and MPSE (mean square predictor error) was the BUP followed by the CMP whereas the MLP did not perform in a satisfactory manner. Finally, Bayesian predictors based on informative prior had a better performance than those based on non-informative priors and Bayesian prediction intervals were wider than classical prediction intervals.

Singh, Tripathi, and Wu (2015) have studied parameter estimation of lognormal distribution under progressive censored samples. First of all, the (EM) algorithm had to be used in order to find the estimates of the parameters. The observed information matrix was obtained as well as the asymptotic variance covariance matrix of the estimated parameters where the asymptotic variance covariance matrix is the inverse of the observed information matrix. The asymptotic confidence intervals were formed. The approximate maximum likelihood estimation was performed by using Taylor expansion around a certain point to get the AMLEs of the parameters. Also Bayes estimation of the parameters was suggested under squared error loss function and since again the estimators could not be obtained explicitly then Lindley's method was suggested. A simulation study has been run to compare the performance of the parameters in different methods. It was noticed that the MLEs and the AMLEs have almost a similar behaviour in terms of Bias and MSE values. Finally, the coverage probabilities for the interval estimation of the parameters were not satisfactory perhaps the reason behind that was the small sample sizes.

Lawless (1971) has considered the prediction of an observation from a sample of n units that follow an exponential distribution. That is if the observed failure times are $X_1 \leq X_2 \leq \cdots \leq X_k$ then, we can predict the r^{th} observation X_r $(k < r \leq n)$ by finding an interval estimate of X_r that will help us predict the remaining time left for the experiment. Then based on the first k failures, we can obtain such a prediction interval. The author considered the probability function of the random variate $U = \frac{X_r - X_k}{S_k}$ where $S_k = \sum_{i=1}^{k} X_i + (n-k)X_k$. To illustrate the above, one can use the probabilistic statement $P(U \leq a) = \gamma$ which is equivalent to $P\left(\frac{X_r - X_k}{S_k} \leq a\right) = \gamma$ and then $P(X_r \leq aS_k + X_k) = \gamma$ and by replacing $a, S_k,$ and X_k in the above probabilistic statement we can be $(1-\gamma)100\%$ confident that the r^{th} failure will occur before the time $aS_k + X_k$.

Basak, Basak, and Balakrishnan (2006) have discussed predicting future time failures of units that are censored in progressive censored samples. For the failure times $X_1, X_2, \ldots \ldots, X_n$ of n units that are put into life testing and observed $Y =$

$(Y_1, Y_2, Y_3, \dots \dots, Y_m)$ is the m progressive type II order statistic, the authors worked on predicting $Y_{j:r_i}$ $(j = 1 \dots \dots r_m, i = 1 \dots . m)$. where $Y_{j:r_i}$ is the j^{th}order statistic out of r_i removed units at stage $i = 1 \dots . n$. They considered that the density of $Y_{j:r_i}$ is the same as the density of the j^{th}order statistic out of r_i units from a population of density $\frac{g(x)}{1-G(Y_i)}$; $x \geq Y_i$ (left truncated at Y_i). The best linear unbiased predictor (BLUP), the maximum likelihood predictor (MLP) and the conditional median predictor (CMP) were considered. The authors proved the existence and uniqueness of the (MLP) for exponential distribution and that the modified (CMP) compares very well with the other predictors in terms of (MSPE). They also formed prediction intervals for exponential and extreme value distributions and came up with the conclusion that the prediction intervals for the (BLUP) and (CMP) are close and the modified (CMP) is well comparable with the other predictors.

Ebrahimi (1992) has studied the prediction of future failures and prediction intervals in exponential distribution with parameter λ under hybrid censoring. When n units are put into life testing with lifetimes $X_1, X_2, \dots \dots X_n$ with number of fixed failures r in advance and time T, the author has predicted X_s, where $s > \min(r, m)$.

Where m represents the number of failures up to time T .The author has found the (MLP) of $\widetilde{X_s}$ and the parameter λ was replaced by the predictive maximum likelihood estimate (PMLE) and the predictor $\widehat{X_s}$ where λ was replaced by the (MLE). Also, prediction intervals were found. The (MLP) performed better when $s < 2r$ and worst when $s > 2r$. Finally, the (MLP) performed better than $\widehat{X_s}$ in terms of MSE.

As we have seen in the literature review section, not so much work has been done on the prediction of future failures when the underlying distribution is the Log-Logistic distribution. Moreover, in this thesis we will build our work on the past researches conducted on prediction of future failures as the statistical and scientific work is an ongoing process, trying to add some concepts such as obtaining the MLEs of the parameters and extending the model to include left censored data in addition to the observed and right censored values. (Hybrid censoring, log-logistic distribution)

1.5 The Specific Problem

It is noticeable that the log-Logistic distribution has been used by statisticians for modelling life time data, its importance comes from the fact that it might be a good replacement of the well-known Weibull distribution to analyse lifetime data.

In this thesis we will extend the work of the authors reviewed in the previous part to the case when the observations follow the Log-Logistic distribution when the data is left censored and with hybrid censoring on the right. This includes several types of censoring as special cases including left censoring, type-I and type-II censoring. Thus, extending and generalizing either results on this problem.

1.6 Objectives of the Study

The main objective of the study is to predict of future failures of log-Logistic distribution model based on Hybrid-Censored Data. We will derive the maximum likelihood estimator in addition to several point predictors such as Maximum Likelihood predictor, Conditional Mean Predictor and Median Predictor. The Performance of such predictors will be investigated using Simulation and the results will be applied to real or simulated data sets.

1.6.1 Specific Objectives

In order to achieve our main objective, we will some specific objectives as follows:

1. Construct the likelihood function when the data is left censored at t_0 and hybrid censored (type-I and type-II), likelihood equations, and MLEs of the parameters α and β.
2. Calculate the bias and MSE for each estimator.
3. Will obtain the predictive likelihood function, and the MLP (Maximum Likelihood Predictor).
4. Consider other predictors like the conditional mean predictor and median predictor.
5. Investigate and compare the performance of the predictors.
6. Prediction Intervals
7. Comparison of the intervals
8. Apply the results to real or simulated data.

CHAPTER 2: ESTIMATION AND PREDICTION USING THE LIKELIHOOD FUNCTION

2.1 The Likelihood equations and the MLE

2.1.1 Overview of Maximum Likelihood

The MLE technique is used to estimate the parameters of a distribution by maximizing the probability of an observed sample. And since the maximum of a function is occurred at the maximum of its logarithm, then it is sufficient to maximize the logarithm of the likelihood.

The likelihood equations can be obtained by deriving the log-likelihood function with respect to the parameters and solving for the parameters when we set the partial derivatives equal to 0.

But sometimes the solutions of these equations cannot be obtained implicitly due to the non-linearity of such equations and hence some numerical approaches must be done in order to estimate the solutions or perimeters for these equations.

One of the most important approaches that can be done is the Taylor Series Expansion around a certain point. The Taylor series expansion can be explained in the following illustration:

If a function f is differentiable in a neighbourhood of a point a then $f(x)$ can be written as $f(x) = f(a) + (x-a)f'(a) + \frac{(x-a)^2}{2!} \cdot f"(a) + \cdots . + \frac{(x-a)^n}{n!} \cdot f^{(n)}(a) + \in$ where $\in$ is a function that tends to zero as $n \to \infty$. So, f can be approximated up to 1 term to $f(a) + (x-a)f'(a)$. Using the Taylor expansion about a certain point, the approximate MLEs can be obtained. Dube et al. (2011). Another numerical technique that could be used to solve a system of simultaneous equations is the Newton Raphson method (NR).

One of the methods used to compute the maximum likelihood estimates when the data comes from an incomplete data is an algorithm called (EM) algorithm which was proposed by Dempster, Laird, and Rubin (1977); the iterations of the algorithm consist of two steps one of which is the expectation and the other is maximization. The method suggests the existence of two sample spaces $S_1$and $S_2$where $S_1$the sample space of non-observed data and S_2 is the sample space of observed data and a correspondence from S_1 to S_2 that maps a non-observed data x into $y(x)$ where y is an observed data and x can be determined indirectly from y.

For a complete data x, incomplete data y, and a vector of parameters θ, there exists the relation between complete data and incomplete data as $f_2(y/\theta) = \int_{S_1(y)} f_1\,(x/\theta)dx$

f_1 and f_2 are the sampling densities of non-observed and observed data respectively. The (EM) algorithm works at finding a value of θ that maximizes $f_2(y/\theta)$ given an observed y and the complete data specification $f_1(x/\theta)$ generates the incomplete data specification $f_2(y/\theta)$ and hence finding $f_1(x/\theta)$ will generate $f_2(y/\theta)$.

2.1.2 Likelihood functions for different types of censored data

According to(Mitra, 2013); for a lifetime variable T whose pdf is $g(t)$, cdf is $G(t)$, and if the censored units belong to set C and the uncensored units belong to set U, then under right censoring the likelihood function $L \propto \prod_{i\in U} g(t_i) \cdot \prod_{i\epsilon C}(1 - G(t_i)$ and such that each right censored observation contributes $(1 - G(t))$ to the whole likelihood. Similarly, for a left censored data the likelihood function $L \propto \prod_{i\in U} g(t_i) \cdot \prod_{i\epsilon C} G(t_i)$ and for interval censored data the likelihood function $L \propto \prod_{i\in U} g(t_i) \cdot \prod_{i\epsilon C}(G(t_{upper}) - G(t_{lower}))$.

A likelihood function can be obtained from type-II censored data and it can be obtained through the following perspective; suppose that n units are put into life testing and r failures have to be obtained and the experiment stops after the r units have been obtained. If t_r represents the smallest r^{th}order statistic of the lifetimes of the n units, then the likelihood function is given by $L \propto \prod_{i=1}^{r} g(t_i) \cdot \{1 - G(t_r)\}^{n-r}$. For a left truncated T random variable and whose point of truncation is t_L, the pdf of the left truncated random variable is $g_{LT}(t) = \frac{g(t)}{1-G(t_L)}$ for $t > t_L$and the cdf is given by $G_{LT}(t) =$

$\frac{G(t)-G(t_L)}{1-G(t_L)}$ for $t > t_L$. Similarly for a right truncated T random variable and whose point of truncation is t_R, the pdf of the right truncated random variable is $g_{RT}(t) = \frac{g(t)}{G(t_R)}$ for $t < t_R$ and the cdf is given by $G_{RT}(t) = \frac{G(t)}{G(t_R)}$ for $t < t_R$.

Finally, for a doubly truncated random variable where the left and right truncation points are respectively t_L and t_R the pdf is given by $g_{DT}(t) = \frac{g(t)}{G(t_R)-G(t_L)}$ for $t_L < t < t_R$ and its cdf is

$$G_{DT}(t) = \frac{G(t)-G(t_L)}{G(t_R)-G(t_L)} \text{ for } t_L < t < t_R$$

2.1.3 The Likelihood equations (MLE) for Log-Logistic distribution

Suppose that n units are put into lifetime test, and of which r units are left censored at time t_0, the likelihood of each left censored unit at time t_0 is:

$L_i(\alpha,\beta) = \frac{(\alpha t_0)^\beta}{1+(\alpha t_0)^\beta}$ and hence the Likelihood for the r left censored units at time t_0 is $\prod_{i=1}^{r} \frac{(\alpha t_0)^\beta}{1+(\alpha t_0)^\beta} = \frac{(\alpha t_0)^{r\beta}}{\left(1+(\alpha t_0)^\beta\right)^r}$.

Now we will consider the Hybrid Censoring (type 1 and type 2). Suppose that the preassigned number of failures is m or that the preassigned time is τ.

Case 1: If we have m observed units where their corresponding ordered failure times are: $t_{r+1}, t_{r+2}, t_{r+3}, \dots\dots\dots\dots\dots, t_{r+m}$ where $t_i < \tau$ for all $i \,\epsilon\{r+1, \dots\dots., r+m\}$.

The contribution of the m observed units or failures to the likelihood in this case is:

$$\prod_{i=r+1}^{r+m} f(t_i) = \prod_{i=r+1}^{r+m} \frac{\alpha\beta(\alpha t_i)^{\beta-1}}{\left(1+(\alpha t_i)^\beta\right)^2}$$

The contribution of the $n-(r+m)$ unobserved data to the likelihood (censored) is:

$\left(1-F(t_{r+m})\right)^{n-(r+m)}$ where t_{r+m} is the $(r+m)^{th}$ smallest order statistic among the lifetimes. So the likelihood is $(1-\frac{(\alpha t_{r+m})^{\beta}}{1+(\alpha t_{r+m})^{\beta}})^{n-(r+m)} = (\frac{1}{1+(\alpha t_{r+m})^{\beta}})^{n-(r+m)}$.

Case 2: Suppose we reach time τ (the preassigned time) with s failures and their corresponding failure times are $t_{r+1}, t_{r+2}, t_{r+3}, \dots\dots\dots\dots\dots, t_{r+s}$ where $s < m$ and $t_{r+s} < \tau < t_{r+s+1}$. The contribution to the likelihood of the s observed units is:

$$\prod_{i=r+1}^{r+s} f(t_i) = \prod_{i=r+1}^{r+s} \frac{\alpha\beta(\alpha t_i)^{\beta-1}}{\left(1+(\alpha t_i)^{\beta}\right)^2}$$

The contribution of the $n-(r+s)$ unobserved data is:

$\left(1-F(\tau)\right)^{n-(r+s)} = (\frac{1}{1+(\alpha\tau)^{\beta}})^{n-(r+s)}$. The likelihood function in case 1 is:

$$L(\alpha,\beta) = \frac{(\alpha t_0)^{r\beta}}{\left(1+(\alpha t_0)^{\beta}\right)^{r}} \cdot \prod_{i=r+1}^{r+m} \frac{\alpha\beta(\alpha t_i)^{\beta-1}}{\left(1+(\alpha t_i)^{\beta}\right)^2} \cdot \left\{\left(\frac{1}{1+(\alpha t_{r+m})^{\beta}}\right)^{n-(r+m)}\right\} \tag{11}$$

Applying the logarithm of the likelihood function:

$$\ln L(\alpha,\beta) = r\beta\ln(\alpha t_0) - r\ln(1+\left(\alpha t_0\right)^{\beta}) + \sum_{i=r+1}^{i=r+m} \ln(\alpha\beta(\alpha t_i)^{\beta-1})$$

$$-2\sum_{i=r+1}^{i=r+m} \ln(\,(1+\left(\alpha t_i\right)^{\beta}) - \left(n-(r+m)\right)\ln(1+\left(\alpha t_{r+m}\right)^{\beta}) \tag{12}$$

The partial derivatives of $\ln L(\alpha,\beta)$ are as follows:

$$\frac{\partial L}{\partial\alpha} = \frac{r\beta}{\alpha} - \frac{r\beta(\alpha t_0)^{\beta-1}t_0}{1+(\alpha t_0)^{\beta}} + \sum_{i=r+1}^{i=r+m}\frac{\beta}{\alpha} - 2\sum_{i=r+1}^{i=r+m}\frac{\beta(\alpha t_i)^{\beta-1}t_i}{1+(\alpha t_i)^{\beta}}$$

$$-\frac{(n-(r+m))\beta(\alpha t_{r+m})^{\beta-1}t_{r+m}}{1+(\alpha t_{r+m})^{\beta}}$$

$$\frac{\partial L}{\partial \alpha} = \frac{r\beta}{\alpha} - \frac{r\beta(\alpha t_0)^\beta}{\alpha(1+(\alpha t_0)^\beta)} + \frac{m\beta}{\alpha} - 2\sum_{i=r+1}^{i=r+m} \frac{\beta(\alpha t_i)^\beta}{\alpha(1+(\alpha t_i)^\beta)} - \frac{\big(n-(r+m)\big)\beta(\alpha t_{r+m})^\beta}{\alpha(1+(\alpha t_{r+m})^\beta)}$$

$$= 0 \qquad (13)$$

$$\frac{\partial L}{\partial \beta} = r\ln(\alpha t_0) - \frac{r(\alpha t_0)^\beta \ln(\alpha t_0)}{(1+(\alpha t_0)^\beta)} + \sum_{i=r+1}^{i=r+m} \frac{1+\beta\ln(\alpha t_i)}{\beta} - 2\sum_{i=r+1}^{i=r+m} \frac{(\alpha t_i)^\beta \ln(\alpha t_i)}{(1+(\alpha t_i)^\beta)}$$

$$- \frac{(n-(r+m))(\alpha t_{r+m})^\beta \ln(\alpha t_{r+m})}{(1+(\alpha t_{r+m})^\beta)} = 0 \qquad (14)$$

Similarly, the likelihood function in case 2 is:

$$L(\alpha,\beta) = \frac{(\alpha t_0)^{r\beta}}{\big(1+(\alpha t_0)^\beta\big)^r} \cdot \prod_{i=r+1}^{r+s} \frac{\alpha\beta(\alpha t_i)^{\beta-1}}{\big(1+(\alpha t_i)^\beta\big)^2} \cdot \left\{\left(\frac{1}{1+(\alpha\tau)^\beta}\right)^{n-(r+s)}\right\} \qquad (15)$$

Applying the logarithm to the likelihood function:

$$\ln L(\alpha,\beta) = r\beta\ln(\alpha t_0) - r\ln(1+\big(\alpha t_0)^\beta\big) + \sum_{i=r+1}^{i=r+s} \ln(\alpha\beta(\alpha t_i)^{\beta-1})$$

$$-2\sum_{i=r+1}^{i=r+s} \ln(\,(1+\big(\alpha t_i)^\beta\big) - \big(n-(r+s)\big)\ln(1+\big(\alpha\tau)^\beta\big) \qquad (16)$$

The partial derivatives of $\ln L(\alpha,\beta)$ are as follows:

$$\frac{\partial L}{\partial \alpha} = \frac{r\beta}{\alpha} - \frac{r\beta(\alpha t_0)^\beta}{\alpha(1+(\alpha t_0)^\beta)} + \frac{s\beta}{\alpha} - 2\sum_{i=r+1}^{i=r+s} \frac{\beta(\alpha t_i)^\beta}{\alpha(1+(\alpha t_i)^\beta)} - \frac{\big(n-(r+s)\big)\beta(\alpha\tau)^\beta}{\alpha(1+(\alpha\tau)^\beta)} = 0 \quad (17)$$

$$\frac{\partial L}{\partial \beta} = r\ln(\alpha t_0) - \frac{r(\alpha t_0)^\beta \ln(\alpha t_0)}{(1+(\alpha t_0)^\beta)} + \sum_{i=r+1}^{i=r+s} \frac{1+\beta\ln(\alpha t_i)}{\beta} - 2\sum_{i=r+1}^{i=r+s} \frac{(\alpha t_i)^\beta \ln(\alpha t_i)}{(1+(\alpha t_i)^\beta)}$$

$$- \frac{(n-(r+s))(\alpha\tau)^\beta \ln(\alpha\tau)}{(1+(\alpha\tau)^\beta)} = 0 \qquad (18)$$

Case 1 and case 2 can be combined as follows:

$$L(\alpha,\beta) = \frac{(\alpha t_0)^{r\beta}}{(1+(\alpha t_0)^{\beta})^r} \cdot \prod_{i=r+1}^{r+f} \frac{\alpha\beta(\alpha t_i)^{\beta-1}}{(1+(\alpha t_i)^{\beta})^2} \cdot \left\{\left(\frac{1}{1+(\alpha\tau^*)^{\beta}}\right)^{n-(r+f)}\right\}$$

$$= \frac{(\alpha t_0)^{r\beta}}{(1+(\alpha t_0)^{\beta})^r} \cdot \prod_{i=r+1}^{r+f} \frac{\alpha\beta(\alpha t_i)^{\beta-1}}{(1+(\alpha t_i)^{\beta})^2} \cdot \left\{\left(1+(\alpha\tau^*)^{\beta}\right)^{(r+f)-n}\right\} \tag{19}$$

where, f is the number of observed failures, τ^* is the time where the experiment stops:

$$f = \begin{cases} m & case\ 1 \\ s & case\ 2 \end{cases} \qquad \tau^* = \begin{cases} t_{r+m} & case\ 1 \\ \tau & case\ 2 \end{cases}$$

The Log-Likelihood function in both cases is:

$$\ln L(\alpha,\beta) = r\beta\ln(\alpha t_0) - r\ln(1+\left(\alpha t_0)^{\beta}\right) + \sum_{i=r+1}^{i=r+f} \ln(\alpha\beta(\alpha t_i)^{\beta-1})$$

$$-2\sum_{i=r+1}^{i=r+f} \ln(\,(1+\left(\alpha t_i)^{\beta}\right) - \left(n-(r+f)\right)\ln(1+\left(\alpha\tau^*)^{\beta}\right) \tag{20}$$

Finally, the partial derivatives of the Log-Likelihood with respect to α and β in both cases is given by:

$$\frac{\partial L}{\partial\alpha} = \frac{r\beta}{\alpha} - \frac{r\beta(\alpha t_0)^{\beta}}{\alpha(1+(\alpha t_0)^{\beta})} + \frac{f\beta}{\alpha} - 2\sum_{i=r+1}^{i=r+f} \frac{\beta(\alpha t_i)^{\beta}}{\alpha(1+(\alpha t_i)^{\beta})} - \frac{\left(n-(r+f)\right)\beta(\alpha\tau^*)^{\beta}}{\alpha(1+(\alpha\tau^*)^{\beta})}$$
$$= 0 \tag{21}$$

$$\frac{\partial L}{\partial\beta} = r\ln(\alpha t_0) - \frac{r(\alpha t_0)^{\beta}\ln(\alpha t_0)}{(1+(\alpha t_0)^{\beta})} + \sum_{i=r+1}^{i=r+f} \frac{1+\beta\ln(\alpha t_i)}{\beta} - 2\sum_{i=r+1}^{i=r+f} \frac{(\alpha t_i)^{\beta}\ln(\alpha t_i)}{(1+(\alpha t_i)^{\beta})}$$
$$- \frac{(n-(r+f))(\alpha\tau^*)^{\beta}\ln(\alpha\tau^*)}{(1+(\alpha\tau^*)^{\beta})} = 0 \tag{22}$$

Note that the likelihood in both cases is reduced to

$$L(\alpha,\beta) = \prod_{i=1}^{f} \frac{\alpha\beta(\alpha t_i)^{\beta-1}}{\left(1+(\alpha t_i)^{\beta}\right)^2} \cdot \left\{\left(1+(\alpha\tau^*)^{\beta}\right)^{f-n}\right\} \tag{23}$$

when $t_0 = 0$ or in other words if there is no left censored units. And the partial derivative equations of the Log-Likelihood function with respect to α and β will be reduced to:

$$\frac{\partial L}{\partial \alpha} = \frac{f\beta}{\alpha} - 2\sum_{i=1}^{i=f} \frac{\beta(\alpha t_i)^{\beta}}{\alpha(1+(\alpha t_i)^{\beta})} - \frac{(n-f)\beta(\alpha\tau^*)^{\beta}}{\alpha(1+(\alpha\tau^*)^{\beta})} = 0 \tag{24}$$

$$\frac{\partial L}{\partial \beta} = \sum_{i=1}^{i=f} \frac{1+\beta\ln(\alpha t_i)}{\beta} - 2\sum_{i=1}^{i=f} \frac{(\alpha t_i)^{\beta}\ln(\alpha t_i)}{(1+(\alpha t_i)^{\beta})} - \frac{(n-f)(\alpha\tau^*)^{\beta}\ln(\alpha\tau^*)}{(1+(\alpha\tau^*)^{\beta})} = 0 \tag{25}$$

It is clear that the solution of equations (21) and (22) cannot be obtained explicitly due to the non-linearity type of the equations and therefore, in order to solve such equations we need to use some numerical techniques such as Newton Raphson method (NR) and the use of this technique is conditional on its convergence as it does not converge all the time, or we can use the (EM) algorithm to solve these equations.

2.2 The Predictive Likelihood Function and the MLP

Our main objective of this thesis is to discuss the prediction of $Y = T_{p+(r+f)}$ where $p = 1, \dots\dots, n-(r+f)$ for all the $n-(r+f)$ unobserved or censored units based on the observed data $T = (T_{r+1}, \dots\dots . T_{r+f})$.

Due to the property of Markov for Censored order statistic, the conditional distribution of Y given $T = t$ is the same as the distribution of the p^{th} order statistic of sample of size $n-(r+f)$ from a population with cumulative distribution function:

$G(y) = \frac{F(y)-F(\tau^*)}{1-F(\tau^*)}$ for all $y > \tau^*$ (left truncated at τ^*). The density function of which is $g(y) = \frac{d}{dy}G(y) = \frac{f(y)}{1-F(\tau^*)}$. Therefore, the conditional density of $Y = T_{p+(r+f)}$

given $T = t = (t_{r+1}, \dots \dots . t_{r+f})$ for all $y > \tau^*$ is given by:

$f(y/t) = \frac{(n-(r+f))!}{(p-1)!(n-(r+f)-p)!}\left(G(y)\right)^{p-1}\left(1 - G(y)\right)^{n-(r+f)-p} g(y)$. Knowing that:

$G(y) = \frac{F(y)-F(\tau^*)}{1-F(\tau^*)}$ $and\ g(y) = \frac{f(y)}{1-F(\tau^*)}$, we get:

$$f(y/t) = \frac{\left(n-(r+f)\right)!}{(p-1)!\,(n-(r+f)-p)!}\left(\frac{F(y)-F(\tau^*)}{1-F(\tau^*)}\right)^{p-1}\cdot\left(1-\left(\frac{F(y)-F(\tau^*)}{1-F(\tau^*)}\right)\right)^{n-(r+f)-p}\cdot\frac{f(y)}{1-F(\tau^*)}$$

$$f(y/t) = p\binom{n-(r+f)}{p}\cdot\left(\frac{(\alpha y)^\beta}{1+(\alpha y)^\beta}-\frac{(\alpha\tau^*)^\beta}{1+(\alpha\tau^*)^\beta}\right)^{p-1}\cdot\left(\frac{1}{1+(\alpha y)^\beta}\right)^{n-(r+f)-p}\cdot\frac{\alpha\beta(\alpha y)^{\beta-1}}{\left(1+\left(\alpha y\right)^\beta\right)^2}\cdot\left(\frac{1}{1+(\alpha\tau^*)^\beta}\right)^{r+f-n}$$

Simplifying $\left(\frac{(\alpha y)^\beta}{1+(\alpha y)^\beta}-\frac{(\alpha\tau^*)^\beta}{1+(\alpha\tau^*)^\beta}\right)^{p-1}$ we get:

$\left(\frac{(\alpha y)^\beta}{1+(\alpha y)^\beta}-\frac{(\alpha\tau^*)^\beta}{1+(\alpha\tau^*)^\beta}\right)^{p-1} = \left[(\alpha y)^\beta-(\alpha\tau^*)^\beta\right]^{p-1}\cdot\left(1+(\alpha y)^\beta\right)^{1-p}\cdot(1+(\alpha\tau^*)^\beta)^{1-p}$ and hence

$$f(y/t) = p\binom{n-(r+f)}{p}\left[(\alpha y)^\beta-(\alpha\tau^*)^\beta\right]^{p-1}\left(1+(\alpha y)^\beta\right)^{-n+(r+f)-1}\cdot\left(1+(\alpha\tau^*)^\beta\right)^{n-(r+f)-p+1}\cdot\alpha\beta(\alpha y)^{\beta-1}$$

for all $y > \tau^*$, $p = 1, \dots \dots ., n-(r+f)$. (26)

2.2.1 Likelihood Prediction Approach

We will apply the Maximum Likelihood to the joint distribution of the prediction of Y and the likelihood of the present distribution. Referring to Valiollahi et al. (2017) The predictive likelihood function (PLF) of $Y\ and\ (\alpha, \beta)$ is given by:

$L(y, \alpha, \beta/t) = f(y/t\,, \alpha, \beta)\cdot g(t/\alpha\,, \beta)$

where $f(y/t\,, \alpha, \beta)$ is the conditional density of Y and $g(t/\alpha\,, \beta)$ is the likelihood of the present log-logistic distribution. Hence

$$L(y, \alpha, \beta) = p\binom{n-(r+f)}{p}\cdot\left[(\alpha y)^\beta-(\alpha\tau^*)^\beta\right]^{p-1}\cdot\left(1+(\alpha y)^\beta\right)^{-n+(r+f)-1}\cdot(1+(\alpha\tau^*)^\beta)^{n-(r+f)-p+1}$$

$$\cdot\,\alpha\beta(\alpha y)^{\beta-1}\cdot\frac{(\alpha t_0)^{r\beta}}{\left(1+(\alpha t_0)^\beta\right)^r}\cdot\prod_{i=r+1}^{r+f}\frac{\alpha\beta(\alpha t_i)^{\beta-1}}{\left(1+\left(\alpha t_i\right)^\beta\right)^2}\cdot\left\{\left(1+(\alpha\tau^*)^\beta\right)^{(r+f)-n}\right\}$$

Which could be simplified further to:

$$L(y,\alpha,\beta) = p\binom{n-(r+f)}{p} \cdot \left[(\alpha y)^{\beta} - (\alpha\tau^{*})^{\beta}\right]^{p-1} \cdot \left(1+(\alpha y)^{\beta}\right)^{-n+(r+f)-1} \cdot (1+(\alpha\tau^{*})^{\beta})^{-p+1}$$

$$\cdot\, \alpha\beta(\alpha y)^{\beta-1} \cdot \frac{(\alpha t_0)^{r\beta}}{(1+(\alpha t_0)^{\beta})^{r}} \cdot \prod_{i=r+1}^{r+f} \frac{\alpha\beta(\alpha t_i)^{\beta-1}}{(1+(\alpha t_i)^{\beta})^{2}} \qquad (27)$$

Applying the Logarithm of the predictive likelihood function and ignoring the constant:

$$\ln L(y,\alpha,\beta) = (p-1)\ln\left[(\alpha y)^{\beta} - (\alpha\tau^{*})^{\beta}\right] + ((r+f)-n-1)\ln(1+(\alpha y)^{\beta}) + (1-p)\ln\left(1+(\alpha\tau^{*})^{\beta}\right)$$

$$+ (f+1)\ln(\alpha\beta) + (\beta-1)\ln(\alpha y) + r\beta\ln(\alpha t_0) - r\ln\left(1+(\alpha t_0)^{\beta}\right) +$$

$$(\beta-1)\sum_{i=r+1}^{i=r+f}\ln(\alpha t_i) - 2\sum_{i=r+1}^{i=r+f}\ln(1+(\alpha t_i)^{\beta}) \qquad (28)$$

We will use the partial derivatives of the Log of the Predictive Likelihood function to obtain the Predictive Likelihood equations (PLEs)

The partial derivatives of $\ln L(y,\alpha,\beta)$ are as follows:

$$\frac{\partial \ln L(y,\alpha,\beta)}{\partial y} = (p-1)\cdot\alpha\beta\cdot\frac{(\alpha y)^{\beta-1}}{(\alpha y)^{\beta}-(\alpha\tau^{*})^{\beta}} + [(r+f)-n-1]\cdot\alpha\beta\cdot\frac{(\alpha y)^{\beta-1}}{1+(\alpha y)^{\beta}} + \frac{\beta-1}{y}$$

$$\frac{\partial \ln L(y,\alpha,\beta)}{\partial y} = \frac{1}{y}\left[\frac{(p-1)\beta(\alpha y)^{\beta}}{(\alpha y)^{\beta}-(\alpha\tau^{*})^{\beta}} + \frac{(r+f-n-1)\beta(\alpha y)^{\beta}}{1+(\alpha y)^{\beta}} + \beta - 1\right] \qquad (29)$$

$$\frac{\partial \ln L(y,\alpha,\beta)}{\partial \alpha} = \frac{(p-1)\left[\beta y(\alpha y)^{\beta-1} - \beta\tau^{*}(\alpha\tau^{*})^{\beta-1}\right]}{(\alpha y)^{\beta}-(\alpha\tau^{*})^{\beta}} + \frac{(r+f-n-1)\beta y(\alpha y)^{\beta-1}}{1+(\alpha y)^{\beta}} + \frac{(1-p)\beta\tau^{*}(\alpha\tau^{*})^{\beta-1}}{1+(\alpha\tau^{*})^{\beta}}$$

$$+\frac{f+1}{\alpha} + \frac{\beta-1}{\alpha} + \frac{r\beta}{\alpha} - \frac{r\beta t_0(\alpha t_0)^{\beta-1}}{1+(\alpha t_0)^{\beta}} +$$

$$(\beta-1)\sum_{i=r+1}^{i=r+f}\frac{1}{\alpha} - 2\sum_{i=r+1}^{i=r+f}\frac{\beta t_i(\alpha t_i)^{\beta-1}}{1+(\alpha t_i)^{\beta}}$$

After some simplifications, we get:

$$\frac{\partial \ln L(y,\alpha,\beta)}{\partial \alpha} = \frac{\beta}{\alpha}\left((p-1) + \frac{(r+f-n-1)(\alpha y)^{\beta}}{1+(\alpha y)^{\beta}} + \frac{(1-p)(\alpha\tau^{*})^{\beta}}{1+(\alpha\tau^{*})^{\beta}} + (f+r+1) \right.$$

$$\left. - \frac{r(\alpha t_0)^{\beta}}{1+(\alpha t_0)^{\beta}} - 2\sum_{i=r+1}^{i=r+f} \frac{(\alpha t_i)^{\beta}}{1+(\alpha t_i)^{\beta}} \right) = 0 \tag{30}$$

Similarly, after deriving $\ln L(y,\alpha,\beta)$ with respect to β with some simplifications, we get:

$$\frac{\partial \ln L(y,\alpha,\beta)}{\partial \beta} = (p-1)\frac{(\alpha y)^{\beta}\ln(\alpha y) - (\alpha\tau^{*})^{\beta}\ln(\alpha\tau^{*})}{(\alpha y)^{\beta} - (\alpha\tau^{*})^{\beta}} + (r+f-n-1)\frac{(\alpha y)^{\beta}\ln(\alpha y)}{1+(\alpha y)^{\beta}}$$

$$+ (1-p)\frac{(\alpha\tau^{*})^{\beta}\ln(\alpha\tau^{*})}{1+(\alpha\tau^{*})^{\beta}} + \frac{(f+1)}{\beta} + \ln(\alpha y) + r\ln(\alpha t_0) - \frac{r(\alpha t_0)^{\beta}\ln(\alpha t_0)}{1+(\alpha t_0)^{\beta}}$$

$$+ \sum_{i=r+1}^{i=r+f}\ln(\alpha t_i) - 2\sum_{i=r+1}^{i=r+f}\frac{(\alpha t_i)^{\beta}\ln(\alpha t_i)}{1+(\alpha t_i)^{\beta}} = 0 \tag{31}$$

Solving equation(29)will give the Maximum Likelihood Predictor of Y;($\widetilde{Y_{MLP}}$)

$\frac{\partial \ln L(y,\alpha,\beta)}{\partial y} = 0$, we get $\frac{(p-1)\beta(\alpha y)^{\beta}}{(\alpha y)^{\beta}-(\alpha\tau^{*})^{\beta}} + \frac{(r+f-n-1)\beta(\alpha y)^{\beta}}{1+(\alpha y)^{\beta}} + \beta - 1 = 0$

$(p-1)\beta(\alpha y)^{\beta}\left[1+(\alpha y)^{\beta}\right] + (r+f-n-1)\beta(\alpha y)^{\beta}\left[(\alpha y)^{\beta} - (\alpha\tau^{*})^{\beta}\right] + (\beta-1)\left[(\alpha y)^{\beta} - (\alpha\tau^{*})^{\beta}\right]\left[1+(\alpha y)^{\beta}\right] = 0$. Let $(\alpha y)^{\beta} = X$ and $(r+f-n-1) = K$

Then $(p-1)\beta X[1+X] + K\beta X\left[X - (\alpha\tau^{*})^{\beta}\right] + (\beta-1)\left[X - (\alpha\tau^{*})^{\beta}\right][1+X] = 0$ which will be reduced to the quadratic equation in X:

$[(p-1)\beta + K\beta + \beta - 1]X^2 + [(p-1)\beta - (\alpha\tau^{*})^{\beta}K\beta + (\beta-1) - (\alpha\tau^{*})^{\beta}]X - (\beta-1)(\alpha\tau^{*})^{\beta} = 0$.

After some simplification we get:

$$[p\beta + K\beta - 1]X^2 + [p\beta - (\alpha\tau^{*})^{\beta}K\beta - 1 - (\alpha\tau^{*})^{\beta}]X - (\beta-1)(\alpha\tau^{*})^{\beta} = 0 \tag{32}$$

We know that $p < n-(r+f)$ and $(r+f-n-1) = K$, then $n-(r+f) = -K-1$ and $p < -K-1,\ p\beta < -K\beta - \beta$, hence $p\beta + K\beta < 0$ and $p\beta + K\beta - 1 < 0$.

Let $A = p\beta + K\beta - 1$, $B = [(p-1)\beta - (\alpha\tau^{*})^{\beta}K\beta + (\beta-1) - (\alpha\tau^{*})^{\beta}]$ and

$C = -(\beta - 1)(\alpha\tau^*)^\beta$. Since $A < 0$ then if we choose $\beta < 1$, C will be positive and $B^2 - 4AC > 0$ and hence the above quadratic equation will have solutions. Choosing the positive solution, we get:

$X = \frac{-B \pm \sqrt{B^2-4AC}}{2A}$, $(\alpha y)^\beta = \frac{-B \pm \sqrt{B^2-4AC}}{2A}$ and so

$$\widetilde{Y_{MLP}} = \frac{1}{\alpha}[\frac{-B \pm \sqrt{B^2 - 4AC}}{2A}]^{\frac{1}{\beta}} \tag{33}$$

(If $\beta \geq 1$, $\widetilde{Y_{MLP}}$ could be obtained iff $B^2 - 4AC > 0$)

Since we do not know the values of α and β we may replace them with the predictive maximum likelihood estimates $\tilde{\alpha}$ and $\tilde{\beta}$.

2.3 Other Predictors

2.3.1 Conditional Prediction Approach

According to Valiollahi et al. (2017), a conditional predictor could be used to predict the future values of $Y = T_{p+(r+f)}$ where $p = 1, \dots \dots, n - (r + f)$ using the conditional distribution of $Y = T_{p+(r+f)}$ given $T = (T_{r+1}, \dots \dots . T_{r+f})$. This conditional predictor Y_{Cond} is the best unbiased predictor (BUP) of Y which means that the mean of $(Y_{Cond} - Y)$ is zero (unbiased predictor) and the variance Var $(Y_{Cond} - Y)$ is less than or equal Var $(Y_{Predictor} - Y)$ for all unbiased predictor values $Y_{Predictor}$ of Y. This (BUP) of Y is given by: $Y_{Cond} = E(Y/T) = \int_{\tau^*}^{\infty} y \cdot f(y/t, \alpha, \beta) dy$ where τ^* is the time where the experiment stops.

$$Y_{Cond} = \int_{\tau^*}^{\infty} y \cdot p \binom{n-(r+f)}{p} \cdot \left[(\alpha y)^\beta - (\alpha\tau^*)^\beta\right]^{p-1} \cdot \left(1 + (\alpha y)^\beta\right)^{-n+(r+f)-1} \cdot$$

$$(1 + (\alpha\tau^*)^\beta)^{n-(r+f)-p+1} \cdot \alpha\beta(\alpha y)^{\beta-1} \cdot dy$$

$$= p \binom{n-(r+f)}{p} \cdot (1 + (\alpha\tau^*)^\beta)^{n-(r+f)-p+1} \cdot I_1$$

where $I_1 = \int_{\tau^*}^{\infty} y \cdot \left[(\alpha y)^{\beta} - (\alpha\tau^*)^{\beta}\right]^{p-1} \cdot \left(1 + (\alpha y)^{\beta}\right)^{-n+(r+f)-1} \cdot \ \alpha\beta(\alpha y)^{\beta-1} \cdot dy$

Using the binomial expansion:

$$\left[(\alpha y)^{\beta} - (\alpha\tau^*)^{\beta}\right]^{p-1} = \sum_{k=0}^{p-1} \binom{p-1}{k} (-1)^{p-1-k} \cdot \left((\alpha y)^{\beta}\right)^{k} \cdot \left((\alpha\tau^*)^{\beta}\right)^{p-1-k} \quad (34)$$

Hence $I_1 = \int_{\tau^*}^{\infty} y \cdot \sum_{k=0}^{p-1} \binom{p-1}{k} (-1)^{p-1-k} \cdot \left((\alpha y)^{\beta}\right)^{k} \cdot \left((\alpha\tau^*)^{\beta}\right)^{p-1-k} \cdot$

$$\left(1 + (\alpha y)^{\beta}\right)^{-n+(r+f)-1} \cdot \ \alpha\beta(\alpha y)^{\beta-1} \cdot dy$$

$$I_1 = \sum_{k=0}^{P-1} \binom{p-1}{k} (-1)^{p-1-k} \cdot \left((\alpha\tau^*)^{\beta}\right)^{p-1-k} \cdot \beta \int_{\tau^*}^{\infty} \left((\alpha y)^{\beta}\right)^{k+1} . \left(1 + (\alpha y)^{\beta}\right)^{-n+(r+f)-1} dy \quad (35)$$

We know that $1 - \frac{1}{1+(\alpha y)^{\beta}} = \frac{(\alpha y)^{\beta}}{1+(\alpha y)^{\beta}}$, let $t = \frac{1}{1+(\alpha y)^{\beta}}$ then $(\alpha y)^{\beta} = \frac{1-t}{t}$ and $1 + (\alpha y)^{\beta} = \frac{1}{t}$ and $dy = \frac{-dt}{\alpha\beta(\alpha y)^{\beta-1}t^2}$ as $y = \tau^*, t = \frac{1}{1+ (\alpha\tau^*)^{\beta}}$, as $y \to \infty, t \to 0$ and the integral $I_2 = \int_{\tau^*}^{\infty} \left((\alpha y)^{\beta}\right)^{k+1} \cdot \left(1 + (\alpha y)^{\beta}\right)^{-n+(r+f)-1} dy$ will become

$$I_2 = \int_{\frac{1}{1+ (\alpha\tau^*)^{\beta}}}^{0} \left(\frac{1-t}{t}\right)^{k+1} \cdot \left(\frac{1}{t}\right)^{-n+(r+f)-1} \cdot \frac{-dt}{\alpha\beta(\alpha y)^{\beta-1}t^2} .$$

Replacing $(\alpha y)^{\beta-1}$ with$\left(\frac{1-t}{t}\right)^{\frac{\beta-1}{\beta}}$, then

$$I_2 = \int_{\frac{1}{1+ (\alpha\tau^*)^{\beta}}}^{0} (1-t)^{k+1} \cdot (t)^{-k-1} \cdot (t)^{n-(r+f)+1} \cdot (-dt) \cdot \frac{1}{\alpha\beta} \cdot (t)^{-2} \cdot (t)^{\frac{\beta-1}{\beta}} \cdot (1-t)^{\frac{1-\beta}{\beta}}$$

$$I_2 = \frac{1}{\alpha\beta} \cdot \int_{0}^{\frac{1}{1+ (\alpha\tau^*)^{\beta}}} (t)^{n-(r+f)-k-1-\frac{1}{\beta}} \cdot (1-t)^{k+\frac{1}{\beta}} dt \quad (36)$$

$$Y_{Cond} = p\binom{n-(r+f)}{p} \cdot \left(1+(\alpha\tau^*)^\beta\right)^{n-(r+f)-p+1} \cdot \sum_{k=0}^{p-1} \binom{p-1}{k} (-1)^{p-1-k}$$

$$\cdot \left((\alpha\tau^*)^\beta\right)^{p-1-k} \cdot \frac{1}{\alpha} \cdot \int_0^{\frac{1}{1+(\alpha\tau^*)^\beta}} (t)^{n-(r+f)-k-1-\frac{1}{\beta}} \cdot (1-t)^{k+\frac{1}{\beta}} dt \quad (37)$$

and since α and β are unknown, one can estimate them by their MLEs to obtain the Y_{Cond}.

Note that, I_2 can be expressed as $\frac{1}{\alpha\beta} \cdot B(\frac{1}{1+(\alpha\tau^*)^\beta}; n-(r+f)-k-\frac{1}{\beta}, k+\frac{1}{\beta}+1)$

where $B(z; a, b)$ is the incomplete beta function defined as:

$B(z; a, b) = \int_0^z (t)^{a-1} \cdot (1-t)^{b-1} dt$ for $0 \le z < 1$

2.3.2 Conditional Median Predictor

According to Valiollahi et al. (2017), another predictor that could be used for the prediction of future values of Y is the conditional median predictor Y_{med} which is the median of the conditional distribution of Y given $T = (T_{r+1}, \dots \dots . T_{r+f})$. That is:

$P(Y \le Y_{med} / T = (T_{r+1}, \dots \dots . T_{r+f})) = P(Y \ge Y_{med} / T = (T_{r+1}, \dots \dots . T_{r+f}))$.

Consider the distribution $B(y) = \frac{F(y)-F(\tau^*)}{1-F(\tau^*)}$ where F is the CDF of the Log-Logistic distribution.

$$\mathrm{B(y)} = \left(\frac{(\alpha y)^\beta}{1+(\alpha y)^\beta} - \frac{(\alpha\tau^*)^\beta}{1+(\alpha\tau^*)^\beta}\right) \cdot \frac{1}{1-\mathrm{F}(\tau^*)} = \left(\frac{(\alpha y)^\beta}{1+(\alpha y)^\beta} - \frac{(\alpha\tau^*)^\beta}{1+(\alpha\tau^*)^\beta}\right) \cdot (1+(\alpha\tau^*)^\beta)$$

$$\mathrm{B(y)} = \frac{(\alpha y)^\beta - (\alpha\tau^*)^\beta}{1+(\alpha y)^\beta} \quad (38)$$

Following the result of Asgharzadeh et al. (2015) and Valiollahi et al. (2017), $B(y)$ can be considered as a Betta distribution $B(p, n-r-f-p+1)$. It is well known that if X has Beta distribution with parameters a and b then the pdf of X is given by:

$f(x) = \begin{cases} \frac{x^{a-1}(1-x)^{b-1}}{B(a,b)} & for\ x, a, b > 0 \\ 0 & otherwise \end{cases}$ and $B(a,b) = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}$ and Γ is the Gamma function.

$$P(Y \leq Y_{med}/T = (T_{r+1}, \dots \dots . T_{r+f}))$$

$$= P(\frac{(\boldsymbol{\alpha Y})^{\boldsymbol{\beta}} - (\boldsymbol{\alpha\tau}^*)^{\boldsymbol{\beta}}}{\mathbf{1}+(\boldsymbol{\alpha Y})^{\boldsymbol{\beta}}} \leq \frac{(\boldsymbol{\alpha} Y_{med})^{\boldsymbol{\beta}} - (\boldsymbol{\alpha\tau}^*)^{\boldsymbol{\beta}}}{\mathbf{1}+(\boldsymbol{\alpha} Y_{med})^{\boldsymbol{\beta}}}/T = (T_{r+1}, \dots \dots . T_{r+f}))$$

which is equivalent to $P\left(B \leq \frac{(\boldsymbol{\alpha} Y_{med})^{\boldsymbol{\beta}} - (\boldsymbol{\alpha\tau}^*)^{\boldsymbol{\beta}}}{\mathbf{1}+(\boldsymbol{\alpha} Y_{med})^{\boldsymbol{\beta}}}\right) = 0.5$ and hence $\frac{(\boldsymbol{\alpha} Y_{med})^{\boldsymbol{\beta}} - (\boldsymbol{\alpha\tau}^*)^{\boldsymbol{\beta}}}{\mathbf{1}+(\boldsymbol{\alpha} Y_{med})^{\boldsymbol{\beta}}}$ =Med(B)

(Median of the Betta distribution) So, $(\alpha Y_{med})^{\beta} = \frac{\text{Med}(B)+(\boldsymbol{\alpha\tau}^*)^{\boldsymbol{\beta}}}{1-\text{Med}(B)}$ and

$$Y_{med} = \frac{1}{\alpha} \cdot \left[\frac{\text{Med}(B)+(\boldsymbol{\alpha\tau}^*)^{\boldsymbol{\beta}}}{1-\text{Med}(B)}\right]^{\frac{1}{\beta}} \qquad (39)$$

To obtain the conditional median predictor Y_{med}, we need to substitute α and β by their MLEs as both of them are unknown.

2.4 Prediction Intervals

2.4.1 Pivotal Method

We will find prediction intervals (PI) for $Y = T_{p+(r+f)}$ based on the hybrid censored sample $T = (T_{r+1}, \dots \dots . T_{r+f})$. Consider the random variable

$$Z = B(y) = \frac{(\boldsymbol{\alpha y})^{\boldsymbol{\beta}} - (\boldsymbol{\alpha\tau}^*)^{\boldsymbol{\beta}}}{\mathbf{1}+(\boldsymbol{\alpha y})^{\boldsymbol{\beta}}} \ .$$

As mentioned before Z has a Betta distribution $B(p, n-r-f-p+1)$ the distribution of Z does not depend on the parameters α and β and it could be used as a pivotal quantity to obtain $(1-\gamma)100\%$ prediction interval of Y (PI).

If B_{γ} is the $100^{\gamma th}$ percentile of $B(p, n-r-f-p+1)$ then $(1-\gamma)100\%$ (PI) of is (a_1, b_1) where

$$a_1 = \frac{1}{\alpha} \cdot \left[\frac{B_{\frac{\gamma}{2}} + (\alpha\tau^*)^\beta}{1 - B_{\frac{\gamma}{2}}}\right]^{\frac{1}{\beta}} \qquad (40)$$

$B_{\frac{\gamma}{2}}$ stands for the $100\frac{\gamma}{2}^{th}$ percentile and

$$a_2 = \frac{1}{\alpha} \cdot \left[\frac{B_{1-\frac{\gamma}{2}} + (\alpha\tau^*)^\beta}{1 - B_{1-\frac{\gamma}{2}}}\right]^{\frac{1}{\beta}} \qquad (41)$$

In order to obtain the lower and upper bounds of the prediction intervals, the parameters α and β have to be replaced by their MLEs.

2.4.2 Highest Density Interval

Another Prediction interval approach that can be used to predict $Y = T_{p+(r+f)}$ is the highest density interval approach which can be briefly described as the interval that contains all points that have a probability density higher than that of points outside the interval.

The following figure shows the difference between 80% Highest Density interval (HDI) and 80% Symmetrical Density interval (as an example)

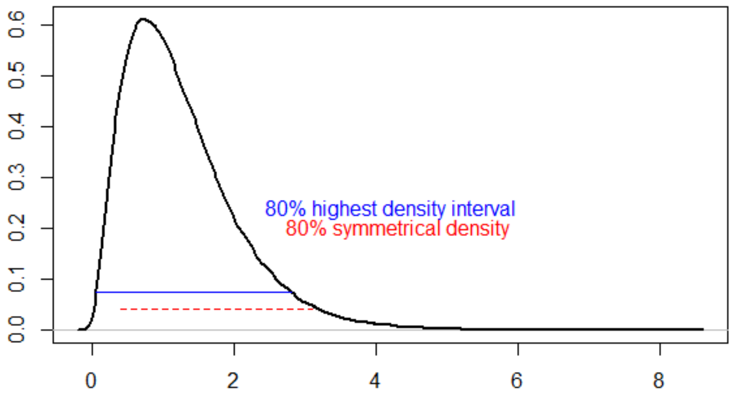

Figure 3: Difference between (HDI) and Symmetrical Density Interval

The (HDI) can be used for unimodal distributions and it could be considered as the narrowest interval containing the required density. Since the distribution of $B(y)$ is beta distribution $B(p, n-r-f-p+1)$ which is unimodal then the (HDI) prediction approach can be used to obtain prediction interval for $Y = T_p$.

The $(1-\gamma)100\%$ (HDI) for prediction of Y is (a_2, b_2) where a_2 and b_2 are the lower and upper limits respectively of the (HDI) prediction interval and are defined by:

$$a_2 = \frac{1}{\alpha} \cdot \left[\frac{w_1 + (\boldsymbol{\alpha\tau^*})^{\boldsymbol{\beta}}}{1 - w_1}\right]^{\frac{1}{\beta}} \qquad (42)$$

and $b_2 = \frac{1}{\alpha} \cdot \left[\frac{w_2 + (\boldsymbol{\alpha\tau^*})^{\boldsymbol{\beta}}}{1 - w_2}\right]^{\frac{1}{\beta}}$ (43)

and w_1, w_2 are defined as:

$\int_{w_1}^{w_2} g(z)dz = 1 - \gamma$ and $g(w_1) = g(w_2)$

$$\int_{w_1}^{w_2} g(z)dz = \int_{0}^{w_2} g(z)dz - \int_{0}^{w_1} g(z)dz$$

$$\int_0^{w_2} \frac{z^{p-1}(1-z)^{n-r-f-p}}{B(p, n-r-f-p-1)} dz - \int_0^{w_1} \frac{z^{p-1}(1-z)^{n-r-f-p}}{B(p, n-r-f-p-1)} dz = 1 - \gamma$$

$$B_{w_2}(p, n-r-f-p+1) - B_{w_2}(p, n-r-f-p+1) = 1 - \gamma \qquad (44)$$

where $B_u(m, n) = \frac{1}{B(m,n)} \int_0^u y^{m-1} \cdot (1-y)^{n-1} dy$ is the incomplete beta integral.

Solving $g(w_1) = g(w_2)$, we get $(\frac{w_1}{w_2})^{p-1} = (\frac{1-w_2}{1-w_1})^{n-r-f-1}$ (45)

it is clear that if $p = 1$, we get $(\frac{1-w_2}{1-w_1})^{n-r-f-1} = 1$ which gives $w_1 = w_2$ and then no prediction interval can be obtained.

In order to obtain the lower and upper bounds of the (HDI), w_1 and w_2 are the solutions of equations (44) and (45); the parameters α and β could be replaced by their MLEs.

CHAPTER 3: SIMULATION STUDY AND SOME EXAMPLES

3.1 Simulation Design and the Simulation Algorithm

It is almost impossible to study the performance of different predictors theoretically, and hence a simulation study will help us investigate such performance and come up with conclusions. In this chapter we will carry out a simulation study using the R programming software Ihaka and Gentleman (1996). The R software that is used to produce graph of the pdf of Log-Logistic function for different parameters, perform calculations and produce results.

In addition, some packages such as (optimr), (zipfR),Evert, Baroni, and Evert (2006) and (HDInterval) Meredith and Kruschke are used. The (optimr) package is used to maximize functions being given a restriction on the parameters, the (zipfR) package is used to evaluate beta and incomplete beta integrals, and the (HDInterval) package is used to find the Highest Density Intervals. The simulation is based on Hybrid Censoring Scheme type-I and Type-II and could be illustrated by the following algorithm:

A sample of size n is generated from a Log-Logistic distribution with parameters α and β. The sample is generated using the uniform distribution and (CDF) of Log-Logistic distribution such that $F(t) = u, t = F^{-1}(u) = \frac{1}{\alpha} \cdot \left(\frac{u}{1-u}\right)^{\frac{1}{\beta}}$ (46)

where F is the CDF of the Log-Logistic distribution and u represents generated numbers from uniform distribution. r values will be left censored at t_0, m values will be observed failures, and the $n - r - m$ remaining units will be right censored at t_1. The preassigned number of failures is m_1.

Different choices of n, t_0, t_1, and m_1 are considered. The ordered pair (α, β) is taken respectively as $(1,1)$ and $(3,2)$, the sample size n is taken as 30,50 and 80. The choice of t_0, t_1, and m_1 is as follows:

t_0 is equal to 0 (no left censored units) or $F(t_0) = 0.1$ that is $\frac{(\alpha t_0)^\beta}{1+(\alpha t_0)^\beta} = 0.1, (\alpha t_0)^\beta = \frac{1}{9}$

t_1 is chosen so that $F(t_1) = \begin{cases} 0.5 \quad Heavy\ Censoring \\ 0.7 \quad Moderate\ Censoring \end{cases}$

i) $F(t_1)$ =0.5, $\frac{(\alpha t_1)^\beta}{1+(\alpha t_1)^\beta} = 0.5$ which will be reduced to $(\alpha t_1)^\beta = 1$

ii) $F(t_1)$ =0.7, $\frac{(\alpha t_1)^\beta}{1+(\alpha t_1)^\beta} = 0.7$ which will be reduced to $(\alpha t_1)^\beta = \frac{7}{3}$

For each value of t_1, m_1 will be chosen such that $m_1 = n\big(F(t_1) - F(t_0)\big)$ and so $m_1 = n(F(t_1) - 0.1)$ (Here m_1 is the expected number of failures between t_0 and t_1)

When $(\alpha, \beta) = (1,1)$, $t_0 = \frac{1}{9}$

$t_1 = 1$, the corresponding pairs (n, m_1) will be $(30,12), (50,20) and\ (80,32)$

$t_1 = \frac{7}{3}$, the corresponding pairs (n, m_1) will be $(30,18), (50,30) and\ (80,48)$

When $(\alpha, \beta) = (3,2)$, $t_0 = \frac{1}{9}$

$t_1 = \frac{1}{3}$, the corresponding pairs (n, m_1) will be $(30,12), (50,20) and\ (80,32)$

$t_1 = \sqrt{\frac{7}{27}}$, the corresponding pairs (n, m_1) will be $(30,18), (50,30) and\ (80,48)$

A special case is to be considered for a few particular runs of simulations when $n = 30$ $t_0 = \frac{1}{9}$, $t_1 = \frac{7}{3}$ and $m_1 = 18$ for $p = 4\ , 5$. Since the (BUP) involves the incomplete beta integral $B(\frac{1}{1+(\alpha\tau^*)^\beta}; n-(r+f)-k-\frac{1}{\beta}, k+\frac{1}{\beta}+1)$, and both the (CMP), and Prediction intervals involve the beta integral $B(p, n-r-f-p+1)$; to avoid the problem of obtaining the parameters of the above incomplete beta and beta distributions as negative values, we can replace t_0 with $\frac{1}{18}$ to decrease the value of r (number of left censored values). Moreover, it would be interesting to study the performance of different point predictors and prediction intervals for small sample sizes mainly for $n = 20$, we will stick to $t_0 = 0$, $m_1 = 8$ and $m_1 = 12$, t_1 will take the same values as designed through all simulation design.

Based on each sample we obtain from the i^{th} iteration, $i = 1,,,,,2000,$ we will predict the future values of $Y = T_{p+(r+f)}$,$p = 1,2 \ldots n - (r + f)$. Replicate the process for 2000 times and report the bias and mean square error (MSE) for α and β, (MLP), (BUP), (CMP) and will find the prediction intervals.

In tables 1 and 2, we will report the Bias and MSE of α and β when $(\alpha, \beta) = (1,1) and\ (3,2)$ respectively. The Bias and MSE of (α, β) are calculated as follows:

Suppose that $\widetilde{\theta_i}$ is the estimator value of $\theta = \alpha$ or $\theta = \beta$ obtained from the i^{th} iteration, ($i = 1,,,,,N = 2000$)

then Bias= $\frac{1}{N}(\sum_{i=1}^{N}(\widetilde{\theta_i} - \theta))$ (47)

and MSE= $\frac{1}{N}(\sum_{i=1}^{N}((\widetilde{\theta_i} - \theta)^2)$ (48)

In tables 3 to 12, we will report the Bias and MSE of each point predictor (MLP, BUP and CMP) when $(\alpha, \beta) = (1,1) and\ (3,2)$ respectively. Moreover, the convergence probabilities and the lengths of the predictive intervals will be reported.

Similarly, if $\widetilde{y_i}$ is the value of the predictor of $Y = T_{p+(r+f)}$ obtained from the i^{th} iteration,($i = 1,,,,,N = 2000$) then Bias= $\frac{1}{N}(\sum_{i=1}^{N}(\widetilde{y_i} - Y))$ (49)

and MSE= $\frac{1}{N}(\sum_{i=1}^{N}((\widetilde{y_i} - Y)^2)$ (50)

In addition to the point prediction estimates, we will obtain the interval estimation. The intervals obtained are the classical intervals obtained from pivotal approach and high-density intervals. The simulation study will estimate the convergence probability and the length of the interval. The convergence probability we are interested in estimating is:

Converge Probability= $P(L \leq Y \leq U) = 1 - \gamma$ where L and U are the lower and upper bounds of the interval, $\gamma = 5\%$ and the length of the interval we want to estimate is length= $U - L$.

The above converge probability and length are estimated in the following algorithm:

Let (L_i, U_i) be the interval obtained from the i^{th} iteration, ($i = 1,,,,, N = 2000$)

And let the indicator function I_i be defined as follows:

$I_i = \begin{cases} 1 & if\ L_i \leq Y \leq U_i \\ 0\ otherwise \end{cases}$ and then $P(L \leq Y \leq U) = \frac{1}{N} \cdot \sum_{i=1}^{N} I_i$ and this can be expressed as the number of intervals that capture the true value of Y divided by $N = 2000$

The length of the interval can be estimated by $\frac{1}{N} \cdot \sum_{i=1}^{N} (U_i - L_i)$which the average length of all intervals is divided by $N = 2000$.

3.2 The Results

In tables 1 and 2, we will report the Bias and MSE of α and β when $(\alpha, \beta) = (1,1) and\ (3,2)$ respectively.

Table 1

Bias and MSE of the Parameters of Log-Logistic Distribution when $(\alpha, \beta) = (1,1)$

n	t_0	t_1	m_1		$\boldsymbol{\alpha = 1}$	$\boldsymbol{\beta = 1}$
30	1/9	1	12	**Bias**	0.097545	0.116178
				MSE	0.197643	0.138666
30	0	1	12	**Bias**	0.170724	0.149191
				MSE	0.283160	0.140554
50	1/9	1	20	**Bias**	0.045967	0.055532
				MSE	0.102250	0.062089

50	0	1	20	**Bias**	0.099848	0.087444
				MSE	0.152339	0.066794
80	1/9	1	32	**Bias**	0.032568	0.038773
				MSE	0.063537	0.035274
80	0	1	32	**Bias**	0.064772	0.054471
				MSE	0.076238	0.034497
30	1/9	7/3	18	**Bias**	0.070046	0.077895
				MSE	0.132020	0.063179
30	0	7/3	18	**Bias**	0.092989	0.081473
				MSE	0.155860	0.063289
50	1/9	7/3	30	**Bias**	0.048125	0.033046
				MSE	0.075044	0.033630
50	0	7/3	30	**Bias**	0.046414	0.043833
				MSE	0.083330	0.030888
80	1/9	7/3	48	**Bias**	0.014646	0.025621
				MSE	0.042546	0.018627
80	0	7/3	48	**Bias**	0.035613	0.030364
				MSE	0.046684	0.018715
20	0	1	8	**Bias**	0.291171	0.264914
				MSE	0.614476	0.322403
20	0	7/3	12	**Bias**	0.136200	0.138709
				MSE	0.279931	0.122507

Table 2

Bias and MSE of the Parameters of Log-Logistic Distribution when $(\alpha, \beta) = (3,2)$

n	t_0	t_1	m_1		$\boldsymbol{\alpha = 3}$	$\boldsymbol{\beta = 2}$
30	1/9	1/3	12	**Bias**	0.076703	0.223737
				MSE	0.421016	0.593744
30	0	1/3	12	**Bias**	0.157618	0.308816
				MSE	0.492402	0.610075
50	1/9	1/3	20	**Bias**	0.040952	0.121221
				MSE	0.220630	0.261818
50	0	1/3	20	**Bias**	0.104775	0.196636
				MSE	0.263885	0.295064
80	1/9	1/3	32	**Bias**	0.031876	0.058661
				MSE	0.137230	0.133005
80	0	1/3	32	**Bias**	0.064550	0.098158
				MSE	0.152831	0.133872
30	1/9	$\sqrt{\frac{7}{27}}$	18	**Bias**	0.059731	0.131678
				MSE	0.267180	0.258162

30	0	$\sqrt{\frac{7}{27}}$	18	**Bias**	0.062043	0.171791
				MSE	0.279396	0.260169
50	1/9	$\sqrt{\frac{7}{27}}$	30	**Bias**	0.027327	0.095494
				MSE	0.157110	0.146275
50	0	$\sqrt{\frac{7}{27}}$	30	**Bias**	0.060411	0.104568
				MSE	0.168519	0.136454
80	1/9	$\sqrt{\frac{7}{27}}$	48	**Bias**	0.017251	0.048773
				MSE	0.098348	0.071689
80	0	$\sqrt{\frac{7}{27}}$	48	**Bias**	0.021057	0.067343
				MSE	0.098120	0.076841
20	0	1/3	8	**Bias**	0.279768	0.547642
				MSE	0.868763	1.394588
20	0	$\sqrt{\frac{7}{27}}$	12	**Bias**	0.106705	0.277820
				MSE	0.446681	0.521039

From the simulation study in tables 1and 2 it is observed that:

- For fixed t_0 and t_1 the Bias and MSE of both α and β decrease as n increases
- For fixed n, t_1, and m_1 the Bias and MSE of both α and β increase as t_0 decreases from $\frac{1}{9}$ to 0
- For fixed n and t_0 the Bias and MSE of β decrease as m_1 increases
- For fixed n and t_0 the Bias and MSE of α almost always decrease as m_1 increases

Table 3

Point Predictors and 95% *PI for* $p = 1$ *and* $(\alpha, \beta) = (1,1)$

					Point Predictors			**Interval Prediction**		
n	t_0	t_1	m_1		**MLP**	**BUP**	**CMP**		**Pivotal**	**HDI**
30	1/9	1	12	**Bias**	-0.1653411	0.0088773	-0.0373782	**Cov.Prob**	0.9335	-
				MSE	0.0546398	0.0000788	0.0013971	**Length**	0.5241	-
30	0	1	12	**Bias**	-0.1583430	-0.0048037	-0.0358420	**Cov.Prob**	0.9110	-
				MSE	0.0682651	0.0000231	0.0012846	**Length**	0.3714	-

50	1/9	1	20	**Bias**	-0.2292700	0.0024348	-0.0227377	**Cov.Prob**	0.9350	-
				MSE	0.0700625	0.0000059	0.0005170	**Length**	0.2936	-
50	0	1	20	**Bias**	-0.0823001	-0.0011181	-0.0194644	**Cov.Prob**	0.9235	-
				MSE	0.0367704	0.0000013	0.0003789	**Length**	0.2106	-
80	1/9	1	32	**Bias**	0.0414881	0.0009631	-0.0141138	**Cov.Prob**	0.9460	-
				MSE	0.0142557	0.0000009	0.0001992	**Length**	0.1795	-
80	0	1	32	**Bias**	0.0583238	0.0005410	-0.0105410	**Cov.Prob**	0.9340	-
				MSE	0.0249204	0.0000003	0.0001111	**Length**	0.1315	-
30	1/9	7/3	18	**Bias**	-0.4152790	0.0066044	-0.1220861	**Cov.Prob**	0.9385	-
				MSE	0.3520266	0.0000436	0.0149050	**Length**	1.4085	-
30	0	7/3	18	**Bias**	-0.2473566	-0.0036681	-0.0808269	**Cov.Prob**	0.9255	-
				MSE	0.2910252	0.0000135	0.0065330	**Length**	0.8597	-
50	1/9	7/3	30	**Bias**	0.5398556	0.0108213	-0.0591195	**Cov.Prob**	0.9360	-
				MSE	0.4113920	0.0001171	0.0034951	**Length**	0.7890	-
50	0	7/3	30	**Bias**	0.5841602	-0.0040112	-0.0465912	**Cov.Prob**	0.9315	-
				MSE	0.5029379	0.0000161	0.0021707	**Length**	0.4988	-
80	1/9	7/3	48	**Bias**	-0.1199604	-0.0052065	-0.0472660	**Cov.Prob**	0.9475	-
				MSE	0.0990764	0.0000271	0.0022341	**Length**	0.4876	-
80	0	7/3	48	**Bias**	-0.0128922	0.0011160	-0.0245192	**Cov.Prob**	0.9520	-
				MSE	0.1102201	0.0000012	0.0006012	**Length**	0.3021	-
20	0	1	8	**Bias**	-0.3534305	-0.0007041	-0.0508772	**Cov.Prob**	0.8805	-
				MSE	0.1800746	0.0000005	0.0025885	**Length**	0.5507	-
20	0	7/3	12	**Bias**	0.4364375	-0.0003411	-0.1239414	**Cov.Prob**	0.9115	-
				MSE	0.4881803	0.0000001	0.0153615	**Length**	1.3763	-

In table 3 we observe that the (BUP) has the best performance in terms of Bias and MSE. It provides the least Bias and MSE. The second best predictor is the (CMP) and the (MLP) does not perform well and there is no fixed pattern in the performance of (MLP) as we change the values of $n, t_0, t_1, and\ m_1$. The coverage probability of the Prediction interval gets close to the nominal level of 95% as n increases for fixed t_0 while the average length decreases.

Table 4

Point Predictors and 95% *PI for* $p = 1$ *and* $(\alpha, \beta) = (3,2)$

					Point Predictors				**Interval Prediction**	
n	t_0	t_1	m_1		**MLP**	**BUP**	**CMP**		**Pivotal**	**HDI**
30	1/9	1/3	12	**Bias**	-0.0356428	-0.0001214	-0.0067327	**Cov.Prob**	0.9170	-
				MSE	0.0023130	0.0000000	0.0000453	**Length**	0.0794	-
30	0	1/3	12	**Bias**	0.0209951	-0.0010612	-0.0064391	**Cov.Prob**	0.9065	-
				MSE	0.0023301	0.0000011	0.0000415	**Length**	0.0649	-
50	1/9	1/3	20	**Bias**	-0.0491196	0.0003674	-0.0036448	**Cov.Prob**	0.9285	-
				MSE	0.0030274	0.0000001	0.0000133	**Length**	0.0470	-
50	0	1/3	20	**Bias**	-0.0180144	-0.0005035	-0.0038210	**Cov.Prob**	0.9200	-
				MSE	0.0016416	0.0000003	0.0000146	**Length**	0.0394	-
80	1/9	1/3	32	**Bias**	-0.0279471	-0.0000879	-0.0025565	**Cov.Prob**	0.9425	-
				MSE	0.0011871	0.0000000	0.0000065	**Length**	0.0297	-
80	0	1/3	32	**Bias**	-0.0021231	-0.0002719	-0.0023719	**Cov.Prob**	0.9380	-
				MSE	0.0009112	0.0000001	0.0000056	**Length**	0.0251	-
30	1/9	$\sqrt{\frac{7}{27}}$	18	**Bias**	-0.0988914	-0.0021778	-0.0141224	**Cov.Prob**	0.9380	-
				MSE	0.0126730	0.0000047	0.0001994	**Length**	0.1428	-
30	0	$\sqrt{\frac{7}{27}}$	18	**Bias**	-0.0961920	-0.0007020	-0.0094334	**Cov.Prob**	0.9345	-
				MSE	0.0136949	0.0000005	0.0000890	**Length**	0.1030	-
50	1/9	$\sqrt{\frac{7}{27}}$	30	**Bias**	-0.0430364	-0.0005838	-0.0076966	**Cov.Prob**	0.9260	-
				MSE	0.0037569	0.0000003	0.0000592	**Length**	0.0830	-
50	0	$\sqrt{\frac{7}{27}}$	30	**Bias**	-0.1058517	0.0001221	-0.0051086	**Cov.Prob**	0.9385	-
				MSE	0.0140694	0.0000000	0.0000261	**Length**	0.0616	-
80	1/9	$\sqrt{\frac{7}{27}}$	48	**Bias**	-0.0072073	-0.0008286	-0.0052486	**Cov.Prob**	0.9445	-
				MSE	0.0012743	0.0000007	0.0000275	**Length**	0.0520	-
80	0	$\sqrt{\frac{7}{27}}$	48	**Bias**	0.0077276	-0.0000515	-0.0033197	**Cov.Prob**	0.9440	-
				MSE	0.0021962	0.0000001	0.0000110	**Length**	0.0390	-
20	0	1/3	8	**Bias**	-0.0073401	-0.0030650	-0.0107117	**Cov.Prob**	0.893	-
				MSE	0.0026192	0.0000094	0.0001147	**Length**	0.1039	-
20	0	$\sqrt{\frac{7}{27}}$	12	**Bias**	0.0370314	-0.0003054	-0.0132941	**Cov.Prob**	0.9025	-
				MSE	0.0070952	0.0000001	0.0001767	**Length**	0.0937	-

In table 4 we also observe that the (BUP) has the best performance in terms of Bias and MSE. And the second-best predictor is the (CMP), the coverage probability of the Prediction interval gets close to the nominal level of 95% as n increases for fixed t_0 while the average length decreases.

Table 5

Point Predictors and 95% *PI for* $p = 2$ *and* $(\alpha, \beta) = (1,1)$

					Point Predictors			Interval Prediction		
					MLP	**BUP**	**CMP**		**Pivotal**	**HDI**
n	t_0	t_1	m_1							
30	1/9	1	12	**Bias**	0.1238545	0.0250110	-0.0356993	**Cov.Prob**	0.8975	0.8840
				MSE	0.0543504	0.0006255	0.0012744	**Length**	0.8769	0.7434
30	0	1	12	**Bias**	0.0585212	0.0015198	-0.0380157	**Cov.Prob**	0.8855	0.8675
				MSE	0.0593781	0.0000023	0.0014452	**Length**	0.6024	0.5164
50	1/9	1	20	**Bias**	-0.5453093	0.0020687	-0.0282673	**Cov.Prob**	0.9240	0.9165
				MSE	0.3203246	0.0000043	0.0007990	**Length**	0.4566	0.3956
50	0	1	20	**Bias**	-0.3688397	-0.0033507	-0.0241865	**Cov.Prob**	0.9180	0.9055
				MSE	0.1721850	0.0000112	0.0005850	**Length**	0.3283	0.2857
80	1/9	1	32	**Bias**	0.1333461	0.0031152	-0.0141857	**Cov.Prob**	0.9280	0.9225
				MSE	0.0320699	0.0000097	0.0002012	**Length**	0.2730	0.2387
80	0	1	32	**Bias**	-0.1518207	0.0004090	-0.0116876	**Cov.Prob**	0.9295	0.9205
				MSE	0.0479051	0.0000002	0.0001366	**Length**	0.1978	0.1733
30	1/9	7/3	18	**Bias**	-0.7310354	0.0209757	-0.1670016	**Cov.Prob**	0.9000	0.8885
				MSE	0.8051026	0.0004400	0.0278895	**Length**	2.5112	2.0944
30	0	7/3	18	**Bias**	0.3077984	0.0101719	-0.0941888	**Cov.Prob**	0.9105	0.9005
				MSE	0.4000714	0.0001035	0.0088715	**Length**	1.4623	1.2386
50	1/9	7/3	30	**Bias**	-0.3068474	0.0141647	-0.0752611	**Cov.Prob**	0.9350	0.9270
				MSE	0.2483905	0.0002006	0.0056642	**Length**	1.3048	1.1168
50	0	7/3	30	**Bias**	-0.2441730	-0.0020880	-0.0543395	**Cov.Prob**	0.9315	0.9300
				MSE	0.2458407	0.0000044	0.0029528	**Length**	0.7951	0.6862
80	1/9	7/3	48	**Bias**	-0.4434387	0.0019518	-0.0475851	**Cov.Prob**	0.9475	0.9365
				MSE	0.2937690	0.0000038	0.0022643	**Length**	0.7558	0.6559

80	0	7/3	48	**Bias**	0.1373340	0.0037369	-0.0261495	**Cov.Prob**	0.9375	0.9335
				MSE	0.1386088	0.0000140	0.0006838	**Length**	0.4628	0.4035
20	0	1	8	**Bias**	0.1128964	-0.0068497	-0.7770131	**Cov.Prob**	0.8470	0.8305
				MSE	0.0939428	0.0000469	0.0060375	**Length**	0.9803	0.8237
20	0	7/3	12	**Bias**	-0.2820765	0.0446107	-0.1579553	**Cov.Prob**	0.8885	0.8770
				MSE	0.5010694	0.0019901	0.0249497	**Length**	2.6007	2.1496

In Table 5 we observe that the (BUP) has the best performance in terms of Bias and MSE. It provides the least Bias and MSE. The second-best predictor is the (CMP) and the (MLP) does not perform well. The probability of the prediction interval based on pivotal quantity is slightly bigger than that of the HDI interval and the length of the HDI is less than that of the pivotal quantity interval for all combinations of n, t_0, t_1, and m_1

Table 6

Point Predictors and 95% *PI for* $p = 2$ *and* $(\alpha, \beta) = (3,2)$

					Point Predictors			**Interval Prediction**		
					MLP	**BUP**	**CMP**		**Pivotal**	**HDI**
n	t_0	t_1	m_1							
30	1/9	1/3	12	**Bias**	-0.0798906	-0.0003420	-0.0078126	**Cov.Prob**	0.9040	0.8970
				MSE	0.0077366	0.0000001	0.0000610	**Length**	0.1228	0.1080
30	0	1/3	12	**Bias**	-0.1098029	-0.0041781	-0.0097948	**Cov.Prob**	0.8820	0.8600
				MSE	0.0142840	0.0000175	0.0000959	**Length**	0.0961	0.0852
50	1/9	1/3	20	**Bias**	-0.0255834	0.0004877	-0.0038237	**Cov.Prob**	0.9115	0.9090
				MSE	0.0014081	0.0000002	0.0000146	**Length**	0.0697	0.0616
50	0	1/3	20	**Bias**	-0.0105547	-0.0014895	-0.0050050	**Cov.Prob**	0.9145	0.9085
				MSE	0.0015578	0.0000022	0.0000250	**Length**	0.0582	0.0516
80	1/9	1/3	32	**Bias**	-0.0081806	0.0001233	-0.0024999	**Cov.Prob**	0.9345	0.9355
				MSE	0.0004983	0.0000000	0.0000062	**Length**	0.0437	0.0387
80	0	1/3	32	**Bias**	0.0481446	-0.0007660	-0.0029591	**Cov.Prob**	0.9220	0.9140
				MSE	0.0032718	0.0000006	0.0000088	**Length**	0.0370	0.0328

30	1/9	$\sqrt{\frac{7}{27}}$	18	**Bias**	-0.1722274	-0.0010769	-0.0159727	**Cov.Prob**	0.9210	0.9125
				MSE	0.0336022	0.0000012	0.0002551	**Length**	0.2291	0.1994
30	0	$\sqrt{\frac{7}{27}}$	18	**Bias**	-0.0629317	-0.0018666	-0.0118709	**Cov.Prob**	0.9035	0.8940
				MSE	0.0094738	0.0000035	0.0001409	**Length**	0.1602	0.1404
50	1/9	$\sqrt{\frac{7}{27}}$	30	**Bias**	0.0727522	-0.0019253	-0.0100715	**Cov.Prob**	0.9370	0.9315
				MSE	0.0074687	0.0000037	0.0001014	**Length**	0.1290	0.1131
50	0	$\sqrt{\frac{7}{27}}$	30	**Bias**	0.0462383	-0.0008137	-0.0065900	**Cov.Prob**	0.9255	0.9185
				MSE	0.0054594	0.0000007	0.0000434	**Length**	0.0933	0.0822
80	1/9	$\sqrt{\frac{7}{27}}$	48	**Bias**	0.0729730	-0.0011922	-0.0060945	**Cov.Prob**	0.9300	0.9345
				MSE	0.0067363	0.0000014	0.0000371	**Length**	0.0791	0.0696
80	0	$\sqrt{\frac{7}{27}}$	48	**Bias**	-0.0040232	-0.0003281	-0.0038653	**Cov.Prob**	0.9385	0.9380
				MSE	0.0022064	0.0000001	0.0000149	**Length**	0.0577	0.0510
20	0	1/3	8	**Bias**	0.0247661	-0.0043042	-0.0130384	**Cov.Prob**	0.8445	0.8265
				MSE	0.0034740	0.0001852	0.0001700	**Length**	0.1447	0.1277
20	0	$\sqrt{\frac{7}{27}}$	12	**Bias**	-0.0055823	-0.0057162	-0.0217901	**Cov.Prob**	0.9105	0.8940
				MSE	0.0072028	0.0000327	0.0000474	**Length**	0.2511	0.2185

In Table 6 we notice that the (BUP) has the best performance in terms of Bias and MSE, The second-best predictor is the (CMP).

The (MLP) performs well. The pivotal quantity interval and the HDI interval perform almost the same but the HDI interval length is slightly narrower and both coverage probabilities get closer to the nominal level as n increases.

Table 7

Point Predictors and 95% *PI for* $p = 3$ *and* $(\alpha, \beta) = (1,1)$

n	t_0	t_1	m_1		Point Predictors MLP	BUP	CMP	Interval Prediction	Pivotal	HDI
30	1/9	1	12	**Bias**	-0.2549160	0.0394335	-0.0400133	**Cov.Prob**	0.8755	0.8680
				MSE	0.1277635	0.0015550	0.0016011	**Length**	1.2462	1.1008
30	0	1	12	**Bias**	-0.3619757	0.0025867	-0.0473229	**Cov.Prob**	0.8450	0.8360
				MSE	0.2068615	0.0000067	0.0022395	**Length**	0.8316	0.7411
50	1/9	1	20	**Bias**	-0.0472683	0.0037886	-0.0307795	**Cov.Prob**	0.8995	0.8920
				MSE	0.0334774	0.0000144	0.0009474	**Length**	0.6067	0.5468
50	0	1	20	**Bias**	-0.2076484	-0.0041888	-0.0277416	**Cov.Prob**	0.9080	0.8975
				MSE	0.0858963	0.0000175	0.0007696	**Length**	0.4285	0.3879
80	1/9	1	32	**Bias**	-0.1688761	0.0049930	-0.0140516	**Cov.Prob**	0.9285	0.9275
				MSE	0.0454471	0.0000249	0.0001974	**Length**	0.3517	0.3199
80	0	1	32	**Bias**	-0.2485550	-0.0020691	-0.0153844	**Cov.Prob**	0.9095	0.9090
				MSE	0.0893068	0.0000043	0.0002367	**Length**	0.2505	0.2285
30	1/9	7/3	18	**Bias**	-0.3044236	0.0677450	-0.2267696	**Cov.Prob**	0.9150	0.9090
				MSE	0.4913320	0.0045894	0.0514245	**Length**	4.0087	3.5692
30	0	7/3	18	**Bias**	-0.3726691	-0.0211598	-0.1606330	**Cov.Prob**	0.9055	0.8935
				MSE	0.5407066	0.0004477	0.0258030	**Length**	2.1365	1.8926
50	1/9	7/3	30	**Bias**	-0.2618255	0.0254532	-0.0866332	**Cov.Prob**	0.9400	0.9300
				MSE	0.2729537	0.0006479	0.0075053	**Length**	1.8353	1.6371
50	0	7/3	30	**Bias**	-0.0257220	-0.0100282	-0.0715004	**Cov.Prob**	0.9110	0.9110
				MSE	0.2344724	0.0001006	0.0051123	**Length**	1.0527	0.9459
80	1/9	7/3	48	**Bias**	-0.2831964	0.0006305	-0.0565767	**Cov.Prob**	0.9265	0.9260
				MSE	0.1971060	0.0000004	0.0032009	**Length**	0.9989	0.9020
80	0	7/3	48	**Bias**	-0.3559214	-0.0005226	-0.0336028	**Cov.Prob**	0.9325	0.9215
				MSE	0.2710168	0.0000003	0.0011291	**Length**	0.6095	0.5529
20	0	1	8	**Bias**	0.0801076	0.0064554	-0.0943852	**Cov.Prob**	0.8010	0.7855
				MSE	0.1276503	0.0000416	0.0089085	**Length**	1.4674	1.2859
20	0	7/3	12	**Bias**	-0.4136681	0.0509599	-0.2645827	**Cov.Prob**	0.8675	0.8590
				MSE	0.8074782	0.0025969	0.0700040	**Length**	4.3019	3.7767

In table 7 the (BUP) shows the best performance in terms of Bias and MSE and the second-best predictor is the (CMP).

The (MLP) does not perform well. The (HDI) intervals are narrower than the pivotal quantity intervals and the coverage probability of the pivotal quantity is slightly bigger than that of the (HDI) but they are close to the nominal level only when n increases.

Table 8

Point Predictors and 95% *PI for* $p = 3$ *and* $(\alpha, \beta) = (3,2)$

					Point Predictors			**Interval Prediction**		
n	t_0	t_1	m_1		**MLP**	**BUP**	**CMP**		**Pivotal**	**HDI**
30	1/9	1/3	12	**Bias**	-0.0002865	0.0009938	-0.0073408	**Cov.Prob**	0.8860	0.8860
				MSE	0.0018606	0.0000010	0.0000539	**Length**	0.1590	0.1458
30	0	1/3	12	**Bias**	0.0402289	-0.0034564	-0.0094984	**Cov.Prob**	0.8550	0.8440
				MSE	0.0044099	0.0000119	0.0000902	**Length**	0.1230	0.1135
50	1/9	1/3	20	**Bias**	0.0218879	-0.0001042	-0.0046385	**Cov.Prob**	0.9100	0.9050
				MSE	0.0013513	0.0000000	0.0000215	**Length**	0.0892	0.0822
50	0	1/3	20	**Bias**	0.0449275	-0.0017215	-0.0053526	**Cov.Prob**	0.8890	0.8820
				MSE	0.0037440	0.0000030	0.0000287	**Length**	0.0735	0.0680
80	1/9	1/3	32	**Bias**	-0.0304175	0.0007796	-0.0019608	**Cov.Prob**	0.9290	0.9240
				MSE	0.0014383	0.0000006	0.0000038	**Length**	0.0550	0.0507
80	0	1/3	32	**Bias**	-0.0116306	-0.0003288	-0.0025965	**Cov.Prob**	0.9220	0.9125
				MSE	0.0011558	0.0000001	0.0000067	**Length**	0.0459	0.0424
30	1/9	$\sqrt{\frac{7}{27}}$	18	**Bias**	-0.1929948	-0.0002218	-0.0189897	**Cov.Prob**	0.8985	0.9040
				MSE	0.0426268	0.0000000	0.0003606	**Length**	0.3256	0.2989
30	0	$\sqrt{\frac{7}{27}}$	18	**Bias**	-0.0178158	-0.0037171	-0.0153040	**Cov.Prob**	0.9010	0.8940
				MSE	0.0067576	0.0000138	0.0002342	**Length**	0.2153	0.1970
50	1/9	$\sqrt{\frac{7}{27}}$	30	**Bias**	0.0327140	-0.0006912	-0.0098018	**Cov.Prob**	0.9150	0.9075
				MSE	0.0038957	0.0000005	0.0000961	**Length**	0.1711	0.1564

50	0	$\sqrt{\frac{7}{27}}$	30	**Bias**	0.0235009	-0.0025488	-0.0087723	**Cov.Prob**	0.9200	0.9110
				MSE	0.0041981	0.0000065	0.0000770	**Length**	0.1199	0.1101
80	1/9	$\sqrt{\frac{7}{27}}$	48	**Bias**	-0.0471465	-0.0000941	-0.0053663	**Cov.Prob**	0.9345	0.9405
				MSE	0.0038174	0.0000000	0.0000288	**Length**	0.1007	0.0924
80	0	$\sqrt{\frac{7}{27}}$	48	**Bias**	0.0519966	-0.0011418	-0.0048357	**Cov.Prob**	0.9350	0.9335
				MSE	0.0050858	0.0000013	0.0000234	**Length**	0.0735	0.0676
20	0	1/3	8	**Bias**	0.0110992	-0.0083584	-0.0180271	**Cov.Prob**	0.8135	0.7960
				MSE	0.0044274	0.0000699	0.0003249	**Length**	0.1904	0.1751
20	0	$\sqrt{\frac{7}{27}}$	12	**Bias**	0.0567831	-0.0043021	-0.0249301	**Cov.Prob**	0.8715	0.8550
				MSE	0.0126722	0.0000185	0.0006215	**Length**	0.3603	0.3306

In table 8 we notice that the (BUP) has the best performance in terms of Bias and MSE.

The second-best predictor is the (CMP). The (MLP) performs well. Both intervals are close to each other in terms of coverage probability and length but the (HDI) interval is slightly narrower than the pivotal quantity interval. Both probabilities get closer to the nominal level only when n is big enough.

Table 9

Point Predictors and 95% *PI for* $p = 4$ *and* $(\alpha, \beta) = (1,1)$

					Point Predictors			Interval Prediction		
					MLP	**BUP**	**CMP**		**Pivotal**	**HDI**
n	t_0	t_1	m_1							
30	1/9	1	12	**Bias**	0.2810341	0.0445622	-0.0572407	**Cov.Prob**	0.8470	0.8495
				MSE	0.1713002	0.0019858	0.0032765	**Length**	1.6541	1.5015
30	0	1	12	**Bias**	-0.0894944	-0.0018370	-0.0605059	**Cov.Prob**	0.8445	0.8370
				MSE	0.1105016	0.0000034	0.0036610	**Length**	1.0343	0.9460
50	1/9	1	20	**Bias**	-0.2741408	0.0124115	-0.0281113	**Cov.Prob**	0.8880	0.8845
				MSE	0.1145347	0.0001540	0.0007902	**Length**	0.7539	0.6944

50	0	1	20	**Bias**	0.3305809	-0.0034037	-0.0303375	**Cov.Prob**	0.8845	0.8750
				MSE	0.1643515	0.0000116	0.0009204	**Length**	0.5228	0.4836
80	1/9	1	32	**Bias**	-0.1268224	0.0012604	-0.0194786	**Cov.Prob**	0.9195	0.9195
				MSE	0.0361029	0.0000016	0.0003794	**Length**	0.4226	0.3926
80	0	1	32	**Bias**	-0.1617194	-0.0013939	-0.0157517	**Cov.Prob**	0.9140	0.9090
				MSE	0.0565256	0.0000019	0.0002481	**Length**	0.3043	0.2833
30	1/18	7/3	18	**Bias**	-0.5043125	0.0900003	-0.2419323	**Cov.Prob**	0.8965	0.8955
				MSE	0.9248591	0.0081005	0.058519	**Length**	6.3597	6.2668
30	0	7/3	18	**Bias**	-1.4497140	-0.0070811	-0.1972037	**Cov.Prob**	0.8885	0.8795
				MSE	2.6260650	0.0000501	0.0388893	**Length**	3.0382	2.7771
50	1/9	7/3	30	**Bias**	0.0293435	0.0294928	-0.1113588	**Cov.Prob**	0.9265	0.9270
				MSE	0.2656450	0.0008698	0.0124008	**Length**	2.4628	2.2556
50	0	7/3	30	**Bias**	0.2778015	0.0014436	-0.0722895	**Cov.Prob**	0.9140	0.9050
				MSE	0.3687683	0.0000021	0.0052258	**Length**	1.3714	1.2604
80	1/9	7/3	48	**Bias**	-0.1224487	0.0029378	-0.0626491	**Cov.Prob**	0.9275	0.9230
				MSE	0.1575907	0.0000086	0.0039249	**Length**	1.2541	1.1573
80	0	7/3	48	**Bias**	-1.5639390	-0.0041911	-0.0407972	**Cov.Prob**	0.9300	0.9235
				MSE	2.6068580	0.0000176	0.0016644	**Length**	0.7375	0.6833
20	0	1	8	**Bias**	0.5204436	-0.0036463	-0.1449950	**Cov.Prob**	0.7825	0.7710
				MSE	0.4496156	0.0000132	0.0210236	**Length**	2.2214	1.9880
20	0	7/3	12	**Bias**	-0.1900704	0.0886910	-0.4643450	**Cov.Prob**	0.8540	0.8485
				MSE	1.0632540	0.0078661	0.2156169	**Length**	7.2680	6.8442

In table 9 we notice that the (BUP) has the best performance in terms of Bias and MSE. The second-best predictor is the (CMP). The (MLP) cannot be used at all where it shows a bad performance in terms of Bias and MSE. The (HDI) is narrower than the pivotal quantity interval, both probabilities are close to each other and get closer to the nominal level only when n is big enough.

Table 10

Point Predictors and 95% *PI for* $p = 4$ *and* $(\alpha, \beta) = (3,2)$

n	t_0	t_1	m_1		MLP	BUP	CMP		Pivotal	HDI
30	1/9	1/3	12	**Bias**	-0.0185014	0.0020521	-0.0073217	**Cov.Prob**	0.8425	0.8320
				MSE	0.0028885	0.0000042	0.0000536	**Length**	0.2963	0.2524
30	0	1/3	12	**Bias**	-0.0409078	-0.0028027	-0.0095292	**Cov.Prob**	0.8200	0.8135
				MSE	0.0048456	0.0000079	0.0000908	**Length**	0.1481	0.1396
50	1/9	1/3	20	**Bias**	-0.0602771	0.0010774	-0.0037363	**Cov.Prob**	0.8930	0.8935
				MSE	0.0048413	0.0000012	0.0000140	**Length**	0.1080	0.1015
50	0	1/3	20	**Bias**	-0.0370754	-0.0013812	-0.0050453	**Cov.Prob**	0.8660	0.8605
				MSE	0.0031402	0.0000019	0.0000255	**Length**	0.0866	0.0818
80	1/9	1/3	32	**Bias**	-0.0463508	-0.0000284	-0.0028394	**Cov.Prob**	0.9175	0.9095
				MSE	0.0027504	0.0000000	0.0000081	**Length**	0.0648	0.0610
80	0	1/3	32	**Bias**	0.0160618	-0.0006086	-0.0029046	**Cov.Prob**	0.9165	0.9015
				MSE	0.0013727	0.0000004	0.0000084	**Length**	0.0540	0.0510
30	1/9	$\sqrt{\frac{7}{27}}$	18	**Bias**	0.0666684	-0.0009970	-0.0268499	**Cov.Prob**	0.8965	0.8955
				MSE	0.0124180	0.0000010	0.0007209	**Length**	0.4369	0.4231
30	0	$\sqrt{\frac{7}{27}}$	18	**Bias**	-0.0052266	-0.0037944	-0.0175511	**Cov.Prob**	0.8990	0.8845
				MSE	0.0077357	0.0000144	0.0003080	**Length**	0.2733	0.2570
50	1/9	$\sqrt{\frac{7}{27}}$	30	**Bias**	-0.0102363	-0.0021888	-0.0124229	**Cov.Prob**	0.9165	0.9100
				MSE	0.0036402	0.0000048	0.0001543	**Length**	0.2127	0.1993
50	0	$\sqrt{\frac{7}{27}}$	30	**Bias**	0.0053242	-0.0028934	-0.0096412	**Cov.Prob**	0.9165	0.9140
				MSE	0.0042642	0.0000084	0.0000930	**Length**	0.1464	0.1374
80	1/9	$\sqrt{\frac{7}{27}}$	48	**Bias**	-0.0208940	-0.0003266	-0.0059089	**Cov.Prob**	0.9310	0.9355
				MSE	0.0022968	0.0000001	0.0000349	**Length**	0.1220	0.1143
80	0	$\sqrt{\frac{7}{27}}$	48	**Bias**	0.0346585	-0.0000556	-0.0039685	**Cov.Prob**	0.9325	0.9280
				MSE	0.0035567	0.0000000	0.0000157	**Length**	0.0875	0.0822
20	0	1/3	8	**Bias**	-0.0029654	-0.0061570	-0.0178502	**Cov.Prob**	0.7745	0.7615
				MSE	0.0057341	0.0003790	0.0003186	**Length**	0.2411	0.2274
20	0	$\sqrt{\frac{7}{27}}$	12	**Bias**	0.0155442	-0.0103898	-0.0390597	**Cov.Prob**	0.8640	0.8570
				MSE	0.0134823	0.0001081	0.0015256	**Length**	0.5034	0.4855

Table 10 shows that (BUP) has the best performance in terms of Bias and MSE. The second-best predictor is the (CMP). The (MLP) performs well. Both intervals are close to each other in terms of coverage probability and length but the (HDI) interval is slightly narrower than the pivotal quantity interval. Both probabilities get closer to the nominal level only when n is big enough.

Table 11

Point Predictors and 95% *PI for* $p = 5$ *and* $(\alpha, \beta) = (1,1)$

					Point Predictors			Interval Prediction		
n	t_0	t_1	m_1		**MLP**	**BUP**	**CMP**		**Pivotal**	**HDI**
30	1/9	1	12	**Bias**	0.1824793	0.0879572	-0.0488863	**Cov.Prob**	0.8125	0.8175
				MSE	0.1646084	0.0077365	0.0023899	**Length**	2.2607	2.0909
30	0	1	12	**Bias**	-0.4629519	0.0211571	-0.0565517	**Cov.Prob**	0.8170	0.8095
				MSE	0.3458682	0.0004476	0.0031981	**Length**	1.3649	1.2685
50	1/9	1	20	**Bias**	0.3734983	0.0226218	-0.0228893	**Cov.Prob**	0.8790	0.8780
				MSE	0.1920756	0.0005117	0.0005239	**Length**	0.9252	0.8642
50	0	1	20	**Bias**	-0.0897199	-0.0063791	-0.0363289	**Cov.Prob**	0.8625	0.8595
				MSE	0.0726516	0.0000407	0.0013198	**Length**	0.6412	0.6007
80	1/9	1	32	**Bias**	-0.1208330	0.0031577	-0.0193609	**Cov.Prob**	0.9065	0.9075
				MSE	0.0385493	0.0000100	0.0003748	**Length**	0.4980	0.4686
80	0	1	32	**Bias**	-0.0109245	-0.0005775	-0.0160837	**Cov.Prob**	0.9050	0.8975
				MSE	0.0331991	0.0000003	0.0002587	**Length**	0.3495	0.3296
30	1/18	7/3	18	**Bias**	0.2502458	0.1965792	-0.3251284	**Cov.Prob**	0.8880	0.8865
				MSE	1.2153840	0.03864361	0.1057084	**Length**	7.0655	7.1528
30	0	7/3	18	**Bias**	0.1499252	0.0514196	-0.2194319	**Cov.Prob**	0.8790	0.8760
				MSE	0.7573855	0.0026440	0.0481504	**Length**	4.2034	3.9594
50	1/9	7/3	30	**Bias**	-0.3966119	0.0730512	-0.1039453	**Cov.Prob**	0.9075	0.9110
				MSE	0.5434928	0.0053365	0.0108046	**Length**	3.1244	2.9248
50	0	7/3	30	**Bias**	0.2860448	0.0033778	-0.0838477	**Cov.Prob**	0.9025	0.9005
				MSE	0.4217468	0.0000114	0.0070304	**Length**	1.6976	1.5853
80	1/9	7/3	48	**Bias**	-0.3590155	0.0057516	-0.0693776	**Cov.Prob**	0.9300	0.9280
				MSE	0.2980865	0.0000331	0.0048132	**Length**	1.5301	1.4321

80	0	7/3	48	**Bias**	-0.3000585	0.0060851	-0.0349530	**Cov.Prob**	0.9210	0.9195
				MSE	0.2608277	0.0000370	0.0012217	**Length**	0.8761	0.8225
20	0	1	8	**Bias**	-0.7640827	0.0513039	-0.1532457	**Cov.Prob**	0.7540	0.7495
				MSE	0.8441595	0.0026320	0.0234842	**Length**	3.0854	2.8730
20	0	7/3	12	**Bias**	0.5364827	0.0415299	-0.9317452	**Cov.Prob**	0.8510	0.8540
				MSE	1.9266670	0.0017247	0.8681491	**Length**	12.8663	13.655

In table 11 we notice that the (BUP) has the best performance in terms of Bias and MSE. The second-best predictor is the (CMP). The (MLP) does not perform well where it shows a bad performance in terms of Bias and MSE. Both intervals are close to each other in terms of coverage probability and length but the (HDI) interval is slightly narrower than the pivotal quantity interval. We can also notice that the length of both intervals is quite big as compared to other cases for the different values of p from $p = 1$ to $p = 4$.

Table 12

Point Predictors and 95% *PI for* $p = 5$ *and* $(\alpha, \beta) = (3,2)$

					Point Predictors			**Interval Prediction**		
n	t_0	t_1	m_1		**MLP**	**BUP**	**CMP**		**Pivotal**	**HDI**
30	1/9	1/3	12	**Bias**	-0.0470967	0.0067358	-0.0042570	**Cov.Prob**	0.8205	0.8245
				MSE	0.0055950	0.0000454	0.0000181	**Length**	0.2437	0.2323
30	0	1/3	12	**Bias**	-0.0408560	-0.0082227	-0.0152693	**Cov.Prob**	0.8085	0.7980
				MSE	0.0054219	0.0000676	0.0002332	**Length**	0.1780	0.1701
50	1/9	1/3	20	**Bias**	-0.1075914	0.0011915	-0.0038696	**Cov.Prob**	0.8900	0.8875
				MSE	0.0129409	0.0000014	0.0000150	**Length**	0.1230	0.1172
50	0	1/3	20	**Bias**	0.0852557	-0.0030114	-0.0068347	**Cov.Prob**	0.8680	0.8680
				MSE	0.0092859	0.0000091	0.0000467	**Length**	0.0999	0.0955
80	1/9	1/3	32	**Bias**	-0.0513937	-0.0000106	-0.0029109	**Cov.Prob**	0.9075	0.9075
				MSE	0.0033344	0.0000000	0.0000085	**Length**	0.0734	0.0699

80	0	1/3	32	**Bias**	0.0518168	-0.0010618	-0.0033688	**Cov.Prob**	0.9005	0.8920
				MSE	0.0038619	0.0000011	0.0000113	**Length**	0.0609	0.0581
30	1/18	$\sqrt{\frac{7}{27}}$	18	**Bias**	-0.1788175	0.0019009	-0.01897	**Cov.Prob**	0.8805	0.8770
				MSE	0.0460049	0.0000036	0.0003599	**Length**	0.4102	0.3980
30	0	$\sqrt{\frac{7}{27}}$	18	**Bias**	-0.0976312	-0.0064467	-0.0229588	**Cov.Prob**	0.8790	0.8755
				MSE	0.0195021	0.0000416	0.0005271	**Length**	0.3494	0.3359
50	1/9	$\sqrt{\frac{7}{27}}$	30	**Bias**	-0.0913452	0.0006618	-0.0111125	**Cov.Prob**	0.9130	0.9110
				MSE	0.0126530	0.0000004	0.0001235	**Length**	0.2585	0.2466
50	0	$\sqrt{\frac{7}{27}}$	30	**Bias**	0.0176153	-0.0014071	-0.0088233	**Cov.Prob**	0.9185	0.9150
				MSE	0.0051903	0.0000020	0.0000779	**Length**	0.1734	0.1650
80	1/9	$\sqrt{\frac{7}{27}}$	48	**Bias**	-0.0451054	-0.0003309	-0.0063283	**Cov.Prob**	0.9345	0.9300
				MSE	0.0040047	0.0000001	0.0000400	**Length**	0.1424	0.1353
80	0	$\sqrt{\frac{7}{27}}$	48	**Bias**	-0.0206750	-0.0000623	-0.0041645	**Cov.Prob**	0.9275	0.9225
				MSE	0.0028336	0.0000000	0.0000173	**Length**	0.1008	0.0959
20	0	1/3	8	**Bias**	-0.0867412	-0.0097039	-0.0237129	**Cov.Prob**	0.7380	0.7320
				MSE	0.0149435	0.0000941	0.0005622	**Length**	0.2871	0.2764
20	0	$\sqrt{\frac{7}{27}}$	12	**Bias**	0.0310069	-0.0263866	-0.0668689	**Cov.Prob**	0.8520	0.8625
				MSE	0.0201124	0.0006963	0.0044715	**Length**	0.7130	0.7369

Table 12 shows that the (BUP) has the best performance in terms of Bias and MSE. The second-best predictor is the (CMP). The (MLP) performs well. Both intervals are close to each other in terms of coverage probability and length but the (HDI) interval is slightly narrower than the pivotal quantity interval. Probabilities are close to nominal level only for big values of n.

3.3 Real Data Analysis Example

In this section, we will consider a real data set from Log-Logistic distribution to compare the performance of different point predictors and prediction intervals when the distribution follows Log-Logistic distribution. Referring to Lawless (2011) , Schmee and Nelson (1977) , the data below represents the number of miles in thousands to failure of different 96 locomotive controls. The life test involving such controls was terminated after 135,000 miles and the failure times of the 37 failed units were recorded as follows:

22.5, 37.5, 46.0, 48.5, 51.5, 53.0, 54.5, 57.5, 66.5, 68.0, 69.5, 76.5, 77.0, 78.5, 80.0, 81.5, 82.0, 83.0, 84.0, 91.5, 93.5, 102.5, 107.0, 108.5, 112.5, 113.5, 116.0, 117.0, 118.5, 119.0, 120.0, 122.5, 123.0, 127.5, 131.0, 132.5, 134.0.

Samples from the data will be obtained according to four different sampling schemes, and based on each sample we will predict T_p for $p = 1$, 2, 3, 4 and 5. The different point predictors and different prediction intervals will be constructed.

In order to proceed, we need to check whether the Log-Logistic model is a good fit to the data set. We notice that the Kolmogorov-Smirnov (K-S) distance and the p-value are as follows: K-S= 0.1214 and the p-value= 0.5879 which means that the Log-Logistic is a good fit to the data.

The different sampling schemes, and the corresponding samples are listed below:

- Scheme 1: $t_0 = 50$, $t_1 = 135$ and $m_1 = 20$

Sample 1: 51.5, 53.0, 54.5, 57.5, 66.5, 68.0, 69.5, 76.5, 77.0, 78.5, 80.0, 81.5, 82.0, 83.0, 84.0, 91.5, 93.5, 102.5, 107.0, 108.5.

The MLEs for Sample 1 are $\hat{\alpha} = 0.0059$ and $\hat{\beta} = 2.4081$

- Scheme 2: $t_0 = 40$, $t_1 = 135$ and $m_1 = 25$

Sample 2: 46.0, 48.5, 51.5, 53.0, 54.5, 57.5, 66.5, 68.0, 69.5, 76.5, 77.0, 78.5, 80.0, 81.5, 82.0, 83.0, 84.0, 91.5, 93.5, 102.5, 107.0, 108.5, 112.5, 113.5, 116.0

The MLEs for Sample 2 are $\hat{\alpha} = 0.0059$ and $\hat{\beta} = 2.4719$

- Scheme 3: $t_0 = 0$, $t_1 = 135$ and $m_1 = 25$

Sample 3: 22.5, 37.5, 46.0, 48.5, 51.5, 53.0, 54.5, 57.5, 66.5, 68.0, 69.5, 76.5, 77.0, 78.5, 80.0, 81.5, 82.0, 83.0, 84.0, 91.5, 93.5, 102.5, 107.0, 108.5, 112.5.

The MLEs for Sample 3 are $\hat{\alpha} = 0.0059$ and $\hat{\beta} = 2.4575$

- Scheme 4: $t_0 = 40$, $t_1 = 135$ and $m_1 = 30$

Sample 4: 46.0, 48.5, 51.5, 53.0, 54.5, 57.5, 66.5, 68.0, 69.5, 76.5, 77.0, 78.5, 80.0, 81.5, 82.0, 83.0, 84.0, 91.5, 93.5, 102.5, 107.0, 108.5, 112.5, 113.5, 116.0, 117.0, 118.5, 119.0, 120.0, 122.5.

The MLEs for Sample 4 are $\hat{\alpha} = 0.0061$ and $\hat{\beta} = 2.4081$

Based on the above 4 sampling schemes, the results are reported in tables 13 to 15 respectively.

Table 13

The values of Point Predictors and 95% *PI – Scheme 1*

	Exact Value	MLP	BUP	CMP	PIVOTAL	HDI
$\boldsymbol{p = 1}$	112.5	110.8	111.0	110.2	(108,117)	-
$\boldsymbol{p = 2}$	113.5	111.9	113.3	112.6	(109,121)	(109,120)
$\boldsymbol{p = 3}$	116.0	113.4	115.8	115.0	(110,125)	(109,123)
$\boldsymbol{p = 4}$	117.0	115.6	118.1	117.4	(111,129)	(110,127)
$\boldsymbol{p = 5}$	119.0	117.2	120.5	119.8	(113,132)	(112,130)

Table 14

The values of Point Predictors and 95% PI – Scheme 2

	Exact Value	MLP	BUP	CMP	PIVOTAL	HDI
$p = 1$	117.0	116.5	118.0	117.7	(116,125)	-
$p = 2$	118.5	117.2	120.7	120.0	(116,129)	(116,127)
$p = 3$	119.0	117.8	123.0	122.0	(118,132)	(117,131)
$p = 4$	120.0	120.1	125.0	124.8	(119,136)	(118,134)
$p = 5$	122.5	122.7	127.0	127.2	(120,140)	(119,138)

Table 15

The values of Point Predictors and 95% PI – Scheme 3

	Exact Value	MLP	BUP	CMP	PIVOTAL	HDI
$p = 1$	113.5	113.0	114.8	113.5	(113,121)	-
$p = 2$	116.0	114.6	117.0	115.2	(113,125)	(113,123)
$p = 3$	117.0	115.7	119.0	116.8	(114,129)	(113,127)
$p = 4$	118.5	117.2	121.0	118.4	(115,133)	(114,131)
$p = 5$	119.0	117.4	124.0	120.0	(117,136)	(117,136)

Table 16

The values of Point Predictors and 95% PI – Scheme 4

	Exact Value	MLP	BUP	CMP	PIVOTAL	HDI
$p = 1$	123.0	122.8	124.8	124.2	(123,131)	-
$p = 2$	127.5	125.7	127.2	126.5	(123,135)	(123,134)
$p = 3$	131.0	129.9	129.7	128.9	(124,139)	(123,138)
$p = 4$	132.5	132.6	132.1	131.3	(125,143)	(124,141)
$p = 5$	134.0	134.2	134.5	133.8	(127,146)	(126,145)

From tables 13 to 16, we noticed that the three different Point Predictors are close to the exact or observed value. Moreover, both of the Prediction Intervals capture the true observation and the HDI intervals have a width slightly less than that of the Pivotal interval.

CHAPTER 4: SUMMARY, CONCLUSIONS AND FURTHER RESEARCH

4.1 Summary

In this study, we have used the maximum Likelihood approach to estimate the parameters of the Log-Logistic distribution under hybrid censored data. The predictive likelihood function was obtained and the predictive likelihood equations were formed. Different point predictors of future failures were obtained such as the (MLP), (BUP), and (CMP). Classical prediction intervals were constructed, one of which is the interval based on pivotal quantity and the other is (HDI) interval. A simulation study based on 2000 replications was run to investigate and compare between the performance of all point predictors in terms of Bias and MSE. Moreover, we compared the two prediction intervals in terms of coverage probability and width. Finally, an example based on real data was used to illustrate the study.

4.2 Conclusions

Based on the simulation study for different values of p and different input values of n, t_0 , t_1, and m_1. We come up with the following conclusions:

- The (BUP) has the best performance in terms of Bias and MSE, the second-best predictor is the (CMP).

- There is no fixed pattern of the performance of the (MLP) in terms of Bias and MSE.
- Under the effect of left censoring, for fixed values of $n \geq 30$ the length of both intervals gets slightly bigger while the coverage probabilities get closer to the nominal level and There is no fixed pattern in the performance of point predictors (in terms of bias and MSE).
- For fixed value of n, as m_1 and t_1 increase it is noticed that the Bias and MSE of all point predictors are almost the same or slightly increasing for that of the CMP, the length of both intervals get slightly bigger while the coverage probability of both intervals get closer to the nominal level.
- The (HDI) interval cannot be used when $p = 1$.
- The interval based on pivotal quantity when $p = 1$ has a convergence probability close to the nominal level for all combinations of n, t_0 , t_1, and m_1. $(n \geq 30)$
- As the value of p gets bigger, the probability of both intervals gets close to the nominal level only for large values of n.
- The width of the (HDI) interval is narrower than that of the interval based on pivotal quantity for $p > 1$.
- We recommend the best point predictor is the (BUP) as it has the least bias and MSE, and the best interval prediction method is the (HDI) method for $p > 1$.

4.3 Further Research

A further research or study might be suggested in the following sense; first of all, the same study might be repeated with another lifetime distribution. Or we can use progressive Censoring scheme for samples from Log-Logistic distribution. Also, same study could be done with left truncation instead of left censoring, and the addition of Bayesian point predictors and Bayesian prediction intervals, where different choices of loss functions and prior distributions might be considered. Furthermore, one can find an approximate enclosed form of the (MLEs).

Moreover, further studies might focus more on the different aspects of the hazard function and its relation with the failure rate in both survival analysis and reliability studies. Finally, it would be quite interesting to study the scenarios that would minimize both sample sizes and time of termination of the experiment as the latter two play an important role in minimizing the cost of the experiment.

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APPENDIX

APPENDIX A: R CODE FOR THE GRAPH OF (PDF) LOG-LOGISTIC

```
par(mfrow=c(2,2))

x = seq(0, 10, length = 1001)

y = 1/(1+x)^2

plot(x,y,type="l",xlim=c(0, 10), ylim = c(0, 1),xlab = "t",ylab = "(α=1,β=1)", main =
"probability density function")

x = seq(0, 10, length = 1001)

y = (0.5*(x^-0.5))/(1+(x^0.5)^2)

plot(x,y,type="l",xlim=c(0, 10), ylim = c(0, 2),xlab = "t",ylab = "(α=1,β=0.5)", main =
"probability density function")

x = seq(0, 10, length = 1001)

y = (18*x)/(1+(3*x)^2)^2

plot(x,y,type="l",xlim=c(0, 10), ylim = c(0, 2),xlab = "t",ylab = "(α=3,β=2)", main =
"probability density function")

x = seq(0, 10, length = 1001)


a=0.6;b=5

y = (a*b*((a*x)^(b-1)))/(1+(a*x)^b)^2

plot(x,y,type="l",xlim=c(0, 10), ylim = c(0, 0.8),xlab = "t",ylab = "(α=0.6,β=5)", main =
"probability density function")
```

```
title(xlab = "t",ylab = "(α=0.6,β=5)", main = "probability density function")
```

APPENDIX B: R CODE FOR THE GRAPH OF HAZARD FUNCTION OF LOG-LOGISTIC DISTRIBUTION

```
par(mfrow=c(2,2))

x = seq(0, 10, length = 1001)

a=1;b=1

y = a*b*((a*x)^(b-1))/(1+(a*x)^b)

plot(x,y,type="l",xlim=c(0, 10), ylim = c(0, 0.7),xlab = "t",ylab = "(α=1,β=1)", main =
"Hazard function")

x = seq(0, 10, length = 1001)

a=1;b=0.5

y = a*b*((a*x)^(b-1))/(1+(a*x)^b)

plot(x,y,type="l",xlim=c(0, 10), ylim = c(0, 3),xlab = "t",ylab = "(α=1,β=0.5)", main =
"Hazard function")

x = seq(0, 10, length = 1001)

a=3;b=2

y = a*b*((a*x)^(b-1))/(1+(a*x)^b)

plot(x,y,type="l",xlim=c(0, 10), ylim = c(0,4),xlab = "t",ylab = "(α=3,β=2)", main =
"Hazard function")

x = seq(0, 10, length = 1001)

a=0.6;b=5

y = a*b*((a*x)^(b-1))/(1+(a*x)^b)

plot(x,y,type="l",xlim=c(0, 10), ylim = c(0, 2),xlab = "t",ylab = "(α=0.6,β=5)", main =
"Hazard function")
```

#######

APPECDIX C: R CODE FOR THE (MLE)

```
X=function(Nsim,n,t0,t1,m1){
  mat=matrix(0,ncol=2,nrow=Nsim)#2000by2 matrix empty matrix
  for (j in 1:Nsim){
    u=runif(n)
    #t=u/(1-u)
    a=1;b=1
    t=(1/a)*((u/(1-u))^(1/b))
    t=c(sort(t))

    G1=subset(t,t<t0)
    r=length(G1)
    G2=subset(t,t>=t0 & t<=min(t1,t[m1+r]))
    G3=subset(t,t>min(t1,t[m1+r]))

     m=length(G2)
   # print(j)
    #print(r)
    #print(m)
   # print(min(t1,t[m1+r]))
   # print("kkkkk")
```

```
    ti=c(G2)#the values of G2
    L=ti[length(ti)]#the last one in G2
    Case1=function(B){#function that gives the minimum of alpha and beta
      a <- B[1]
      b <- B[2]
      -1*((r * b * log(a * t0))-(r * log(1 + (a * t0)^b))+(sum(log(a * b * ((a * ti)^(b-1))))-(2 *
sum(log(1 + (a * ti)^b)))-((n-(r+m)) * log(1+(a * L)^b))))
    }# logliklihood function for case 1
    Case2=function(B){
      a <- B[1]
      b <- B[2]
      -1*((r * b * log(a * t0))-(r * log(1 + (a * t0)^b))+(sum(log(a * b * ((a * ti)^(b-1))))-(2 *
sum(log(1 + (a * ti)^b)))-((n-(r+m)) * log(1+(a * t1)^b))))
    }
    if (length(ti)<m1){
      mat[j,]=c(optim(c(a,b), Case2 , method = "L-BFGS-B",lower = 0.00000001, upper
= Inf)$par)#start with alpha and beta equal 1 and estimate max for case 1
    }else{
      mat[j,]=c(optim(c(a,b), Case1 , method = "L-BFGS-B",lower = 0.00000001, upper
= Inf)$par)
    }
  }
  alpha_h=c(mat[,1])
  beta_h=c(mat[,2])
```

```
  mean(alpha_h)
  bias_alpha=(mean(alpha_h)-a)
  mse_alpha=sum((alpha_h-a)^2)/Nsim
  mean(beta_h)
  bias_beta=(mean(beta_h)-b)
  mse_beta=sum((beta_h-b)^2)/Nsim
  print(bias_alpha)
  print(mse_alpha)
  print(bias_beta)
  print(mse_beta)
}
X(2000,30,0.0001,1,12) ### when t0=0 (no left censored units)we replace it with 0.0001##
#m1=20
#n=50
```

APPENDIX D: R CODE FOR THE (MLP)

```
library(optimr)
X=function(Nsim,n,p,t0,t1,m1){
  mat=matrix(0,ncol=3,nrow=Nsim)#2000by2 matrix empty matrix
  for (j in 1:Nsim){
    u=runif(n)
    #t=u/(1-u)
```

```
a=1;b=1
t=(1/a)*((u/(1-u))^(1/b))
t=c(sort(t))
G1=subset(t,t<t0)
r=length(G1)
G2=subset(t,t>=t0 & t<=min(t1,t[m1+r]))
G3=subset(t,t>min(t1,t[m1+r]))

m=length(G2)
# print(j)
# print(r)
# print(m)
#print(min(t1,t[m1+r]))
#print("kkkkk")
ti=c(G2)#the values of G2
L=ti[length(ti)]#the last one in G2
Case1=function(B){#function that gives the minimum of alpha and beta
  a <- B[1]
  b <- B[2]
  y <- B[3]
  -1*((p-1)*log((a*y)^b - (a*L)^b)+(((r+m)-n-1)*log(1+(a*y)^b))+((1-p)*log(1+(a*L)^b))+((m+1)*log(a*b))+((b-1)*log(a*y))+(r*b*log(a*t0))-(r*log(1+(a*t0)^b))+((b-1)*sum(log(a*ti)))-2*sum(log(1+(a*ti)^b)))
```

```
  }# logliklihood function for case 1
  Case2=function(B){
    a <- B[1]
    b <- B[2]
    y <- B[3]
    -1*((p-1)*log((a*y)^b - (a*t1)^b)+(((r+m)-n-1)*log(1+(a*y)^b))+((1-p)*log(1+(a*t1)^b))+((m+1)*log(a*b))+((b-1)*log(a*y))+(r*b*log(a*t0))-(r*log(1+(a*t0)^b))+((b-1)*sum(log(a*ti)))-2*sum(log(1+(a*ti)^b)))
  }
  if (length(ti)<m1){
    mat[j,]=c(optimr(c(a,b,G3[p]), Case2 , method = "L-BFGS-B", lower=c(0.0001,0.0001,t1+0.0001))$par)
  }else{
    mat[j,]=c(optimr(c(a,b,G3[p]), Case1 , method = "L-BFGS-B", lower=c(0.0001,0.0001,L+0.0001))$par)
  }
 }
 alpha_h=c(mat[,1])
 beta_h=c(mat[,2])
 y_h=c(mat[,3])
 mean(alpha_h)
 bias_alpha=(mean(alpha_h)-a)
 mse_alpha=sum((alpha_h-a)^2)/Nsim
 mean(beta_h)
```

```
  bias_beta=(mean(beta_h)-b)

  mse_beta=sum((beta_h-b)^2)/Nsim

  mean(y_h)

  bias_y=(mean(y_h)-(G3[p]))

  mse_y=sum((y_h-G3[p])^2)/Nsim


  print(bias_y)

  print(mse_y)

 }

n=80

p=2

X(2000,80,2,1/9,1,32) ### t0 is replaced with 0.0001 when t0=0###
```

APPENDIX E: R CODE FOR THE (BUP)

```
library(zipfR)

X=function(Nsim,n,t0,t1,m1,p){

  mat=data.frame(matrix(nrow = Nsim, ncol = 7))

  colnames(mat)=c("a","b","case","L","m","r","G3")

  for (j in 1:Nsim){

    u=runif(n)

    #t=u/(1-u)

    a=1;b=1

    t=(1/a)*((u/(1-u))^(1/b))
```

```
    t=c(sort(t))
    G1=subset(t,t<t0)
      r=length(G1)
    G2=subset(t,t>=t0 & t<=min(t1,t[m1+r]))
    G3=subset(t,t>min(t1,t[m1+r]))
    m=length(G2)
#print(j)
#print(r)
#print(m)
#print(min(t1,t[m1+r]))
#print("kkkkk")
    ti=c(G2)#the values of G2
    L=ti[length(ti)]#the last one in G2
    Case1=function(B){#function that gives the minimum of alpha and beta
      a <- B[1]
      b <- B[2]
      -1*((r * b * log(a * t0))-(r * log(1 + (a * t0)^b))+(sum(log(a * b * ((a * ti)^(b-1))))-(2 *
sum(log(1 + (a * ti)^b)))-((n-(r+m)) * log(1+(a * L)^b))))
    }# logliklihood function for case 1
    Case2=function(B){
      a <- B[1]
      b <- B[2]
```

```
   -1*((r * b * log(a * t0))-(r * log(1 + (a * t0)^b))+(sum(log(a * b * ((a * ti)^(b-1))))-(2 * sum(log(1 + (a * ti)^b)))-((n-(r+m)) * log(1+(a * t1)^b))))
  }
  if (length(ti)<m1){
   mat[j,]=c(optim(c(a,b), Case2 , method = "L-BFGS-B",lower = c(0.00000001,0.00000001), upper = Inf)$par,2,t1,m,r,G3[p])#it saves different values of t1,m,r in each row or one sim##
  }else{
   mat[j,]=c(optim(c(a,b), Case1 , method = "L-BFGS-B",lower = c(0.00000001,0.00000001), upper = Inf)$par,1,L,m,r,G3[p])
  }
 }
 alpha_h=c(mat[,1])
 beta_h=c(mat[,2])
 mean(alpha_h)
 bias_alpha=(mean(alpha_h)-a)
 mse_alpha=sum((alpha_h-a)^2)/Nsim
 mean(beta_h)
 bias_beta=(mean(beta_h)-b)
 mse_beta=sum((beta_h-b)^2)/Nsim
 print(bias_alpha)
 print(mse_alpha)
 print(bias_beta)
 print(mse_beta)
```

```
  print(mat)
}
Mat=X(2000,50,1/9,1,20,1) ### ###
#p=5
#n=30
Mat
ycon = numeric(2000)
for (i in 1:2000){
  a = Mat[i,1]
  b = Mat[i,2]
  L = Mat[i,4]
  m = Mat[i,5]
  r = Mat[i,6]
  x = 1/(1+(a*L)^b)
  for (k in 0:(p-1))
    ycon[i]=ycon[i]+(p*(choose(n-r-m,p))*((1+(a*L)^b)^(n-r-m-p+1))*(sum((choose(p-1,k))*(-1)^(p-1-k)*((a*L)^b)^(p-1-k)*(1/a)))*(Ibeta(x,(n-r-m-k-(1/b)),(k+(1/b)+1))))
}
#print(ycon)
ych=mean(ycon)
#ycon[2000]
yc=mean(Mat$G3,na.rm=TRUE) # The mean excluding NA results
#mean(ycon)-mean(Mat$G3,na.rm=TRUE)
```

```
#sum(is.na(Mat$G3)) # number of NA results

#print(yc)

#print(ych)

bias=ych-yc

mse=mean((ych-yc)^2)

print(bias)

print(mse)
```

APPENDIX F: R CODE FOR BOTH (BUP) AND CONDITIONAL MEDIAN PREDICTOR

```
library(zipfR)

X=function(Nsim,n,t0,t1,m1,p){

  mat=data.frame(matrix(nrow = Nsim, ncol = 7))

  colnames(mat)=c("a","b","case","L","m","r","G3")

  for (j in 1:Nsim){

    u=runif(n)

    a=3;b=2

    t=(1/a)*((u/(1-u))^(1/b))

    #t=u/(1-u)

    t=c(sort(t))

      G1=subset(t,t<t0)

      r=length(G1)

    G2=subset(t,t>=t0 & t<=min(t1,t[m1+r]))
```

```
    G3=subset(t,t>min(t1,t[m1+r]))


    m=length(G2)
#print(j)
#print(r)
#print(m)
#print(min(t1,t[m1+r]))
#print("kkkkk")
    ti=c(G2)#the values of G2
    L=ti[length(ti)]#the last one in G2
    Case1=function(B){#function that gives the minimum of alpha and beta
      a <- B[1]
      b <- B[2]
      -1*((r * b * log(a * t0))-(r * log(1 + (a * t0)^b))+(sum(log(a * b * ((a * ti)^(b-1))))-(2 *
sum(log(1 + (a * ti)^b)))-((n-(r+m)) * log(1+(a * L)^b))))
    }# logliklihood function for case 1
    Case2=function(B){
      a <- B[1]
      b <- B[2]
      -1*((r * b * log(a * t0))-(r * log(1 + (a * t0)^b))+(sum(log(a * b * ((a * ti)^(b-1))))-(2 *
sum(log(1 + (a * ti)^b)))-((n-(r+m)) * log(1+(a * t1)^b))))
    }
    if (length(ti)<m1){
```

```
    mat[j,]=c(optim(c(a,b), Case2 , method = "L-BFGS-B",lower = c(0.00000001,0.00000001), upper = Inf)$par,2,t1,m,r,G3[p])#it saves different values of t1,m,r in each row or one sim##
  }else{
    mat[j,]=c(optim(c(a,b), Case1 , method = "L-BFGS-B",lower = c(0.00000001,0.00000001), upper = Inf)$par,1,L,m,r,G3[p])
  }
 }
 alpha_h=c(mat[,1])
 beta_h=c(mat[,2])
 mean(alpha_h)
 bias_alpha=(mean(alpha_h)-a)
 mse_alpha=sum((alpha_h-a)^2)/Nsim
 mean(beta_h)
 bias_beta=(mean(beta_h)-b)
 mse_beta=sum((beta_h-b)^2)/Nsim
 print(bias_alpha)
 print(mse_alpha)
 print(bias_beta)
 print(mse_beta)
 print(mat)
 }
Mat=X(2000,80,1/9,1,32,5)
Mat
```

```
p=5

n=80

ycon = numeric(2000)

ymed = numeric(2000)

for (i in 1:2000){

  a = Mat[i,1]

  b = Mat[i,2]

  L = Mat[i,4]

  m = Mat[i,5]

  r = Mat[i,6]

  x = 1/(1+(a*L)^b)

  z = qbeta(0.5,p,n-r-m-p+1)

  ymed[i]=((1/a)*((z+(a*L)^b)/(1-z))^(1/b))

  for (k in 0:(p-1))

    ycon[i]=ycon[i]+(p*(choose(n-r-m,p))*((1+(a*L)^b)^(n-r-m-p+1))*(sum((choose(p-
1,k))*(-1)^(p-1-k)*((a*L)^b)^(p-1-k)*(1/a)))*(Ibeta(x,(n-r-m-k-(1/b)),(k+(1/b)+1))))

}

#print(ycon)

ych=mean(ycon)

yph=mean(ymed)

#ycon[2000]

yc=mean(Mat$G3,na.rm=TRUE) # The mean excluding NA results

#mean(ycon)-mean(Mat$G3,na.rm=TRUE)
```

```
#sum(is.na(Mat$G3)) # number of NA results

#print(yc)

#print(ych)

bias_yc=ych-yc

mse_yc=mean((ych-yc)^2)

print(bias_yc)

print(mse_yc)

bias_ymed=yph-yc

mse_ymed=mean((yph-yc)^2)

print(bias_ymed)

print(mse_ymed)
```

APPENDIX G: R CODE FOR THE PIVOTAL METHOD INTERVAL

```
library(zipfR)

X=function(Nsim,n,t0,t1,m1,p){

  mat=data.frame(matrix(nrow = Nsim, ncol = 7))

  colnames(mat)=c("a","b","case","L","m","r","G3")

  for (j in 1:Nsim){

    u=runif(n)

    a=1;b=1

    t=(1/a)*((u/(1-u))^(1/b))

   # t=u/(1-u)

    t=c(sort(t))
```

```
G1=subset(t,t<t0)
r=length(G1)
G2=subset(t,t>=t0 & t<=min(t1,t[m1+r]))
G3=subset(t,t>min(t1,t[m1+r]))
m=length(G2)
#print(j)
#print(r)
# print(m)
#print(min(t1,t[m1+r]))
#print("kkkkk")
ti=c(G2)#the values of G2
L=ti[length(ti)]#the last one in G2
Case1=function(B){#function that gives the minimum of alpha and beta
  a <- B[1]
  b <- B[2]
  -1*((r * b * log(a * t0))-(r * log(1 + (a * t0)^b))+(sum(log(a * b * ((a * ti)^(b-1))))-(2 *
sum(log(1 + (a * ti)^b)))-((n-(r+m)) * log(1+(a * L)^b))))
}# logliklihood function for case 1
Case2=function(B){
  a <- B[1]
  b <- B[2]
```

```
    -1*((r * b * log(a * t0))-(r * log(1 + (a * t0)^b))+(sum(log(a * b * ((a * ti)^(b-1))))-(2 * sum(log(1 + (a * ti)^b)))-((n-(r+m)) * log(1+(a * t1)^b))))
   }
   if (length(ti)<m1){
    mat[j,]=c(optim(c(a,b), Case2 , method = "L-BFGS-B",lower = c(0.00000001,0.00000001), upper = Inf)$par,2,t1,m,r,G3[p])#it saves different values of t1,m,r in each row or one sim##
   }else{
    mat[j,]=c(optim(c(a,b), Case1 , method = "L-BFGS-B",lower = c(0.00000001,0.00000001), upper = Inf)$par,1,L,m,r,G3[p])
   }
  }
  alpha_h=c(mat[,1])
  beta_h=c(mat[,2])
  mean(alpha_h)
  bias_alpha=(mean(alpha_h)-a)
  mse_alpha=sum((alpha_h-a)^2)/Nsim
  mean(beta_h)
  bias_beta=(mean(beta_h)-b)
  mse_beta=sum((beta_h-b)^2)/Nsim
  print(bias_alpha)
  print(mse_alpha)
  print(bias_beta)
  print(mse_beta)
```

```
  print(mat)

}
Mat=X(2000,80,0.0001,sqrt(7/27),48,1)
Mat
p=1
n=80
lbound = numeric(2000)
ubound = numeric(2000)
capture_interval= numeric(2000)
#width_interval= numeric(2000)
for (i in 1:2000){
  a = Mat[i,1]
  b = Mat[i,2]
  L = Mat[i,4]
  m = Mat[i,5]
  r = Mat[i,6]
  z1 = qbeta(0.025,p,n-r-m-p+1)
  z2 = qbeta(0.975,p,n-r-m-p+1)

  lbound[i]= ((1/a)*((z1+(a*L)^b)/(1-z1))^(1/b)) ## lower bound for the interval from each sample#
  ubound[i]= ((1/a)*((z2+(a*L)^b)/(1-z2))^(1/b)) # upper bound#
```

```
  if(lbound[i] < Mat$G3[i] & Mat$G3[i] < ubound[i]) (capture_interval[i]=1) else
(capture_interval[i]=0)
  cat(i, lbound[i],Mat$G3[i],ubound[i],"\n")
}
conv_prob= mean(capture_interval)
print(conv_prob)
lf= mean(lbound)
uf= mean(ubound)
avwidth=uf-lf
print(avwidth)
```

APPENDIX H: R CODE FOR THE HIGH-DENSITY INTERVAL AND PIVOTAL QUANTITY INTERVAL

```
library(HDInterval)
library(zipfR)
X=function(Nsim,n,t0,t1,m1,p){
  mat=data.frame(matrix(nrow = Nsim, ncol = 7))
  colnames(mat)=c("a","b","case","L","m","r","G3")
  for (j in 1:Nsim){
    u=runif(n)
    #t=u/(1-u)
    a=3;b=2
```

```
t=(1/a)*((u/(1-u))^(1/b))
t=c(sort(t))


G1=subset(t,t<t0)
r=length(G1)
G2=subset(t,t>=t0 & t<=min(t1,t[m1+r]))
G3=subset(t,t>min(t1,t[m1+r]))


m=length(G2)
#print(j)
#print(r)
#print(m)
# print(min(t1,t[m1+r]))
#print("kkkkk")
ti=c(G2)#the values of G2
L=ti[length(ti)]#the last one in G2
Case1=function(B){#function that gives the minimum of alpha and beta
  a <- B[1]
  b <- B[2]
  -1*((r * b * log(a * t0))-(r * log(1 + (a * t0)^b))+(sum(log(a * b * ((a * ti)^(b-1))))-(2 *
sum(log(1 + (a * ti)^b)))-((n-(r+m)) * log(1+(a * L)^b))))
}# logliklihood function for case 1
Case2=function(B){
```

```
a <- B[1]
b <- B[2]
-1*((r * b * log(a * t0))-(r * log(1 + (a * t0)^b))+(sum(log(a * b * ((a * ti)^(b-1))))-(2 * sum(log(1 + (a * ti)^b)))-((n-(r+m)) * log(1+(a * t1)^b))))
}
if (length(ti)<m1){
mat[j,]=c(optim(c(a,b), Case2 , method = "L-BFGS-B",lower = c(0.00000001,0.00000001), upper = Inf)$par,2,t1,m,r,G3[p])#it saves different values of t1,m,r in each row or one sim##
}else{
mat[j,]=c(optim(c(a,b), Case1 , method = "L-BFGS-B",lower = c(0.00000001,0.00000001), upper = Inf)$par,1,L,m,r,G3[p])
}
}
alpha_h=c(mat[,1])
beta_h=c(mat[,2])
mean(alpha_h)
bias_alpha=(mean(alpha_h)-a)
mse_alpha=sum((alpha_h-a)^2)/Nsim
mean(beta_h)
bias_beta=(mean(beta_h)-b)
mse_beta=sum((beta_h-b)^2)/Nsim
print(bias_alpha)
print(mse_alpha)
```

```
  print(bias_beta)
  print(mse_beta)
  print(mat)
  }
Mat=X(2000,80,0.0001,sqrt(7/27),48,5)
Mat
p=5
n=80
lbound = numeric(2000)
ubound = numeric(2000)
capture_interval= numeric(2000)
l2 = numeric(2000)
u2 = numeric(2000)
hdIlow = numeric(2000)
hdIupp = numeric(2000)
capt_interval2 = numeric(2000)
#width_interval= numeric(2000)
for (i in 1:2000){
  a = Mat[i,1]
  b = Mat[i,2]
  L = Mat[i,4]
  m = Mat[i,5]
  r = Mat[i,6]
```

```
  z1 = qbeta(0.025,p,n-r-m-p+1)
  z2 = qbeta(0.975,p,n-r-m-p+1)

  lbound[i]= ((1/a)*((z1+(a*L)^b)/(1-z1))^(1/b)) ## lower bound for the interval from each sample#
  ubound[i]= ((1/a)*((z2+(a*L)^b)/(1-z2))^(1/b)) # upper bound#

  if(lbound[i] < Mat$G3[i] & Mat$G3[i] < ubound[i]) (capture_interval[i]=1) else (capture_interval[i]=0)
  cat(i, lbound[i],Mat$G3[i],ubound[i],"\n")
  print(i)
  w=hdi(qbeta,0.95,shape1=p,shape2=n-r-m-p+1)

  l2[i]= w[1]
  u2[i]= w[2]

  hdIlow[i]=(1/a)*((w[1]+(a*L)^b)/(1-w[1]))^(1/b)## the lower bound from each run#
  hdIupp[i]=(1/a)*((w[2]+(a*L)^b)/(1-w[2]))^(1/b)
  if(hdIlow[i]< Mat$G3[i] & Mat$G3[i] < hdIupp[i]) (capt_interval2[i]=1) else (capt_interval2[i]=0)
  cat(i, hdIlow[i],Mat$G3[p],hdIupp[i],"\n")
}
lf= mean(lbound)
uf= mean(ubound)
```

```
conv_prob= mean(capture_interval)

print(conv_prob)

avwidth=uf-lf
```