AN ALGEBRAIC OPPORTUNITY TO DEVELOP PROVING ABILITY

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ABSTRACT

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Set-based reasoning and conditional language are two critical components of deductive argumentation and facility with proof. The purpose of this qualitative study was to describe the role of truth value and the solution set in supporting the development of the ability to reason about classes of objects and use conditional language. This study first examined proof schemes – how students convince themselves and persuade others – of Algebra I students when justifying solutions to routine and non-routine equations. After identifying how participants learned to use set-based reasoning and conditional language in the context of solving equations, the study then determined if participants would employ similar reasoning in a geometrical context.

As a whole, the study endeavored to describe a possible trajectory for students to transition from non-deductive justifications in an algebraic context to argumentation that supports proof writing. First, task-based interviews elicited how participants became absolutely certain about solutions to equations. Next, a teaching experiment was completed to identify how participants who previously accepted empirical arguments as proof shifted to making deductive arguments. Last, additional task-based interviews in which participants reasoned about the relationship between Varignon Parallelograms and Varignon Rectangles were conducted.

The first set of task-based interviews found that a majority of participants displayed ritualistic proof schemes – they viewed equations as prompts to execute processes and solutions as results, or "answers." Approximately half of participants employed empirical proof schemes; they described convincing themselves or others using a range of arguments that do not constitute valid proof. One particularly noteworthy finding was that no participants initially used deductive justifications to reach absolute certainty. Participants successfully adopted set-based reasoning and learned to use conditional language by progressively accommodating a series of understandings. They later utilized their new ways of reasoning in the geometrical context. Participants employed the implication structure, discriminated between necessary and sufficient conditions, and maintained a disposition of doubt toward empirical evidence. Finally, implications of these findings for pedagogues and researchers, as well as future directions for research, are discussed.

DEDICATION

For Edward, whose curiosity and laughter bring me great joy.

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INTRODUCTION

This chapter introduces my dissertation study. First, I describe the need for this study by situating it within extant literature. Second, I state the purpose of the study and the research questions I endeavored to answer. Third, I provide an overview of the procedures used to answer the research questions.

Need for Study

Proof serves a central role in both the creation of new mathematics and the learning of previously discovered mathematics (Haimo, 1995). From a mathematician's viewpoint, proof is the mechanism through which new mathematics is codified. From a pedagogue's perspective, proof can serve an explanatory role and has the potential to facilitate student understanding of mathematical concepts (Hanna, 2000; Knuth, 2002a). Schoenfeld (1994) argued that "proof is not a thing separable from mathematics…it is an essential component of doing, communicating, and recording mathematics" (p. 76).

Despite the importance of proof in the field of mathematics and the learning opportunities it can facilitate in classrooms, it is well documented that students struggle with proof (Coe & Ruthven, 1994; Harel & Sowder, 2007; Kloosterman, 2010; Weber, 2001). Students have varied conceptions of proof, and there are numerous reasons why students struggle to produce proofs (Healy & Hoyles, 2000; Herbst & Brach, 2006; Knuth, Slaughter, Choppin, & Sutherland, 2002). For example, Tinto (1988) documented that some students believe proof only serves to verify facts already known. Even when students understand the nature of proof, their lack of strategic knowledge or ability to recall definitions may prevent them from producing proofs (Weber, 2001). However, previous studies report that a majority of students at all levels use empirical arguments when asked to provide a proof (Balacheff, 1988; Chazan, 1993; Harel & Sowder, 2007). An empirical argument uses an arbitrary number of specific cases to make a claim. In contrast, a deductive argument reasons about classes or sets; only deductive arguments are considered valid mathematical proofs (Weber, 2008).

Harel and Sowder (1998, 2007) align valid mathematical proof with what they call an analytical proof scheme. The development of an analytical proof scheme includes a number of critical milestones. An analytical proof scheme requires students to use and interpret implication statements, be familiar with the role of quantifiers, and reason about a class of objects.

Student success with proof requires the ability to use and interpret implication statements. Implication is crucial to proof writing because theorems are often expressed as conditional statements. Proofs frequently employ modus ponens, a deduction rule that relies on an understanding of implication (Morash, 1991; Rodd, 2000; Vellman, 2006). Hoyles and Küchemann (2002) report that students struggle to fundamentally understand implications; a large portion of students in their study failed to differentiate between a conditional statement and its converse. Students need conditional statements to describe mathematical relationships; fluency with conditional language greatly aids the development of the analytical proof scheme (Epp, 2003).

Student familiarity with the role of quantifiers is another milestone in the development of an analytical proof scheme. According to Durand-Guerrier (2003), students' struggles with implications are due in part to the implicit quantifiers they contain. For example, the statement "if a quadrilateral is a square, then the quadrilateral is a rectangle" conveys that *all* squares are rectangles. Previous studies find that students struggle to create and decode statements involving quantifiers (Dubinsky & Yiparaki, 2000; Selden & Selden, 1995). Epp (1999) suggested that successfully navigating the form and language of mathematical proof requires fluency with the quantifiers of predicate calculus.

A third feature in the development of an analytical proof scheme is students' abilities to reason – and express their reasoning – about a class of objects. A class of objects, sometimes referred to as a set of objects, is a category of objects defined by their shared properties. A class can consist of actual objects – rectangles, for example, are a class of objects because of their common, defining properties. The objects, however, can also be of a different nature. Critical for understanding equations, the solution set is such as a class. The solution set is the set of values that make an equation (or other mathematical statement) true when substituted into an equation. For example, {-2, 2} is the solution set for $x^2 = 4$. Some sets contain an infinite number of objects; proofs are often concerned with the truth of a statement pertaining to an infinite set (Vellman, 2006). In order to make an argument about an infinite set of objects, one needs to be able to fully understand and articulate the properties of the set. That is, understanding and conveying necessary and sufficient conditions is itself a necessary condition for successful proof writing (Moore, 1994).

In the United States, geometry is often considered the secondary school course that explicitly prepares students to write proofs (Herbst, 2002). However, students continue to struggle with proof after this course, and it appears the instruction students receive in geometry frequently is not sufficient to develop an analytical proof scheme (Alcock & Weber, 2005; Recio & Godino, 2001; Thompson, 1991). A geometry course may have limited efficacy in developing proving ability if students enter the course with an insufficient understanding of implication. Specifically, a student's ability to write proofs at the end of a geometry course has been found to correlate with the proclivity to reason about classes of objects at the start of the course (Senk, 1989). To build student capacity for proof, it stands to reason that instruction – particularly about the nature of implication statements – is needed prior to geometry. The National Council for Teachers of Mathematics (NCTM, 2000) explicitly makes this claim: "Reasoning and proof are not special activities reserved for special times or special topics in the curriculum but should be a natural, ongoing part of classroom discussions, no matter what topic is being studied" (p. 342). Furthermore, the Common Core State Standards strongly advocate for student engagement in reasoning and proving activities across grade levels (National Governors Association Center for Best Practices, 2010).

Algebra is often the course preceding geometry in the secondary school mathematics sequence. As such, it provides an opportunity to develop aspects of students' proving and reasoning abilities immediately before they are formally required to write proofs in their geometry courses. Hoyles (1997) found that different curricula shape students' approaches to proof. Yet recent studies find that algebra textbooks do not offer sufficient opportunities for students to develop their proving abilities (Dituri, 2013; Thompson, Senk, & Johnson, 2012). Algebra students would benefit from opportunities to internalize the structure of conditional statements, part of which requires they learn to reason about classes or sets.

Purpose for Study

The purpose of this qualitative study is to describe the role of truth value and the solution set in supporting the development of the ability to reason about classes of objects and use conditional language. To support this goal, the study adopts Harel and Sowder's (2007) framework to identify the proof schemes that students use when justifying solutions to equations. In addition, this study aims to determine whether or not students who reason conditionally about solution sets are also able to reason about classes of geometrical objects.

Research Questions

1. How do proof schemes differ, if at all, when students justify solutions to different types of algebraic equations?
2. Can students learn to reason about classes of objects and use conditional language when considering the truth value of algebraic equations? If so, how?
3. Are students who reason conditionally about solution sets also able to reason about classes of geometrical objects?

Procedures for Study

This study used task-based interviews (Goldin, 2000) to address the first and third research questions. Task-based interviews are structured such that participants interact with a preplanned task environment. The goal of the researcher during the interview is to elicit a "complete, coherent verbal reason for each of the child's responses, and a coherent external representation constructed by the child" (Goldin, 2000, p. 312).

This study also used a teaching experiment methodology (Steffe & Thompson, 2000) to address the second research question. While conducting a teaching experiment, one sets out to experience firsthand students' mathematical reasoning and learning. The researcher builds and tests viable models of students' mathematical thinking in order to explain the schemes and operations behind student behavior when solving equations. It is important to note the distinction between scheme and proof scheme. Scheme refers to the underlying structure of students' mathematical thinking (Steffe & Thompson, 2000), while proof scheme refers to the type of justification provided (Harel & Sowder, 2007).

Participants

The study was conducted in a large suburban high school with freshmen enrolled in Algebra I. At the time the study was conducted, potential participants had previous experiences solving equations; however, they had no prior experience solving equations with multiple solutions or extraneous solutions. The initial pool of potential participants of approximately 400 students allowed for a carefully considered selection of participants.

There were multiple purposes behind the initial sampling of participants (Creswell & Poth, 2018). First, a basic skill assessment simplified the study by focusing the investigation on a homogenous set of participants. The participants were homogenous in the sense that all participants were able to perform a common set of algebraic procedures (solve two-step equations, use the distributive property, simplify algebraic expressions, and evaluate exponents) before the start of the study. Second, conversations with participants' current algebra teachers ensured that participants would highlight what is typical or average. Last, the participants were identified by their teachers as being potentially information rich cases that manifested information "intensely but not extremely" (Miles & Huberman, 1994, p. 28). That is, participants who were considered typical by their teachers, except for their willingness to share and explain their thinking, were considered for the study.

The number of participants in the study was not predetermined. After the initial pool of potential participants was identified, the first set of task-based interviews commenced. Participants were selected for the task-based interview one at a time until a satisfactory degree of data saturation was achieved. Consistent with Lincoln and Guba's (1985) description of purposeful sampling, redundancy was the criterion for sampling – interviews were terminated when no new information was forthcoming. Then, a subset of participants was selected to participate in the teaching experiment. These participants were selected because they frequently employed empirical arguments during the task-based interview. The teaching experiment set out to demonstrate these participants could learn not to rely on empirical arguments and instead use set-based reasoning and conditional language.

Methods

To answer the first research question, participants were presented one equation at a time during a task-based interview and prompted with the question “can you tell me about the solution to this equation?” The equations presented progressed from familiar to non-routine in order to determine if the types of justifications varied. For example, $4x + 2 = 10$ is an equation a first-year algebra student will likely find routine, while equations with multiple solutions (e.g., $x^3 = 4x$) and no solutions (e.g., $x + 1 = x + 2$) are likely to be non-routine and might cause students to employ different proof schemes. In addition to looking at justifications within tasks, participants were asked to reflect on their thinking across tasks. A set of equivalent equations ($4x + 2 = 10$, $4x + 1 = 9$, and $7x + 1 = 3(x + 3)$) was presented in a single task to offer the opportunity for participants to reason about the relationships among equations.

The data were analyzed using open, axial, and thematic coding (Strauss & Corbin, 2015) to answer the first research question. The interviews were transcribed and read through multiple times before the process of open coding generated a set of codes for every statement participants made. The open codes were then constantly compared within a transcript and across transcripts in order to group codes together and form axial codes (Strauss & Corbin, 2015). This process used Harel and Sowder’s (2007) framework as a lens when analyzing statements that potentially offered insight into the proof scheme behind participants’ justification. Careful attention was paid to the level of certainty participants conveyed in justifications so that it was clear whether or not participants were absolutely certain of their claims (Weber & Mejia-Ramos, 2015). The axial codes were constantly compared and distilled into themes that described any differences in proof schemes used while solving different types of equations.

To answer the second research question, a teaching experiment was conducted in which participants were presented with a series of equations and asked if they are true or false. In order to build viable models of participants' mathematical thinking, participants were asked follow-up questions such as "how do you know?" "is it always true?" "are there other values that make it true?" and "how can you be sure?" The hypothesis of the teaching experiment was that this line of questioning would facilitate the development of reasoning ability about classes of objects and use of conditional language. Participants engaged in tasks until there was sufficient evidence that they had developed the ability to reason about classes of objects and use conditional language. This was confirmed during a post-interview where participants specifically demonstrated that they both i) understood the difference between equations that are always true and sometimes true and ii) were able to reason conditionally about the solution set.

The data from the teaching experiment were analyzed continuously during and in-between the teaching sessions, and retrospectively at the conclusion of the experiment. The ongoing analysis during the experiment monitored the evolution of the models of participants' mathematics; it guided the development of tasks and follow-up questions presented to participants so that the models of participants' mathematics evolved to include the ability reason about classes of objects and use conditional language. The ongoing analysis was influenced by the aim of collecting evidence of participants' ability to i) interpret equations as having a truth value or solution set, ii) distinguish between equations that are universally and conditionally true, iii) use the implication structure to link equivalent equations, and iv) use the implication structure to relate equations where one solution set is a subset of another solution set. The following is an example of using the implication structure: $4x = 8$ is open sentence that could either be true or false, yet if $4x + 1 = 9$ is true, then $4x = 8$ must be true as well; if $4x + 1 = 9$ is false, then $4x = 8$ must also be false.

The retrospective analysis of the data included the constant comparison of statements within a transcript of an individual participant as well as across the transcripts of all participants to produce open, axial, and thematic codes (Strauss & Corbin, 2015). The interpretation of statements during the coding process was consistent with a conceptual analysis (Thompson, 2008). The retrospective analysis sought to determine what participants knew and comprehended in specific contexts, what ways of knowing were propitious or problematic for participants learning to reason about classes of objects and use conditional language, and whether the analyses previously performed during the teaching experiment, taken as a whole, afforded a coherent understanding of participants' mathematics over time.

To answer the third research question, participants were prompted to reason about the relationship between an arbitrary quadrilateral and its Varignon Parallelogram. A Varignon Parallelogram is formed by connecting the midpoints of any quadrilateral. Furthermore, the Varignon Parallelogram is a rectangle if and only if the original quadrilateral has perpendicular diagonals. To elicit participants' reasoning about classes of quadrilaterals, a reasoning task using dynamic geometry software (i.e., Geogebra) was adopted from Lacmy and Koichu (2014).

Participants were first asked to reason about the conditions sufficient for claiming the Varignon Parallelogram is a rectangle. Certain familiar quadrilaterals – squares, rhombi, and kites – have perpendicular diagonals. As a result, knowing a quadrilateral is one of these quadrilaterals is sufficient to claim its Varignon Parallelogram is a rectangle. Similar to Lacmy and Koichu’s (2014) analysis, the data analysis sought to determine if participants reasoned analytically (i.e., about properties) or empirically (i.e., by verifying an arbitrary number of cases). Participants were also prompted to reason about the necessary condition of perpendicular diagonals. The analysis specifically considered whether or not participants attended to unfamiliar, arbitrary quadrilaterals and, in doing so, fully described Varignon Parallelograms as a class of objects.

Chapter II

LITERATURE REVIEW

This chapter provides a review of the literature on proof. First, I discuss the role of proof in mathematics and its relationship to the values and norms of mathematicians. Second, I provide an overview of students' conceptions of proof. In this section, I specifically use Harel and Sowder's (2007) framework of proof schemes as a lens. Third, I provide a review of the literature on specific components of proof: conditional language, quantifiers, and set-based reasoning. Last, I highlight the literature detailing the role algebra serves in the development of proving ability.

The Role of Proof

According to Schoenfeld (2009), "If problem solving is the 'heart of mathematics,' then proof is its soul…This dualism of exploration and confirmation, the lifeblood of mathematics, is the everyday work of every mathematician" (p. xii). Given the importance of proof in the field of mathematics, it is not a surprise that there is "a general consensus on the fact that the development of a sense of proof constitutes an important objective of mathematics education" (Mariotti, 2006, p. 173). Proof is the focus of a substantial amount of research in mathematics education because of the many powerful purposes proof can serve. Hanna's (2000) survey of proof literature produced the following list describing the various roles of proof:

- *Verification* that a statement is true
- *Explanation* of why a statement is true
- *Systematization* of the results into an organized, deductive system of axioms, theorems and major concepts
- *Discovery* of new results
- *Communication* (or transmission) of mathematical knowledge
- *Construction* of an empirical theory
- *Exploration* of the meaning of a definition or consequences of an assumption
- *Incorporation* of a well-known fact into a new framework allowing for a new perspective

Mathematicians pursue truth; they investigate conjectures and use proofs to verify that results are in fact true. Whereas evidence suggests an assertion may be true, a mathematical proof is a robust argument that not only convinces its author, but should be able to convince any skeptic (Bell, 1976). In searching for a proof that verifies the truth of a conjecture, Mason, Burton, and Stacey (2010) offered the following advice: "convince yourself, convince a friend, convince an enemy" (p. 87). Proof is imbued with certainty, and this makes proof especially useful when encountering non-intuitive or doubtful results (de Villiers, 1990).

While verification is a hallmark of proof, de Villiers (1990) adopted the view that verification is not necessarily the most important aspect of proof and other roles, especially explanation, are equally, if not more, important. Yet, Hanna (2000) remarked that a proof does not necessarily serve all of the purposes in her list. While Hanna (2000) argued the explanation of why a statement is true is one of proof's most important functions, she also stated that "some proofs by their nature are more explanatory than others" (p. 8). Hanna continued on to assert that explanatory proofs – proofs that reveal why a statement is true – make power pedagogical tools. Consider, for example, Hsu's (2010) study (n = 621) that found students do better with geometric calculations after engaging with related proof tasks.

Mathematicians seek to build upon the current body of mathematical knowledge, and proof is the primary way they discover and create new results (Hanna, 2000). Similarly, proving can be an activity in which students learn new mathematics. Edwards (2010) interviewed doctoral students about their experiences with proof and found that they often evoked the metaphor of proof as a journey. Continuing this metaphor, proof not only allows one to set out to verify and explain a particular result, proof provides opportunities to take detours into new, unanticipated, mathematics. Consider a typical proof of the quadratic formula that requires one to complete the square. Prior to that point, a student may not have needed to know the technique of completing the square, but the proof naturally engenders the need to introduce the procedure in a way that connects it to previously established mathematics (Hanna & Barbeau, 2008).

The communication of mathematics is another purpose of proof – proof is the format in which mathematics is codified (Hanna, 2000). Yet proofs do not exist in a vacuum; proofs need to be written and read by mathematicians and mathematics students. Proof is a form of social interaction, and proof is only useful as a communication tool to the extent that the different parties have shared understandings (Yackel & Cobb, 1996).

Yackel and Cobb (1996) used the term "sociomathematical norms" to capture how the notion of proof is shaped by the learning community. The development of normative interpretations of proof and what it means to prove a conjecture are developed through social interactions.

Stylianides (2007a) argued that proof in a particular mathematics community requires the use of: i) a common set of statements accepted without any need for further justification, ii) forms of reasoning that are valid and known to the community, and iii) modes of argument representation that are appropriate and known to the community. Stylianides' (2007a) argument is supported by the proving behavior of mathematicians. Weber (2011) interviewed nine mathematics professors and found they present proofs differently based on their audience. Professors attend to explanatory details if the proof is pedagogical in nature, whereas certain details are assumed to be understood when writing for academic purposes (i.e., writing for other mathematicians). Some mathematicians also believe that proofs in lectures should include diagrams, whereas diagrams are often omitted when writing for other mathematicians (Weber, 2011).

Mathematical Values and Norms

In the classroom, mathematical values and norms are developed through social interaction; the teacher mediates this process as a representative of the mathematics community (Yackel & Cobb, 1996). Dawkins and Weber (2017) offered a framework in which they explicated the values and norms of the mathematics community vis-à-vis proof. Deductive reasoning and proving, the actions one takes, represent norms of doing mathematics. Moreover, "one would not call someone a 'deductive mathematician' as it would be redundant" (Dawkins & Weber, 2017, p. 128). Values influence norms, and proof, the product of mathematical activity, can be thought of as encapsulating the values of mathematicians. Dawkins and Weber (2017) submitted the following values are the driving force behind the mathematics community's norms for proof:

> 1. Mathematical knowledge is justified by a priori arguments.
> 2. Mathematical knowledge and justifications should be a-contextual and specifically be independent of time and author.
> 3. Mathematicians desire to increase their understanding of mathematics.
> 4. Mathematicians desire a set of consistent proof standards. (p.128)

According to Paseau (2011), "Mathematicians prefer deductive evidence and actively look for it even in the presence of overwhelming inductive evidence. The reason for this is that they are mathematicians and as such value deduction" (p. 144). Paseau continued on to draw the analogy of running a marathon. If the goal was to simply get to the end of the course, one might consider driving the 26.2 miles. Similarly, one reason mathematicians write proofs is to see if they can support their claims with deductive evidence.

The values and norms of a mathematics classroom – while not necessarily the same as the values and norms of the greater mathematics community – serve to uphold the values and norms of mathematicians (Dawkins & Weber, 2017). To that end, Herbst and Balacheff (2009) argued the practice of mathematicians (i.e., mathematicians' proving activities) should inform and constrain what happens in mathematics classrooms. Specifically, although students may not engage with proof exactly like mathematicians do, students' classroom experiences should not leave them with a distorted sense of how mathematicians prove. If students do not share the values of mathematicians (especially deduction), they may struggle to comply with norms of proof production. Dawkins and

Cook (2017) suggested the values and norms of students could be more like that of mathematicians given the right experiences with mathematics: “We anticipate that students are more likely to accept an epistemic aim if they experience that aim being achieved” (p. 138). If proofs are intended to provide convincing and explanatory evidence, students need to experience proofs that are convincing and explanatory.

Conceptions of Proof

Harel and Sowder (2007) offered a summary of proof literature through the lens of one’s proof scheme. Harel and Sowder defined one’s proof scheme as how one ascertains (i.e., convinces oneself) and persuades (i.e., convinces others). Hammack (2013) wrote, “A proof of a theorem is written verification that shows that the theorem is definitely and unequivocally true. A proof should be understandable and convincing to anyone who has the requisite background and knowledge” (p. 87). This means that, ideally, students seek out proof and view it as the most convincing type of evidence – yet, this is often not the case (Stylianides, Stylianides, & Weber, 2017).

Proof and the act of proving mean different things to different students. The notion of mathematical proof is often conflated with “proving” in everyday life because of the multiple ways the word “proof” is used. Tall (1989) observed proof for a jury means “beyond a reasonable doubt,” a scientist finds “empirical proof,” and a statistician may view proof probabilistically. Johnson-Laird (2010) stated that everyday reasoning and mathematical proof are different:

> Human reasoning is not simple, neat, and impeccable. It is not akin to a proof in logic. Instead, it draws no clear distinction between deduction, induction, and abduction, because it tends to exploit what we know. Reasoning is more a

> simulation of the world fleshed out with all our relevant knowledge than a formal manipulation of the logical skeletons of sentences. (p. 18249)

Empirical Proof Schemes

"In an empirical proof scheme, conjectures are validated, impugned, or subverted by appeals to physical facts or sensory experiences" (Harel & Sowder, 1998, p. 252). The empirical proof scheme is important because students are often convinced by empirical or inductive arguments (Bell, 1976; Harel & Sowder, 2007). According to Harel and Sowder (1998), an inductive proof scheme – a proof scheme in which students are convinced by examining specific cases – is a type of empirical proof scheme. The impact of possessing an empirical proof scheme on one's mathematical behavior is significant. When asked to produce proofs, students with an empirical proof scheme demonstrate they are primarily interested in verification, not uncovering the relationships that give rise to the conditions (e.g., Coe & Ruthven, 1994; Recio & Godino, 2001). Nevertheless, students more readily accept justifications that help them understand why a statement is true, and reasoning from examples is often explanatory for students (Bieda & Lepak, 2014).

Balacheff (1988) distinguished between empirical and deductive arguments using the terms "pragmatic proofs" and "conceptual proofs." Balacheff's framework delineated pragmatic and conceptual proofs into four types of arguments:

1. Naïve empiricism
2. The crucial experiment
3. The generic example
4. The thought experiment

Inductive arguments commonly fall into the category of naïve empiricism. With this type of reasoning, students verify an arbitrary number of cases (usually not very many). A critical component of Balacheff's taxonomy is the distinction between the crucial experiment and the generic example. In a crucial experiment, a student picks a seemingly arbitrary case and "…poses explicitly the problem of generality and resolves it by staking all on the outcome of a particular case that she [recognizes] to be not too special" (Balacheff, 1988, p. 219). The crucial experiment employs the use of a prototype that may not be representative of the class of objects, whereas the generic example uses a prototype that was selected after considering the properties of the class. The transition between these two stages represents a shift from inductive to deductive thought (Balacheff, 1988).

One might conclude the prevalence of empirical arguments in lieu of deductive arguments is a result of students' inability to write proofs and think deductively. However, employing inductive arguments in one context does not mean one cannot reasoning deductively in another. As a testament to how sensitive argumentation can be to context, consider Lacmy and Koichu's (2014) study that examined student reasoning with quadrilaterals using dynamic geometry software. The researchers documented that students sometimes prove a biconditional statement by proving one conditional statement empirically and its converse deductively.

Even when students are able to write a proof, it is often the case that they still employ empirical strategies to convince themselves the statement is true. Fischbein and Kedem (1982) were among the first researchers to report that students verify a number of cases after successfully writing a correct proof. In addition, some students will try to construct geometric figures that they just proved were impossible to construct (Schoenfeld, 1989). Many students appear to treat proof as simply evidence, whereas (empirical) evidence is accepted as proof (Chazan, 1993). Raman (2003) claimed these students may not have access to the key ideas of the proof. Key ideas are the insights that would allow students to bridge the gap from the empirical intuition that they find convincing to accepting a formal proof.

Student struggles to navigate the mathematical legitimacy of inductive and deductive arguments may be due, in part, to their teachers. Pre-service teachers sometimes conflate deductive and inductive arguments and find empirical arguments convincing (Martin & Harel, 1989; Morris, 2002). Additionally, pre-service teachers also struggle to validate proofs. They rely on their overall feeling of whether the argument is plausible and focus on surface features of the proof instead of the evaluating the substance of the argument (Selden & Selden, 2003). Knuth (2002b) interviewed sixteen in-service teachers and found most did not view proof to be entirely convincing – some stated it was possible for a proof and a counterexample to coexist. Not only did the teachers Knuth (2002b) interviewed fail to appreciate the role proof plays in verifying a conjecture, they also failed to recognize and talk about the potential of proof to help develop understanding of mathematical concepts. It appears, somewhat troublingly, that not all teachers believe a proof is the best way to convince students something is true (Miyakawa & Herbst, 2007).

On one hand, there is evidence that the production of a proof does not guarantee its author recognizes its role in verifying the truth of a statement (Chazan, 1993; Knuth, 2002b). On the other hand, it appears the opposite is also true. A. Stylianides and

Stylianides (2009) examined how pre-service teachers constructed proofs and validated their own arguments. The teachers in the study produced empirical arguments, but then went on to acknowledge their proofs were not valid. They offered empirical arguments because they were unable to coordinate all the pieces required to write a proof. Weber, Lew, and Mejia-Ramos (2019) documented that students who submit empirical arguments still have doubts about the truth of the conjecture and would be further convinced by a proof. They did not produce a proof because they either lacked i) the ability or confidence to search for a deductive proof, or ii) the motivation to try to find one. Students who successfully produce proofs may not appreciate the nature of proof, but a lack of proof production does not necessarily mean one does not understand the role of proof.

It may be tempting to think that mathematicians are purely deductive thinkers and the use of empirical arguments means one is not behaving like a mathematician. However, mathematicians also rely on empirical arguments to increase their confidence that a conjecture is true. Mathematicians accept statements within a proof if the examples they generate produce a high degree of certainty, especially if the overall argument seems plausible (Mejia-Ramos & Weber, 2014; Weber, 2008; Weber & Mejia-Ramos, 2011). Physicists also use empirical and deductive reasoning in combination. They work with mathematical models to learn about the physical world, and the physical world can provide insights about the mathematics (Paseau, 2015). Witten (2002) wrote that physicists have no trouble accepting the truth of a conjecture before seeing the mathematical proof:

> A mathematical proof that quantum Yang-Mills theory exists in four dimensions would be a milestone in coming to grips mathematically with twentieth century theoretical physics. The reaction of physicists, however, would be that…one already understands why this theory exists, and that mathematicians would have merely succeeded in supplying the ε's and δ's. (p. 25)

Similarly, de Villiers (1990) remarked that one may be convinced a conjecture is true before a proof exists; without conviction in place, one might not be willingly to spend months attempting to prove the conjecture.

Weber, Inglis, and Mejia-Ramos (2014) also argued that mathematicians use deductive and empirical evidence in tandem. The two sources of evidence do not need to be treated as though they are in opposition to each other. Similar to Paseau's (2015) observation, Hanna (2000) wrote that "no physicist, for example, would accept a fact as true on the basis of a theoretical deduction alone" (p. 19). Muis (2008) found that students who strictly had empiricist views performed worse than who held a hybrid view concerning the role of empirical and deductive evidence. Weber et al. (2014) claimed the presence of empirical beliefs is not the problem, but rather the true problem is the lack of value placed on deductive reasoning.

External Proof Schemes

A number of epistemological models have a lower level where knowledge originates outside the self; a key development in many models is the transition to the point where learners consider themselves as a knower in their own right (Hofer, 2000; Muis 2004). Harel and Sowder (1998, 2007) defined the authoritative proof scheme as a type of external proof scheme – a scheme where doubt is removed by a source external to the learner. This is often the classroom teacher or course textbook. In fact, students often trust what their teachers tell them but not their own reasoning (Yackel & Cobb, 1996). Harel and Sowder (1998) warned that students who rely on their teacher are unlikely to gain confidence in their ability to create mathematics:

> The first and most common expression of this proof scheme is students' insistence on being told the procedure to solve their homework problems, and when proofs are emphasized, they expect to be told the proof rather than take part in its construction. (p. 247)

Muis (2007, 2008) argued epistemological beliefs can affect the types of learning goals students set for themselves. These learning goals in turn influence the types of learning strategies students adopt. Muis and Franco (2009) analyzed the performance of 201 undergraduate students in an educational psychology course and found evidence that students who believe knowledge resides outside of themselves set learning goals and adopt learning strategies that result in lower achievement.

According to Weber et al. (2014), the largest problem with the authoritative proof scheme is not that students will come to believe things that are not true; rather, they will miss out on the opportunity to understand why things are true. Students may come to view mathematics as a "collection of truths, with little or no concern and appreciation for the origin of these truths" (Harel & Sowder, 1998, p. 247). Deductive proof does not necessarily convince, and social acceptance – especially by an authority figure – contributes to the conviction gained from the proof (Weber et al., 2014). While this may not seem like mathematical behavior, mathematicians, however, also gain conviction by accepting results published in respected journals (Mejia-Ramos & Weber, 2014).

A ritualistic proof scheme is another type of external proof scheme (Harel & Sowder, 1998, 2007). In this proof scheme, rituals (the appearance of the argument) are the basis for conviction. The rituals are a source of conviction that does not originate from within the learner, but from outside the learner. In a ritualistic proof scheme, students are concerned with form over content – the ritual of *how* the proof is presented is very important to them. For example, some students think proofs have to use symbols or be in a two column format, but cannot be written in a paragraph (Martin & Harel, 1989). With an emphasis on form instead of content, some proofs in secondary school can be taught and learned in a rote manner, not fulfilling any mathematical purpose (Herbst & Brach, 2006). This is ironic because two-column proofs were originally introduced into geometry classrooms to show that reasoning was being taught (Herbst, 2002). It appears the act of writing down statements and reasons alone is not sufficient to claim one is engaging with proof. As an illustration of this point, Herbst (2002) offered a comment from one of his undergraduate students, "'We did proofs in school, but we never proved anything'" (p. 307).

Healy and Hoyles (2000) found that students sometimes select one type of argument as most convincing, but when asked to select the justification that they thought would receive the best grade from their teacher, they selected arguments with a formal appearance. Other studies have also found students, including undergraduates, rely on surface features (the form) when validating a proof (Inglis & Alcock, 2012; Selden & Selden, 2003; Weber, 2010). Ritualistic arguments are particularly problematic because they can appear legitimate. Vinner (1997) provided the example of a student proficiently using the zero-product property to solve equations of the form $(x – a)(x – b) = 0$ and thus giving the impression that they reasoned mathematically. However, a student with a ritualistic proof scheme may adapt the ritual for $(x – a)(x – b) = 2$ by setting $x – a = 2$ and x – b = 2. The interplay between rituals and actual mathematical reasoning can be subtle. Housman and Porter (2003) conducted a small study with high performing undergraduates and found students can possess more than one proof scheme. When students relied on rituals, they were generally unsuccessful. It appears that students resort to ritualistic arguments when lacking the necessary mathematical knowledge (Houseman & Porter, 2003; Vinner, 1997).

A variation of the ritualistic proof scheme is the symbolic proof scheme. In the symbolic proof scheme, the symbols "…possess a life of their own without reference to their possible functional or quantitative reference" (Harel & Sower, 1998, p. 250). This can be problematic or powerful, as illustrated by Weber and Alcock (2004). In what the authors called syntactic proof production, one unwraps definitions and moves symbols into place. Students are able to successfully employ this strategy because they appeal to the structure or "flow" of a proof. This formulaic approach may be productive, but also may leave the writer without an understanding of the concepts in the proof (Weber & Alcock, 2004). Symbols, while mathematically potent, can provide an illusion of generality and engender a specious sense of conviction. For example, in a study involving fifty secondary mathematics teachers, less than half of the participants correctly invalidated non-general arguments that were presented symbolically (Tsamir, Tirosh, Dreyfus, Barkai, & Tabach, 2008).

Analytical Proof Schemes

Harel and Sowder (1998, 2007) aligned mathematical proof with what they call an analytical proof scheme.

> Simply stated, an analytical proof scheme is one that validates conjectures by means of logical deductions. By this, however, we mean much more that what is commonly referred to as the 'method of mathematical demonstration' – a procedure involving a sequence of statements deduced progressively by certain logical rules from a set of statements (i.e., a set of axioms). (1998, p. 258)

Logic and deduction alone are not enough to qualify as analytical reasoning (Weber & Alcock, 2004). An analytical proof scheme requires sense making and understanding the mathematical content. Furthermore, Harel and Sowder (1998) stated that a critical attribute of an analytical proof scheme is the ability to apply goal-orientated operations to transform objects or images while anticipating the results (e.g., by identifying invariant properties).

The development of an analytical proof scheme, and a rejection of non-analytical proof schemes, requires students to not only be able to reason deductively, but also value deductive justifications. Brown (2014) demonstrated students can develop a disposition of doubt of empirical evidence even though a particular set of empirical data is convincing. That is, students can learn to be skeptical of convincing arguments because of the form of the argument. There are two ways in which this can occur: i) experience with empirical arguments that fail and ii) exposure to culture and norms that are set by an authority (i.e., the teacher values deduction and does not allow empirical arguments).

G. Stylianides and Stylianides (2009) developed a teaching sequence in which they attempted to lead students to distrust empirical evidence. The authors had students work in contexts that lent themselves to empirical arguments but would lead to false conclusions. For example, the expression $1 + 1141n^2$ (where n is a natural number) is not a perfect square for the natural numbers from 1 to 30,693,385,322,765,657,197,397,207. Those convinced by empirical evidence were surprised that the next natural number fails to uphold the pattern. G. Stylianides and Stylianides (2009) used this and other similar contexts to have their students realize empirical evidence, even when overwhelming, cannot be trusted to produce absolute certainty.

The development of an analytical proof scheme does not wait until one interacts with proof, and the ability to generalize is a critical skill that needs to be in place before students work with formal proofs (Stylianides & Stylianides, 2017). Students can learn how to generalize, invalidate false generalizations and explain the "why" of a generalization in algebraic contexts (Mata-Pereira & da Ponte, 2017). Algebra offers the opportunity to use and highlight the importance of deductive reasoning, a goal articulated in *Catalyzing Change in High School Mathematics: Initiating Critical Conversations* (NCTM, 2018).

> Proof is essential in algebra and should not be reserved only for geometry. Proofs of statements in algebra rely on reasoning deductively from definitions and properties that are accepted as true…by a sequence of statements that justify the reasoning. (p. 49)

Further, the ability to generalize in elementary grades is an important development so that the notion of proof does not appear to be an abrupt shift in secondary school (Stylianides, 2007b). Stylianides (2007b) analyzed a classroom episode originally documented by Ball (1993). During this episode, a third grade class investigated the conjecture that the sum of two odd numbers is always an even number. At least one student was convinced by empirical evidence, and another student argued "…the class cannot prove the conjecture for all pairs of odd numbers, because odd numbers and even numbers 'go on forever' and so one 'cannot prove that all of them work'" (Stylianides, 2007b, p. 6). However, one student, Betsy, was able to produce an analytical argument.

The crux of Betsy's argument was consistent with Balacheff's (1988) notion of a thought experiment since she was able to express operations on an abstract object (the class of odd numbers). Stylianides (2007b) argued that because the thought experiment is deductive in nature, Betsy's argument both i) qualified as proof in elementary school and ii) was propitious for the long-term development of her proving ability.

The Development of an Analytical Proof Scheme

Stylianides, Bieda, and Morselli (2016) called for "...productive ways for assessing students' capacities to not only engage in proof, but also to engage in processes that are 'on the road' to proof" (p. 344). The following are three categories of processes that are 'on the road' to proof:

- The expression and interpretation of conditional language
- The expression and interpretation of quantifiers
- The expression and interpretation of classes of objects

Conditional language requires the use and interpretation of the phrases "if, then" and "if and only if" and understanding the mathematical notion of material implication. The quantifiers of predicate logic entail the use and interpretation of the following words or phrases: all, some, none, for all, and there exists. Reasoning and expressing reasoning about classes of objects (i.e., sets) requires understanding necessary and sufficient properties and conceptualizing a large number of objects (sometimes infinite) as a single entity.

Conditional Language

Theorems are often written as conditional statements and recognizing the logical form of the theorem is crucial to writing a proof (Vellman, 2006). Weber (2010) found that even when undergraduates were able to reject empirical arguments, they were still limited by the way they read proofs. The participants were seemingly unaware of the local links between statements within a proof. Linking together statements in a proof usually requires the use of modus ponens, a deduction rule that confirms the conclusion of a conditional statement when arguing from a true premise (Morash, 1991; Rodd, 2000; Vellman, 2006). Undergraduates sometimes also struggle to interpret conditional language when validating a proof as a whole. For example, students may check a proof line by line only to fail to notice that the proof does not prove the intended theorem, but the converse of theorem (Selden & Selden, 2003).

Implication statements, even outside of the context of proof, are a source of great struggle for students (Hoyles & Küchemann, 2002; Markovits & Doyon, 2011; Yu, Chin, & Lin, 2004). Hoyles and Küchemann (2002) found that students commonly conflate a conditional statement with its converse; they interpret a conditional statement as being the same as a biconditional statement. It is possible to produce conclusions that appear to reflect correct reasoning when in fact they are based on misinterpretations of conditional language (Markovits & Doyon, 2011). For example, when both the premise and conclusion of a conditional statement are true, the entire statement is true whether or not one interprets the statement as a conditional or biconditional.

Epp (2003) argued that subtle differences between mathematical and everyday language usage need to be explicitly addressed in the classroom and, if not attended to, will result in difficulties for students. For example, the word “or” has two interpretations: inclusive and exclusive. In mathematics, the inclusive interpretation is generally used, whereas everyday language tends to use the exclusive interpretation. In mathematical parlance, “or” might be used inclusively to state, “negative two or positive two is a solution to $x^2 = 4$.” When using “or” colloquially, one might say they are considering having steak or fish for dinner – this usage would be understood as exclusive. Legal documents avoid ambiguity by using “and/or” to communicate the inclusive or.

Epp (2003) went on to articulate that student struggles with implication statements are due, in part, to uses of conditional language in everyday language that are not consistent with mathematical usage. A conditional statement with a false premise is an instance in which interpretations of everyday language and mathematical language diverge. Mathematically, conditional statements with a false premise are true, yet within the context of everyday language, these statements are interpreted as “not applicable” (Braine, 1978; Inglis & Attridge, 2017). A particularly problematic difference in the interpretation between colloquial and mathematical language occurs when considering the distinction between conditional and biconditional statements. For example, a parent probably would not say to their child, “You can go to the movie if, and only if, you finish your homework” (Epp, 2003, p. 889). Rather, they might use the phrase, “If you finish your homework, then you can go to the movie,” and this would be interpreted as “if I don’t finish my homework, then I can’t go to the movie” (Epp, 2003, p. 889). This interpretation of the conditional statement is different from the interpretation used in mathematics and may partly explain the high incidence of converse error.

Wason's (1968) selection task is perhaps the most used task to evaluate logical reasoning ability. In the original task, a rule, "If there is a D on one side of any card, then there is a 3 on its other side" (p. 275) is presented to a participant. Four cards are placed in front of the participant and each card has a letter on one side and a number on the other (the four cards show D, K, 3, 7). Each card corresponds to a part of the conditional rule: D is a true antecedent, K is a false antecedent, 3 is a true consequent, and 7 is a false consequent. The participant is prompted to select the cards, but only those cards, which would need to be turned over in order to find out whether the rule is true or false. In Wason's (1968) experiment, participants generally sought out confirming evidence and only 10% demonstrated a response consistent with the contrapositive. This result supports the argument that students tend to seek out empirical verification (e.g., Coe & Ruthven, 1994; Recio & Godino, 2001) and highlights the difficulty students have reasoning with the contrapositive (see also Inglis & Attridge, 2017; Stylianides, Stylianides, & Philippou, 2004).

There are a number of follow-up studies to Wason's (1968) original study because participants' performance can be manipulated by changing the abstract number/letter rule to other contexts (e.g., Cheng & Holyoak, 1985; Cosmides & Tooby, 1992; Gigerenzer & Hug, 1991). Griggs and Cox (1982) famously showed participants perform significantly better when the rule and corresponding cards are changed to "If a person is drinking beer, then the person must be over [21] years of age" (p. 415). None of the participants produced the correct response for the original abstract task, but 73% selected the correct cards with the new rule. Cummins (1999) showed that, using the same rule and cards, performance changes given the perspective of the subject. For example, given a rule concerning a business, the cards selected correlated with whether the participant adopted the perspective of the employee or employer. Given the variability and malleability of what is supposed to a logical reasoning task, it is clear that context matters (Ahn & Graham, 1999). The influence of context, and the difficultly students have internalizing a generalizable interpretation of conditional statements, can be confounding given that mathematics requires a consistent interpretation of conditional statements.

Poor performance with abstract rules may be one explanation why interventions that attempt to improve proving ability by teaching abstract logic rules (i.e., truth tables) are generally ineffective (e.g., Deer, 1969; Leighton, 2006; Markovits & Doyon, 2011; Mueller, 1975). Dawkins and Cook (2017) conducted a series of teaching experiments in which students were prompted to systematically examine the use of logical connectives and develop generalizable heuristics to assess truth values. The authors argued:

> We think it [reasoning about logic] constitutes a necessary direction for mathematics education research on proof-oriented mathematics instruction because students must be trained to consciously impose normative logical structure in their reasoning about mathematical content. For students, logic is an emergent structure that often requires guidance and reflection. (p. 255)

Consistent with the results of Wason's (1968) selection task, Dawkins and Cook (2017) found that most of the participants initially had inconsistent interpretations of logical statements that varied across contexts (i.e., their interpretation was domain specific). By guiding students to reflect on their language and interpretation, they were able to develop consistent, normative interpretations of logical connectives.

Quantifiers

Understanding and using quantifiers is an integral part of verifying and writing proofs (Epp, 1999). For example, proofs are often concerned with proving a statement is true for all cases, and as a result it is imperative to distinguish between the quantifiers of "all" and "some" (Vellman, 2006). Epp (1999) drew attention to instances where the use of quantifiers can cause confusion. Similar to the confusion regarding conditional and biconditional statements, students tend to erroneously reverse universal statements (i.e., "all A are B" is interpreted as "all B are A"). Students also have difficultly negating "all" and "none" statements (Epp, 1999). For example, the negation of "all A are B" is "some A are not B," but some students think the negation is "no A are B." Similarly, the negation of "no A are B" is often thought to be "all A are B," when in fact it is only the case that "some A are B." In mathematics, "some" can mean "all", but in everyday language, "some" is generally interpreted to mean "not all" (Epp, 1999).

Durand-Guerrier (2003) made the case that student struggles with conditional language can be traced back, in part, to the implicit quantifiers they contain. Durand-Guerrier contended there are two notions of implication: the propositional connective (one particular instance) and the generalized conditional (for a set of instances). Whether or not an implication statement needs to work for all cases is an underlying source of confusion for students. The author provided an example of an implication statement concerning person X. Durand-Guerrier found students determined the truth value based on a single person (named X), whereas teachers interpreted the implication as a generalized conditional, even though X was not a variable and there was no referent population. The ambiguity resulting from implicit quantifiers needs to be clarified in order for students to successfully use and interpret conditional statements.

Usiskin, Peressini, Marchisotto, and Stanley (2003) drew attention to the significance of the existential quantifier for beginning algebra students learning to solve equations. To illustrate their point, they offered the following task: solve for x where 6 + x = 6x. This task contains an implicit quantifier and is actually stating *there exists an x* such that 6 + x = 6x, identify such an x. Without making this inference, 6 + x = 6x is open sentence without meaning. Further, if one were to assume the statement to implicitly convey the universal quantifier, the statement would now be mathematically false and the meaning of the task completely different.

Dubinsky and Yiparaki (2000) found undergraduate students struggle to interpret the existential quantifier. Specifically, students have difficultly differentiating between "for every x, there exists a y," and "there exists an x such that y." The authors also found that context matters – students have less difficulty when the statements are about real life scenarios. A subsequent study of six undergraduate illustrated that misunderstanding the existential quantifier can inhibit students' ability to prove statements (Piatek-Jimenez, 2010).

Classes of Objects

According to Moore (1994), the ability to reason about classes of objects (i.e., sets) is a critical skill that one needs to have in order to write proofs. This includes the ability to reason about infinite sets since proofs are generally concerned with determining if a statement is true for all cases (Morash, 1991). For example, the proof of the base angle theorem for isosceles triangles stipulates that each pair of base angles are congruent for every single isosceles triangle in the infinite set of isosceles triangles. The ability to conceive of a set of objects as a single entity is major milestone in one's mathematical development (Balacheff, 1988; Crowley, 1998; Harel & Sowder, 1998). In turn, the ability to reason with classes of objects greatly affects one's ability to use and interpret conditional statements (Durand-Guerrier, 2003; Moore, 1994).

Hub and Dawkins (2018) conducted a teaching experiment and illustrated how one student developed his notion of the generalizable truth conditional by relying on set-based reasoning. Hub and Dawkins (2018) argued the transition to the generalizable truth conditional is significant because it reflects the ability to consistently interpret statements across contexts.

> We use the term *generalizable* to acknowledge that students may or may not have considered whether their interpretation of a single statement would work on other statements…We want to help students consider whether their criteria for truth apply viably to other conditionals, which is why we pursue student development of *conscious* truth conditions. (p. 91)

Hub and Dawkins (2018) documented that the transition can occur by providing a student multiple opportunities to: read mathematical statements and assign a truth value based on their understanding, make connections across tasks and representations, reframe a representation with a different task, and take a representation (e.g., an Euler diagram) and connect it to multiple tasks. In essence, the development of set-based reasoning and ability to relate sets to conditional statements requires multiple opportunities to reflect on one's thinking and then transfer one's knowledge to new contexts (Hub & Dawkins, 2018).

Dawkins and Cook (2017) found that the development of logical reasoning ability is inhibited by a lack of set-based reasoning. In particular, the use of negations can be problematic for students. The introduction of a negation into a conditional statement results in an increase in non-normative interpretations because the negation effectively overtaxes one's working memory (Inglis & Attridge, 2017; Politzer, 1981). Without fully developed set-based reasoning, some students reduce the working memory load by substituting the negation of a property with an affirmative property. For example, 'not acute' is interpreted as obtuse, and 'not rectangle' is interpreted as parallelogram (Dawkins, 2017). Similarly, students often reduce their working memory load by using a prototype as a representative member of the set; this can be problematic because students often fail to use prototypes that are representative of the entire set (Alcock & Simpson, 2002; Cummins, 1995).

Since one cannot imagine every individual element of an infinite set, reasoning about classes of objects requires the ability to understand, express, and reason about the necessary and sufficient conditions for the elements in the set. Example selection is one way to evaluate how students understand necessary and sufficient conditions; how examples or counterexamples are chosen matter can be evidence of deductive thought (Balacheff, 1988; Buchbinder & Zaslavsky, 2009; Marrades & Gutiérrez, 2000). Mason and Pimm (1984) used the notion of a "generic example" to discuss the role of specific examples in developing a sense of generality. Examples can seem like specific cases to students, yet well-chosen examples can highlight the generality of the attributes of the specific case presented and help develop set-based reasoning. Stylianides, Bieda, and

Morselli (2016) claimed that presenting a range of examples and having students generate their own examples are key ingredients for successful example usage.

Counterexamples also play an important role in a student's ability to develop set-based reasoning. Counterexamples are powerful because a universal statement is false if a single counterexample is found. Teaching students to eliminate the possibility of counterexamples can be a viable strategy to develop proving ability (Yopp, 2017). Zazkis and Chernoff (2008) illustrated the notion of an exemplary counterexample. Counterexamples are only counterexamples to students if the instances are counterexamples in one's personal example space. That is, how one conceives of the class – the necessary and sufficient and conditions for objects to belong to the class – determines whether or not a counterexample produces the necessary cognitive conflict to be considered a counterexample. Given an incorrect conception of a class, a typical counterexample may not seem like a counterexample. It is important to consider how a student actually conceives of the mathematical object, not the correct conception alone.

The importance of considering how one chooses examples and counterexamples can compared to the importance of considering one's concept image (Tall & Vinner, 1981). Concept image is a widely used notion to describe one's "cognitive structure that is associated with the concept, which includes all the mental pictures and associated properties and processes" (Tall & Vinner, 1981, p. 152). Vinner (1992) asserted that if one knows a student's concept image, then the results of a task can be predicted. There is a strong link between students' conceptions and their actions. Bingolbali and Monaghan (2008) documented that mechanical engineering students and students majoring in mathematics had different concept images for the concept of tangent. As a result, the two groups produced different types of responses given the same task. In addition, Kontorovich (2018) showed the necessity of a fully formed concept image that is transferable between domains. A certain concept image may be sufficient in one domain, but problematic in another. For example, a tangent line in geometry is thought of a line and a circle or ellipse with exactly one point of intersection. In calculus, this image is problematic when considering the derivative of x^3 at $x = 0$.

Supporting Proof in Algebra

Mathematical Processes as Objects

Using conditional language to reason about classes of objects plays a major role in developing an analytical proof scheme (Harel & Sowder, 1998). The use of the word "objects" might suggest one is reasoning about geometrical objects and thus is an activity reserved for geometrical contexts. Not only geometers, however, see objects while doing mathematics. The mathematician Henry Pollack once said in an interview (Albers & Alexanderson, 2008),

> Mathematicians often argue whether mathematics is discovered or invented. I certainly had the feeling in that particular case that I was discovering it and not inventing it…We couldn't have invented all that. We had discovered a structure that must have been there. At least that's the feeling I had; it hung together too well. (p. 249)

Consider, for example, the case of functions. They are not tangible, yet in many ways they are discussed and treated as though they are actual objects. At the same time, they are more than just objects. In order to "see" functions as objects, one needs to be able to conceive of the object as an encapsulation of a related process (i.e., a rule mapping one quantity to another) (Kieran, 1992).

Sfard (1991) argued that mathematical concepts are both operational and structural, similar to physicists' conception of the wave-particle duality. The operational notion is "dynamic, sequential, and detailed" whereas the structural notion is "timeless, instantaneous, and integrative" (Sfard, 1991, p. 4). Consider, for example, geometrical reflections. On one hand, a line reflection can be thought of as a process mapping one half of a figure onto the other half, but a structural view affords the static concept of line symmetry. The operational and structural duality of mathematics is captured in a number of conceptual frameworks (e.g., Beth & Piaget, 1966; Greeno, 1983; Steffe & Cobb, 1988). Furthermore, the underlying notion of process/object duality can be traced back to Marx (1867/1992), the German philosopher and economist, and his use of the term *Verdinglichung* – which roughly translates to "making into a thing." Marx used the process/object duality to critique the transformation of labor, the process of working, into a commodity (an entity which is bought and solid like a physical good).

APOS theory (Dubinsky & McDonald, 2001) specifically describes how mathematical conceptions develop. First, one sees a mathematical idea as an action or a set of externally-based instructions. The action becomes a process when the action is internalized and one no longer needs external prompts to complete the action. At this point, one can mentally execute the action without actually carrying it out. The process transitions into an object when one realizes the process can be transformed. For example, a linear function can be transformed by shifting the y-intercept. Despite this transformation, the function still describes the same action (namely, a constant rate of change between two quantities). Lastly, similar to the idea of concept image (Tall &

Vinner, 1981), one develops a scheme when one links related processes and objects together and can apply them to solve problems.

Additionally, there is an iterative nature in mathematics where lower level processes turn into objects, and then these mathematical objects are used in more sophisticated processes (Sfard, 1991). Transitions from mathematical processes to mathematical objects first occur in elementary school and continue through higher level mathematics. The concept of cardinality represents the first time in a child's mathematical learning where a process (i.e., counting) is conceived of as an object (i.e., number) (Greenes, Ginsburg, & Balfanz, 2004). In fact, the set of objects – the counting numbers – is named after the process that generated them. Subsequently, the process of dividing two natural numbers becomes the object of rational number; the process of taking a square root of certain rational numbers becomes the object of irrational number (Sfard, 1991). Additionally, abstract algebra is highly structured and the result of processes conceived of as sophisticated objects: groups, rings, and fields (Sfard & Linchevski, 1994).

Sfard (1991) explicated the operational/structural duality with her theory of reification. The theory of reification has three stages: interiorization, condensation, and reification. The first stage, interiorization, is marked by the ability to perform mental operations instead of executing physical operations. Condensation, the second stage, occurs when one shortens the sequences of actions into units and thinks of the process as a whole without needing to delve into all the specifics. Consider how a baker might conceive of baking a cake not as a long sequence of specific actions, but a smaller set of tasks where each task represents multiple actions (e.g., making the filling, batter, and frosting). Similarly, complex mathematical procedures can be conceived of as a handful of smaller units. The final stage in Sfard's (1991) theory, reification, describes the transformation of a process into an object. Prior stages are gradual, but reification is instantaneous and marked by "an ontological shift – a sudden ability to see something in a totally new light" (p. 19). Sfard (1991) made the crucial point that the ability to see a process as an object allows for one to relate processes with the concept of classes and sets:

> The new entity is soon detached from the process which produced it and begins to draw its meaning from the fact of its being a member of a certain category. At some point, this category rather than any kind of concrete construction becomes the ultimate base for claims on the new object's existence. (p. 20)

The critical shift to understanding an object as belonging to a set vis-à-vis its properties affords one the ability to reason with classes of objects and make deductive arguments (Balacheff, 1988; Crowley, 1998; Harel & Sowder, 1998).

The theory of reification can be connected to the usefulness of metaphors (Sfard, 1994). Metaphors allow for abstract notions to come into being and have meaning. For example, "cognitive strain" is a metaphor that compares the mind to a container and allows for the mental processes one experiences to be conceived of as a meaningful abstract entity (Sfard, 1994). In algebra, solving equations is often taught with the metaphor of a balanced scale. This metaphor aids in the shift from solving equations as a process to a structural conception involving objects (Kieran, 1992). Functions – objects that represent processes that map inputs to outputs – can be described by the metaphor of a vending machine. A vending machine illustrates the process of mapping an input (i.e., a button) with an output (e.g., a can of soda). Additionally, the metaphor also allows for the process to be conceived of as an object – the vending machine itself.

Equations as Classes of Objects

Harel and Kaput (1991) noted that students often effectively manage symbolic notation by "…looking to translate from their natural language-based encoding of encapsulated *process* to algebra" (p. 90). Students can be successful so long as the numbers involved are positive whole numbers, although this strategy ceases to be effective when negative integers and rational numbers are introduced. Sfard and Linchevski (1994) contended that "algebraic symbols do not speak for themselves" (p. 192) and that in the long term, one's competence in algebra depends on "…our mind's eye's ability to envision the result of processes as permanent entities in their own right" (p. 194). Merging operational and structural conceptions while working with algebraic symbols can be difficult for students (Harel & Kaput, 1991). Cañadas, Molina, and del Rio (2018) conducted a study in which participants were given an algebraic statement and prompted to pose a problem that would match the statement. Participants struggled to match the syntax of the statement, and this suggested it was difficult for them to assign meaning to symbolic statements. Participants had less difficulty with addition prompts as compared to multiplication prompts. This is noteworthy because it is consistent with the notion that the structural conception of addition comes into place before structural conception of multiplication.

Students' struggle to transition from verbal, process-based algebra to symbolic algebra can be attributed, in part, to curriculum. Nathan, Long, and Alibali (2002) performed an analysis of the organization of algebra textbooks. Although the authors found that textbooks published after 1990 have fewer symbol-only sections compared to older textbooks, they also found that symbolic problems are generally presented before verbal problems. The placement of symbolic problems before verbal problems is contrary to what research suggests is conducive for student understanding, but in line with what teachers prefer (Nathan et al., 2002). Sherman, Walkington, and Howell (2016) set out to determine if the introduction of “Common Core aligned” textbooks had adjusted the tendency of textbooks to present symbolic problems before verbal problems in order to better support student learning. The authors found that the problematic presentation of symbolic problems before verbal problems still persisted. The reverse ordering was less pronounced in “reform” texts, but still evident. These findings suggest current curricula, especially traditional textbooks, are at odds with how students naturally come to understand algebraic symbols (Sherman et al., 2016).

The ordering of sections within algebra textbooks might lead one to believe that the introduction of symbols is what leads to a structural conception of equations. This is not the case – the introduction of symbols alone is necessary, but not sufficient, for a student to achieve reification (Sfard & Linchevski, 1994). Similarly, while learning to solve equations presents an opportunity to internalize symbolic meanings, learning to solve equations does not guarantee one’s operational conception will evolve into a structural perspective. Linsell’s (2009) study showed students use a variety of strategies to solve linear equations. Students sometimes solve equations without using inverse operations, and sometimes they partially work backwards but then guess and check. Linsell (2009) noted how difficult the transition to a structural view of equations can be.

> However, this study has confirmed just how difficult the strategy of using transformations [of objects] is for students. Using transformations requires seeing an equation as an object to be acted on (Sfard, 1991), but it is clear that most students see equations as processes. (p. 41)

The pathway from arithmetic to algebra begins in elementary school and algebra can only be meaningful for students if certain elementary structural conceptions are in place (Kieran, 1992). Possessing a relational view of the equal sign is a major milestone in a child's development of structural conceptions (Kieran, 1981; Rittle-Johnson & Alibali, 1999). In a relational view of the equal sign, one sees the equal sign as comparing two quantities. Prior to this conception, students possess an operational view of the equal sign and view the equal sign as an operator that signals "the answer is coming up" (Behr, Erlwanger, & Nichols, 1980). Baroody and Ginsburg (1983) noted an operational conception can cause students to believe certain equations are written backwards (e.g., 6 = 8 – 2) and use multiple equal signs incorrectly (e.g., 2 + 3 = 5 + 9 = 14). It takes time and access to multiple contexts for children to fully develop a relational view of the equal sign (Seo & Ginsburg, 2003).

Ideas in mathematics are structured such that the same ideas can be viewed differently from different perspectives (Sfard & Linchevski, 1994). Harel and Kaput (1991) refer to this as the "vertical growth" of mathematics (p. 83). One way this vertical growth can be seen is in the way students interpret expressions and equations. Sfard and Linchevski (1994) described the transition from generalized arithmetic to functional algebra. Consider the statement x + 3. With an operational view, x + 3 is two numbers, and x + 3 represents the process of adding them. The perspective of generalized arithmetic, also known as algebra of a fixed value, views x + 3 as one quantity – the sum of x and 3 – and it just happens to be unknown (Sfard & Linchevski, 1994). However, this notion gives way to another, more sophisticated understanding known as functional algebra. In functional algebra, $x + 3$ can also be interpreted as $f(x) = x + 3$ with real numbers as the replacement set (Sfard & Linchevski, 1994). Now, $x + 3$ no longer represents a single unknown quantity, but an infinite number of numbers.

Dubinsky (1991) linked the transition from propositional calculus to predicate calculus with the concept of function. This transition introduces the notion that expressions and equations can be conceived of as classes of objects. The transition from $p \rightarrow q$, a statement concerning two fixed (but possibly unknown) quantities, to $P(x) \rightarrow Q(x)$ requires that the unknown p be represented by a class; it is the function $P(x)$ that maps an object from the class into the implication statement (Durand-Guerrier, 2003). Continuing the use of $x + 3$ as an example, if $x + 3$ is a fixed value, then it is treated as a proposition. However, if $x + 3$ is viewed through the lens of functional algebra, it can be considered a predicate; as a result, $x + 3$ can be reasoned with as though it is a class of objects (Dubinsky, 1991).

Functional algebra, where each expression on both sides of an equation represents a class of objects, allows for equations to be treated as objects themselves (Dubinsky, 1991). To employ a familiar metaphor: an equation, like a balanced scale, is object containing other objects. Further, the solution set is also now an object – an object related to the truth value of the equation – and no longer limited to the role of "the answer." Consider the equation $x + 3 = 4x$. The equation represents two infinite classes: $x + 3$ is one set of numbers, and $4x$ is one set of numbers. The solution set, $\{1\}$, is the set that identifies two equivalent subsets in $x + 3$ and $4x$. With this notion, equivalent equations (e.g., $x + 3 = 4x$ and $x + 4 = 4x + 1$) are defined by their common solution set (Steinberg, Sleeman, & Ktorza, 1991).

Reasoning Conditionally with Equations

A structural conception of equations not only allows one to efficiently solve equations; it is necessary in order to proficiently solve certain types of equations, especially quadratic equations. Tall, de Lima, and Healy (2014) found that students struggle to solve quadratic equations. In particular, the participants were limited by their operational conceptions of solving linear equations that did not transfer into the new context of solving quadratic equations. Senk and Thompson (2006) asked second-year algebra students to compare $y = 4(x + 3)^2 +$ and $y = 4x^2 + 24x$. The majority of participants set the equations equal to zero and attempted to find solutions in order to make a comparison. This demonstrated that students had an operational view of equations – the equations were prompts for the solving process (Senk & Thompson, 2006). Students with a structural conception used more efficient methods to make a comparison (e.g., examined the graphs or multiplied the binomial). The use of additional strategies to ascertain equivalence is a significant step in the development of student understanding of equations and the solution set (Knuth, Stephens, McNeil, & Alibali, 2006; Steinberg et al., 1991).

In order to improve student understanding of quadratic equations, Tall et al. (2014) argued there needs to be "…a growing awareness of the crystalline structure of mathematical concepts that enable them to be grasped and manipulated as mental entities with flexible meaningful links between them" (p. 12). Proficiently solving quadratic equations requires the use of the zero-product property (ZPP), a property that is structural in nature and is expressed with conditional language (Cook, 2018; Tall et al., 2014). The ZPP – the conditional statement, "if ab = 0, then a = 0 or b = 0" – makes up half of the biconditional statement, for all real numbers, ab = 0 if and only if a = 0 or b = 0. The ZPP allows quadratic equations (e.g., (x + 2)(x + 3) = 0) to be solved by setting each factor equal to zero to obtain the solution set. However, many students do not proficiently use the ZPP. For example, given a quadratic equation in factored form, some students will inefficiently multiply the binomials together and then use the quadratic formula (Ochoviet & Oktaç, 2009; Vaiyavutjamai & Clements, 2006).

Even when students appear to use to the ZPP when solving quadratic equations, students may in fact only have a pseudo-analytical understanding (Vinner, 1997). For example, when solving the equation (x + 2)(x + 3) = 0, students may set each factor to zero and appear to have a structural understanding. Yet, when asked to justify this step, students will sometimes cite the converse of the ZPP – namely, if the factors are zero, then the product is zero (Cook, 2018; Ochoviet & Oktaç, 2009). Because the ZPP is part of a biconditional statement, this reasoning error may seem inconsequential or go unnoticed. However, it is evidence of an understanding – of both solving quadratic equations and conditional language – that is not fully developed.

Cook (2018) reported on a teaching experiment that supports the following claims: i) conditional language provides the opportunity to highlight the structural nature of the solution set, and ii) solving equations is a viable context to internalize conditional language. The author specifically examined how one student came to develop an understanding of solving quadratic equations by attending to the use of conditional language. Cook's (2018) intervention made use of the fact that although the ZPP is true for the set of real numbers, it does not hold in the group Z_{12} (the set of integers from 0 to 11, inclusive). When employing modular arithmetic in the group Z_{12}, a product of zero does not necessarily imply one of the factors is zero (e.g., 3*4 = 0). This property of the group has the potential to create cognitive conflict and can draw attention to the conditional nature of the ZPP. For example, when solving 3(x – 2) = 0, Cook's (2018) participant identified two as a solution, verified the solution by substituting into the original equation, and then erroneously claimed that two is the solution set. Because 3*4 = 0 and 3*8 = 0 in Z_{12}, six and ten are also solutions. After careful, guided reflection, the participant corrected his error by attending to the conditional relationship between equations.

Chapter III

METHODOLOGY

This chapter provides an account of the methods and procedures used to collect and analyze data for the study. I begin with a restatement of the purpose of the study and the research questions I set out to answer, followed by the methods I used to answer the research questions. I then describe the research setting and participants. Last, I provide a detailed description of the methods and techniques used to collect and analyze data at each stage of the study in order to answer each research question.

Research Questions

The purpose of this qualitative study was to describe the role of truth value and the solution set in supporting the development of the ability to reason about classes of objects and use conditional language. To support this goal, the study adopted Harel and Sowder's (2007) framework to identify the proof schemes that students use when justifying solutions to equations. In addition, this study aimed to determine whether or not students who reason conditionally about solution sets are also able to reason about classes of geometrical objects.

1. How do proof schemes differ, if at all, when students justify solutions to different types of algebraic equations?
2. Can students learn to reason about classes of objects and use conditional language when considering the truth value of algebraic equations? If so, how?
3. Are students who reason conditionally about solution sets also able to reason about classes of geometrical objects?

Research Approach

This study used a combination of technology-based methods and qualitative methods to answer the research questions. I selected technology-based methods – a Livescribe Smartpen and Geogebra dynamic geometry software – to completely capture participants' verbal reasoning in conjunction with their visual work simultaneously in real time. I selected qualitative methods because I wanted to generate a rich, in-depth description of student reasoning (Creswell & Poth, 2018). In particular, qualitative approaches afforded insights into how student reasoning could be developed to support their future interactions with proof.

I first used task-based interviews (Goldin, 2000) to uncover the proof schemes participants initially used when reasoning about a variety of equations. Second, I conducted a teaching experiment (Steffe & Thompson, 2000) to document how some participants developed set-based reasoning to make claims while solving equations. The qualitative nature of a teaching experiment provided insights not only into changes in student reasoning, but the circumstances that precipitated those changes. Last, I conducted additional task-based interviews (Goldin, 2000) to determine if participants, after the completion of the teaching experiment, attempted to develop an analytical proof scheme in a geometrical context. I again used a qualitative approach to describe how each participant weighed evidence in their attempts to make mathematical claims with certainty.

Research Setting and Participants

Research Setting

The study was conducted in a natural setting (Creswell & Poth, 2018) in the sense that students often experience making sense of mathematics during class or in other school environments. This study took place over the course of five months (Table 1).

Table 1. *Timeline for Research Activities*

Research Activity	Date
Recruit participants	September 2018
Task-Based Interview #1	September – October 2018
Teaching Experiment	November – December 2018
Task-Based Interview #2	January 2019

I conducted approximately two task-based interviews per week during the first stage of the study. I then conducted a teaching experiment with three participants, meeting with each participant individually once a week for five weeks. I concluded the study by conducting a second task-based interview with each participant who completed the teaching experiment.

This study took place in a large, suburban, and diverse high school with a student body that is approximately 45% Hispanic, 25% African-American, 25% White, and 5%

Asian-American. Approximately 55% of students qualify for free or reduced-price lunch. Both the size and diversity of the high school made it an attractive research site because it afforded a robust pool of prospective participants.

The high school offers four tracked mathematics courses for incoming freshmen: Geometry, Geometry Honors, Algebra I, or Sheltered Algebra. The majority of ninth grade students take Algebra I, while a small percentage enroll in Sheltered Algebra. Sheltered Algebra is a two-year algebra course designed to meet the needs of students whose first language is not English. A fixed number of seats are made available for students to enroll in Geometry Honors every year. In order to qualify for this course, students must meet the following criteria: have been selected by their teachers to take accelerated mathematics in middle school, successfully completed Algebra I in eighth grade, and have a final course average that places them in the top 120 students. Students who study algebra in eighth grade but do not qualify for Geometry Honors enroll in Geometry. The tracking of students at the research site is noteworthy because it signifies a traditional mathematics program that in many ways can be considered typical for a high school in the United States.

Participant Selection

The tracking of students at the research site resulted in a preliminary de facto selection of participants. High achieving students selected to take algebra in middle school were not part of the pool of potential participants because I was only interested in recruiting students enrolled in Algebra I at the high school. After taking this into account, I selected participants for this study in two stages. Twelve participants, herein referred to as “Group A participants,” were selected to participate in the first task-based interview examining student proof schemes for solving equations. Three “Group B participants,” selected from the set of Group A participants, continued on to subsequent parts of the study. In other words, only a subset of Group A participants was selected to participate in the teaching experiment and final task-based interview.

Group A participants. This first set of participants was recruited from Algebra I classes. At the time of the study, students had previous experience solving equations, but limited or no experience solving equations with multiple or extraneous solutions. There were multiple purposes (Creswell & Poth, 2018) when considering prospective participants for Group A. The first purpose was to simplify the study by identifying a homogenous set of participants. The participants were homogenous in the sense they were all able to perform a common set of algebraic procedures (solve two-step equations, use the distributive property, simplify algebraic expressions, and evaluate exponents) before the study commenced. Second, participants’ current algebra teachers were consulted with the intention of identifying participants that teachers thought would highlight what is typical or average. Last, prospective participants were identified by their teachers as being potentially information rich cases that manifested information “intensely but not extremely” (Miles & Huberman, 1994, p. 28). That is, I set out to recruit participants willing to share and explain their thinking.

The number of Group A participants was not predetermined. After prospective participants were identified, I invited these students to participate in the study by having their Algebra I teacher hand them a flyer and parental consent form in class. Students who wished to participate had one week to return the parental consent form. Flyers were distributed to all students on the same day, and as a result there was a uniform deadline for returning consent forms. Considering only students who met this deadline, I listed students alphabetically by their last name. I conducted task-based interviews according to the order of my list and continued conducting interviews until a satisfactory degree of data saturation was achieved. Consistent with Lincoln and Guba's (1985) description of purposeful sampling, redundancy was the criterion for sampling – I terminated task-based interviews when no new information was forthcoming. As a result, I conducted twelve task-based interviews and did not interview every student who wished to participate.

Group B participants. This second set of participants was identified after the conclusion of the task-based interviews with Group A participants. The purpose of the task-based interviews was to identify the proof schemes of students for solving a variety of equations. Participants who consistently and strongly exhibited empirical proof schemes during the task-based interview were selected as Group B participants. This resulted in the selection of four participants, one of whom declined to further participate in the study due to scheduling conflicts. Collectively, the three Group B participants formed an instrumental case (Creswell & Poth, 2018) that illustrated how students can adopt set-based reasoning to make claims while solving equations. In other words, Group B participants illustrated how students can learn to reach absolute certainty with deductive justifications despite their initial predilection for empirical evidence.

Methods of Data Collection and Analysis

The overarching goal of this study was to document that solving equations is a viable context to develop ways of reasoning that support proof. As such, I wanted to see if students could both: i) use conditional language to develop a set-based, analytical conception of the solution set, and ii) seek out analytical ways of reasoning in a geometrical context by adopting a set-based perspective. The primary data source for this study was transcripts of audio recordings from each task-based interview and teaching experiment session. The interviews and teaching sessions took place immediately after the end of regular school hours in a classroom at the research site. This allowed for a seamless transition where participants continued their school day. During the teaching experiment, I did not instruct participants or impose ways of thinking. Rather, the goal was for participants to spontaneously, through insights afforded by the tasks and questions posed, make accommodations to their mathematics so that their reasoning could support future interactions with proof.

The audio recordings for the first task-based interview and teaching experiment sessions were made using a Livescribe Smartpen. A Smartpen writes like a regular pen, but it has two additional features: it makes an audio recording, and it captures what participants write in digital format. Participants' written work from reasoning about equations was saved so that I could refer back to exactly what they wrote instead of having to recall by memory. In addition, the contemporaneous audio recording and digitized handwriting were linked together. This allowed me to go back and examine what participants wrote as they were speaking as a congruous set of data.

Participants did not write on paper for the second task-based interview. Instead, they used dynamic geometry software (i.e., Geogebra) on a computer. To record the data, a screencast was made while participants used the software. A screencast records mouse movement on the computer screen while simultaneously recording audio. Similar to the use of a Smartpen, a screencast allowed me to examine what participants did and said at the same time.

In addition, memos (i.e., my notes) were another source of data in this study. Immediately after each interview or teaching session, I would reflect and write down my initial thoughts and impressions of participants' reasoning as well as any significant interactions I wanted to revisit and examine further. I specifically made sure to include any decisions I made as a result of unexpected responses. I also wrote memos when analyzing the audio recordings. These memos were used document my understanding of participants' reasoning about solutions at a given point in time.

Stage 1: Task-Based Interview #1

The first set of data was collected and analyzed to answer the first research question: *How do proof schemes differ, if at all, when students justify solutions to different types of algebraic equations?*

Task-based interviews are structured such that participants interact with a preplanned task environment. For this study, the task consisted of presenting a sequence of equations, one equation at a time, and prompting each participant to describe the solution for each equation and their process for arriving at the solution. The goal of the interview was to elicit the proof scheme(s) each participant relied on when justifying solutions. Specifically, using Harel and Sowder's (2007) framework as a lens, I gathered evidence that participants relied on external, empirical, and/or analytical schemes to convince themselves that their solution was correct. In addition, the size of the solution set was varied to determine if a participant's proof scheme changed depending on the nature of the equation.

Interview procedures. The data from the first round of task-based interviews (Goldin, 2000) consisted of three parts: the solutions participants provided for equations, the justification for their solutions, and their level of certainty in their solution. To elicit solutions, I prompted participants to "tell me about the solution for this equation." I avoided the phrases "find x" and "solve for x" because these phrases overtly ask for a process, whereas I was interested in uncovering a structural conception of the solution set. To elicit justifications, I asked, "How do you know?" after participants provided a solution and I prompted them to show and explain their thinking in as much detail as possible. To elicit participants' level of certainty, I asked them to rate how certain they were on a scale from one to five. If participants expressed doubt, I asked them what further information they would require to be fully convinced. If participants were completely certain of their solution, I asked them why they were absolutely sure and what they would say to a classmate who was not fully convinced.

Participants were prompted to describe solutions to a variety of equations. They were first presented with equations I expected participants to find routine: $4x + 2 = 10$, $4x + 1 = 9$, $7x + 1 = 3(x + 3)$. I verified this was true by asking, "Is this a type of equation you are familiar with?" After describing the solutions, providing justifications, and conveying their level of certainty for the three aforementioned equations, I presented the equations side by side and asked, "Why do you think these three equations have the same solution?" The goal of this question was to elicit whether or not students were able to identify equivalent equations.

Additional data were collected by presenting participants with equations I anticipated they would find non-routine. I verified they were non-routine by asking if they remembered similar equations from past experience. The first non-routine equation I presented was $x = 2$. Solving this equation is trivial and as a result I thought it unlikely participants were previously asked to reason about the solution to an equation of this form. I then presented participants with an equation with no solution ($x + 1 = x + 2$), an equation that might be mistaken as having no solution ($1x = 2x$), and an equation with an infinite number of solutions ($2(x + 1) – 2 = 2x$). Next, I presented participants with equations with multiple solutions: $x^2 = 4$ and $x^3 = 4x$. While participants may have previously encountered $x^2 = 4$, I included it before presenting $x^3 = 4x$ in order to activate any prior knowledge about the possibility of multiple solutions. In addition, I wanted to provide an opportunity for participants to relate non-equivalent equations that share some, but not all, solutions (in this case, -2 and 2).

The task-based interview concluded by explicitly seeking evidence of set-based reasoning and the use of conditional language. The previously presented equation $4x + 1 = 9$ was again presented to participants, and this time I simply asked, "True or false?" I followed up by asking if it was always true (or false), how confident they were, and how they would convince somebody who disagreed with them. For participants who claimed that the equation was always true because x had to equal two, I asked if and why they were sure and how they would respond to a classmate who claimed x could equal a different value.

Data analysis. I analyzed the data using open, axial, and thematic coding (Strauss & Corbin, 2015) to answer the first research question. After concluding each interview, I transcribed the interview and reread the transcript multiple times before generating a set of open codes. Open codes consist of a single word or phrase that characterize a segment of a participant’s statements (Strauss & Corbin, 2015). For example, whenever I interpreted a participant making a statement that conveyed why they believed their response was correct, the statement or set of statements was coded as “justification.” Similarly, instances in which participants communicated whether or not they were familiar with a given equation were coded as “familiar” or “unfamiliar.” Additionally, I coded occurrences as “relate” every time participants attempted to relate, compare, or contrast equations and solution sets.

The open codes were then constantly compared within a transcript, grouped together or separated, and used to form axial codes (Strauss & Corbin, 2015). For example, all the instances of “justification” codes from a single transcript were analyzed as a whole to identify any similarities or differences among the instances. This process used Harel and Sowder’s (2007) framework as a lens to differentiate justifications. When participants selected examples as their justification, I referred to Balacheff’s (1988) taxonomy to determine whether the justification was empirical or deductive in nature.

To continue the example of “justification codes,” open codes that captured instances of participants using rituals to justify a statement were grouped together under the axial code of “ritualistic argument.” Likewise, the justification codes were also partitioned into the axial codes of “ascertain” and “persuade” depending on whether participants were describing why they were absolutely convinced or how they would lead a classmate to absolute certainty. Finally, instances identified with the open code of “relate” were separated into the axial codes of “implication” or “undeveloped conditional language" depending on whether or not participants' language was consistent with the notion of mathematical implication.

Instances coded as the expression of conviction were analyzed to determine whether or not participants were absolutely certain of their responses. Weber and Mejia-Ramos (2015) differentiated absolute conviction from relative conviction:

> An individual who has absolute conviction in a claim has a stable psychological feeling of indubitability about that claim…[whereas] an individual has relative conviction if the subjective level of probability that one attributes to that claim being true exceeds a certain threshold to provide a warrant for some future actions. (p. 16)

Harel and Sowder's (1998, 2007) proof scheme framework relies upon absolute certainty. As per the recommendation of Weber and Mejia-Ramos (2015), the analysis pertaining to proof schemes and conviction attended to phrases that conveyed participants' level of conviction. Specifically, I flagged phrases such as "I know" and "I am sure," while taking care not to conflate relative conviction with absolute conviction when participants used phrases such as "I think" and "I believe." I always checked with participants in any ambiguous instance to clarify their level of certainty.

I also constantly compared open and axial codes across transcripts to develop thematic codes (Strauss & Corbin, 2015). This allowed for my previous analysis to inform my analysis going forward and, in addition, the opportunity to return to previously coded transcripts and make revisions. Specifically, after each interview I first generated open and axial codes by examining the transcript in isolation. I then returned to previous transcripts and adjusted my codes for the current and previous transcript so that, taken together, the codes were consistent and coherent. This allowed me to continuously refine my codes as I conducted subsequent interviews and identify themes as they emerged. As

I previously mentioned when describing participant selection, this process eventually yielded no new information (Lincoln & Guba, 1985). I reached the end of the coding process once I identified themes that coherently accounted for the data from all the participants. At this point, I discontinued the selection of participants and ceased conducting interviews.

Stage 2: The Teaching Experiment

The second set of data was collected and analyzed to answer the second research question: *Can students learn to reason about classes of objects and use conditional language when considering the truth value of algebraic equations? If so, how?*

I conducted a teaching experiment (Steffe & Thompson, 2000) with three participants who consistently relied on non-analytical proof schemes. In particular, they relied on empirical and ritualistic proof schemes to reach conviction; they demonstrated empirical proof schemes more frequently than ritualistic proof schemes during the first task-based interview. I attempted to develop participants' reasoning supportive of an analytical proof scheme – namely, reasoning about classes of objects that used conditional language.

Data description. According to Steffe and Thompson (2000), an essential part of a teaching experiment is looking behind what students say and do to identify rational grounds for their mathematical reality. In addition, a teaching experiment allows the researcher to build and test viable models of students' mathematical schemes and operations in order to explain the thinking behind their behavior. Scheme refers to the underlying structure of students' mathematical thinking; I set out to build models explaining participants' initial behavior while solving routine and non-routine equations. The participants selected for the teaching experiment did not employ analytical proof schemes when solving equations during the first task-based interview. As such, I built and refined models of student thinking that would explain participants' reasoning about solution sets while accounting for their non-analytical proof schemes. I specifically made sure to probe beyond correct solutions in order to ensure the reasoning participants' used to arrive at correct solutions was consistent with an analytical proof scheme.

A teaching experiment consists of teaching episodes (Steffe & Thompson, 2000) in which the teacher-researcher guides participants to make accommodations to their mathematics. That is, once I could describe a model explaining a participant's behavior, I introduced tasks and asked questions so that the participant would make accommodations to their schemes and operations. Changes in participants' conceptions of solutions resulted in changes in behavior, and I aimed to notice and understand these changes so that I could adjust my models accordingly. The ultimate goal of the series of teaching episodes was for participants' schemes and operations – and my models describing them – to evolve over time to include reasoning about classes of objects and the use of conditional language. While conducting a teaching experiment, one sets out to experience firsthand students' mathematical reasoning and learning (Steffe & Thompson, 2000).

I chose to conduct a teaching experiment because I wanted to discover and document how students learn to reason about classes of objects and use conditional language in the context of solving equations. I did not know ahead of time how the teaching experiment would unfold. However, my review of the literature and experience teaching Algebra I allowed me to develop a hypothetical learning trajectory (HLT) (Simon, 1995) that described my prediction as to how participant learning might proceed.

According to Simon (1995), a HLT includes “the learning goal that defines the direction, the learning activities, and the hypothetical learning process – a prediction of how the students’ thinking and understanding will evolve in the context of the learning activities” (p. 136). My initial HLT envisioned participants would come to use set-based reasoning and conditional language while solving equations by progressively developing the following understandings: the equals sign is a relational symbol (Kieran, 1981), implicit quantifiers determine truth value (Durand-Guerrier, 2003; Usiskin et al., 2003), the truth value of conditional statements should be determined with a generalizable heuristic (Hub & Dawkins, 2018), and equivalent equations are defined by their common solution set (Knuth et al., 2006). After the teaching experiment commenced, this trajectory was refined to match my models describing participant thinking.

Identifying and understanding participants’ mistakes is a central component to model building and participant learning during a teaching experiment. Specifically, Steffe and Thompson (2000) articulated the need to identify *essential mistakes* – mistakes that persist despite the researcher’s efforts to eliminate them. These mistakes are a result of the participants’ failure to adapt their knowledge to new circumstances. However,

> rather than believing that a student is absolutely wrong or that the student’s knowledge is immature or irrational, the teacher-researcher must attempt to understand what the student can do; that is, the teacher-research must construct a frame of reference in which what the student can do seems rational. (Steffe & Thompson, 2000, p. 277)

I aimed to identify persistent errors when participants attempted to reason about solutions or relate equivalent (or non-equivalent) equations. I used my HLT as a guide to formulate tasks and questions that would guide participants away from their errors and toward an analytical conception of equations. I continuously probed participants' thinking to ensure their errors were actually corrected. When my plan to develop participants' thinking did not result in the adoption of the ways of thinking I intended, I adjusted my anticipated learning trajectory to account for participants' reactions (or lack thereof).

The models I developed were never intended to be one-to-one representations of participants' thinking. My mathematical understandings, the lens I adopted for the study (i.e., proof schemes), and my learning goals (i.e., conditional language and set-based reasoning) influenced my models (Steffe & Thompson, 2000). Although the models developed during a teaching experiment may not precisely reflect the mathematical reality of participants, my models were tested and refined until I thought I could adequately explain participants' behavior while reasoning about solutions. I considered the models viable in the sense that Steffe and Thompson (2000) describe: "Because the models that we formulate are grounded in our interactions with students, we fully expect that the models will be useful to us as we engage in further interactive mathematical communication with other students" (p. 295). By grounding my models of participants' thinking in what they said and did in a variety of situations, I ensured that the models would be useful in other contexts despite being shaped by my perspective as the researcher.

Data analysis. There are two major phases of analysis during a teaching experiment: during the experiment and after the conclusion of the experiment. During the teaching experiment, I conducted ongoing analysis (Cobb, 2000). Ongoing analysis included analysis conducted while interacting with participants as well as in between sessions. After the teaching experiment concluded, I conducted retrospective analysis (Steffe & Thompson, 2000). I developed and refined models of participants' thinking through my ongoing and retrospective analysis. In addition, the analysis allowed me to document how participants' understandings of equations evolved into ways of thinking that included reasoning about classes of objects and using conditional language.

Ongoing analysis. I conducted ongoing analysis throughout the teaching experiment. Specifically, there were three phases of ongoing analysis: during sessions with participants, processing the audio recordings and written work immediately after each session, and preparing for a subsequent teaching session.

A central component of ongoing analysis is building and testing a model of participant's thinking in real time. My model building was consistent with Thompson's (2008) conceptual analysis in that I intended to describe "what students actually know at some specific time and what they comprehend in specific situations" (p. 45). I accomplished this by evaluating responses and asking follow-up questions to confirm my interpretations of participants' behavior. In addition, I adopted the goal of continuously identifying questions or prompts that might evoke responses contradicting my current model of participants' thinking. In other words, I did not just attempt to confirm what I thought; I also played the role of devil's advocate to test the strength of my model.

A significant portion of ongoing analysis happens "on the fly" while interacting with participants (Steffe & Thompson, 2000). While I prepared tasks and questions for participants ahead of time, I did not know exactly how they would response (if I did, I would not need to ask in the first place). Although I had a sense of how participants might respond, inevitably there were responses that I did not anticipate. In order to proceed, I analyzed the response and decided in the moment how to respond in order to continue building and testing a model of the participant's thinking.

The second phase of data analysis occurred immediately after each session when I wrote memos that documented my initial impressions of the session. The point of these memos was to capture my working models of participants' thinking. The memos served as a starting point – an encapsulation of my perspective before delving deeper into the data. I then transcribed the audio recording. By transcribing the audio recording myself, I gained familiarly with the data and further developed my impression of what transpired during the session. I revisited my original memos and appended any additional thoughts about the session.

The third and final phase of ongoing analysis occurred after the transcription of the audio recording of a session, but before the next session took place. I collectively analyzed the transcript, participant's work, and my memos. This allowed me to take stock of my current model of participant's thinking, consider ways I might test it further, and compare the participant's current conceptions of equations with the end goal of using set-based reasoning and conditional language. In particular, I evaluated the extent to which my model of the participant's thinking matched my initial hypothetical learning trajectory. This process allowed me to plan the next teaching session and anticipate ways in which the participant might respond.

Retrospective analysis. After the conclusion of the teaching experiment, I conducted retrospective analysis (Steffe & Thompson, 2000). I started this process by rereading the transcripts of each participant from start to finish. I then revisited my memos and each participant's written work. The retrospective analysis was similar to the ongoing analysis, with the key distinction that I now had a different perspective because I had completed the entire experiment. Steffe and Thompson described this new perspective as both retrospective and prospective – prospective in the sense that, unlike the ongoing analysis, I knew what participants would say and do next.

My retrospective analysis, consistent with Thompson's (2008) conceptual analysis, allowed me to describe the ways participants learned to reason about classes of objects and use conditional language. I identified instances that could be used to describe "ways of knowing that might be propitious for students' mathematical learning, and…ways of knowing that be deleterious to students' understanding of important ideas and…ways of knowing that might be problematic in specific situations" (Thompson, 2008, p. 45). To perform this analysis, I read the transcripts of a participant's sessions and highlighted instances that seemed significant. Specifically, I identified instances that captured ways of knowing that I thought facilitated or prevented the participant from developing understandings about classes of objects and conditional language. While performing this analysis, I adopted a broad view of each participant's understandings by constantly comparing (Strauss & Corbin, 2015) the participant's ways knowing throughout the entire experiment. Once I completed this analysis for each individual participant, I also constantly compared all noteworthy instances across participants in order to strengthen my analysis.

Stage 3: Task-Based Interview #2

The third set of set of data was collected to answer the third research question: *Are students who reason conditionally about solution sets also able to reason about classes of geometrical objects?*

After the conclusion of the teaching experiment, I conducted a second task-based interview (Goldin, 2000) with each participant who, as a result of the teaching experiment, demonstrated they could reason about classes of objects and use conditional language. The purpose of this second task-based interview was to determine if participants who successfully developed set-based reasoning and used conditional language to solve and relate equations would also seek out analytical justifications in a geometrical context. In other words, did participants continue to exhibit behaviors consistent with an analytical proof scheme? It would be significant if participants approached geometrical objects with the perspective of relating individual objects to a broader class of objects (regardless of whether or not they could properly articulate the class) (Crowley, 1998). My goal was not necessarily to have participants exhibit an analytical proof scheme – this is unlikely when students first interact with new contexts (Weber & Mejia-Ramos, 2015). Rather, I aimed to determine whether or not participants maintained a disposition of doubt toward empirical evidence and sought out analytical ways of reasoning.

Interview procedures. The task-based interview consisted of a task adopted from Lacmy and Koichu (2014) and required participants to reason about the relationship between an arbitrary quadrilateral and its Varignon Parallelogram. A Varignon Parallelogram is formed by connecting the four midpoints of the sides of any quadrilateral. Furthermore, a Varignon Parallelogram is a rectangle if and only if the external quadrilateral has perpendicular diagonals.

First, participants were prompted to consider an arbitrary quadrilateral and its Varignon Parallelogram and then asked, "Which external quadrilaterals have a rectangle as their internal quadrilateral?" Certain familiar quadrilaterals – squares, rhombi, and kites – have perpendicular diagonals. As a result, knowing a quadrilateral is a square, rhombus, or kite is sufficient to claim its Varignon Parallelogram is a rectangle.

I used a line of questioning similar to what I described for the first task-based interview. I asked participants how sure they were and what additional evidence, if any, they would like to consider. I specifically asked participants, "Do you think you always get an internal rectangle when you connect the four midpoints of the sides of any square?" My goal was to both i) elicit reasoning that conveyed squares are a class of objects, and ii) confirm that participants did not employ an empirical proof scheme in this context. To ensure I understood participants' responses, I rephrased the question and asked, "Do all squares have an internal rectangle when you connect the four midpoints?"

Next, participants considered the condition necessary for a Varignon Parallelogram to be a rectangle: an external quadrilateral with perpendicular diagonals. To do this, I prompted participants to consider an arbitrary quadrilateral and its Varignon Parallelogram and said, "We are given that an internal quadrilateral of some external quadrilateral is a rectangle. What, if anything, do you know for sure about the external quadrilateral?" While the sufficient conditions (the external shape is a square, rhombus, or kite) guarantee an internal rectangle, they are not necessary.

In order to determine if participants would consider unfamiliar quadrilaterals and adopt a broader view beyond special cases, I used a tracing feature of the dynamic geometry software. When turned on, the feature visually recorded each position a participant moved the vertices to. This not only kept track of each attempt to identify a Varignon Rectangle, it also generated the perpendicular diagonals once participants were able to move the vertices while preserving the Varignon Rectangle. In other words, the tracing feature highlighted the invariant property of perpendicular diagonals as participants considered multiple quadrilaterals with Varignon Rectangles.

Data analysis. My analysis for the second task-based interview was similar to my analysis for the first task-based interview. First, I transcribed the interview and reread the transcript multiple times before generating a set of open codes (Strauss & Corbin, 2015). Reasoning about a conditional statement and its converse afforded participants the opportunity to articulate the necessary and sufficient conditions that demarcated set-based reasoning, conditional language, and behaviors consistent with an analytical proof scheme. I constantly compared my open codes to develop axial codes (Strauss & Corbin, 2015) that grouped together instances that supported or refuted the claim that participants used set-based reasoning and conditional language.

Similar to my analysis of the first task-based interview, I identified phrases or statements that conveyed absolute certainty (Weber & Mejia-Ramos, 2015). I was especially careful when participants considered empirical evidence. Dynamic geometry software allows for the easy creation of many cases and presents the temptation to rely on empirical evidence. Participants' consideration of empirical evidence, however, does not necessarily mean they employed an empirical proof scheme. I sought to determine if participants were convinced by empirical evidence (i.e., had an empirical proof scheme), or if they merely considered it alongside other forms of evidence.

To this end, I constantly compared (Strauss & Corbin, 2015) the axial codes for participants' level of certainty and the axial codes describing participants' reasoning and language. This allowed me to develop thematic codes and determine whether or not participants employed a non-analytical proof scheme to make claims about Varignon Parallelograms. If they were not absolutely sure of their conjectures (i.e., had no proof scheme), I identified the ways they attempted to gain certainty and whether or not their behaviors were consistent with an analytical proof scheme.

Chapter IV

RESULTS

This chapter describes the results from the task-based interviews and teaching experiment used to answer the research questions. First, I provide details from the first set of task-based interviews and answer the first research question. Next, I describe the results of the teaching experiment and answer the second research question. Last, I provide results of the geometrical task-based interviews and answer the third research question. In all three sections, I include excerpts of dialogue that notably convey participants' thinking. I bold key phrases and responses to alert the reader to significant moments that capture participants' sense making and shifts in their understandings.

Stage 1: Task-Based Interview #1

In this section, I first describe the proof schemes participants exhibited and the contexts that gave rise to those proof schemes. Participants often exhibited multiple proof schemes, and I use primary and secondary proof schemes to categorize and describe each participant's justifications over the course of the entire interview. Last, I answer the first research question and specifically describe the similarities and differences among participants when justifying solutions for particular types of equations.

Proof Schemes for Equations

As a whole, participants exhibited three proof schemes: empirical, ritualistic, and authoritative. Each participant, however, only exhibited one or two proof schemes over the course of the first task-based interview. In this next section, I describe salient instances of each proof scheme.

Empirical proof scheme. Participants employed an empirical proof scheme in a variety of contexts. The predominant display of an empirical proof scheme for some participants was to verify a solution obtained from a solving process. This was commonly referred to as "the check" or "plugging in." In addition, some participants used an empirical proof scheme to identify solutions to equations that were unfamiliar. In these instances they did not know of the applicable solving procedures; they instead used an empirical proof scheme in which they searched for and verified potential solutions.

Importance of the check. A key characteristic of an empirical proof scheme is achieving absolute certainty (i.e., ascertaining) through verification. Participants were able to identify correct solutions while solving familiar equations (e.g., $4x + 2 = 10$ and $7x + 1 = 3(x + 3)$). However, participants reported being absolutely sure only *after* completing the check. The following exchanges demonstrates how Harry ascertained two is a solution for $7x + 1 = 3(x + 3)$. His accompanying work is shown in Figure 1.

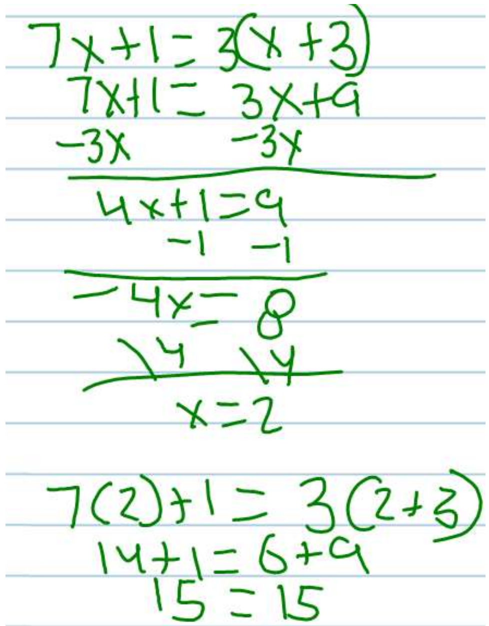

Figure 1. Harry ascertained only after the check

Julius: How sure are you, on a scale of one to five?
Harry: **Four.**
Julius: Okay, so why aren't you all the way sure?
Harry: I mean, because this one looks a little bit harder than that one, so **I'm not sure if I did it right.**
Julius: Okay, so is there anything you could do to be sure?
Harry: **You can check it.**
Julius: Okay.
Harry: So that equals the same, 15 and 15. So if you get the right answer, **if they match, that's how you know it's correct.**
Julius: Okay, so on a scale of one to five?
Harry: **Five now.**

In a very similar fashion, the exchange with Isabel also conveys the role of the check:

Julius: I guess I'm going to ask specifically, on a scale of one to five, how sure are you right now that x equals two?
Isabel: Four.
Julius: Four. Okay, so what do you have to do to be absolutely sure?
Isabel: **You can plug it back into the original equation.**
Julius: Okay.
Isabel: I have 15 equals 15.

Julius: Okay, so now you're 100% sure?
Isabel: Yeah.

I later asked Isabel, "Just to be clear, are you sure before you check or only after?" She replied, "I'm sure before I check, but not 100 percent." Julia made a similar comment: "So I'm not *really* [emphasis] sure if that's the answer, so I would check." These statements captured the role of empirical evidence – it enabled participants to achieve absolute certainty. Additionally, the check was not always written down, but its role remained the same. Edgar, for example, said he was convinced "if I do the check step in my mind."

The importance of verification and the check step was also evident when participants described how they would persuade a classmate. Isabel's work solving $4x + 2 = 10$ is shown in Figure 2. The subsequent exchange demonstrates the role of the check step in persuading a classmate.

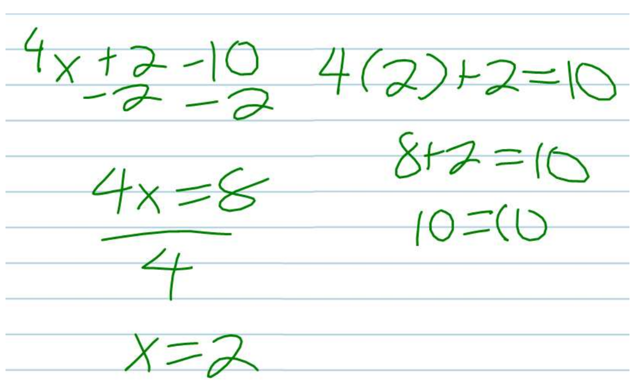

Figure 2. Isabel used the check to persuade

Julius: Alright, so on a scale of one to five, when five is absolutely 100% sure and one is not sure at all, how sure are you that you're right?
Isabel: Five.

Julius: Five, okay. If one of your classmates didn't believe you, how would you convince them that you're right?
Isabel: **I would do the check**, and then like **I would show them that I plugged in the number I got and it works.**
Julius: When you say it works, what do you mean by it works?
Isabel: **I plugged in the two**, and because I plugged in the two, I got eight plus two equals ten, **and then ten equals ten**.

Edgar described persuading in a very similar way: "I would tell them to do the check step." When I asked Julia what she would do to persuade a classmate, she said, "I would *prove it* [emphasis added] to them from checking my solution." Julia's use of "prove" is significant because it illustrates the check is central to her proof scheme. In addition, I wanted to clarify that participants were not simply using a check to persuade because it was convenient, but because they found it most convincing. I asked Harry, "How would you convince them that it was two for this equation?" He responded, "I don't know anything else besides plugging in the answer I got."

Searching for evidence. Instead of verifying a solution obtained from a solving procedure, some participants used an empirical proof scheme to search for and verify potential solutions when they did not know the applicable solving procedure. Specifically, some participants identified two as a solution to the equations $x^2 = 4$ and $x^3 = 4x$ without using inverse operations to isolate x. Denise's work in Figure 3 demonstrates the focus on verifying the solution for $x^2 = 4$ without any attempt manipulate $x^2 = 4$ and isolate x.

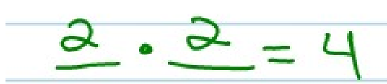

Figure 3. Denise verified two is a solution

The following exchange illustrates that this constituted ascertaining and persuading:

> Julius: What can you tell me about the solution to this equation?
> Denise: It's going to be with…an exponent is basically the number times itself, so it'd be a number times a number equals four.
> Julius: Okay. Can you tell me anything else about the solution?
> Denise: The solution would be two.
> Julius: Okay, how do you know?
> Denise: **Because two times two is four**.
> Julius: Okay. And on a scale of one to five, how sure are you?
> Denise: Five.
> Julius: One of your classmates wasn't convinced that it was two. What would you do?
> Denise: Well, I'd explain what an exponent is to them and write out **blank times blank is four**. Then **give them options**. So it would be two times two is four.

Denise's use of the phrase "give them options" is noteworthy. This phrase conveys the focus on verification and highlights the potential to fail to identify additional solutions – the identification of all solutions is dependent on the set of potential values initially identified. Indeed, she did not identify negative two as a solution.

As was the case with Denise, I asked Edgar if there was a way to be sure of the solution without knowing the solving procedure. He said, "Yes, yes, check step. I replace the x with two, and two to the second power equals four." Luis articulated a similar justification when reasoning about $x^3 = 4x$.

> Julius: How sure are you that x equals two?
> Luis: I would go with a five. Because since **I proved in my head by plugging in a number**, so I know in my head that with the number two it works. If it works with the number two, and **we also tried it with the number five, and the five doesn't work**. Since I tried the **two, and I know that it does work, that's the one solution** that I can stick to and say, "Hey, this is the solution to this equation."
> Julius: Okay, so one of your classmates, they say, "Okay I believe you, two is the solution. I believe that it equals eight, but I found a different number."
> Luis: Okay. **I would ask them to test it,** and maybe **give me the numbers so that I can test it myself**, like a peer review, if you will. And see if the

> answers are the same. **I'm doing it just to make sure**. Because**, I don't want to say something's not true until I actually try it.**

Luis is convinced through verification ("I proved in my head by plugging in…"). Moreover, his strategy is intended to identify a single solution and does not eliminate the possibility of additional solutions. His comment, "I don't want to say something's not true until I actually try it," conveys his reliance on verification to determine whether or not additional values are in fact solutions.

Similar to the empirical strategy of searching for and verifying values to identify a solution, Harry employed a similar strategy to rule out of the possibility of a counterexample. Figure 4 shows Harry's work to identify a solution to an equation that is always true.

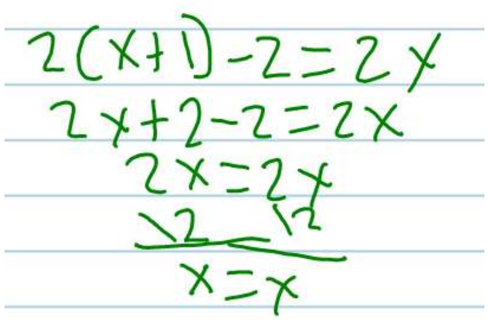

Figure 4. Harry's attempt to solve 2(x + 1) – 2 = 2x

As was the case when Harry solved other equations, his written work suggests that he identified the solution set through his algebraic manipulation (i.e., identifying x = x to determine the equation is an identity). However, his empirical proof scheme is highlighted in the following exchange:

Julius: And will it work for any number?
Harry: Yeah.
Julius: What did you just do in your head before you answered that?
Harry: **I was trying to think of a number that wouldn't work**.
Julius: And you couldn't find any?
Harry: No.
Julius: Okay. How sure are you right now?
Harry: Around four.
Julius: Four. So how could you get to five? What would you want to do to be absolutely sure that it works for any number?
Harry: I mean, the numbers have to be the same, that's the only thing I can think of right now. The numbers have to be the same or else it won't work, because if you have different numbers they're not going to equal to each other.
Julius: Okay. But if they're going to be the same numbers, will it work for any number?
Harry: Yeah.
Julius: Okay, and how sure are you about that?
Harry: Five now.
Julius: And what changed? Do you know what changed, or it just makes sense?
Harry: **I couldn't find any number.**
Julius: Okay, and if I gave you, so you looked for other, can you say that again?
Harry: **I was looking for other numbers that wouldn't work, but I couldn't think of any.**
Julius: You think or you know that any other numbers wouldn't work?
Harry: **I know no other numbers won't work.**

While Harry does not initially reach absolute certainty by verifying values, he eventually does. After some arbitrary number of attempts ("I was looking for other numbers…"), he determines that the equation will be satisfied by any value ("I know no other numbers won't work.").

Another instance of an empirical proof scheme was Harry's use of a crucial experiment (Balacheff, 1988) to incorrectly assert $1x = 2x$ is never true. Harry verifies arbitrary values and claims that the results of those values are representative of all values.

Julius: You said there's no solution? Okay, on a scale of one to five, you're sure there's no solution?
Harry: Yeah, five.

Julius: And so one of your classmates doesn't believe you. They say, "There's supposed to be an answer. I think there's an answer. I don't believe you." How do you convince them that there's no solution?

Harry: **Well, you can tell them no matter what number you put**, you can put a number that replaces the X and it will equal the same, because **even if you do one**, one times one is one, and then two times one is two, **so you won't get the same number ever, no matter what number you do.**

Julius: Okay, so you just put in one, right? Any special reason one, or just-

Harry: **It's just a simple number you can use. But if you want to use 10, 10 times 1 is 10 and then 10 times 2 is 20, so it's not the same number no matter what number you put.**

Julius: Okay. Did you test other numbers, or one and 10 is sort of enough to let you know?

Harry: I mean, besides zero, **because they both [sides] give you, if you use zero, two times zero is zero and then one times zero is zero as well**. So that's the only number that works.

Julius: So before you said there's no number that works, and then all of a sudden you said zero works.

Harry: **Oh, so zero's the only number that works, but the answer's always going to be zero.** Any number besides zero does not fit.

Harry's use of the phrase, "a simple number," conveys that he chose to verify the values of one and ten as representatives of other numbers that would be onerous to verify. Although Harry eventually identifies zero as the solution, he initially stated he was absolutely sure there was no solution. Even when he correctly identified zero, it is not the result of relating equations with inverse operations, but by verifying additional values ("…if you use zero…"). Ashley also used a very similar line of reasoning but used five as her test value: "If you were to do one times five equals to two times five, which would be *hypothetically* [emphasis added] saying if that would be what the x equals to, five equals to ten." Ashley use of "hypothetically" conveys that five is an arbitrary value she chose to conduct her crucial experiment; her hypothesis is that if five fails to satisfy the equation, no values will satisfy the $1x = 2x$. As a result, she erroneously concluded there is no solution to the equation.

Summary. Participants exhibited empirical proof schemes in four ways. First, they ascertained and persuaded by completing a check to verify a value obtained from a solving procedure. Second, when participants did not know the solving procedures for quadratic and cubic equations, they searched for and verified potential solutions. This often resulted in a failure to completely identify the solution set. Third, Harry argued an equation has infinite solutions by searching for and failing to identify a counterexample. Fourth, some participants concluded there was no solution for a particular equation because they did not identify a solution after verifying values. Specifically, they conducted a crucial experiment in which arbitrarily selected test values were used as representative values for all potential solutions.

Ritualistic proof scheme. With a ritualistic proof scheme, participants achieved absolute certainty because they executed a familiar procedure to identify a solution. However, they did not appear to attach mathematical meaning to their actions; there was no evidence that participants reached conviction because of deduction. Instead, their conviction stemmed from the familiarity of the context – the previously ingrained ritual of solving. Participants also expected that the form of a solution conform to the result of the application of their solving ritual. It was often the case that they either reached an erroneous conclusion or failed to achieve absolute certainty when the form of a solution differed from what was expected.

A prescription for conviction. Participants used a ritualistic proof scheme to ascertain by comparing the solving process at hand to a familiar recipe previously established. In the following exchange, Kim describes why she is certain two is the solution for $7x + 1 = 3(x + 3)$.

Julius: Okay, same question, how do you know that's the solution?

Kim: Because the right side, it's three times x plus three, which are in parentheses. So, **to get rid of** the parentheses **you need to** distribute three to x, and positive three. With that you get three x plus nine. You **put down** the seven x plus one, same side on the left, and then since there's variables on both sides, **you should** ... What I did first is that since it was the smallest number, and since it was probably the easiest to move over, I did minus one. Since positive one minus one is zero – that **cancels out by itself**. Nine minus one is eight. You **bring down** three x, and on the left side, seven x alone, because you **got rid of** the one.

Julius: Mm-hmm.

Kim: So then you have seven x equals three x plus eight, so then since the next smallest number to that variable is three x, you subtract it. Since positive three x minus three x **cancels out by itself, you just get rid of that.** So, seven x minus three x, is four x. Then you **bring down** the eight and now it's four x equals eight. Since the opposite of multiplication is divide, you divide four, by four x. The four **just cancels out by itself** but you still have the x. So, eight divided by four is two, now you have **the answer, x equals two.**

Kim's assertions that "you need to" and "you should" stand out because they communicate a prescription to obtain the solution. Although there are numerous ways to solve an equation, Kim's conviction stems from solving the equation with a procedure she can reference. In addition, Kim's use of certain phrases illustrate that the steps of her solving process are rooted in a ritual to obtain "the answer" and not necessarily predicated on relevant mathematical properties. For example, she uses the phrase "got rid of" to describe the use of the distributive property. She also uses "cancels out" as well as "get rid of" to describe the use of additive and multiplicative inverses. These phrases do not serve to link equivalent equations through deductive justifications, but instead reveal a process that yields "the answer." In turn, Kim was certain, not because she attended to the truth value of each equation, but because a familiar process – her ritual for solving – linked the original equation and her result.

Since participants viewed the solution as the end result of a ritual, their conviction, or lack thereof, corresponded to recognizing a familiar solving procedure. If the equation was unfamiliar and they could not identify an applicable procedure, they did not achieve conviction. They did not verify a potential solution to increase certainty or obtain absolute certainty. Instead, they described the need to see the process that yielded the solution.

Julius: And then, how would they convince you that it's actually the solution?
Gaby: **By just doing it,** like, so I could **see it.**
Julius: See it, okay.
Gaby: **See the steps.**

In describing what she would require to ascertain, Gaby associated the equation with a process. Her remark, "see the steps," demonstrates that she sought a relevant ritual to identify a solution. Similarly, when I asked Isabel how she could be convinced, she replied, "I would want to see them go through the steps that they did to get where they are." Like Gaby, Isabel expressed the need to see the process used to obtain the solution. In particular, the phrase "get where they are" conveys that the solution is a destination or a result, instead of an object that encapsulates the truth value of the equation.

Just as ascertaining in a ritualistic proof scheme is about recognizing a familiar process, persuading is about demonstrating the execution of a familiar prescription. Participants specifically described convincing classmates by being clear about the steps taken to obtain a solution. In the following exchange, Luis conveys that he would persuade by showing the process to isolate the variable.

Julius: Okay, and one of your classmates doesn't believe you. What do you do?
Luis: Well, I would ask them what they think **the answer** is, and find **a way to prove it incorrect or prove a way** that they could understand it better.
Julius: Can you tell me a little more what you mean by prove that it's incorrect?

> Luis: Say they think it stops at four x is equal to eight. I can explain to them how that's not correct because if you're trying **to get x by itself** you can't have another, I believe it's a coefficient with it. Just try to **show them how that's not where you need to be** for the…say if you're trying to simplify it to x is equal to something instead of a coefficient with the x. **Just proving to them how you need to take it all the way to the simplest it can be.**

Luis' use of the phrase "the answer" in conjunction with his comment, "…show them how that's not where you need to be…," illustrates his view in this instance that the solution to an equation requires the execution of a process. To that end, he would persuade by completing his solving ritual and obtaining a result.

In Luis' ritualistic proof scheme, he would not persuade by demonstrating a non-solution fails to satisfy an equation. Instead, he articulated the need for the solving procedure to match his ritual for solving equations. He said, "You figure out where they went wrong, and then you try to explain to them, 'Hey, this is where you went wrong, and this is how to *properly* [emphasis] do it.'" Luis' emphasis on properly executing a procedure is another reference to his ritual for solving equations. Other participants made similar comments. For example, when I asked Fernando how he would convince a classmate of a solution, he said, "Do the same steps all over again." In a ritualistic proof scheme, the solving ritual – not deductive justifications or empirical evidence – are the basis to persuade others.

Obtaining a result of x equals a constant was part and parcel of participants' ritualistic proof schemes. Since the "answer" was a result of a ritual and not deductive justifications, participants did not recognize that equations with the same solution were equivalent. Instead, they merely considered it a coincidence that equations shared a solution. For example, the following exchange captures Catherine's reasoning about 4x + 2 = 10, 4x + 1 = 9, and 7x + 1 = 3(x + 3).

Julius: When you looked at these last three equations, the ones we just did, did you notice anything?
Catherine: **Same answer.**
Julius: The same answer. What do you mean by same answer?
Catherine: x = 2
Julius: Okay, so you noticed the answer for all of these is x = 2. Why do you think that is? You're shrugging your shoulders.
Catherine: I guess **that's how the equations go. I don't know.**

Catherine's remark that the equations have the "same answer" conveys her view that a solution is the result of a process. Aside from the comparison of her results, Catherine did not compare the equations in any other way or identify them as equivalent. This is despite creating equivalent equations while solving. Her work for to identify a solution to 7x + 1 = 3(x + 3) is shown in Figure 5.

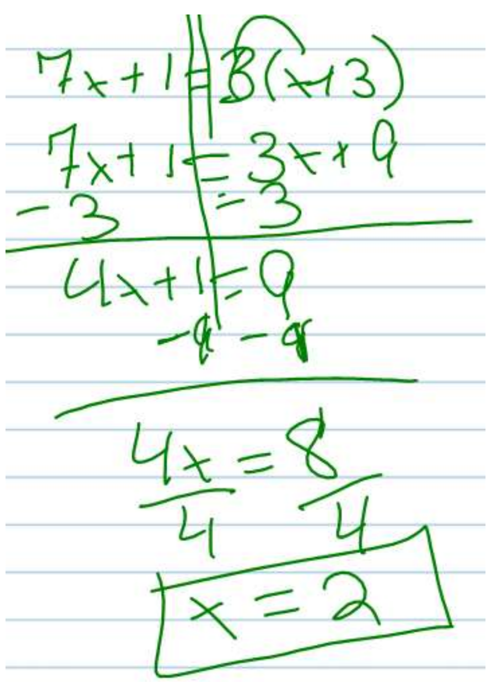

Figure 5. Catherine created 4x + 1 = 9 while solving

Even though Catherine generated 4x + 1 = 9 during her solving process, she did not identify 7x + 1 = 3(x + 3) and 4x + 1 = 9 as equivalent. Moreover, she generated 4x = 8 while solving 4x + 2 = 10, 4x + 1 = 9, and 7x + 1 = 3(x + 3), yet she only related the equations by comparing the result of x = 2. This highlights that the focus of a ritualistic proof scheme is the result of the solving ritual and not the preservation of the solution set throughout the solving process.

Focus on form. In a ritualistic proof scheme, the form of a solution is critical for conviction. Participants were convinced by the familiarity of the solving process, and part of the process involved obtaining a result with an anticipated appearance. Specifically, participants expected the result of their solving ritual to be in the form of x equals a constant. When I presented the equation x = 2 on its own and asked about the solution, Denise said, "It's already solved…there's nothing else to do to it." Her response indicated that she identified x = 2 as the solution not because two is equal to itself, but because the form of the equation matched what her solving ritual required ("there's nothing else to do"). Similarly, Fernando immediately identified x = 2 as the solution because "x is by itself."

Some participants identified the entire equation of x = 2, not the value of two, as the solution because of their view that the form of the solution matches the result of a solving process. Other participants claimed x = 2 has no solution because there was no process to execute. For example, Ashley said, "It can't be solved because there's *only* [emphasis added] x is equal to two," and Brittany said, "This one I'm confused on because normally you would have a problem to go with this." For both Ashley and Brittany, their rituals for solving equations required the execution of a process to obtain a result. They reasoned that since x = 2 does not afford a process to carry out, there is no result to obtain and therefore no solution.

Gaby, too, initially made a similar claim: “Well, it’s just x equals two, so there’s really no solution to that.” However, when I asked her to justify her response she changed her mind and concluded x = 2 is the solution. Her reasoning is shown in her work in Figure 6.

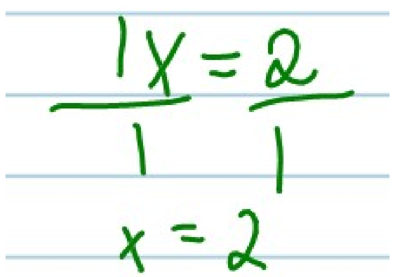

Figure 6. Gaby rewrote x = 2 as 1x = 2

Gaby’s ritualistic proof scheme required the application of her solving ritual. In order to make x = 2 conform to her ritual, she rewrote x = 2 as 1x = 2. This afforded her an action to execute in order to obtain a result. Gaby concluded with absolute certainty that x = 2 is the solution only after carrying out the step of dividing both sides of the equation by one.

In a ritualistic proof scheme, participants relied on the form of the solution. This resulted in erroreneous reasoning and incorrect conclusions for certain equations. For example, some participants declared that 1x = 2x has no solution because their solving process did not yield the expected result of x equals a constant. Kim’s work is shown in Figure 7.

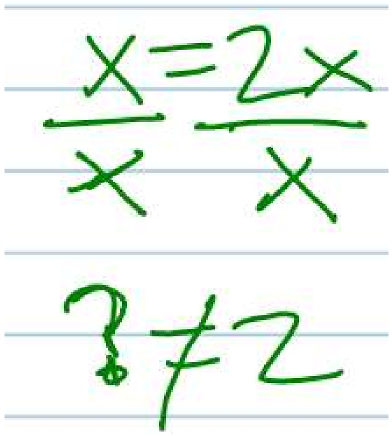

Figure 7. Kim "canceled" the x

The absence of a deductive justification and Kim's reliance on her ritual of "canceling" left her with a result in a form inconsistent with her expectation. She explained in the following exchange.

> Kim: **So you can't divide it**, x divided by x would be nothing. **So it would be no solution.**
> Julius: Okay, so x divided by x disappears and so because there's no x-
> Kim: **Yeah, there's no solution.**
> Julius: On a scale of one to five, how sure are you?
> Kim: Sure. Five.

Luis arrived at the same conclusion with very similar reasoning. After writing x = 2x, he said, "It just doesn't work out at all since we're trying to isolate x and find a numerical answer." A ritualistic proof scheme requires a result of an expected form. The absence of a conforming result led Kim and Luis to speciously conclude there is no solution.

Participants also falsely asserted that an equation with an infinite number of solutions had no solution. Participants argued that without a result in the form of x equals a constant, there is no solution for 2(x + 1) – 2 = 2x. Ashely's work is shown in Figure 8.

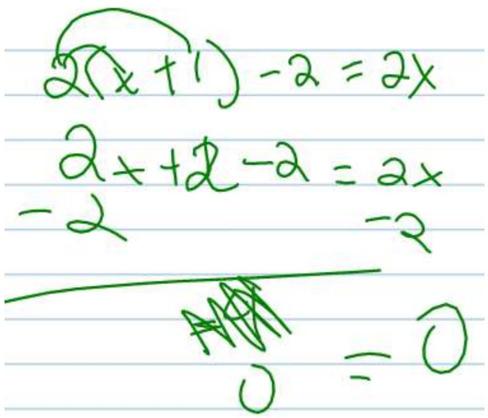

Figure 8. Ashley claimed there is no solution

Ashely's written work does not reveal her errorenous reasoning. When I asked Ashley what she could say about the solution, she replied, "There's no variable, and there's no answer." Similarly, Brittany said, "You can't just find x by itself…you won't get the right answer." Brittany's reference to "the right answer" signals that she expected a result of a certain form. To that point, Kim wrote x = x and asserted, "You can't have a solution…you need a number on at least one of the sides." Participants based their conclusions on the lack of a result in a familiar form. Participants did not account for the truth value of each equation or the deductive reasons that linked equivalent equations. Instead, their reliance on their ritualistic proof scheme to obtain an "answer" left them unable to recongize that $2(x + 1) – 2 = 2x$ is an identity and has an infinite number of solutions.

The focus on the form of a result also led some participants to reach true conclusions based on flawed reasoning rooted in ritualistic justifications. Edgar declared that x + 1 = x + 2 has no solution, but the following exchange demonstrates his ritualsitic proof scheme.

> Julius: What can you tell me about the solution to this equation?
> Edgar: It has no solution.
> Julius: So what do you mean by it has no solution?
> Edgar: Because **I did minus x on both sides and then took out the x completely** and one equals the two so that would be no solution.
> Julius: So one equals two. When you see one equals two, you say no solution? So why does that mean no solution to you?
> Edgar: **It doesn't show what x is.**
> Julius: Okay. So how do you know what you just said, that there is no solution, there is no x equal to. How do you know that?
> Edgar: Like I said, **I did minus x on both sides and that just completely takes out x**.

Edgar's remark, "It doesn't show what x is," illustrates the need for a result in an expected form. Similarly, Kim asserted there is no solution because, "There's no x, and with *a problem like this* [emphasis added] you need to find it, you need to have an x." Edgar and Kim sought to employ a ritual in order to obtain an expected result of x equals a constant. Their true conclusion that there is no solution was based, not on a deductive justification, but on the failure of their solving process to yield a result that conformed to their solving ritual.

Summary. Participants used ritualistic proof schemes to ascertain and persuade. Moreover, they viewed equations as prompts to execute a familar process. They were convinced by using their solving ritual – a familiar prescription of steps to isolate x. While participants' written work gave the apperarnce of deduction, participants' conviction was predicated upon the familiarity of the context and their ability to use a process to obtain a result in the form of x equals a constant. Participants' expectation that their solving rituals would produce results of a certain form proved problematic.

Specifically, some participants claimed an identity (i.e., an equation with an infinite number of solutions) had no solution. They also correctly identified an equation that is never true as having no solution, but by employing a spurious argument.

Authoritative proof scheme. Participants demonstrated an authoritative proof scheme as a contingency proof scheme when their ritualistic proof scheme was insufficient for certain equations. That is, when unsure of a solving ritual, participants described the need for an authority to provide clarity on the appropriate process that would lead to a legitimate result. They only described an authortiatve proof scheme as the way they could be absolutely sure after failing to achieve conviciton on their own, not a way they actually achieved conviction during the interview. Both teachers and other students were identified as authority figures that could be called upon to achieve absolute conviction.

Participants demonstrated an authoriative proof scheme because they failed to reach absolute certainty on their own and described the need for an authority to certify their solution as correct. In the following exchange, Catherine is not sure of the solution and describes the need to check with an authority – namely, her teacher.

Julius:	Okay, what are you out of five in terms of being sure?
Catherine:	I'm probably like a three.
Julius:	And even though we just went through all the steps and we checked all your steps, you're still not sure?
Catherine:	**I just like to check it with my teacher.**
Julius:	So you want a teacher. Okay. And do you always want check in with your teacher or is there something about this equation in particular that makes you want to go ask?
Catherine:	When I get my answer, **I normally raise my hand and check with the teacher**. I just like double checking.
Julius:	With a teacher, right?
Catherine:	Yeah.

Julius:	When you check with a teacher, then are you always a five after you check in with your teacher?
Catherine:	Yeah. **Because I'm sure** of what I'm doing.
Julius:	After the teacher?
Catherine:	Yeah.

Catherine's lack of absolute certainty in this specific instance was rooted in her classroom behavior. Her comment, "Because I'm sure of what I'm doing [after asking the teacher]," demonstrates that an authority's approval can be the basis for absolute conviction. Similarly, when I asked Gaby what would be required to be absolutely sure, she replied, "If somebody else were to do it…a teacher or somebody like that."

As was previously the case, participants' proof schemes were not necessarily evident from their written work. Catherine correctly identified two as the solution for 7x + 1 = 3(x + 3). Her written work is shown in Figure 9.

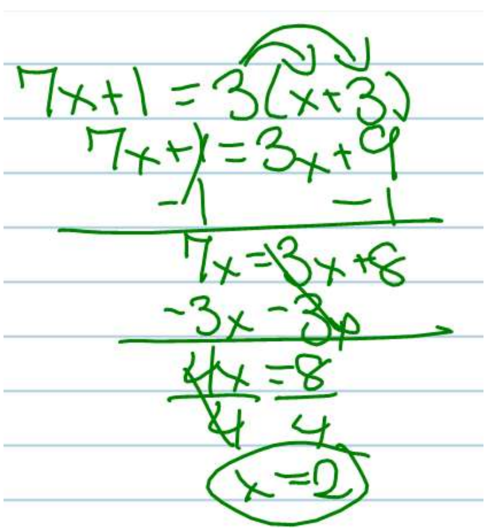

Figure 9. Catherine identified the solution for 7x + 1 = 3(x + 3)

Catherine, however, again reported not be absolutely sure. When I asked her how she could be absolutely sure, she replied, “Well, because sometimes I know I mess up, so *I need somebody else* [emphasis added]…I’ll ask somebody what did you get? And if we get the same answer, then I know I’ll be good.” I followed up by asking if it mattered who she asked and she replied, “Well, somebody who shows that they’re smarter than the rest of the class.” Her description of the classmate conveys that a person’s past record of success is the basis of their authority and, consequently, her certainty. Even though Catherine executed a written solving process to correctly identify the solution, she did not reach absolute conviction. Instead, she described absolute certainty based externally in the form of an authority’s approval.

Summary. Participants described authoritative proof schemes when unable to reach absolute conviction on their own. They described an authority’s approval as the way they could be absolutely sure. They specifically called upon an authority to either certify their solution as correct or to validate their solving process. In other words, there were instances in which participants had no other means to achieve absolute conviction besides calling upon an authority. While participants often named a teacher as the authoritative source to obtain absolute conviction, other students were also identified as potential authoritative sources.

Primary and Secondary Proof Schemes

Participants often exhibited more than one proof scheme, although they typically relied on a predominate proof scheme. I determined which proof scheme was primary and secondary based on the number of instances a participant employed each proof scheme during the interview. In this next section, I describe how participants switched between proof schemes depending on a number of different factors. First, participants sometimes used a certain proof scheme to reason about a particular equation, but then used a different proof scheme when reasoning about another equation. Second, participants sometimes used one proof scheme to ascertain, but when asked how they would persuade a classmate, they described a different proof scheme. Third, participants did not always reach absolute conviction. When they did not, they described additional evidence that could lead them to absolute conviction – the desired proof scheme they described was not always the same as the proof scheme they previously demonstrated.

Ten of twelve participants demonstrated multiple proof schemes (see Table 2). When participants only employed one proof scheme, I categorized the single proof scheme as both the primary and secondary proof scheme.

Table 2. *Primary and Secondary Proof Scheme of Each Participant*

Participant	**Primary Proof Scheme**	**Secondary Proof Scheme**
Ashley	Ritualistic	Empirical
Brittany	Ritualistic	Ritualistic
Catherine	Ritualistic	Authoritative
Denise	Ritualistic	Empirical
Edgar	Empirical	Ritualistic
Fernando	Ritualistic	Ritualistic
Gaby	Ritualistic	Authoritative
Harry	Empirical	Ritualistic
Isabel	Empirical	Ritualistic
Julia	Empirical	Ritualistic
Kim	Ritualistic	Authoritative
Luis	Ritualistic	Empirical

Ritualistic-Ritualistic. Brittany and Fernando exclusively relied on ritualistic proof schemes to ascertain and persuade. They consistently described equations as processes that need to be executed or demonstrated. Their processes always served to isolate x; an equation of the form x equals a constant was the expected result. Brittany and Fernando referred to their resulting equation as “the answer.” Their ritualistic proof schemes were very limiting because they often did not identify a solution when presented with unfamiliar equations. In addition, they did not evaluate to determine whether or not a potential solution satisfied a given equation (i.e., consider empirical evidence). They frequently did not achieve absolute certainty because of their reliance on referencing a ritual and the absence of additional strategies to increase conviction.

Brittany and Fernando consistently described a solving ritual as their basis to ascertain and persuade. The solution to a familiar equation was portrayed solely as a process. For example, when I asked Fernando about the solution to $7x + 1 = 3(x + 3)$, he said,

> I had to distribute, and three times x equals 3x, and three times three equals nine. 7x plus one equals 3x plus nine. You have to subtract 3x from 3x and from 7x and you’ll get 4x equals eight. Divide by two and you get x equals two.

Fernando was not simply describing what he wrote, but justifying why he was absolutely certain of the solution. Brittany often said she was sure because she “did the math.” The execution of the familiar process itself served as their justification. They described persuading classmates similarly. Brittany said, “I would tell how I did it and really show them how I really did it,” and Fernando stated, “I would do the same steps all over again.” The demonstration of the process – the steps of the solving ritual – constituted sufficient evidence to persuade a classmate.

The use of a ritualistic proof scheme required the application of a known ritual, and this was only possible if participants recognized that they had previously solved a similar equation. In other words, participants' ritualistic proof schemes were used in familiar contexts. When they encountered unfamiliar equations, however, Brittany and Fernando did not identify a potential solution and rarely reached absolute conviction. Their ritualistic proof schemes were limiting because when they did not recall an applicable solving process they had no other ways to reason about the equation. In following exchange, Brittany describes how she would want to be shown the solving procedure for 2(x + 1) – 2 = 2x.

Brittany:	You can't do two x divided by two x, and then you won't get the right answer.
Julius:	Okay. And on a scale of one to five, how sure are you?
Brittany:	Probably a two or three.
Julius:	So you're not sure? Okay.
Brittany:	No.
Julius:	Is there something else you'd want to see?
Brittany:	I don't really know.
Julius:	How could I, so if I said I really want you to be more sure. What would you want me to show you? Or your math teacher, or your friend?
Brittany:	**I want them to show me how they get an answer this question**, to this problem.
Julius:	Right, they would explain…
Brittany:	**Like explain how they planned it out. Like how they did it.**

Brittany transformed the equation into 2x = 2x, but at that point her solving ritual was no longer applicable because it would not result in an equation of the form x equals a constant. She went on to describe the need to be shown the process used to solve the equation. Fernando reasoned about the equation similarly and was unable to describe a solution. He said, "I'm getting stuck," when I asked him if there was anything else he could try or want to know in order to identify the solution. Participants' ritualistic proof schemes not only lacked deductive justifications – they left participants unable to identify a solution or reach conviction for unfamiliar equations.

Ritualistic-Authoritative. Catherine, Gaby, and Kim frequently cited the need for a ritual or an authority to be absolutely sure of a solution. They primarily employed ritualistic proof schemes; their reasoning was comparable to Brittany's and Fernando's. Catherine, Gaby, and Kim, however, also conveyed an authoritative proof scheme when describing how they could reach absolute conviction. The authoritative proof scheme served as a contingency proof scheme – participants described the need for an authority when their ritualistic proof schemes were insufficient. Specifically, the role of the authority was to provide or validate a solving ritual. Like participants who exclusively used a ritualistic proof scheme, participants who demonstrated both ritualistic and authoritative proof schemes were limited because they both: i) often failed to achieve absolute conviction, and ii) struggled to correctly identify solutions to unfamiliar equations.

When reasoning about familiar equations, Catherine, Gaby, and Kim ascertained and persuaded by referencing the steps and result of their solving ritual. They also cited the need for an authority's affirmation. The following exchange illustrates both Catherine's ritualistic and authoritative proof schemes.

Julius:	Okay, so how sure are you?
Catherine:	Like 4.5.
Julius:	So not all the way. So why not? What else do you want to see here? What could…how would you get to being a five [absolutely sure]?
Catherine:	Well, because sometimes I know I mess…when I have **equations like that** I mess up. So **I need somebody else** to…I'll ask somebody what did you get? And **if we get the same answer, then I know I'll be good.**

Julius:	So if you're at home studying by yourself, there would be no way for you to be sure?
Catherine:	I'd just **redo it.**
Julius:	Redo it, right? But that wouldn't…so if you redid it though, you might be a little more sure, but would you be a five? Would you be absolutely sure if you redid it?
Catherine:	I mean, **if I get the same answer, then yeah.**
Julius:	So if you got the same answer a second time, you would be sure?
Catherine:	Yeah, or I'll just **review my steps**.

Catherine describes absolute conviction based on the steps of her solving ritual. Her ritualistic proof scheme is evident from her comments, "the same answer," "I'd just redo it," and "review my steps." Catherine viewed solving equations as a process and the solution as a result. She also described persuading a classmate in a similar manner: "What I normally do is I see what they wrote, and then I show them my work, and we compare." On the other hand, her authoritative proof scheme manifested in her remark, "I need somebody else…if we get the same answer, then I'll know I'll be good." Absolute conviction, in this instance, was predicated upon somebody else's affirmation.

Participants executed a ritual to isolate x and did not reach absolute conviction because of deductive justifications. They did not recognize that certain equations were equivalent even though they shared a common solution of $x = 2$. Gaby described $4x + 2 = 10$, $4x + 1 = 9$, and $7x + 1 = 3(x + 3)$ as "different problems, but the solution is the same." Catherine thought it was a coincidence that "the answer" was the same for all three equations. Kim noted that the solving process for all three equations generated $4x = 8$. However, when I asked her if she needed to do all the steps to be sure the solution is the same, she replied, "If you want to be 100% sure that you're correct then yeah, you should." Completing a process is central to the ritual – absolute certainty is firmly established in obtaining a result from a procedure.

Ritualistic proof schemes often proved inadequate when participants were unfamiliar with an equation. In unfamiliar contexts, they could not apply a solving procedure and therefore could not obtain a result or achieve certainty. In the following exchange, Gaby is uncertain about the solution to $1x = 2x$ because her solving ritual was not sufficient to isolate x.

Gaby: I'm trying to do something, and **I'm kind of getting stuck**.
Julius: Show me what it is that you're doing.
Gaby: I'm trying to do, like, division and stuff, but **it's not working out.**
Julius: If one of your friends said they did it, would they have to do something in order for you to believe them? Or, you just wouldn't be sure?
Gaby: They would have to, like, **show their work in order for me to believe them.**
Julius: Okay, so when a friend says, 'I found a solution,' what it would take for you to believe them would be clearly showing the work?
Gaby: Yes.
Julius: And is there another way they could convince you? Or that's how you would be convinced?
Gaby: **No, that's how I'd be convinced.**

The visible execution of a procedure – the steps of the solving ritual – are critical for Gaby to reach absolute certainty. When I later asked how she would persuade a classmate, she said, "I'd just show my work." Likewise, Kim said she would convince a classmate by "showing them how I did the problem." Any consideration of empirical evidence was completely absent; participants did not describe substituting values to verify solutions. The steps of the solving procedure in and of themselves were the basis for conviction for Catherine, Gaby, and Kim.

In addition, participants displayed an authoritative proof scheme by describing the need for an authority to affirm or provide a relevant ritual for an unfamiliar equation. Kim believed, with relative certainty, that $x^3 = 4x$ had no solution because she could not obtain a result in the form x equals a constant. She conspicuously stated an authority is required for absolute conviction.

> Kim: **I'm not entirely sure** but I think it would be a **no solution.**
> Julius: Tell me why you think – it's totally fine that you're not sure, but you think it would be no solution. What's making you say that?
> Kim: I just don't think you'd be able to divide x cubed. **I don't think you'd be able to do x cubed divided by x** because you don't know what the number is.
> Julius: Okay. So you don't think there's a step we can do here?
> Kim: No.
> Julius: So there's no solution?
> Kim: Yes.
> Julius: Okay. What would it take, what would you like to see, what could somebody do or show you, so that you'd be sure there's no solution? How could somebody convince you there's no solution?
> Kim: **Be a teacher.** Because even the best students in math class could make a mistake, so **I would definitely trust someone who's done this multiple years or a teacher.**

There are two noteworthy parts of this exchange. First, Kim's uncertainty is rooted in the lack of an applicable procedure. This is illustrated by her comment, "I don't think you'd be able *to do* [emphasis added] x cubed divided by x." Second, Kim stated that she could be convinced by "someone who's done this multiple years or a teacher." For Kim, her classmates do not have the necessary authority to persuade her, but the authority of somebody with experience or that of a teacher would be sufficient for her to be absolutely certain. Gaby similarly remarked she could be convinced "if someone else were to do it…a teacher or somebody like that." Catherine, too, alluded to the authority of the teacher when not absolutely sure of a solution: "I just like to check in with my teacher…I normally raise my hand and check with the teacher." When confronted with unfamiliar equations, participants' unsatisfied need of an authority's affirmation frequently resulted in a lack of absolute conviction.

Ritualistic-Empirical. Ashley, Denise, and Luis used ritualistic proof schemes to reason about familiar equations. Their justifications were comparable to those of other participants with ritualistic proof schemes. For unfamiliar equations, however, they employed a variety of empirical proof schemes. In addition, participants employed empirical proof schemes to varying degrees. Denise and Luis used empirical proof schemes multiple times, but Ashley only demonstrated an empirical proof scheme once. Participants demonstrated that they understood the key idea that a solution makes both sides of an equation equal. Yet for familiar equations, they generally did not use this reasoning. This was a testament to the strength of their ritualistic proof schemes.

Ashley, Denise, and Luis viewed equations they were familiar with as processes to be executed. Their certainty was rooted in the familiarity of the solving process. When explaining why she was certain about the solution to $4x + 1 = 9$, Ashely said, “Because if you do the math, you’d subtract one from both sides. You would subtract one – nine minus one, which is eight. You’re left with 4x. You divide eight by four, you get two. The answer is two.” When I asked Denise and Luis how they could be certain of a solution, Denise said, “You’re going to need to find the value of x,” and Luis replied, “We’re trying to isolate x, to find a numerical answer.” As was the case for other participants with ritualistic proof schemes, Ashley, Denise, and Luis relied on the familiarity of the procedure, not deductive justifications, to achieve absolute conviction. They did not consider empirical evidence by substituting a solution to verify it satisfied an equation.

Participants also persuaded by referencing their solving rituals. Denise, for example, said she would convince a classmate by “Just trying to show them how, the steps I took more clearly." Luis similarly said, "Just try to show them….proving to them you need to take it [the equation] all the way to the simplest it can be." These comments reflect that participants viewed the demonstration of a procedure as sufficient evidence to persuade – Luis even used the phrase "proving to them." Luis also remarked that persuading is about figuring out "where they went wrong." Ashley similarly commented, "I would tell them that's the wrong answer," and, "They did the math wrong." Their references to being "wrong" illustrate their view that a solving ritual was not properly applied, rather than the concept that a non-solution fails to satisfy an equation. Denise, however, adopted an empirical proof scheme to persuade. Although she ascertained with a ritualistic proof scheme (e.g., "I solved it."), when I asked her how she would convince a classmate, she said, "You'd have to try it [the solution] out." This comment conveys that Denise viewed verification as sufficient evidence to persuade.

Participants' responses when describing the solution for x = 2 provided further evidence that they employed ritualistic proof schemes, and not deductive justifications, to achieve absolute certainty. When participants were prompted to reason about x = 2 as a standalone equation, they brought to bear their process-based view of equations. For example, Denise stated x = 2 is the solution because "there is nothing else to do it." Likewise, Luis said x = 2 is the solution because "it's as simple as it can be, you can't really do anything – you can't manipulate it in any way to get a different answer." For Denise and Luis, x = 2 was the solution not because two satisfied the equation, but because x = 2 represented the result of a solving ritual. Ashley, on the other hand, said x = 2 "can't be solved." Although her conclusion was different, her reasoning was similar because she also viewed equations as processes. Ashley argued that since x = 2 does not afford a process to execute, there is no result and thus no solution.

Although Ashley, Denise, and Luis primarily employed ritualistic proof schemes, they also demonstrated empirical proof schemes when reasoning about solutions to equations they were unfamiliar with. The equations were unfamiliar in the sense that participants could not employ a solving process to identify a solution. Denise and Luis described achieving absolute conviction as a result of searching for and verifying potential solutions. For example, Luis said, "I proved in my head by plugging in a number," to explain why he was absolutely sure two was the solution for $x^3 = 4x$. His strategy resulted in him failing to identify zero and negative two as solutions.

After using an empirical proof scheme to identify two as the solution for $x^2 = 4$, Denise also used an empirical strategy, albeit unsuccessfully, to reason about the solution for $x^3 = 4x$ (Figure 10).

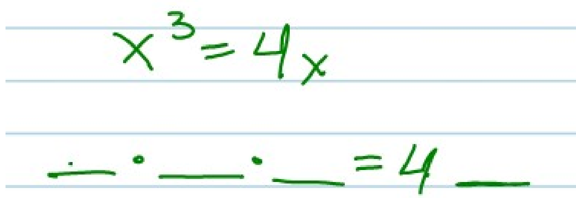

Figure 10. Denise's empirical strategy to identify a solution

Denise describes her reasoning behind her work in the following exchange.

Julius: So what do the blanks mean?
Denise: It's **the possibilities of numbers**. So then **I'd go for which one would make sense.**

Julius: And when you say make sense, what do you mean make sense?
Denise: **I'd try out the possibilities.**
Julius: And the numbers that go in these blanks…
Denise: Are x.
Julius: x. So can they be different?
Denise: No.
Julius: So you're using the same number in every blank?
Denise: Yes.

Denise conveys a relational understanding of the equals sign as she endeavors to find the possibility that "makes sense." She later searched for and failed to identify any solutions. She declared, with relative certainty, that there was no solution. When I asked her what she would require to be absolutely sure, she replied, "If they would give me options of what numbers I would be able to use, and then I'll figure it out from there." This description is consistent with an empirical proof scheme: Denise would reach absolute conviction through verification.

Luis and Ashley described a crucial experiment (Balacheff, 1988) to be absolutely certain of solutions. Luis initially was relatively certain 2(x + 1) – 2 = 2x has infinite solutions. In following exchange, he achieves absolute conviction by verifying a seemingly arbitrary value (x = 4).

Julius: So what can you tell me about the solution to this equation?
Luis: **I believe it's infinite solutions.** Wait. Yeah, because really any number could fill in for x, and it'd still be true.
Julius: Okay.
Luis: No matter what.
Julius: So you don't sound sure. Are you sure that it's any number? On a scale of one to five?
Luis: About four, **I'm fairly certain.**
Julius: So you're fairly certain that any number would work right?
Luis: Yeah.
Julius: Okay. What would you need to see to be 100% sure? What could somebody show you or tell you for you to be absolutely sure?
Luis: **If I just plugged in say, four is equal to x.** I would get eight, I would get the four plus one is equal to five. Then you'd distribute the two into it,

which would get you 10 minus two, that would be eight. Then you just multiply the two by the four, that's eight, **so eight equals eight. That proves the rationale.**

Julius: So you said proves. So are you convinced now?
Luis: **It does prove.** Yes.
Julius: So you're now a five [absolutely sure]?
Luis: Yes.
Julius: So you plug in four, you saw that eight equals eight, and now you are 100% convinced?
Luis: Yeah.

This episode constituted a crucial experiment because Luis verified $x = 4$ as a representative for all potential values. However, he did not consider deductive justifications (e.g., state $2(x - 1) - 2x$ simplifies to $2x$) and so his strategy constituted an empirical proof scheme. This particularly evident from his remark, "that proves the rationale." Ashley employed a similar argument to conclude $1x = 2x$ has no solution. She verified five was not a solution and erroneously concluded no value satisfied the equation.

To summarize, Ashley, Denise, and Luis employed ritualistic proof schemes when they were familiar with an equation and switched to empirical proof schemes to reach conclusions about unfamiliar equations. Except for instances where Denise ascertained with a solving ritual and persuaded through verification (i.e., a check), there was generally little overlap between participants' ritualistic and empirical proof schemes. The empirical proof schemes illustrated participants' relational view of the equals sign and their understanding that a solution satisfies an equation. However, this reasoning was reserved for equations without a known solving procedure. To ascertain in familiar contexts, participants' ritualistic proof schemes took precedent and they did not verify solutions.

Empirical-Ritualistic. Edgar, Harry, Isabel, and Julia displayed both empirical and ritualistic proof schemes. Unlike Ashley, Denise, and Luis, these four participants employed empirical proof schemes more often than they did ritualistic proof schemes. The check step served a central role to ascertain and persuade and was primarily used when participants reasoned about familiar equations. However, they described empirical proof schemes to varying extents in other contexts as well. The ritualistic proof scheme was secondary because it appeared less often. It typically occurred when participants confronted unfamiliar equations. Edgar, Harry, Isabel, and Julia demonstrated ritualistic proof schemes in one of two ways: i) ascertaining by referencing the form of the solution, and ii) when not absolutely sure, stating the need to be told a ritual in order to achieve absolute certainty. In addition, when participants did not achieve absolute conviction, they sometimes described the need for both a ritual and empirical evidence.

Verification of a solution with a check was the primary way participants demonstrated an empirical proof scheme. For example, Julia said, "Since I'm not sure if x is the answer, I would check." For familiar equations, participants executed a solving procedure and then reached absolute conviction after completing a check. When I asked Harry what he would need to do to be sure, he replied, "You have to plug in two [the solution] instead of x…and if the answer matches, then it's correct." Participants persuaded in a similar manner. Isabel, for example, said in order to convince a classmate, "I would go over the check." Likewise, Edgar said, "I would tell them to do the check step."

Julia, too, conveyed that a check could be used to persuade, but also indicated the steps of her solving ritual would suffice: "[I would check] or I would talk them through what I did." In addition to illustrating Julia's use of verification to persuade, her use of the phrase, "what I did," indicates a process-based view of equations in which absolute certainty is also rooted in a previously ingrained solving ritual. Still, participants primarily conveyed empirical proof schemes overall because of the number of times they verified solutions with a check.

Participants usually did not employ empirical proof schemes for equations they were unfamiliar with. Specifically, some participants reached or described reaching absolute conviction with ritualistic proof schemes for $1x = 2x$, $x + 1 = x + 2$, $2(x + 1) – 2 = 2x$, and $x^3 = 4x$. Isabel was relatively certain there was no solution for $1x = 2x$, and when I asked what she would require to be absolutely sure, she replied, "I would want to them go through the steps that they did to get where they are." Similarly, in order to reach absolute certainty for $x^3 = 4x$, she said. "I would want to see how they get the x to the third into just x." Although Isabel previously ascertained and persuaded through verification, she did not describe substituting potential solutions to reach conviction. Instead, she stated the need to see a ritual, a solving procedure used to isolate x.

Unlike Isabel, Edgar was absolutely certain $x + 1 = x + 2$, $1x = 2x$, and $2(x + 1) – 2 = 2x$ have no solution. However, he also described a ritualistic proof scheme. In particular, he relied on the form of the result obtained from his solving procedure. He stated $x + 1 = x + 2$ has no solution, not because it is impossible for a value of x to satisfy the equation, but because his solving procedure "doesn't show what x is." Since Edgar's solving ritual required a solution in the form of x equals a constant, he concluded there is no solution when this solving process did not generate a result in the expected form. He employed a similar argument to state 1x = 2x and 2(x + 1) – 2 = 2x have no solution as well: "I'm used to seeing x equals to a number and I'm not used to seeing this [x =2x or x = x]."

Harry had an especially predominant empirical proof scheme and it extended to some equations he was unfamiliar with. Whereas Edgar and Isabel described ritualistic proof schemes for 1x = 2x and 2(x + 1) – 2 = 2x, Harry used empirical strategies to identify solutions with absolute certainty. For 1x = 2x, he conducted a crucial experiment (Balacheff, 1988) to initially determine there was no solution. He verified values of one and ten as representatives of all numbers; he argued that since these values do not satisfy the equation, no values do. However, while explaining his strategy of using "simple numbers" to test if the equation was ever true, he accidentally discovered that zero is a solution. He also used an empirical proof scheme to argue that 2(x + 1) – 2 = 2x has infinite solutions: "I was looking for other numbers that wouldn't work, but I couldn't think of any."

Harry demonstrated a ritualistic proof scheme on one occasion. Similar to Edgar, he resorted to justifying the solution to x = 2 based on form. Harry did not consider empirical evidence but stated x = 2 was the solution because "you can't do anything to it because it tells you the answer…it's already simplified." Isabel similarly said, "You have to isolate x to get the solution, and x is already isolated." These instances demonstrated ritualistic proof schemes because the form of the equation invoked the result of a process, the ritual typically used to generate a solution.

Julia failed to reach absolute conviction while attempting to identify solutions to unfamiliar equations. Just as she used both ritualistic and empirical proof schemes to persuade classmates about solutions to familiar equations, Julia described the need to see both ritualistic and empirical evidence to be absolutely certain about solutions for unfamiliar equations. For $x + 1 = x + 2$, $1x = 2x$, and $2(x + 1) – 2 = 2x$, she stated she could be convinced “if they showed me the steps and they also checked.” Although Julia demonstrated an empirical proof scheme more frequently than she employed a ritualistic proof scheme, she considered empirical and ritualistic evidence almost equally.

Answer to First Research Question

How do proof schemes differ, if at all, when students justify solutions to different types of algebraic equations?

Most participants displayed more than one proof scheme. All participants employed ritualistic proof schemes to some extent; roughly half demonstrated empirical proof schemes. No participants displayed an analytical proof scheme – deductive justifications were not the basis for absolute conviction. Participants’ proof schemes largely depended on their familiarity with an equation.

For each equation, participants reported whether or not they recalled prior experiences solving similar equations. Participants recognized $4x + 2 = 10$, $4x + 1 = 9$, and $7x + 1 = 3(x + 3)$ as familiar and routine. They reported being less familiar or completely unfamiliar with equations that have no solution ($x + 1 = x + 2$), multiple solutions ($x^2 = 4$ and $x^3 = 4x$), and infinite solutions ($2(x + 1) – 2 = 2x$). They also viewed $x = 2$ as unfamiliar because it did not require the execution of a solving process.

The majority of participants demonstrated a ritualistic proof scheme for familiar equations. These participants conveyed a processed-based view of equations and viewed the solution as a result. They often referred to the solution as "the answer." Their conviction was rooted in the familiarity of the context and the execution of a previously ingrained solving procedure. When participants confronted unfamiliar equations, they either: i) failed to reach absolute certainty and stated the need to be told a solving procedure, ii) reached absolute conviction with a ritualistic proof scheme based on the form of the solution, or iii) implemented empirical proof schemes.

For unfamiliar equations, participants were limited in their ability to reach conclusions because their solving rituals were no longer applicable. Instead, they articulated the need for a new solving procedure (i.e., a series of steps that would constitute a new ritual). In this vein, some participants described an authoritative proof scheme as their contingency proof scheme. They required somebody else – typically a reliable classmate or a teacher – to be absolutely sure.

Some participants attempted to transfer their ritualistic proof scheme to unfamiliar equations. They specifically relied on the form of the solution to reach conclusions. These conclusions were often incorrect. In particular, they expected the form of the solution to match the anticipated result of their solving rituals: an equation in the form of x equals a constant. When their solving process did not yield a result in the expected form, participants declared there was no solution.

Three participants demonstrated empirical proof schemes instead of ritualistic proof schemes to reason about unfamiliar equations. These participants previously did not consider empirical evidence; they did not verify solutions through the use of a check. Their reasoning in familiar contexts did not reveal that they necessarily had a relational view of the equals sign. However, their empirical strategies to reach conclusions about unfamiliar equations unveiled a more sophisticated understanding of equations than their responses in familiar contexts suggested.

Four participants displayed robust empirical proof schemes to reason about familiar equations. In order to reach absolute conviction, these participants consistently verified solutions with a check. When these participants encountered unfamiliar equations, the range of outcomes was consistent with the strategies of other participants described earlier. Even though they demonstrated a relational view of the equals sign and understood that a solution satisfies an equation, they did not consider the truth value of each equation generated during the solving process. In other words, unfamiliar equations illuminated that participants did not use deductive justifications to link equivalent equations and preserve the solution set throughout the solving process.

Stage 2: The Teaching Experiment

In this section, I first restate the goal of the teaching experiment and summarize the participants' initial ways of reasoning described in the first task-based interviews. Second, I provide an overview of the phases of the teaching experiment. Third, I provided a detailed description of the data collected in each phase of the teaching experiment. Last, I answer the second research question.

Overview

Purpose. The purpose of the teaching experiment was to determine how students can come to reason about classes of objects and use conditional language when considering the truth value of algebraic equations. In particular, the teaching experiment set out to document how participants moved away from the view of the solution to an equation as representing "the answer" and instead adopted a set-based perspective in which a solution describes the truth value of an equation. The development of a set-based perspective also required participants to relate the equations generated during the solving process to the preservation of, or any changes in, the solution set.

The first task-based interview served as the pre-interview for the teaching experiment. As I described in the previous section, the three participants chosen for the teaching experiment did not describe solutions as a class of objects during their first task-based interview. Furthermore, they did not discuss the truth value of equations or use conditional language to relate equations. The participants instead relied on a combination of empirical and ritualistic strategies to generate solutions. While participants' non-analytical strategies sometimes generated correct solutions, participants were often limited by these strategies. The teaching experiment allowed participants to develop reasoning that supported an analytical conception of equations and solutions.

Phases of the teaching experiment. The teaching experiment unfolded in four phases. In the first phase, participants came to discriminate equations that are always true from equations that are sometimes true. During the second phase, participants used conditional language to relate non-equivalent equations. In doing so, they differentiated between a conditional statement and its converse. In the third phase, participants linked solving processes with the solution set in order to identify equivalent equations. The final phase consisted of a post-interview similar to the first task-based interview in order to confirm changes in participants' ways of reasoning.

The number of sessions in each phase was determined by the progress participants made (Table 3). When the tasks and questioning used in a session did not sufficiently develop a participant's understandings, I used my models of their understandings to adjust the tasks and attempted again to develop their ways of reasoning. Once I made adjustments for one participant, I used my revised tasks and questions with the other participants in subsequent sessions.

Table 3. *Number of Sessions in Each Phase of Teaching Experiment*

Session	**Phase**	**Participant**	**Sufficient Evidence to Move to Next Phase?**
1	I	Edgar	No
2	I	Isabel	Yes
3	I	Edgar	Yes
4	I	Harry	Yes
5	II	Isabel	Yes
6	II	Harry	Yes
7	II	Edgar	Yes
8	III	Harry	No
9	III	Edgar	Yes
10	III	Isabel	Yes
11	III	Harry	Yes
12	IV	Edgar	Yes
13	IV	Isabel	Yes
14	IV	Harry	Yes

Phase I: Distinguishing Between Universal and Conditional Statements

The purpose of the first phase was to understand how participants describe equations as true or false, and while doing so, start to develop their sense of when equations are true (i.e., the solution set). In order for this to happen, participants needed to differentiate between equations that are always true (i.e., universally) and equations are sometimes true (i.e., conditionally). In this section, I describe how Edgar and Brianna initially struggled with this idea and the intervention I used to develop their understanding. I also describe how Harry was initially able to identify equations as sometimes true but still struggled to connect the solving process to the ideas of truth value and the solution set. His struggles helped shape the subsequent phase of the teaching experiment.

Edgar (Session 1). Equations similar to $4x + 1 = 9$ are only true when x is the value of the solution obtained from the solving process. If x is a value different from the solution, then the statement is false. In other words, equations of this form are not always true, but only true when x takes on the value of the solution. However, Edgar initially described the equation as always true and never false. When asked for his reasoning, he stated "x equals to two, so I plugged in the two where the x was. Four times two equals eight, plus one equals nine. So nine would equal to nine." His assertion appeared to be based on both: i) his ritual of obtaining a solution from his solving process, and ii) his validation that both sides of the equation are the equal when the value is substituted for x.

To probe Edgar's thinking, I prompted him with similar equations and asked if they were true or false. Anticipating a response of "always true," my initial plan was to challenge him by asking, "what if x equals a different value?" For example, when asked if $4x + 2 = 22$ is true, he provided a ritualistic explanation by explaining the steps he took to obtain the solution. I asked Edgar, "What if x is four?" He responded that it would not be true, but goes on to say x cannot be four. When asked why x cannot be four, he provided a circular argument that x is not four because it makes the statement false.

Edgar clearly recognized that a value other than five makes the equation is false, and in the following discussion I attempted to have him acknowledge the statement is not always true because there are values that make the statement false.

Julius: Okay, so what would it take for this not to be true?
Edgar: Let's say, take x equals five out of the picture, saying x is equal to six or four or three.
Julius: So, if x is something besides five you would say this is false?
Edgar: Yes.
Julius: Okay. And if you say x is five then you would say this is true?
Edgar: Mm-hmm.
Julius: Okay. Maybe I asked you, sorry. Is this always true now?
Edgar: **Yes. Wait. Yes and no.**
Julius: So yes and no. So tell me what you mean yes and no.
Edgar: Like you were saying, what if it was something else, **probably be wrong, but here, it is right. Because x is equal to five.**
Julius: So if x equals to five this true, right?
Edgar: Yeah.
Julius: But I said, always true, right?
Edgar: Then yeah, it's always true.
Julius: It's always true, because x is always equal to five?
Edgar: Mm-hmm.
Julius: Even though you just said x is equal to six sometimes?
Edgar: **Then, it's always right, but not wrong.**
Julius: What do you mean always right but not wrong?
Edgar: Like I said before, if x is equal to six then, let's forget all about that, because **it's always right**. Because x always equals the five either way.
Julius: Okay, so it's always right because it's absolutely equal to five and we're just going to pretend like x can't be six?
Edgar: Mm-hmm.

This exchange revealed that although Edgar understood values other than five make the equation false, he has eliminated the possibility of those values because they are "wrong" and the equation is "always right." He aligned his response to his ritual for solving, and values that do not match his ritual are wrong and therefore not allowed.

My next step during this session was to better understand Edgar's insistence that equations are always true. I prompted him to consider whether $11 = 3x - 1$ is true or false.

He again claimed the equation is always true and repeated a similar argument that relied on the ritual of the solving process. Anticipating a similar response, I asked if 11 = 3? – 1 is true or false. However, he responded, "I don't know if it's true or false…because it's a mystery number." I probed this remark by asking if x is different from question mark. He responded, "No, it's not different. Because you don't know what x is, and they're both mystery numbers until you solve it. But then again it's different because the question mark is not a variable." There are three insights from this response. First, Edgar's view of solving equations was consistent with the notion of generalized arithmetic – that is, solving an equation reveals the unknown (single) value of x. Second, Edgar associated 'x' with his solving ritual but did not attempt to isolate '?' in a similar manner. Third, Edgar's use of the word "variable" was inconsistent with his actions. Even though he called x a variable, he treated it like an unknown (i.e., a single value); further, he said the question mark was not a variable, but treated it as such (i.e., question mark could be multiple values).

In attempt to have Edgar consider the possibility of multiple values when solving equations, I had him consider the equations 2x = 6 and 2x = 2x side by side. He once again repeated the fallacious argument that 2x = 6 is always true because x is always the result from solving and other values are not allowed because they are "wrong." When asked why 2x = 2x is always true, he responded that any value of x would make the statement true. I prompted Edgar to consider if both equations were "always true" in the same way, and in regard to 2x = 2x, he responded "I learned about this…it's an infinite solution." Yet, he was not able to explain what he meant by infinite solution. Although Edgar was able to conceive of the different solution sets for the two equations, he still claimed both equations were always true. My attempt to have Edgar identify an equation as sometimes true was not successful.

Next, I prompted Edgar to consider whether 2 + x = 2x was true or false. My intention was for the difference in operations (addition versus multiplication) to help him realize the equation is not always true. He initially declared that the equation is false, but without prompting, wrote down the equation and isolated x. He then changed his response to “always true…because x is always equal to two.” When I challenged him to consider that the equation is not always true because one side of the equation is addition and the other side is multiplication, he acknowledged the operations were different. Yet, his solving ritual took precedent in his reasoning and he once again claimed that although other values of x make the equation false, they are not allowed because they are wrong. This exchange provided further evidence that Edgar viewed the equations as generalized arithmetic. In his view, the statement 2 + x = 2x does not claim that any number plus two is equal to any number times two, but rather only concerns the specific value of two. Yet, he maintained the equation is always true.

In summary, the first session with Edgar was not successful because he continued to state that equations are always true and failed to identify certain equations as only sometimes true. His progress was inhibited by two factors: his reliance on his solving ritual, and his view of equations as generalized arithmetic. He understood that replacing x with a non-solution would make the equation false. However, he believed that doing so is “wrong” and “not allowed.” In order for Edgar to identify equations are sometimes true, his conception of equations needed to develop so that both solutions and non-solutions can be substituted into the equation.

Isabel (Session 2). I started my first session with Isabel as I did with Edgar. I asked her if 4x + 1 = 9 is true or false and she, too, stated that it is always true because, "if you solve it, then you would get x equals two, and it's true, because if you plug two back in, it would equal nine." Also, similar to Edgar, she treated 11 = 3x – 1 and 11 = 3? – 1 differently. For Isabel, the equation with x evoked a conception of generalized arithmetic and as a result she said the equation was always true. The question mark, however, was treated as a variable and consequently she stated the equation was sometimes true.

Julius: Okay. And then eleven equals three times some mystery number minus one. Is that true or false?
Isabel: **It depends on what the number is.**
Julius: So tell me what you mean, it depends on what the number is.
Isabel: **If you put 167, then no, it's not going to be true.** But if you put in four like before, then it would be true.
Julius: So you would say this equation as always true?
Isabel: No.
Julius: Sometimes true?
Isabel: Yeah.
Julius: Okay. So now it's 11 = 3x - 1. Would you say this equation is true or false?
Isabel: Well...it's true.
Julius: Is it always true?
Isabel: Yeah.
Julius: Okay. So when you look at the question mark and you look at the x, here you told me that it's sometimes true, it depends, but here now it has to be four.
Isabel: Yeah, because **x represents a number and the question mark could represent anything.**
Julius: Okay. So x doesn't represent anything, it represents one specific number?
Isabel: Yeah.
Julius: Okay. And if one of your classmates said, 'But I thought x was a variable. It could be whatever you want.'
Isabel: Well, no, because **when it's in that equation, x is equal to the number, because when you solve it, x is only equal to one thing.**

Isabel employed the notion of a generic example ("If you put in 167…") to explain that the equation with a question mark is not always true. However, for a typical equation with an x, she changed her response to always true. Isabel, like Edgar, was constrained by her solving ritual and view of equations as generalized arithmetic.

To have Isabel reason about equations with an x in the same manner she reasoned about equations with a question mark, I introduced a task with the goal of having her treat expressions with x as functions that generate multiple values. Specifically, I presented the expression "2x" and asked, "How many numbers do you see?" Isabel stated that there are two numbers, and "after you solve it," there is only one number. She misused the word "solve" and instead described the process of multiplying the two numbers to yield the product. When asked if she knew which number the product is, Isabel stated, "No, because it's not an equation, it's just by itself." Additionally, she provided a few examples of what the product could potentially be, and, crucially, indicated the list of possibilities was infinite (e.g., "It could be any number.").

Now that Isabel demonstrated her ability to treat an expression with x as a function that generates multiple values, I set out to have her adopt this perspective while reasoning about equations. I used the expressions x – 4 and 5x and, by asking how many numbers she saw, elicited that each expression could be considered either two numbers (i.e., the process of subtraction or multiplication), one number (i.e., the difference or product), or an infinite number of numbers (i.e., the set of numbers generated by treating each expression as a function). After Isabel indicated that each expression could represent an infinite number of numbers, I asked, "I have x – 4 and 5x, and I want to know if this number [x – 4] and this number [5x] can represent different numbers." Isabel stated that they can be different and provided the example of x = 10 to show the expressions can have different values.

Next, I provided the expressions x – 4 and 5x side by side and asked Isabel if she thought the expressions could ever represent the same number. She responded, "No, I don't think so." More importantly, after being given time to think, she did not realize that setting up and solving the equation x – 4 = 5x would provide an answer to the question. The disconnect between the process of solving equations and finding the value of x that makes two expressions equal suggested her understanding of the solution set was still developing. I then prompted her to consider negative one for the value of x, and she proceeded to evaluate both expressions. Upon seeing that the two expressions have the same value, she stated that the two expressions are, "Sometimes the same and [sometimes] different."

With the goal of having Isabel connect the solving process to the truth value of an equation, I inserted an equals sign in between the expressions x – 4 and 5x to produce the equation x – 4 = 5x. Isabel, without any prompting, immediately began to solve to equation. Seeing the equation was a prompt for her to execute the procedure of isolating x, even though I did not ask a question. Furthermore, after obtaining x = -1, she appeared surprised and said, "It's the same…oh wait, x is negative one, but x was negative one in this one, too." She went on to articulate that x – 4 = 5x is sometimes true because, "They're [both sides] are equal to each other." This marked a shift in Isabel's behavior from the beginning of the session where she described 4x + 1 = 9 and 11 = 3x – 1 as always true. Additionally, Isabel stated the equation is sometimes false because, "Everything except negative one would be wrong." I clarified that wrong and false meant the same thing to her. Her use of “wrong” to describe non-solutions was similar to Edgar’s description of why equations are always true.

For both Edgar and Isabel, their rituals of solving equations influenced the way they determined the truth value of equations. Isabel, however, moved toward a perspective in which certain equations are sometimes true. The intervention of asking Isabel how many numbers she saw when looking at an expression allowed Isabel to adopt a functional view of algebra (as opposed to the view of generalized arithmetic) and state that some equations are sometimes true.

To ensure Isabel’s conception had evolved from seeing certain equations as always true to only sometimes true, I prompted her with a few follow-up tasks. First, for $3x - 2 = 2x$, she stated the equation was sometimes false because, “They’re not the same when you plug in 10.” She also stated that the equation was sometimes true and in order to find out when she would “have to solve this without plugging in the ten.” Her decision to solve the equation indicated she saw the solving process as connected to the task of determining when the equation was true. Second, I returned to the task of determining if $4x - 1 = 9$ is true or false. In our exchange below, Isabel offered a response different from her original interpretation.

Julius: Okay. So let’s go back to where we started. So I’ll ask you again, when I say true or false?
Isabel: Oh, yeah. True when x is two.
Julius: Okay.
Isabel: But sometimes false.
Julius: When?
Isabel: When x isn’t two.

Third, I prompted Isabel to determine whether $2 + x = 2x$ is true or false. She initially indicated the equation is false, but given time to determine if she was certain, offered that solving the equation would produce a value that makes the equation sometimes true. Last, I asked Isabel to create a true equation, and she created 2x = 2. When asked if her equation is always true, she responded, “I didn’t make one that was always true,” and went on to create 6 = 6 as an example of an equation that is always true.

Edgar (Session 3). During Edgar’s first session, he continued to state that equations similar to 4x – 1 = 9 are always true because the solution is the only value that is allowed to be substituted for x. During the previous session with Isabel, she also initially stated equations are always true using a similar argument. However, Isabel was able to move past her initial conception of equations as rituals and generalized arithmetic and instead adopt the view of functional algebra when determining if equations are true or false. By prompting Isabel to consider that an expression can be viewed as a function that generates an infinite number of values, she stated that certain equations are sometimes (not always) true at the conclusion of the session. The goal of the third session was to determine if a similar approach would develop Edgar’s understanding as well.

I started the session with Edgar by confirming my previous model of his understanding – a reliance on his solving ritual and a view of equations as generalized arithmetic. I again asked if 11 = 3x – 1 and 11 = 3? – 1 were true or false. 11 = 3x – 1 is always true, he said, “Because I solved the equation and x would be equal to four.” He viewed 11 = 3? – 1, however, as neither true nor false “…because I don’t know what the number is.” Edgar treated the question mark like a variable that can be replaced by any number, and as a result said it was not possible for the equation to be always true or always false. In addition, he offered the unsolicited comment that 11 = 3x – 1 was not the way he was taught and preferred 3x – 1 = 11. However, he said that the way it was written did not change his view that each equation is always true. This remark offered further evidence of the role of his solving ritual ("I wasn't taught that way."). In addition, this comment suggested, at least in this instance, an operational view of the equals sign in which 11 represents the result. This view was consistent with my understanding that he viewed equations as generalized arithmetic – that is, a process on a single, unknown number that produced a result of 11.

After confirming that Edgar's reasoning matched his responses from the first session, I introduced the tasks I used with Isabel to have him understand that equations of the form 3x – 1 = 11 are sometimes, and not always, true. I first asked him how many numbers he saw in the expression "2x." He initially responded that x is not a number, but "it's soon turned into a number – depends on the equation." This comment again supported my understanding that Edgar viewed expressions and equations as generalized arithmetic. To develop his view into one that allowed for expressions to be treated as functions, I prompted him to start a list of all the ways he can multiply two by another number (i.e., the two times table). This allowed him to see that the product – a single value – results from the process of multiplying two values. Furthermore, he stated that this list of possibilities is infinite and that 2x represented this set of possibilities. To confirm this view, I repeated the process with the expression "x + 5" and he once again considered the expression either two numbers (i.e., the process of addition), one number (i.e., the sum), or an infinite number of numbers (i.e., the set of numbers generated by treating the expression as a function).

Next, I prompted Edgar to separately consider the expressions "x – 4" and "5x." For each expression, he again stated the expression was, "…two numbers, [and] you combine those two numbers it would turn into one number," as well as an infinite list of possibilities. With this understanding in place, I prompted Edgar to consider if the two expressions could ever have different values. He offered ten as an example to show the expressions are not necessarily the same. In the discussion below, Edgar considered if the two expressions can ever be the same.

Julius: Okay. Do you think they ever are the same?
Edgar: No.
Julius: No. So what are you doing in your head just then?
Edgar: Because **I was trying to replace a number to see if they would both be the same.**
Julius: How many numbers did you try? Or what numbers were you trying in your head?
Edgar: I was trying five, I was trying one, but none of them seemed to match up.
Julius: Okay. So you tried a couple numbers and are you sure that there's no number or you just sort of gave up?
Edgar: **I sort of gave up.**
Julius: Okay. So the reason I'm asking you is because on a scale of one to five, are you absolutely sure there's no number? Or it's just your instinct?
Edgar: My instinct.
Julius: Okay. And so you don't think there are any time they're the same but you're not sure?
Edgar: **Yeah, I'm not sure.**
Julius: Okay. And there's no way that's coming to your mind of how you could figure this out or you could be sure?
Edgar: **Wait. Well, yeah. I could figure out one way. If I'd make that into an equation.**
Julius: Okay, show me what you mean.
Edgar: It'd be negative one. Wait. Yeah. It would be negative one…because like I would replace this with negative one, it'd be negative five, then ... Yeah. Yeah. It'd be negative one.

There are two noteworthy events during this exchange. First, Edgar's initial decision to search for the value that makes the expressions equal and his resulting skepticism when he did not easily find the value he searched for. While he did not gain absolute certainty and therefore did not employ an empirical proof scheme, the instance still illustrated the use of an empirical strategy. Second, he spontaneously – without my prompting –

realized that solving the associated equation would determine when the two expressions have the same value.

As was the case for Isabel, linking the solving process to the task of determining when two expressions have the same value allowed Edgar to state $x - 4 = 5x$ is sometimes true and sometimes false. Additionally, this task marked a shift in Edgar's language when talking about equations. First, when asked about the significance of x equals negative one, he said, "That there is *some* [emphasis] number that makes them similar [the same]." The use of the quantifier, "some," was further evidence that Edgar now saw the equations as sometimes true. In addition, he used conditional language for the first time when he said, "Well, what I'm trying to say is that *if* [emphasis] x doesn't equal to negative one, then it would be false. But *if* [emphasis] x is equal to negative one, yeah, it would be true." To confirm this shift in reasoning and language, I asked Edgar if three more equations were true or false: $3x + 2 = 2x$, $4x + 1 = 9$, and $2 + x = 2x$. He again stated they are sometimes true and sometimes false and responded in a similar manner (e.g., "It depends if x doesn't equal two.").

Harry (Session 4). The purpose of Harry's first session was to determine his conception of the solution set and, in particular, how he related it to the solving process. Linking the solving process to the solution set and truth value of an equation requires recognizing that certain equations are sometimes, and not always, true. Harry, unlike Edgar and Isabel, stated from the outset that $4x + 1 = 9$ is sometimes true and sometimes false. As a result, there were three goals for this session. First, confirm Harry's view that equations similar to $4x + 1 = 9$ are sometimes, and not always, true. Second, determine if he recognized equations that are always true and differentiated them from equations that are sometimes true. Third, build a model of Harry's conception of the solution set that took into account his ability to discriminate equations that are always true from equations that are sometimes true.

I started the session with the goal of confirming Harry's view that certain equations are sometimes true and sometimes false. The following exchange highlights his understanding:

Julius: 11 = 3x – 1. Would you say this is true or false?
Harry: **It'd be true. The answer is four.**
Julius: Okay. Is it always true?
Harry: Only when four is plugged in as x.
Julius: Okay. So, if I said this is always true, how would you evaluate my claim that this is always true?
Harry: **Then you'd have to plug in any number and it would have to work every time, if it's always true.**
Julius: And is that the case?
Harry: No.

Harry was able to state that the equation is sometimes true and explain it is not always true because it would need to "work every time." In addition, he referred to the solution as "the answer." This suggested that, although he understood the equation is sometimes true, his notion of solution is still connected to his solving ritual. Harry made similar comments (e.g., "You have to subtract the one to get your answer.") when justifying other equations were also sometimes true.

Given Harry's ability to recognize certain equations are not always true, I proceeded to determine if he could recognize equations that are always true. When prompted to consider if 2x = 2x is true or false, he said it was always true because, "If you plug in, the numbers have to be the same for them to be true. So, if you plug into 2x, x equals four for both sides, and then they both equal eight." I followed up and asked why x equals four and he replied, "No, it's just a random number…every number works." His use of four to show the equation is always true is consistent with the notion of a generic example – four is a representative for all the possible values that could be substituted in.

At this point in the session, Harry had demonstrated that certain equations are conditionally true and 2x = 2x is universally true. I wanted to see if he would also describe an equation as universally false. I asked him what it meant for an equation to be false and he said, "Both sides are different. They're not the same." I followed up and asked what it meant for an equation to always be false, and he replied, "No number will ever work. It will *always* [emphasis added] be a different value on each side." Next, I prompted Harry to create equations that are true and he offered 6x – 2 = 10 as "sometimes true, not always," and 10x = 10x as "always true because no matter what number you put in [for x]…it's the same number [on each side]."

I aimed to further understand Harry's conception of the solution set and the extent to which he related the solving process to the truth value of equations. I prompted him to consider whether 2 + x = 2x is true or false. Even though Harry previously differentiated between equations that are sometimes true and always false, the following discussion illustrates that his reasoning was not always connected to the solving process.

Harry: It's not true.
Julius: Is it ever true?
Harry: No. Because the different operations, you're going to get a different value every time.
Julius: So you would say this is always false?
Harry: Yeah.
Julius: Because there's different operations?
Harry: Different operations. So you get a different kind of value every time you put in a number.

Julius: And so you're saying always, for all these numbers? Let me ask you this. How sure are you on a scale of one to five that there's no other number?
Harry: Five.
Julius: So you're sure?
Harry: Yeah.
Julius: But why are you so sure? What have you done to be sure?
Harry: **Because simple numbers are always a good example to use, like five, two, ten.**
Julius: So when those don't work, you don't feel the need to test other ones because they don't work?
Harry: Yeah.
Julius: Okay. And do you usually do that, test those simple numbers?
Harry: **Yeah, I usually test those simple numbers that would make the equation work.**
Julius: So, you're a hundred percent sure there's no other number?
Harry: Yeah.

Although Harry understood what it meant for an equation to be sometimes or always true, this exchange critically captured that he employed, at least on some occasions, an empirical proof scheme – and not the solving process – to determine whether the equation was sometimes true. To support my model of his understanding, I prompted him with $2x = 6x$. Again, he did not solve the equation but instead used an empirical strategy to determine the equation was sometimes true.

To summarize this session, Harry initially could discriminate equations that are sometimes true from equations that are always true. While he connected the ideas of solution and the truth value, he did not appear to completely connect the solving process to the task of determining if an equation is sometimes true. Harry sometimes used an empirical proof scheme in which he searched for values that made an equation true. When he failed to find a value, he erroneously claimed the equation was never true. At this point, Harry's conception of truth value and his solving ritual were somewhat disparate ideas.

Summary of Phase I. The objective of the first phase was to determine how participants related the solving process to the ideas of solution and truth value. Whether or not participants could discriminate equations that are always true from equations that are sometimes true was of particular interest. Edgar and Isabel initially used a spurious argument to claim equations are always true because x can only take on the value of the solution produced by the solving ritual. A functional view of expressions allowed Edgar and Isabel to understand equations with a single solution are sometimes, and not always, true. Harry initially understood that certain equations are sometimes true but did not necessarily connect his solving process to the truth value of equations. At the end of the first phase, all three participants were able to differentiate between the notions of always true and sometimes true. However, their conceptions of truth value and solution did not extend to equations generated during the solving process; they did not use the solution as an object to relate equations.

Phase II: Using the Implication Structure to Relate Equations

All three participants responded similarly to the tasks I used during this phase. For the convenience of the reader, I present the aggregate results of this phase. That is, even though the three sessions of this phase were conducted with each participant individually, I describe the results of their progress together. All three participants reasoned about the same five equations (Table 4).

Table 4. *Prompts for Phase II of Teaching Experiment*

Task	Equation
#1	$x^2 = 4$
#2	$2x = 4$
#3	$x^2 = 2x$
#4	$x^3 = 4x$
#5	$4x + 2 = 6$

A solution set for every equation. When asked to reason about the solution for $x^2 = 4$, each participant immediately stated two was the solution. Further, their solution only took the form of the value of two, and none of the participants said or wrote the equation $x = 2$. When prompted to write down the equation $x = 2$, Edgar asked, “Are we doing a separate equation?” In the previous phase and at the start of this phase, it appeared that participants did not naturally consider the truth value of more than one equation at a time.

To have participants consider the truth value of multiple equations, I had participants write out their solution as another equation under each initial equation (e.g., $x = 2$ under $x^2 = 4$). I then proceeded to phrase questions in the form of conditional statements. For example, I asked, “If $x = 2$ is true, is $x^2 = 4$?” I always followed up by rephrasing the question as the converse of my original question (e.g., if $x^2 = 4$ is true, is $x = 2$?). When participants failed to identify all the solutions in a solution set, I prompted them to consider the missing solution(s). For example, after it became clear that each of the participants did not see negative two as a solution to $x^2 = 4$, I said, “What if x is negative two?” After they acknowledged the missing solution, I repeated my questions phrased as conditional statements. This led participants to keep track of the different solution sets for each equation. I encouraged participants to write down their thought process and show their work; they started to list the solutions next to each equation. As an example, Edgar's work for $x^2 = 2x$ is displayed in Figure 11.

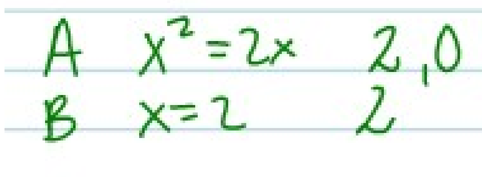

Figure 11. Edgar wrote out each solution set in Task 3

I repeated this process for each task, and when a task produced intermediate equations, I asked participants to relate those as well. For example, I prompted participants to relate $x^3 = 4x$ to $x^2 = 4$, and they again made use of the solution set in response to my questions regarding the truth value of the equations. Isabel's work is displayed in Figure 12.

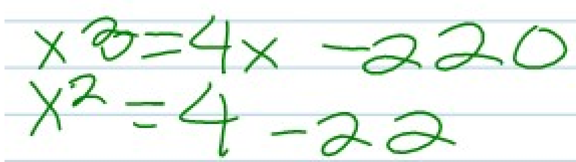

Figure 12. Isabel wrote out each solution set in Task 4

My line of questioning allowed participants not only to recognize that each equation has a solution set, but solution sets can be compared and contrasted.

Fluency with conditional language. In order to adopt the implication structure when relating equations, participants needed to interpret and use conditional language. I was initially unsure if they would be able to do this; I was particularly wary of their ability to distinguish a conditional statement from its converse. To my surprise, all three participants were generally able to interpret and use conditional language when relating equations. The following exchange with Harry illustrates the ease with which participants used conditional language.

> Julius: I'm saying this is true, x^3 equals 4x is true, if this is true, can you say that x^2 equals 4?
> Harry: Not anymore.
> Julius: Why not?
> Harry: If 0 was the value, it wouldn't work.
> Julius: If I walk in the room and I say x^3 equals 4x is true, you wouldn't say that x^2 equals 4?
> Harry: No, I wouldn't, because it could be 0.
> Julius: If I walk in the room and say x^2 equals 4 is true, if I say this [pointing to $x^2 = 4$] is the thing that's true, would you feel comfortable saying A [$x^3 = 4x$] is definitely true?
> Harry: Yeah.
> Julius: Why?
> Harry: Because 2 and -2 would work for both, and they'd both be true.

This exchange demonstrates that Harry's ability to distinguish between a conditional statement and its converse. Additionally, Edgar created his own terminology and began to refer to this distinction as "the reverse."

Implication statements are only considered false if the premise of the statement is true and the conclusion of the statement is false. Statements with a false premise are considered true, regardless of whether or not the conclusion of the statement is true or false. On two occasions, I observed participants conflate a false statement and a true statement with a false conclusion. For example, I asked Edgar, "If $x^2 = 4$ is true, is it possible that x is anything else besides two?" He offered that x could be five. Similarly, on one occasion, Harry offered a value that was not a counterexample, but rather, made both equations false. When this occurred, I asked Edgar and Harry if the equation in the premise was true, and this allowed them to change course. Specifically, Harry modified his response to include the phrase "in this instance" to signal that his conclusion was based on the truth value of another equation.

Empirical strategies. Participants used conditional language to correctly relate two solution sets they had described. However, participants sometimes articulated a solution set that did not match the actual solution set for a particular equation; it was often the case that they reasoned with incomplete solution sets. This was a consequence of their reliance on empirical strategies whereby they would search for solutions and, after a certain number of attempts, declare there were no more solutions. These strategies often resulted in participants failing to identify one or more solutions. Even when participants had successfully identified all solutions, they would often continue to test additional values before deciding the solution set was complete.

Edgar articulated his modus operandi for constructing a solution set while working on Task 4 (Figure 13). He said, "I would usually do a range from zero to ten and if none of the numbers work except for one or two, then yeah, those numbers are the only ones that can make it true." He went on to add that this is sufficient information for him to be convinced.

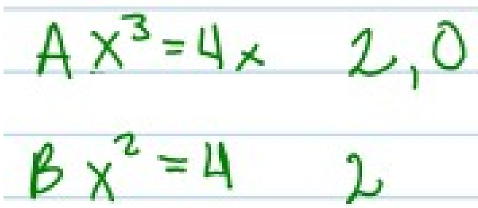

Figure 13. Edgar's empirically-constructed solution sets for Task 4

Edgar's strategy limited him to non-negative solutions and he failed to identify negative two as a solution as a result. However, it is important to note that his conditional reasoning was correct. While reasoning about Task 4, Edgar stated if $x^2 = 4$ is true, then $x^3 = 4x$ is also true. Even though he failed to completely identify the solution sets, his reasoning was correct given the solution sets he constructed in Figure 13.

Harry and Isabel were also limited by the empirical strategies they employed and as a result often reached erroneous conclusions as well. For example, Harry only identified zero as the solution for $x^3 = 4x$. When I asked him why he was sure zero was the only member of the solution set, he responded, "I couldn't come up with any other numbers that work for this." Similarly, Isabel did not identify zero as a solution when reasoning about $x^2 = 2x$. When I asked how she knew two was the solution set, she responded, "I plugged two back in and it works." As was the case for Edgar, Harry and Isabel arrived at incorrect conclusions, not based on faulty logic, but because of their incomplete, empirically-constructed solution sets.

It occurred to me that participants' reliance on empirical strategies might be a result of their unfamiliarity with quadratic and cubic equations. To see if this was the case, I prompted them reason with two equations they certainly were familiar with:

$2x = 4$ and $4x + 2 = 6$. While participants were able to identify the correct solution sets for these routine equations, it was a result of empirical strategies. For example, all three participants verified negative two was not a solution to $2x = 4$ before claiming they were certain that $2x = 4$ implied $x = 2$. For $4x + 2 = 6$, the last task, Harry verified that none of the previous solutions to other equations were also solutions. Despite his reliance on verifying many solutions, his conditional reasoning was still correct. He articulated that $4x + 2 = 6$ and $x = 1$ implied each other and observed that “they’re true together.”

It is important to note that participants did not necessarily employ an empirical proof scheme every time they employed an empirical strategy. While it was often the case that participants were absolutely convinced by their empirical verification of potential solutions, there were occasions they were skeptical that they had identified the complete solution set. Generally, Edgar and Harry proceeded to verify additional values in order to reach absolute conviction (i.e., used an empirical proof scheme in the end). At a certain point, however, it became clear Isabel realized that empirically searching for and verifying potential solutions was not sufficient to guarantee a complete solution set. She described this strategy as unreliable and said, “You don’t know how long you would have to keep going,” and added she would have to keep searching because there might be additional solutions. This appeared to be a promising development. However, when determining whether $4x + 2 = 6$ implies $x = 1$, Isabel stated there might an additional solution and extended her skepticism to familiar, single-solution equations. This reveled that, at least in this instance, Isabel had no method to identify the complete solution set with absolute certainty.

Summary of Phase II. During this phase, participants extended their developing notion of the solution set to the group of related equations generated during the solving process. To relate equations, all three participants adopted the implication structure and used conditional language (e.g., if $x = 2$, then $x^2 = 4$). While participants fluently related two equations by relating two sets, they often used incorrect sets in their reasoning. All three participants often reached erroneous conclusions, not because of their logic, but because they relied on empirical methods – not solving operations – to construct their image of a solution set.

Phase III: Linking Process and Object

The third phase consisted of four sessions, and I present the results of this phase in three sections. First, I describe Session 8. During this session, Harry continued to rely on empirical strategies when reasoning about the solution set. In the second section, I describe my new tasks and questions in an attempt to adapt to Harry's responses. In the third section, I describe the aggregated results from using my new tasks with all three participants. Just as was the case in the previous phase, all sessions were conducted individually, but I present the results from the three sessions (9, 10, and 11) together because all three participants reasoned about the same tasks and responded similarly.

Harry (Session 8). In Harry's previous session, he used the implication structure to relate two equations vis-à-vis their solution sets. However, he often reached incorrect conclusions because he reasoned with incomplete sets – sets not obtained through solving operations, but by searching for and verifying solutions. My main objective in this session was to elicit evidence Harry could anticipate changes in, or the preservation of, the solution set by relating the solution set to solving operations. In the event I could not to do this, I intended to further develop my model of Harry's understandings to describe why he relied on empirical evidence and not solving operations when constructing solution sets.

Preservation of the solution set. In earlier sessions, Edgar and Isabel responded differently when I presented them equations with a question mark instead of an x. At the start of this session with Harry, I decided to do the same in an attempt to elicit a different response from him. Further, I replaced some of the expressions in the equations with a figure of a box and an oval. The purpose was to allow Harry to relate the truth value of two equations without substituting a value into the equations. Harry's related work with the equation $4x + 1 = 9$ from Session 4 is shown in Figure 14. The first two tasks I used with Harry in this session are shown in Figure 15 and Figure 16.

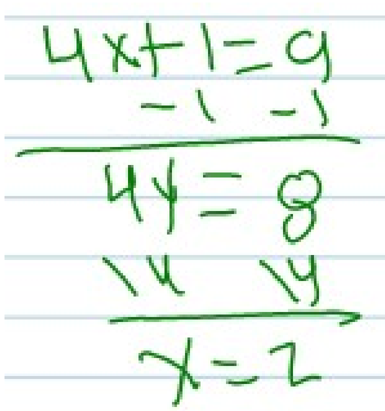

Figure 14. Harry's work solving $4x + 1 = 9$ from Session 4

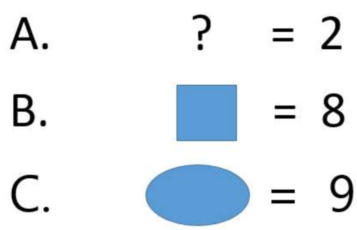

Figure 15. Task 6 from Session 8

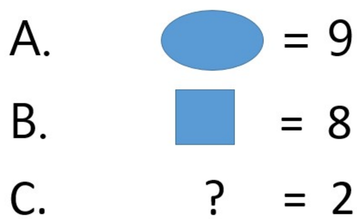

Figure 16. Task 7 from Session 8

I presented the two similar tasks for two reasons. First, I wanted to gather additional evidence that Harry could distinguish a conditional statement from its converse. Second, I was interested to see if Harry would use different operations to relate two equations. For example, if he were to read down, equation A and equation B in Task 6 are related by multiplication, whereas those same equations, reading down, are related by division in Task 7. I thought these tasks might give Harry the opportunity to connect reversible operations with the fact that two equations could be linked by a true conditional statement and a true converse (i.e., a true biconditional statement).

While reasoning about the truth value of the equations in the two tasks, Harry attended to the direction of his implication statements (i.e., avoided conflating a conditional statement with its converse). More significantly, he linked each pair of equations with two conditional statements in both tasks. For example, he said, "If four [times] question mark is eight, then the question mark will always be two. If the question mark is two, then four times question mark equals eight, then that will always be true." However, Harry's responses from Task 6 and Task 7 did not conclusively support my second objective. One hand, his responses suggested that he could link equations and the preservation of the solution set to the solving operations. On the other hand, it appeared the he employed an empirical strategy and continued to verify solutions.

Harry made a number of comments where he related the truth value of two equations through operations and did not substitute values. When relating equations A and B in Task 7, he said the oval is not always nine, but it is "if you'd find a box in the oval and the box is equal to eight, and…[also] a value of one, so eight plus one, that would make nine." I specifically asked him if he needed to know the value of the question mark in order to be sure, and he replied that he did not because "eight is the only number that adds with one that equals nine." He provided a similar response after rewriting Task 7 (Figure 17).

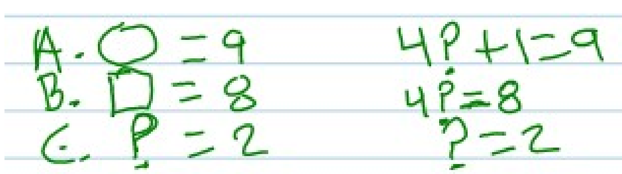

Figure 17. Harry's work for Task 7

I asked Harry to relate 4? + 1 = 9 and 4? = 8 and he said, “Even if you don’t know the value [of ?], if you know that 4? is eight… that’s the only number you would add by one to get nine.” He reversed his argument to correctly describe the converse: “4? = 8 does not have the plus one, it has to be one less [than 4? + 1 = 9].” Taken together, these comments provided some evidence that Harry could use operations to relate the truth value of equations.

Despite Harry’s comments that suggested he could reason about a solution set by relating one equation to another, he made a number of additional comments that suggested he still used an empirical strategy to identify a solution set. The following exchange highlights two comments that made me question whether or not he was in fact still relying on a strategy of searching for and verifying solutions.

> Julius: Tell me how, if 4? + 1 = 9 is true, why are you sure the other things are true?
> Harry: If you figure out the value [of question mark], it all really depends on the value of question mark.
> Julius: And if one [equation] is false, why are you sure that the rest would have to be false?
> Harry: Let’s say question mark is actually equal to three…they’d all be false.
> Julius: Is there any reason you chose three?
> Harry: It’s just a random number.
> Julius: So you could have chosen any other number?
> Harry: Yeah.

This exchange alone was not sufficient to conclude Harry used an empirical strategy. His comment that “it all really depends on the value [of question mark]” could simply indicate his understanding that the solution is what makes a statement true. Similarly, when he substituted in a value of three, it could be that three is a generic example (i.e., a representative of the set of non-solutions). However, there were other instances throughout the course of the session when his justifications seemed to include an empirical strategy (e.g., “They’re both true if you plug in two.”).

Changes in the solution set. During the second half of this session, I set out to determine if Harry could relate the truth value of two non-equivalent equations without substituting a value into the equations. I was specifically interested in his ability to identify whether a solution of zero was gained or lost while transforming one equation into another through the multiplication or division of both sides of an equation by x. To accomplish this, I asked Harry to reason about $x^2 = 5x$ (Task 9) and the equations in Task 10 (Figure 18).

A. $x^3 = 9x$

B. $x^2 = 9$

C. $x = -3, 3$

Figure 18. Task 10 from Session 8

For this session, my objective in using these tasks was to elicit evidence that Harry could use solving operations, and not an empirical strategy, to identify changes in the solution set.

Before using Tasks 9 and 10 with Harry, I attempted to have him connect changes in the solution set with the multiplication of both sides of an equation by zero. In Task 8, I prompted him to multiply both sides of a true equation (2 = 2) as well as a false equation (1 = 2). He first multiplied both sides by a number of his choosing, and then I prompted him to multiply both sides by zero. His resulting equations are shown in Figure 19.

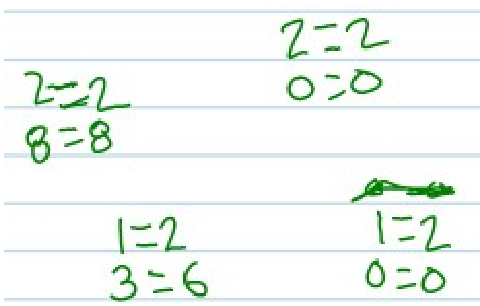

Figure 19. Harry's work from Task 8

For each pair of equations, I asked Harry to describe the truth value of each equation. I then asked him if any pair seemed different than the others. He identified 1 = 2 and 0 = 0 and explained that the false equation (1 = 2) became a true equation (0 = 0). When I asked him why he thought that was the case, he responded, "Well zero is kind of a special number. If you multiply anything by zero, you would always get zero equals zero." Next, I proceeded to have Harry reason about the equations in Tasks 9 and 10.

Similar to Harry's responses earlier in the session, Tasks 9 and 10 elicited some evidence that Harry could indeed relate the solution sets of two equations to the operation that linked the respective equations. The following exchange describes Harry's reasoning about $x^2 = 5x$ and $x = 5$ in Task 9.

Julius: You said you're sure, and why are you sure?
Harry: If x equals zero on Equation A [$x^2 = 5x$], then that makes B [$x = 5$] false. If you plug in x to A [$x^2 = 5x$], zero squared is zero and five times zero is zero. Then for B [$x = 5$], if you plug in zero instead of x, zero does not equal to five.
Julius: Okay. Where do you think the zero comes from?
Harry: Since anything that's multiplied by zero is going to be zero, **A has multiplication involved in it, and B does not.**
Julius: Say again, what's the connection here between A and B?
Harry: A involved multiplication and B does not.

This exchange captured three noteworthy moments: i) Harry identified zero as a solution to one equation but not the other, ii) Harry correctly articulated the relationship between the equations using the implication structure, and iii) Harry appeared to identify a solution of zero as a result of relating the equations with multiplication. While Harry's responses to this task gave me the impression he had constructed his solution set through analytical – not empirical – reasoning, I wanted to confirm my new model of Harry's understandings with another task.

I prompted Harry to reason about the three equations in Task 10. By providing the statement $x = -3, 3$, I started the construction of the solution set for him and intended to see if he would consider other solutions or simply verify the two solutions presented to him. Instead of relating the equations with operations, as he did in the previous task, he resorted to an empirical strategy and simply verified the two solutions. As a result, he did not identify zero as a solution to $x^3 = 9x$. The following exchange conveys how his empirical strategy led him to his incorrect conclusion:

Julius: Okay. If one of these equations is true, are the other equations true?
Harry: Yes.
Julius: How do you know?
Harry: Because well, if you use B [$x^2 = 9$] for example. If x is either three or negative three, that makes the equation nine. If that's right, it would work for A [$x^3 = 9x$] and C [$x = -3, 3$].

Julius: Yeah, so when you said 'right,' I think you meant if $x^2 = 9$ is true, you would say A is true, right?
Harry: Yeah.
Julius: Okay, so if B [$x^2 = 9$] is true then A [$x^3 = 9x$] is true. How did you know if B [$x^2 = 9$] is true then A [$x^3 = 9x$] is true?
Harry: **Because if you plug in negative three or three for A [$x^3 = 9x$]**, it will always equal 27, which is what it's equal to.
Julius: Okay. What if A is true?
Harry: That makes B true. And C.
Julius: Always?
Harry: Always.
Julius: Okay. Again, how do you know that A [$x^3 = 9x$] makes the other ones [equations] true?
Harry: Because, they both, **the value of x both works for either equation.**

The task did not provide additional evidence supporting my model of Harry's ability to relate equations and solution sets through operations. Instead, the exchange revealed that Harry's reliance on the verification of solutions prevented him from identifying the complete solution set for $x^3 = 9x$. In addition, I prompted Harry to consider zero as a solution once he conveyed he was convinced there were no additional solutions. Upon realizing he missed a solution, Harry tested other values to ensure there were no additional solutions.

In summary, Harry sometimes articulated reasoning that suggested he could relate equations and their respective solution sets with operations. However, he did not consistently reason in this manner and instead resorted to the empirical verification of solutions. This led him to produce incomplete solution sets and reach incorrect conclusions. At the end of the session, my new model of Harry's understanding was that although he sometimes linked equations with operations, the appeal of empirical verification often took precedent in his reasoning.

Revised tasks. In this next section, I describe the sequence of three tasks that I developed in response to Harry's reliance on empirical verification when relating equations. The tasks still had the primary objective of eliciting evidence that participants could anticipate the preservation of, or any changes in, the solution set by relating the solution set to solving operations. I hypothesized that I could elicit evidence of analytical reasoning by providing pairs of equivalent equations that had two features: i) an easily identifiable scale factor between the two equations (e.g., $2x = 5$ and $4x = 10$), and ii) a non-integer solution that would be onerous to verify with substitution (participants did not have access to a calculator during the tasks). In Session 8, even when it appeared Harry related equations analytically, he often resorted to empirical verification. The purpose of using equations with solutions that would be difficult to verify was to isolate participants' ability to reason with non-empirical strategies.

Task 11. This new task centered on the equation $11x = 50$. First, I asked participants if the equation was true or false. The solution to this equation is an unfamiliar fraction and a repeating decimal. I hypothesized participants would not attempt to substitute this value into the original equation to verify the solution. Second, I presented the equations $11x = 50$ and $22x = 100$ together and asked if they were true or false. The two equations are equivalent and share a common solution. My intention was for participants to recognize that one equation implies the other – by stating that $11x = 50$ and $22x = 100$ are related by multiplication – and not by substituting the common solution into the equations.

For the next part of the task, I presented a set of equations equivalent to $11x = 50$ (Figure 20) and again asked if they were true or false.

$$11x = 50$$

$$22x = 100$$

$$33x = 150$$

$$44x = 200$$

$$55x = 250$$

$$110x = 500$$

Figure 20. A set of equivalent equations used in Task 11

The purpose of presenting the set of equations together was to see if participants would reason that equivalent equations can be related by factors other than two. In addition, I thought the equations would allow participants to relate equations to each other, not just to $11x = 50$. For example, $110x = 500$ is not only equivalent to $11x = 50$, but to $55x = 250$ as well. As was the case with only $11x = 50$ and $22x = 100$, the intention was to elicit reasoning that conveyed these equations had the same solution, not because a value was substituted and verified, but because the equations can be related with multiplication or division.

For the last part of Task 11, I aimed to see if participants would correctly generalize that $11x = 50$ can be multiplied by any non-zero factor to generate an equivalent equation. To accomplish this, I presented $11x = 50$ and $11x^2 = 50x$ together and asked if they were true or false. I specifically sought evidence that participants saw $11x^2 = 50x$ as related to $11x = 50$ through the multiplication of both sides by x. In addition, I was interested in determining if participants would anticipate $11x^2 = 50x$ gaining a solution of zero. In other words, although $11x = 50$ implies $11x^2 = 50x$, I intended to determine if participants recognized that $11x^2 = 50x$ does not necessarily imply $11x = 50$. Critically, I wanted to see if participants attributed the change in the solution set to the multiplication of both sides of $11x = 50$ by x, or if they again resorted to an empirical strategy of verifying zero.

Task 12. The purpose of Task 12 was twofold. First, I aimed to gather additional evidence that participants could relate equations and their respective solution sets through operations and without verifying solutions. Second, I wanted to see if participants would recognize that multiplying both sides of an equation by x does not necessarily result in a change in the solution set. Figure 21 shows the set of equations I used for this task.

	True	False
$2x = 5$		
$4x = 10$		
$40x = 100$		
$2x^2 = 5x$		
$2x^3 = 5x^2$		

Figure 21. Equations related to $2x = 5$ used in Task 12

The equivalent equations $2x^2 = 5x$ and $2x^3 = 5x^2$ can be related through multiplication. The intention behind these equations was to determine if participants could reason that multiplying $2x^2 = 5x$ by x preserves the solution set (unlike $2x = 5$). The two columns labeled “True” and “False” were included in this task to prompt participants to keep track of the sets of solutions and non-solutions. I thought this would result in participants attending to changes in the solution set as each equation was transformed into another.

As was the case in the previous task, I simply asked if the equations presented were true or false and how participants knew. I hypothesized that this would allow participants to use the implication structure to relate equations. Initially, I did not present the entire set of equations shown in Figure 21. I first presented $2x = 5$ and $4x = 10$ and the two columns for participants to keep track of the solution and non-solution sets. Subsequently, I included one additional equation at a time until I generated the entire list of equations shown in Figure 21. By introducing one new equation at a time, I thought it would be more likely that participants would successfully reason about any changes in the solution set.

Task 13. In order to gather additional evidence that would support my findings from the previous two tasks, I developed one additional task similar to the previous two. Figure 22 shows the set of equations used in Task 13.

A. $10x^3 = 5x^2$

B. $10x^2 = 5x$

C. $10x = 5$

Figure 22. Set of equations used in Task 13

The three equations were presented together, and the two columns used to assist participants keep track of solutions in the previous task were not included. Similar to the previous task, the equations in Task 13 prompted participants to reason about the preservation of, or changes in, the solution set when relating equations by multiplication or division by x. Unlike the previous task, this task did not offer equations related by an integer scale factor. In other words, this task offered only an abstract opportunity to relate equations and solution sets.

My approach and questions for this task were similar to that of the previous tasks. I asked if the equations were true or false and how participants knew. I anticipated participants would use the implication structure to relate equations and solution sets. I was interested to see if participants would relate equations and their respective solution sets without resorting to the verification of solutions. Although $10x = 5$ has a solution that can be easily verified, $10x^2 = 5x$ and $10x^3 = 5x^2$ do not allow for the easy verification of the shared solution. Consequently, I thought this task would allow participants to demonstrate they could relate the truth value of equations, not through empirical verification, but by linking solution sets to the operations that related the equations (i.e., with analytical reasoning).

Results (Sessions 9-11). In this next section, I present the aggregated results using the three new tasks. I conducted the sessions individually with Edgar, Isabel, and Harry, and each participant reasoned about all three tasks in a single session. I present the results together because all three participants responded similarly.

Task 11. This task prompted participants to determine when $11x = 50$ and other equivalent equations were true or false. The goal was not for participants to identify a solution; the aim was for participants to recognize that an equation implies an equivalent equation. Harry had previously resorted to substituting values into equations to be certain equations are equivalent. I used $11x = 50$ because I thought the non-integer solution would be onerous to verify. All three participants were able to identify equations as sharing a solution by reasoning analytically and not empirically. That is, they were able to identify equivalent equations by relating pairs of equations through multiplication and division to reason the solution set was persevered. They did not resort to substituting values in order to verify solutions. However, participants failed to anticipate zero as a solution when relating $11x = 50$ and $11x^2 = 50x$.

All three participants identified $11x = 50$ and $22x = 100$ as statements that are sometimes true and share a common solution. The following exchange with Harry represents typical reasoning defending this assertion.

Julius: If you look at these equations together, $22x = 100$, and $11x = 50$. Are these true or false?
Harry: They're sometimes true.
Julius: Okay, they're both sometimes true. Do you know anything else about that sometimes?

Harry: I figured the number would always be the same, x for 11 and 22.
Julius: So, the first equation and second equation?
Harry: **X would be the same value because the numbers are getting doubled on each side.**
Julius: Mm-hmm. So the numbers are getting doubled on each side. So you said these are sometimes true, right?
Harry: Yeah. When x is the same it's going to be equal to 50 for 11x. And for 22x, if x is the same as 11x, it would become 100.
Julius: And you know this even though you don't know the number, right?
Harry: Yeah.
Julius: Okay. You're sure that it would be the same, right?
Harry: Mm-hmm.
Julius: If one of these is false, what would the other one be?
Harry: It'd be false as well.
Julius: Why is that?
Harry: Because, **if the number is the same for both equations** and you use a different number for both equations, it wouldn't be true because **there's only one number that makes the equation true.**

Harry identified the equations as sharing a common solution without verifying, or even identifying, a value. In a similar manner, Isabel said the equations are true when "x equals the same thing." Edgar, too, said, "The x would always be the same." All three participants articulated that knowing $11x = 50$ or $22x = 100$ is false implies that the other equation is false. All three participants also identified other equations in the task (e.g., $33x = 150$) as equivalent by articulating that knowing one equation is true implies the other is true because the solution has to remain the same.

Throughout the task, Edgar, Isabel, and Harry repeatedly made the argument that pairs of equations were equivalent and shared a common solution by relating two equations with multiplication and division. Harry, for example, said, "The numbers that are multiplied by x are getting bigger every time. On the other side as well." Importantly, participants were able to reverse their reasoning and state that a particular equation can be related to another through division. For example, when I asked Harry how he knew that

$44x = 200$ meant $22x = 100$ is true, he replied, "The half of $44x = 200$ equals $22x = 100$." Isabel and Edgar made similar comments to argue one equation being true meant another is true. When I asked Edgar if there was another instance that would imply $11x = 50$ is true, he described dividing both sides of $110x = 500$ by ten to reason that the solution for $110x = 500$ is also the solution for $11x = 50$. It is significant that participants related equations through both multiplication and division because it highlighted their ability to distinguish a conditional statement from its converse while maintaining that both statements are true.

One notable development during this task was that participants began explicitly referring to equivalent equations as the same. When I asked Edgar if knowing that $11x = 50$ is true meant that $22x = 100$ had to be true, he responded, "Well, yeah, because it's simply the same thing. You just multiply it by two." When I asked him a similar question later in the task, he again responded, "Because they're the same, but in different numbers. You're just multiplying it by two, three, [and] four." Similarly, Isabel stated that if one equation is true, others must be true because "they're all the same equation." These comments marked a shift in which participants identified an equation not by coefficients, but its solution set; equivalent equations were identified through their common solution set. These comments were consistent with an object-based (rather than process-based) view of equations. The fact that participants were convinced by this object-based view supported my conclusion that they reasoned analytically – not empirically or ritualistically.

In the final part of the task, I included $11x^2 = 50x$ alongside the equations equivalent to $11x = 50$. This equation was included in the task to see if participants would recognize that multiplying both sides of an equation by x results in a different a solution set (multiplying $11x = 50$ by x generates zero as an additional solution). All three participants did not anticipate the change in the solution set. Instead, they again argued that $11x^2 = 50x$ is equivalent to $11x = 50$ because it was related by multiplication. I directly sought evidence of this misconception and asked if there was another solution that might make $11x^2 = 50x$ true but not $11x = 50$. All three failed to identify zero as a value. I rephrased the question and asked, "Is there is ever a time when one of them [$11x = 50$ or $11x^2 = 50x$] is true but the other one isn't?" Again, all three participants said that both equations are either simultaneously true or false, and they failed to identify zero as the additional solution to $11x^2 = 50x$.

Participants' failure to identify the correct solution set for $11x^2 = 50x$ was not the result of a failed empirical strategy – there was no evidence that participants substituted and verified potential solutions. Rather, participants overgeneralized their strategy of using multiplication and division to identify equivalent equations. I ended the task by suggesting zero was a solution to $11x^2 = 50x$ and all participants changed their responses to indicate that $11x^2 = 50x$ was not equivalent to the other equations.

This task successfully elicited evidence that participants could reason about equivalent equations without resorting to an empirical strategy. In particular, participants began referring to equivalent equations as "the same." However, their inability to anticipate a change in the solution set when multiplying an equation by x suggested their understanding was not yet fully developed. I ended this task knowing that participants may or may not repeat this error in the next task.

Task 12. Similar to the previous task, this task prompted participants to reason about equations that shared a solution (but not necessarily a solution set) with $2x = 5$. This task successfully elicited additional evidence that all three participants could reason about solution sets by relating equations with multiplication and division and without employing a verification strategy. Isabel and Harry were able to identify that zero was introduced to the solution set when reasoning about $2x^2 = 5x$ and $2x^3 = 5x^2$. Edgar recognized that this task was similar to the previous task, but again did not identify zero as an additional solution. In addition, this task generated significant evidence that participants had fully adopted conditional language to fluently relate equations.

Although 2.5 is a value that can be substituted into many of the equations, participants did not verify this solution. Instead, they related equations as equivalent by recognizing that one equation was a multiple of another. Edgar solved $2x = 5$ and identified 2.5 as the solution for each of the equations. When asked how he knew, he said he solved the “easiest one” and the others “must be true…because I just multiplied.” Isabel and Harry did not identify 2.5 as a solution, but they still identified the set of equations as sharing a common solution. Isabel said, “Even though I don’t know what x is, x is going to be the same in all of them.” She went on to articulate that she knew this because of the multiplication between equations. Harry said, “There’s going to be a decimal for the value of x here,” while pointing to the entire set of equations. He went on to say, “Because you can just divide [an equation] to get the same number,” when asked how he knew they shared a solution. In order to facilitate the conversation, I told Isabel and Harry that 2.5 was the shared solution. All three participants indicated that 2.5 was a common solution for $2x = 5$, $4x = 10$, and $40x = 100$. In addition, they stated that all three equations were false when x was a value other than 2.5.

Although $2x^2 = 5x$ shares a solution with $2x = 5$, the equations are not equivalent because the solution set for $2x^2 = 5x$ includes zero while $2x = 5$ does not. However, $2x^3 = 5x^2$ is equivalent to $2x^2 = 5x$. $2x^3 = 5x^2$ was included to ensure participants reasoned correctly about the effect of multiplying an equation by x – multiplication by x does not necessarily modify a solution set. Isabel and Harry reasoned analytically to identify correct solution sets. Isabel and Harry recognized $2x^2 = 5x$ as sharing a solution with $2x = 5$. When I asked Isabel how she knew this, she said, "You multiply them both [sides of $2x = 5$] by x." Isabel and Harry also recognized that the solution set of $2x^2 = 5x$ also contained zero. When I asked Harry how he knew, he said, "Because you have to multiply both sides by x, and if x is equal to zero, you would get zero for both of them [each side of the equation]." Isabel reasoned about the solution set for $2x^3 = 5x^2$ by identifying it as $2x = 5$ multiplied by x^2. Harry recognized $2x^3 = 5x^2$ as $2x^2 = 5x$ multiplied by x again. Edgar failed to identify zero as a solution to $2x^2 = 5x$ and $2x^3 = 5x^2$. He recognized the equations are not equivalent to $2x = 5$, but when I asked him why, he said, "Because when we were discussing this one [the previous task], we both said that if this one's [pointing to $2x^2 = 5x$] true, this [$2x = 5$] could be false." He realized the set of equations in this task were similar to the previous task, but, unlike Isabel and Harry, was not able to attribute that change to the multiplication by x.

This task provided significant additional evidence that all participants fluently related multiple equations with conditional language. Isabel's work (Figure 23) shows how all three participants kept track of solutions and non-solutions.

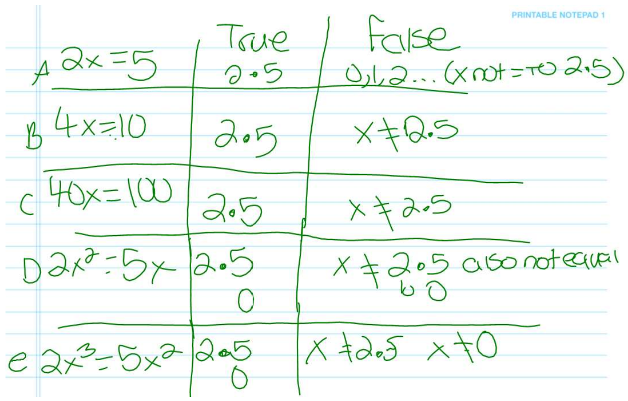

Figure 23. Isabel kept track of solutions and non-solutions

Edgar and Harry made similar charts. The following exchange shows how Isabel's chart allowed her to easily relate pairs of equations using the implication structure.

> Julius: Okay, so if the other equations are true, any of them, A, B, C, or D, is this last equation [E] definitely true?
> Isabel: Yeah if any of those are true then this one is true.
> Julius: Even if we're talking about D. If I tell you D is true, you're sure E is true?
> Isabel: Yeah
> Julius: A hundred percent sure?
> Isabel: Yeah
> Julius: Okay and then what about if E is true, the last one. Are the others true?
> Isabel: **D is always going to be true in this one because it has the same solutions as E**, but C, B, and A, they would only be true if it's two point five.
> Julius: Which is always or not always?
> Isabel: Not always.
> Julius: And you used the word solutions, what did you mean by that?
> Isabel: What x is equal to.
> Julius: But I thought x could be lots of things? Like x can be seven.
> Isabel: Well, when x works.

This exchange highlights Isabel's ability to employ the implication structure. She is able to state that equations A, B, and C imply equation E, but that the converse is not true. She specifically uses "always" to differentiate equation D from equations A, B, and C. In addition, Isabel previously stated equation D is sometimes true and included values in both the true and false columns. However, she said, “D is always going to be true *in this one* [emphasis added].” Isabel indicated that equation D is always true when the premise of my conditional question is true (i.e., when equation E is true). This further suggested her full adoption of the implication structure when relating equations.

Harry and Edgar also displayed their ability to use the implication structure while relating equations. In particular, even though Edgar failed to identify the correct solution set for $2x^2 = 5x$ and $2x^3 = 5x^2$ on his own, he was still able to fluently relate equations. After I prompted him to include zero in his chart, his responses to my questions matched those of Isabel and Harry. When I asked him if $2x^2 = 5x$ implies $2x = 5$, $4x = 10$, and $40x = 100$, he replied, “No. Not necessarily. Sometimes.” I asked him what he could say about $2x^3 = 5x^2$ and he stated that the equation is the same “as this equation [$2x^2 = 5x$], and this equation *only* [emphasis].”

In conclusion, this task generated additional evidence that all three participants could reason about solution sets by relating equations with multiplication and division and without employing a verification strategy. Isabel and Harry, but not Edgar, were able to identify a change in the solution set and attribute that change to the multiplication of $2x = 5$ by x on both sides. Also, this task allowed all three participants demonstrated their ability to employ the implication structure to fluently relate equations using conditional language.

Task 13. This task had two goals: generate additional evidence supporting conclusions reached in the previous tasks, and allow Edgar the opportunity to anticipate the change in the solution set when multiplying an equation by x. This task was successful in achieving both goals. I was able to elicit additional evidence that all three participants reasoned about the truth value of equations without verifying solutions and instead related equations with multiplication and division. Edgar successfully anticipated the change in the solution set when relating $10x = 5$ and $10x^2 = 5x$.

All three participants identified .5 as the common solution for all three equations by solving $10x = 5$. Importantly, none of the participants verified by substitution into $10x^2 = 5x$ and $10x^3 = 5x^2$. They instead argued that the two equations generated by multiplying an equation by x must share the solution of .5. Isabel even made this assertion before identifying .5 as the common solution. When describing why $10x^2 = 5x$ and $10x^3 = 5x^2$ are equivalent, Isabel said, "They're the same equation, but multiplied." Similarly, Harry argued, "They're the same equation, just different variations. You just multiply the equation or divide to get different versions of the equation."

Edgar successfully, on his own, identified $10x^2 = 5x$ and $10x^3 = 5x^2$ as gaining zero as a solution. His work is shown in Figure 24.

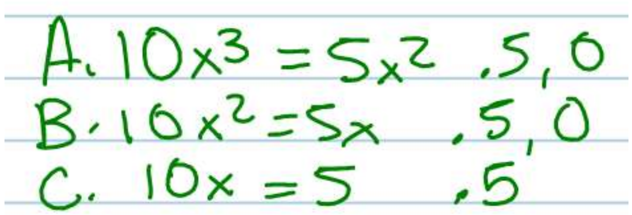

Figure 24. Edgar's work for Task 13

Unlike the previous two tasks, Edgar spontaneously, without my prompting, included zero in the solution sets for $10x^2 = 5x$ and $10x^3 = 5x^2$. He explained his reasoning in the following exchange:

Julius: So, if one of them is true, are the others true? If one of them is false is the other false? What's going through your mind? How are you trying to figure it out?
Edgar: What I'm trying to do is solving the equation for C, and x would be .5.
Julius: Okay. That's equation C. What about equation A and B?
Edgar: They'd be also .5, but **then they'd also be zero.**
Julius: You don't have a calculator. Tell me how you know they're also .5.
Edgar: Because you're just **multiplying B by x and then again for A**.
Julius: Okay. That's how you did it, and that's how you know the .5 is there. How do you know the zero is there now? Where did that come from?
Edgar: I mean they'll always be equal to zero, so **both sides would be zero**.
Julius: Are you sure that it's zero? Do you believe that it's zero?
Edgar: Yeah.
Julius: Because I told you before or because you're plugging in?
Edgar: I'm plugging it in.
Julius: Okay. Does it make sense that it's zero, that all of a sudden there's a zero? Can you explain why there's a zero for these, but not this one?
Edgar: Because if you plug in a zero for equation C, zero wouldn't be equal to five. But if you plugged in for A and B, then **both sides would be equal to zero**.
Julius: Is that a coincidence about this problem or is there something about what we did in this problem? Or you're not sure?
Edgar: It's something we did with this problem.

During this exchange, Edgar related the equations through multiplication. He argued that .5 is a common solution for all three equations. He recognized that both sides of $10x^2 = 5x$ and$10x^3 = 5x^2$, but not $10x = 5$, are zero when x is zero. This allowed him to identify a change in the solution set for $10x^2 = 5x$ and$10x^3 = 5x^2$.

Isabel articulated reasoning similar to Edgar's. The following exchange shows how her reasoning was consistent with her reasoning from the previous tasks:

Julius: And why are you so sure about what we were just saying?
Isabel: Because I know that when you multiplied zero for these [$10x^2 = 5x$ and $10x^3 = 5x^2$] it's still going to be zero equals zero, but if you put zero in here [$10x = 5$] it's not going to work, and then one half works for all of them all the time.
Julius: Okay, and then last question. If one of them is false, are the other ones false?
Isabel: Yeah. Well no, because C [$10x = 5$] can be false, but zero for…so these two [$10x^2 = 5x$ and $10x^3 = 5x^2$] might be true, but this [$10x = 5$] is false,

but if these [$10x^2 = 5x$ and $10x^3 = 5x^2$] are false then this [$10x = 5$] is always false.
Julius: C [$10x = 5$] would always be false?
Isabel: Yeah.

This exchanged further supported my previous conclusions that Isabel could anticipate a change in the solution set and employ the implication structure to relate equations. Harry similarly related the three equations. I asked him what he could conclude if $10x^3 = 5x^2$ was true. He replied, "B [$10x^2 = 5x$] will always be true but C [$10x = 5$] will sometimes be true…because zero works on A and B [$10x^3 = 5x^2$ and $10x^2 = 5x$] but doesn't work on C [$10x = 5$].

In summary, this task gave participants the opportunity to reason about the truth value of equations analytically (through multiplication and division) without resorting to a verification strategy. Edgar benefited from a third opportunity to anticipate a change in the solution set; he was able to anticipate and justify the change on his own. In addition, all three participants continued to demonstrate their ability to employ the implication structure and fluently relate equations with conditional language.

Summary of Phase III. The tasks used in Session 8 did not yield conclusive evidence supporting Harry's ability to relate equations analytically. He often resorted to a verification strategy in which he substituted values into equations. As a result, I developed a new set of tasks to move participants away from a verification strategy and toward analytical arguments. These tasks specifically used sets of equations with two key features: an easily identifiable multiplicative scale factor between equations, and solutions that are onerous to verify without a calculator. All three participants were able to produce analytical arguments using the new tasks; a shift in language occurred in which participants began explicitly referring to equivalent equations as the same. However, participants initially overgeneralized their argument to incorrectly reason that the multiplication of an equation by x produced an equivalent equation. Participants subsequently avoided this error when presented with a similar follow-up task.

Phase IV: Post-Interview

The fourth and final phase replicated the initial task-based interview in order to document changes in participants' reasoning as a result of the teaching interventions. Taken together, four major shifts constituted participants' new ability to reason about classes of objects and use conditional language when considering the truth value of algebraic equations. First, participants began describing a solution as a value that satisfies an equation instead of a result of a process. Second, instead of verifying solutions by substituting values, participants justified solutions by relating the truth value of equations generated from the solving process. This shift was particularly evident when participants described equations with no solution and infinite solutions. Third, participants adopted conditional language to describe the truth value of equations; they employed the implication structure to relate equations. Last, participants shifted away from searching for and verifying solutions (i.e., empirical strategies). Instead, they relied on deductive justifications – namely, operations that link equations – to reason about the preservation of, or changes in, the solution set.

The first shift occurred when participants moved away from referring to a solution as a result and instead started describing it as a value that makes an equation true. When I asked Harry about the solution for $4x + 1 = 9$ during the initial task-based interview, he said, “the answer is two.” Edgar similarly said, “I did it in my mind and got x equals two…four x equals eight and you divide both sides by four and what *you end up with* [emphasis added] is x equals two.” Participants’ initial responses did not consider truth value. When I prompted Edgar and Isabel to consider the truth value, they viewed equations with a single solution as always true. For example, Edgar said, “It’s always true because the x always equals two.” Harry specifically said, “You know it’s correct,” – not, the statement is true – when verifying a solution. Their initial view that equations are always true was tied to their process-based view of equations that required a correct, invariant result.

During the post-interview, I asked participants about the solution to $11 = 3x - 1$. In addition to describing the solution as a value that satisfies the equation, participants also articulated that non-solutions do not. In other words, participants began to view equations in terms of truth value instead of prompts to obtain a result. This was evident from Edgar’s response describing the solution. He said, “It’s sometimes true…because if it’s four, then yeah, it’d be true, but if it’s another number other than four, then it’s false, so it’s sometimes true.” In a similar manner, Harry responded, “Only when x is four. Sometimes true, sometimes false.” These comments indicate participants came to view equations as statements with truth value and solutions as values that satisfy equations.

The second shift was marked by a change in how participants sought to increase certainty. They initially verified solutions by substitution into the original equation (i.e., the check step) and did not attend to the truth value of the intermediate equations generated from the solving process. For example, Isabel said, “Well, you can check it. You plug back into the [original] equation.” Harry similarly said, “Because if you plug in the answer you got instead of x and you do the equation [evaluate]…you know it's correct." These comments demonstrate that participants viewed solutions as values that make the original equation true. However, they did not relate the preservation of the solution to their solving process. Participants only verified that the solution satisfied the original equation and made no mention that each equation generated from the solving process was also satisfied.

During the post-interview, however, participants recognized that each equivalent equation resulting from the solving process was satisfied by the solution. Harry's justification for the solution to 11 = 3x – 1 saliently illustrates this point.

> Julius: How do you know?
> Harry: Because if you plug into **any of the other equations**, it would make it true as well.
> Julius: How do you know that's the case here that if one true, the others are true?
> Harry: Because, they're both the same equation, they're just in different numbers.

Harry's responses includes two critical choice of words. First, his use of the phrase "make it true" instead of "correct" demarcates the first shift previously described. Second, he verified by substituting into "any of the other equations," not just the original. His phrase, "the same equation," alludes to the fact that the solving process generated equivalent equations (i.e., persevered the solution set). This exchange highlights the second shift: the notion that all equations have truth value, not just the original equation presented.

The shift to assigning a truth value to all equations generated from a solving process was particularly evident when participants reasoned about equations that were always false or always true. During the initial task-based interview, Edgar made an argument based on the form of the equation: "Because I did minus x on both sides and then took out the x completely and one equals the two so that would be no solution." In other words, Edgar argued that since there was no x in the resulting equation, there was no solution. During the post-interview, Edgar made an argument relating the truth value of equations to conclude x + 2 = x + 3 has no solution (Figure 25).

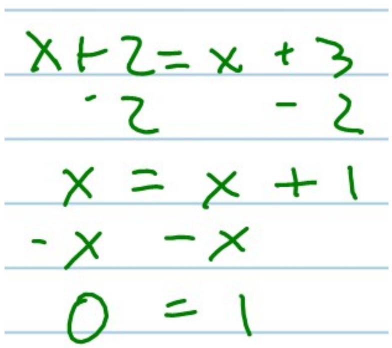

Figure 25. Edgar related the truth value of equations

> Julius: Now, x plus two equals x plus three, what can you tell me about the solution to that equation?
> Edgar: Yeah. So zero equals one.
> Julius: So what can you tell me about the solution?
> Edgar: **It's false.**
> Julius: It's false. So what's false?
> Edgar: Zero is not equal to one.
> Julius: Okay. And what about the original equation, x plus two equals x plus three?
> Edgar: It'd be false, because if this is false, the other one is.
> Julius: You just pointed to something. I asked you if x plus two equals x plus three, and then you said?
> Edgar: **If this one's false [0 = 1], then this one [x + 2 = x + 3] has to be false,** because zero isn't equal to one.

Edgar's argument that there is no solution not only includes a reference to truth value, the truth value of an equation (x + 2 = x + 3) is now predicated upon the truth value of another equation (0 = 1) generated from his solving process.

Isabel also shifted her argument to similarly relate the truth value of equations. She initially made an empirical argument to argue x + 1 = x + 2 is always false: "Because I can tell them to try to plug in a number and I can show them that there's no way that it would work. No matter what number they tried, it would still come out to no solution." In the post-interview, Isabel said, "There's no solution…because after you solve it you get zero equals one, and that's not true." Harry similarly said, "It's false for any value of x…it goes for both [equations]." In a change from their responses to the initial task of x + 1 = x + 2, all three participants argued during the post-interview that x + 2 = x + 3 has no solution because a related equation is always false.

Participants also shifted their justifications to include the truth value of related equations during the post-interview when they argued that 3x = x + x + x is always true. Previously, Harry used an empirical strategy and verified potential counterexamples when he argued that 2(x + 1) – 2 = 2x has infinite solutions. He said, "I was trying to think of a number that it wouldn't work for." During the post-interview, Harry based his argument in the truth value of a related equation: "It's always true because 0 = 0." Edgar, too, shifted his justification to include the truth value of a related equation. In the initial task-based interview, similar to his argument for x + 1 = x + 2, he argued 2(x + 1) – 2 = 2x has no solution because a resulting equation did not include x. Edgar said there was no solution because "I'm used to seeing x equals a number." During the post-interview, he connected his solving procedure to the truth value of the equivalent equations: "Let me just solve it. And then you take away 3x on both sides. Zero would be equal to zero. When you solve this, x could be anything."

The shift to considering the truth value of related equations engender another shift: the adoption of conditional language and the implication structure. Conditional language first appeared when participants described solutions and non-solutions in terms of truth value. For example, Edgar said, “This can be sometimes true, because it [x] can be four-fifths, but *if* [emphasis] it’s zero, then it would be false.” Likewise, Isabel said,

> If you plug four over five back into any of these [equations] it would work, and if you plug zero into [equations] A or B, then that would work, too. But for [equations] C and D, if you plugged in zero, then it wouldn’t work.

After participants adopted conditional language to relate a value (i.e., a solution or non-solution) to the truth value of an equation, they proceeded to adopt the implication structure to relate equations. Harry, for example, said, “If [equation] A is true, then [equation] B is true, and [equation] C is always false.” Edgar and Isabel responded with similar language when relating equations. Importantly, all three participants differentiated between a conditional statement and its converse.

The final shift occurred when participants related equations with absolute certainty as a result of deductive, not empirical, justifications. Even though participants related equations by comparing solution sets, they initially were not certain they reasoned with the correct sets. For example, I asked Isabel if she was sure that $4x + 2 = 6$ did not have a second solution. She replied, “There probably isn’t. There might be.” In the post-interview, uncertainty was eliminated as a result of participants’ ability to use deductive justifications to link equations and corresponding solution sets. Edgar’s work (Figure 26) shows his deductive construction of the solution sets.

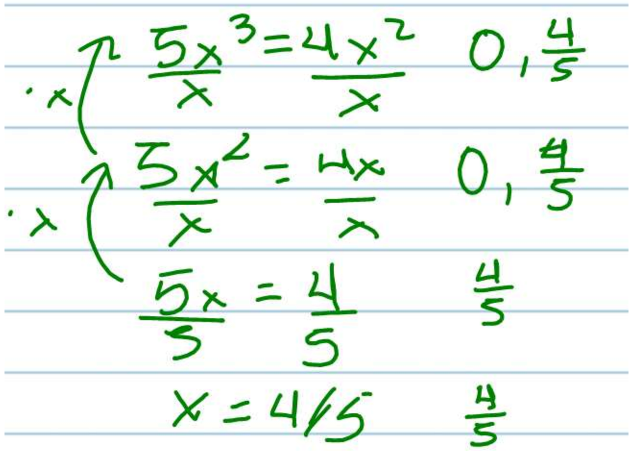

Figure 26. Edgar related equations with multiplication

In the following exchange, Edgar explains that he certain four-fifths is a solution to all equations because they are linked with multiplication.

> Julius: When we talk about the solution to the original equation [$5x^3 = 4x^2$], is there another solution besides zero?
> Edgar: Yeah, four-fifths.
> Julius: So tell me why you say that or how you know.
> Edgar: Because it's the same thing, you're just multiplying it [the equation] by x.

Isabel similarly said, "Because I got it down to this equation. When I solved it I got x equals four over five. These [equations] are just times x." Although participants previously compared solutions sets to relate equations, these episodes illustrate the final shift of relating equations and corresponding solution sets with operations. The construction of solution sets through deductive, not empirical, justifications allowed participants to relate equations and reason about solution sets with certainty.

Answer to Second Research Question

Can students learn to reason about classes of objects and use conditional language when considering the truth value of algebraic equations? If so, how?

Yes. During the teaching experiment, participants came to reason about classes of objects and use conditional language by progressively adopting four key understandings. First, equations are mathematical sentences that have truth value; the solution describes when an equation is true. Second, equations are either universally true (always true), universally false (never true), or conditionally true (sometimes true). Third, equations generated during a solving process, not just the original equation, have truth value – any pair of equations can be related using the implication structure. Last, the operations between equations determine whether or not the solution set is preserved.

First, participants came to understand that equations are mathematical sentences with a truth value. This required a shift from process-based to object-based view (Dubinsky & McDonald, 2001; Sfard, 1991). In a process-based view, equations are prompts to execute procedures, and solutions are results or answers. As a mathematical objects, equations convey information about the solution set without necessarily invoking a process. To facilitate this shift, rather than asking participants to solve for x or find the solution, I instead simply asked, “True or false?” This brought the relational nature of the equals sign into focus. The solution became the object that determines truth value.

Second, participants developed the understanding that equations are always, sometimes, or never true. This required a shift from generalized arithmetic to functional algebra (Sfard & Linchevski, 1994). With a functional view of algebra, participants came to see expressions as generators of infinite sets instead of a representation of a specific, unknown number. When participants used all real numbers as the replacement set, they recognized an equation was always true when the equation was an identity. Participants were then able to identify conditionally true equations (i.e., equations with a finite solution set) as sometimes, not always, true.

To shift participants to a functional view of algebra, I presented an expression and asked, “How many numbers do you see?” I started with an expression that can be used to generate a times table (e.g., 5x) before using other expressions (e.g., x – 4). If participants did not immediately identify the expression as a generator of an infinite set, I elicited a functional view of algebra by asking if they thought a classmate might answer differently. 5x can either represent two numbers (the process of multiplying), one number (the product), or an infinite number of numbers (the infinite set of numbers generated by multiplying any real number by 5). Recognizing 5x as a representation of an infinite set is contingent upon a functional view of algebra. With a functional view of algebra in place, I then elicited the distinction between equations that are universally and conditionally true. When participants claimed an equation was always true, I verified this understanding by asking, “Always?” and, “Are there any values that make it false?”

Third, participants eventually understood that every equation generated during a solving process, not just the original, has truth value. Further, they used the implication structure to relate equations. With their functional view of algebra, participants differentiated between propositional and predicate calculus and expressed their reasoning about necessary and sufficient conditions (Dubinsky, 1991; Durand-Guerrier, 2003). For example, the statement, “if $x^3 = 4x$, then $x^2 = 4$” is true as a proposition when $x = 2$. With a predicate view, however, the statement is false ($x^3 = 4x$ does not imply $x^2 = 4$ because if $x^3 = 4x$ is true, $x^2 = 4$ is not necessarily true).

To engender this shift, I presented multiple equations and asked, “If this equation is true, then is this equation true or false?” I followed-up by asking, “Always?” to ensure a predicate, not propositional, interpretation of implication statements. I also followed-up by rephrasing question in the form of the converse. I used pairs of equations with identical solution sets as well pairs where one solution set was a subset of the other. This allowed participants to internalize the nature of the converse.

Last, participants learned that the preservation of a solution set depends on the operations that link equations. This is consistent with an analytical proof scheme (Harel & Sowder, 1998) because participants learned to transform equations with goal-oriented operations while anticipating the result. In other words, participants were certain the solution set was or was not preserved without having to verifying solutions.

To move participants away from the empirical construction of a solution set to an analytical approach, I prompted participants with sets of equations that had an easily identifiable multiplicative scale factor but solutions that were onerous to verify without a calculator. For example, $11x = 50$ and $22x = 100$ are equivalent because they share a common solution. Instead of solving each equation and comparing solutions, participants relied on the deductive justification that $11x = 50$ and $22x = 100$ must have the same solution set because one equation can be transformed into the other. I included instances where an operation does not preserve the solution set. In particular, I used equations related by a multiple of x (e.g., $x = 2$ and $x^2 = 2x$) to elicit that the solution set is not always preserved because zero can be gained or lost as a solution.

Stage 3: Task-Based Interview #2

The following results concern the relationship between an arbitrary quadrilateral and the internal quadrilateral (namely, a Varignon Parallelogram) formed by connecting the midpoints of the initial quadrilateral. Participants used dynamic geometry software (i.e., Geogebra) to explore this relationship; I include images of participants' work that illustrate their reasoning. In this next section, I report the relevant results in three subsections. First, I describe participants' reasoning for the first part of the Varignon Parallelogram task in which they attempted to determine which external quadrilaterals form a Varignon Rectangle. Next, I describe how participants reasoned about the converse of the first task (i.e., given a Varignon Rectangle, what is necessarily true about the external quadrilateral?). Last, I answer the third research question.

Similar Shapes as a Class of Objects

During the first part of the task, participants were asked to identify quadrilaterals that form a Varignon Rectangle. In their responses, all three participants recognized similar shapes as a class of objects. They specifically articulated similar shapes would share the property of forming a Varignon Rectangle. For example, Isabel said, "If you made it bigger or smaller, it's the same thing…everything would be the same, there'd still be a rectangle." Harry similarly said, "It does the same thing as this one because you can just make this [the external quadrilateral] bigger and it'll still be a square." Harry's use of the phase, "it does the same thing," communicates his view of invariant properties – including the preservation of the Varignon Rectangle inside the square. Edgar made similar comments and specifically mentioned an invariant property when reasoning about similar shapes. When I asked Edgar why he thought different size squares would still produce a Varignon Rectangle, he responded, “Because all sides of the [external quadrilateral] are the same, so since all sides are the same, the internal would be the same thing.”

Participants did not identify all possible quadrilaterals that form a Varignon Rectangle. In addition to squares, kites and rhombuses form Varignon Rectangles because they also have perpendicular diagonals. None of the participants identified kites or rhombuses. Furthermore, non-special quadrilaterals with perpendicular diagonals also form Varignon Rectangles. Only Harry identified a non-special quadrilateral as containing a Varignon Rectangle (Figure 27).

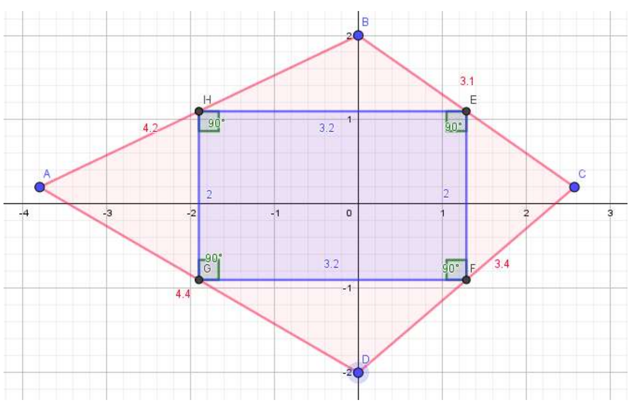

Figure 27. Harry’s non-special quadrilateral containing a Varignon Rectangle

Describing a New Class of Objects

Participants identified a sufficient condition (the external quadrilateral is a square) for Varignon Rectangles without identifying the necessary condition (perpendicular diagonals). The second part of the task was designed to elicit participants’ reasoning about this distinction. Since Edgar and Isabel initially only identified a square as containing a Varignon Rectangle, asking them if a Varignon Rectangle requires an external square enabled me to evaluate their understanding of the implication structure and ability to avoid a converse error. Even though Harry identified a non-special quadrilateral as containing a Varignon Rectangle, he did not identify the underlying property of perpendicular diagonals. The second part of the task allowed him to identify additional non-special quadrilaterals and describe them as a set.

All three participants identified the class of objects that form a Varignon Rectangle: quadrilaterals with perpendicular diagonals. Participants, however, did not explicitly articulate this property. Rather, they used empirical evidence and inductive reasoning to construct the set. This is illustrated in the following exchange with Isabel.

> Julius: Do you notice anything else?
> Isabel: **It makes a line.** If you pull these [vertices] all out the same way you pull those out, it'd keep working.
> Julius: You say you pull those out. What are you pulling out the same way?
> Isabel: You see, I'm keeping it [the vertex] on this line, so I did that same line with these two.
> Julius: With points B and D, right?
> Isabel: Yeah, if I tried to drag that one out, it would still work. **Any of these, if I dragged them in or out and I kept them on the line, it would work.**

Isabel noticed that the vertices trace out a line along the diagonals when the Varignon Rectangle is persevered. Her construction of the class of objects was based on verifying multiple cases and is shown in Figure 28.

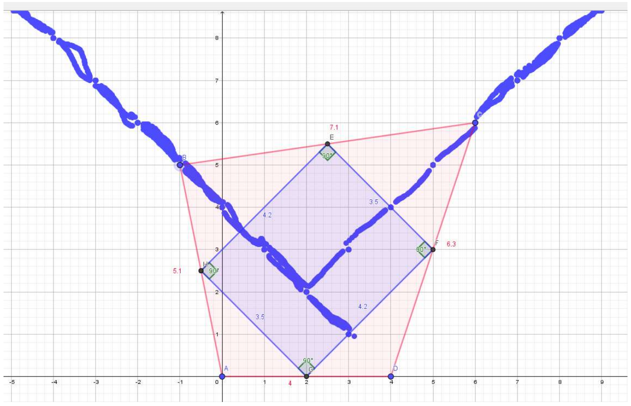

Figure 28. Isabel moved vertices to empirically identify a class of objects

Edgar was also able to empirically construct the class of quadrilaterals. In Figure 29, he moves a vertex along the diagonal to identify external quadrilaterals containing a Varignon Rectangle.

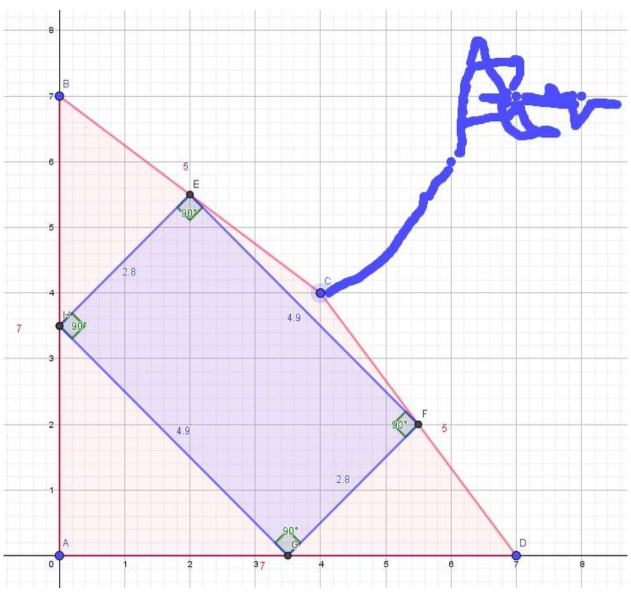

Figure 29. Edgar tested multiple cases to identify the class of objects

Edgar experimented with a number of quadrilaterals before discovering that those along the diagonal produce Varignon Rectangles. Even though Edgar did not verbalize this class of objects, he immediately used this property to identify additional quadrilaterals when moving another vertex (Figure 30).

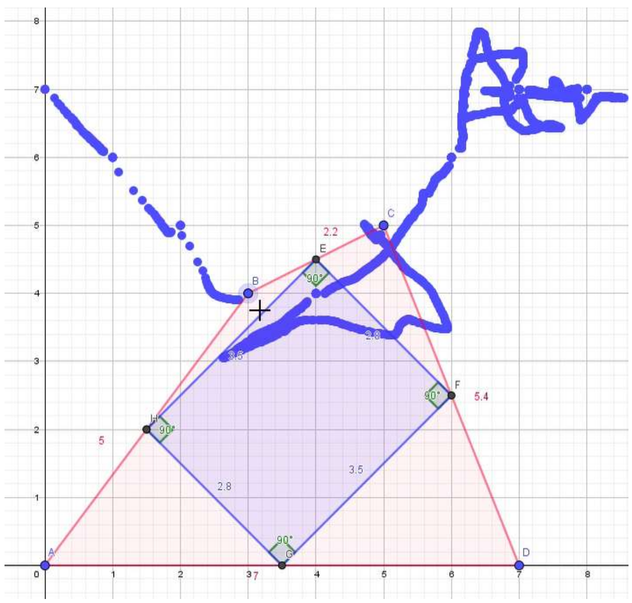

Figure 30. Edgar immediately moved vertex B along the diagonal

The fact that Edgar moved vertex B directly along the diagonal demonstrates that he realized the necessary location of a vertex in order to maintain a Varignon Rectangle.

Harry also identified the class of quadrilaterals that produce a Varignon Rectangle. Figure 31 shows that Harry tested a number of instances before discovering the set of quadrilaterals along the diagonals.

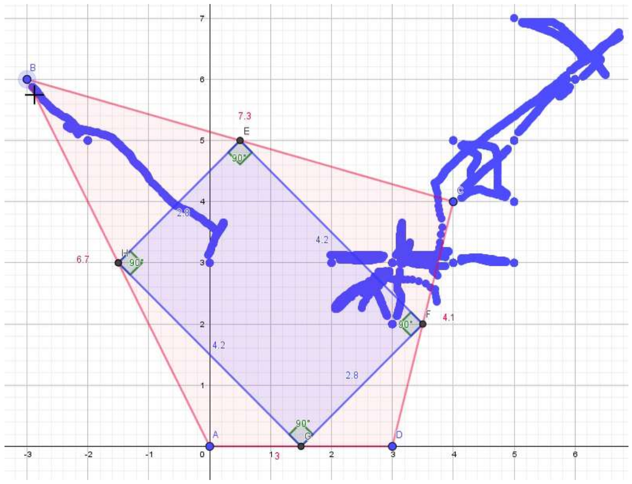

Figure 31. Harry eventually identified the quadrilaterals along the diagonals

Harry was able to describe the class of quadrilaterals: "Well, they [the vertices] end up on like specific points on the line [diagonal]." Harry's set of quadrilaterals, like Edgar's and Isabel's, was constructed empirically. Harry, however, explicitly acknowledged the empirical nature of his set. He said, "Well, just like the pattern was working, so I kept going."

Even though participants empirically constructed the class of objects, it did not constitute an empirical proof scheme. Participants maintained a disposition of doubt and did not reach absolute certainty. For example, even though Harry explicitly used a pattern, he did not reach absolute conviction: "I'm *not really sure* [emphasis added] if the pattern [along the diagonal] would *always* [emphasis original] work." Isabel was similarly not absolutely certain she had found all possibilities and said, "I don't know how to find it but there probably is more shapes that have an internal rectangle." I asked Edgar if he would believe a classmate who claimed there were no additional quadrilaterals. He replied,

> Well, since I already see here that there can be multiple shapes, I'd try to discuss with them saying that oh, we've already tried this, there is multiple shapes. Yes, maybe yours may work, but *there's still other shapes that may work* [emphasis added] as well.

Edgar, too, was not absolutely certain that his empirically constructed class of objects included all possibilities.

In addition, participants did not demonstrate empirical proof schemes because they sought to identify properties that would explain the patterns. When I asked Isabel why she thought the diagonal produced Varignon Rectangles, she said, "There is a reason for it. I don't know what. Maybe certain angles on the pink one [external quadrilateral] could make the 90 degrees angles on the inside [quadrilateral]." Edgar made a similar attempt to explain the relationship and said, "The big one [external quadrilateral] might have to be in a specific angle." Harry specifically articulated that the external quadrilateral did not need to be a special quadrilateral: "Well, *it's not a specific shape* [emphasis added], but like, it can be any shape that works to make a blue [interior] rectangle."

Answer to Third Research Question

Are students who reason conditionally about solution sets also able to reason about classes of geometrical objects?

Yes. Participants articulated similar figures as a class of objects and used similarity to justify their claims. They identified quadrilaterals with perpendicular diagonals as a class that forms Varignon Rectangles. In doing so, participants distinguished between necessary and sufficient conditions. This allowed participants to correctly employ the implication structure; they successfully discriminated a conditional statement from its converse.

Although participants constructed the set of quadrilaterals with perpendicular diagonals empirically, they were aware of the limitation of their reasoning and did not reach absolute certainty. They sought out analytical reasons but were limited by their lack of geometrical knowledge. Participants were inclined to reason about individual shapes vis-à-vis their membership in a class of objects. Consistent with a deductive argument, they specifically described the need for the construction of the set to be based on properties.

Chapter V

CONCLUSIONS

In this final chapter I draw conclusions from the study. First, I summarize the study and its principal findings. Then, I discuss the implications of these findings for both pedagogues and researchers. Next, I acknowledge the limitations of the study. Finally, I suggest a few directions for future research.

Summary

The purpose of this qualitative study was to describe the role of truth value and the solution set in supporting the development of the ability to reason about classes of objects and use conditional language. To support this goal, the study adopted Harel and Sowder's (2007) framework to identify the proof schemes that students use when justifying solutions to equations. I specifically aimed to answer the following research questions:

1. How do proof schemes differ, if at all, when students justify solutions to different types of algebraic equations?
2. Can students learn to reason about classes of objects and use conditional language when considering the truth value of algebraic equations? If so, how?

3. Are students who reason conditionally about solution sets also able to reason about classes of geometrical objects?

I conducted a series of task-based interviews (Goldin, 2000) to answer the first research question. To do so, I first identified a pool of prospective participants at a large, suburban high school. Only freshmen enrolled in Algebra I were considered (i.e., I excluded accelerated students). I then culled a homogenous set of potential participants in the sense that they could all solve two-step equations, use the distributive property, simplify algebraic expressions, and evaluate exponents. Drawing from the pool of potential participants, I conducted interviews with one participant at a time until data saturation was reached (Lincoln & Guba, 1985). To uncover participants' proof schemes, I determined how participants ascertained (convinced themselves) and persuaded (would convince their classmates) when justifying solutions to equations (Harel & Sowder, 2007; Weber & Mejia-Ramos, 2015). During the interview, I presented a range of equations, one at time, to determine proof schemes. The prompts progressed from familiar equations with one solution to equations participants were less familiar with (equations with multiple solutions, infinite solutions, and no solution).

Next, I carried out a teaching experiment (Steffe & Thompson, 2000) to answer the second research question. After conducting the first set of task-based interviews, I selected three participants to also participate in the teaching experiment. The participants were selected because they primarily relied on empirical proof schemes when justifying solutions during their task-based interview. Empirical proof schemes are problematic because even though students often adopt empirical arguments, they do not constitute valid mathematical proof (Harel & Sowder, 2007; Weber, 2008). The purpose of the teaching experiment was to determine how students can adopt set-based reasoning and conditional language when considering the truth value of equations – that is, how students can move away from empirical arguments and toward deductive justifications.

Last, I conducted an additional set of task-based interviews to answer the third research question. These task-based interviews were conducted with the three participants who completed the teaching experiment. I aimed to determine if they would continue to seek out deductive justifications and maintain a disposition of doubt toward empirical evidence in a geometrical context. The task prompted participants to reason about the relationship between Varignon Parallelograms (see Figure 32) and Varignon Rectangles (see Figure 33).

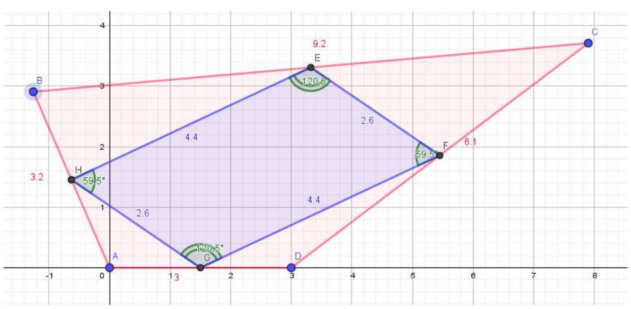

Figure 32. A Varignon Parallelogram

A Varignon Parallelogram is formed by connecting the midpoints of the four sides of any quadrilateral. A Varignon Rectangle is formed when the external quadrilateral has perpendicular diagonals. Since certain classes of quadrilaterals (i.e., squares, rhombi, and kites) form Varignon Rectangles, the task provided participants the opportunity to demonstrate set-based reasoning while distinguishing between a conditional statement and its converse.

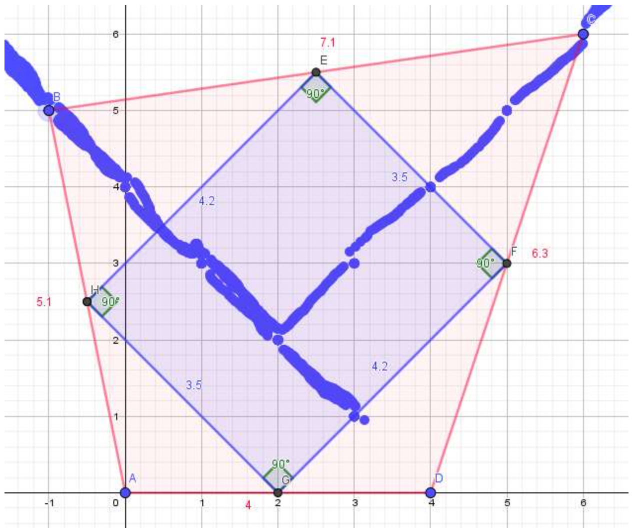

Figure 33. A Varignon Rectangle

Additionally, the task was presented using dynamic geometry software to provide the temptation for participants to consider many cases and adopt empirical arguments.

Results

I conducted twelve task-based interviews to address the first research question. Almost all participants displayed more than one proof scheme. A majority displayed ritualistic proof schemes and about half displayed empirical proof schemes. Participants did not employ deductive justifications to reach absolute certainty. Proof schemes varied depending on whether or not participants were familiar with the equation. They recognized $4x + 2 = 10$, $4x + 1 = 9$, and $7x + 1 = 3(x + 3)$ as routine. They were less familiar or completely unfamiliar with equations with no solution ($x + 1 = x + 2$), multiple solutions ($x^2 = 4$ and $x^3 = 4x$) and infinite solutions ($2(x + 1) - 2 = 2x$). Also, participants were generally unfamiliar with $x = 2$ and $1x = 2x$ and struggled to identify correct solutions.

The majority of participants used ritualistic proof schemes when reasoning about solutions to familiar equations. Participants who predominately used ritualistic proof schemes viewed equations as prompts to execute processes and solutions as results, or “answers.” Their certainty was rooted in the familiarity of the process; the expected form of the solution, x equals a constant, was critical to achieve absolute certainty. Although some of these participants displayed empirical proof schemes when confronted with unfamiliar equations, most participants were limited by their reliance on rituals and either failed to achieve absolute certainty or were absolutely certain about an incorrect conclusion. A few participants displayed authoritative proof schemes when they asserted that a teacher or trusted classmate was necessary to be absolutely sure.

Some participants relied on an empirical proof scheme – verification through substituting values – to achieve absolute certainty for familiar equations. When participants considered empirical evidence, they were only concerned with the truth value of the original equation presented. Moreover, they did not consider the truth value of each equation generated by their solving process, and as a result, did not use deductive justifications to link equivalent equations or examine the preservation of the solution set. Some participants also displayed empirical proof schemes for unfamiliar equations. When reasoning about equations with no solution or an infinite number of solutions, some participants searched for a counterexample by verifying an arbitrary number of values. They reached absolute certainty after failing to identify a counterexample. On a few occasions, participants also verified an arbitrary value with a crucial experiment (Balacheff, 1988). The value verified, although not actually representative of all possibilities, was viewed as such by participants.

When addressing the second research question, I found that participants learned to reason about classes of objects and use conditional language by progressively adopting four key understandings. First, instead of simply seeing equations as prompts to execute procedures and obtain a result, participants came to see equations as mathematical sentences that have truth value. That is, participants began to transition from a process-based view to an object-based view of equations (Dubinsky & McDonald, 2001). I facilitated this shift by bringing the relational nature of the equals sign into focus – I asked, "True or false?" and followed up with "Always?"

Second, participants' view of equations transitioned from generalized arithmetic to functional algebra (Sfard & Linchevski, 1994). With their initial view of generalized arithmetic, participants were prevented from reasoning conditionally because they insisted that x only represents a single value and as a result equations (e.g., $4x = 8$) are always true. Participants accommodated the view of functional algebra after I prompted them with expressions (e.g., $5x$) and asked, "How many numbers do you see?" This allowed them to see expressions as both representations of a single, unknown quantity as well as functions that generate an infinite set of values. As result, they could discriminate between equations that are always true and conditionally true.

Third, participants began to use the implication structure to relate the truth value of one equation to another. Once participants saw equations as statements that can be sometimes true, they were able to relate an instance in which one equation is true to the truth value of another equation. In order to have participants adopt the implication structure, I specifically asked, “If this equation is true, then is this equation true or false?” I always followed up by asking if it was always true (or false) and by reformulating the question in terms of its converse. This line of questioning afforded participants the understanding that all equations generated during the solving process have a truth value and can be related using conditional language. Their success at this stage was also marked by correctly identifying equations with no solution or an infinite number of solutions.

Last, participants realized that the preservation of the solution set is dependent on the operations that link equations. When initially relating the truth value of equations using the implication structure, participants constructed their solution sets empirically. As a result, the form of their reasoning was correct, but since they relied on a “search and verify” strategy, they often reasoned with incomplete sets and reached erroneous conclusions. In order to have participants adopt an analytical approach that included all possibilities, I presented pairs of equations that had an easily identifiable multiplicative scale factor (e.g., $11x = 50$ and $22x = 100$) but solutions that were onerous to verify without a calculator. In the end, participants transformed equations with goal-orientated operations while anticipating the result. In other words, consistent with an analytical proof scheme (Harel & Sowder, 1998), they employed deductive justifications to achieve absolute certainty.

In addressing the third research question, I found that participants demonstrated set-based reasoning and used conditional language in a geometrical context. They identified similar figures as a class of objects and used similarity to justify claims. They also identified quadrilaterals with perpendicular diagonals as a class of objects that form Varignon Rectangles. While recognizing that certain special quadrilaterals and arbitrary quadrilaterals with perpendicular diagonals form Varignon Rectangles, participants differentiated between necessary and sufficient conditions. Participants efficaciously employed the implication structure to communicate the relationship between the set of quadrilaterals with perpendicular diagonals and the subset of special cases. They specifically discriminated between a conditional statement and its converse.

However, using the multitude of cases produced by the dynamic geometry software, participants constructed the set of quadrilaterals with perpendicular diagonals empirically. They seemed to lack the necessary geometrical knowledge to produce deductive justifications. Nonetheless, they acknowledged the limitations of their reasoning and did not reach absolute conviction – they did not use empirical proof schemes. Participants identified and reasoned about individual objects through their membership in a class of objects. Akin to a deductive argument and analytical proof scheme, they specifically cited the need for membership in the class to be predicated on shared properties.

Implications for Pedagogues

This study has several implications for teaching, particularly the teaching of secondary algebra and geometry. First, the data show that students develop conceptions of proof in algebra before learning to formally write proofs in geometry. Even though participants were not asked to prove statements or write proofs, they did reach absolute conviction on numerous occasions. It was clear they considered their ritualistic, authoritative, and empirical arguments – arguments that are not mathematically valid – to constitute proof. This is significant for the teaching of formal proof, especially in a secondary geometry class where proof is typically first introduced. If students do not have experience with or see the need for deductive arguments, they will likely struggle to adopt the aim of writing formal proofs to demonstrate deductive justifications (Dawkins & Weber, 2017).

In addition, the large portion of participants demonstrating ritualistic or authoritative proof schemes suggests that many students will arrive in their geometry classes with the expectation of being told how to write proofs, rather than learning to construct arguments on their own (Harel & Sowder, 1998). Secondary geometry teachers would be well served to acknowledge students' previous experiences ascertaining or persuading with non-deductive arguments. Before setting out to have students write formal proofs, teachers should plan interventions in which students reach absolute certainty through deductive justifications and come to embrace the role of deduction in mathematical argumentation.

A second implication of the study is that solving equations is a viable context to develop students' sense of proof. This is noteworthy because this occurs in algebra, typically prior to any formal proof experiences. Developing student ability to employ deductive justifications before writing formal proofs would likely increase their success writing proofs (Senk, 1989). The teaching experiment illustrates a possible trajectory to support the development of deductive arguments while solving equations. In particular, the development of a functional algebra understanding was a critical piece of the instructional sequence. This study provides further evidence that the development of a functional algebra view does not require the teaching of functions in the traditional sense. In fact, it can be developed as early as elementary school (Chimoni, Pitta-Pantazi, & Christou, 2018). Teachers should approach expressions, equations, and germane pre-algebra content with this in mind and seek out opportunities to have students see expressions as more than just generalized arithmetic.

A third implication of this study is that "How certain are you?" and "True or false?" are valuable questions to build models of student understanding. These questions can reveal misconceptions that might otherwise go unnoticed. For example, some participants correctly claimed x + 1 = x + 2 had no solution. It was only because I followed by asking whether or not they were certain did they reveal their erroneous reasoning (e.g., the equation has no solution because there is no x). Asking "True or false?" can also uncover pertinent understandings that could be leveraged to further student knowledge (e.g., a relational view of the equal sign or a functional view of algebra).

Epp (2003) asserted that students need experience with logic prior to formal proof writing. Another indication of this study is that students can gain useful experience with the implication structure without using truth tables and prior to writing proofs in geometry. Moreover, solving equations provides an opportunity to develop conditional language that is already part of most, if not all, algebra curricula. Students can internalize logical structures when the mathematics motivates the careful consideration of the meaning of the logical words we use (Dawkins & Cook, 2016). Algebra teachers would make a significant impact on student readiness to read and write proofs by highlighting the implication structure as students learn to solve equations. This is especially true for solving procedures that generate extraneous solutions or "lose" solutions because they illustrate the distinction between conditional and biconditional statements.

Implications for Researchers

There are three significant implications of this study for the research community. First, the teaching experiment addresses the call for research "on the road to proof" (Stylianides, Bieda, & Morselli, 2016). It provides an example of a type of study that could be conducted to identify other instructional tasks that also advance the call for research in this domain. In addition, the study illustrates that research on the foundation of proving ability is twofold: articulating the knowledge needed prior to formal proof instruction, and identifying propitious mathematical content to situate the learning of this knowledge. This study addresses the latter. Students can internalize the implication structure in algebraic contexts and without explicit instruction in logic and truth tables. Additional content could be examined to identify further opportunities to increase student familiarity with logical structures prior to their formal proof experiences.

Second, this study bolsters the case of Weber and Mejia-Ramos (2015): Researchers investigating proof schemes should attend to participants' level of conviction. My data illustrate that the distinction between absolute and relative conviction is relevant in ascertaining proof schemes. The study also supports the idea that students may consider one type of evidence to increase their conviction but rely on another type of evidence to reach absolute conviction. Moreover, their consideration of non-analytical evidence might not engender absolute certainty – students who employ empirical justifications do not necessarily use empirical proof schemes. Future research on mathematical argumentation should bear in mind that the use of empirical evidence does not always constitute a misconstrued sense of proof.

Third, this dissertation illustrates that studies take place in the context of ongoing conversations within the research community. In particular, I learned that these conversations do not pause while one collects data and processes results. From the time I reviewed literature to the time I synthesized my results, a number of articles related to proof were published. A few of these, were they published earlier, would have altered my perspective during my study. For example, Czocher and Weber (2020) offered a new paradigm to conceive of proof: proof as a cluster category, instead of a classical category. Their framework, while not directly at odds with Harel and Sowder's (2007) classification structure, would have influenced my approach delineating components of participants' arguments. Another recent publication drew attention to the role of representation systems in demarcating convincing arguments from explanatory arguments (Lockwood, Caughman, & Weber, 2020). With this in mind, I would have identified whether participants' arguments varied given opportunities to use different representations. These recent developments highlight the need for researchers to forge relationships with their colleagues in order to remain abreast of research underway and the current direction of their field.

Limitations of the Study

One should consider the findings of this study with its limitations in mind. The first limitation of this study is that my interpretations as a researcher influenced both the collection and analysis of the data. My 14 years of experience as an educator in secondary classrooms impacted my perspective when constructing tasks, considering potential responses, evaluating responses to create models of participants' understandings, and effecting the development those understandings. Another researcher with a different set of experiences would likely approach the situation differently – the study provides only one example of how to ascertain the proof schemes of algebra students. Likewise, the teaching experiment is just one possible example of how students can learn to reason about classes of objects and use conditional language; it is possible another researcher with a different lens could develop another possible learning trajectory. Nonetheless, my experiences shaped the data analysis only to a point. It is reasonable to assume another researcher with experiences similar to mine would arrive at similar conclusions from my data because my analysis was constrained by the coding process (Strauss & Corbin, 2015).

The second limitation of this study is that the population of participants did not include accelerated students. I endeavored to increase knowledge about typical Algebra I students and excluded students who completed Algebra I in middle school. It is possible that accelerated students employ proof schemes differently. In particular, they might use deductive justifications to reach absolute certainty or rely on ritualistic or authoritative proof schemes less often. Additionally, the learning progression identified in answering the second research question applies specifically to the three participants who completed the teaching experiment. One cannot assume that other students would necessarily respond in the same way. The teaching experiment simply identified one example of students adopting set-based reasoning and employing conditional language; it is intended to serve as a model when identifying other possible learning trajectories.

The third limitation of this study is that the geometry task used to answer the third research question did not require a written proof. An underlying aim of the study was to improve participants' set-based reasoning in order to better support their facility with proof. Although the task elicited participants' ability to reason about classes of geometrical objects and highlighted their adroitness with conditional language, it did not require a written argument similar to what students might be asked to produce or evaluate in a classroom setting. Additionally, participants benefited from the multitude of cases generated by the dynamic geometry software. Their ability to recognize and describe the class of Varignon Rectangles might not necessarily be the same in a classroom that does not utilize similar software.

Future Directions

This study naturally engenders a number of directions for additional research. First, another study should attempt to replicate the answer to the first research question at another research site. In many ways, the site in the study is a typical high school; one would expect similar results at similar sites. Likewise, future research at dissimilar sites could address whether certain factors (e.g., demographic makeup, curriculum enacted, etc.) lead to different results. Additionally, it would be of particular interest to conduct the study with a population of accelerated students. Students are purportedly accelerated because of their mastery of previous content and advanced understanding. One could compare accelerated students to their non-accelerated peers to determine the extent to which accelerated students use deductive justifications to reach absolute conviction – are they less reliant on ritualistic, authoritative, or empirical proof schemes?

A second direction for future research is to conduct additional teaching experiments that identify other contexts in which students can learn to reason about classes of objects and employ conditional language. Solving equations is just one context where students can internalize the implication structure; students would benefit from multiple opportunities across a variety of content to build fluency with conditional language. Furthermore, future research should investigate the feasibility of implementing the results of the teaching experiment on a larger scale. The answer to the second research question outlined a learning progression that allowed participants in this study to adopt set-based reasoning and use conditional language. Since a classroom is different from the environment of this study, testing out the learning progression in a typical instructional setting would be an appropriate next step.

Third, another study should examine whether or not students transfer their analytical proof scheme for algebra content to tasks typical of a geometry class. The task used to answer the third research question only examined participants' ability to reason about classes of objects and use conditional language; it did not require the construction of a written proof. Future study should specifically examine student ability to apply their familiarity with set-based reasoning and knowledge of the implication structure when reading and writing proofs in their geometry classes. In other words, do students maintain their disposition of doubt toward empirical evidence and continue to seek out deductive justifications when asked to read and write proofs typical for a secondary geometry class?

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