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PALISADES, NEW YORK

Lamont Geological Observatory

Technical Report on Seismology

No. 8

Propagation of Elastic Waves in a Floating Ice Sheet

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Technical Report No. 8

by

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The research reported in this document has been made possible through support and sponsorship extended by the Geophysical Research Directorate of the Cambridge Field Station, AMC, U.S. Air Force, under Contract W-28-099 ac-396. It is published for technical information only and does not represent recommendations or conclusions of the sponsoring agency.

November 1950

ABSTRACT

The characteristic equation for the propagation of elastic waves in a floating ice sheet is derived and discussed, with special emphasis on the evaluation of phase velocity for very large and very small wavelengths. The solutions cannot be reduced to symmetric and anti-symmetric modes as in the case of a plate in a vacuum or a liquid. Among the waves discussed are compressional waves, shear waves, longitudinal waves, flexural waves, Rayleigh waves, Stoneley waves, and Love waves.

INTRODUCTION

This paper was undertaken as a result of current interest in the utilization of elastic waves transmitted through a floating ice sheet for determining the thickness and mechanical strength of the ice, for position fixing and long range signalling. The ice sheet is treated as a two dimensional wave guide and the characteristic equation giving the functional dependence between phase velocity and period is derived. From an examination of this equation one can identify the possible waves and determine whether or not they are attenuated. In the derivations which follow the atmosphere is replaced by a vacuum. Air coupling of flexural waves in floating ice has been treated in other papers by Press, Crary, Oliver, and Katz (1950) and by Press and Ewing (1950).

INFINITE PLATE IN A VACUUM AND LIQUID

The analogous problems of wave propagation in an infinite elastic plate were solved by Lamb (1917) for a plate in a vacuum and by Osborne and Hart (1945) for a plate in a liquid having infinite extent and considerably smaller sound velocity than that for compressional or shear waves in the plate. These writers did not include SH waves (shear waves polarized horizontally) in their solutions, but fully discussed all other elastic wave types.

In the solutions of Lamb, and Osborne and Hart the period equations take the form of fourth and sixth order determinants respectively. These reduce respectively to two second and third order determinants representing motions of the plate which are symmetrical and antisymmetrical about the median plane. For wavelengths small compared to the thickness of the plate the symmetric and antisymmetric solutions reduce to Rayleigh waves transmitted along the surfaces. These surface waves are attenuated for the case of a plate in a liquid because of losses by acoustic radiation. For short wavelengths there is also a Stoneley type wave travelling along the plate-liquid interfaces with a velocity close to but less than the velocity of sound in the liquid.

For wavelengths large compared to the thickness of the plate the antisymmetric solution reduces to flexural waves and the symmetric solution becomes longitudinal waves of an elastic plate. The effect of the liquid on the propagation of the longitudinal waves is to add a small attenuation which increases as the frequency. The flexural waves, travelling with a velocity less than the speed of sound in the liquid, remain unattenuated in the presence of the liquid, although the dispersion is somewhat changed. For long wavelengths the presence of the liquid introduces an additional solution for the symmetric case in which the waves are propagated without attenuation with a velocity very close to that of sound in the liquid.

For intermediate wavelengths solutions exist for the plate in a liquid which are closely related to the corresponding solutions for a plate in a vacuum. The effect of the liquid is to introduce attenuation caused by radiation loss for those branches in which phase velocity exceeds the speed of sound in the liquid. The presence of a liquid introduces two additional branches representing unattenuated propagation at velocities less than the speed of sound in the liquid.

FLOATING ICE SHEET

The theory of elastic waves in a floating ice sheet will now be discussed in greater detail. The procedure will follow that of Lamb and Osborne and Hart whose solutions were used as a basis for solving the present problem.

Let us consider the propagation of elastic waves in an infinite, floating ice sheet of thickness 2H, density ρ_1, in which the velocities of compressional and distortional waves are respectively α_1 and β_1. The underlying liquid having density ρ_2 and compressional wave velocity v_2 is taken to be infinitely deep. λ_1, μ_1 are Lamé's constants for the ice and λ_2 is the incompressibility of the liquid related to the elastic wave velocities as follows:

$$\alpha_1^2 = (\lambda_1 + 2\mu_1)/\rho_1 \ , \ \beta_1^2 = \mu_1/\rho_1 \ , \ v_2^2 = \lambda_2/\rho_2 \tag{1}$$

The cartesian coordinate system is chosen with the x-axis in the median plane of the ice sheet parallel to the direction of propagation and the z-axis vertically downward. The subscripts 1 and 2 refer to the ice and liquid respectively. We introduce the functions φ (x,z,t) and ψ (x,z,t) defined by the equations

$$
\begin{aligned}
u_1 &= \partial\varphi_1/\partial x + \partial\psi_1/\partial z \\
w_1 &= \partial\varphi_1/\partial z - \partial\psi_1/\partial x \\
u_2 &= \partial\varphi_2/\partial x \\
w_2 &= \partial\varphi_2/\partial z
\end{aligned} \tag{2}
$$

where u and w are the horizontal and vertical particle displacements. SH waves are not included in this discussion and will be treated more simply in a later section. The vertical stress p_{zz} and the tangential stress p_{xz} can be expressed in terms of φ, ψ and the elastic constants as follows:

$$
\left.
\begin{aligned}
p_{zz} &= \lambda\nabla^2\varphi + 2\mu(\partial^2\varphi/\partial z^2 - \partial^2\psi/\partial x\partial z) \\
p_{xz} &= \mu(\partial^2\psi/\partial z^2 - \partial^2\psi/\partial x^2 + 2\,\partial^2\varphi/\partial x\partial z)
\end{aligned}
\right\} \tag{3}
$$

It is required that the functions φ and ψ satisfy the wave equations

$$
\left.
\begin{aligned}
&\partial^2\varphi_1/\partial t^2 = \alpha_1^2\,\nabla^2\varphi_1 && \text{in the ice} \\
&\partial^2\psi_1/\partial t^2 = \beta_1^2\,\nabla^2\psi_1 && \text{in the ice} \\
&\partial^2\varphi_2/\partial t^2 = V_2^2\,\nabla^2\varphi_2 && \text{in the liquid}
\end{aligned}
\right\} \tag{4}
$$

and the boundary conditions

$$
\left.
\begin{aligned}
&(p_{zz})_1 = 0 && \text{at } z = -H \\
&(p_{zx})_1 = 0 && \text{at } z = -H \\
&(p_{zz})_1 = (p_{zz})_2 && \text{at } z = H \\
&(p_{zx})_1 = 0 && \text{at } z = H \\
&w_1 = w_2 && \text{at } z = H
\end{aligned}
\right\} \tag{5}
$$

It can readily be verified that solutions of (4) are of the form:

$$\left.\begin{aligned}
\varphi_1 &= \left[A\sinh(\xi z) + B\cosh(\xi z)\right]\exp\left[i(kx-\omega t)\right]\\
\psi_1 &= \left[C\sinh(\eta z) + D\cosh(\eta z)\right]\exp\left[i(kx-\omega t)\right]\\
\varphi_2 &= E\exp(-\zeta z)\exp\left[i(kx-\omega t)\right]
\end{aligned}\right\} \quad (6)$$

where

$$\left.\begin{aligned}
\xi^2 &= k^2(1-c^2/\alpha_1^2)\\
\eta^2 &= k^2(1-c^2/\beta_1^2)\\
\zeta^2 &= k\ \ (1-c^2/V_2^2)
\end{aligned}\right\} \quad (7)$$

and $c = \omega/k$. In equation (6) φ_1 , ψ_1 , and φ_2 make up a system of waves progressing in the x direction with a phase velocity c and wavelength L related to the wave number k by $k = 2\pi/L$. The frequency f is given by $f = \omega/2\pi = ck/2\pi$. φ_2 is taken to decrease exponentially with depth in the liquid since we are particularly interested in the case where no energy is lost by refraction into the liquid.

If equations (6) are substituted in the boundary conditions (5), using (2) and (3), five independent homogeneous equations in the unknown amplitude coefficients A, B, C, D, E, are obtained. Any four of the unknowns can be found in terms of the fifth. For a solution of all five to exist, however, the fifth order determinant of their coefficients must vanish. The characteristic equation is the condition for the determinant to vanish. Poisson's assumption is made at this point to simplify our equations. Thus we take Poisson's constant $\sigma = 0.25$ for ice, so that $\lambda_1 = \mu_1$. Although observed values of σ for lake ice run higher, the simplification involved justifies this assumption since we are interested primarily in identifying the possible waves and only order of magnitude calculations are made. The period equation can then be written as follows:

$$P\left[Q + \delta \cosh(\xi H)\cosh(\eta H)\right] + Q\left[P + \delta \sinh(\xi H)\sinh(\eta H)\right] = 0 \tag{8}$$

where

$$\left.\begin{aligned}
\delta &= \rho_2 V_2^2(\eta^2-\kappa^2)(\zeta^2-\kappa^2)\xi/\rho_1\beta_1^2\zeta \\
Q &= (\eta^2+\kappa^2)^2\sinh(\xi H)\cosh(\eta H) - 4\xi\eta\kappa^2\cosh(\xi H)\sinh(\eta H) \\
P &= (\eta^2+\kappa^2)^2\cosh(\xi H)\sinh(\eta H) - 4\xi\eta\kappa^2\sinh(\xi H)\cosh(\eta H)
\end{aligned}\right\} \tag{9}$$

If these equations are compared to the characteristic equations given by Lamb it will be seen that P = 0 and Q = 0 represent respectively the symmetric and antisymmetric solutions for a plate in a vacuum. Osborne and Hart obtained for the case of a plate in a liquid $P+\delta\sinh(\xi H)\sinh(\eta H)=0$ and $Q+\delta\cosh(\xi H)\cosh(\eta H)=0$ which represent respectively the symmetric and antisymmetric solutions. It is evident that unlike these cases, the motion in a floating ice sheet cannot be reduced to purely symmetric and antisymmetric modes. It can also be seen that δ is a correction term introduced by the presence of the liquid, and vanishes as the density of the liquid approaches zero.

The period equation (8) defines an implicit relation between the phase velocity c and the wave number k. The dependence of c on ω can be obtained by substituting $k = \omega/c$ in (8). Real values of c and k which satisfy (8) correspond to propagation without attenuation. Complex values of c or k for real ω correspond to propagation with attenuation, the degree of attenuation increasing with the magnitude of the imaginary component. The energy loss associated with damped propagation is due to radiation from the ice sheet into the liquid.

From equation (8) it is seen that $c = \alpha_1$ and $c = \beta_1$ are solutions which satisfy the characteristic equation for all frequencies.

EVALUATION OF PHASE VELOCITY FOR VERY SMALL WAVELENGTHS

For wavelengths small compared to the thickness of the ice sheet (kH large) P = Q and equation (8) reduces to

$$(\eta^2 + K^2)^2 - 4\xi\eta K^2 = 0 \tag{10}$$

and

$$(\eta^2 + K^2)^2 - 4\xi\eta K^2 + \delta = 0 \tag{11}$$

Equation (10) is the same expression derived by Lamb for a plate in a vacuum and corresponds to Rayleigh waves propagated along the free surface of a semi-infinite solid medium. The algebraic solution of (10) for c gives the well known result for Rayleigh waves $c_R = .9194\beta_1$. Equation (11) is identical to the expression derived by Osborne and Hart for a plate in a liquid and corresponds to Rayleigh waves propagated along the interface between a semi-infinite liquid and solid medium. Following Osborne and Hart the phase velocity can be deduced approximately from (11) by substituting $c = c_R (1 + \epsilon)$, ($\epsilon < 1$) in (11). If δ is small we may write approximately

$$\epsilon = -i \frac{(\rho_2/\rho_1)(1 - c_R^2/\alpha_1^2)^{1/2} c_R^4/\beta_1^4}{4(c_R^2/v_2^2 - 1)^{1/2}\left[2\left(1-(1/2)c_R^2/\beta_1^2\right)c_R^2/\beta_1^2 - \left(\frac{1-c_R^2/\beta_1^2}{1-c_R^2/\alpha_1^2}\right)^{1/2} c_R^2/\alpha_1^2 - \left(\frac{1-c_R^2/\alpha_1^2}{1-c_R^2/\beta_1^2}\right)^{1/2} c_R^2/\beta_1^2\right]} \tag{12}$$

Using $\rho_1 = .917\,\rho_2$, $\beta_1 = 6300$ ft/sec, $v_2 = 4800$ ft/sec, $\alpha_1 = \sqrt{3}\,\beta_1$, we obtain approximately $\epsilon = .25\,i$.

The reduction of the period equation (8) into the algebraic expressions (10) and (11) for wavelengths small compared to the plate thickness is simply interpreted. For these wavelengths the ice sheet is effectively infinitely thick and the propagation reduces to Rayleigh waves propagated without attenuation at the free surface and attenuated Rayleigh waves transmitted along the bottom surface, continuously radiating energy to the liquid.

Equation (11) can also be satisfied by c real and less than v_2. This can be seen by rewriting (11) in the following form:

$$(1-c^2/v_2^2)^{1/2}\left[(2-c^2/\beta_1^2)^2 - 4(1-c^2/\alpha_1^2)^{1/2}(1-c^2/\beta_1^2)^{1/2}\right] + (1-c^2/\alpha_1^2)^{1/2}\rho_2 c^4/\rho_1\beta_1^4 = 0 \tag{13}$$

The first and second terms of (13) are of opposite sign, so that c less than v_2 is a possible solution. For the constants listed above we obtain $c = .87\, v_2$ as a root of (13). This is the speed of a boundary wave travelling along the ice-water interface, with an amplitude which decreases with distance from the ice-water interface. Stoneley (1924) described an analogous boundary wave which can be transmitted under certain conditions along the interface between two semi-infinite solids.

EVALUATION OF PHASE VELOCITY FOR VERY LARGE WAVELENGTHS

For wavelengths which are long compared to the thickness of the ice sheet (kH small), equation (8) can be reduced to the following forms:

$$P + Q\delta \sinh(\xi H) \sinh(\eta H) / 2Q + \delta = 0 \qquad (15)$$

$$Q + \delta/2 = 0 \qquad (16)$$

Equations (15) and (16) will give respectively the velocities of longitudinal waves and flexural waves in a floating ice sheet over deep water. The factor $\sinh(\xi H) \sinh(\eta H)$ is neglected in equation (16) and carried in (15) in order to keep the high order imaginary term giving the attenuation of longitudinal waves. Replacing the hyperbolic functi ons of (15) by linear terms or unity we can derive the approximate expression:

$$\frac{c}{\beta_1} = 2(1-\beta_1^2/\alpha_1^2)^{1/2} \left\{ 1 + i(kH)^3 (2\beta_1/v_1)(\rho_1/\rho_2)(1-\beta_1^2/\alpha_1^2)^{1/2} \left[1 - (1-\beta_1^2/\alpha_1^2) 4\beta_1^2/\alpha_1^2 \right] \right\} \qquad (17)$$

If (17) is compared with the solution $c/\beta_1 = 2(1-\beta_1^2/\alpha_1^2)^{1/2}$ for a plate in a vacuum it is seen that the presence of the liquid hardly affects the real part of the phase velocity, but adds a small attenuation, which for large wavelengths increases as the inverse cube of wavelength. This is in agreement with the experimental work of Ewing, Crary, and Thorne (1934) whose determination of the velocity of longitudinal waves in floating lake ice checks the velocity given by the real part of (17).

For long wavelengths (16) gives for the velocity of flexural waves:

$$\frac{c^2}{\beta_1^2} = (8/3)(\rho_1/\rho_2)(\kappa H)^3(1-\beta_1^2/\alpha_1^2)\Big/(1+2\kappa H\rho_1/\rho_2) \tag{18}$$

In obtaining (18) third order terms must be included in the expansion of the hyperbolic functions. For a plate in a vacuum the corresponding solution is $c^2/\beta_1^2 = (4/3)(\kappa H)^2(1-\beta_1^2/\alpha_1^2)$. It is interesting to note that (18) can be reduced to the corresponding expression for a plate in a liquid simply by replacing ρ_2 by $2\rho_2$.

The steady state plane wave theory of long flexural waves in an ice sheet on water of either finite or infinite depth was given by Ewing and Crary (1934). The results for the latter case were extended by Press and Ewing (1951) for waves originating in an impulsive point source. The observed dispersion and amplitude of explosion generated flexural waves on lake ice are shown to be in reasonable agreement with theory in these papers. At certain frequencies flexural waves in ice are strongly coupled to the atmosphere. This phenomenon has been discussed by Press, Crary, Oliver, and Katz (1951) and Press and Ewing (1951).

EVALUATION OF PHASE VELOCITY FOR INTERMEDIATE WAVELENGTHS

Unlike the results for a plate in a vacuum or a liquid, the characteristic equation does not reduce to symmetric and antisymmetric modes. We are left with a complicated expression (equation (8)) which makes the evaluation of phase velocity as a function of wavelength or frequency an exceedingly difficult task for intermediate wavelengths. Such calculations would indicate how the high and low frequency values of phase velocity discussed above connect up. In general for $c > v_1$, the phase velocity has an imaginary component, indicating attenuation due to radiation from the sheet into water. Both real and imaginary components of phase velocity are frequency dependent, the waves being dispersive as well as selectively attenuated. For $c < v_1$ no energy losses due to radiation into the water occur and the waves are propagated as a dispersive, unattenuated train.

LOVE WAVES IN A FLOATING ICE SHEET

The theory of Love waves in a floating ice sheet can be derived by finding a solution of the wave equation for SH waves which satisfies appropriate boundary conditions. It is convenient and instructive, however, to derive the characteristic equation geometrically, for this method offers a clear physical picture of its origin.

SH waves incident on a free surface or a liquid interface are completely reflected without a change of phase (see for example Jeffreys (1926)). Consequently a floating ice sheet behaves essentially as a wave guide for the propagation of SH waves.

In Figure 1, ABCE represents the ray-path of a plane SH wave of length 1 (measured along the ray-path) undergoing multiple reflections from the boundaries at an angle of incidence θ. As the wave front (shown by the dashed line) moves a distance ED $= \beta_1$ t in t seconds, a point on the wave front moves a distance ct in the horizontal direction where the phase velocity $c = \beta_1 / \sin\theta$ After many reflections the amplitude will be greatest for those waves for which constructive interference occurs between the multiply reflected rays. The characteristic equation is simply the mathematical statement of this condition. Referring to Figure 1 again, it is seen that if the wave front at AE is to interfere constructively with the coincident wavefront which has traversed the additional path ABCE, it is required that

$$\overline{ABCE} = 2H\cos\theta = n\ell \qquad n = 1, 2, 3 \cdots \tag{19}$$

The integers n determine the normal modes of propagation. For each value of n, there is a fixed relationship between 1 and θ (or c) for which constructive interference occurs. The propagation of constructively interfering SH waves is dispersive, the dependence of phase velocity on wavelength being given by (19) and $c = \beta_1 / \sin\theta$. A consequence of dispersion is that the energy associated with each frequency is propagated with the group velocity U given by the well known formula $U = c + \kappa\, dc/d\kappa$

$$= c + \kappa (dc/d\theta)(d\theta/d\kappa)$$

where $\kappa = (2\pi/\ell) \sin\theta$ is the wave number. Carrying out the indicated differentiation one obtains

$$U = \beta_1 \sin\theta = \beta_1^2/c \tag{20}$$

U/β_1 and c/β_1 are plotted for the first and second modes as functions of the dimensionless parameter $\gamma = H/\ell = Hf/v_1$ in Figure 2. The higher modes of Love waves are simply higher harmonics of the first or fundamental mode.

The sequence of waves arriving at a given point from a distant disturbance can be described with the aid of the group velocity curve of Figure 2. The first arrivals are high frequency waves which travel with the velocity β_1. As time progresses the frequency of the arrivals decreases. This corresponds to moving down the group velocity curve. The wave train is infinitely long, the last wave having the lowest frequency given by $f_c = \beta_1/2H$

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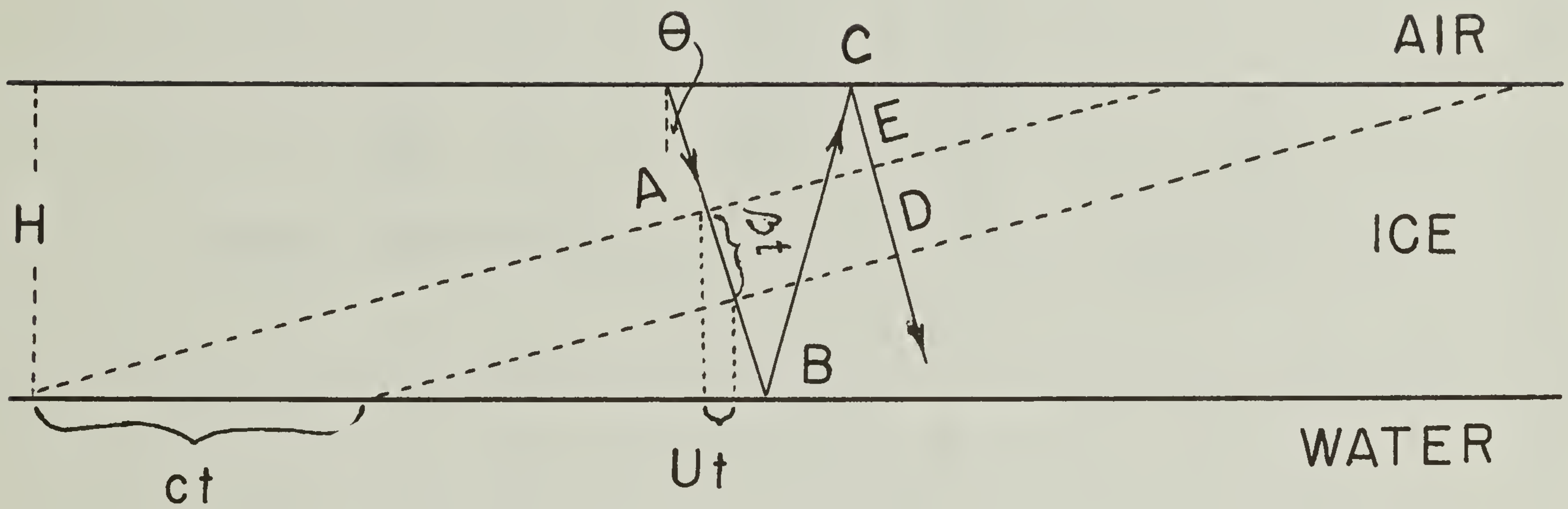
Figure 1. Ray path diagram of SH waves in an ice sheet on water.

Figure 2. Phase and group velocity curves of SH waves in an ice sheet on water.



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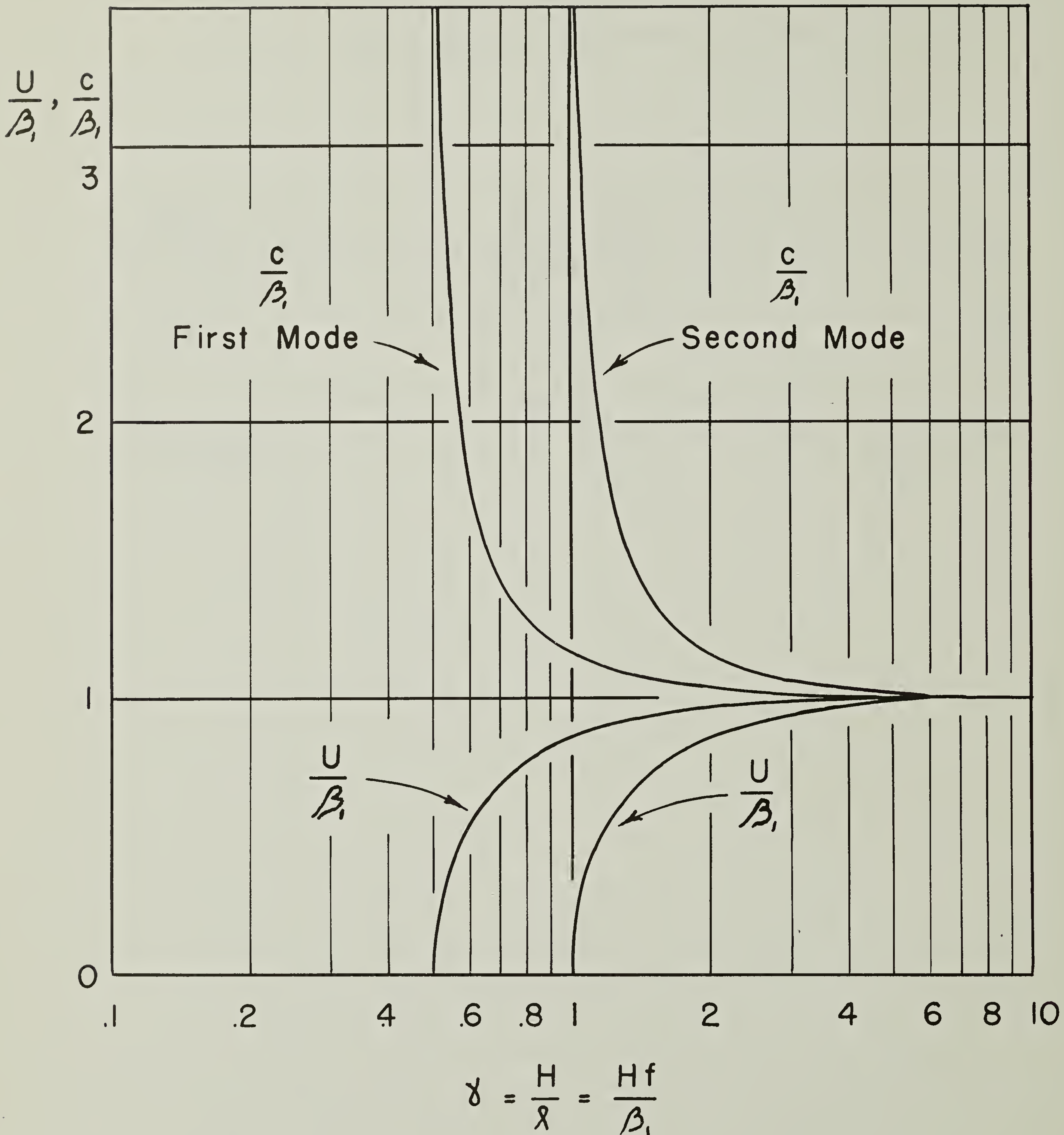

U/β₁, c/β₁
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γ = H/λ = Hf/β₁