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TOTAL ASYMPTOTICALLY NONEXPANSIVE MAPPINGS IN UNIFORMLY CONVEX METRIC SPACES

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We approximate common fixed point of a pair of total asymptotically nonexpansive mappings in the setting of a uniformly convex metric space. The proposed algorithm is computationally simpler than the existing ones in the literature of metric fixed point theory. Our results are new and are valid in Hilbert spaces, CAT(0) spaces and uniformly convex Banach spaces satisfying Opial's property, simultaneously.

Keywords: Convex metric space, total asymptotically nonexpansive mapping, jointly demiclosed principle, common fixed point, iterative algorithm, convergence.

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1. Introduction

Let C be a nonempty subset of a metric space X and $T : C \to C$ a mapping. A point $x \in C$ is a fixed point of T if $Tx = x$. Denote the set of all fixed points of T by $F(T)$. We say that the mapping T is:
(i) contraction if there exists $k \in (0,1)$ such that $d(Tx,Ty) \leq kd(x,y)$ for all $x, y \in C$ (ii) nonexpansive if $d(Tx,Ty) \leq d(x,y)$ for all $x,y \in C$ (iii) asymptotically nonexpansive mapping if there is a nonnegative real sequence $\{k_n\}$ such that $k_n \to 0$ and $d(T^n x, T^n y) \leq (1+k_n)\, d(x,y)$ for all $x, y \in C,\ n \geq 1$ (iv) generalized asymptotically nonexpansive if there are nonnegative real sequences $\{k_n^1\}$ and $\{k_n^2\}$ with $k_n^1 \to 0$ and $k_n^2 \to 0$ such that $d(T^n x, T^n y) \leq (1+k_n^1)\, d(x,y) + k_n^2$ for all $x, y \in C,\ n \geq 1$ (v) asymptotically nonexpansive in the intermediate sense if it is continuous and $\limsup_{n\to\infty} \sup_{x,y\in C} (d(T^n x, T^n y) - d(x,y)) \leq 0$ (vi) total asymptotically nonexpansive [2] if there exist nonnegative real sequences $\{k_n^1\}, \{k_n^2\}$ with $k_n^1 \to 0, k_n^2 \to 0$ and a strictly increasing continuous function $\psi : [0,\infty) \to [0,\infty)$ with $\psi(0) = 0$ and $d(T^n x, T^n y) \leq d(x,y) + k_n^1 \psi\,(d(x,y)) + k_n^2$ for all $x, y \in C,\ n \geq 1$ (vii) uniformly $L-$Lipschitzian if $d(T^n x, T^n y) \leq Ld(x,y)$ for all $x, y \in C,\ n \geq 1$

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(vii) uniformly continuous if for each $\varepsilon > 0$, there exists $\delta(\varepsilon) > 0$ such that $d(Tx, Ty) < \varepsilon$ whenever $d(x, y) < \delta$.

Every uniformly $L-$Lipschitzian mapping is uniformly continuous but the converse is not true in general. The function $T(x) = \sqrt{x}$ is uniformly continuous on $[0, \infty)$ but not Lipschitz.The class of total asymptotically nonexpansive mappings is the most general as it includes the classes of mappings mentioned in (ii)-(vi).

The Banach contraction principle is of metrical nature and its proof hings on Picard iterations. This principle is applicable to a variety of subjects such as integral equations, partial differential equations and image process. Picard iterative algotithm fails to converge for nonexpansive mappings on a Banach space. Krasnoselskii, Mann and Ishikawa iterative algorithms are employed for the approximation of fixed points of the classes (ii)-(vi) in Hilbert spaces, Banach spaces, CAT(0) spaces and convex metric spaces (see for example, [2, 4, 6, 13, 26]).

To approximate common fixed point of two asymptotic nonlinear mappings $T_1, T_2 : C \to C$ in a linear domain, many authors have used the following modified Ishikawa's iterative algorithm[8]:

$$\begin{aligned} &x_1 = x \in C, \\ &x_{n+1} = (1 - \alpha_n) x_n + \alpha_n T_1^n y_n \\ &y_n = (1 - \beta_n) x_n + \beta_n T_2^n x_n \end{aligned} \tag{1}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $(0, 1)$ (also see [5, 9]).

Abbas et al. [1] introduced a new one$-$step iterative algorithm to compute common fixed point of two asymptotically nonexpansive mappings in uniformly convex Banach spaces. For two asymptotically nonexpansive mappings $T_1, T_2 : C \to C$, they defined the following iterative algorithm:

$$\begin{aligned} &x_1 = x \in C, \\ &x_{n+1} = \alpha_n T_1^n x_n + (1 - \alpha_n) T_2^n x_n \end{aligned} \tag{2}$$

where $\{\alpha_n\}$ is a sequence in $(0, 1)$.

It is worth to mention that algorithm (2) is of independent interest and is computationally simpler than the algorithm (1) to approximate common fixed point of two asymptotic nonlinear mappings. Neither (1) implies (2) nor conversely. However, when $T_1 = I$ (the identity mapping)$, T_2 = T$, both (1) and (2) reduce to the following Mann's iterative algorithm:

$$\begin{aligned} &x_1 = x \in C, \\ &x_{n+1} = \alpha_n x_n + (1 - \alpha_n) T^n x_n. \end{aligned} \tag{3}$$

A mapping $W : X^2 \times [0, 1] \to X$ is a convex structure on a metric space X [16] if it satisfies the following inequality

$$d(u, W(x, y, \alpha)) \leq \alpha d(u, x) + (1 - \alpha) d(u, y)$$

for all $u, x, y \in X$ and $\alpha \in [0,1]$. A subset C of X is convex if $W(x, y, \alpha) \in C$ for all $x, y \in X$ and $\alpha \in [0,1]$.

A convex metric space X is uniformly convex [14] if for any $\varepsilon > 0$, there exists $\delta(\varepsilon) > 0$ such that $d\left(z, W\left(x, y, \frac{1}{2}\right)\right) \leq r(1 - \delta(\varepsilon)) < r$ for all $r > 0$ and $x, y, z \in X$ with $d(z, x) \leq r, d(z, y) \leq r$ and $d(x, y) \geq r\varepsilon$.

Uniformly convex Banach space is linear while CAT(0) space is a nonlinear uniformly convex metric space. An example of a convex metric space due to Goebel and Reich [7] is stated as follows:

Let B_H be the open unit ball in a general complex Hilbert space H and k_{B_H} a metric on B_H (known as Kobayashi distance) defined as

$$k_{B_H}(x, y) = \tanh^{-1}(1 - \sigma(x, y))^{\frac{1}{2}},$$

where

$$\sigma(x, y) = \frac{\left(1 - \|x\|^2\right)\left(1 - \|y\|^2\right)}{|1 - \langle x, y \rangle|^2} \text{ for all } x, y \in B_H.$$

The open unit ball B_H together with the metric k_{B_H} is named as a Hilbert ball. One can define a convex structure W for the corresponding convex metric space (B_H, k_{B_H}).

In a convex metric space, (2) becomes:

$$x_1 = x \in C, \; x_{n+1} = W\left(T_1^n x_n, T_2^n x_n, \alpha_n\right) \text{ for all } n \geq 1 \tag{4}$$

where $\max(\alpha_n, 1 - \alpha_n) \leq \delta$ for some $\delta \in (0,1)$.

When $T_2 = I, T_1 = T$ in (3), it becomes the following Mann iterative algorithm[10]:

$$x_{n+1} = W\left(T^n x_n, x_n, \alpha_n\right) \text{ for all } n \geq 1. \tag{5}$$

The fixed point theory of nonexpansive mappings and its various generalizations majorly depends on the geometrical characteristics of the under consideration space. The class of nonexpansive mappings enjoys the fixed point property(FPP) and the approximate fixed point property(AFPP) in various settings of spaces, see for example [11] for the later property for the class of nonexpansive mappings. Therefore, it is natural to extend such results to generalized classes of nonexpansive mappings as a mean of testing the limit of the theory of nonexpansive mappings. It is remarked that FPP and AFPP of various generalized classes of nonexpansive mappings are still developing in a linear and nonlinear domains. The class of uniformly convex metric space is endowed with rich geometric structures which are helpful to obtain new results. Metric fixed point theory of nonlinear mappings in a general setup of convex metric spaces is a fascinating field of research in nonlinear functional analysis. Moreover, iterative algorithms are the only main tool to study fixed point problems of nonexpansive mappings and its generalized classes in spaces of non-positive sectional curvature.

Our purpose in this paper is to approximate common fixed point of a pair of total asymptotically nonexpansive mappings through $\triangle-$convergence and strong convergence of iterative algorithm (4) in the general setup of convex metric spaces. Our new setting includes, as special cases, Hilbert spaces, uniformly convex Banach spaces with Opial's property and $CAT\,(0)$ spaces, simultaneously.

2. **Preliminaries**

In this section, we give some required definitions and state some needed results.

For a bounded sequence $\{x_n\}$ in a metric space X, set $r(x, \{x_n\}) = \limsup_{n\to\infty} d(x, x_n)$ for all $x \in X$.

The asymptotic radius of $\{x_n\}$ with respect to $C \subseteq X$ is defined as

$$r(\{x_n\}) = \inf_{x\in C} r\left(x, \{x_n\}\right).$$

A point $y \in C$ is called the asymptotic center of $\{x_n\}$ with respect to $C \subseteq X$ if

$$r\left(y, \{x_n\}\right) \le r(x, \{x_n\}) \text{ for all } x \in C.$$

The set of all asymptotic centers of $\{x_n\}$ is denoted by $A(\{x_n\})$.

A sequence $\{x_n\}$ in X, $\triangle-$converges to $x \in X$ if x is the unique asymptotic center of $\{y_n\}$ for every subsequence $\{y_n\}$ of $\{x_n\}$ .A mapping $T : C \to C$ satisfies demiclosed principle if a sequence $\{x_n\}$ in C that $\triangle-$converges to a point $x \in C$ and $\lim_{n\to\infty} d(x_n, Tx_n) = 0$, then $x \in F\left(T\right)$. A pair of mappings $T_1, T_2 : C \to C$ satisfies the jointly demiclosed principle [12] if $\{x_n\}$ $\triangle-$converges to a point $x \in C$ and $\lim_{n\to\infty} d(T_1x, T_2x) = 0$, then $x \in F\left(T_1\right) \cap F\left(T_2\right)$.

Let $\ell^2\left(\mathbb{N}\right) = \left\{ w = (w_1, w_2, ..., w_n, ...) : \sum\limits_{n=1}^{\infty} \|w_n\|^2 < \infty \right\}$ with $\|w\| = \left(\sum\limits_{n=1}^{\infty} \|w_n\|^2 \right)^{\frac{1}{2}}$.

Naraghirad has shown in [12] that there exists mappings $T_1, T_2 : \ell^2\left(\mathbb{N}\right) \to \ell^2\left(\mathbb{N}\right)$ which satisfy the jointly demiclosed principle but T_1 does not satisfy demiclosed principle.

Let $h : [0, \infty) \to [0, \infty)$ be a nondecreasing function with $h(0) = 0$ and $f(h) > 0$ for every $h > 0$. Then the mappings $T_1, T_2 : C \to C$ with $F = F\left(T_1\right) \cap F\left(T_2\right) \neq \phi$, satisfy condition(J) if

$$d(T_1x, T_2x) \ \ge h(d(x, F)) \text{ for all } x \in C$$

and condition(D) if

$$\max\left(d(x, T_1x), d(x, T_2x)\right) \ \ge h(d(x, F)) \text{ for all } x \in C$$

where $d(x, F) = \inf_{z\in F} d\left(x, z\right)$.

Note that condition (J) and condition(D) becomes condition(A) [15] if either T_1 (or T_2) $= I$ (the identity mapping).

In the sequel, the following lemmas will be needed.

Lemma 2.1. [17] *If* $\{a_n\}, \{b_n\}$ *and* $\{c_n\}$ *are nonnegative real sequences satisfying*

$$a_{n+1} \leq (1+b_n)a_n \ + c_n \ \ \text{for all } n \geq 1, \sum_{n=1}^{\infty} b_n < \infty \text{ and } \sum_{n=1}^{\infty} c_n < \infty,$$

then $\lim_{n\to\infty} a_n$ *exists.*

Lemma 2.2. [3] *Let* C *be a nonempty, closed and convex subset of a complete and uniformly convex metric space* X*. Then every bounded sequence* $\{x_n\}$ *in* X *has a unique asymptotic center with respect to* C *that lies in* C*.*

Lemma 2.3. [4] *Let* X *be a uniformly convex metric space. Let* $x \in X$ *and* $\{a_n\}$ *be a sequence in* $[b_1, b_2]$ *for some* $b_1, b_2 \in (0,1)$*. If* $\{u_n\}$ *and* $\{v_n\}$ *are sequences in* X *such that* $\limsup_{n\longrightarrow\infty} d(u_n, x) \leq r,\ \limsup_{n\longrightarrow\infty} d(v_n, x) \leq r$ *and* $\lim_{n\longrightarrow\infty} d(W(u_n, v_n, a_n), x) = r$*for some* $r \geq 0$*, then* $\lim_{n\to\infty} d(u_n, v_n) = 0$.

3. Convergence Analysis

We start with the following technical lemma.

Lemma 3.1. *Let* C *be a nonempty, closed and convex subset of a uniformly convex metric space* X*. Let* $T_i\,(i = 1,2) : C \to C$ *be total asymptotically nonexpansive mappings where sequences* $\{k^1_{n,i}\}, \{k^2_{n,i}\}$ *and functions* ψ_i *satisfy the following conditions:*
(C1): $\sum\limits_{n=1}^{\infty} k^1_{n,i} < \infty$ *and* $\sum\limits_{n=1}^{\infty} k^2_{n,i} < \infty$;
(C2): there exist constants $a_i, b_i > 0$ *such that* $\psi_i(t) \leq a_i t$ *for all* $t \geq b_i$*.*
If $F = F(T_1) \cap F(T_2) \neq \phi$ *and* $\{x_n\}$ *is the sequence in (4), then we have the followings assertions:*
(i) $\lim_{n\to\infty} d(x_n, x)$ *exists for each* $x \in F$
(ii) $\lim_{n\to\infty} d(T^n_1 x_n, T^n_2 x_n) = 0$
(iii) $\lim_{n\to\infty} d(x_{n+1}, T^n_j x_n) = 0$ *for* $j = 1, 2$.

Proof. By (C2) and the strictly increasing function ψ_i, it follows that

$$\psi_i(t) \leq \psi_i(b_i) + a_i t \text{ for } i = 1,2. \tag{6}$$

With the help of (6), we calculate for $x \in F$ that

$$\begin{aligned}
d(x_{n+1},x) &= d\left(W\left(T_1^n x_n, T_2^n x_n, \alpha_n\right), x\right) \\
&\leq \alpha_n d\left(T_1^n x_n, x\right) + (1-\alpha_n)\, d\left(T_2^n x_n, x\right) \\
&\leq \alpha_n \left[d(x_n,x) + k_{n,1}^1 \psi_1\left(d(x_n,x)\right) + k_{n,1}^2\right] \\
&\quad + (1-\alpha_n)\left[d(x_n,x) + k_{n,2}^1 \psi_2\left(d(x_n,x)\right) + k_{n,2}^2\right] \\
&\leq \alpha_n \left[d(x_n,x) + k_{n,1}^1 \left[\psi_1(b_1) + a_1 d(x_n,x)\right] + k_{n,1}^2\right] \\
&\quad + (1-\alpha_n)\left[d(x_n,x) + k_{n,2}^1 \left[\psi_2(b_2) + a_2 d(x_n,x)\right] + k_{n,2}^2\right] \\
&= \left[1 + a_1 \alpha_n k_{n,1}^1 + a_2(1-\alpha_n) k_{n,2}^1\right] d(x_n,x) \\
&\quad + \alpha_n k_{n,1}^1 \psi_1(b_1) + (1-\alpha_n) k_{n,2}^1 \psi_2(b_2) \\
&\quad + \alpha_n k_{n,1}^2 + (1-\alpha_n) k_{n,2}^2 \\
&\leq \left[1 + a\delta\left(k_{n,1}^1 + k_{n,2}^1\right)\right] d(x_n,x) \\
&\quad + \delta a\left(k_{n,1}^1 + k_{n,2}^1\right) + \delta\left(k_{n,1}^2 + k_{n,2}^2\right)
\end{aligned}$$

where $a = \max_{1 \leq i \leq 2}\left(a_i, \psi_i(b_i)\right)$ and $\max\left(\alpha_n, 1-\alpha_n\right) \leq \delta$.

By Lemma 2.1, we see that $\lim_{n\to\infty} d(x_n,x)$ exists for each $x \in F$, thus proving (i).

Next, let $\lim_{n\to\infty} d(x_n,x) = c$. For $c = 0$, there is nothing to prove. Suppose $c > 0$. Since

$$\limsup_{n\to\infty} d(T_1^n x_n, x) \leq c, \limsup_{n\to\infty} d(T_2^n x_n, x) \leq c$$

and

$$\lim_{n\to\infty} d\left(W\left(T_1^n x_n, T_2^n x_n, \alpha_n\right), x\right) = c,$$

therefore by Lemma 2.3, we get that

$$\lim_{n\to\infty} d(T_1^n x_n, T_2^n x_n) = 0, \tag{7}$$

that is (ii).

To prove (iii), we use the definition of $\{x_n\}$ to get that

$$\begin{aligned}
d\left(x_{n+1}, T_j^n x_n\right) &= d\left(W\left(T_1^n x_n, T_2^n x_n, \alpha_n\right), T_j^n x_n\right) \\
&\leq (1-\alpha_n)\, d(T_1^n x_n, T_2^n x_n) \\
&\leq \delta d(T_1^n x_n, T_2^n x_n).
\end{aligned}$$

Finally with the help of (7), we have that

$$\lim_{n\to\infty} d(x_{n+1}, T_j^n x_n) = 0 \text{ for } j = 1, 2.$$

□

Now we are in a position to approximate common fixed point of the mappings T_1 and T_2 through $\triangle-$convergence of the sequence $\{x_n\}$ defined in (4). Our first result in this direction uses $L-$Lipschitzian property of the mappings and the second one uses uniform continuity.

Theorem 3.1. *Let C be a nonempty, closed and convex subset of a complete and uniformly convex metric space X. Let $T_i\,(i=1,2):C\to C$ be uniformly $L-$Lipschitzian and total asymptotically nonexpansive mappings satisfying jointly demiclosed principle and conditions $(C1)-(C2)$ given in Lemma 3.1. If $F\neq\phi$ and $\{x_n\}$ is the sequence given in (4) with $\lim_{n\to\infty} d\,(x_{n+1},x_n)=0$, then $\{x_n\}$ $\triangle-$converges to an element of F.*

Proof. Note that

$$\begin{aligned} d(T_1x_{n+1},T_2x_{n+1}) &\leq d(T_1x_{n+1},T_1^{n+1}x_{n+1})+d(T_1^{n+1}x_{n+1},T_2^{n+1}x_{n+1}) \\ &\quad +d(T_2^{n+1}x_{n+1},T_2^{n+1}x_n)+d(T_2^{n+1}x_n,T_2x_{n+1}) \\ &\leq Ld(x_{n+1},T_1^{n}x_{n+1})+d(T_1^{n+1}x_{n+1},T_2^{n+1}x_{n+1}) \\ &\quad +Ld(x_{n+1},x_n)+Ld(T_2^{n}x_n,x_{n+1}) \\ &\leq L\left[d(x_{n+1},T_1^{n}x_n)+d\left(T_1^{n}x_n,T_1^{n}x_{n+1}\right)\right]+Ld(x_{n+1},x_n) \\ &\quad +d(T_1^{n+1}x_{n+1},T_2^{n+1}x_{n+1})+Ld(T_2^{n}x_n,x_{n+1}) \\ &\leq L\left[d(x_{n+1},T_1^{n}x_n)+d(x_{n+1},T_2^{n}x_n)\right] \\ &\quad +d(T_1^{n+1}x_{n+1},T_2^{n+1}x_{n+1})+L\,(1+L)\,d(x_{n+1},x_n). \end{aligned}$$

This inequality together with Lemma 3.1 (ii)-(iii) and $\lim_{n\to\infty} d\,(x_{n+1},x_n)=0$ gives that

$$\lim_{n\to\infty} d(T_1x_n,T_2x_n)=0. \tag{8}$$

Suppose that T_1 and T_2 satisfy jointly demiclosed principle. Let $\{y_n\}$ be any subsequence of $\{x_n\}$ such that $A(\{y_n\})=\{y\}$. As $\{y_n\}$, $\triangle-$converges to y and

$$\lim_{n\to\infty} d(T_1y_n,T_2y_n)=0,$$

so $y\in F$. Therefore $\lim_{n\to\infty} d(x_n,y)$ exists by Lemma 6. If $x\neq y$, then by the uniqueness of asymptotic centres(Lemma 2.2), we have

$$\begin{aligned} \limsup_{n\to\infty} d(y_n,y) &< \limsup_{n\to\infty} d(y_n,x) \\ &\leq \limsup_{n\to\infty} d(x_n,x) \\ &< \limsup_{n\to\infty} d(x_n,y) \\ &= \limsup_{n\to\infty} d(y_n,y), \end{aligned}$$

a contradiction. Hence $x=y$.
Therefore, $A(\{y_n\})=\{x\}$ for all subsequences $\{y_n\}$ of $\{x_n\}$. This proves that $\{x_n\}$, $\triangle-$*converges to an element of* F.

□

Theorem 3.2. *Let C be a nonempty, closed and convex subset of a complete and uniformly convex metric space X. Let $T_i\,(i=1,2):C\to C$ be uniformly continuous and total asymptotically nonexpansive mappings satisfying the inequality: $d\,(x_n,T_1^n x_n)\leq d(T_1^n x_n,T_2^n x_n)$ and conditions$(C1)-(C2)$ given in*

Lemma 3.1. If $F \neq \phi$ and $\{x_n\}$ is the sequence in (4), then $\{x_n\}$ $\triangle-$converges to an element of F.

Proof. The given inequality

$$d\left(x_n, T_1^n x_n\right) \leq d(T_1^n x_n, T_2^n x_n)$$

together with Lemma3.1 (ii) provides that

$$\lim_{n\to\infty} d\left(x_n, T_1^n x_n\right) = 0. \tag{9}$$

Next the inequality

$$d\left(x_n, T_2^n x_n\right) \leq d\left(x_n, T_1^n x_n\right) + d(T_1^n x_n, T_2^n x_n),$$

Lemma 3.1 (ii) and (9) all together imply that

$$\lim_{n\to\infty} d\left(x_n, T_2^n x_n\right) = 0. \tag{10}$$

Also the following inequality

$$d\left(x_{n+1}, x_n\right) \leq d(x_{n+1}, T_2^n x_n) + d(x_n, T_2^n x_n),$$

Lemma 3.1 (iii) and (10) all together provide that

$$\lim_{n\to\infty} d\left(x_{n+1}, x_n\right) = 0. \tag{11}$$

Finally the inequality

$$\begin{aligned} d\left(x_{n+1}, T_j x_{n+1}\right) &\leq d\left(x_{n+1}, T_j^{n+1} x_{n+1}\right) + d\left(T_j^{n+1} x_{n+1}, T_j^{n+1} x_n\right) \\ &\quad + d\left(T_j\left(T_j^n x_n\right), T_j x_{n+1}\right) \end{aligned}$$

with the help of Lemma 3.1 (iii), (9)-(11) and uniform continuity of T_j yields that

$$\lim_{n\to\infty} d\left(x_n, T_j x_n\right) = 0 \text{ for } j = 1, 2. \tag{12}$$

It has been taken in Theorem 3.1 that $A(\{x_n\}) = \{x\}$ and $A(\{y_n\}) = \{y\}$ for any subsequence $\{y_n\}$ of $\{x_n\}$. Also for the subsequence $\{y_n\}$, we have

$$\lim_{n\to\infty} d(y_n, T_j y_n) = 0 \text{ for } j = 1, 2. \tag{13}$$

Define a sequence $\{z_i\}$ in C by $z_i = T_1^i y$. In the presence of strictly increasing function ψ_1,(C2) and uniformly $L-$Lipschitzian mapping T_1, we calculate that

$$\begin{aligned} d(z_i, y_n) &\leq d(T_1^i y, T_1^i y_n) + d(T_1^i y_n, T_1^{i-1} y_n) + \cdots + d(T_1 y_n, y_n) \\ &\leq d\left(y, y_n\right) + k_{n,1}^1 \psi_1\left(d\left(y, y_n\right)\right) + k_{n,1}^2 + \sum_{r=0}^{i-1} d(T_1^r y_n, T_1^{r+1} y_n) \\ &\leq \left(1 + k_{n,1}^1 a_1\right) d(y, y_n) + k_{n,1}^1 \psi_1\left(b_1\right) + k_{n,1}^2 + \sum_{r=1}^{i-1} d(T_1^r y_n, T_1^{r+1} y_n) \\ &\leq \left(1 + k_{n,1}^1 a_1\right) d(y, y_n) + k_{n,1}^1 \psi_1\left(b_1\right) + k_{n,1}^2 + iLd(T_1 y_n, y_n). \end{aligned}$$

This estimate together with (13) implies that

$$r(z_i, \{y_n\}) = \limsup_{n\to\infty} d(z_i, y_n) \leq \limsup_{n\to\infty} d(y, y_n) = r(y, \{y_n\}).$$

That is, $|r(z_i, \{y_n\}) - r(y, \{y_n\})| \to 0$ as $i \to \infty$. It follows from Lemma 2.2 that $\lim_{i\to\infty} T_1^i y = y$. Utilizing the uniform continuity of T_j, we have that $T_j(y) = T_j(\lim_{i\to\infty} T_j^i y) = \lim_{i\to\infty} T_j^{i+1} y = y$.Therefore $y \in F(T_1)$. Similarly, we can show that $y \in F(T_2)$. That is, $y \in F$. The rest of the proof is the same as carried out in Theorem 3.1. □

We now prove a strong convergence theorem in general convex metric space.

Theorem 3.3. *Let C be a nonempty, closed and convex subset of a convex metric space X. Let $T_i\,(i = 1, 2) : C \to C$ be total asymptotically nonexpansive mappings satisfying conditions $(C1) - (C2)$ given in Lemma 3.1. If $F \neq \phi$, then $\{x_n\}$ given in (4), strongly converges to an element of F if and only if* $\liminf_{n\to\infty} d(x_n, F) = 0$.

Proof. Necessity is obvious. Conversely, suppose that $\liminf_{n\to\infty} d(x_n, F) = 0$. In the proof of Lemma 3.1, we have shown that

$$\begin{aligned} d(x_{n+1}, x) \leq & \left[1 + a\delta\left(k_{n,1}^1 + k_{n,2}^1\right)\right] d(x_n, x) \\ & + \delta a\left(k_{n,1}^1 + k_{n,2}^1\right) + \delta\left(k_{n,1}^2 + k_{n,2}^2\right). \end{aligned} \tag{14}$$

On setting $d_n^1 = a\delta\left(k_{n,1}^1 + k_{n,2}^1\right)$ and $d_n^2 = \delta a\left(k_{n,1}^1 + k_{n,2}^1\right) + \delta\left(k_{n,1}^2 + k_{n,2}^2\right)$, we note that $\sum_{n=1}^{\infty} d_n^1 < \infty$ and $\sum_{n=1}^{\infty} d_n^2 < \infty$. Hence (14) becomes

$$d(x_{n+1}, x) \leq \left(1 + d_n^1\right) d(x_n, x) + d_n^2. \tag{15}$$

By taking $\inf_{x\in F}$ on both sides of (15), we obtain that

$$d(x_{n+1}, F) \leq \left(1 + d_n^1\right) d(x_n, F) + d_n^2.$$

Applying Lemma 2.1 to (15), we get that $\lim_{n\to\infty} d(x_n, F)$ exists; but by the hypothesis $\liminf_{n\to\infty} d(x_n, F) = 0$, we conclude that $\lim_{n\to\infty} d(x_n, F) = 0$. Next, we claim that $\{x_n\}$ is a Cauchy sequence. Assume that $\sum_{n=1}^{\infty} d_n^1 = d^0$ and hence $\prod_{n=1}^{\infty}(1 + d_n^1) = d^0$. For $\varepsilon > 0$,there exists $n_0 \geq 1$ such that $d(x_{n_0}, F) < \frac{\varepsilon}{4d^0+4}$ and $\sum_{n=n_0}^{\infty} d_n^2 < \frac{\varepsilon}{4d^0}$. Let $m > n \geq n_0$. Then with the help of (15), we obtain

$$\begin{aligned} d(x_m, x_n) &\leq d(x_m, F) + d(x_n, F) \\ &\leq \prod_{i=n_0}^{m-1} \left(1+d_i^1\right) d(x_{n_0}, F) + \prod_{i=n_0}^{m-1} \left(1+d_i^1\right) \sum_{n=n_0}^{m-1} d_i^2 \\ &\quad + \prod_{i=n_0}^{n-1} \left(1+d_i^1\right) d(x_{n_0}, F) + \prod_{i=n_0}^{n-1} \left(1+d_i^1\right) \sum_{n=n_0}^{n-1} d_i^2 \\ &\leq \prod_{i=n_0}^{\infty} \left(1+d_i^1\right) d(x_{n_0}, F) + \prod_{i=n_0}^{\infty} \left(1+d_i^1\right) \sum_{n=n_0}^{\infty} d_i^2 \\ &\quad + \prod_{i=n_0}^{\infty} \left(1+d_i^1\right) d(x_{n_0}, F) + \prod_{i=n_0}^{\infty} \left(1+d_i^1\right) \sum_{n=n_0}^{\infty} d_i^2 \\ &< 2\left[\left(1+d^0\right) \frac{\varepsilon}{4d^0+4} + \frac{\varepsilon d^0}{4d^0}\right] = \varepsilon. \end{aligned}$$

This proves that $\{x_n\}$ is a Cauchy sequence in C. Let $\lim_{n\to\infty} x_n = q$. Then $d(q, F) = d(\lim_{n\to\infty} x_n, F) = \lim_{n\to\infty} d(x_n, F) = 0$. As F is closed, so we obtain $q \in F$. Hence $\{x_n\}$ strongly converges to a point of F. □

Our next theorems are applications of Theorem 3.3 and make use of condition(J) and condition(D) .

Theorem 3.4. *Let C be a nonempty, closed and convex subset of a complete and uniformly convex metric space X. Let $T_i\,(i = 1, 2) : C \to C$ be uniformly $L-$Lipschitzian and total asymptotically nonexpansive mappings satisfying condition(J) and conditions $(C1) - (C2)$ as given in Lemma 3.1. If $\{x_n\}$ is the sequence in (4) with $\lim_{n\to\infty} d(x_n, x_{n+1}) = 0$, then $\{x_n\}$ strongly converges to an element of F.*

Proof. In the proofs of Theorem 3.1 and Theorem 3.3, we have shown that $\lim_{n\to\infty} d(T_1x_n, T_2x_n) = 0$ and $\lim_{n\to\infty} d(x_n, F)$ exists, respectively.

By using condition (J), we have that

$$\lim_{n\to\infty} h(d(x_n, F)) \leq \lim_{n\to\infty} d(T_1x_n, T_2x_n) = 0.$$

Since h is a nondecreasing function and $h(0) = 0$,therefore $\lim_{n\to\infty} d(x_n, F) = 0$. Now, applying Theorem 3.3, we get the required conclusion. □

Theorem 3.5. *Let C be a nonempty, closed and convex subset of a complete and uniformly convex metric space X. Let $T_i\,(i = 1, 2) : C \to C$ be uniformly continuous and total asymptotically nonexpansive mappings satisfying condition(D) and conditions $(C1)-(C2)$ given in Lemma 3.1. If $\{x_n\}$ is the sequence in (4) with $d(x_n, T_1^n x_n) \leq d(T_1^n x_n, T_2^n x_n)$, then $\{x_n\}$ strongly converges to an element of F.*

Proof. In the proof of Theorem 3.2, we have shown that $\lim_{n\to\infty} d(x_n, T_1x_n) = 0 = \lim_{n\to\infty} d(x_n, T_2x_n)$. Also $\lim_{n\to\infty} d(x_n, F)$ exists as shown in Theorem 9.

By using condition (D), we have that

$$\lim_{n\to\infty} h(d(x_n, F)) \leq \max\left[\lim_{n\to\infty} d(x_n, T_1x_n), \lim_{n\to\infty} d(x_n, T_2x_n)\right] = 0.$$

The rest of the proof is the same as the proof of Theorem 3.4. □

4. **Conclusions**

We conclude that:

(i) Hilbert spaces and CAT (0) spaces are uniformly convex metric spaces, therefore our results hold in these spaces immediately.

(ii) Nonexpansive mappings, asymptotically nonexpansive mappings, generalized asymptotically nonexpansive mappings and asymptotically nonexpansive mappings in the intermediate sense all are total asymptotically nonexpansive, therefore our theorems hold for these mappings straightforward.

(iii) When $T_1 = I$(the identity mapping), $T_2 = T$, all the above theorems remain valid for the Mann's iterative algorithm(5).

(vi) One can easily establish results of this paper for nonself total asymptotically nonexpansive mappings in CAT(0) spaces. The new results will be analogue of the results of Zhou et al. [26].

(v) The variational inequality problem and split feasibility problem in certain situations can be converted into a fixed point problem, therefore it is expected that our results will be helpful to address these types of problems; for instance, see [21, 22, 23, 24, 25].

(vi) The established results are interesting for applied mathematics and can be utilized for further research studies; to explore more in this direction, we refer the reader to consult [18, 19, 20, 21, 22, 23, 24, 25].

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