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The Gelfand-Kirillov dimension of rank 2 Nichols algebras of diagonal type

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Contents

1 | Introduction

Most interests in the theory of Nichols algebras emerged from the the theory of pointed Hopf algebras. A pointed Hopf algebra H over an arbitrary field $\mathbb{K}$ has a coradical that is isomorphic to a group algebra $\mathbb{K}G$. Examples to be emphasized include universal enveloping algebras of semi-simple Lie algebras and their q-deformations. In the study of such Hopf algebras an important tool to be used is the coradical filtration. By [28] the corresponding associated graded Hopf algebra $\operatorname{gr} H$ can be decomposed into a smash product

$$\operatorname{gr} H \quad \cong \quad R \# \mathbb{K}G\,.$$

Here R is a graded Hopf algebra in the category of Yetter-Drinfeld modules over $\mathbb{K}G$ [1]. Much information about this is contained in the subalgebra $\mathcal{B}(V)$ generated by the vector space V of primitive elements. This subalgebra is called the Nichols algebra of V [2]. As group algebras are well understood, they are finite dimensional if the corresponding group G is finite. Moreover, there is an equivalence between $\mathbb{K}G$ being of finite Gelfand-Kirillov dimension, G being of finite growth and G being virtually nilpotent, see [23] for details. Hence the lifting method of Andruskiewitsch and Schneider states that for the question of classifying finite (Gelfand-Kirillov-) dimensional pointed Hopf algebras under some finiteness conditions it is an essential step to classify finite (Gelfand-Kirillov-) dimensional Nichols-algebras [2].

For the first time Nichols algebras have been studied in their own interest in [27] as bialgebras of type one. Later they have been discussed in various settings [25, 26, 29, 30, 32, 33]. In the last years much progress has been made in their understanding. Especially, those of diagonal type which yielded interesting applications, for example as the positive part $U_q^+(\mathfrak{g})$ of the quantized enveloping algebra of a simple finite-dimensional Lie algebra $\mathfrak{g}$. Moreover, these Nichols algebras inherit a simple braiding. Therefore, these could be analyzed in great detail and many powerful tools have been developed to handle these, for example the vector space basis of ordered products of monomials indexed by Lyndon words [21]. Finite-dimensional Nichols algebras of diagonal type have been classified in a series of papers [12, 11, 14, 17, 18]. One important step for this has been the introduction of the root-system $\triangle$ of $\mathcal{B}(V)$ and the Weyl groupoid under some weak finiteness assumptions [13]. In this context the following implications were observed:

$$\dim \mathcal{B}(V) < \infty \quad \overset{(1)}{\Rightarrow} \quad \#\triangle < \infty \quad \overset{(2)}{\Rightarrow} \quad \mathrm{GKdim}\,(\mathcal{B}(V)) < \infty\,.$$

For (1) the converse is true if the height of all restricted PBW-generators is finite and the converse of (2) has been conjectured to be true [4]. Later finite dimensional Nichols algebras of diagonal type have been described using generators and relations [6]. Recently, the topic of finite Gelfand-Kirillov dimensional Nichols algebras has received increased attention. In particular rank 2 Nichols algebras of diagonal type with finite Gelfand-Kirillov dimension over a field of characteristic zero have been classified [5] and were used to also classify finite Gelfand-Kirillov-dimensional Nichols algebras over abelian groups [4].

The goal of this work is to extend this result to any characteristic. Note that there are more braidings yielding a finite root system in positive characteristic, especially there are examples with simple roots α yielding $\chi(\alpha, \alpha) = 1$ where χ denotes the corresponding bicharacter. Roots of this kind imply infinite Gelfand-Kirillov dimension in characteristic zero [5]. Hence new tools have to be developed generalizing the results for characteristic zero in addition.

We start this work by proving that in ordered products of monomials forming a vector space basis the monomials can be rearranged under some technical assumptions in the following chapter. This applies to the vector space basis granted by [21] and is used to check the freeness of certain sub-algebras later in this work.

Chapter 3 introduces the Gelfand-Kirillov dimension and we will recollect some general results on the topic concerning universal constructions of algebras. We conclude this chapter by proving a valuable inequality between the Gelfand-Kirillov dimensions of an algebra and of certain sub-quotients needed for the progress of this work before introducing Nichols algebras in Chapter 4. Due to organizational reasons we will mainly revise Nichols algebras of diagonal type and refer interested reader in a comprehensive introduction to [2]. Well-known features such as the PBW-basis and the root system will be discussed. Finally, we sketch arguments regarding properties of the Weyl groupoid and Weyl equivalence, as extended details can be found in [13].

Our approach requires considerable knowledge of root systems. The main problem when working with Nichols algebras of diagonal type is that little is known about the corresponding root system. In Chapter 5 we recall the results from [19, 34] describing preconditions for the existence of roots of the form $m\alpha_1 + 2\alpha_2$ for some positive integer m. Moreover, we show some additional results on the root system, especially a procedure to prove the existence of infinitely many roots.

With the knowledge of the root system in mind we will be able to give conditions on a given rank 2 Nichols algebra of diagonal type to be of infinite Gelfand-Kirillov dimension in Chapter 6. Our approach draws inspiration from two main sources:

- ▷ Proving the existence of roots of the form $k\alpha + \beta$ for all $k \in \mathbb{N}$ where $\alpha, \beta \in \mathbb{Z}^2$.
- ▷ Constructing an infinite chain of "included" Nichols algebras having strictly decreasing Gelfand-Kirillov dimension.

It is known that the first source is applicable for Nichols algebras of affine Cartan-type [5]. For those braidings for which we constructed infinitely many roots in the preceding chapter we get an analogous result. Using the latter source we reproduce the main argument used in [5] under additional assumptions.

Finally, chapter 7 is dedicated to the step-by-step proof of our main result applying the developed arguments. As the converse is clearly true we will prove the following

Theorem 7.1 Let $\mathbb{K}$ be an arbitrary field and $\mathcal{B}(V)$ a rank two Nichols algebra of diagonal type over $\mathbb{K}$. If $\mathcal{B}(V)$ is of finite Gelfand-Kirillov dimension, then the corresponding root system is finite.

The reader of this work is assumed to be familiar with concepts of Lie algebras, Kac-Moody algebras and Hopf algebra. Such theories will not be reiterated and we refer to standard literature for elementary details such as [8, 20, 31].

1.1 Notation

Unless otherwise stated $\mathbb{K}$ is an arbitrary field of characteristic p and $\mathbb{K}^\times$ denotes the multiplicative group of $\mathbb{K}$. All vector spaces and (co)algebras are over $\mathbb{K}$. Furthermore, by algebra we mean a unital algebra. We use Sweedler notation when working with coalgebras and comodules. Fix a Hopf algebra $(H, \cdot, 1_H, \Delta_H, \epsilon_H, S_H)$ with bijective antipode S_H. For n a positive integer we denote $\mathbb{G}_n$ (resp. $\mathbb{G}'_n$) the set of n^{th} roots of unity (resp. the set of primitive n^{th} roots of unity). The assumption of $p \nmid n$ is implicit when dealing with $\mathbb{G}'_n$. The set $\{\alpha_1, \dots, \alpha_n\}$ denotes the standard basis of $\mathbb{Z}^n$. Finally, we define q-numbers as

$$(k)_q = \sum_{i=0}^{k-1} q^i, \quad (k)_q^! = \prod_{i=1}^{k} (i)_q, \quad \binom{m}{k}_q = \frac{\prod_{i=m-k+1}^{m} (i)_q}{(k)_q^!}.$$

for $q \in \mathbb{K}^\times$, $k, m \in \mathbb{N}$, $k \le m$. Recall the following equations for $k, m \in \mathbb{N}$:

$$(k)_q + q^k (m)_q = (m+k)_q$$

and

$$\binom{m-1}{k}_q + q^{m-k} \binom{m-1}{k-1}_q = \binom{m}{k}_q.$$

2 | Rearranging ordered products of monomials

When working with possibly infinite-dimensional algebras knowledge of the underlying vector space structure can be essential. For example the vector space basis of ordered products of monomials provided by the PBW theorem is an important tool in the context of the universal enveloping algebra of a Lie algebra, see [8]. Similar vector space bases appear in other fields of algebraic Lie theory, e.g. for Hopf algebras generated by skew-primitive semi-invariants, see [22]. For technical proofs rearranging the monomials can be necessary.

In this chapter we prove in a general setting that the given ordering may be exchanged by another under some technical preconditions. We will later apply this result to Nichols algebras, see Corollary 6.2 and Corollary 6.9.

First, we fix the setting. Let $n \in \mathbb{N}$, $\mathbb{S}_n$ denote the symmetric group of degree n, A be a $\mathbb{K}$-algebra, I an index set and $<$ and $\ll$ total orderings on I. Additionally, let $b : I \to \mathbb{N} \cup \{\infty\}$ be a map and $X = \{x_i\}_{i \in I}$ be a family of elements in A and $\prec$ a partial ordering on the set $\mathbb{X}$ of monomials over X such that the following properties are satisfied:

(K1) $\forall x, y, z \in \mathbb{X}$ and $i, j \in I : xx_ix_jy \prec z \Rightarrow xx_jx_iy \prec z$ and
$$z \prec xx_ix_jy \Rightarrow z \prec xx_jx_iy\,.$$

(K2) $\forall x \in \mathbb{X} : \#\{y \in \mathbb{X} \,|\, y \prec x\} < \infty\,.$

(K3) $\forall x, y, z \in \mathbb{X} : x \prec y \Rightarrow xz \prec yz$ and $zx \prec zy\,.$

(B1) The family

$$\mathcal{B} := \left(x_{i_1}^{a_1} \cdots x_{i_k}^{a_k} \,|\, k \in \mathbb{N},\, i_j \in I,\, i_1 > i_2 > \cdots > i_k,\, 1 \le a_j < b(i_j)\, \forall\, 1 \le j \le k\right)$$

constitutes a vector space basis for $A\,.$

(B2) $\forall i \in I$ with $b(i) \in \mathbb{N} : x_i^{b(i)} \in \left\langle y \in \mathbb{X} \,|\, y \prec x_i^{b(i)} \right\rangle_{\mathbb{K}}$.

(B3) $\forall i < j \,\exists \lambda_{ij} \in \mathbb{K}^* :\; x_i x_j - \lambda_{ij} x_j x_i \in \langle y \in \mathbb{X} \,|\, y \prec x_i x_j \text{ and } y \prec x_j x_i \rangle_{\mathbb{K}}$.

Moreover, let

$$\mathcal{C} := \left(x_{i_1}^{a_1} \cdots x_{i_k}^{a_k} \,|\, k \in \mathbb{N},\, i_j \in I,\, i_1 \gg i_2 \gg \cdots \gg i_k,\, 1 \leq a_j < b(i_j) \,\forall\, 1 \leq j \leq k \right)$$

be a family of elements in A.

Our aim is to prove that $\mathcal{C}$ as well constitutes a vector space basis for A. Note that there is no relation between $<$ and $\ll$. Those are only used as "parameters" of the vector space bases $\mathcal{B}$ and $\mathcal{C}$. Examples for such an algebra A with ordered vector space basis $\mathcal{B}$ include Lie algebras and Nichols algebras with the corresponding PBW-bases.

Lemma 2.1 For $x \in \mathbb{X} \setminus \{0\}$ there are representations

$$x = \bar{x} + \sum_{\ell \in L} \lambda_\ell \bar{x}_\ell = \tilde{x} + \sum_{\ell \in L'} \mu_\ell \tilde{x}_\ell,$$

where L, L' are finite index sets, $\bar{x}, \bar{x}_\ell \in \mathcal{B}$ with scalars $\lambda_\ell \in \mathbb{K}$ for all $\ell \in L$ and $\tilde{x}, \tilde{x}_\ell \in \mathcal{C}$ with $\mu_\ell \in \mathbb{K}$ for all $\ell \in L'$. These satisfy $\bar{x}_\ell, \tilde{x}_\ell \prec x$ and

$$\{y \in \mathbb{X} \,|\, y \prec x\} = \{y \in \mathbb{X} \,|\, y \prec \bar{x}\} = \{y \in \mathbb{X} \,|\, y \prec \tilde{x}\} .$$

Proof. We prove the existence of $\bar{x}, \bar{x}_\ell \in \mathcal{B}$. The existence of $\tilde{x}, \tilde{x}_\ell \in \mathcal{C}$ can be proved analogously. The main idea here is that the factors of x can be reordered using (B3) to satisfy the ordering in $\mathcal{B}$. This can be used inductively on the additionally appearing summands.

Let x be given by $x = x_{i_1} \cdots x_{i_k}$ with $x_{i_j} \in X$, $M := \{y \in \mathbb{X} \,|\, y \prec x\}$ and $\sigma \in \mathbb{S}_k$ such that $\sigma(i_1) > \sigma(i_2) > \cdots > \sigma(i_k)$. We will argue inductively on $|M|$. This is possible due to (K2).

First, if $x \in \mathcal{B}$, then $\bar{x} := x$ and $L = \emptyset$.

If $|M| = 0$ holds, then applying (B3) to x does not produce additional terms. Therefore, we can iteratively swap factors of x by application of (B3) until we obtain $\bar{x} := x_{\sigma(i_1)} \cdots x_{\sigma(i_k)} \in \mathcal{B}$. Since (B2) holds we may assume the exponents of $\bar{x}$ to be bounded by b. Since (K1) holds we have

$$\{y \in \mathbb{X} \,|\, y \prec x\} = \{y \in \mathbb{X} \,|\, y \prec \bar{x}\} .$$

If $|M| > 0$ holds, we can again apply (B3) on x iteratively to obtain a decomposition

$$x = \underbrace{x_{\sigma(i_1)} \cdots x_{\sigma(i_k)}}_{=:\bar{x} \in \mathcal{B}} + \sum_{\ell \in L} \lambda_\ell x_\ell \,,$$

where L is a finite index set, $\lambda_\ell \in \mathbb{K}$, $x_\ell \in \mathbb{X}$ for all $\ell \in L$. Again, due to (B2) we may assume the exponents of $\bar{x}$ to be bounded by b and due to (K1) and (K3) we have $x_\ell \prec x$ for all $\ell \in L$. Now, obviously $x_\ell \neq x$. Hence $x_\ell \prec x$ implies

$$\{y \in \mathbb{X} \,|\, y \prec x_\ell\} \subsetneq \{y \in \mathbb{X} \,|\, y \prec x\} \,.$$

Thus, we can apply the induction hypothesis on the x_ℓ and get a finite index set L', $\lambda'_\ell \in \mathbb{K}$ and $\bar{x}_\ell \in \mathbb{X} :\ \bar{x}_\ell \prec x$ for all $\ell \in L'$ such that

$$x = \bar{x} + \sum_{\ell \in L'} \bar{x}_\ell \,.$$

Finally, the equation

$$\{y \in \mathbb{X} \,|\, y \prec x\} = \{y \in \mathbb{X} \,|\, y \prec \bar{x}\}$$

holds due to (K1). □

Proposition 2.2 The family $\mathcal{C}$ constitutes a vector space basis for A.

Proof. The family $\mathcal{C}$ spans A due to Lemma 2.1. It remains to prove that the elements in $\mathcal{C}$ are linearly independent.

Suppose there exists $k \in \mathbb{N}$, pairwise distinct elements $y_1, \ldots, y_k \in \mathcal{C}$ and $\lambda_1, \ldots, \lambda_k \in \mathbb{K}^\times$ such that

$$0 = \sum_{i=1}^{k} \lambda_i y_i \,.$$

We consider the partial ordering $\prec$ on $\{y_i\}_{1 \leq i \leq k}$. Since the number of vectors is finite there is at least one element among the y_i which is maximal with respect to $\prec$. Let this without loss of generality be y_1. By Lemma 2.1 there are monomials $\bar{y}_i \in \mathcal{B}$ and families $(y_\ell)^{(i)}_{\ell \in L_i} \in \mathcal{B}$ for all $1 \leq i \leq k$ such that

$$y_i = \bar{y}_i + \sum_{\ell \in L_i} y^{(i)}_\ell \,.$$

Because $\prec$ is independent of the order of the factors and we constructed $\bar{y}_i$ by rearranging the factors of y_i, the vector $\bar{y}_1$ has to be maximal among the the vectors $\bar{y}_i$, $1 \leq i \leq k$. Now, since $\mathcal{B}$ is a vector space basis there has to be some $i \in \{2, \cdots, k\}$ such that $\bar{y}_1 = \bar{y}_i$. However, the arrangement of the factors for elements in $\mathcal{B}$ or $\mathcal{C}$ is fixed and hence the equation $y_1 = y_i$ holds, a contradiction. Thus, the elements of $\mathcal{C}$ are linearly independent and, therefore, constitute a basis for A. $\square$

3 | Preliminaries on the Gelfand-Kirillov dimension

The Gelfand-Kirillov dimension is an often-considered invariant in the study of non-commutative algebras describing the growth of the structure. Since it's introduction in a paper by Borho and Kraft [7] many properties and applications have been developed, for example in the study of universal enveloping algebras of finite dimensional Lie algebras. In this section we introduce the notion of the Gelfand-Kirillov dimension and prove an important result that will be central in our study of Nichols algebras. For the basics we follow [23]. Let A be an (unital) $\mathbb{K}$-algebra.

Definition 3.1 Let $W \subseteq A$ be a vector space and

$$d_W(n) = \dim_{\mathbb{K}} \left(\mathbb{K} + W + W^2 + \cdots + W^n \right) .$$

The **Gelfand-Kirillov dimension** $\operatorname{GKdim}(A)$ of A is given by

$$\operatorname{GKdim}(A) = \sup_W \overline{\lim} \log_n d_W(n) ,$$

where the supremum is taken over all finite dimensional subspaces W of A.

Remark 3.2 If A is finitely generated and W is a generating subspace, i.e. $A = \sum_{i=1}^{\infty} W^i$, then the equation

$$\operatorname{GKdim}(A) = \overline{\lim} \log_n d_W(n)$$

holds and the actual choice of such W does not matter for calculating $\operatorname{GKdim}(A)$.

Example 3.3 [23, 1.2, 1.6]

(i) Let A be a locally finite dimensional algebra, i.e. every finitely generated subalgebra is finite dimensional. Then $\operatorname{GKdim}(A) = 0$ holds since for every finite dimensional subspace $W \subset A$ there is a $N \in \mathbb{N}$ such that the sequence $(\dim_{\mathbb{K}} W^n)_{n \geq N}$ is stable.

(ii) Let A be the free algebra on two generators x and y. Consider the finite dimensional subspace $W = \langle x, y \rangle_{\mathbb{K}}$. Then we have

$$d_W(n) = \dim_{\mathbb{K}} \left(\sum_{i=1}^{n} W^i \right) = \sum_{i=1}^{n} 2^i = 2^{n+1} - 1$$

and, consequently, $\operatorname{GKdim}(A) = \infty$.

(iii) Let $A = \mathbb{K}[X_1, \ldots, , X_m]$ be the polynomial ring in m Variables $X_1, \ldots, X_m$ and $W = \langle X_1, \ldots, X_m \rangle_{\mathbb{K}}$. Then W^n is spanned by monomials of degree $\leq n$ and $\dim_{\mathbb{K}}(W^n) = \binom{n+m}{m}$ is a polynomial in n of degree m. Since A is generated by W due to remark Remark 3.2 this yields $\operatorname{GKdim}(A) = m$.

Next, we want to give some results on how the Gelfand-Kirillov dimension interacts with algebraic constructions of a given algebra. When proving that an algebra is of infinite Gelfand-Kirillov dimension it is often helpful to consider subalgebras for which it is easy to prove that they are of infinite Gelfand-Kirillov. This is the approach that we will excessively use when studying the Gelfand-Kirillov dimension of Nichols algebras of diagonal type later on. With the following lemma we can conclude that the initial algebra has infinite Gelfand-Kirillov dimension as well.

Lemma 3.4 [23, 3.1] If B is a subalgebra or a homomorphic image of A, then

$$\operatorname{GKdim}(B) \leq \operatorname{GKdim}(A).$$

If no subalgebras of infinite Gelfand-Kirillov dimension can be found, the following lemma yields an approach to prove that a given algebra is of infinite Gelfand-Kirillov dimension relying on knowledge about the vector space structure of the algebra.

Lemma 3.5 [29, Lemma 19] Let $A = \bigoplus_{n \geq 0} A^n$ be a finitely generated graded algebra with $A^0 = \mathbb{K}$. Let $(y_k)_{k \geq 0}$ be a family of homogeneous elements of A such that $(y_{i_1} \ldots y_{i_k})_{0 \leq i_1 < \cdots < i_k}$ is a family of linearly independent elements. If there exist $m, p \in \mathbb{N}$ such that for all $i \in \mathbb{N}$ the inequality $\deg(y_i) \leq mi + p$ holds, then $\operatorname{GKdim} A = \infty$.

In the context of Nichols algebras when searching for families of linearly independent homogeneous elements needed for the application of above lemma it is natural to consider the vector space basis of ordered products of root vectors, see Corollary 6.2. Hence information on the set of roots will be crucial in our discussion.

The next result gives a precondition for the Gelfand-Kirillov dimensions of a filtered algebra and the associated graded algebra to coincide. We will need this for some technical proof, see Lemma 6.10.

Proposition 3.6 [23, 6.6] Suppose A has a filtration $\{A_i\}_{i\in\mathbb{N}_0}$ such that the subspaces A_i are finite dimensional for all $i \in \mathbb{N}_0$ and the associated algebra $\operatorname{gr} A$ of A is finitely generated. Then,

$$\operatorname{GKdim}(\operatorname{gr} A) = \operatorname{GKdim}(A).$$

Next, one might expect that the equation

$$\operatorname{GKdim}(A \otimes B) = \operatorname{GKdim}(A) + \operatorname{GKdim}(B)$$

holds for $\mathbb{K}$-algebras A and B. However, this is not true in general. There are plenty of proofs for this statement under diverse preconditions. We will use the following proposition which yields a sufficient result in our context. It is a variation of [23, 3.13].

Proposition 3.7 Let $A_1, A_2 \subseteq A$ be subalgebras of A such that the multiplication of A induces an isomorphism $A_1 \otimes A_2 \to A_1 A_2$. Suppose there is a finite dimensional subspace W of A_2 and a convergent strictly positive sequence $g : \mathbb{N} \to \mathbb{R}$ such that $\log_n d_W(n) \geq g(n)$ for all $n \in \mathbb{N}$. Then the following inequality holds:

$$\operatorname{GKdim}(A) \geq \operatorname{GKdim}(A_1) + \lim_{n\to\infty} g(n).$$

Proof. Let W be as in the claim, V be a finite dimensional subspace of A_1 and $U = V + W$. Now, $\operatorname{GKdim}(A)$ can be estimated via

$$\begin{aligned} \operatorname{GKdim}(A) &\overset{\text{Def}}{=} \sup_{V'} \overline{\lim} \log_n d_{V'}(n) \\ &\overset{\text{special VS}}{\geq} \overline{\lim} \log_n d_U(n) \\ &\overset{\text{reduced series}}{\geq} \overline{\lim} \log_n d_U(2n) \\ &\overset{\text{Def}}{=} \overline{\lim} \log_n \dim_{\mathbb{K}} \left(\sum_{i=0}^{2n} U^i \right). \end{aligned}$$

Because the multiplication of A induces an isomorphism $A_1 \otimes A_2 \to A_1 A_2$, every

$u \in U^i$ can be written as

$$u = \sum_{j \in J} \prod_{\ell \in L_j} v^{(\ell)} w^{(\ell)}$$

where J and L_j are finite index sets and $v^{(\ell)} \in V^{k_\ell}, w^{(\ell)} \in W^{m_\ell}$ for some $k_\ell, m_\ell \in \mathbb{N}_0$ satisfying $\sum_{\ell \in L_j}(k_\ell + m_\ell) \leq i$ for all $j \in J$. Restricting to those vectors where $|L_j| = 1$ for all $j \in J$ we obtain

$$\begin{aligned}
\overline{\lim} \log_n \dim_{\mathbb{K}} \left(\sum_{i=0}^{2n} U^i \right) \quad & \geq \quad \overline{\lim} \log_n \dim_{\mathbb{K}} \left(\sum_{i=0}^{2n} \sum_{j+k=i} \left(V^j \otimes W^k \right) \right) \\
& \overset{\text{restriction}}{\geq} \quad \overline{\lim} \log_n \dim_{\mathbb{K}} \left(\left(\sum_{j=0}^{n} V^j \right) \otimes \left(\sum_{k=0}^{n} W^k \right) \right) \\
& \overset{V,W \text{ fin. dim.}}{=} \quad \overline{\lim} \log_n \dim_{\mathbb{K}} \left(\sum_{j=0}^{n} V^j \right) \dim_{\mathbb{K}} \left(\sum_{k=0}^{n} W^k \right) \\
& \overset{\text{Def}}{=} \quad \overline{\lim} \log_n \left(d_V(n) d_W(n) \right) \\
& \overset{\text{claim}}{\geq} \quad \overline{\lim} \left(\log_n d_V(n) + g(n) \right) \\
& \overset{g(n) \text{ conv.}}{=} \quad \overline{\lim} \log_n d_V(n) + \lim_{n \to \infty} \log_n g(n) \, .
\end{aligned}$$

This completes the proof. □

4 | Introduction to Nichols algebras

4.1 Preliminaries on Lyndon words

Before studying Nichols algebras we recall the basics of Lyndon words. For a more comprehensive discussion we refer interested readers to [24]. Lyndon words will play an important role in the construction of a vector space basis for Nichols algebras of diagonal type. In this section we introduce the basic notions and give examples that will be of interest later on.

Let M be a finite set with a fixed ordering $<$. Denote by $\mathbb{M}$ and $\mathbb{M}^\times$ the sets of words and non-empty words with letters in M resp. and by $<_{\text{lex}}$ the lexicographical ordering on $\mathbb{M}$ extending $<$, i.e.

$$w <_{\text{lex}} w' \quad :\Leftrightarrow \quad \begin{array}{l} w' = w \cdot v \text{ for some } v \in \mathbb{M}^\times \text{ or} \\ \exists w'', v, v' \in \mathbb{M}, a, a' \in M, a < a' : \; w = w''av, \; w' = w''a'v' \,. \end{array} \tag{4.1}$$

Let $\ell : \mathbb{M} \to \mathbb{N}_0$ denote the usual length function on $\mathbb{M}$, i.e. $\ell(m_1 \cdots m_k) = k$.

Definition 4.1 A word $w \in \mathbb{M}^\times$ is called a **Lyndon word** if for any decomposition $w = v \cdot v'$ with $v, v' \in \mathbb{M}^\times$ the relation $w <_{\text{lex}} v' \cdot v$ holds. The set of Lyndon words over M will be denoted by $\mathcal{L}(M)$.

Proposition 4.2 [24] Let $w \in \mathcal{L}(M)$ be a Lyndon word with $\ell(w) \geq 2$. Then there is an unique decomposition $w = v \cdot v'$ of w into the product of two Lyndon words v and v' such that $\ell(v)$ is minimal.

Definition 4.3 The decomposition (v, v') of $w \in \mathcal{L}(M)$ with $\ell(w) \geq 2$ introduced in Proposition 4.2 is called the **Shirshov decomposition** of w .

We give some examples that will appear later in this work.

Example 4.4 Let $I = \{1, 2\}$ be a set with natural ordering. Note that the word 1 is minimal in $\mathbb{I}^\times$. Therefore every Lyndon word ends with 2 . Let $w = i_1 \cdots i_k \in \mathcal{L}(I)$

be a Lyndon word with

$$\#\{i_j | 1 \le j \le k,\ i_j = 2\} = m\,.$$

If $m = 1$ holds, then $w = 1^k 2$ is the only possible Lyndon word with this degree. The Shirshov decomposition is given by $(1, 1^{k-1}2)$.

If $m = 2$ holds, then $w = 1^{\ell_1} 2 1^{\ell_2} 2$ such that $\ell_1 > \ell_2$, $\ell_1 + \ell_2 = k$. If $\ell_1 = \ell_2 + 1$, then the Shirshov decomposition is $(1^{\ell_1}2, 1^{\ell_2}2)$. Otherwise it is given by $(1, 1^{\ell_1 - 1} 2 1^{\ell_2} 2)$.

Finally, we want to collect some examples with $m = 3$ that will appear later in our discussion. For $w = 112122$ the Shirshov decomposition is given by $(1, 12122)$.

For $w = 111211212$ the Shirshov decomposition is given by $(1, 11211212)$.

For $w = 111212112$ the Shirshov decomposition is given by $(111212, 112)$.

For $k, t \in \mathbb{N}$ and $w = 1^{k+1}2(1^k 2)^t$ the Shirshov decomposition is given by $(1^{k+1}2(1^k 2)^{t-1}, 1^k 2)$.

4.2 On braided vector spaces and related structures

In this chapter a brief introduction to braided vector spaces and Yetter-Drinfeld modules is given following [2]. Those will be the input data for the definition of Nichols algebras. Hence the Nichols algebra structure is determined by the corresponding braiding. This section is devoted to the introduction of the category we are working in and the kind of braidings we will restrict to later.

Definition 4.5 Let V be a vector space and $c : V \otimes V \to V \otimes V$ be a bijective linear map. The tuple (V, c) is called a **braided vector space** if c satisfies

$$(c \otimes \mathrm{id}_V)(\mathrm{id}_V \otimes c)(c \otimes \mathrm{id}_V) = (\mathrm{id}_V \otimes c)(c \otimes \mathrm{id}_V)(\mathrm{id}_V \otimes c)\,. \tag{4.2}$$

The map c is called the **braiding** of (V, c).

Usually if the context is clear, a braided vector space (V, c) will be denoted by V. If different braidings appear, we use indexes to indicate the corresponding braided vector space, e.g. c_V for (V, c_V). Now, we want to introduce the kind of braidings that are of interest in our further study.

Example 4.6 Let n be a natural number and V a n-dimensional braided vector space. If there is a vector space basis $\{x_i\}_{1\leq i\leq n}$ of V and $Q = (q_{ij})_{1\leq i,j\leq n} \in \mathbb{K}^{\times n\times n}$ such that

$$c : V \otimes V \to V \otimes V, \quad x_i \otimes x_j \mapsto q_{ij}\, x_j \otimes x_i\,,$$

then (V, c) is a braided vector space. In this case (V, c) is called a braided vector space **of diagonal type**. The matrix $Q = (q_{ij})_{1\leq i,j\leq n}$ is called the **braiding matrix of Vcorresponding to** $\{x_i\}_{1\leq i\leq n}$.

Next, we recall the notion of Yetter-Drinfeld modules. In the following the structures to consider are contained in the category of Yetter-Drinfeld modules. Let H be a Hopf algebra with bijective antipode S.

Definition 4.7 A **Yetter-Drinfeld module over** H is a triple $(V, \cdot, \delta)$ where

- $(V, \cdot)$ is an H-module.
- (V, δ) is an H-comodule.
- $\delta(h \cdot x) = h_{(1)}x_{(-1)}S(h_{(3)}) \otimes h_{(2)} \cdot x_{(0)}$.

Morphisms of Yetter-Drinfeld-modules over H are linear maps commuting with $\cdot$ and δ. The category of Yetter-Drinfeld-modules is denoted by ${}^H_H\mathcal{YD}$.

Remark 4.8 [2] ${}^H_H\mathcal{YD}$ is a braided monoidal category. The braiding of two Yetter-Drinfeld modules V and W is given by

$$c_{V,W} : V \otimes W \to W \otimes V, \quad c_{V,W}(x \otimes y) = x_{(-1)} \cdot y \otimes x_{(0)}$$

for all $x \in V$ and $y \in W$.

Example 4.9 Let G be a group, $H = \mathbb{K}G$ and $V \in {}^H_H\mathcal{YD}$. Then, for all $g \in G$ and $x \in V$ there are $x_g \in V$ such that

$$\delta(x) = \sum_{g\in G} g \otimes x_g\,.$$

Moreover, one checks

$$\sum_{g\in G} g \otimes g \otimes x_g = \sum_{g\in G} \Delta_H(g) \otimes x_g = \sum_{g\in G} g \otimes \delta(x_g)$$

using coassociativity. This and the equation $x = \sum_{g\in G} \epsilon_H(g)x_g = \sum_{g\in G} x_g$ imply that V is G-graded. Set $V_g = \{x \in V \mid \delta(x) = g \otimes x\}$.

Now, assume G is abelian. Then the equality

$$\delta(h \cdot v) = hgh^{-1} \otimes h \cdot x = g \otimes h \cdot x$$

holds for all $g, h \in G$, $x \in V_g$. Thus, V_g is a G-module for all $g \in G$.

Assume, that for every $g \in G$ the action of G on V_g is given by characters, that is $V = \bigoplus_{g \in G, \zeta \in \hat{G}} V_g^\zeta$ where $\hat{G}$ denotes the group of multiplicative characters of G and

$$V_g^\zeta = \{x \in V_g \mid h \cdot x = \zeta(h)x \text{ for all } h \in G\}.$$

In this case V is called a **Yetter-Drinfeld module of diagonal type**.

Now, let n a natural number and assume V is n-dimensional with vector space basis $\{x_1, \ldots, x_n\} \subset V$. Let $g_1, \ldots, g_n$ elements of G and $\zeta_1, \ldots, \zeta_n \in \hat{G}$ be given by $x_i \in V_{g_i}^{\zeta_i}$. Then the following holds:

$$c_{V,V}(x_i \otimes x_j) = (g_i \cdot x_j) \otimes x_i = \zeta_j(g_i)x_j \otimes x_i .$$

Hence V induces a braided vector space $(V, c_{V,V})$ of diagonal type with braiding matrix $Q = (q_{ij})_{1 \le i,j \le n}$ corresponding to $\{x_i \,|\, 1 \le i \le n\}$ where $q_{ij} = \zeta_j(g_i)$.

Example 4.10 [2] Let (V, c) be a braided vector space of diagonal type. Then there is a vector space basis $x_1, \ldots, x_n$ of V and $Q = (q_{ij})_{1 \le i,j \le n} \in {\mathbb{K}^\times}^{n \times n}$ such that $c(x_i \otimes x_j) = q_{ij} x_j \otimes x_i$ for all $1 \le i, j \le n$. For $1 \le i \le n$ identify $\alpha_i \in \mathbb{Z}^n$ with the automorphism of V given by $x_j \mapsto q_{ij} x_j$ for all $1 \le j \le n$. Then V is a Yetter-Drinfeld module over $\mathbb{K}\mathbb{Z}^n$ with coaction $\delta(x_i) = \alpha_i \otimes x_i$. The braiding induced by the Yetter-Drinfeld module structure coincides with c.

Next, we introduce algebras and coalgebras in ${}^H_H\mathcal{YD}$.

Definition 4.11 An **algebra in** ${}^H_H\mathcal{YD}$ is a triplet $(V, m_V, 1_V)$ where $V \in {}^H_H\mathcal{YD}$, $m_V : V \otimes V \to V$ is an associative multiplication with unit $1_V : \mathbb{K} \to V$ and $m_V, 1_V$ are morphisms in ${}^H_H\mathcal{YD}$.

Analogously, a **coalgebra in** ${}^H_H\mathcal{YD}$ is a triplet $(V, \Delta_V, \epsilon_V)$ where $V \in {}^H_H\mathcal{YD}$, $\Delta_V : V \to V \otimes V$ is a coassociative coproduct with counit $\epsilon_V : V \to \mathbb{K}$ and Δ_V, ϵ_V are morphisms in ${}^H_H\mathcal{YD}$.

Let $(V, m_V, 1_V)$ be an algebra in ${}^H_H\mathcal{YD}$. There is a "twisted" algebra structure on $V \otimes V$ given by $V \underline{\otimes} V = (V \otimes V, \underline{m}_{V \otimes V}, \underline{1}_{V \otimes V})$ where

$$\underline{m}_{V \otimes V} = (m_V \otimes m_V)(\mathrm{id}_V \otimes c \otimes \mathrm{id}_V)$$

and

$$\underline{1}_{V\otimes V} = 1_{V\otimes V} = 1_V \otimes 1_V \,.$$

Here, c replaces the usual swap.

Definition 4.12 A **bialgebra in** ${}^H_H\mathcal{YD}$ is the quintuple $(V, m_V, 1_V, \Delta_V, \epsilon_V)$ such that

- $(V, m_V, 1_V)$ is an algebra in ${}^H_H\mathcal{YD}$.
- $(V, \Delta_V, \epsilon_V)$ is a coalgebra in ${}^H_H\mathcal{YD}$.
- $\Delta : V \to V\underline{\otimes}V$ and $\epsilon_V : V \to \mathbb{K}$ are morphisms of algebras.

A **Hopf algebra in** ${}^H_H\mathcal{YD}$ is a bialgebra V in ${}^H_H\mathcal{YD}$ such that the identity is convolution invertible in $\mathrm{End}(V)$. The corresponding inverse is called the **antipode of** V.

Finally we recall a well-known result for the construction of subquotients of a given bialgebra in ${}^H_H\mathcal{YD}$ which are bialgebras.

Proposition 4.13 [10] Let B be a bialgebra in ${}^H_H\mathcal{YD}$ and let K be a right coideal subalgebra of B in ${}^H_H\mathcal{YD}$. For any coideal I of B in ${}^H_H\mathcal{YD}$ such that I is an ideal of K and

$$\Delta(K) \subseteq K \otimes K + I \otimes B \,,$$

the structure maps of B induce a bialgebra structure on K/I in ${}^H_H\mathcal{YD}$.

4.3 Preliminaries on Nichols Algebras

Nichols algebras first appeared in [27]. Today, various equivalent characterizations are known. In this chapter we recall the one that will be most convenient for the further discussion. Note that this introduction is not meant to be comprehensive. For more details refer to [2]. In the following let V be a Yetter-Drinfeld module over H.

Proposition 4.14 [2] The tensor algebra

$$T(V) = \bigoplus_{m=0}^{\infty} V^{\otimes m}$$

constitutes an $\mathbb{N}_0$-graded Hopf algebra in ${}^H_H\mathcal{YD}$ where for all $x \in V$ we have

$$\Delta_{T(V)}(x) = 1 \otimes x + x \otimes 1 \,, \quad \epsilon_{T(V)}(x) = 0 \,.$$

Definition 4.15 Let $\mathfrak{S}$ denote the set of homogeneous Hopf ideals of $T(V)$ with trivial intersection with $\mathbb{K} \oplus V$. For all $J \in \mathfrak{S}$ the quotient $T(V)/J$ is called **pre-Nichols algebra of**V.

Remark 4.16 [4] A $\mathbb{N}_0$-graded bialgebra A in ${}^H_H\mathcal{YD}$ generated by $A(1)$ consisting of primitive elements is a pre-Nichols algebra of $A(1)$.

There is a partial ordering on $\mathfrak{S}$ given by inclusion $\subset$.

Proposition 4.17 [2] The set $\mathfrak{S}$ has a maximal element $\mathcal{J}$ with respect to $\subset$.

Definition 4.18 Let $\mathcal{J}$ be as in Proposition 4.17. The quotient $T(V)/\mathcal{J}$ is called the **Nichols algebra of** V and is denoted by $\mathcal{B}(V)$. The dimension $\dim V$ is called the **rank** of $\mathcal{B}(V)$. $\mathcal{B}(V)$ is called **Nichols algebra of diagonal type** if V is of diagonal type.

Theorem 4.19 [2] $\mathcal{B}(V)$ is a $\mathbb{N}_0$-graded Hopf algebra in ${}^H_H\mathcal{YD}$ and the primitive elements $P(\mathcal{B}(V)) = \mathcal{B}(V)(1) = V$.

The greatest problem when working with Nichols algebras is that in general $\mathcal{J}$ can not be determined explicitly although there are several equivalent characterizations of this ideal [2]. Hence, in general it remains a difficult task to describe the vector space structure of a Nichols algebra. For those of diagonal type many tools have been developed to cope with this kind of problems. Moreover, finite-dimensional Nichols algebras of diagonal type have been classified in a series of papers [12, 11, 14, 17, 18]. Therefore, in the following we restrict to this case.

4.4 Nichols Algebras of diagonal type

We restrict to the case of Nichols algebras of diagonal type. Those are the most accessible and many strong tools have been developed to work in this setting. In this chapter we collect some well-known results. Some of those might be true in a more general context. The notions are mainly taken from [19]. In the following we might switch between braided vector spaces and Yetter-Drinfeld modules if necessary.

Let n be a natural number, $I = \{1, \dots, n\}$ and V be an n-dimensional Yetter-Drinfeld module of diagonal type. If the context is clear, the braiding will be denoted by c. Fix a vector space basis $X = \{x_i\}_{i \in I}$ satisfying

$$c(x_i \otimes x_j) = q_{ij} x_j \otimes x_i$$

for any $i, j \in I$ and $Q = (q_{ij})_{i,j\in I} \in (\mathbb{K}^\times)^{n\times n}$. First, we give another characterization of the defining ideal $\mathcal{J}$ in the definition of a Nichols algebra.

Remark 4.20 The $\mathbb{Z}^n$-grading of V given by Example 4.9 extends to an $\mathbb{Z}^n$-grading on $T(V)$ and $\mathcal{B}(V)$.

In the following we will denote the degree function corresponding to the $\mathbb{N}_0$-grading with $\deg_{\mathbb{N}}$. For homogeneous elements x in $T(V)$ or $\mathcal{B}(V)$ with respect to the $\mathbb{Z}^n$-grading we denote the degree by $\deg(x)$. By definition we have $\deg(x_i) = \alpha_i$. Let χ be the bicharacter on $\mathbb{Z}^n$ satisfying $\chi : \mathbb{Z}^n \times \mathbb{Z}^n \to \mathbb{K}^\times, \quad \chi(\alpha_i, \alpha_j) = q_{ij}$ for all $i, j \in I$.

Proposition 4.21 [2, 2.8] For all $i \in I$ there is a unique skew-derivation ∂_i of $T(V)$ satisfying

$$\partial_i(x_j) = \delta_{ij} \text{ (Kronecker Delta)}, \quad \partial_i(xy) = x\partial_i(y) + \chi(\alpha_i, \alpha)\partial(x)y$$

for any $j \in I$, $x, y \in T(V)$ with $\deg(y) = \alpha$. These skew-derivations induce skew-derivations of any pre-Nichols algebra of V which will be denoted by the same symbol.

Moreover, for any pre-Nichols algebra $T(V)/I$ the following holds:

$$\bigcap_{i\in I} \ker \partial_i = \mathbb{K} \quad \Leftrightarrow \quad I = \mathcal{J}\,.$$

This characterization of $\mathcal{B}(V)$ is very useful to decide whether some expression is 0 in $\mathcal{B}(V)$. This is not a trivial task since the ideal $\mathcal{J}$ is hard to determine in general. The following corollary recapitulates this fact.

Corollary 4.22 [2] For $x \in \mathcal{B}(V)$ the following equivalence holds:

$$x = 0 \iff \partial_i(x) = 0 \text{ for all } i \in I\,.$$

When determining the Gelfand-Kirillov dimensions for all rank two Nichols algebras of diagonal type the following property will yield cases to be treated identically.

Definition 4.23 Let $Q = (q_{ij})_{i,j\in I}$ and $Q' = (q'_{ij})_{i,j\in I}$ be matrices over $\mathbb{K}^\times$ and V and V' be Yetter-Drinfeld modules of diagonal type with vector space bases B of V and B' of V' such that Q is a braiding matrix of V corresponding to B and Q' is a braiding matrix of V' corresponding to B'. V and V' are called **twist-equivalent** if $q_{ii} = q'_{ii}$ and $q_{ij}q_{ji} = q'_{ij}q'_{ji}$ for all $i, j \in I$.

Lemma 4.24 [3, 1.2] Let V and V' be twist-equivalent Yetter-Drinfeld modules. Then $\mathcal{B}(V)$ and $\mathcal{B}(V')$ are isomorphic as vector spaces and $\dim(V^{\otimes m}) = \dim(V'^{\otimes m})$ for all $m \in \mathbb{N}_0$. In particular, the equation $\operatorname{GKdim} \mathcal{B}(V) = \operatorname{GKdim} \mathcal{B}(V')$ holds.

Next, we develop the framework of tools usually applied in the work with Nichols algebras, that is a vector space basis similar to the one of universal enveloping algebras of a Lie algebra together with so called reflections. These are excessively used in the classification of finite dimensional Nichols algebras of diagonal type and will be essential in our deduction as well.

Let $\mathbb{I}$ denote the set of non-empty words with letters in I. We fix the natural ordering on I and extend it to $\mathbb{I}$ via (4.1). Moreover, set

$$O_\alpha = \begin{cases} \{1, \operatorname{ord}(\chi(\alpha,\alpha)), \infty\} & \text{if } \operatorname{ord}(\chi(\alpha,\alpha)) = \infty \text{ or } p = 0\,. \\ \{1, \operatorname{ord}(\chi(\alpha,\alpha))\, p^k, \infty | k \in \mathbb{N}_0\} & \text{if } \operatorname{ord}(\chi(\alpha,\alpha)) < \infty,\ \operatorname{char}(\mathbb{K}) = p\,. \end{cases},$$

where $\operatorname{ord}(\chi(\alpha,\alpha))$ denotes the multiplicative order of $\chi(\alpha,\alpha)$. Additionally, set

$$\mathcal{N}(I) := \{w^k | w \in \mathcal{L}(I), k \in \mathbb{N}\} \text{ and } \mathfrak{R}(V) := \{w^k | w \in \mathcal{L}(I), k \in O_\alpha \setminus \{\infty\}\}\,.$$

Remark 4.25 The set $\mathfrak{R}(V)$ is the set of root vector candidates in [19].

Definition 4.26 We define a mapping $[\,\cdot\,] : \mathcal{N}(I) \to \mathcal{B}(V)$ inductively as follows:

$$\begin{aligned} [i] &= x_i \text{ for } i \in I\,. \\ [w] &= [v][v'] - \chi(\deg(v), \deg(v'))\,[v'][v] \text{ if } w \in \mathcal{L}(I), \ell(w) \geq 2 \text{ and} \\ &\quad (v, v') \text{ is the Shirshov decomposition of } w\,. \\ [w^k] &= [w]^k \text{ for any } w \in \mathcal{L}(I) \text{ and } k \geq 2\,. \end{aligned}$$

For $w \in \mathcal{N}(I)$ the vector $[w]$ is called **superletter of** w.

The lexicographical ordering of $\mathbb{I}$ can be extended to an ordering of superletters via

$$[w] <_{\text{lex}} [v] \iff w <_{\text{lex}} v.$$

For $w \in \mathfrak{R}(V)$ let $\mathcal{B}^{>w}$ denote the vector space spanned by products

$$[v_m]^{k_m} \cdots [v_1]^{k_1}$$

where $m \in \mathbb{N}_0$, $k_1, \ldots, k_m \in \mathbb{N}$ and $v_1, \ldots, v_m \in \mathfrak{R}(V)$ with $w <_{\text{lex}} v_1 <_{\text{lex}} \cdots <_{\text{lex}} v_m$ and $\deg([v_m]^{k_m} \cdots [v_1]^{k_1}) = \deg(w)$.

Definition 4.27 Let w be an element in $\mathfrak{R}(V)$. $[w]$ is a **root vector of $\mathcal{B}(V)$** if $[w] \notin \mathcal{B}^{>w}$.

Remark 4.28 If $[w]$ is not a root vector for $w \in \mathfrak{R}(V)$, then $w \in \mathcal{B}^{>w}$. Let $[v_\ell]^{t_\ell} \cdots [v_1]^{t_1}$ be one summand in the representation of $[w]$ in $\mathcal{B}^{>w}$. Then

$$w <_{\text{lex}} v_1^{t_1} \cdot \cdots \cdot v_\ell^{t_\ell} .$$

Remark 4.29 [21, Lemma 5] Let $w = i_1 \cdots i_k$ be a word in $\mathbb{I}$ for $k \in \mathbb{N}$. We set $x_w = x_{i_1} x_{i_2} \cdots x_{i_k}$. Then $<_{\text{lex}}$ can be extended to the vectors x_w via

$$x_w <_{\text{lex}} x_v :\Leftrightarrow w <_{\text{lex}} v$$

for any $w, v \in \mathbb{I}$. Thus, in the representation of $[w]$ with $w \in \mathfrak{R}(V)$ after application of $[\,\cdot\,]$ the minimal term with respect to $<_{\text{lex}}$ is x_w with coefficient 1.

A consequence of this fact is the following: Let $m > 1, k_1, \ldots, k_m$ non-negative integers and $w = 1^{k_1} 2 \cdots 1^{k_m} 2 \in \mathcal{L}(I)$. For every $1 \leq j < m$ the following holds:

$$[[1^{k_1} 2 \cdots 1^{k_j} 2], [1^{k_{j+1}} 2 \cdots 1^{k_m} 2]]_c \in [w] + \mathcal{B}^{>w} .$$

Especially, if $[w]$ is a root vector, this implies $[[1^{k_1} 2 \cdots 1^{k_j} 2], [1^{k_{j+1}} 2 \cdots 1^{k_m} 2]]_c \neq 0$ for any $j \in \{1, \ldots, m\}$.

Theorem 4.30 [21] Let $L = \{y \in \mathfrak{R}(V) \mid y \text{ is a root vector}\}$ be the set of root vectors. Then the elements

$$\begin{aligned} [y_k]^{m_k} \cdots [y_1]^{m_1}, \quad & k \in \mathbb{N}_0, y_1, \ldots, y_k \in L, y_1 <_{\text{lex}} \cdots <_{\text{lex}} y_k , \\ & 0 < m_i < \min(O_{\deg(y_i)} \setminus \{1\}) \text{ for any } 1 \leq i \leq k , \end{aligned}$$

form a vector space basis of $\mathcal{B}(V)$.

Corollary 4.31 The total ordering $<_{\text{lex}}$ can be exchanged by any total ordering $\ll$ on L in Theorem 4.30.

Proof. Consider the following partial ordering on the set of monomials $[v_k]^{m_k} \cdots [v_1]^{m_1}$ with $k \in \mathbb{N}_0, m_1, \ldots, m_k \in \mathbb{N}, v_i \in L$:

$$[v_k]^{m_k} \cdots [v_1]^{m_1} \prec [v'_\ell]^{m'_\ell} \cdots [v'_1]^{m'_1} :\Leftrightarrow (c_1, \ldots, c_n) \leq_{lex} (c'_1, \ldots, c'_n)$$

where $k, \ell \in \mathbb{N}$ and $c_i, c'_i \in \mathbb{N}_0$ count the number of appearances of x_i in $[v_k]^{m_k} \cdots [v_1]^{m_1}$ and $[v'_k]^{m'_k} \cdots [v'_1]^{m'_1}$ respectively.

It is easy to see that (K1)-(K3) and (B1)-(B3) in 2 are satisfied by $\prec$. Thus, we can apply Proposition 2.2 in our case to exchange the total ordering $<_{\text{lex}}$ by another one. $\square$

Definition 4.32 Let L be the set from Theorem 4.30. The sets

$$\triangle_+ = \{\deg([u]) \,|\, u \in L\} \subset \mathbb{Z}^n \text{ and } \triangle = \triangle_+ \cup -\triangle_+$$

are called **set of positive roots of $\mathcal{B}(V)$** and the **root system of $\mathcal{B}(V)$**, respectively. For any $\alpha \in \triangle_+$ the **multiplicity of α** is given by

$$\text{mult}(\alpha) = \#\{u \in L |\, \deg([u]) = \alpha\}.$$

As the name suggests root systems of Nichols algebras are similar to roots systems of Lie algebras. In fact the former are a generalization of the latter ones, see [9]. There are also reflections for the root systems of Nichols algebras, but in general these will not form a group, but a groupoid. We do not develop the full theory here because this would lead to far. We recall some results that will be applied later in this work. See [16] for a more comprehensive introduction to Weyl groupoids.

Definition 4.33 Let $P = (p_{jk})_{j,k \in I} \in (\mathbb{K}^\times)^{n \times n}$ be a matrix. P is called ***i*-finite** for $i \in I$ iff the set

$$\left\{k \in \mathbb{N}_0 \,|\, (k+1)_{p_{ii}}(1 - p_{ii}^k p_{ij} p_{ji}) = 0\right\}$$

is non-empty.

In the following assume Q is i-finite for all $i \in I$.

Definition 4.34 A matrix $C = (c_{ij})_{i,j \in I} \in \mathbb{Z}^{n \times n}$ is called a **generalized Cartan matrix** iff

- $c_{ii} = 2$ and $c_{jk} \leq 0$ for all $i, j, k \in I, j \neq k$.
- $c_{ij} = 0 \Rightarrow c_{ji} = 0$ for all $i, j \in I$.

Lemma 4.35 [13, Lemma 3] Set $c_{ii} = 2$ and

$$c_{ij} = -\min\left\{k \in \mathbb{N}_0 \,|\, (k+1)_{q_{ii}}(1 - q_{ii}^k q_{ij} q_{ji}) = 0\right\}$$

for all $i, j \in I$. Then the matrix $C^V = (c_{ij})_{i,j \in I}$ is a generalized Cartan matrix.

Definition 4.36 The matrix C^V in the above lemma will be called the **Cartan matrix of $\mathcal{B}(V)$**.

If the context is clear, we denote C^V by C.

Theorem 4.37 [5, 2.4] For any $i \in I$ there is a n-dimensional Yetter-Drinfeld module $\mathcal{R}^i(V)$ with vector space basis $\{y_1 \,|\, 1 \leq i \leq n\}$ satisfying

$$c_{\mathcal{R}^i(V),\mathcal{R}^i(V)}(y_j \otimes y_k) = p_{jk} p_{ik}^{-c_{ij}} p_{ji}^{-c_{ik}} p_{ii}^{c_{ij}c_{ik}} \; y_k \otimes y_j$$

and $\text{GKdim}\,(\mathcal{B}(V)) = \text{GKdim}\,(\mathcal{B}(\mathcal{R}^i(V)))$.

Let $\triangle_+^W$ and $\triangle^W$ denote the set of positive roots and the root system of $\mathcal{B}(W)$ for a braided vector space (W, c_W), respectively.

Theorem 4.38 [13] For $i \in I$ there is a $\mathbb{Z}$-linear bijective map $s_i^V : \triangle^V \to \triangle^{\mathcal{R}^i(V)}$ given by

$$s_i(\alpha_j) = \alpha_j - c_{ij}\alpha_i.$$

This map satisfies

$$s_i^V(\triangle_+^V \setminus \{\alpha_i\}) = \triangle_+^{\mathcal{R}^i(V)} \setminus \{\alpha_i\}, \quad s_i^V(\triangle^V) = \triangle^{\mathcal{R}^i(V)}.$$

Moreover, the multiplicities of α and $s_i(\alpha)$ coincide for any $\alpha \in \triangle^V$.

If the context is clear, we denote s_i^V by s_i.

Note that if V and $\mathcal{R}^i(V)$ are twist-equivalent, then we can identify $\triangle^V$ and $\triangle^{\mathcal{R}^i(V)}$. Hence s_i^V is an automorphism of $\triangle^V$ is this case. Next, we consider one special case.

Definition 4.39 A Nichols algebra $\mathcal{B}(V)$ of diagonal type is **of Cartan-type** iff for all $i, j \in I$ there is some $k \in \mathbb{N}$ such that $q_{ij}q_{ji} = q_{ii}^k$.

Moreover, we say that $\mathcal{B}(V)$ is **of finite or affine Cartan-type** if it is of Cartan-type and the corresponding Cartan matrix is of finite type or affine type resp. (for details refer to [20]).

The next theorem motivates the notion we just introduced. Let $\mathcal{B}(V)$ be of Cartan-type. Then V and $\mathcal{R}^i(V)$ are twist-equivalent for all $i \in I$. Hence the automorphisms s_i generate a group.

Theorem 4.40 [9, Thm. 3.3] Let $\mathcal{B}(V)$ be of Cartan-type with Cartan matrix C. The group generated by the s_i is isomorphic to the Weyl-group associated with C.

5 | On the root system of rank two Nichols algebras of diagonal type

We have seen in the last chapter that for the classification of finite Gelfand-Kirillov dimensional Nichols algebras of diagonal type it will suffice to identify twist equivalent Yetter-Drinfeld modules. This motivates the following definition.

Definition 5.1 Let $q, r, s \in \mathbb{K}^\times$ and $b(q,r,s)$ denote the full subcategory of ${}^{\mathbb{K}\mathbb{Z}^2}_{\mathbb{K}\mathbb{Z}^2}\mathcal{YD}$ consisting of those objects V that have a vector space basis (x_1, x_2) such that

$$c_{V,V}(x_i \otimes x_j) = q_{ij} x_j \otimes x_i$$

where $q_{11} = q$, $q_{12}q_{21} = r$ and $q_{22} = s$. We call $b(q,r,s)$ the **category of braided vector spaces distinguished by** (q,r,s). For $V \in \mathrm{Ob}(b(q,r,s))$ a vector space basis (x_1, x_2) as above is called **basis distinguished by** (q,r,s).

We declare objects in this category via $V \in b(q,r,s)$.

Remark 5.2 If $V \in b(q,r,s)$, then obiviously $V \in b(s,r,q)$ by exchanging the indices of the basis vectors.

From now on let $q, r, s \in \mathbb{K}^\times$ and $V \in b(q,r,s)$ with basis (x_1, x_2) distinguished by (q,r,s). Let $Q = (q_{ij})_{1 \le i,j \le 2}$ denote the corresponding braiding matrix of V with respect to (x_1, x_2) and $\mathcal{B}(V)$ the corresponding Nichols algebra. Moreover, we will use the notation

$$q_{k\alpha_1 + \ell\alpha_2} = \chi(k\alpha_1 + \ell\alpha_2, k\alpha_1 + \ell\alpha_2) = q^{k^2} r^{k\cdot\ell} s^{\ell^2}\,.$$

For the analysis of $\mathrm{GKdim}\,(\mathcal{B}(V))$ knowledge on $\triangle_+$ will be crucial. If $\triangle_+$ is finite, it is well-known that all roots have multiplicity 1 and for $\alpha \in \triangle_+$ the cardinality of $\{k\alpha \,|\, k \in \mathbb{N}\}$ is one, see [15]. This leads to the approach to use roots of multiplicity 2 or roots with multiples to prove $\mathrm{GKdim}\,(\mathcal{B}(V)) = \infty$. As stated before, calculating whether a given root exists or not is very much depending on c. Thus, there are not many general results on the existence of roots. In this chapter we collect the information we will need in the following discussion.

When working with rank two Nichols algebras of diagonal type a set of special vectors proved to be useful. For $x \in \mathcal{B}(V)$ let ad_x denote the adjoint action of $\mathcal{B}(V)$ on itself, i.e.

$$\mathrm{ad}_x(y) = [x, y]_c = xy - \chi(\deg(x), \deg(y))\, yx\,.$$

Here, $[\,\cdot\,,\,\cdot\,]_c$ is called braided commutator.

$$\text{Set } u_0 = x_2 \text{ and } u_{k+1} = \mathrm{ad}_{x_1}(u_k) \text{ for all } k \in \mathbb{N}.$$

Note that $\deg(u_k) = k\alpha_1 + \alpha_2$. In this context for $k \in \mathbb{N}$ the elements

$$b_k = \prod_{i=0}^{k-1}(1 - q^i r) \in \mathbb{K}$$

appear frequently.

5.1 Calculus with u_k

First, we will collect some well known properties of the vectors u_k:

$$\partial_1(u_k) = 0\,, \quad \partial_2(u_k) = b_k x_1^k\,,$$

$$\Delta(u_k) = u_k \otimes 1 + 1 \otimes u_k + \sum_{i=0}^{k-1} \binom{k}{i}_q \frac{b_k}{b_i}\, x_1^{k-i} \otimes u_i\,. \tag{5.1}$$

Lemma 5.3 Let m be a natural number and $k_1, \ldots, k_m \in \mathbb{N}_0$. Then the following holds:

$$\begin{aligned} x_1 u_{k_1} u_{k_2} \cdots u_{k_m} \;=\; & \left(\sum_{i=1}^{m} q^{\sum_{j=1}^{i-1} k_j} q_{12}^{i-1}\, u_{k_1} \cdots u_{k_{i-1}} u_{k_i+1} u_{k_{i+1}} \cdots u_{k_m} \right) \\ & + q^{\sum_{j=1}^{m} k_j} q_{12}^{n}\, u_{k_1} \cdots u_{k_m} x_1\,. \end{aligned}$$

Proof. We argue by induction on m.

For $m = 1$ by definition the equation

$$u_{k+1} = x_1 u_k - q^k q_{12}\, u_k x_1$$

holds. By rearrangement of the terms this yields

$$x_1 u_k = u_{k+1} + q^k q_{12}\, u_k x_1 \,.$$

This was claimed.

For arbitrary $m \geq 2$ we calculate

$$\begin{aligned}
& x_1 u_{k_1} \cdots u_{k_m} \\
= \quad & \big(x_1 u_{k_1} \cdots u_{k_{m-1}}\big)\, u_{k_m} \\
\overset{\text{induction hyp.}}{=} \quad & \left(\sum_{i=1}^{m-1} q^{\sum_{j=1}^{i-1} k_j} q_{12}^{i-1}\, u_{k_1} \cdots u_{k_{i-1}} u_{k_i+1} u_{k_{i+1}} \cdots u_{k_{m-1}} u_{k_m}\right) \\
& + q^{\sum_{j=1}^{m-1} k_j} q_{12}^{m-1}\, u_{k_1} \cdots u_{k_{m-1}} x_1 u_{k_m} \\
= \quad & \left(\sum_{i=1}^{m-1} q^{\sum_{j=1}^{i-1} k_j} q_{12}^{i-1}\, u_{k_1} \cdots u_{k_{i-1}} u_{k_i+1} u_{k_{i+1}} \cdots u_{k_m}\right) \\
& + q^{\sum_{j=1}^{m-1} k_j} q_{12}^{m-1}\, u_{k_1} \cdots u_{k_{m-1}} u_{k_m+1} + q^{\sum_{j=1}^{m} k_j} q_{12}^{m}\, u_{k_1} \cdots u_{k_m} x_1 \\
= \quad & \left(\sum_{i=1}^{m} q^{\sum_{j=1}^{i-1} k_j} q_{12}^{i-1}\, u_{k_1} \cdots u_{k_{i-1}} u_{k_i+1} u_{k_{i+1}} \cdots u_{k_m}\right) \\
& + q^{\sum_{j=1}^{m-1} k_j} q_{12}^{n}\, u_{k_1} \cdots u_{k_m} x_1 .
\end{aligned}$$

□

Since $\mathcal{B}(V)$ is defined as a quotient in general it is not clear whether some product or commutator of elements is 0. The next statements address this problem. It is a variation of [5, 4.7].

Lemma 5.4 Let m, k, j be natural numbers with $m \leq k$. Then the following equation holds:

$$\partial_1^m \partial_2(u_k^j) = b_k (k)_q \cdot \ldots \cdot (k-m+1)_q \sum_{i=0}^{j-1} (q^{k \cdot m} q_{12}^m q_{21}^k s)^i \; u_k^{j-i-1} x_1^{k-m} u_k^i \,.$$

For the special case $m = k$ we get $\partial_1^k \partial_2(u_k^j) = b_k (k)_q^! (j)_{q^{k^2} r^k s}\, u_k^{j-1}$.

Proof. This follows from direct application of the skew derivatives. □

Corollary 5.5 [29, Lemma 14] The following holds: $u_k = 0$ iff $b_k (k)_q^! = 0$.

Lemma 5.6 [19, 4.1] Let k be a natural number such that $u_k \neq 0$. Then $u_k^2 = 0$ iff $q^{k^2 r^k s} = -1$ and $u_{k+1} = 0$.

Finally, we prove some formulas on the comultiplication of the u_k. These will be used to find primitive elements in quotients of certain subalgebras of $\mathcal{B}(V)$.

Lemma 5.7 For $k \geq 2$ the following holds:

$$\begin{aligned}\Delta([u_{k+1}, u_{k-1}]_c) &= 1 \otimes [u_{k+1}, u_{k-1}]_c + [u_{k+1}, u_{k-1}]_c \otimes 1 \\ &\quad +(1 - q^k r)(k+1)_q q^{k(k-1)} r^{k-1} s q_{12}\, u_k \otimes u_k \\ &\quad + \text{terms } x \otimes y \text{ with } \deg(x) = a_1\alpha_1 + a_2\alpha_2,\ \frac{a_1}{a_2} > k\,.\end{aligned}$$

Proof. Recall (5.1). We calculate

$$\begin{aligned}&\Delta([u_{k+1}, u_{k-1}]_c) \\ =\ & \Delta(u_{k+1})\Delta(u_{k-1}) - q^{k^2-1} r^{k-1} s q_{12}^2\, \Delta(u_{k-1})\Delta(u_{k+1}) \\ =\ & \left(1 \otimes u_{k+1} + u_{k+1} \otimes 1 + \sum_{i=1}^{k} \binom{k+1}{i}_q \frac{b_{k+1}}{b_i}\, x_1^{k+1-i} \otimes u_i\right) \Delta(u_{k-1}) \\ & -q^{k^2-1} r^{k-1} s q_{12}^2\, \Delta(u_{k-1}) \left(1 \otimes u_{k+1} + u_{k+1} \otimes 1 + \sum_{i=1}^{k} \binom{k+1}{i}_q \frac{b_{k+1}}{b_i}\, x_1^{k+1-i} \otimes u_i\right) \\ =\ & \Big(1 \otimes u_{k+1}u_{k-1} + q^{k^2-1} r^{k-1} s q_{12}^2\, u_{k-1} \otimes u_{k+1} \\ & +u_{k+1}u_{k-1} \otimes 1 + q^{k(k-1)} r^{k-1} s q_{12} (1 - q^k r)(k+1)_q\, x_1 u_{k-1} \otimes u_k \\ & + \text{terms } x \otimes y \text{ with } \deg(x) = a_1\alpha_1 + a_2\alpha_2,\ \frac{a_1}{a_2} > k\Big) \\ & -q^{k^2-1} r^{k-1} s q_{12}^2\ \Big(1 \otimes u_{k-1}u_{k+1} + u_{k-1} \otimes u_{k+1} \\ & +u_{k-1}u_{k+1} \otimes 1 + (1 - q^k r)(k+1)_q\, u_{k-1}x_1 \otimes u_k \\ & + \text{terms } x \otimes y \text{ with } \deg(x) = a_1\alpha_1 + a_2\alpha_2,\ \frac{a_1}{a_2} > k\Big) \\ =\ & 1 \otimes [u_{k+1}, u_{k-1}]_c + [u_{k+1}, u_{k-1}]_c \otimes 1 \\ & +(1 - q^k r)(k+1)_q q^{k(k-1)} r^{k-1} s q_{12}\, \underbrace{(x_1 u_{k-1} - q^{k-1} q_{12}\, u_{k-1} x_1)}_{=u_k} \otimes u_k \\ & + \text{terms } x \otimes y \text{ with } \deg(x) = a_1\alpha_1 + a_2\alpha_2,\ \frac{a_1}{a_2} > k\,.\end{aligned}$$

This was claimed. ∎

Lemma 5.8 For $k, m \in \mathbb{N}$ the following holds:

$$\begin{aligned}\Delta(u_k^m) &= \sum_{i=0}^{m} \binom{m}{i}_{q^{k^2} r^k s} u_k^i \otimes u_k^{m-i} \\ &\quad + \text{terms } x \otimes y \text{ with } \deg(x) = a_1\alpha_1 + a_2\alpha_2,\ \frac{a_1}{a_2} > k\,.\end{aligned}$$

Proof. We argue by induction on m. For $m = 1$ this was stated in (5.1).

Now, assume $m \geq 2$. Then the following equations holds:

$$\begin{aligned}\Delta(u_k^m) &= \Delta(u_k^{m-1})\Delta(u_k) \\ &\overset{\text{ind. hyp.}}{=} \Bigg(\sum_{i=0}^{m-1} \binom{m-1}{i}_{q^{k^2} r^k s} u_k^i \otimes u_k^{m-1-i} \\ &\qquad + \text{terms } x \otimes y \text{ with } \deg(x) = a_1\alpha_1 + a_2\alpha_2,\ \frac{a_1}{a_2} > k\Bigg) \\ &\qquad \left(u_k \otimes 1 + 1 \otimes u_k + \sum_{i=0}^{k-1} \binom{k}{i}_q b_k\, x_1^{k-i} \otimes u_i\right) \\ &= \left(\sum_{i=0}^{m-1} \binom{m-1}{i}_{q^{k^2} r^k s} u_k^i \otimes u_k^{m-1-i}\right)(u_k \otimes 1 + 1 \otimes u_k) \\ &\qquad + \text{terms } x \otimes y \text{ with } \deg(x) = a_1\alpha_1 + a_2\alpha_2,\ \frac{a_1}{a_2} > k \\ &= \sum_{i=0}^{m-1} (q^{k^2} r^k s)^{m-1-i} \binom{m-1}{i}_{q^{k^2} r^k s} u_k^{i+1} \otimes u_k^{m-1-i} \\ &\qquad + \sum_{i=0}^{m-1} \binom{m-1}{i}_{q^{k^2} r^k s} u_k^i \otimes u_k^{m-i} \\ &\qquad + \text{terms } x \otimes y \text{ with } \deg(x) = a_1\alpha_1 + a_2\alpha_2,\ \frac{a_1}{a_2} > k \\ &= u_k^m \otimes 1 + 1 \otimes u_k^m \\ &\qquad + \sum_{i=1}^{m-1} \left((q^{k^2} r^k s)^{m-i} \binom{m-1}{i-1}_{q^{k^2} r^k s} + \binom{m-1}{i}_{q^{k^2} r^k s}\right) u_k^i \otimes u_k^{m-i} \\ &\qquad + \text{terms } x \otimes y \text{ with } \deg(x) = a_1\alpha_1 + a_2\alpha_2,\ \frac{a_1}{a_2} > k \\ &= \sum_{i=0}^{m} \binom{m}{i}_{q^{k^2} r^k s} u_k^i \otimes u_k^{m-i} \\ &\qquad + \text{terms } x \otimes y \text{ with } \deg(x) = a_1\alpha_1 + a_2\alpha_2,\ \frac{a_1}{a_2} > k\,.\end{aligned}$$

This was claimed. □

5.2 Existence of roots

Although rank two Nichols algebras of diagonal type have been studied intensively in general little is known about the corresponding root system for a given braiding. Recently, there was progress in this direction. For our deduction it will be essential to know whether for some $m \in \mathbb{N}$ there exists an $k \geq 2$ such that $k(m\alpha_1+\alpha_2) \in \triangle_+$. For general $k \geq 3$ no general result in this direction is known so far. For $k = 2$ this was discussed in [19] and [34]. First, we recall those results. Later we develop some new results on the existence of roots.

First, we need to connect the vectors u_k to Lyndon words. For any $k \in \mathbb{N}_0$ by definition $u_k = [1^k2]$ as superletter due to the Shirshov decomposition of 1^k2. Moreover, $\mathcal{B}^{>1^k2} = \{0\}$ since 1 is minimal in $\mathbb{I}$ with respect to $<_{\text{lex}}$. Hence $u_k \neq 0$ iff $k\alpha_1+\alpha_2$ is a root by Example 4.4. Moreover, $k\alpha_1+\alpha_2 \in \triangle_+ \;\Rightarrow\; (k-1)\,\alpha_1+\alpha_2 \in \triangle_+$ for all $k \in \mathbb{N}$.

Lemma 5.9 [19, 3.5,4.1] If u_k^2 is a root vector with $k \in \mathbb{N}$, then

$$u_{k+1} \neq 0 \text{ and } q^{k^2}r^ks = -1\,.$$

Lemma 5.10 [19, 4.3] Let k, ℓ be natural numbers such that $k > \ell$ and $u_k \neq 0$. If $[1^k21^\ell2]$ is not a root, then $[1^{k+1}21^{\ell-1}2]$ is not a root.

The preceding lemma shows that the superletters of maximal Lyndon words with respect to $<_{\text{lex}}$ are more likely root vectors than smaller ones. However, such a strong dependency does not hold for superletters of Lyndon words of degree $m_1\alpha_1 + m_2\alpha_2$ with $m_1, m_2 \geq 3$. We give an example later on, see Lemma 5.23.

Next, we recall the classification of roots $m\alpha_1 + 2\alpha_1$ with $k \in \mathbb{N}$ from [34].

Definition 5.11 [34, 3.1,3.6] Let $\mathbb{J} \subset \mathbb{N}_0$ be the set of those $j \in \mathbb{N}_0$ satisfying one of the following

(1) $q^{j(j-1)/2}(-r)^js = -1$ and $q^{k+j-1}r^2 \neq 1$ for all $k \in \mathbb{J}$, $k < j$.

(2) there exists some $k \in \mathbb{J}$ with $k < j$ such that

$$\begin{cases} q^{k+j-1}r^2 = 1 \text{ and } 2p \,|\, (j-k), & \text{if } j-k \text{ is even.} \\ q^{(k+j-1)/2}r = -1 \text{ and } p \,|\, (j-k), & \text{if } j-k \text{ is odd.} \end{cases}$$

Remark 5.12 [34, 3.6] Note that in case (2) of the above definition the relation $q^{j(j-1)/2}(-r)^js = -1$ is also fulfilled.

The following lemma gives a restriction to the cardinality of $\mathbb{J} \cap [0, k]$. Note that this will essentially be the reason for Nichols algebras with $u_k \neq 0$, $k \geq 8$ to be of infinite Gelfand-Kirillov dimension, see Lemma 7.3.

Lemma 5.13 [34, 3.5] For $j \in \mathbb{J}$ we have $j+1, j+2 \notin \mathbb{J}$. In particular, for any $k \in \mathbb{N}_0$ the following inequality holds:

$$\#\left(\mathbb{J} \cap [0, k]\right) \leq \frac{k}{3} + 1 \,.$$

Theorem 5.14 [34, 3.23] Let $m \in \mathbb{N}_0$ such that $u_m \neq 0$. Then the multiplicity of $m\alpha_1 + 2\alpha_2$ is given by

$$m' - \#\left(\mathbb{J} \cap [0, m]\right) \,,$$

where m' depends on m as follows

$$m' = \begin{cases} (m+1)/2 & \text{if } m \text{ is odd,} \\ m/2 & \text{if } m \text{ is even and } q^{m^2/4} r^{m/2} s \neq 1 \,, \\ m/2 + 1 & \text{if } m \text{ is even and } q^{m^2/4} r^{m/2} s = -1 \,. \end{cases}$$

Remark 5.15 By Remark 5.2 the above theorem implies an analogous result for roots of the form $2\alpha_1 + m\alpha_2$.

As stated before such a simple answer to the question whether $m_1\alpha_1 + m_2\alpha_2$ with $m_1, m_2 \geq 3$ can not be expected due to the more complex set of corresponding Lyndon words. In the following we prove some special results in this direction. As stated above it appears natural to check Lyndon words that are large with respect to $<_{\text{lex}}$. This is due to the fact that the dimension of $\mathcal{B}^{>w}$ is smaller for such w.

Lengthy formulas where shifted to Appendix B to improve readability.

Remark 5.16 Assume $w = 1^{k+1}21^{k-1}2$. Then $\mathcal{B}^{>w} = \langle u_k^2 \rangle_{\mathbb{K}}$. This follows directly from Remark 4.28. Hence if $2k\alpha_1 + 2\alpha_2$ is not a root, this implies $[w] \in \mathbb{K}\, u_k^2$ by Lemma 5.10.

First, we prove under special conditions on q, r and s that [112122] is a root vector.

Lemma 5.17 Let $w = 112122 \in \mathfrak{R}(V)$. Then

$$\mathcal{B}^{>w} = \langle [112212],\, u_1[1122],\, u_1^3 \rangle_{\mathbb{K}} \,.$$

Proof. Let $[v_\ell]^{t_\ell} \cdots [v_1]^{t_1}$ be a generator of $\mathcal{B}^{>w}$. Then $w <_{\text{lex}} v_1^{t_1} \cdot \cdots \cdot v_\ell^{t_\ell} =: v$ by Remark 4.28. Now, if v starts with 12, then there are two 1's and two 2's to be arranged. Since $v_1 <_{\text{lex}} v_i$ for all $2 \le i \le \ell$ we conclude $v = (12)^3$. Thus, $u_1^3 \in \mathcal{B}^{>w}$. Furthermore, if v started with 1112, this yields $v <_{\text{lex}} w$, a contradiction. Thus, assume $v = 112v'$ with $v' \in \mathbb{I}$. If v' starts with 1, then due to $w <_{\text{lex}} v$ it starts with 122 and hence $v = w$, a contradiction. Thus, v' starts with 2 and cannot end with 1 due to $1 <_{\text{lex}} w$. This yields $v = 112212$. There are only two decompositions into Lyndon words satisfying the assumptions on v, namely $1122 \cdot 12$ and 112212. The claim follows. □

Corollary 5.18 Let $w = 112122 \in \mathfrak{R}(V)$. If [1122] is not a root vector, then $[w] \in \mathcal{B}^{>w}$ iff $[w] \in \mathbb{K}\, u_1^3$.

Proof. First, note $\mathcal{B}^{>w} = \langle [112212],\, u_1[1122],\, u_1^3 \rangle_{\mathbb{K}}$. Now, [1122] is not a root vector and, consequently, $[1122] = \lambda u_1^2$ for some $\lambda \in \mathbb{K}$ by Remark 5.16. Therefore, $u_1[1122]$ and $[112212] = [[1122], u_1]_c$ are multiples of u_1^3. Thus, $\mathcal{B}^{>w} = \mathbb{K}u_1^3$. □

Lemma 5.19 Assume $u_3 \neq 0$, $[122] \neq 0$, $(3)^!_{qrs} \neq 0$ and $qr^2s+1 = 0$. Then [112122] is a root vector.

Proof. Let $w = 112122 \in \mathfrak{R}(V)$ and assume $[w]$ is not a root vector. First note that $u_1^2 \neq 0$ since $(3)^!_{qrs} \neq 0$. Moreover, $qr^2s + 1 = 0$ and $[122] \neq 0$ imply $2 \in \mathbb{J}$ and, consequently, [1122] is not a root vector by Theorem 5.14. Thus, there exists $\lambda \in \mathbb{K}$ such that $[112122] = \lambda u_1^3$ by Corollary 5.18. Moreover, let $\mu \in \mathbb{K}$ such that $\partial_2([12122]) = \mu u_1^2$ by Lemma B.3(i). We construct a contradiction by application of skew-derivations using $qr^2s = -1$.

By Lemma B.1 and Lemma B.3 comparing $\partial_1\partial_2([112122])$ and $\partial_1\partial_2(\lambda u_1^3)$ implies

$$\mu q^2 q_{12}^2 b_1 u_1^2 = \partial_1\partial_2([112122]) = \partial_1\partial_2(\lambda u_1^3) = \lambda (3)_{qrs} b_1 u_1^2 \,.$$

Hence $\lambda = \frac{\mu q^2 q_{12}^2}{(3)_{qrs}}$. Moreover, comparison of $\partial_2^2([112122])$ and $\partial_2^2(\lambda u_1^3)$ yields

$$\begin{aligned} & \mu b_1 \left(q_{21} s u_3 + (1 + qrs + q^2rs - q^2r^3s) u_2 x_1 + (2)_{qrs} q q_{12} (1 - qr^2) u_1 x_1^2 \right) \\ = \ & \partial_2^2([112122]) = \partial_2^2(\lambda u_1^3) \\ = \ & (2)_s b_1^2 q_{21} \left(q_{21}^2 s^2 u_3 + (1 + qrs + q^2rs) q_{21} q_{22} u_2 x_1 + (3)_{qrs} u_1 x_1^2 \right) \,. \end{aligned}$$

Here, all vectors are basis vectors due to Theorem 4.30 and $u_3 \neq 0$ was assumed. Comparing the coefficients of u_3 using the above solution for λ we conclude

$$\mu b_1 q_{21} s = b_1^2 q_{21}^3 s^2 (2)_s \frac{\mu q^2 q_{12}^2}{(3)_{qrs}} \,,$$

that is

$$\begin{aligned} 0 &= (3)_{qrs} - (1-r)q^2r^2s(2)_s \\ &= 1 + qrs - qs + q + qs - qr - qrs \\ &= 1 + q - qr\,. \end{aligned}$$

Additionally, by Lemma B.1 and Lemma B.3 comparison of $\partial_2^3(u_1^3)$ and $\partial_2^3([112122])$ implies

$$\left((3)_{qrs}\left(b_3s + b_2(1+qrs+q^2rs-q^2r^3r) + b_1(2)_{qrs}qr(1-qr^2)\right) - q^2r^2b_1^2(3)_s^!\right)x_1^3 = 0\,.$$

Note that $x_1^3 \neq 0$ since $u_3 \neq 0$. Then simplification using $qr^2s+1=0$ and $1+q-qr=0$ yields

$$-q^2s - 2qs - rs - s = -s\frac{(2)_q(3)_q}{q} = 0\,.$$

But $s \in \mathbb{K}^\times$ and $u_3 \neq 0$, i.e. $b_3(2)_q(3)_q \neq 0$, a contradiction. □

For two special braidings we give additional result. In those cases all other tools will not be applicable.

Lemma 5.20 Let $w = 111212112 \in \mathfrak{R}(V)$. Then $\mathcal{B}^{>w} = \langle u_2^3\rangle_{\mathbb{K}}$.

Proof. Let $[v_\ell]^{t_\ell}\cdots[v_1]^{t_1} \in \mathcal{B}^{>w}$. Then $w <_{\text{lex}} v_1^{t_1}\cdot\cdots\cdot v_\ell^{t_\ell} =: v$ by Remark 4.28. Now, if v starts with 112, then there are four $1's$ and two $2's$ to be arranged. Since $v_1 <_{\text{lex}} v_i$ for all $2 \leq i \leq \ell$ we conclude $v = (112)^3$. Thus, $u_2^3 \in \mathcal{B}^{>w}$.

If v starts with 11122, then $v = 111221112$ since $1 <_{\text{lex}} w <_{\text{lex}} v_1$. Then $v \notin \mathcal{L}(I)$ and, consequently, $v_1 = 11122$. This yields a contradiction to $v_1 <_{\text{lex}} \cdots <_{\text{lex}} v_\ell$.

If v starts with 11121, then $v = 111212112 = w$ since $1 <_{\text{lex}} w <_{\text{lex}} v_1$. This contradicts $w <_{\text{lex}} v$. The same applies if v starts with 1111. □

Lemma 5.21 Let $w = 111211212 \in \mathfrak{R}(V)$ and assume $4\alpha_1 + 2\alpha_2$ is not a root. Then $\mathcal{B}^{>w} = \langle u_2^3\rangle_{\mathbb{K}}$.

Proof. The argumentation is similar to the one in the proof of Lemma 5.20. The difference is the case where v starts with 11121. Here, $v = 111212112$ is possible. Then the products [111212][112] and [111212112] are possible decompositions into products of superletters. Note that $[111212112] = [[111212], u_2]_c$. Now, $4\alpha_1 + 2\alpha_2$ not a root and hence $[111212] \in \mathbb{K}u_2^2$ by Remark 5.16. Therefore, both possible products are in $\mathbb{K}\,u_2^3$. This implies the claim. □

Lemma 5.22 Let $s = q \in \mathbb{G}'_{12}$ be a primitive 12-th root of unity and $r = q^8$. Then [111212112] is a root vector.

Proof. Let $w = 111212112 \in \mathfrak{R}(V)$. First, note that (q, r, s) is symmetric with $u_4 \neq 0$, $q^4 r = rs^4 = 1$. One easily checks $0, 1 \notin \mathbb{J}$ and $2 \in \mathbb{J}$ due to $qr^2 s = -1$. Now, $\mathcal{B}^{>w} = \langle u_2^3 \rangle_{\mathbb{K}}$ by Lemma 5.20. Thus, if $[w]$ is not a root vector, then there is some $\lambda \in \mathbb{K}$ such that $[w] = [[111212], u_2]_c = \lambda u_2^3$. Consequently, comparison of $\partial_1^2 \partial_2([w])$ and $\partial_1^2 \partial_2(u_2^3)$ yields

$$\begin{aligned}
&\lambda b_2 (2)_q (3)_{q^4 r^2 s}\, u_2^2 \\
=& \partial_1^2 \partial_2 (\lambda\, u_2^3) \\
=& \partial_1^2 \partial_2 ([w]) \\
=& \partial_1^2 \partial_2 ([111212] u_2 + u_2 [111212]) \\
=& b_2 (2)_q\, [111212] + q^4 r^2 s\, \partial_1^2 \partial_2 ([111212]) u_2 \\
&+ u_2 \partial_1^2 \partial_2 ([111212]) + q^8 r^4 s^2 b_2 (2)_q\, [111212] \\
=& 0\,.
\end{aligned}$$

This implies $\lambda = 0$ since $b_2 (2)_q (3)_{q^4 r^2 s} u_2^2 \neq 0$ due to $q^4 r^2 s = -q^3 \in \mathbb{G}'_4$ and the assumptions. Using this conclusion and comparing the application of ∂_2^3 on $[w]$ and u_2^3 we obtain

$$\begin{aligned}
0 &= \lambda b_2^3 (3)_s^! q_{21}^6\, x_1^6 \\
&= \partial_2^3 (\lambda u_2^3) \\
&= \partial_2^3 ([w]) \\
&= \partial_2^3 ([111212] u_2 + u_2 [111212]) \\
&= \partial_2^2 (b_2\, [111212] x_1^2 + q_{21}^2 s\, \partial_2 ([111212]) u_2 \\
&\quad + u_2 \partial_2 ([111212]) + q_{21}^4 s^2 b_2\, x_1^2 [111212]) \\
&= b_2 q_{21}^4 (3)_s\, \partial_2^2 ([111212]) x_1^2 + b_2 q_{21}^4 (3)_s\, x_1^2 \partial_2^2 \\
&= 2 b_1 b_2^2 (2)_q q_{21}^5 (3)_s (1 - q^3 r^2)\, x_1^6\,.
\end{aligned}$$

But since $p \notin \{2, 3\}$ due to $q \in \mathbb{G}'_{12}$ and $b_1 b_2^2 (2)_q q_{21}^5 (3)_s (1 - q^3 r^2) \neq 0$ this implies $x_1^6 = 0$. Now, $\partial_1^6 (x_1^6) = (6)_q^! = \frac{\prod_{i=1}^6 (q^i - 1)}{(q-1)^6} \neq 0$ due to $q \in \mathbb{G}'_{12}$. This is a contradiction. □

Lemma 5.23 Let $q \in \mathbb{G}'_{18}$ be a primitive 18-th root of unity, $s = q^5$ and $r = q^{13}$ or $q \in \mathbb{G}'_9$, $s = q^5$, $r = q^4$, $p = 2$. Then [111211212] is a root vector.

Proof. Let $w = 111211212$. First, note that $rs = 1$, $u_5 \neq 0$, $q^5r = 1$ and $1, 4 \in \mathbb{J}$. Now, $\mathcal{B}^{>w} = \langle u_2^3 \rangle_{\mathbb{K}}$ by Lemma 5.21. Thus, if $[w]$ is not a root vector, then $[w] = [x_1, [11211212]]_c = [x_1, [u_2, [11212]]_c]_c = \lambda u_2^3$ for some $\lambda \in \mathbb{K}$. Consequently, we obtain the following equation by comparison of $\partial_1^2\partial_2(\lambda\, u_2^3)$ and $\partial_1^2\partial_2([w])$

$$\begin{aligned}
&\lambda b_2 (2)_q (3)_{q^4r^2s}\, u_2^2 \\
=&\partial_1^2\partial_2(\lambda\, u_2^3) \\
=&\partial_1^2\partial_2([w]) \\
=&\partial_1^2\partial_2(x_1[11211212] - q^5q_{12}^3\,[11211212]x_1) \\
=&\partial_1^2\partial_2(x_1u_2[11212] - q^6r^3s^2q_{12}\,x_1[11212]u_2 \\
&- q^5q_{12}^3\,u_2[11212]x_1 + q^{11}r^4s^2q_{12}^4\,[11212]u_2x_1) \\
=&0\,.
\end{aligned}$$

Hence $\lambda = 0$ since $b_2(2)_q(3)_{q^4r^2s}\,u_2^2 \neq 0$. Now, we conclude

$$\begin{aligned}
0 =&\lambda b_2^3 (3)_s^! q_{21}^6\, x_1^6 \\
=&\partial_2^3(\lambda\, u_2^3) \\
=&\partial_2^3([w]) \\
=&\partial_2^3(x_1u_2[11212] - q^6r^3s^2q_{12}\,x_1[11212]u_2 \\
&- q^5q_{12}^3\,u_2[11212]x_1 + q^{11}r^4s^2q_{12}^4\,[11212]u_2x_1) \\
=&q_{21}^4(3)_s^!(2)_q b_1 b_2^2 (1 - q^6r^4s^2 - q^5r^3 + q^{11}r^7s^2)\, x_1^6\,.
\end{aligned}$$

But since $q_{21}^4(3)_s^!(2)_q b_1 b_2^2 x_1^6 \neq 0$ this implies

$$0 = (1 - q^6r^4s^2 - q^5r^3 + q^{11}r^7s^2) = (1 - q^6r^4s^2)(1 - q^5r^3) = (1 - qr)(1 - r)(1 + r).$$

This is a contradiction to the assumptions. □

5.3 A construction of infinitely many roots

We will see that our usual approach using roots with multiples will not work if $V \in b(q, r, s)$ with $q, r, s \in \mathbb{G}_4$, see Corollary 6.16 and Corollary 6.18. Especially for those cases we introduce an approach to prove the existence of infinitely many roots in this section. This will be done in a more general setting mostly using linear algebra. Note that for general q, r, s and p this approach only gives limited results, but could be used in $p = 0$ to greater extent.

Let t, k be natural numbers such that $u_{k+1} \neq 0$. Moreover, let $w_{k,t} = 1^{k+1}2(1^k2)^t \in \mathbb{I}$. The following lemma is inspired by [5, 4.9].

Lemma 5.24 The vector space $\mathcal{B}^{>w_{k,t}}$ is trivial, i.e. $\mathcal{B}^{>w_{k,t}} = \{0\}$.

Proof. Let $[v_\ell]^{t_\ell} \cdots [v_1]^{t_1} \in \mathcal{B}^{>w_{k,t}} \setminus \{0\}$ and $1 \leq i \leq \ell$. First, note that $1 <_{\text{lex}} w_{k,t}$ and, consequently, $v_i \neq 1$. Thus, since $v_i \in \mathfrak{R}(V)$ we conclude $v_i = 1^{m_1}2 \cdots 1^{m_{\ell_i}}2$ for some $\ell_i, m_j \in \mathbb{N}_0$, $1 \leq j \leq \ell_i$ with $m_j \leq m_1 \leq k+1$ since $w_{k,t} <_{\text{lex}} v_i$.

Let $M = \{1 \leq j \leq \ell_i \,|\, m_j = k+1\}$ and assume $M \neq \emptyset$. Hence $1 \in M$ since $v_i \in \mathfrak{R}(V)$. Then decompose $v_i = \prod_{j \in M} v_i^{(j)}$ where $v_i^{(j)} = 1^{m_j}21^{m_{j+1}}2 \cdots 1^{m_{j'-1}}2$. Here, j' denotes the successor of j in M. Now, $v_i^{(1)} \neq 1^{k+1}2(1^k2)^N$ for some $0 \leq N \leq \ell_i$ since otherwise $v_i \leq_{\text{lex}} w_{k,t}$. Thus,

$$\deg(v_i^{(1)}) = k_1^{(1)}\alpha_1 + k_2^{(1)}\alpha_2$$

with $k_1^{(1)} \leq k_2^{(1)}$. Since $v_i \in \mathfrak{R}(V)$ we have $v_i^{(1)} \leq_{\text{lex}} v_i^{(j)}$ for all $j \in M$. Hence for any $j \in M$ we conclude $\deg(v_i^{(j)}) = k_1^{(j)}\alpha_1 + k_2^{(j)}\alpha_2$ with $k_1^{(j)} \leq k_2^{(j)}$ and, consequently, $\deg(v_i) = k_1\alpha_1 + k_2\alpha_2$ with $k_1 \leq k_2$.

The same holds if $m_j \leq k$ for all $1 \leq j \leq \ell_i$. Since i was arbitrary we conclude

$$\deg\left([v_\ell]^{t_\ell} \cdots [v_1]^{t_1}\right) \neq \deg([w_{k,t}]).$$

This is a contradiction. □

Our aim will be to prove that $[w_{k,t}]$ is a root vector for all $t \in \mathbb{N}$. Note that by above lemma $w_{k,t}$ is the maximal Lyndon word with that degree with respect to $<_{\text{lex}}$. Hence this is a promising candidate as stated before.

Corollary 5.25 If $[w_{k,t}] \neq 0$, then it is a root vector.

Proof. Assume $[w_{k,t}]$ is not a root vector. This implies $[w_{k,t}] \in \mathcal{B}^{>w_{k,t}}$. By Lemma 5.24 $\mathcal{B}^{>w_{k,t}} = \{0\}$, a contradiction. □

Now, proving that $w_{k,t} \neq 0$ for all $t \in \mathbb{N}$ seems to be hard to decide at first. We want to reduce it to the question whether some vectors are linearly independent. Then we can apply the rich theory of linear algebra to solve this question.

Corollary 5.26 Let $k \in \mathbb{N}$, $u_{k+1} \neq 0$ and $\left(u_k^{t-j}u_{k+1}u_k^j \,|\, 0 \leq j \leq t\right)$ be a set of linearly independent vectors. Then $((t+1)k+1)\alpha_1 + (t+1)\alpha_2$ is a root.

Proof. Note that by Example 4.4 for all $m \in \mathbb{N}$ we have

$$[w_{k,m}] = [[w_{k,m-1}], u_k]_c = \sum_{j=0}^{t} \lambda_j \, u_k^{m-j} u_{k+1} u_k^j$$

for some $\lambda_j \in \mathbb{K}$, $1 \le j \le t$. In this representation the coefficient of $u_{k+1}u_k^t$ equals 1 by definition of $[\,\cdot\,,\,\cdot\,]_c$. By assumption these vectors are linearly independent, so $[w_{k,t}] \neq 0$. Hence $[w_{k,t}]$ is a root vector by Corollary 5.25 and, consequently, $\deg([w_{k,t}]) = ((t+1)k+1)\alpha_1 + (t+1)\alpha_2$ is a root. □

In the following proposition we use the notation $u_k^{-1} = 0$.

Lemma 5.27 Let $k, t, j \in \mathbb{N}$ such that $j \le t$ and $u_{k+1} \neq 0$. The following equation holds:

$$\begin{aligned} \partial_1^k \partial_2 (u_k^{t-j} u_{k+1} u_k^j) \;=\; & -b_k (k)_q^! (j)_{q^{k^2} r^k s} q^k q_{12} u_k^{t-j+1} x_1 u_k^{j-1} \\ & + b_k (k)_q^! \Big((j)_{q^{k^2} r^k s} + (q^{k^2} r^k s)^j (1 - q^k r)(k+1)_q \\ & \qquad - (q^{k^2} r^k s)^j q^{k(k+2)} r^{k+1} s (t-j)_{q^{k^2} r^k s} \Big) u_k^{t-j} x_1 u_k^j \\ & + b_k (k)_q^! (q^{k^2} r^k s)^j q^{k(k+1)} r^k s q_{21} (t-j)_{q^{k^2} r^k s} u_k^{t-j-1} x_1 u_k^{j+1} \, . \end{aligned}$$

Proof. We prove this equality by direct application of the skew-derivations.

$$\begin{aligned} & \partial_1^k \partial_2 (u_k^{t-j} u_{k+1} u_k^j) \\ = & \partial_1^k \big(u_k^{t-j} u_{k+1} \partial_2 (u_k^j) \\ & \quad + (q_{21}^k q_{22})^j u_k^{t-j} \partial_2 (u_{k+1}) u_k^j \\ & \quad + (q_{21}^k q_{22})^j q_{21}^{k+1} q_{22} \partial_2 (u_k^{t-j}) u_{k+1} u_k^j \big) \\ = & u_k^{t-j} u_{k+1} \partial_1^k \partial_2 (u_k^j) \\ & + (q_{11}^{k^2} q_{12}^k q_{21}^k q_{22})^j u_k^{t-j} \partial_1^k \partial_2 (u_{k+1}) u_k^j \\ & + (q_{11}^{k^2} q_{12}^k q_{21}^k q_{22})^j q_{11}^{k(k+1)} q_{12}^k q_{21}^{k+1} q_{22} \partial_1^k \partial_2 (u_k^{t-j}) u_{k+1} u_k^j \\ \overset{Lemma\ 5.4}{=} & b_k (k)_q^! (j)_{q^{k^2} r^k s} u_k^{t-j} u_{k+1} u_k^{j-1} \\ & + (q^{k^2} r^k s)^j b_{k+1} (k+1)_q^! u_k^{t-j} x_1 u_k^j \\ & + (q^{k^2} r^k s)^j q^{k(k+1)} r^k s q_{21} b_k (k)_q^! (t-j)_{q^{k^2} r^k s} u_k^{t-j-1} u_{k+1} u_k^j \\ \overset{\text{split } u_{k+1}}{=} & b_k (k)_q^! (j)_{q^{k^2} r^k s} u_k^{t-j} (x_1 u_k - q^k q_{12} u_k x_1) u_k^{j-1} \\ & + (q^{k^2} r^k s)^j b_{k+1} (k+1)_q^! u_k^{t-j} x_1 u_k^j \\ & + (q^{k^2} r^k s)^j q^{k(k+1)} r^k s q_{21} b_k (k)_q^! (t-j)_{q^{k^2} r^k s} u_k^{t-j-1} (x_1 u_k - q^k q_{12} u_k x_1) u_k^j \end{aligned}$$

$$
\begin{aligned}
=&b_k(k)^!_q(j)_{q^{k^2}r^ks}u_k^{t-j}x_1u_k^j\\
&-b_k(k)^!_q(j)_{q^{k^2}r^ks}q^kq_{12}u_k^{t-j+1}x_1u_k^{j-1}\\
&+(q^{k^2}r^ks)^jb_{k+1}(k+1)^!_qu_k^{t-j}x_1u_k^j\\
&+(q^{k^2}r^ks)^jq^{k(k+1)}r^ksq_{21}b_k(k)^!_q(t-j)_{q^{k^2}r^ks}u_k^{t-j-1}x_1u_k^{j+1}\\
&-(q^{k^2}r^ks)^jq^{k(k+2)}r^{k+1}sb_k(k)^!_q(t-j)_{q^{k^2}r^ks}u_k^{t-j}x_1u_k^j\\
=&-b_k(k)^!_q(j)_{q^{k^2}r^ks}q^kq_{12}u_k^{t-j+1}x_1u_k^{j-1}\\
&+b_k(k)^!_q\Big((j)_{q^{k^2}r^ks}+(q^{k^2}r^ks)^j(1-q^kr)(k+1)_q\\
&\qquad-(q^{k^2}r^ks)^jq^{k(k+2)}r^{k+1}s(t-j)_{q^{k^2}r^ks}\Big)u_k^{t-j}x_1u_k^j\\
&+b_k(k)^!_q(q^{k^2}r^ks)^jq^{k(k+1)}r^ksq_{21}(t-j)_{q^{k^2}r^ks}u_k^{t-j-1}x_1u_k^{j+1}. \qquad\square
\end{aligned}
$$

For $k,t\in\mathbb{N}_0$ let

$$\mathcal{B}_{k,t}=\{u_k^{t-j}u_{k+1}u_k^j\,|\,0\le j\le t\},\quad \mathcal{C}_{k,t}=\{u_k^{t-j}x_1u_k^j\,|\,0\le j\le t\}$$

and $V_{k,t}$ and $W_{k,t}$ be the vector spaces generated by $\mathcal{B}_{k,t}$ and $\mathcal{C}_{k,t}$ resp. By the above lemma $\partial_1^k\partial_2$ can be restricted to a linear map

$$\varphi_{k,t}:=\partial_1^k\partial_2\big|_{V_{k,t}}:V_{k,t}\to W_{k,t}$$

for all $k,t\in\mathbb{N}_0$.

Proposition 5.28 Let $t\in\mathbb{N}$ arbitrary and $k\in\mathbb{N}$ such that $u_{k+1}\neq 0$. If $\varphi_{k,m}$ is an isomorphism for all $0\le m\le t$, then $\mathcal{B}_{k,t}$ is a set of linearly independent vectors.

Proof. If $\varphi_{k,m}$ is an isomorphism of vector spaces, then the dimensions of $V_{k,m}$ and $W_{k,m}$ need to coincide. We want to prove $\dim W_{k,m}=m+1$ by induction over m. For $m=0$ the equation $\dim W_{k,0}=\dim(\langle x_1\rangle_{\mathbb{K}})=1$ holds since $x_1\neq 0$. For arbitrary $m\le t$ the vector space $W_{k,m}$ is given by $\langle u_k^{m-j}x_1u_k^j\,|\,0\le j\le m\rangle_{\mathbb{K}}$. Here, the following holds

$$
\begin{aligned}
u_k^{m-j}x_1u_k^j &\overset{Lemma\ 5.3}{=} u_k^{m-j}\left(\left(\sum_{i=0}^{j-1}q^{ik}q_{12}^iu_k^iu_{k+1}u_k^{j-1-i}\right)+q^{jk}q_{12}^ju_k^jx_1\right)\\
&= \left(\sum_{i=0}^{j-1}q^{ik}q_{12}^iu_k^{m-j+i}u_{k+1}u_k^{j-1-i}\right)+q^{jk}q_{12}^ju_k^mx_1
\end{aligned}
$$

for any $0 \leq j \leq m$ by Lemma 5.3. We want to check that the vectors

$$\left\{ \left(\sum_{i=0}^{j-1} q^{ik} q_{12}^{i} u_k^{m-j+i} u_{k+1} u_k^{j-i} \right) \,|\, 0 \leq j \leq m \right\}$$

are linearly independent. Note that $q^{ik} q_{12}^{i}$ are non zero for all $0 \leq i \leq j$. Additionally, the matrix composed by the coordinate vectors of these vectors with respect to $\mathcal{B}_{k,m-1}$ is an upper triangular matrix with non-zero entries along the diagonal. Thus, $W_{k,m} = V_{k,m-1} + \mathbb{K}\, u_k^m x_1$.

Using the induction hypothesis the dimension of $V_{k,m-1}$ is given by $V_{k,m-1} = W_{k,m-1} = m$. Note that the sum of vector spaces is actually a direct sum since $\partial_1(V_{k,t}) = 0$ and $\partial_1(u_k^m x_1) = u_k^m$. Now, the vector u_k^m is non-zero since otherwise $u_k^{m-1} u_{k+1} = u_k^{m-1} x_1 u_k$ and $u_{k+1} u_k^{m-1} = u_k x_1 u_k^{m-1}$ due to $u_{k+1} = [x_1, u_k]_c$. Hence $V_{k,m-1} \subset \langle u_k^{m-j} x_1 u_k^j \,|\, 1 \leq j \leq m-1 \rangle$ and, consequently, $\dim V_{k,m-1} \leq m-1$, a contradiction to the induction hypothesis. Thus, $\dim V_{k,m} = \dim W_{k,m} = m+1$ for all $m \leq t$. Since $V_{k,t} = \langle \mathcal{B}_{k,t} \rangle_{\mathbb{K}}$ is generated by $t+1$ vectors, those need to be linearly independent. This proves the claim. □

Finally, let $D_{k,t}$ denote the transformation matrix of $\varphi_{k,t}$ corresponding to $\mathcal{B}_{k,t}$ and $\mathcal{C}_{k,t}$. $D_{k,t}$ inherits the structure of a so-called tridiagonal matrix by Lemma 5.27. We will see that under some weak assumptions this matrix decomposes into blocks. This will make it possible to check $\det(D_{k,t}) \neq 0$ for all $t \in \mathbb{N}$ with reasonable effort.

For a set $M \subset \{1, \ldots, m\}$ and $A \in \mathbb{K}^{m \times m}$ let $A_{(M)}$ denote the matrix given by canceling all rows i and columns j of A where $i, j \notin M$. Assume in the following that $q^{k^2 r^k s} \in \mathbb{G}'_N$ with $N \geq 2$. For $t \in \mathbb{N}$ let $1 \leq \bar{t} \leq t$ be minimal such that

$$t \equiv \bar{t} \mod N\,.$$

Lemma 5.29 For $t \geq \bar{t} + N$ the following holds

$$\det\left(D_{k,t+mN}\right) \neq 0 \text{ for all } m \in \mathbb{N}_0 \Leftrightarrow \det\left((D_{k,t})_{(\bar{t}+1,\ldots,\bar{t}+N)} \right) \neq 0\,.$$

Proof. By Lemma 5.27 the matrix $D_{k,t} = b_k(k)^!_q(d_{i,j})_{1\le i,j\le t+1}$ is given by

$$d_{ij} = \begin{cases} 0, & |i-j|\ge 2\,, \\ -(j)_{q^{k^2}r^ks}q^kq_{12}, & i+1=j\,, \\ (j)_{q^{k^2}r^ks} + (q^{k^2}r^ks)^j(1-q^kr)(k+1)_q & \\ \qquad -(q^{k^2}r^ks)^jq^{k(k+2)}r^{k+1}s(t-j)_{q^{k^2}r^ks}, & i=j\,, \\ (q^{k^2}r^ks)^jq^{k(k+1)}r^ksq_{21}(t-j)_{q^{k^2}r^ks}, & i=j+1\,. \end{cases}$$

Note that d_{ij} does not essentially depend on k but on $k \mod N$. We can easily check $(t-\bar{t})_{q^{k^2}r^ks} = 0$ and $(q^{k^2}r^ks)^{t-\bar{t}} = 1$. Especially $d_{(\bar{t}+1)\,\bar{t}} = 0$. Hence $D_{k,t}$ consists of blocks

$$D_{k,t} = \begin{pmatrix} D^{(1)}_{k,t} & 0 & 0 & \cdots & 0 \\ 0 & D^{(2)}_{k,t} & 0 & \cdots & 0 \\ \vdots & & \ddots & & \vdots \\ 0 & & & 0 & D^{(\ell)}_{k,t} \end{pmatrix}$$

where $\ell = \min\{i\in\mathbb{N}_0 | \bar{t}+i\cdot N \ge t+1\}+1$ is the number of blocks and

$$D^{(i)}_{k,t} = D_{k,t(\{\bar{t}+1,\cdots,\bar{t}+N\})} = D^{(2)}_{k,t} \text{ for } 2\le i\le \ell$$

and $D^{(1)}_{k,t} = D_{k,t(\{1,\cdots,\bar{t}\})}$.

First note that for $\bar{t}\neq 0$ the matrix $D_{k,\bar{t}} = D^{(1)}_{k,t} = \left(D^{(2)}_{k,t}\right)_{(\{N,\cdots,\bar{t}+N\})}$ is a block of $D^{(2)}_{k,t}$ due to $d_{(N-1)\,N} = (N)_{q^{k^2}r^ks}q^kq_{12} = 0$. Next, the blocks that appear in the matrices $D^{(2)}_{k,t}$ and $D^{(2)}_{k,t+N}$ are identical due to $\bar{t} = \bar{t}+N$ and $(t+N-\bar{t})_{q^{k^2}r^ks} = (t-\bar{t})_{q^{k^2}r^ks}$. Now, $b_k(k)^!_q \neq 0$ since $u_{k+1}\neq 0$ and, consequently, $\det\left(D^{(2)}_{k,t}\right)\neq 0$ if and only if $\det\left(D_{k,\bar{t}+m\cdot N}\right)\neq 0$ for all $m\in\mathbb{N}_0$. □

For $t\ge N$ the matrix $(D_{k,t})_{(\bar{t}+1,\ldots,\bar{t}+N)} = D^{(2)}_{k,t}$ that appeared in the preceding proof will be denoted in the following with $F_{k,\bar{t}}$.

Corollary 5.30 The following holds:

$$\varphi_{k,t} \text{ is an isomorphism for all } t\in\mathbb{N} \text{ iff } \det(F_{k,\bar{t}})\neq 0 \text{ for all } 1\le\bar{t}\le N\,.$$

Proof. This follows directly from Lemma 5.29. □

Remark 5.31 It turns out that for given $k\in\mathbb{N}$ and $q^{k^2r^ks}\in\mathbb{G}'_N$ the determinants of the matrices $F_{k,\bar{t}}$ coincide for all $0\le\bar{t}<N$.

By construction of the matrix $F_{k,\bar{t}}$ its determinant is in $\mathbb{Z}[q,r,s]$. In the cases we determine such determinants the elements q, r and s of $\mathbb{K}$ are related such that $\det(F_{k,\bar{t}}) \in \mathbb{Z}[q]$ and $q \in \mathbb{G}'_N$ for some $N \in \mathbb{N}$. Since we work over an arbitrary field it is hard to prove $\det(F_{k,\bar{t}}) \neq 0$ independent from p since we do not know the minimal polynomial of q. There is one special case known where this can be accomplished.

Proposition 5.32 Let $k \in \mathbb{N}$ and $q, r, s \in \mathbb{K}$ such that $u_{k+1} \neq 0$ and $q^{k^2} r^k s = -1$. Then $((t+1)k+1)\alpha_1 + (t+1)\alpha_2$ is a root for all $t \in \mathbb{N}_0$.

Proof. We calculate $\det(F_{k,\bar{t}})$ for $\bar{t} \in \{0,1\}$. First, assume $\bar{t} = 0$. If j is even, then $(j)_{q^{k^2} r^k s} = (t-j)_{q^{k^2} r^k s} = 0$ and we obtain

$$\partial_1^k \partial_2 (u_k^{t-j} u_{k+1} u_k^j) = b_k (k)_q^! (1 - q^k r)(k+1)_q u_k^{t-j} x_1 u_k^j = b_{k+1} (k+1)_q^! u_k^{t-j} x_1 u_k^j .$$

If j is odd, then $(j)_{q^{k^2} r^k s} = (t-j)_{q^{k^2} r^k s} = 1$ and hence we obtain

$$\begin{aligned} \partial_1^k \partial_2 (u_k^{t-j} u_{k+1} u_k^j) \quad = \quad & -b_k (k)_q^! q^k q_{12} u_k^{t-j+1} x_1 u_k^{j-1} \\ & + b_k (k)_q^! \left(1 - (1 - q^k r)(k+1)_q - q^{2k} r\right) u_k^{t-j} x_1 u_k^j \\ & + b_k (k)_q^! q^k q_{21} u_k^{t-j-1} x_1 u_k^{j+1} . \end{aligned}$$

Thus, the determinant of $F_{k,0}$ is the product of the diagonal entries, that is

$$\det(F_{k,t}) \quad = \quad b_{k+1} (k+1)_q^! b_k (k)_q^! \left(1 - (1 - q^k r)(k+1)_q - q^{2k} r\right) .$$

Now, the equations

$$\begin{aligned} & 1 - (1 - q^k r)(k+1)_q - q^{2k} r \\ = & 1 - \sum_{i=0}^{k} q^i + r \sum_{i=0}^{k} q^{k+i} - q^{2k} r \\ = & - \sum_{i=1}^{k} q^i + r \sum_{i=0}^{k-1} q^{k+i} \\ = & - q(k)_q + q^k r (k)_q \\ = & q(k)_q \left(q^{k-1} r - 1\right) \neq 0 \end{aligned}$$

hold since $u_k \neq 0$. Therefore, $\det(F_{k,0}) \neq 0$ since $u_{k+1} \neq 0$.

On the other hand consider $\bar{t} = 1$. If j is even, then $(j)_{q^{k^2} r^k s} = 0$ and $(t -$

$j)_{q^{k^2}r^ks} = 1$. Hence we obtain

$$\begin{aligned}\partial_1^k\partial_2(u_k^{t-j}u_{k+1}u_k^j) &= b_k(k)_q^!\left((1-q^kr)(k+1)_q+q^{2k}r\right)u_k^{t-j}x_1u_k^j\\ &\quad -b_k(k)_q^!q^{2k}q_{21}u_k^{t-j+1}x_1u_k^{j+1}.\end{aligned}$$

If j is odd, then $(j)_{q^{k^2}r^ks} = 1$ and $(t-j)_{q^{k^2}r^ks} = 0$. We obtain

$$\begin{aligned}\partial_1^k\partial_2(u_k^{t-j}u_{k+1}u_k^j) &= -b_k(k)_q^!q^kq_{12}u_k^{t-j+1}x_1u_k^{j-1}\\ &\quad +b_k(k)_q^!\left(1-(1-q^kr)(k+1)_q\right)u_k^{t-j}x_1u_k^j.\end{aligned}$$

Thus, we conclude

$$F_{k,1} := b_k(k)_q^!\begin{pmatrix}(1-q^kr)(k+1)_q+q^{2k}r & -q^kq_{12}\\ -q^kq_{21} & 1-(1-q^kr)(k+1)_q\end{pmatrix}.$$

Now, we calculate

$$\begin{aligned}\det(F_{k,1}) =& b_k^2\left((k)_q^!\right)^2\left((1-q^kr)(k+1)_q+q^{2k}r\right)\left(1-(1-q^kr)(k+1)_q\right)-q^{2k}r\\ =& b_k^2\left((k)_q^!\right)^2\Big(1-q^kr)(k+1)_qu_k^{t-j}x_1u_k^j((1-q^kr)(k+1)_q+q^{2k}r\\ &+(1-q^kr)^2(k+1)_q-q^{2kr}(1-q^kr)(k+1)_q-q^{2k}r\Big)\\ =& b_{k+1}b_k(k+1)_q^!(k)_q^!\left(1-(1-q^kr)(k+1)_q-q^{2k}r\right)\neq 0\end{aligned}$$

analogously to the case where t was even. Thus, $\det(F_{k,\bar t})\neq 0$ for arbitrary $t\in\mathbb{N}_0$ and hence $B_{k,t}$ is a set of linearly independent vectors for all $t\in\mathbb{N}_0$ by Proposition 5.28. Thus, Corollary 5.26 completes the proof. □

We want to give one more example with a given braiding. This as well is a case, where our other tools will not apply.

Example 5.33 Let $p\geq 3$, $q=-r=1$, $s\in\mathbb{G}_4'$. Then $u_2\neq 0$ and $qrs=-s\in\mathbb{G}_4'$. We determine $F_{1,\bar t}$ for $0\leq \bar t\leq 3$ and calculate the corresponding determinants.

$$F_{1,0} = \begin{pmatrix} s^4-s^3+s^2-4s+1 & -s+1 & 0 & 0\\ s^4-s^3+s^2 & s^4-s^3+4s^2-s+1 & s^2-s+1 & 0\\ 0 & s^4-s^3 & s^4-4s^3+s^2-s+1 & 0\\ 0 & 0 & s^4 & 4\end{pmatrix}$$

$$= \begin{pmatrix} -3s+1 & -s+1 & 0 & 0 \\ s & -2 & -s & 0 \\ 0 & s+1 & 3s+1 & 0 \\ 0 & 0 & 1 & 4 \end{pmatrix}.$$

$$F_{1,1} = \begin{pmatrix} -s^5+s^4-s^3+4s^2-s+1 & s^2-s+1 & 0 & 0 \\ -s^5+s^4-s^3 & -s^5+s^4-4s^3+s^2-s+1 & 0 & 0 \\ 0 & -s^5+s^4 & -s+4 & 1 \\ 0 & 0 & -s & -4s+1 \end{pmatrix}$$

$$= \begin{pmatrix} -s-2 & -s & 0 & 0 \\ 1 & 2s+1 & 0 & 0 \\ 0 & -s+1 & -s+4 & 1 \\ 0 & 0 & -s & 4 \end{pmatrix}.$$

$$F_{1,2} = \begin{pmatrix} s^6-s^5+s^4-4s^3+s^2-s+1 & 0 & 0 & 0 \\ s^6-s^5+s^4 & s^2-s+4 & 1 & 0 \\ 0 & s^2-s & s^2-4s+1 & -s+1 \\ 0 & 0 & s^2 & 4s^2-s+1 \end{pmatrix}$$

$$= \begin{pmatrix} 2s & 0 & 0 & 0 \\ -s & -s+3 & 1 & 0 \\ 0 & -s-1 & -4s & -s+1 \\ 0 & 0 & -1 & -s-3 \end{pmatrix}.$$

$$F_{1,3} = \begin{pmatrix} -s^3+s^2-s+4 & 1 & 0 & 0 \\ -s^3+s^2-s & -s^3+s^2-4s+1 & -s+1 & 0 \\ 0 & -s^3+s^2 & -s^3+4s^2-s+1 & s^2-s+1 \\ 0 & 0 & -s^3 & -4s^3+s^2-s+1 \end{pmatrix}$$

$$= \begin{pmatrix} 3 & 1 & 0 & 0 \\ -1 & -3s & -s+1 & 0 \\ 0 & s-1 & -3 & -s \\ 0 & 0 & s & 3s \end{pmatrix}.$$

Then for any $0 \leq \bar{t} \leq 3$ the we calculate $\det(F_{1,\bar{t}}) = -64 \neq 0$ due to $p \neq 2$.

6 | On the Gelfand-Kirillov dimension of rank two Nichols algebras of diagonal type

In this chapter we develop tools to decide whether the Gelfand-Kirillov dimension of $\mathcal{B}(V)$ is infinite. At first, this is not a trivial task due to the fact that we don't know the defining ideal $\mathcal{J}$ of $\mathcal{B}(V)$ or the set of root vectors. We show that in most cases little knowledge of the set of root vectors suffices. It is known that the Gelfand-Kirillov dimension of a Nichols algebra is finite if the set of roots is, see [13]. Recall the classification of Nichols algebras of diagonal type with finite root system.

Theorem 6.1 [18, thm. 5.1] The following are equivalent:

- $\#\triangle^{\mathrm{re}} < \infty$.
- (q, r, s) or (s, r, q) appears in A.

We will show that in any non-finite case the Gelfand-Kirillov dimension is infinite. We need to develop tools to prove this. Therefore, Lemma 3.5 will play an important role. We start with some simple applications.

Corollary 6.2 If there are $\alpha, \beta \in \mathbb{Z}^2$ such that $k\alpha + \beta \in \triangle_+$ for all $k \in \mathbb{N}$, then $\mathrm{GKdim}\,(\mathcal{B}(V)) = \infty$.

Proof. By assumption there is a root vector y_k of degree $k\alpha + \beta$ for any $k \in \mathbb{N}$. We define an ordering on $\mathfrak{R}(V)$. For $v_i, v_j \in \mathfrak{R}(V)$ let

$$v_j \ll v_i :\Leftrightarrow \begin{cases} \ell(v_i) < \ell(v_j)\,. \\ \ell(v_i) = \ell(v_j), v_i <_{\mathrm{lex}} v_j\,. \end{cases}$$

This can be extended to the corresponding superletters in Theorem 4.30 as we did with $<_{\mathrm{lex}}$.

Then

$$(y_{k_1}\cdots y_{k_\ell}\,|\,\ell\in\mathbb{N}, k_1<\cdots<k_\ell)$$

forms a family of homogeneous basis vectors of $\mathcal{B}(V)$ after rearrangement of the factors in Theorem 4.30 using Corollary 4.31. Thus, they are linearly independent. Moreover, assuming

$$\alpha=\alpha^{(1)}\alpha_1+\alpha^{(2)}\alpha_2,\quad \beta=\beta^{(1)}\alpha_1+\beta^{(2)}\alpha_2$$

we conclude $\deg_{\mathbb{N}}(y_k)=k(\alpha^{(1)}+\alpha^{(2)})+\beta^{(1)}+\beta^{(2)}$. Consequently, $\mathcal{B}(V)$ is of infinite Gelfand-Kirillov dimension by Lemma 3.5. □

Corollary 6.3 [2, 3.7] If $\operatorname{GKdim}(\mathcal{B}(V))<\infty$, then $(k)_q b_k=0$ for some $k\in\mathbb{N}$. In particular, if $\operatorname{GKdim}(\mathcal{B}(V))<\infty$, then Q is i-finite for all $i\in I$.

Proof. For any $k\in\mathbb{N}$ the following holds

$$(k)_q b_k\neq 0\Leftrightarrow u_k\neq 0\Leftrightarrow k\alpha_1+\alpha_2\in\triangle_+.$$

Thus, if $(k)_q b_k\neq 0$ for all $k\in\mathbb{N}$, then Corollary 6.2 is applicable. The last part follows from Remark 5.2. □

Since we want to prove $\operatorname{GKdim}(\mathcal{B}(V))=\infty$ we can in the following assume that Q is i-finite for all $i\in I$ if $\#\triangle_+=\infty$ by the above corollary. In the last chapter we constructed another set of roots which allows the application of Corollary 6.2. For $k,\bar t\in\mathbb{N}$ we use the notation $F_{k,\bar t}$ from the preceding chapter.

Corollary 6.4 Let k be a natural number and $q,r,s\in\mathbb{K}$ such that $u_{k+1}\neq 0$ and $q^{k^2}r^ks\in\mathbb{G}'_N$. If $\det(F_{k,\bar t})\neq 0$ for all $0\le\bar t<N$, then $\operatorname{GKdim}(\mathcal{B}(V))=\infty$.

Proof. The assumption implies that $\{u_k^{t-j}u_{k+1}u_k^j\}_{t\in\mathbb{N},0\le j\le t}$ are linearly independent by Corollary 5.30 and Proposition 5.28. Therefore, $t(k\alpha_1+\alpha_2)+(k+1)\alpha_1+\alpha_2$ is a root for all $t\in\mathbb{N}$ by Corollary 5.26. Thus, $\operatorname{GKdim}(\mathcal{B}(V))=\infty$ by Corollary 6.2.□

The above corollary especially holds for $q^{k^2}r^ks=-1$. This follows from Proposition 5.32. This special case will be the main application of the statement.

As discussed before the structure generated by the reflections of roots of a Nichols algebra of Cartan-type with Cartan matrix C coincides with the Weyl group $W(C)$. The next application fully utilizes the knowledge of the corresponding root systems developed in [20]. We use the well-known notions of real roots and imaginary roots, resp.

Proposition 6.5 [4, 3.1] If the Nichols algebra $\mathcal{B}(V)$ is of affine Cartan-type, then $\mathrm{GKdim}\,(\mathcal{B}(V)) = \infty$.

Proof. Let $\triangle^{\mathrm{re}}$ denote the set of real roots of $\mathcal{B}(V)$. There exists a positive imaginary root δ such that $\triangle^{\mathrm{re}} + \delta = \triangle^{\mathrm{re}}$, see [20, Prop. 6.3 (d)]. Let h be the height of δ and let α be a simple root. Then for all $k \geq 0$ there exists a root vector y_k of $\mathbb{N}_0$-degree $k \cdot h + 1$. Hence $\mathrm{GKdim}\,(\mathcal{B}(V)) = \infty$ by Corollary 6.2. □

To use above results one needs specific knowledge of Q. Next we develop the main tool that only depends on the existence of roots that are multiples of others or have multiplicity bigger 1. With this result we will be able to approach general Q. We introduce certain quotients of subalgebras of $\mathcal{B}(V)$ as it was done in [5]. Note that the general approach shown here and in [5] both work with this kind of quotients, but the argument that yields infinite Gelfand-Kirillov dimension ultimately is different.

Let $\mathcal{B}(V)^{(a_1,a_2)}$ denote the homogeneous component of $\mathcal{B}(V)$ of degree $a_1\alpha_1+a_2\alpha_2$. For any $d \in \mathbb{Q}_{\geq 0}$ we set

$$\begin{aligned} B_{\geq d} &:= \bigoplus_{\substack{(a_1,\, a_2) \,\in\, \mathbb{Z}^2 \\ a_1 \,\geq\, d a_2}} \mathcal{B}(V)^{(a_1,a_2)} . \\ K_{\geq d} &:= \{ y \in \mathcal{B}(V) \,|\, \Delta(y) \in B_{\geq d} \otimes \mathcal{B}(V) \} , \\ K_{>d} &:= K_{\geq d} \cap B_{>d} . \end{aligned}$$

First observe that for $d, e \in \mathbb{Q}_{\geq 0}$, $d < e$ the following inclusions hold:

$$K_{\geq e} \subseteq K_{>d}, \; K_{>d} \subseteq K_{\geq d} \subseteq B_{\geq d} .$$

The last inclusion follows via application of the counit.

Lemma 6.6 [5, 3.9] $K_{\geq d}$ is a right coideal subalgebra of $\mathcal{B}(V)$ in ${}^{\Bbbk\mathbb{Z}^2}_{\Bbbk\mathbb{Z}^2}\mathcal{YD}$. Moreover, $K_{>d}$ is an ideal of $K_{\geq d}$ and a coideal of $\mathcal{B}(V)$ in ${}^{\Bbbk\mathbb{Z}^2}_{\Bbbk\mathbb{Z}^2}\mathcal{YD}$ such that

$$\Delta(K_{\geq d}) = K_{\geq d} \otimes K_{\geq d} + K_{>d} \otimes \mathcal{B}(V) .$$

Proposition 6.7 [5, 3.9] The bialgebra structure of $\mathcal{B}(V)$ induces a bialgebra structure on $K_{\geq d}/K_{>d}$.

Remark 6.8 The quotient $K_{\geq d}/K_{>d}$ is isomorphic to

$$K_{\geq d} \cap \bigoplus_{a \in \mathbb{Z}} \mathcal{B}(V)^{(a,da)}$$

as an algebra due to the $\mathbb{Z}^2$-algebra grading on $\mathcal{B}(V)$.

The following corollary will be very important to prove infinite Gelfand-Kirillov dimension in many cases of Proposition 6.15 and Proposition 6.17. This replaces the argument used in [5] which does not work for arbitrary fields.

Corollary 6.9 Let $d, e, f \in \mathbb{Q}_{\geq 0}$, $d \leq e, f$ and $e \neq f$ and assume $\operatorname{GKdim}(K_{\geq f}/K_{>f}) \geq 1$. Then the following holds:

$$\operatorname{GKdim}(B_{\geq d}) \geq \operatorname{GKdim}(K_{\geq e}/K_{>e}) + 1\,.$$

Proof. Assume $f < e$ without loss of generality. Moreover, let $\operatorname{cnt} : I \times \mathbb{I} \to \mathbb{N}_0$ be the map

$$\operatorname{cnt}(i, w) = \#\{1 \leq j \leq \ell(w) \,|\, w = i_1 \cdots i_{\ell(w)},\ i_j = i\}$$

counting the appearances of i in w. We define an ordering $\ll$ on $\mathbb{I}$:

$$v \ll w :\Leftrightarrow \begin{cases} \frac{\operatorname{cnt}(1,v)}{\operatorname{cnt}(2,v)} < \frac{\operatorname{cnt}(1,w)}{\operatorname{cnt}(2,w)} \\ \frac{\operatorname{cnt}(1,v)}{\operatorname{cnt}(2,v)} = \frac{\operatorname{cnt}(1,w)}{\operatorname{cnt}(2,w)},\ v <_{\text{lex}} w\,. \end{cases}$$

Now, by Proposition 2.2 and Corollary 4.31 we can exchange the arrangement of the factors in Theorem 4.30 with $\ll$.

Thus, we get a vector space basis of $B_{\geq d}$ by restricting the PBW-bases of $\mathcal{B}(V)$ to those generators satisfying the constraint on the degree in $B_{\geq d}$. Now by Remark 6.8 we can identify $K_{\geq e}/K_{>e}$ and $K_{\geq f}/K_{>f}$ as subalgebras of $B_{\geq d}$. Recall the inclusion $K_{\geq f} \subset K_{>e}$. It follows from the arrangement of the generators that the multiplication of $B_{\geq d}$ induces an isomorphism

$$K_{\geq e}/K_{>e} \otimes K_{\geq f}/K_{>f} \to K_{\geq e}/K_{>e}K_{\geq f}/K_{>f}\,.$$

Thus, we can apply Proposition 3.7. This completes the proof. □

Due to $B_{\geq d} \subseteq \mathcal{B}(V)$ and Lemma 3.4 the above inequlity extends to

$$\operatorname{GKdim}(\mathcal{B}(V)) \geq \operatorname{GKdim}(K_{\geq e}/K_{>e}) + 1.$$

To be able to apply the above result we need information on $\operatorname{GKdim}(K_{\geq e}/K_{>e})$. Below we give a way to construct an "included" Nichols-algebra. Then we can reuse results for special braidings especially those of affine Cartan-type.

Lemma 6.10 Let $x, y \in K_{\geq d}/K_{>d}$ be linearly independent primitive elements. Then there is a Nichols algebra $\mathcal{B}(W)$ with $W \in b(q'_{11}, q'_{12}q'_{21}, q'_{22})$ where

$$Q' = (q'_{ij})_{1\leq i,j\leq 2} = \begin{pmatrix} \chi(\deg(x), \deg(x)) & \chi(\deg(x), \deg(y)) \\ \chi(\deg(y), \deg(x)) & \chi(\deg(y), \deg(y)) \end{pmatrix}$$

such that $\operatorname{GKdim}(K_{\geq d}/K_{>d}) \geq \operatorname{GKdim}(\mathcal{B}(W))$.

Proof. Let $A \subset K_{\geq d}/K_{>d}$ be the subalgebra generated by x and y. Define the following filtration $\{F_i\}_{i\in\mathbb{N}_0}$ of A:

$$F^0 = \mathbb{K},\ F^1 = \mathbb{K} + \langle x, y\rangle_{\mathbb{K}},\ F^n = \langle y_1 \cdots y_m \,|\, y_i \in F^1,\ 1 \leq i \leq m\rangle_{\mathbb{K}}.$$

Note that $\dim F^i < \infty$ for all $i \in \mathbb{N}_0$. This filtration is obviously an algebra filtration. Moreover, it is a coalgebra filtration since x and y are primitive in $K_{\geq d}/K_{>d}$. Consequently, the associated graded algebra $\bar{A}$ is an $\mathbb{N}_0$-graded bialgebra in ${}^{\mathbb{K}\mathbb{Z}^2}_{\mathbb{K}\mathbb{Z}^2}\mathcal{YD}$. Note that for $m \geq 1$ the homogenous component $\bar{A}(m) = F^m/F^{m-1}$ is generated by $\bar{A}(1)$ consisting of the primitive cosets corresponding to x and y. Thus, $\bar{A}$ is generated by primitives and hence it is a pre-Nichols algebra by Remark 4.16 and there is a projection $\bar{A} \to \mathcal{B}(W)$ where W is the vector space generated by the cosets of x and y. The braiding matrix is induced by construction. The last claim follows from Proposition 3.6. □

Finally, we give ways to construct primitive elements in $K_{\geq d}/K_{>d}$. With those at hand Corollary 6.9 and Lemma 6.10 yield a new argument to prove that $\mathcal{B}(V)$ is of infinite Gelfand-Kirillov dimension.

Remark 6.11 Let $x \in \mathcal{B}(V)$ be a nonzero homogeneous element of degree $m_1\alpha_1 + m_2\alpha_2$ with $m_1, m_2 \in \mathbb{N}_0$. If $\gcd(m_1, m_2) = 1$, then x is primitive in $K_{\geq \frac{m_1}{m_2}}/K_{>\frac{m_1}{m_2}}$. This follows from the fact that the $\mathbb{Z}^2$-grading of $\mathcal{B}(V)$ is a coalgebra-grading.

Lemma 6.12 Let $x \in \mathcal{B}(V) \setminus \{0\}$ be an homogeneous element of degree $\deg(x) = m(k\alpha_1 + \alpha_2)$ such that x and u_k^m are linearly independent and satisfy

$$\begin{aligned} \Delta(x) \in\ & 1 \otimes x + x \otimes 1 + \sum_{i=1}^{m-1} \lambda_i\, u_k^i \otimes u_k^{m-i} \\ & + B_{>k} \otimes \mathcal{B}(V) \end{aligned}$$

where $\lambda_i \in \mathbb{K}$ for $1 \leq i \leq m-1$. If $(m)^!_{q^{k^2}r^k s} \neq 0$, then there is a homogeneous element $y \in \mathcal{B}(V)$ with $\deg(y) = \deg(x)$ which is primitive in $K_{\geq k}/K_{>k}$.

Proof. If $u_k = 0$, then the coset of x in $K_{\geq k}/K_{>k}$ satisfies the claim. Otherwise, using Lemma 5.4 we calculate

$$\left(\partial_1^k \partial_2\right)^m (u_k^m) = \left(b_k (k)_q^!\right)^m (m)_{q^{k^2} r^k s}^! \neq 0$$

due to $u_k \neq 0$ and the assumptions. Note that $(m)_{q^{k^2} r^k s}^! \neq 0$ by assumption. Thus, let y be the coset of $x - \lambda u_k^m$ in $K_{\geq k}/K_{>k}$ where $\lambda = \frac{\lambda_1}{(m)_{q^{k^2} r^k s}}$. This vector satisfies

$$\Delta(y) \;=\; 1 \otimes y + y \otimes 1 + \sum_{i=2}^{m-1} \left(\lambda_i - \binom{m}{i}_{q^{k^2} r^k s} \frac{\lambda_1}{(m)_{q^{k^2} r^k s}} \right) u_k^i \otimes u_k^{m-i}$$

in $K_{\geq k}/K_{>k}$ due to Lemma 5.8. We want to prove $\left(\lambda_i - \binom{m}{i}_{q^{k^2} r^k s} \frac{\lambda_1}{(m)_{q^{k^2} r^k s}} \right) = 0$ for all $i \in \{2, \cdots, m-2\}$. This will be accomplished using coassociativity.

First, due to the distribution of degrees and the fact that the $\mathbb{Z}^2$-grading is a grading of coalgebras the term $(\Delta \otimes \mathrm{id})(y)$ represented as a sum of tensors whose factors are products of some u_ℓ has a summand

$$(i)_{q^{k^2} r^k s} \left(\lambda_i - \binom{m}{i}_{q^{k^2} r^k s} \frac{\lambda_1}{(m)_{q^{k^2} r^k s}} \right) u_k \otimes u_k^{i-1} \otimes u_k^{m-i}$$

by Lemma 5.8. Here $(i)_{q^{k^2} r^k s}\, u_k \otimes u_k^{i-1} \otimes u_k^{m-i} \neq 0$ since $i < m$, $(m)_{q^{k^2} r^k s}^! \neq 0$ and $u_k^m \neq 0$.

On the other hand, using the same argumentation $(\mathrm{id} \otimes \Delta)(y)$ can not have a summand with this distribution of degrees since the $\mathbb{Z}^2$-grading is a grading of bialgebras. Consequently, we obtain

$$\left(\lambda_i - \binom{m}{i}_{q^{k^2} r^k s} \frac{\lambda_1}{(m)_{q^{k^2} r^k s}} \right) = 0$$

for any $i \in \{2, \ldots, m-1\}$. Hence y satisfies $\deg(y) = \deg(x)$ and is primitive in $K_{\geq k}/K_{>k}$. □

Corollary 6.13 Let $m \in \mathbb{N}$ be such that $m\alpha_1 + \alpha_2$, $2m\alpha_1 + 2\alpha_2 \in \triangle_+$. Then there is some nonzero homogeneous $y \in \mathcal{B}(V)$ with $\deg(y) = 2m\alpha_1 + 2\alpha_2$ which is primitive in $K_{\geq m}/K_{>m}$.

Proof. Since $2m\alpha_1 + 2\alpha_2$ is a root there exists a corresponding root vector x such that $\deg(x) = 2m\alpha_1 + 2\alpha_2$. Due to [34, 3.22] we can assume

$$x = \left[1^{m+1}21^{m-1}2\right] \text{ or } x = [1^m 2]^2 .$$

If $x = u_m^2$ is a root vector, then by Lemma 5.9 and Lemma 5.8 x is primitive in $K_{\geq m}/K_{>m}$ and there is nothing else to prove.

Otherwise, $x = [1^{m+1}21^{m-1}2]$. By Lemma 5.7 we know

$$\begin{aligned}\Delta(x) &= x \otimes 1 + 1 \otimes x \\ &\quad +(m+1)_q(1-q^m r)\chi(\beta_{m+1}, \beta_{m-1}) u_m \otimes u_m \\ &\quad +B_{>m} \otimes \mathcal{B}(V) .\end{aligned}$$

Consequently, by Lemma 6.12 there is an element $y \in \mathcal{B}(V)$ such that $\deg(y) = \deg(x)$ that is primitive in $K_{\geq m}/K_{>m}$. This completes the proof. □

Corollary 6.14 Assume that one of the following holds:

(i) $u_3 \neq 0$, $[122] \neq 0$, $(3)^!_{qrs} \neq 0$, $qr^2s + 1 = 0$ and $\gamma = qrs$.

(ii) $s = q \in \mathbb{G}'_{12}$, $r = q^8$ and $\gamma = q^4 r^2 s$.

(iii) $q \in \mathbb{G}'_{18}$, $s = q^5$, $r = q^{-5}$ and $\gamma = q^4 r^2 s$.

(iv) $q \in \mathbb{G}'_9$, $s = q^5$, $r = q^4$, $p = 2$ and $\gamma = q^4 r^2 s$.

Then there exists $W \in b(\gamma, \gamma^6, \gamma^9)$ such that $\operatorname{GKdim}(\mathcal{B}(V)) \geq \operatorname{GKdim}(\mathcal{B}(W))$.

Proof. Under the given preconditions Lemma 5.19, Lemma 5.22 or Lemma 5.23 are applicable. Then Lemma 6.12 can be applied due to Lemma B.4, Lemma B.5 or Lemma B.6 resp.. Then Lemma 6.10 yields the claim. □

Proposition 6.15 Assume $r = s = q^4$ and $q \in \mathbb{G}'_N$ where $N \notin \{1, 2, 4\}$. Then $\mathcal{B}(V)$ satisfies $\operatorname{GKdim}(\mathcal{B}(V)) = \infty$.

Proof. First note that $\mathcal{B}(V)$ is of Cartan-type. Moreover, if $N \in \{3, 5, 6, 8\}$, then $\mathcal{B}(V)$ is of affine Cartan type. Thus, Proposition 6.5 implies $\operatorname{GKdim}(\mathcal{B}(V)) = \infty$. For the remaining cases we differentiate three cases:

In the following note that $c_{12} = -N + 4 \leq -3$. Especially, 0 and 1 are not contained on $\mathbb{J}$. First, if $N \in \{2^{k+3}, 3^{k+1} \,|\, k \in \mathbb{N}\}$, then $c_{21} = -N \leq -7$ holds.

This yields $\mathrm{mult}(2\alpha_1 + 4\alpha_2) \geq 1$ by Theorem 5.14 and Lemma 5.13 due to $q \neq -1$, $qr = q^5 \neq 1$. Then we obtain

$$q' := q_{\alpha_1+2\alpha_2} = qr^2s^4 = q^{25} \in \mathbb{G}'_N$$

since $\gcd(25, N) = 1$. Therefore, there is a Nichols algebra $\mathcal{B}(W_{q'})$ with $W_{q'} \in b(q', q'^4, q'^4)$ due to Lemma 6.10 and Corollary 6.13. Now the Gelfand-Kirillov dimensions of $\mathcal{B}(W_{q'})$ and $\mathcal{B}(V)$ coincide since they are twist-equivalent up to the choice of the primitive N-th root of unity.

Furthermore, we have $qr^2s = q^{13} \neq -1$ and $q^3r^3s = q^{19} \neq -1$. Thus, the multiplicity of $3\alpha_1 + 2\alpha_2$ equals two by Theorem 5.14 and $\hat{q} := q_{3\alpha_1+2\alpha_2} = q^9r^6s^4 = q^{49} \in \mathbb{G}'_N$ since $\gcd(49, N) = 1$. Now the root vectors for $3\alpha_1 + 2\alpha_2$ are linearly independent and primitive in $K_{\geq\frac{3}{2}}/K_{>\frac{3}{2}}$ due to Remark 6.11. Thus, there is a Nichols algebra $\mathcal{B}(W_{\hat{q}})$ with $W_{\hat{q}} \in b(\hat{q}, \hat{q}^2, \hat{q})$ by Lemma 6.10 and $\mathrm{GKdim}(\mathcal{B}(W_{\hat{q}})) \geq 1$ by Theorem 6.1. Consequently, we obtain

$$\mathrm{GKdim}(\mathcal{B}(V)) \geq \mathrm{GKdim}(B_{\geq\frac{1}{2}}) \geq \mathrm{GKdim}(\mathcal{B}(W_{q'})) + 1 = \mathrm{GKdim}(\mathcal{B}(V)) + 1$$

by Corollary 6.9. Therefore, $\mathcal{B}(V)$ is of infinite Gelfand-Kirillov dimension.

If $N \in \{7^k \mid k \in \mathbb{N}\}$, we can again construct the Nichols algebra $\mathcal{B}(W_{q'})$ as above. Here $qr^2s \neq -1$ and $qr^3s^3 = q^{25} \neq 1$. Thus, the multiplicity of $2\alpha_1 + 3\alpha_2$ equals two by Theorem 5.14 and $\hat{q} := q_{2\alpha_1+3\alpha_2} = q^4r^6s^9 = q^{64} \in \mathbb{G}'_N$ since $\gcd(64, N) = 1$. The root vectors for $2\alpha_1 + 3\alpha_2$ are linearly independent and primitive in $K_{\geq\frac{2}{3}}/K_{>\frac{2}{3}}$ due to Remark 6.11. Thus, there is a Nichols algebra $\mathcal{B}(W_{\hat{q}})$ with $W_{\hat{q}} \in b(\hat{q}, \hat{q}^2, \hat{q})$ by Lemma 6.10 and $\mathrm{GKdim}(\mathcal{B}(W_{\hat{q}})) \geq 1$ by Theorem 6.1. Consequently, we obtain

$$\mathrm{GKdim}(\mathcal{B}(V)) \geq \mathrm{GKdim}(B_{\geq\frac{1}{2}}) \geq \mathrm{GKdim}(\mathcal{B}(W_{q'})) + 1 = \mathrm{GKdim}(\mathcal{B}(V)) + 1$$

by Corollary 6.9. Therefore, $\mathcal{B}(V)$ is of infinite Gelfand-Kirillov dimension.

Finally, let $N \notin \{5, 6, 2^{k+2}, 3^k, 7^k \mid k \in \mathbb{N}\}$. Then we have $N > 9$. In this case the multiplicity of $4\alpha_1 + 2\alpha_2$ is at least one by Lemma 5.13 and Theorem 5.14 due to $s \neq -1$, $rs = q^8 \neq 1$. Furthermore, we have $q' := q_{2\alpha_1+\alpha_2} = q^4r^2s = q^{16} \notin \mathbb{G}_4$ by assumption on N. Again, there is a Nichols algebra $\mathcal{B}(W_{q'})$ with $W_{q'} \in b(q', q'^4, q'^4)$ due to Lemma 6.10 and Corollary 6.13 with $\mathrm{GKdim}(\mathcal{B}(W_{q'})) = \mathrm{GKdim}(\mathcal{B}(V))$.

Now, the equations $qr^2s = q^{13} = -1$ and $q^{10}r^5s = q^{34} = 1$ can not hold simultaneously since $q \notin \mathbb{G}_8$. Thus, the multiplicity of $5\alpha_1 + 2\alpha_2)$ is at least two and

$\hat{q} := q_{5\alpha_1+2\alpha_2} = q^{81} \notin \mathbb{G}_3$ by assumption on N. As before we construct a Nichols algebra $\mathcal{B}(W_{\hat{q}})$ with $W_{\hat{q}} \in b(\hat{q}, \hat{q}^2, \hat{q})$ like above. If $\hat{q} \notin \mathbb{G}_2$, then $\mathcal{B}(W_{\hat{q}})$ is infinite dimensional by Theorem 6.1. Hence $\operatorname{GKdim}(K_{\geq \frac{5}{2}}/K_{> \frac{5}{2}}) \geq 1$.

If $q^{81} \in \mathbb{G}_2$, then $qr^2s = q^{13} \neq -1$ due to $1 = q^{162} = q^{156}q^6$ and $q \notin \mathbb{G}_6$. Moreover, $qr^3s^3 = q^{25} \neq 1$ since otherwise $1 = q^{162-150} = q^{12}$ and $1 = q^{162-13\cdot 12} = q^6$, a contradiction to the assumptions on q. Thus, $\operatorname{mult}(2\alpha_1 + 3\alpha_2) = 2$. We set $\bar{q} := q_{2\alpha_1+3\alpha_2} = q^{64}$. Now, $\bar{q} \notin \mathbb{G}_2$ due to the assumptions on q. Furthermore, if $\bar{q} \in \mathbb{G}_3$, then $1 = q^{192-162} = q^{30}$ and, consequently, $1 = q^{162-150} = q^{12}$. This yields a contradiction as seen before. Therefore, the Nichols algebra $\mathcal{B}(W_{\bar{q}})$ with $W_{\bar{q}} \in b(\bar{q}, \bar{q}^4, \bar{q}^4)$ constructed as before is infinite dimensional by Theorem 6.1. Hence $\operatorname{GKdim}(K_{\geq \frac{2}{3}}/K_{> \frac{2}{3}}) \geq 1$.

In any case we obtain

$$\begin{aligned} \operatorname{GKdim}(\mathcal{B}(V)) &\overset{\textit{Lemma } 3.4}{\geq} \operatorname{GKdim}(B_{\geq \frac{2}{3}}) \\ &\overset{\textit{Corollary } 6.9}{\geq} \operatorname{GKdim}(\mathcal{B}(W_{q'})) + 1\,. \end{aligned}$$

We can use the above construction iteratively since $q' = q^{16} \notin \mathbb{G}_4$ for $q \notin \mathbb{G}_{2^k}$ for any $k \in \mathbb{N}_0$. If at any point the constructed $q' \in \mathbb{G}_M$ for $M \in \{5, 6, 2^{k+2}, 3^k, 7^k \mid k \in \mathbb{N}\}$, then $\operatorname{GKdim}(\mathcal{B}(V)) = \infty$ by the above. Otherwise, we have an chain of Nichols-algebras $(B^{(k)})_{k\in\mathbb{N}}$ with

$$B^{(1)} = \mathcal{B}(V) \text{ and } \operatorname{GKdim}(B^{(k)}) \geq \operatorname{GKdim}(B^{(k+1)}) + 1\,.$$

Since in this construction q' can never be in $\mathbb{G}_4$ due to the assumption this chain is infinite and we conclude $\operatorname{GKdim}(\mathcal{B}(V)) = \infty$. □

Corollary 6.16 Let $m \in \mathbb{N}$ such that $m\alpha_1 + \alpha_2$, $2m\alpha_1 + 2\alpha_2 \in \triangle_+$ and $q_{m\alpha_1+\alpha_2} \notin \mathbb{G}_4$. Then $\operatorname{GKdim}(\mathcal{B}(V)) = \infty$.

Proof. By assumption here is a root vector x of degree $m\alpha_1 + \alpha_2$ which is primitive in $K_{\geq m}/K_{>m}$ by Remark 6.11. Moreover, there is a homogeneous element y of degree $2m\alpha_1 + 2\alpha_2 \in \triangle_+$ which is primitive in $K_{\geq m}/K_{>m}$ due to Corollary 6.13. Application of Lemma 6.10 yields a braided vector space $W \in b(q', q'^4, q'^4)$ with $q' = q_{m\alpha_1+\alpha_2}$ satisfying $\operatorname{GKdim}(\mathcal{B}(V)) \geq \operatorname{GKdim}(\mathcal{B}(W))$. Then the claim follows from Proposition 6.15. □

Proposition 6.17 Suppose $\mathcal{B}(V)$ is of Cartan type with Cartan matrix

$$C = \begin{pmatrix} 2 & -N \\ -N & 2 \end{pmatrix}$$

for some $N \geq 2$. Then $\operatorname{GKdim}(\mathcal{B}(V)) = \infty$.

Proof. Note that we can assume $q, s \neq 1$ since Cartan-type would imply $r = 1$ and hence $N = 0$, a contradiction. For $N = 2$ the claim is true since $\mathcal{B}(V)$ is of affine Cartan-type and we apply Proposition 6.5.

Thus, consider $N = 3$. Since $\mathcal{B}(V)$ is of Cartan-type the equations $q^3r = rs^3 = 1$ hold by assumption. Assume $(qr^2s + 1)(q^4r^4s^4 - 1) \neq 0$. Then $2 \notin \mathbb{J}$ and hence $\operatorname{mult}(2\alpha_1 + 2\alpha_2) = 1$ by Theorem 5.14. Then $\operatorname{GKdim}(\mathcal{B}(V)) = \infty$ by Corollary 6.16 since $q_{\alpha_1+\alpha_2} \notin \mathbb{G}_4$. Thus, assume $(qr^2s + 1)(q^4r^4s^4 - 1) = 0$.

First, suppose $qr^2s = -1$. We obtain

$$-1 = (qr^2s)^3 = (q^3r)r^4(rs^3) = r^4.$$

If $p = 2$, this implies $r = 1$, a contradiction to $N = 3$. Otherwise, we conclude $r \in \mathbb{G}'_8$. Assume $q' := qrs = -r^{-1} \in \mathbb{G}'_8$. Then by Corollary 6.14 there exists a Nichols algebra $\mathcal{B}(W)$ with $W \in b(q', q'^6, q'^9)$ and $\operatorname{GKdim}(\mathcal{B}(V)) \geq \operatorname{GKdim}(\mathcal{B}(W))$. In any case this is of affine Cartan-type. Hence $\operatorname{GKdim}(\mathcal{B}(V)) = \infty$.

Now, suppose $q^4r^4s^4 = 1$ and $qr^2s \neq -1$. Then with $q^3r = rs^3 = 1$ we conclude $1 = q^4r^4s^4 = qr^2s$. As above we obtain $r^4 = 1$, $1 = (qr^2s)^2 = q^2s^2$ and $(q^3r)^4 = q^{12} = 1$, analogously for s. We want to calculate the multiplicity of $3\alpha_1 + 2\alpha_2$. Therefore, $(q^3r^3s)^3 = r^5 = r \neq 1$. Hence $q^3r^3s \neq 1$ and, consequently, the multiplicity of $3\alpha_1 + 2\alpha_2$ is two by Theorem 5.14. Let $q' := q_{3\alpha_1+2\alpha_2} = q^9r^6s^4 = r^2s = s^7$. Now, Lemma 6.10 yields a Nichols algebra $\mathcal{B}(W)$ with $W \in b(q', q'^2, q')$. Here $s \in \mathbb{G}'_4 \cup \mathbb{G}'_6 \cup \mathbb{G}'_{12}$ since $N = 3$. If $s \in \mathbb{G}'_4$, then $\mathcal{B}(W)$ is of affine Cartan-type and hence $\operatorname{GKdim}(\mathcal{B}(V)) \geq \operatorname{GKdim}(\mathcal{B}(W)) = \infty$. Otherwise, the Cartan matrix of $\mathcal{B}(W)$ has entries $c_{12}^{\mathcal{B}(W)} = c_{21}^{\mathcal{B}(W)} \in \{-4, -5\}$. These cases are implicitly treated below.

Now, let $N \geq 4$. Due to $s \neq 1$, $rs \neq 1$ we have $0, 1 \notin \mathbb{J}$ and hence the multiplicity of $4\alpha_1 + 2\alpha_1$ is at least one by Theorem 5.14. If $q_{2\alpha_1+\alpha_2} \notin \mathbb{G}_4$, then we conclude $\operatorname{GKdim}(\mathcal{B}(V)) = \infty$ by Corollary 6.16.

Otherwise, we can additionally assume $(qr^2s + 1)(q^4r^4s^4 - 1) = 0$ as above. If $p = 2$, then $q_{2\alpha_1+\alpha_2} = 1$ and, consequently, $1 = q_{2\alpha_1+\alpha_2} = q^4r^2s \in \{q^3r, q^3\}$, in any way a contradiction to $N \geq 4$.

Now, let $p \neq 2$. If $qr^2s = -1$, then $q_{2\alpha_1+\alpha_2} = q^4r^2s = -q^3 \in \mathbb{G}_4$. Thus, $q^{12} = 1$. Due to $N \geq 4$ we conclude $q \in \mathbb{G}'_6 \cup \mathbb{G}'_{12}$. If $q \in \mathbb{G}'_6$, then $r \in \{q, q^2\}$ again since $N \geq 4$. In any case $1 = -(qr^2s)^3 = s^3$. This is a contradiction to the assumption. If $q \in \mathbb{G}'_{12}$, let $r = q^i$ for some $i \in \{1, \ldots, 8\}$ due to $N \geq 4$. Then $-1 = qr^2s = q^{1+2i}s$ is equivalent to $q^{5-2i} = s$ and $N = 12 - i$. Since $rs^N = 1$ we conclude $i \in \{2, 8\}$. In any case $s = q$. For $i = 2$ we have $\mathrm{mult}(6\alpha_1 + 2\alpha_2) \geq 1$ by Theorem 5.14 and $q_{3\alpha_1+\alpha_2} = q^9r^3s = q^4 \notin \mathbb{G}_4$. Thus, $\mathrm{GKdim}\,(\mathcal{B}(V)) = \infty$ by Corollary 6.16. Hence assume $i = 8$ and $q' := q_{2\alpha_1+\alpha_2} = q^4r^2s = -q^3 = q^9 \in \mathbb{G}'_4$. Then by Corollary 6.14 there is a Nichols algebra $\mathcal{B}(W)$ with $W \in b(q', q'^2, q')$ satisfying $\mathrm{GKdim}\,(\mathcal{B}(V)) \geq \mathrm{GKdim}\,(\mathcal{B}(W))$. This is of affine Cartan-type. Hence $\mathrm{GKdim}\,(\mathcal{B}(V)) = \infty$.

Next, let $qrs \in \mathbb{G}_4$ and assume $qr^2s \neq -1$. As above

$$1 = q_{2\alpha_1+\alpha_2}^4 = q^{16}r^8s^4 = q^{12}r^4(qrs)^4 = q^{12-4N}\,.$$

Thus, $q \in \mathbb{G}_{4(N-3)}$ and, consequently, $N \geq 5$. Analogously, we obtain $s \in \mathbb{G}_{8N-12}$. Moreover $1 = q^{16}r^8s^4 = (qrs)^8(q^8s^{-4}) \Leftrightarrow s^4 = q^8$. Combining these equations we get

$$1 = s^{8N-12} = s^{4(2N-3)} = q^{8(2N-3)} = q^{2(4N-12)}q^{8N}.$$

From above calculations we conclude $q^{24} = r^8 = s^{12} = 1$. Since we are in Cartan-type $r = s^{-N}$, that is $r \in \mathbb{G}_8 \cap \mathbb{G}_{12} \setminus \{1\} = \mathbb{G}_4 \setminus \{1\}$. Due to $N \geq 5$ we know $s \in \mathbb{G}'_6 \cup \mathbb{G}'_{12}$. In the first case this together with the restriction on r yields a contradiction to $N \geq 5$. Thus, $s \in \mathbb{G}'_{12}$ and $N \in \{6, 9\}$.

If $N = 9$, then $1 = q^Nr = q^9s^3 = qs^7$ implies $q = s^5$. Now, the multiplicity of $6\alpha_1 + 2\alpha_2$ is greater than 1 by Theorem 5.14 and $q_{3\alpha_1+\alpha_2} = q^9r^3s = s^7$. Thus, $\mathrm{GKdim}\,(\mathcal{B}(V)) = \infty$ by Corollary 6.16.

If $N = 6$, then $1 = q^4r^4s^4 = q^4s^4$ and $1 = q^{16}r^8s^4 = q^{12}$. Thus, $q = s^i$ for some $0 \leq i \leq 11$. In any case $\mathrm{mult}(6\alpha_1 + 2\alpha_2) \geq 1$ and $q_{3\alpha_1+\alpha_2} = q^9r^3s = -s^{9i+1} = (-s^{9i})s$. This is not in $\mathbb{G}_4$ due to the fact that $(-s^{9i}) \in \mathbb{G}_4$ and $s \in \mathbb{G}'_{12}$. Thus, $\mathrm{GKdim}\,(\mathcal{B}(V)) = \infty$ by Corollary 6.16. This completes the proof. $\square$

Corollary 6.18 Let $\alpha = m_1\alpha_1 + m_2\alpha_2 \in \triangle_+$ be such that $\gcd(m_1, m_2) = 1$, $\mathrm{mult}(\alpha) = 2$ and $q_\alpha \in \mathbb{G}_N$ for some $N > 3$. Then $\mathrm{GKdim}\,(\mathcal{B}(V)) = \infty$.

Proof. There are root vectors x and y of degree α. Those are linearly independent by definition and primitive in $K_{\geq \frac{m_1}{m-2}}/K_{> \frac{m_1}{m_2}}$ due to Remark 6.11. Then application of Lemma 6.10 yields a braided vector space $W \in b(q', q'^2, q')$ with $q' = q_\alpha$ satisfying $\mathrm{GKdim}\,(\mathcal{B}(V)) \geq \mathrm{GKdim}\,(\mathcal{B}(W))$. Finally, the claim follows from Proposition 6.17 since $\mathcal{B}(W)$ is obviously of Cartan-type. □

7 | Infinite Gelfand-Kirillov dimensional Nichols algebras

This section is devoted to the step-by-step proof of our main result.
Theorem 7.1 Let $\mathbb{K}$ be an arbitrary field and $\mathcal{B}(V)$ a rank two Nichols algebra of diagonal type over $\mathbb{K}$. If $\mathcal{B}(V)$ is of finite Gelfand-Kirillov dimension, then the corresponding root system is finite.

Note that the converse is known to be true [13]. We recall a statement from [5]. Together with above theorem and Theorem 6.1 one can compute the Gelfand-Kirillov dimension of $\mathcal{B}(V)$ in case it is finite.
Proposition 7.2 [5] Let L be as in Theorem 4.30. For $\ell \in L$ we set

$$N_\ell = \min\{k \in \mathbb{N} \,|\, (k)_{q_{\deg(\ell)}} = 0\} \in \mathbb{N} \cup \infty \,.$$

Then $\operatorname{GKdim}(\mathcal{B}(V)) = \#\{\ell \in L \,|\, N_\ell = \infty\}$.

The proof of Theorem 7.1 follows from below lemmata proving the statement step-by-step. The main idea of the proof is to exhaust the knowledge of

$$\triangle_+ \cap \{m\alpha_1 + 2\alpha_2 \,|\, m \in \mathbb{N}\}$$

stated in Theorem 5.14 and apply Corollary 6.16 and Corollary 6.18 to conclude $\operatorname{GKdim} \mathcal{B}(V) = \infty$. The remaining cases satisfy $\#\triangle < \infty$ or will be treated individually mainly using Corollary 6.4.

We stick to the notation from the previous chapters. By Corollary 6.3 we can assume $\mathcal{B}(V)$ to be i-finite for all $i \in \{1,2\}$. Thus, we can assign a Cartan matrix $C = (c_{ij})_{1\le i,j\le 2}$ to $\mathcal{B}(V)$. Furthermore, by Remark 5.2 we can assume $c_{12} \le c_{21}$. In case $c_{12} = c_{21}$ we use Remark 5.2 to reduce the number of cases to be considered. Finally, we assume $r \neq 1$ since otherwise the set of roots and, consequently, by

Theorem 6.1 the Gelfand-Kirillov dimension are finite.

We denote $\mathcal{B}(V)$ to be of finite type k if $\#\triangle < \infty$ and $\mathcal{B}(V) \in b(q, r, s)$ with q, r, s as in the row in A.1. Moreover, in the proofs below if $1 \in \{q, s\}$, we implicitly assume $p > 0$ since we assumed i-finiteness.

Lemma 7.3 Suppose $u_6 \neq 0$. Then $\mathcal{B}(V)$ is of infinite Gelfand-Kirillov dimension $p \neq 7$.

If $p = 7$, then

$$\operatorname{GKdim} \mathcal{B}(V) < \infty \Leftrightarrow \mathcal{B}(V) \text{ is of finite type } 18.$$

Proof. The proof splits in three big parts:

(**A**) $6\alpha_1 + 2\alpha_2 \notin \triangle_+$,

(**B**) $6\alpha_1 + 2\alpha_2 \in \triangle_+$ with $c_{12} \leq -2$,

(**C**) $6\alpha_1 + 2\alpha_2 \in \triangle_+$ with $c_{12} \leq -2$.

Starting with case (**A**) note that $6\alpha_1 + 2\alpha_2 \notin \triangle_+$ if and only if $0, 3, 6 \in \mathbb{J}$ by Lemma 5.13 and Theorem 5.14.

Assume $0, 3, 6 \in \mathbb{J}$. It follows that

$$\begin{aligned} &s = -1, \\ &q^3 r^3 s = 1, \qquad qr \neq -1 \text{ or } (qr = -1 \text{ and } p - \mid 3) \\ &q^{15} r^6 s = -1, \quad q^5 r^2 \neq 1 \text{ or } (q^5 r^2 = 1 \text{ and } 2 \cdot p \mid 6) \\ &\qquad\qquad\qquad \text{and } q^4 r \neq -1 \text{ or } (q^4 r = -1 \text{ and } p \mid 3) \end{aligned}$$

We conclude $-1 = q^{15} r^6 s = q^9 (q^3 r^3 s)^2 s^{-1} = -q^9$, that is $q^9 = 1$. It follows that $q \in \{1\} \cup \mathbb{G}'_9$ since $u_6 \neq 0$. Moreover, $-1 = (q^{15} r^6 s)(q^3 r^3 s) = q^{18} r^9 s^2 = r^9$. Note that $r \neq -1$ since otherwise $1 = q^3 r^3 s = q^3$ contradicts $u_6 \neq 0$.

First, we consider $p = 2$. Then $q \neq 1$ due to $u_6 \neq 0$ and $q^9 = r^9 = 1$. So $q \in \mathbb{G}'_9$ and $r = q^i$ for some $1 \leq i \leq 3$ again due to $u_6 \neq 0$. We obtain

$$\begin{aligned} &i = 1 \Rightarrow q^3 r^3 = q^6 \neq 1\,. \\ &i = 2 \Rightarrow q^5 r^2 = r^9 = 1, \text{ but } 2 \cdot p \nmid 6\,. \\ &i = 3 \Rightarrow q^3 r^3 = q^3 \neq 1\,. \end{aligned}$$

All cases contradict the assumptions. Thus, there is no solution for $p = 2$.

Now, assume $p \neq 2$. First, assume $q \in \mathbb{G}'_9$. If $r \in \mathbb{G}'_6$, then $q^{15} r^6 s = -q^6 \neq 1$. Hence $r \in \mathbb{G}'_{18}$ and $q = r^{2i}$ for some $1 \leq i \leq 8$. Then $c_{12} = -8$ and hence

$4\alpha_1+\alpha_2 \in \triangle_+$ and $8\alpha_1+2\alpha_2 \in \triangle_+$ by Theorem 5.14 with $q_{4\alpha_1+\alpha_2} = q^{16}r^4s = r^{13+14i}$. Here $r^{13+14i} \in \mathbb{G}_4$ iff $i = 1$. In this case $q^4r = -1$, but $p \nmid 3$ since otherwise $r \in \mathbb{G}'_{18} = \emptyset$, a contradiction to $6 \in \mathbb{J}$. Hence $q_{4\alpha_1+\alpha_2} \notin \mathbb{G}_4$ and $\operatorname{GKdim} \mathcal{B}(V) = \infty$ by Corollary 6.16.

Next, assume $q = 1$ and $p \geq 7$ since otherwise $u_6 = 0$. This implies $r \neq -1$ and $r^3 = -1$ since $3 \in \mathbb{J}$. If $p > 7$, then $7, 8 \notin \mathbb{J}$ by Lemma 5.13. Consequently, $8\alpha_1+2\alpha_2 \in \triangle_+$ and $q_{4\alpha_1+\alpha_2} = q^{16}r^4s = r \notin \mathbb{G}_4$ by the above. Hence $\operatorname{GKdim} \mathcal{B}(V) = \infty$ by Corollary 6.16. If $p = 7$, then $\mathcal{B}(V)$ is of finite type 18 by Theorem 6.1.

Now, we consider case (**B**) with $6\alpha_1 + 2\alpha_2 \in \triangle_+$ and we assume $q_{3\alpha_1+\alpha_2} \in \mathbb{G}_4$ due to Corollary 6.16.

First suppose $c_{21} < -1$. This implies $s \neq -1, rs \neq 1$ and hence $0, 1 \notin \mathbb{J}$. Consequently, $\#(\mathbb{J} \cap [0, 4]) \leq 1$ by Lemma 5.13. Thus, $4\alpha_1 + 2\alpha_2 \in \triangle_+$ and we assume $q_{2\alpha_1+\alpha_2} = q^4r^2s \in \mathbb{G}_4$ like above. If $p = 2$, then $q^4r^2s = 1$ and $q^9r^3s = 1$ since $\mathbb{G}_4 = \{1\}$ and hence $q^5r = 1$, a contradiction to $u_6 \neq 0$.

Next, assume $p \neq 2$. This divides into the cases (**B.1**) $2 \in \mathbb{J}$ and (**B.2**) $2 \notin \mathbb{J}$.

First, assume (**B.1**) $2 \in \mathbb{J}$ holds, that is $qr^2s = -1$. Then $1 = (q^4r^2s)^4 = (-q^3)^4 = q^{12}$. If $q \in \mathbb{G}_{12} \setminus (\mathbb{G}'_{12} \cup \{1\})$, then $u_6 = 0$, a contradiction. Thus, suppose $q \in \mathbb{G}'_{12}$ first. This implies $1 = (q^9r^3s)^4 = q^{36}r^{12}s^4 = q^{20}r^4(q^4r^2s)^4 = q^8r^4 \Rightarrow r^{12} = 1$. That is $r = q^i$ for some $1 \leq i \leq 6$. Then $1 = (q^9r^3s)^4 = q^{36}r^{12}s^4 = s^4$ and $qr^2s = -1 \Rightarrow s = q^{5-2i} \in \mathbb{G}_4$. Since $q \in \mathbb{G}'_{12}$ this is equivalent to $5 - 2i \equiv 0 \mod 3$, that is $i \in \{1, 4\}$. If $i = 1$, then $c_{12} \leq -8$ and hence $8\alpha_1 + 2\alpha_2 \in \triangle_+$ and $q_{4\alpha_1+\alpha_2} = q^{16}r^4s = q^{11} \in \mathbb{G}'_{12}$. This yields $\operatorname{GKdim} \mathcal{B}(V) = \infty$ by Corollary 6.16. If $i = 4$, then $q^{10}r^5s \neq 1$. This implies $\operatorname{mult}(5\alpha_1 + 2\alpha_2) = 2$ with $q_{5\alpha_1+2\alpha_2} = q^{25}r^{10}s^4 = q^5 \in \mathbb{G}'_{12}$. Then $\operatorname{GKdim} \mathcal{B}(V) = \infty$ by Corollary 6.18.

Now, assume $q = 1$. As above $r^4 = q^{20}r^4 = 1$ and, consequently, $1 = (qr^2s)^2 = s^2$. Since $s \neq -1$ by assumption this yields $s = 1$ and $-1 = qr^2s = r^2$, that is $r \in \mathbb{G}'_4$. Thus, $u_3 \neq 0$ and $q^4r^2s = -1$. This implies $\operatorname{GKdim} \mathcal{B}(V) = \infty$ by Corollary 6.4.

Next, assume (**B.2**) holds, i.e. $2 \notin \mathbb{J}$ and, consequently, $qr^2s \neq -1$ since $0, 1 \notin \mathbb{J}$. Note that by Lemma 5.13 and Theorem 5.14 the multiplicity of $5\alpha_1 + 2\alpha_2$ is at least two. Moreover, we can assume $qrs \in \mathbb{G}_4$ for otherwise Corollary 6.16 was applicable and $q^4r^2s, q^9r^3s \in \mathbb{G}_4$ for the same reason. This implies $1 = (q^4r^2s)^4 = (q^3r)^4(qrs)^4 = q^{12}r^4$ and $1 = (q^9r^3s)^4 = (q^5r)^4(q^4r^2s)^4 = q^{20}r^4$. Hence $q \in \mathbb{G}_8$. Esp. $q \notin \mathbb{G}'_2 \cup \mathbb{G}'_4$ since $u_6 \neq 0$ and, consequently, $q \in \{1\} \cup \mathbb{G}'_8$. Therefore, $1 = q^{12}r^4q^{20}r^4 = q^{32}r^8 = r^8$.

If $p = 2$, then $\mathbb{G}_8 = \{1\}$ and $r = 1$, a contradiction. So assume $p \neq 2$ and

$q \neq 1$. Then $1 = q^{12}r^4 = -r^4$, i.e. $r \in \mathbb{G}'_8$, and since $u_6 \neq 0$ we conclude $q = r$ for otherwise $q^i r = 1$ for $1 \leq i \leq 5$. In addition, $1 = (q^4r^2s)^4 = q^{16}r^8s^4 = s^4$. Hence $q_{5\alpha_1+2\alpha_2} = q^{25}r^{10}s^4 = qr^2 = q^3 \in \mathbb{G}'_8$ which implies $\operatorname{GKdim}\mathcal{B}(V) = \infty$ by Corollary 6.18.

If $q = 1$, then $1 = q^{12}r^4 = r^4$ and $1 = (qrs)^4 = s^4$. We have $s \notin \{-1, r^{-1}\}$ due to $0, 1 \notin \mathbb{J}$. Thus, $s \in \mathbb{G}'_4$ and $r \in \{-1, s\}$ or $s = 1$ and $r = -1$ since $qr^2s \neq -1$. If $r = -1, s \in \mathbb{G}'_4$, then $qrs \in \mathbb{G}'_4$. Note that $p \neq 2$ due to $u_6 \neq 0$. By Example 5.33 $\det(F_{1,\bar{t}}) = -64 \neq 0$ for all $0 \leq \bar{t} \leq 3$. Hence Corollary 6.4 implies $\operatorname{GKdim}\mathcal{B}(V) = \infty$. Otherwise $qrs = -1$ and Corollary 6.4 implies $\operatorname{GKdim}\mathcal{B}(V) = \infty$.

Finally, assume (**C**) holds, that is $c_{21} = -1$ and, consequently, (**C.1**) $s = -1 \Leftrightarrow 0 \in \mathbb{J}$ or (**C.2**) $rs = 1, s \neq -1 \Leftrightarrow 1 \in \mathbb{J}$. We still assume $6\alpha_1 + \alpha_2 \in \triangle_+$ and, thus, $q^9r^3s \in \mathbb{G}_4$ as above.

We start by assuming (**C.2**): $s \neq -1, rs = 1$. Note that $r \neq -1$ since otherwise $s = -1$. Then $1 = (q^9r^3s)^4 = q^{36}r^8$. If $4 \notin \mathbb{J}$, then $\operatorname{mult}(4\alpha_1 + 2\alpha_2) \geq 1$ and $q_{2\alpha_1+\alpha_2} = q^4r^2s = q^4r \in \mathbb{G}'_4$ or $\operatorname{GKdim}\mathcal{B}(V) = \infty$ by Corollary 6.16. This implies $1 = q^{36}r^8 = q^4(q^4r)^8 = q^4$. Then $q = 1$ for otherwise $u_6 = 0$. Hence $p \geq 7$, $1 = (q^4r)^4 = r^4$ and since $r \notin \mathbb{G}_2$ we conclude $r \in \mathbb{G}'_4$. Then $u_4 \neq 0$ and $q_{3\alpha_1+\alpha_2} = q^9r^3s = -1$ which implies $\operatorname{GKdim}\mathcal{B}(V) = \infty$ by Corollary 6.4.

If $4 \in \mathbb{J}$, then $-1 = q^6r^4s = q^6r^3$. If $p = 2$, then $1 = q^9r^3s = q^8r^2$ since $\mathbb{G}_4 = \{1\}$ and, thus, $1 = (q^8r^2)^3 = (q^6r^3)^4r^{-6} = r^{-6}$. This implies $r \in \mathbb{G}'_3$ since $p = 2$. Consequently, $1 = q^6r^3$ implies $q \in \mathbb{G}_3$ and hence $u_6 = 0$, a contradiction.

Thus, assume $p \neq 2$. Then $1 = (q^9r^3s)^4 = q^{36}r^8 = (q^6r^3)^6r^{-10} = r^{-10}$ and hence $r \in \mathbb{G}'_5 \cup \mathbb{G}'_{10}$ since $r \notin \mathbb{G}_2$. Moreover, $-1 = (q^6r^3)^5 = q^{30}r^{15} = (q^{36}r^8)q^{-6}r^7 \Leftrightarrow q^6 = -r^7 \Rightarrow q^{18} = -r$.

If $r \in \mathbb{G}'_5$, then $q^{30} = -1$ and $r = q^{-12}$. Hence we conclude $q \in \mathbb{G}'_{20} \cup \mathbb{G}'_{60}$ since $r \neq 1$ and $u_6 \neq 0$. Thus, $c_{12} = -12$ and $\operatorname{mult}(10\alpha_1 + 2\alpha_2) \geq 1$ by Theorem 5.14 with $q_{5\alpha_1+\alpha_2} = q^{25}r^5s = q^{25}(q^{-12})^2(rs) = q \notin \mathbb{G}_4$. Then $\operatorname{GKdim}\mathcal{B}(V) = \infty$ by Corollary 6.16.

If $r \in \mathbb{G}'_{10}$, then $q^{30} = 1$ and $q^{16}r^4s = q^{16}r^3 = -q^{10}$. We conclude $q \in \mathbb{G}'_{10} \cup \mathbb{G}'_{15} \cup \mathbb{G}'_{30}$ since $u_6 \neq 0$. In the first case, $u_5 \neq 0$, $q^{16}r^4s = -1$. Thus, $\operatorname{GKdim}\mathcal{B}(V) = \infty$ by Corollary 6.4. In the latter cases $c_{12} \leq -8$ due to $-r = q^{18}$ and $q_{4\alpha_1+\alpha_2} = q^{16}r^4s = -q^{10} \notin \mathbb{G}_4$. Hence $\operatorname{GKdim}\mathcal{B}(V) = \infty$ by Corollary 6.16.

Finally, assume (**C.1**), that is $s = -1$, $q^9r^3s = -q^9r^3 \in \mathbb{G}_4$. This implies $0 \in \mathbb{J}, 1, 2 \notin \mathbb{J}$.

If $p = 2$, then $q^9r^3 = 1$ since $\mathbb{G}_4 = \{1\}$. Assume $3 \in \mathbb{J}$. This implies $q^3r^3 = 1$

and hence $1 = q^9r^3 = q^6$, a contradiction to $u_6 \neq 0$.

Assume $4 \in \mathbb{J}$. Then $1 = q^6r^4s = q^6r^4$. Thus, $1 = (q^9r^3)^2 = (q^6r^4)^3r^{-6} = r^{-6}$ and, consequently, $r^3 = 1$ since $p = 2$. Therefore, $1 = q^9r^3 = q^9$ and $1 = q^6r^4 = q^6r$. This implies $r = q^3$. Then $\mathcal{R}^2(V) \in (q^4, q^6, 1)$ and $c_{12}^{\mathcal{B}(\mathcal{R}^2(V))} = -3$, $c_{21}^{\mathcal{B}(\mathcal{R}^2(V))} = -1$. Note that $V = \mathcal{R}^1(V)$ and $\mathcal{R}^2(V) = \mathcal{R}^1(\mathcal{R}^2(V))$. We conclude that $\sigma_1\sigma_2\sigma_1\sigma_2$ is an automorphism of $\triangle^V$. We calculate

$$\begin{aligned} \sigma_1\sigma_2\sigma_1\sigma_2(3\alpha_1+\alpha_2) &= \sigma_1\sigma_2\sigma_1(3\alpha_1+2\alpha_2) \\ &= \sigma_1\sigma_2(3\alpha_1+2\alpha_2) \\ &= \sigma_1(3\alpha_1+\alpha_2) \\ &= 3\alpha_1+\alpha_2 \end{aligned}$$

and

$$\begin{aligned} \sigma_1\sigma_2\sigma_1\sigma_2(\alpha_1) &= \sigma_1\sigma_2\sigma_1(\alpha_1+\alpha_2) \\ &= \sigma_1\sigma_2(2\alpha_1+\alpha_2) \\ &= \sigma_1(2\alpha_1+\alpha_2) \\ &= 4\alpha_1+\alpha_2 \\ &= \alpha_1+(3\alpha_1+\alpha_2)\,. \end{aligned}$$

Hence by linearity of the σ_i we conclude $k(3\alpha_1 + \alpha_2) + \alpha_1 \in \triangle_+^V$ for all $k \in \mathbb{N}_0$. Consequently, $\mathrm{GKdim}\,(\mathcal{B}(V)) = \infty$ by Corollary 6.2.

If $3, 4 \notin \mathbb{J}$, then $\mathrm{mult}(4\alpha_1+2\alpha_2) \geq 1$. Consequently, we assume $q_{2\alpha_1+\alpha_2} = q^4r^2s \in \mathbb{G}_4 = \{1\}$. Then $1 = q^9r^3s = q^5r$, contradicting $u_6 \neq 0$.

Now, assume $p \neq 2$. We want to to discuss the following cases:

(**C.1.1**) $3 \in \mathbb{J}$,

(**C.1.2**) $4 \in \mathbb{J}$,

(**C.1.3**) $3, 4 \notin \mathbb{J}$.

First, assume (**C.1.1**), i.e. $3 \in \mathbb{J}$ and, consequently, $q^3r^3s = 1$. Then $1 = (q^9r^3)^4 = (-q^6)^4 = q^{24}$ and $1 = (q^3r^3)^8 = r^{24}$. Thus, $q \in \{1\} \cup \mathbb{G}'_8 \cup \mathbb{G}'_{12} \cup \mathbb{G}'_{24}$ since $u_6 \neq 0$.

First, assume $q \in \mathbb{G}'_{24}$. Then $q^3r^3 = -1$ implies $r^3 = q^9 \in \mathbb{G}'_8$. If $r \in \mathbb{G}'_{24}$, then $r = q^i$ with $i \in \{1, 5, 7, 11, 13, 17, 19, 23\}$. If $i > 18$, then $u_6 = 0$, a contradiction.

If $i \neq 11$, then $q^3r^3 \neq -1$, a contradiction to $3 \in \mathbb{J}$. If $r = q^{11}$, we have $q^2r^2 = 1$. Hence $3 \in \mathbb{J}$ iff $p\,|\,3-0=3$. This is a contradiction to $q \in \mathbb{G}'_{24}$.

Otherwise, $r \in \mathbb{G}'_8$. Then $-1 = (q^3r^3)^3 = q^9r$ implies $r = q^3$. Consequently, $c_{12} \leq -8$, $8\alpha_1 + 2\alpha_2 \in \triangle_+$ and $q_{4\alpha_1+\alpha_2} = q^{16}r^4s = q^{16} \in \mathbb{G}'_3$. Hence $\operatorname{GKdim}\mathcal{B}(V) = \infty$ by Corollary 6.16.

Next, assume $q \in \mathbb{G}'_{12}$. This implies $r^3 = (q^3r^3)q^9 = -q^9 = q^3 \in \mathbb{G}'_4$. Thus, $r \in \mathbb{G}'_4 \cup \mathbb{G}'_{12}$. In the first case $-1 = q^3r^3 = -q^3r$ yields $q^3r = 1$, a contradiction to $u_6 \neq 0$. In the latter case $r = q^i$ with $i \in \{1,5,7,11\}$. Here $i > 6$ implies $u_6 = 0$, a contradiction to $u_6 \neq 0$. If $r = q^5$, then $q^2r^2 = 1$. Hence $3 \in \mathbb{J}$ iff $p\,|\,3-0=3$, a contradiction to $q \in \mathbb{G}'_{12}$. Consequently, $r = q$ and $c_{12} \leq -8$. Thus, $8\alpha_1 + 2\alpha_2 \in \triangle_+$ by Theorem 5.14 and $q_{4\alpha_1+\alpha_2} = q^{16}r^4s = q^8 \in \mathbb{G}'_3$. Corollary 6.16 implies $\operatorname{GKdim}\mathcal{B}(V) = \infty$.

Now, assume $q \in \mathbb{G}'_8$. Then $r^3 = (q^3r^3)q^{-3} = -q^{-3} = q \in \mathbb{G}'_8$. Hence $r \in \mathbb{G}'_8 \cup \mathbb{G}'_{24}$. In the former case $r = (r^3)^3 = q^3$ and hence $q^5r = 1$, a contradiction to $u_6 \neq 0$. In the latter case $\mathcal{R}^1(V) \in b(r^3, r^5, r^2 2)$ with $c_{12}^{\mathcal{B}(\mathcal{R}^1(V))} = -7, c_{21}^{\mathcal{B}(\mathcal{R}^1(V))} < -1$. This was treated above and satisfies $\operatorname{GKdim}\mathcal{B}(V) = \operatorname{GKdim}(\mathcal{B}(\mathcal{R}^1(V))) = \infty$.

Next, assume $q = 1$. This implies $q^3r^3 = r^3 = -1$. If $r = -1$, then $q^2r^2 = 1$ and hence $3 \in \mathbb{J}$ iff $p\,|\,3-0=3$. Then $u_6 = 0$, a contradiction. Thus, $r \in \mathbb{G}'_6$ and, consequently, $6 \in \mathbb{J}$ and $6\alpha_1 + 2\alpha_2 \notin \triangle_+$. This is a contradiction.

For the next step assume $p \neq 2$, $s = -1$, $3 \notin \mathbb{J}$ and $q^9r^3s \in \mathbb{G}_4$.

Next assume (**C.1.2**): $4 \in \mathbb{J}$. Then $q^6r^4 = 1$ and hence $1 = (q^9r^3s)^4 = q^{18}(q^6r^4)^3 = q^{18}$ and $1 = (q^6r^4)^3 = q^{18}r^{12} = r^{12}$. Esp. $q \in \{1\} \cup \mathbb{G}'_9 \cup \mathbb{G}'_{18}$ for otherwise $u_6 = 0$. First, if $q = 1$, then $1 = q^6r^4 = r^4$. If $1 = r^2 = q^3r^2$, then $4 \in \mathbb{J}$ iff $p\,|\,4-0=4$, a contradiction to $p \neq 2$. Thus, $r \in \mathbb{G}'_4$, $q^{16}r^4s = -1$ and $u_5 \neq 0$. Thus, $\operatorname{GKdim}\mathcal{B}(V) = \infty$ by Corollary 6.4.

Now, assume $q \neq 1$, i.e. $q \in \mathbb{G}'_9 \cup \mathbb{G}'_{18}$. Consequently, if $r \neq q^i$ for some $1 \leq i < \operatorname{ord}(q)$, then $c_{12} = -\operatorname{ord}(q) + 1 \in \{8, 17\}$, $8\alpha_1 + 2\alpha_2 \in \triangle_+$ and $q_{4\alpha_1+\alpha_2} = q^{16}r^4s = -q^{10}(q^6r^4) = -q^{10} \in \mathbb{G}'_{18}$. Thus, $\operatorname{GKdim}\mathcal{B}(V) = \infty$ by Corollary 6.16. Therefore, assume $r = q^i$ for some $1 \leq i < \operatorname{ord}(q)$. Assume $q \in \mathbb{G}'_{18}$. If $r \in \mathbb{G}'_2 \cup \mathbb{G}'_6$, then $r \in \{q^3, q^9, q^{15}\}$. Since $r = q^{15}$ yields $u_6 = 0$ we conclude $c_{12} \leq -8$ and $q_{4\alpha_1+\alpha_2} = q^{16}r^4s = q^7r^4 \in \mathbb{G}'_{18}$. Hence $\operatorname{GKdim}\mathcal{B}(V) = \infty$ by Corollary 6.16. If $r \in \mathbb{G}'_3$, then $1 = q^6r^4 = q^6r$ and hence $r = q^{12}$. Here $\mathcal{R}^2(V) \in (q^4, q^6, -1)$ and $c_{12}^{\mathcal{B}(\mathcal{R}^2(V))} = -3$, $c_{21}^{\mathcal{B}(\mathcal{R}^2(V))} = -1$. Note that $V = \mathcal{R}^1(V)$ and $\mathcal{R}^2(V) = \mathcal{R}^1(\mathcal{R}^2(V))$.

We conclude that $\sigma_1\sigma_2\sigma_1\sigma_2$ is an automorphism of $\triangle^V$. We calculate

$$\begin{aligned}\sigma_1\sigma_2\sigma_1\sigma_2(3\alpha_1+\alpha_2) &= \sigma_1\sigma_2\sigma_1(3\alpha_1+2\alpha_2)\\ &= \sigma_1\sigma_2(3\alpha_1+2\alpha_2)\\ &= \sigma_1(3\alpha_1+\alpha_2)\\ &= 3\alpha_1+\alpha_2\end{aligned}$$

and

$$\begin{aligned}\sigma_1\sigma_2\sigma_1\sigma_2(\alpha_1) &= \sigma_1\sigma_2\sigma_1(\alpha_1+\alpha_2)\\ &= \sigma_1\sigma_2(2\alpha_1+\alpha_2)\\ &= \sigma_1(2\alpha_1+\alpha_2)\\ &= 4\alpha_1+\alpha_2\\ &= \alpha_1+(3\alpha_1+\alpha_2)\,.\end{aligned}$$

Hence by linearity of the σ_i we conclude $k(3\alpha_1+\alpha_2)+\alpha_1 \in \triangle_+^V$ for all $k \in \mathbb{N}_0$. Consequently, $\operatorname{GKdim}(\mathcal{B}(V)) = \infty$ by Corollary 6.2.

Next, assume $q \in \mathbb{G}'_9$. Then $r \in \mathbb{G}'_3$ and $q^9r^3s = -1$. Since $u_4 \neq 0$ we can apply Corollary 6.4. Thus, $\operatorname{GKdim}\mathcal{B}(V) = \infty$.

Finally, we can assume (**C.1.3**): $3, 4 \notin \mathbb{J}$. By Theorem 5.14 this implies $4\alpha_1 + 2\alpha_2 \in \triangle_+$ and hence we may assume $q_{2\alpha_2+\alpha_2} = q^4r^2s \in \mathbb{G}_4$. Then $1 = (q^9r^3s)^8 = q^{72}r^{24} = q^{24}(-q^4r^2)^{12} = q^{24}$ and hence $1 = (q^4r^2s)^{12} = q^{48}r^{24} = r^{24}$. Again, $q \in \{1\} \cup \mathbb{G}'_8 \cup \mathbb{G}'_{12} \cup \mathbb{G}'_{24}$.

First, consider $q \in \mathbb{G}'_{24}$. Then $-q^{12} = 1 = (q^9r^3s)^4 = q^{36}r^{12} = -r^{12}$ and $r^8 = (q^4r^2s)^4q^{-16} = q^8$. Therefore $r^4 = q^4$ and consequently $r \in \mathbb{G}'_{24}$. Thus, $r = q^i$ with $i \in \{1, 7, 13, 19\}$. Here $i = 19$ implies $q^5r = 1$ and $u_6 = 0$, a contradiction. In any remaining cases $q^{10}r^5s \neq 1$ and hence $5 \notin \mathbb{J}$. Therefore, $\operatorname{mult}(5\alpha_1 + 2\alpha_2) \geq 2$ and $q^2_{5\alpha_1+2\alpha_2} = q^{50}r^{20}s^4 = q^{70} = q^{22} \in \mathbb{G}'_{12}$ and, consequently, $q_{5\alpha_1+2\alpha_2} \notin \mathbb{G}_2 \cup \mathbb{G}_3$. Hence Corollary 6.18 yields $\operatorname{GKdim}\mathcal{B}(V) = \infty$.

Now, assume $q \in \mathbb{G}'_{12}$. Then $1 = q^{36}r^{12} = r^{12}$ and $q^4 = r^4$ like above. We conclude $r \in \mathbb{G}'_3 \cup \mathbb{G}'_6 \cup \mathbb{G}'_{12}$. Due to $q^4 = r^4$ and $u_6 \neq 0$ this implies $c_{12} \leq -8$, $8\alpha_1 + 2\alpha_2 \in \triangle_+$ and $q^{16}r^4s = -q^{20} = q^2 \in \mathbb{G}'_6$. Hence Corollary 6.16 implies $\operatorname{GKdim}\mathcal{B}(V) = \infty$.

Next, consider $q \in \mathbb{G}'_8$. Like above $1 = q^{36}r^{12} = -r^{12}$ and $1 = q^{16}r^8 = r^8$. This implies $r \in \mathbb{G}'_8$ and, thus, $r = q$ due to $u_6 \neq 0$. Then $q^{10}r^5s = -q^{-1} = q^3 \neq 1$. Hence

$\mathrm{mult}(5\alpha_1 + 2\alpha_2) \geq 2$ with $q_{5\alpha_1+2\alpha_2} = q^{25}r^{10}s^4 = q^3 \in \mathbb{G}_8'$. Again, Corollary 6.18 yields $\mathrm{GKdim}\,\mathcal{B}(V) = \infty$.

Finally, assume $q = 1$. Then $1 = (q^9r^3s)^4 = r^{12}$ and $1 = (q^4r^2s)^4 = r^8$. Hence $r^4 = 1$. Therefore, $q^{16}r^4s = -1$ and $u_5 \neq 0$ and by Corollary 6.4 $\mathrm{GKdim}\,\mathcal{B}(V) = \infty$ holds. □

Lemma 7.4 Suppose $-5 \leq c_{12} \leq -4$ and $-5 \leq c_{21} \leq -2$. Then $\mathrm{GKdim}\,\mathcal{B}(V) = \infty$.

Proof. First, note that $c_{21} \leq -2$ implies $0, 1 \notin \mathbb{J}$. We can assume

$$(qr^2s + 1)((qrs)^4 - 1) = 0$$

since otherwise $\mathrm{mult}(2\alpha_1 + \alpha_2) \geq 1$ due to Theorem 5.14 and, consequently, Corollary 6.16 yields the claim. Moreover, $\mathrm{mult}(4\alpha_1 + 2\alpha_2) \geq 1$ due to $0, 1 \notin \mathbb{J}$ and Lemma 5.13 and Theorem 5.14. Hence we assume $q^4r^2s \in \mathbb{G}_4$.

If $p = 2$, then $q^4r^2s = 1$ since $\mathbb{G}_4 = \{1\}$. Thus, $qr^2s \neq 1$ for otherwise $1 = q^4r^2s = q^3$, a contradiction to $c_{12} \leq -4$. Analogously, $qrs \neq 1$ since $1 = q^4r^2s = (qrs)(q^3r)$. This is a contradiction to above assumption. Thus, $\mathrm{GKdim}\,\mathcal{B}(V) = \infty$ if $p = 2$.

Thus, assume $p \neq 2$. The remaining proof splits into the cases (**A**) $c_{12} = -4$ and (**B**) $c_{12} = -5$. Both cases will be further split into the cases $qr^2s \neq -1$ and $qrs \in \mathbb{G}_4$ by the above. Those will be denoted by (**A.1**) and (**A.2**) or, analogously, (**B.1**) and (**B.2**).

We start discussing case (**A.1**): $c_{12} = -4$ and $qr^2s \neq -1$. We consider the cases

$$\text{(a) } q^4r = 1; \quad \text{(b) } q \in \mathbb{G}_5'; \quad \text{(c) } q = 1,\, p = 5\,.$$

In case (a) $1 = (q^4r^2s)^4 = (q^4r)^4(rs)^4 = q^{-4}(qrs)^4 = q^{-4}$, a contradiction to $c_{12} = -4$.

In case (b) $1 = (q^4r^2s)^4 = q^{16}r^8s^4 = q^{12}r^4(qrs)^4 = q^2r^4$. Hence $r^4 = q^3$ and $s^4 = (qrs)^4q^{-4}r^{-4} = q^3$. Then $(q^3r^3s)^4 = q^{12}r^{12}s^4 = q^{12+9+3} = q^4 \in \mathbb{G}_5'$. This implies $q^3r^3s \neq 1$ and hence $3 \notin \mathbb{J}$. We conclude $\mathrm{mult}(3\alpha_1+2\alpha_2) = 2$ with $q_{3\alpha_1+2\alpha_2}^2 = (q^9r^6s^4)^2 = (q^4r^4s^4)^2(q^5r^2)^2 = r^4 = q^3 \in \mathbb{G}_5'$. Thus, $q_{3\alpha_1+2\alpha_2} \notin \mathbb{G}_2 \cup \mathbb{G}_3$ and Corollary 6.18 implies $\mathrm{GKdim}\,\mathcal{B}(V) = \infty$.

In case (c) $q^4r^2s = r^2s \in \mathbb{G}_4$ and $qrs = rs \in \mathbb{G}_4$ imply $r \in \mathbb{G}_4$. Note that $r^2s \neq -1$ since we assumed $r^2s = qr^2s \neq -1$. Moreover, $r \neq 1$ and $s \neq -1$.

If $r^2 s = 1$, then $1 = r^4 s^4 = (r^2 s)^2 s^2 = s^2$ and hence $s = 1$ and $r = -1$. Then $qrs = -1$ and $u_2 \neq 0$ by assumption on c_{12}. Thus, Corollary 6.4 is applicable and $\operatorname{GKdim} \mathcal{B}(V) = \infty$.

If $r^2 s \in \mathbb{G}_4'$, then $-1 = (r^2 s)^2 = (r^4) s^2 = s^2$,i.e. $s \in \mathbb{G}_4'$. Consequently, $r = s$ or $s \in \mathbb{G}_4'$ and $r = -1$. In any case $u_2 \neq 0$ since $c_{12} = -4$. If $r = s$, then $qrs = -1$ and, consequently, $\operatorname{GKdim} (\mathcal{B}(V) = \infty$ by Corollary 6.4. Otherwise $qrs \in \mathbb{G}_4'$ and $\det(F_{1,\bar{t}}) = -64 \neq 0$ by Example 5.33 for all $0 \leq \bar{t} \leq 3$ due to $p = 5$. Hence Corollary 6.4 implies $\operatorname{GKdim} \mathcal{B}(V) = \infty$.

Next, we discuss (**A.2**). This implies$qr^2 s = -1$ and, consequently, $1 = (q^4 r^2 s)^4 = (qr^2 s)^4 q^{12} = q^{12}$. We consider the cases

$$\text{(a) } q^4 r = 1 \text{ and (b) } q = 1,\, p = 5\,.$$

In case (a) $1 = (q^4 r)^3 = q^{12} r^3 = r^3$ and hence $1 = (q^4 r^2 s)^4 = (q^4 r)^4 (rs)^4 = rs^4$. Here $s \neq 1$ since $r \neq 1$. Due to $c_{21} \leq -2$ this implies $rs^i \neq 1$ for $i \leq 4$. Hence $\mathcal{B}(V)$ is of Cartan-type with $c_{12} = c_{21} = -4$. Thus, $\operatorname{GKdim} \mathcal{B}(V) = \infty$ by Proposition 6.17.

In case (b) $u_3 \neq 0$ since $c_{12} = -4$ and $q_{2\alpha_1+\alpha_2} = q^4 r^2 s = qr^2 s = -1$. Then Corollary 6.4 implies $\operatorname{GKdim} \mathcal{B}(V) = \infty$.

In the following we still assume $q^4 r^2 s \in \mathbb{G}_4$.

Assume case (**B.1**) holds, that is $c_{12} = -5$ and $qr^2 s \neq -1$, $2 \notin \mathbb{J}$. Thus, $\operatorname{mult}(5\alpha_1 + 2\alpha_2) \geq 2$ by Lemma 5.13. If $q_{5\alpha_1+2\alpha_2} = q^{25} r^{10} s^4 = q^9 r^2 (q^4 r^2 s)^4 = q^9 r^2 \notin \mathbb{G}_2 \cup \mathbb{G}_3$, then Corollary 6.18 implies $\operatorname{GKdim} \mathcal{B}(V) = \infty$. Therefore assume $q^9 r^2 \in \mathbb{G}_2 \cup \mathbb{G}_3$.

If $q^5 r = 1$, then $q = q^{-9} r^{-2} \in \mathbb{G}_2 \cup \mathbb{G}_3$, contradicting $c_{12} = -5$.

If $q \in \mathbb{G}_6'$, then since $qr^2 s \neq -1$ we assume $q^4 r^4 s^4 = 1$ for otherwise Corollary 6.16 was applicable. Then $1 = q^{16} r^8 s^4 = q^{12} r^4 (qrs)^4 = r^4$. If $qrs = -1$, then Corollary 6.4 was applicable and hence $\operatorname{GKdim} \mathcal{B}(V) = \infty$. If $qrs \neq -1$, then $\mathcal{R}^1(V) \in b(q, q^2 r^3, qrs)$ and the Nichols algebra $\mathcal{B}(\mathcal{R}^1(V))$ has $c_{21}^{\mathcal{B}(\mathcal{R}^1(V))} \leq -2$ and $c_{12}^{\mathcal{B}(\mathcal{R}^1(V))} = -5$. Thus, $\operatorname{mult}(4\alpha_1 + 2\alpha_2) \geq 1$ by Theorem 5.14. In this Nichols algebra the following holds:

$$q_{2\alpha_1+\alpha_2} = q^{16} (q^2 r^3)^8 (qrs)^4 = q^{32} = q^2 \in \mathbb{G}_3'.$$

Thus, $\operatorname{GKdim} (\mathcal{B}(V)) = \operatorname{GKdim} (\mathcal{B}(\mathcal{R}^1(V))) = \infty$ by Corollary 6.16.

Finally, we discuss (**B.2**), that is $c_{12} = -5$ and $qr^2 s = -1$. Then $1 = (q^4 r^2 s)^4 = (-q^3)^4 = q^{12}$. Since $c_{12} = -5$ we conclude $q \in \mathbb{G}_6' \cup \mathbb{G}_{12}'$. First, assume $q \in \mathbb{G}_{12}'$. Then

$q^5r = 1$ for otherwise $c_{12} \neq -5$. Thus, $r = q^7$ and $s = (qr^2s)q^{-1}r^{-2} = -q^{-15} = q^3$. Note that for these parameters q, r, s the preconditions of Lemma 5.19 are met. Thus, Corollary 6.14 yields some $W \in b(q^{11}, -1, q^3)$. The corresponding Nichols algebra $\mathcal{B}(W)$ has $c_{12}^{\mathcal{B}(W)} = -6$. Thus, $\mathrm{GKdim}\,(\mathcal{B}(V)) \geq \mathcal{B}(W) = \infty$ by Lemma 7.3.

Next, if $q \in \mathbb{G}'_6$, then $q^{10}r^5s = q^9(qr^2s)r^3 = r^3 \neq 1$ due to $c_{12} = -5$. Thus, $5 \notin \mathbb{J}$ and $\mathrm{mult}(5\alpha_1 + 2\alpha_2) = 2$. Now, $q^{25}r^{10}s^4 = q^9r^2 = -r^2$. If $-r^2 \notin \mathbb{G}'_2 \cup \mathbb{G}'_3$, then Corollary 6.18 implies $\mathrm{GKdim}\,(\mathcal{B}(V)) = \infty$. If $-r^2 \in \mathbb{G}_2$, then $r^4 = 1$. If $r = -1$, then $1 = q^3r$, contradicting $c_{12} = -5$. Thus, $r \in \mathbb{G}'_4$. Then $q = (qr^2s)r^{-2}s^{-1} = s^{-1}$. Analogously to a case above, $\mathcal{R}^1(V) \in b(q, q^2r^3, r)$ and $c_{12}^{\mathcal{B}(\mathcal{R}^1(V))} = -5$ and $c_{21}^{\mathcal{B}(\mathcal{R}^1(V))} = -3$. Thus, $\mathrm{mult}(4\alpha_1 + 2\alpha_2) \geq 1$ and $q^{16}(q^2r^3)^8r^4 = q^2 \in \mathbb{G}'_3$. Thus, $\mathrm{GKdim}\,(\mathcal{B}(V)) = \mathrm{GKdim}\,(\mathcal{B}(\mathcal{R}^1(V))) = \infty$ by Corollary 6.16.

Finally, if $-r^2 \in \mathbb{G}'_3 \setminus \{1\}$, then $r^6 = -1$ and $s^3 = -r^6s^3 = -1$. We conclude $r \in \mathbb{G}'_{12}$ since $r^2 \neq -1$ and $s \in \mathbb{G}'_6$ since $c_{21} \leq -2$. Now, $-1 = qr^2s$ yields $q = r^2 = s$ since $r^2, s \in \mathbb{G}'_6 = \{q, q^{-1}\}$. Then $(qrs)^2 = q^5 \in \mathbb{G}'_6$ and, consequently, $\gamma := qrs \in \mathbb{G}'_{12}$. Note that for these parameters q, r, s the preconditions of Lemma 5.19 are met. Thus, Corollary 6.14 yields some $W \in b(\gamma, -1, -\gamma^3)$. The corresponding Nichols algebra $\mathcal{B}(W)$ has $c_{12}^{\mathcal{B}(W)} = -6$. Thus, $\mathrm{GKdim}\,(\mathcal{B}(V) \geq \mathcal{B}(W) = \infty$ by Lemma 7.3. This completes the proof □

Lemma 7.5 If $-3 \leq c_{12} \leq c_{21} \leq -2$, then $\mathrm{GKdim}\,\mathcal{B}(V) < \infty$ if and only if $\#\triangle_+ < \infty$.

Proof. Note that $\mathrm{GKdim}\,\mathcal{B}(V) = \infty \Rightarrow \#\triangle_+ = \infty$ is well known. Moreover, $c_{21} \leq -2$ implies $0, 1 \notin \mathbb{J}$. We can assume

$$(qr^2s + 1)((qrs)^4 - 1) = 0$$

since otherwise we could apply Corollary 6.16.

The proof consists of the parts

(**A**) $c_{12} = c_{21} = -3$,

(**B**) $c_{12} = -3,\ c_{21} = -2$,

(**C**) $c_{12} = c_{21} = -2$,

We start discussing (**A**) $c_{12} = c_{21} = -3$. Then $q \neq 1$ for otherwise $p = 4$. Moreover, the case $q^3r = rs^3 = 1$ has already been treated by Proposition 6.17.

Thus, we consider the following cases:

(a) $q^3r = 1$, $s \in \mathbb{G}'_4$, $rs^3 - 1 \neq 0$ and (b) $q, s \in \mathbb{G}'_4$, $q^3r - 1 \neq 0$, $rs^3 - 1 \neq 0$.

Note that in both cases forth root of unity appear. Thus, assume $p \neq 2$. Additionally, if $r \in \mathbb{G}_4$, then $rs^i = 1$ for some $1 \leq i \leq 3$ since $r \neq 1$. This contradicts the assumptions in both cases.

First, we consider (a). If $qrs \in \mathbb{G}_4$, then $1 = q^4r^4s^4 = q^4r^4 = q^{-8}(q^3r)^4 = q^{-8}$. Moreover, $q \notin \mathbb{G}_4$ since otherwise $r \in \mathbb{G}_4$ and hence $q \in \mathbb{G}'_8$. Then $q^3r = 1$ implies $r = q^5$ and $s \in \{q^2, q^6\}$.

If $s = q^2$, then $q^3r^3s = q^{20} = -1 \neq 1$ since $p \neq 2$. Thus, $3 \notin \mathbb{J}$, $\mathrm{mult}(3\alpha_1 + 2\alpha_2) = 2$ by Theorem 5.14 and $q_{3\alpha_1+2\alpha_2} = q^9r^6s^4 = q^{39} = q^{-1} \in \mathbb{G}'_8$. Consequently, $\mathrm{GKdim}\,\mathcal{B}(V) = \infty$ by Corollary 6.18.

If $s = q^6$, then $qrs = q^{12} = -1$ and $u_2 \neq 0$ due to $c_{21} = -3$. Then $\mathrm{GKdim}\,\mathcal{B}(V) = \infty$ by Corollary 6.4.

If $qr^2s = -1$, then $1 = (qr^2s)^6 = q^6r^{12}s^6 = -(q^3r)^2r^{10} = -r^{10}$. Thus, $r \in \mathbb{G}'_{20}$ since $r \notin \mathbb{G}_4$. This implies $1 = (qr^2s)^{10} = -q^{10}$, i.e. $q \in \mathbb{G}'_4 \cup \mathbb{G}'_{20}$. Since $r = q^{-3}$ we conclude $q \in \mathbb{G}'_{20}$. Thus, $s = (qr^2s)(qr^2)^{-1} = -(q^{-5})^{-1} = q^{15}$. Assume $q' := qrs = q^{13} \in \mathbb{G}'_{20}$. Here by Corollary 6.14 there is $W \in b(q', q'^6, q'^9)$ satisfying $\mathrm{GKdim}\,(\mathcal{B}(V)) \geq \mathrm{GKdim}\,(\mathcal{B}(W)) = \infty$ by Lemma 7.3.

Next, consider (b). If $qrs \in \mathbb{G}_4$, then $r^4 = 1$, a contradiction to $r \notin \mathbb{G}_4$. If $qr^2s = -1$, then $1 = (qr^2s)^2 = q^2r^4s^2 = (-1)^2r^4 = r^4$, again a contradiction to $r \notin \mathbb{G}_4$. Thus, this case has already been treated.

Now, assume (**B**) $c_{12} = -3$, $c_{21} = -2$. If $p = 2$, then $q^3r = 1$ and $1 \in \{qrs, qr^2s\}$. In the first case $1 = (qrs)^3 = (q^3r)r^2s^3 = r^2s^3 = (rs)rs^2$. Since $c_{21} = -2$ either $s \in \mathbb{G}'_3$ or else $rs^2 = 1$ holds. By the preceding equation both yields a contradiction to $c_{21} = -2$. In the latter case, first assume $rs^2 = 1$. Then $1 = (qr^2s)^6 = (q^3r)^2r^7(rs^2)^3 = r^7$ and hence $r \in \mathbb{G}'_7$ since $r \neq 1$. Consequently, $q = q(q^3r)(rs^2)^2 = (qr^2s)^4r^{-5} = r^2$ and $s = s(q^3r)(rs^2) = (qr^2s)^3r^{-4} = r^3$. Assume $q' := qrs = r^6 \in \mathbb{G}'_7$. Here by Corollary 6.14 there is $W \in b(q', q'^6, q'^2)$ satisfying $\mathrm{GKdim}\,(\mathcal{B}(V)) \geq \mathrm{GKdim}\,(\mathcal{B}(W))$. This is of affine Cartan-type. Hence $\mathrm{GKdim}\,(\mathcal{B}(V)) = \infty$.

If $s \in \mathbb{G}'_3$, then $1 = (qr^2s)^3 = r^5$. Therefore, $r \in \mathbb{G}'_5$ again due to $r \neq 1$. This implies $q \in \mathbb{G}'_5 \cup \mathbb{G}'_{15}$ since $q^3 = r^{-1}$. In the former case $r = q^2$ and hence $1 = qr^2s = s$, a contradiction to $c_{21} = -2$ because $p = 2$. Thus, $q \in \mathbb{G}'_{15}$ and $r = q^{-3} = q^{12}$, $s = q^{30}s = q^5(qr^2s)^2 = q^5$. Then $\mathcal{B}(V)$ is a Nichols algebra of finite type 16.

Now, assume $p \neq 2$. As stated in the beginning of the proof we differentiate the cases (**B.1**) $qr^2s = -1$ and (**B.2**) $qr^2s \neq -1$, $qrs \in \mathbb{G}_4$. Each case will be further split into the following sub cases:

(a) $q^3r = rs^2 = 1$;

(b) $q^3r = 1$, $s \in \mathbb{G}'_3$;

(c) $q^3r = 1$, $s = 1$, $p = 3$;

(d) $q \in \mathbb{G}'_4$, $rs^2 = 1$;

(e) $q \in \mathbb{G}'_4$, $s \in \mathbb{G}'_3$;

(f) $q \in \mathbb{G}'_4$, $s = 1$, $p = 3$.

We start by discussing (**B.1**) $qr^2s = -1$.

In case (a) the equation $1 = (qr^2s)^6 = (q^3r)^2r^7(rs^2)^3 = r^7$ holds and hence $r \in \mathbb{G}'_7$ since $r \neq 1$. Consequently, $q = q(q^3r)(rs^2)^2 = (qr^2s)^4r^{-5} = r^2$ and $s = s(q^3r)(rs^2) = (qr^2s)^3r^{-4} = -r^3$. Assume $q' := qrs = -r^6 \in \mathbb{G}'_{14}$. Here by Corollary 6.14 there exists $W \in b(q', q'^6, q'^9)$ satisfying $\mathrm{GKdim}\,(\mathcal{B}(V)) \geq \mathrm{GKdim}\,(\mathcal{B}(W)) = \infty$ by Lemma 7.3.

If (b) holds, then $-1 = (qr^2s)^3 = (q^3r)s^3r^5 = r^5$. Thus, $r \in \{-1\} \cup \mathbb{G}'_{10}$. In the first case, $-1 = qr^2s = qs$. Thus, $q \in \mathbb{G}'_6$ and, consequently, $s = q^2$ since $s \in \mathbb{G}'_3$. If $p = 5$, then $\mathcal{B}(V)$ is of finite type 16" where $q = -\zeta$ with $\zeta \in \mathbb{G}'_3$. Otherwise, $\mathcal{R}_1(\mathcal{B}(V)) \in b(-1, q, q^2)$ and $\mathcal{R}_2\mathcal{R}_1(\mathcal{B}(V)) \in b(-1, q^5, 1)$. Since $p \notin \{2, 3, 5\}$ we have $c_{21} \leq -6$. Thus, $\mathrm{GKdim}\,(\mathcal{B}(V)) = \mathrm{GKdim}\,(\mathcal{B}(\mathcal{R}^2(\mathcal{R}^1(V)))) = \infty$ by Lemma 7.3.

If $r \in \mathbb{G}'_{10}$, then $q^3 = r^{-1} \in \mathbb{G}'_{10}$ and $q^5 = -(qr^2s)^5r^{-10}s^{-5} = -s \in \mathbb{G}'_6$. Thus, $q \in \mathbb{G}'_{30}$, $r = q^{27}$ and $s = q^{20}$. Then $\mathcal{B}(V)$ is of finite type 16 where $q = -\zeta$, $\zeta \in \mathbb{G}'_{15}$.

In case (c) the equation $-1 = (qr^2s)^3 = (q^3r)r^5 = r^5$ holds. Since $-1 = qr^2s = qr^2$ this implies $r \in \mathbb{G}'_{10}$ and, consequently, $q = r^3$ by the preceding. Then $\mathcal{B}(V)$ is of finite type 16' where $q = -\zeta^2$, $r = -\zeta^{-1}$ for some $\zeta \in \mathbb{G}'_5$.

For the following cases note that $r \neq -1$ since otherwise $q^2r = 1$, contradicting $c_{12} = -3$.

In case (d) the equation $1 = (qr^2s)^2 = -r^3$ holds and hence $r \in \mathbb{G}'_6$. Now, $s^2 = r^{-1} \in \mathbb{G}'_6$ implies $s \in \mathbb{G}'_{12}$ and $r = s^{10}$. Thus, $q = (qr^2s)r^{-2}s^{-1} = -s^{-21} = s^9$. Assume $q' := qrs = s^8 \in \mathbb{G}'_4$. Then by Corollary 6.14 there exists $W \in b(q', q'^2, q')$ satisfying $\mathrm{GKdim}\,(\mathcal{B}(V)) \geq \mathrm{GKdim}\,(\mathcal{B}(W)) = \infty$ since this is of affine Cartan-type.

In case (e) the equation $1=(qr^2s)^6=-r^{12}$ holds, that is $r\in\mathbb{G}_8'\cup\mathbb{G}_{24}'$. In the first case $1=(qr^2s)^4=s^4=s$, a contradiction. In the latter case the equations $1=(qr^2s)^4=r^8s$ and $-1=(qr^2s)^3=q^3r^6$ hold. Hence $s=r^{16}$ and $q=r^{18}$. Thus, $\mathcal{B}(V)$ is a Nichols algebra of finite type 13.

In case (f), $1=(qr^2s)^2=-r^4$ yields $r\in\mathbb{G}_8'$ and, consequently, $-1=qr^2s=qr^2$, that is $q=r^2$. Therefore, $\mathcal{B}(V)$ is a Nichols algebra of finite type 13'.

Now, assume (**B.2**) $qr^2s\neq -1$, $(qrs)^4=1$. We consider the same cases as above.

In case(a) the equation $1=(qrs)^4=(q^3r)(rs^2)qr=qr$ holds, a contradiction to $c_{12}=-3$.

If (b) holds, then $1=(qrs)^{12}=(q^3r)^4(s^3)^4r^8=r^8$. Thus, $r\in\mathbb{G}_N'$ with $N\in\{2,4,8\}$. In any case $q^3r^3s=r^2s\neq 1$. Therefore, $\{0,1,2,3\}\cap\mathbb{J}=\emptyset$ and, consequently, $\operatorname{mult}(3\alpha_1+2\alpha_2)=2$ with $q_{3\alpha_1+2\alpha_2}=q^9r^6s^4=r^3s\in\mathbb{G}_{3N}'$. Thus, $\operatorname{GKdim}\mathcal{B}(V)=\infty$ by Corollary 6.18.

In case (c) the equation $1=(qrs)^{12}=r^8$ holds. Again, $r\in\mathbb{G}_N'$ with $N\in\{2,4,8\}$. If $r=-1$, then $q^3=-1$ and, consequently, $q=-1$ due to $p=3$, a contradiction to $c_{12}=-3$. Otherwise, $q^3r^3s=r^2\neq 1$ and $q^9r^6s^4=r^3\in\mathbb{G}_N'$. Hence $\operatorname{GKdim}\mathcal{B}(V)=\infty$ by Corollary 6.18.

For the following cases note that $r\neq -1$ since otherwise $q^2r=1$, contradicting $c_{12}=-3$.

If (d) holds, then $1=q^4r^4s^4=r^2$, a contradiction to $r\notin\mathbb{G}_2$.

In case (e) the equation $1=(qrs)^{12}=r^{12}$ holds. Then $(q^3r^3s)^4=s\neq 1$. Hence the multiplicity of $3\alpha_1+2\alpha_2$ is two and $q_{3\alpha_1+2\alpha_2}^2=q^{18}r^{12}s^8=-s^2\in\mathbb{G}_6'$ and, consequently, $q_{3\alpha_1+2\alpha_2}\notin\mathbb{G}_2\cup\mathbb{G}_3$. Thus, $\operatorname{GKdim}\mathcal{B}(V)=\infty$ by Corollary 6.18.

In case (f), $1=(qrs)^4=r^4$ implies $r=q\in\mathbb{G}_4'$ due to $c_{12}=-3$. This case was treated above.

Finally, assume (**C**) $c_{12}=c_{21}=-2$. If $q^2r=rs^2=1$, then $\mathcal{B}(V)$ is of affine Cartan-type and hence $\operatorname{GKdim}(\mathcal{B}(V))=\infty$ by Proposition 6.5.

We consider the following cases

(a) $q^2r=1$, $s\in\mathbb{G}_3'$;

(b) $q^2r=1$, $s=1$, $p=3$;

(c) $q\in\mathbb{G}_3'$, $s\in\mathbb{G}_3'$;

(d) $q=s=1$, $p=3$.

First, assume $p = 2$. Assume (a) holds. If $qr^2s = 1$, then $1 = (qr^2s)^3 = qr^5 = (q^2r)^{-5}qr^5 = q^{-9}$. Hence $r = q^7$ and $s = (q^2r)^2s = q^3(qr^2s) = q^3$. Then $\mathcal{B}(V)$ is a Nichols algebra of finite type 10.

If $1 = qrs$, then $1 = (qrs)^3 = qr^2$ and $1 = (qrs)^6 = r^3$. Hence $q = r = s \in \mathbb{G}'_3$ due to $c_{21} = -2$. This case is of affine Cartan-type.

Now, in case (c) the equations $(qr^2s)^3 = r^6$ and $(qrs)^3 = r^3$ hold. So in any case $q = r = s \in \mathbb{G}'_3$ due to $c_{12} = c_{21} = -2$. The claim follows as above.

Now, assume $p \neq 2$. We discuss the cases (**C.1**) $qr^2s = -1$ and (**C.2**) $(qrs)^4 = 1$, $qr^2s \neq -1$. First, consider (**C.1**) $qr^2s = -1$.

In case (a), $1 = (qr^2s)^2 = r^3s^2$ implies $s = r^3$. Thus, $r \in \mathbb{G}'_9$ and, consequently, $q^2 = r^{-1}$, that is $q \in \mathbb{G}'_9 \cup \mathbb{G}'_{18}$. In the former case $r = q^7$ and $s = r^3 = q^3$. But then $-1 = qr^2s = q^{18} = 1$. A contradiction to $p \neq 2$.

In the latter case the equations $r = q^{-2} = q^{16}$ and $s = r^3 = q^{12}$ hold. Then $\mathcal{B}(V)$ is a Nichols algebra of finite type 10.

If (b) holds, then $1 = (qr^2s)^2 = r^3$. But $r \neq 1$ and $\mathbb{G}_3 = \{1\}$ due to $p = 3$, a contradiction.

Now, in case (c) the equation $-1 = (qr^2s)^3 = r^6$ holds. Hence $r \in \mathbb{G}'_4 \cup \mathbb{G}'_{12}$. In the first case $-1 = qr^2s = -qs$ holds and, consequently, $s = q^2$. Assume $\zeta \in \mathbb{G}'_{12}$ such that $r = \zeta^3$ and without loss of generality $q = \zeta^8$, $s = \zeta^4$. Then $\mathcal{B}(V)$ is a Nichols algebra of finite type 8.

In the latter case the equation $q, s \in \{r^4, r^8\}$ holds. Since $-1 = qr^2s$ we conclude $q = s = r^8$. Thus, $\mathcal{B}(V)$ is a Nichols algebra of finite type 9.

In case (d) the equation $-1 = qr^2s = r^2$ holds. Then $r \in \mathbb{G}'_4$ and $\mathcal{B}(V)$ is a Nichols algebra of finite type 9'.

Finally, assume (**C.2**) $(qrs)^4 = 1$, $qr^2s \neq -1$.

Assume (a) holds. Then $1 = (qrs)^4 = r^2s$, that is $r^2 = s^2 \in \mathbb{G}'_3$. If $r \in \mathbb{G}'_3$, then $r = s$ and $\mathcal{B}(V)$ is of affine Cartan type. Hence $\operatorname{GKdim} \mathcal{B}(V) = \infty$. Thus, assume $r \in \mathbb{G}'_6$. Therefore $q^2 = r^{-1}$ implies $q \in \mathbb{G}'_{12}$, $r = q^{-2} = q^{10}$, $s \in \{q^4, q^8\}$. In the first case we have $\mathcal{R}^2(V) \in b(q, r, s)$. We conclude that $\sigma_1\sigma_2\sigma_1\sigma_2$ is an automorphism of $\triangle^V$. We calculate

$$\begin{aligned}
\sigma_1\sigma_2\sigma_1\sigma_2(\alpha_1 + \alpha_2) &= \sigma_1\sigma_2\sigma_1(\alpha_1 + \alpha_2) \\
&= \sigma_1\sigma_2(\alpha_1 + \alpha_2) \\
&= \sigma_1(\alpha_1 + \alpha_2) \\
&= \alpha_1 + \alpha_2
\end{aligned}$$

and

$$\begin{aligned}\sigma_1\sigma_2\sigma_1\sigma_2(\alpha_1) &= \sigma_1\sigma_2\sigma_1(\alpha_1+2\alpha_2)\\ &= \sigma_1\sigma_2(3\alpha_1+2\alpha_2)\\ &= \sigma_1(3\alpha_1+4\alpha_2)\\ &= 5\alpha_1+4\alpha_2\\ &= \alpha_1+4(\alpha_1+\alpha_2)\,.\end{aligned}$$

Hence by linearity of the σ_i we conclude $k(4\alpha_1+4\alpha_2)+\alpha_1 \in \triangle^V_+$ for all $k \in \mathbb{N}_0$. Consequently, $\operatorname{GKdim}(\mathcal{B}(V)) = \infty$ by Corollary 6.2.

If $s = q^8$, then $\mathcal{R}^2(V) \in b(q^5, -1, q^8)$. Then $\mathcal{B}(\mathcal{R}^2(W))$ has $c_{12}^{\mathcal{B}(\mathcal{R}^2(W))} = -6$. Thus, $\operatorname{GKdim}(\mathcal{B}(V)) = \operatorname{GKdim}(\mathcal{B}(W)) = \infty$ by Lemma 7.3.

If (b) holds, then $1 = (qrs)^4 = r^2$. Thus, $r = -1$ and, consequently, $q \in \mathbb{G}'_4$. Then $\mathcal{R}^2(V) \in b(q, r, s)$. Then this case works analogously to case (a) with $s = q^4$.

In case (c) the equation $1 = (qrs)^4 = qr^4s$ holds. If $q = s^2$, then $r^4 = 1$ and since $-1 \neq qr^2s = r^2$ we conclude $r = -1$. Now, $u_2 \neq 0$ and $qrs = -1$. Then Corollary 6.4 implies $\operatorname{GKdim}\mathcal{B}(V) = \infty$. Otherwise, $q = s$ and, consequently, $r^4 = q$. If $r \in \mathbb{G}'_3$, then $r = q$ and $q^2r = rs^2 = 1$. This case is of affine Cartan-type. Thus, $r \in \mathbb{G}'_6 \cup \mathbb{G}'_{12}$. If $r \in \mathbb{G}'_6$, then $u_2 \neq 0$ and $qrs = r^9 = -1$. Hence Corollary 6.4 implies $\operatorname{GKdim}\mathcal{B}(V) = \infty$. If $r \in \mathbb{G}'_{12}$, then $\mathcal{R}^2(V) \in b(r^{10}, r^7, r^4)$. Then $\mathcal{B}(W)$ satisfies $\operatorname{GKdim}\mathcal{B}(W) = \operatorname{GKdim}\mathcal{B}(V)$ and $c_{12}^{\mathcal{B}(W)} = -5, c_{21}^{\mathcal{B}(W)} = -2$. This was treated before yielding $\operatorname{GKdim}\mathcal{B}(V) = \infty$.

Assume (d) holds. Then $1 = (qrs)^4 = r^4$ and $-1 \neq qr^2s = r^2$. Hence $r \in \mathbb{G}'_4$. Then $\mathcal{R}^2(V) \in b(q, r^{-1}, s)$. We conclude that $\sigma_1\sigma_2\sigma_1\sigma_2$ is an automorphism of $\triangle^V$. We calculate

$$\begin{aligned}\sigma_1\sigma_2\sigma_1\sigma_2(\alpha_1+\alpha_2) &= \sigma_1\sigma_2\sigma_1(\alpha_1+\alpha_2)\\ &= \sigma_1\sigma_2(\alpha_1+\alpha_2)\\ &= \sigma_1(\alpha_1+\alpha_2)\\ &= \alpha_1+\alpha_2\end{aligned}$$

and

$$\begin{aligned}\sigma_1\sigma_2\sigma_1\sigma_2(\alpha_1) &= \sigma_1\sigma_2\sigma_1(\alpha_1+2\alpha_2)\\ &= \sigma_1\sigma_2(3\alpha_1+2\alpha_2)\\ &= \sigma_1(3\alpha_1+4\alpha_2)\\ &= 5\alpha_1+4\alpha_2\\ &= \alpha_1+4(\alpha_1+\alpha_2)\,.\end{aligned}$$

Hence by linearity of the σ_i we conclude $k(4\alpha_1+4\alpha_2)+\alpha_1 \in \triangle_+^V$ for all $k \in \mathbb{N}_0$. Consequently, $\operatorname{GKdim}(\mathcal{B}(V)) = \infty$ by Corollary 6.2.

This completes the proof. □

In the following cases $c_{21} = -1$ and hence $\{0,1\} \cap \mathbb{J} \neq \emptyset$. Therefore, $2\alpha_1 + 2\alpha_2$ is not a root.

Lemma 7.6 If $c_{12} = -5$ and $c_{21} = -1$, then $\operatorname{GKdim}\mathcal{B}(V) < \infty$ if and only if $\#\triangle_+ < \infty$.

Proof. First, assume $p = 2$. Note that $\mathbb{G}_6' = \emptyset$. Hence $q^5r = 1$. Assume $s = 1$. Then $0 \in \mathbb{J}$, $1, 2 \notin \mathbb{J}$ by Lemma 5.13. If $3 \in \mathbb{J}$, then $1 = q^3r^3s = (q^5r)^{-3}q^3r^3 = q^{-12}$. Hence $q \in \mathbb{G}_3$ due to $p = 2$. This is a contradiction to $c_{12} = -5$.

If $4 \in \mathbb{J}$, then $1 = q^6r^4s = (q^5r)^{-4}q^6r^4 = q^{-14}$. Hence $q \in \mathbb{G}_7'$ due to $p = 2$ and, consequently $r = q^2$. Assume $\zeta \in \mathbb{G}_7'$ such that $q = \zeta^{-2}$, $r = q^2 = \zeta^3$. Then $\mathcal{B}(V)$ is a Nichols algebra of finite type 17.

Thus, assume $3, 4 \notin \mathbb{J}$. Then $\operatorname{mult}(4\alpha_1 + 2\alpha_2) \geq 1$. If $q^4r^2s \in \mathbb{G}_4 = \{1\}$, then $q^{-6} = (q^5r)^{-2}(q^4r^2s) = 1$ and, consequently, $q \in \mathbb{G}_3$ due to $p = 2$, a contradiction to $c_{12} = -5$. Thus, $q_{2\alpha_1+\alpha_2} \notin \mathbb{G}_4$. This implies $\operatorname{GKdim}(\mathcal{B}(V)) = \infty$ due to Corollary 6.16.

Now, assume $rs = 1$. Then $0, 2, 3 \notin \mathbb{J}$. If $4 \in \mathbb{J}$, then $1 = q^6r^4s = (q^5r)^{-3}q^6r^3 = q^{-9}$. We conclude $q \in \mathbb{G}_9'$ due to $c_{12} = -5$. Hence $r = q^4, s = q^5$. Assume $q' := q^4r^2s = q^8 \in \mathbb{G}_9'$. Then by Corollary 6.14 there exists $W \in b(q', q'^6, 1)$ with $\operatorname{GKdim}(\mathcal{B}(V) \geq \operatorname{GKdim}(\mathcal{B}(W))$. Now, $\mathcal{R}^2(W) \in b(q'^7, q'^3, 1)$. Thus, $c_{12}^{\mathcal{B}(\mathcal{R}^2(W))} = -6$ and, consequently, $\operatorname{GKdim}(\mathcal{B}(V) \geq \operatorname{GKdim}(\mathcal{B}(\mathcal{R}^2(W))) = \infty$ by Lemma 7.3.

If $4 \notin \mathbb{J}$, then $\operatorname{mult}(4\alpha_1 + 2\alpha_2) \geq 1$ and $q_{2\alpha_1+\alpha_2} = q^4r^2s = q^{-1} \notin \mathbb{G}_4$ since otherwise $c_{12} \neq -5$. Thus, $\operatorname{GKdim}\mathcal{B}(V) = \infty$ due to Corollary 6.16.

Now, assume $p \neq 2$. We consider the following cases

(a) $q^5r = rs = 1$;

(b) $q \in \mathbb{G}'_6$, $rs = 1$;

(c) $q^5 r = 1$, $s = -1$;

(d) $q \in \mathbb{G}'_6$, $s = -1$.

We assume $s \neq -1$ in cases (a) and (b). Thus, $1 \in \mathbb{J}$, i.e. $\{0, 2, 3\} \cap \mathbb{J} = \emptyset$.

Assume (a) holds. If $4 \notin \mathbb{J}$, $\mathrm{mult}(4\alpha_1 + 2\alpha_2) \geq 1$ and $q_{2\alpha_1+\alpha_2} = q^4 r^2 s = q^{-1} \notin \mathbb{G}_4$ due to $c_{12} = -5$. Hence $\mathrm{GKdim}\, \mathcal{B}(V) = \infty$ by Corollary 6.16.

Thus, assume $4 \in \mathbb{J}$, that is $-1 = q^6 r^4 s = (q^5 r)^{-3} q^6 r^3 = q^{-9}$. If $q \in \mathbb{G}'_6$, then $q = r$ and $s = q^5$. But $q^{4+1-1} r^2 = q^6 = 1$ and $p \neq 3$ contradicts $4 \in \mathbb{J}$. If $q \in \mathbb{G}'_{18}$, assume $q' := q^4 r^2 s = q^{-1} \in \mathbb{G}'_{18}$. Then by Corollary 6.14 there exists $W \in b(q', q'^6, -1)$ with $\mathrm{GKdim}\, (\mathcal{B}(V) \geq \mathrm{GKdim}\, (\mathcal{B}(W))$. This implies $\mathrm{GKdim}\, (\mathcal{B}(V)) = \infty$ by Lemma 7.3 due to $c_{12}^{\mathcal{B}(W)} = -12$.

Next, assume (b) holds. If $4 \notin \mathbb{J}$, then $\mathrm{mult}(4\alpha_1 + 2\alpha_2) \geq 1$ and $q_{2\alpha_1+\alpha_2} = q^4 r^2 s = q^4 r \in \mathbb{G}_4$. Thus, $r^{12} = 1$ and $q^4 r^4 = 1$ implies $q = -r^8$. Therefore, $r \in \mathbb{G}'_3 \cup \mathbb{G}'_6 \cup \mathbb{G}'_{12}$. In the first case $r \in \{q^2, q^4\}$, contradicting $c_{12} = -5$. If $r \in \mathbb{G}'_6$, then $q = r$ since $qr \neq 1$ due to $c_{12} = -5$. This case has already been considered. In the last case we have $\mathcal{R}^1(V) \in b(r^2, r^3, -1)$ and $\mathcal{R}^2(\mathcal{R}^1(V)) \in b(r^{11}, r^9, -1)$. Here $c_{12}^{\mathcal{B}(\mathcal{R}^2(\mathcal{R}^1(V))} = -9$. Hence $\mathrm{GKdim}\, (\mathcal{B}(V)) = \mathrm{GKdim}\, (\mathcal{B}(\mathcal{R}^2(\mathcal{R}^1(V))) = \infty$ by Lemma 7.3.

If $4 \in \mathbb{J}$, then $-1 = q^6 r^4 s = r^3$. Thus, $r = q$ due to $rs = 1$, $s \neq -1$ and $c_{12} = -5$. This implies $q^5 r = 1$ and was treated in case (a).

Now, assume $s = -1$. Then $0 \in \mathbb{J}$, $1, 2 \notin \mathbb{J}$.

First, we consider case (c). If $3 \in \mathbb{J}$, then $1 = q^3 r^3 s = (q^5 r)^{-3} q^3 r^3 s = -q^{-12}$. Hence $q \in \mathbb{G}'_8 \cup \mathbb{G}'_{24}$. If $q \in \mathbb{G}'_8$, then $q^2 r^2 = q^{-8} = 1$. Thus, by definition $3 \in \mathbb{J}$ implies $p = 3$. Then $\mathcal{B}(V)$ is of finite type 13'.

If $q \in \mathbb{G}'_{24}$, then $\mathcal{B}(V)$ is a Nichols algebra of finite type 13.

Next, $4 \in \mathbb{J}$ implies $1 = -q^6 r^4 s = (q^5 r)^{-3} q r^3 = q^{-14}$. Due to $4 \in \mathbb{J}$, $q^4 r^2 = q^{-7}$ and $p \nmid 4$ we conclude $q \in \mathbb{G}'_{14}$. Assume $\zeta \in \mathbb{G}'_7$ such that $q = -\zeta^{-2}$. Then $r = q^2 = \zeta^3$ and $\mathcal{B}(V)$ is a Nichols algebra of finite type 17.

If $3, 4 \notin \mathbb{J}$, then $\mathrm{mult}(4\alpha_1 + 2\alpha_2) \geq 1$. Hence we may assume $q_{2\alpha_1+\alpha_2} = q^4 r^2 s \in \mathbb{G}_4$ for otherwise $\mathrm{GKdim}\, \mathcal{B}(V) = \infty$ due to Corollary 6.16. This implies $1 = (q^4 r^2 s)^4 = q r^5$. Thus, $1 = q^{25} r^5 = q^{24}$. Now, $q^{10} r^5 s = -q^9$. Hence $5 \notin \mathbb{J}$ iff $q \notin \mathbb{G}'_6$ due to $c_{12} = -5$. In those cases $q_{5\alpha_1+2\alpha_2} = q^{25} r^{10} s^4 = q^{23} \notin \mathbb{G}_2 \cup \mathbb{G}_3$. This implies $\mathrm{GKdim}\, \mathcal{B}(V) = \infty$ by Corollary 6.18.

If $q \in \mathbb{G}'_6$, then $q = r$, $u_3 \neq 0$ and $q^4 r^2 s = -1$. Hence $\mathrm{GKdim}\, \mathcal{B}(V) = \infty$ by Corollary 6.4.

Finally, assume (d) holds. If $3 \in \mathbb{J}$, then $1 = q^3r^3s = r^3$. Thus, $r \in \mathbb{G}'_3 = \{q^2, q^4\}$. This contradicts $c_{12} = -5$.

If $4 \in \mathbb{J}$, this implies $1 = -q^6r^4s = r^4$. Since $q^3 = -1$ and $q^3r \neq 1$ this yields $r \in \mathbb{G}'_4$. Then $\mathcal{R}^2(V) \in b(-qr, r^{-1}, -1)$. Note that $-qr \in \mathbb{G}'_{12}$ and $(-qr)^3r^{-1} = -1$. Hence $c_{12}^{\mathcal{B}(\mathcal{R}^2(V))} \leq -6$ and $\operatorname{GKdim} \mathcal{B}' = \infty$ by Lemma 7.3.

If $3, 4 \notin \mathbb{J}$, then again assume $q^4r^2s = qr^2 \in \mathbb{G}_4$. If additionally $5 \notin \mathbb{J}$, then $\operatorname{mult}(5\alpha_1 + 2\alpha_2) \geq 2$ and $q_{5\alpha_+2\alpha_2} = q^{25}r^{10}s^4 = (q^{16}r^8s^4)q^9r^2 = -r^2$. Note that $-r^2 \notin \mathbb{G}_4$ due to $qr^2 \in \mathbb{G}_4$, $q \notin \mathbb{G}_4$. If $-r^2 \notin \mathbb{G}_3$, then $\operatorname{GKdim}(\mathcal{B}(V)) = \infty$ by Corollary 6.18. Otherwise $-r^2 \in \mathbb{G}'_3$ implies $-r^6 = 1$ and, consequently, $r \in \mathbb{G}'_{12}$ since $r \notin \mathbb{G}_4$. By the above results $1 = (qr^2)^4 = qr^2$. Then $\mathcal{R}^1(V) \in b(r^{10}, r^7, r^9)$. Note that $c_{12}^{\mathcal{B}(\mathcal{R}^1(V))} = -5$ and $c_{21}^{\mathcal{B}(\mathcal{R}^1(V))} \leq -2$. This has already been treated by the preceding lemmata.

If $5 \in \mathbb{J}$, then $1 = q^{10}r^5s = qr^5$ and $1 = q^4r^8 = -r^3$ by the above results. Due to $c_{12} = -5$ this implies $r = q$. Then $q^4r^2s = -1$ and $u_3 \neq 0$ and, consequently, $\operatorname{GKdim} \mathcal{B}(V) = \infty$ by Corollary 6.4.

This completes the proof. □

Lemma 7.7 If $c_{12} = -4$ and $c_{21} = -1$, then $\operatorname{GKdim} \mathcal{B}(V) < \infty$ if and only if $\#\triangle_+ < \infty$.

Proof. First, note that the case $q^4r = rs = 1$ is of affine Cartan-type. Thus, $\operatorname{GKdim}(\mathcal{B}(V)) = \infty$ due to Proposition 6.5. It remains to consider the following cases:

(a) $q^4r = 1$, $s = -1$;

(b) $q \in \mathbb{G}'_5$, $s = -1$.

(c) $q = 1$, $s = -1$, $p = 5$.

(d) $q \in \mathbb{G}'_5$, $rs = 1$;

(e) $q = 1$, $rs = 1$, $p = 5$;

Assume $p = 2$. Assume (a) holds. If $3 \in \mathbb{J}$, then $1 = q^3r^3s = (q^4r)^{-3}q^3r^3 = q^{-9}$. This implies $q \in \mathbb{G}'_9$ due to $c_{12} = -4$ and, consequently, $r = q^5$. Here $\mathcal{B}(V)$ is a Nichols algebra of finite type 10.

If $4 \in \mathbb{J}$, then the equation $1 = q^6r^4s = (q^4r)^{-4}q^6r^4 = q^{-10}$ implies that $q \in \mathbb{G}'_5$ due to $p = 2$, $c_{12} = -4$. Then $r = q$ and $\mathcal{B}(V)$ is a Nichols algebra of finite type 14.

If $3,4 \notin \mathbb{J}$, then $\mathrm{mult}(4\alpha_1 + 2\alpha_2) \geq 1$ and $q_{2\alpha_1+\alpha_2} = q^4r^2s = r \notin \mathbb{G}_4$ due to $\mathbb{G}_4 = \{1\}$. Hence $\mathrm{GKdim}\,(\mathcal{B}(V)) = \infty$ by Corollary 6.16.

In case (b), if $3 \in \mathbb{J}$, then $1 = q^3r^3s = q^{-2}r^3$. Thus, $q^2 = r^3$ and $r^{15} = 1$, that is $r \in \mathbb{G}'_5 \cup \mathbb{G}'_{15}$. In the former case $r = q^{-1}$ due to $q^2 = r^3$, but then $qr = 1$ contradicts $c_{12} = -4$. In the latter case assume $\zeta \in \mathbb{G}'_{15}$ such that $r = \zeta^2$, $q = \zeta^3$. Then $\mathcal{B}(V)$ is a Nichols algebra of finite type 16.

If $4 \in \mathbb{J}$, then $1 = q^6r^4s = qr^4$ and, consequently, $r \in \mathbb{G}_{20} = \mathbb{G}_5$ due to $p = 2$. Hence $r = q \in \mathbb{G}'_5$. Then $q^4r = 1$ which was treated in case (a).

If $3,4 \notin \mathbb{J}$, then $\mathrm{mult}(4\alpha_1 + 2\alpha_2) \geq 1$ and $q_{2\alpha_1+\alpha_2} = q^4r^2s = q^{-1}r^2$. Suppose $q^{-1}r^2 \in \mathbb{G}_4 = \{1\}$. This implies $r = q^3 \in \mathbb{G}'_5$ due to $p = 2$ and, consequently, $q^2r = 1$, a contradiction to $c_{12} = -4$. Therefore, $\mathrm{GKdim}\,(\mathcal{B}(V)) = \infty$ by Corollary 6.16.

Now, assume (d) holds and $s \neq 1$. If $3 \in \mathbb{J}$, then $1 = q^3r^3s = q^{-2}r^2$. Hence $r \in \mathbb{G}'_5$ due to $p = 2$. This implies $q^4r = 1$ which was treated above. If $4 \in \mathbb{J}$, then $1 = q^6r^4s = qr^3$. That is $r \in \mathbb{G}'_5 \cup \mathbb{G}'_{15}$. In the first case $r = q^3$, contradicting $c_{12} = -4$. Otherwise, assume $\zeta \in \mathbb{G}'_{15}$ such that $r = \zeta^4$, $s = \zeta^{-4}$, $q = r^{-3} = \zeta^3$. Then $\mathcal{B}(V)$ is a Nichols algebra of finite type 16.

If $3,4 \notin \mathbb{J}$, then $\mathrm{mult}(4\alpha_1 + 2\alpha_2) \geq 1$ and $q_{2\alpha_1+\alpha_2} = q^4r^2s = q^{-1}r$. In case $q = r$ holds, $\mathcal{B}(V)$ is of affine-Cartan type. This was treated before. Otherwise, Corollary 6.16 implies $\mathrm{GKdim}\,(\mathcal{B}(V)) = \infty$.

Next, assume $p \neq 2$. In the cases (a), (b) and (c) we have $0 \in \mathbb{J}$ and consequently $1,2 \notin \mathbb{J}$.

Assume (a) holds. If $3 \in \mathbb{J}$, then $1 = q^3r^3s = -q^{-1}r^2$. This implies $1 = q^4r = r^9$, that is $r \in \mathbb{G}'_3 \cup \mathbb{G}'_9$. In the former case $\mathcal{R}^2(V) \in b(1, r^{-1}, -1)$. Note that $p \geq 5$ due to $r \in \mathbb{G}'_3$. Thus, $u_4 \neq 0$ and $1^9(r^{-1})^3(-1) = -1$. Hence $\mathrm{GKdim}\,(\mathcal{B}(V)) = \mathrm{GKdim}\,(\mathcal{B}(\mathcal{R}^2(V))) = \infty$ by Corollary 6.4. In the latter case $\mathcal{B}(V)$ is a Nichols algebra of finite type 10.

If $4 \in \mathbb{J}$, then $1 = -q^6r^4s = q^{-2}r^2 = q^{-10}$. We also conclude $1 = q^4r = r^5$. Thus, $r \in \mathbb{G}'_5$ and $q \in \mathbb{G}'_5 \cup \mathbb{G}'_{10}$. In the former case $q^2 = r^2$ implies $q = r$. This was treated before. In the latter case assume $\zeta \in \mathbb{G}'_{10}$ such that $q = -\zeta^{-2}$ and, consequently, $r = \zeta^{-2}$. Then $\mathcal{B}(V)$ is of finite type 14.

If $3,4 \notin \mathbb{J}$, then $\mathrm{mult}(4\alpha_1 + 2\alpha_2) \geq 1$ and $q_{2\alpha_1+\alpha_2} = q^4r^2s = -r$. Thus, $r \in \mathbb{G}_4 \setminus \{1\}$. If $r = -1$, then we are in affine Cartan-type. Hence assume $r \in \mathbb{G}'_4$. Then $q \in \mathbb{G}'_{16}$. Here $\mathcal{R}^2(V) \in b(q^5, q^4, -1)$. We conclude that $\sigma_1\sigma_2\sigma_1\sigma_2$ induces an automorphism of $\triangle^V$. We calculate

$$\begin{aligned}
\sigma_1\sigma_2\sigma_1\sigma_2(2\alpha_1+\alpha_2) &= \sigma_1\sigma_2\sigma_1(2\alpha_1+\alpha_2)\\
&= \sigma_1\sigma_2(2\alpha_1+\alpha_2)\\
&= \sigma_1(2\alpha_1+\alpha_2)\\
&= 2\alpha_1+\alpha_2
\end{aligned}$$

and

$$\begin{aligned}
\sigma_1\sigma_2\sigma_1\sigma_2(\alpha_1) &= \sigma_1\sigma_2\sigma_1(\alpha_1+\alpha_2)\\
&= \sigma_1\sigma_2(3\alpha_1+\alpha_2)\\
&= \sigma_1(3\alpha_1+2\alpha_2)\\
&= 5\alpha_1+2\alpha_2\\
&= \alpha_1+2(2\alpha_1+\alpha_2)\,.
\end{aligned}$$

Hence by linearity of the σ_i we conclude $k(4\alpha_1+2\alpha_2)+\alpha_1\in\triangle^V_+$ for all $k\in\mathbb{N}_0$. Consequently, $\mathrm{GKdim}\,(\mathcal{B}(V))=\infty$ by Corollary 6.2.

In case (b) if $3\in\mathbb{J}$, then $1=-q^3r^3$ implies $r^3=-q^2\in\mathbb{G}'_{10}$. Thus, $r\in\mathbb{G}'_{10}\cup\mathbb{G}'_{30}$. If $r\in\mathbb{G}'_{10}$, then $1=(q^3r^3s)^7=-qr$. Then $r=-q^4$ yielding $q^2r^2=1$. Thus, $p=3$ by definition of $3\in\mathbb{J}$. Then $\mathcal{B}(V)$ is of finite type 16'. In the last case assume $\zeta\in\mathbb{G}'_{15}$ such that $r=-\zeta^2$ and, consequently, $q=\zeta^3$. Then $\mathcal{B}(V)$ is of finite type 16.

If $4\in\mathbb{J}$, then $1=-q^6r^4s=qr^4$. Therefore, $r\in\mathbb{G}'_5\cup\mathbb{G}'_{10}\cup\mathbb{G}_{20}$. If $r\in\mathbb{G}_{10}$, then $q^3r^2=r^{-12}r^2=1$. Thus, $4\in\mathbb{J}$ implies $p=2$ due to definition of $\mathbb{J}$. This is a contradiction to $p\neq 2$. In the remaining case $q=r^{16}$. Assume $\zeta\in\mathbb{G}'_{20}$ such that $r=\zeta^3$. Then $q=\zeta^{-2}$. Here $\mathcal{B}(V)$ is of finite type 15.

If $3,4\notin\mathbb{J}$, then $\mathrm{mult}(4\alpha_1+2\alpha_2)\geq 1$ and $q_{2\alpha_1+\alpha_2}=q^4r^2s=-q^{-1}r^2\in\mathbb{G}_4$ for otherwise $\mathrm{GKdim}\,(\mathcal{B}(V))=\infty$. Now $\mathcal{R}^1(V)\in b(q,q^2r^{-1},-qr^4)$. If $-qr^4=-1$ holds, then $1=(-q^{-1}r^2)^4=qr^8$ implies $q=1$, contradicting $q\in\mathbb{G}'_5$. Furthermore, assume $(q^2r^{-1})(-qr^4)=-q^3r^3=1$. Thus, $q^3r^3s=1$. Now, if additionally $q^2r^2=1$, then $r^{14}=1$ due to $qr^8=1$ and $r^{10}=1$ due to $q\in\mathbb{G}'_5$. This contradicts $r\notin\mathbb{G}_2$. Thus, $q^2r^2\neq 1$ and, consequently, $3\in\mathbb{J}$, a contradiction. Thus, $c_{21}^{\mathcal{B}(\mathcal{R}^1(V))}\leq -2$. Thus, $\mathrm{GKdim}\,(\mathcal{B}(V))=\mathrm{GKdim}\,(\mathcal{B}(\mathcal{R}^1(V)))=\infty$ by Lemma 7.4.

Assume (c) holds. If $3\in\mathbb{J}$, then $1=q^3r^3s=-r^3$. Thus, $r\in\mathbb{G}'_2\cup\mathbb{G}'_6$. In the former case $u_3\neq 0$ and $q^4r^2s=-1$. Hence $\mathrm{GKdim}\,\mathcal{B}(V)=\infty$ by Corollary 6.4. In the latter case assume $\zeta \in \mathbb{G}'_3$ such that $r = -\zeta^{-1}$. Here $\mathcal{B}(V)$ is of finite type 16".

If $4 \in \mathbb{J}$, then $1 = -q^6r^4s = r^4$. If $r = -1$, then $u_3 \neq 0$ and $q^4r^2s = -1$ imply $\mathrm{GKdim}\,\mathcal{B}(V) = \infty$ via Corollary 6.4. Thus, assume $r \in \mathbb{G}'_4$. We conclude that $\mathcal{B}(V)$ is of finite type 15.

If $3, 4 \notin \mathbb{J}$, then $\mathrm{mult}(4\alpha_1 + 2\alpha_2) \geq 1$ and $q_{2\alpha_1+\alpha_2} = q^4r^2s = -r^2$. Like before we can assume $-r^2 \in \mathbb{G}_4$, that is $r^8 = 1$. If $r = -1$, then $u_3 \neq 0$ and $q^4r^2s = -1$. Hence $\mathrm{GKdim}\,\mathcal{B}(V) = \infty$ by Corollary 6.4. If $r \in \mathbb{G}'_4$, then $q^6r^4s = -1$ and $q^3r^2 = -1 \neq 1$. This contradicts $4 \notin \mathbb{J}$. In the last case we have $\mathcal{R}^1(V) \in b(1, r^{-1}, -1)$. We conclude that $\sigma_1\sigma_2\sigma_1\sigma_2$ induces an automorphism of $\triangle^V$. We calculate

$$\begin{aligned}
\sigma_1\sigma_2\sigma_1\sigma_2(2\alpha_1+\alpha_2) &= \sigma_1\sigma_2\sigma_1(2\alpha_1+\alpha_2)\\
&= \sigma_1\sigma_2(2\alpha_1+\alpha_2)\\
&= \sigma_1(2\alpha_1+\alpha_2)\\
&= 2\alpha_1+\alpha_2
\end{aligned}$$

and

$$\begin{aligned}
\sigma_1\sigma_2\sigma_1\sigma_2(\alpha_1) &= \sigma_1\sigma_2\sigma_1(\alpha_1+\alpha_2)\\
&= \sigma_1\sigma_2(3\alpha_1+\alpha_2)\\
&= \sigma_1(3\alpha_1+2\alpha_2)\\
&= 5\alpha_1+2\alpha_2\\
&= \alpha_1+2(2\alpha_1+\alpha_2)\,.
\end{aligned}$$

Hence by linearity of the σ_i we conclude $k(4\alpha_1 + 2\alpha_2) + \alpha_1 \in \triangle^V_+$ for all $k \in \mathbb{N}_0$. Consequently, $\mathrm{GKdim}\,(\mathcal{B}(V)) = \infty$ by Corollary 6.2.

Next, assume $s \neq -1$, $rs = 1$. Thus, $\{0, 1, 2, 3\} \cap \mathbb{J} = \{1\}$. Note that $r \neq -1$ due to $s \neq -1$.

In case (d) if $4 \in \mathbb{J}$, then $1 = -q^6r^4s = -qr^3$. Thus, $r^{15} = -1$, that is $r \in \mathbb{G}'_6 \cup \mathbb{G}'_{10} \cup \mathbb{G}'_{30}$. In the first case the equation $1 = -qr^3 = q$ holds, a contradiction to $q \in \mathbb{G}'_5$. If $r \in \mathbb{G}'_{10}$, then $1 = (-qr^3)^3 = -q^3r^{-1}$. Thus, $r = -q^3$ and, consequently, $s = -q^{-3}$. Now, $q^4r^2 = q^{10} = 1$. Hence $p = 3$ by definition of $\mathbb{J}$. Then $\mathcal{B}(V)$ is of finite type 16'. In the last case assume $\zeta \in \mathbb{G}'_{15}$ such that $r = -\zeta^4$. Then $1 = -qr^3 = q\zeta^{12}$ implies $q = \zeta^3$. Here $\mathcal{B}(V)$ is of finite type 16.

If $4 \notin \mathbb{J}$, then $\mathrm{mult}(4\alpha_1 + 2\alpha_2) \geq 1$ and $q_{2\alpha_1+\alpha_2} = q^4r^2s = q^{-1}r$. Like before we can assume $q^{-1}r \in \mathbb{G}_4$, that is $1 = q^{-4}r^4 = qr^4$. Then $r \in \mathbb{G}'_5 \cup \mathbb{G}'_{10} \cup \mathbb{G}'_{20}$. In the first case $q = r$ and, consequently, $q^4 r = 1$. Thus, $\mathcal{B}(V)$ is of affine Cartan-type. Otherwise, $\mathcal{R}^1(V) \in b(q, q^2 r^{-1}, qr^3)$. Note that $qr^3 = qr^4 r^{-1} = r^{-1} \neq -1$. If $q^2 = r^2$, then $q^6 r^4 s = -1$ and $q^4 r^2 = q \neq 1$ imply $4 \in \mathbb{J}$, a contradiction. Otherwise, $c_{21}^{\mathcal{B}(\mathcal{R}^1(V))} \leq -2$. This was treated in Lemma 7.4.

Finally, in case (e) if $4 \in \mathbb{J}$, then $1 = -q^6 r^4 s = -r^3$. Assume $\zeta \in \mathbb{G}_3'$ such that $r = -\zeta$. Here $\mathcal{B}(V)$ is of finite type 16".

If $4 \notin \mathbb{J}$, then $\mathrm{mult}(4\alpha_1 + 2\alpha_2) \geq 1$ and $q_{2\alpha_1+\alpha_2} = q^4 r^2 s = r$. Thus, we can assume $r \in \mathbb{G}_4'$. Then $u_4 \neq 0$ and $q^9 r^3 s = -1$. Then Corollary 6.4 implies $\mathrm{GKdim}\, \mathcal{B}(V) = \infty$. □

The last two proofs rely on references to the above cases and the classification of finite root systems.

Lemma 7.8 If $c_{12} = -3$ and $c_{21} = -1$, then $\mathrm{GKdim}\, \mathcal{B}(V) < \infty$ if and only if $\#\triangle_+ < \infty$.

Proof. We consider the following cases:

(a) $q^3 r = rs = 1$;

(b) $q^3 r = 1$, $s = -1$;

(c) $q \in \mathbb{G}_4'$, $rs = 1$;

(d) $q \in \mathbb{G}_4'$, $s = -1$.

First, note that case (a) is of finite type 11.

First, assume $p = 2$. The only case to consider is (b). Here $\mathcal{R}^2(V) \in b(q', r', s')$ where $(q', r', s') := (q^{-2}, q^3, 1)$. Now, $1 \notin \{r', q'r', q'^2 r', q'^3 r', q', q'^2, q'^3, q'^4\}$ due to $c_{12} = -3$ and $p = 2$. Thus, $c_{12}^{\mathcal{B}(\mathcal{R}^2(V))} \leq -4$. Thus, the claim follows using the above lemmata.

Now we assume that $p \neq 2$. Assume (b) holds. Note that $q \neq 1$ due to $r \neq 1$. Now, $\mathcal{R}^2(V) \in b(q', r', s')$ with $(q', r', s') := (-q^{-2}, q^3, -1)$. Here $1 \notin \{r', q'r', q'^2 r'\}$. If $1 = q'^3 r' = -q^3$, then $q \in \mathbb{G}_6'$ since $c_{12} = -3$ and, consequently, $q^3 r = -r = rs = 1$. This has already been discussed. Moreover, $1 \in \{q', q'^2, q'^3, q'^4\}$ implies $q \in \mathbb{G}_4' \cup \mathbb{G}_8' \cup \mathbb{G}_{12}'$. In the first case $q' = 1$. If $p > 3$, then $c_{12}^{\mathcal{B}(\mathcal{R}^2(V))} < -3$. Those cases were treated in above lemmata. If $p = 3$, then $\mathcal{B}(V)$ is of finite type $9'$. If $q \in \mathbb{G}_8'$, then $q' = q^2$ and $r' = -q^{-1}$. Then $\mathcal{B}(\mathcal{R}^2(V))$ and $\mathcal{B}(V)$ are of finite type 12. If $q \in \mathbb{G}_{12}$, assume $\zeta \in \mathbb{G}_{12}'$ such that $q = -\zeta^{-1}$ and, consequently, $r = q^{-3} = -\zeta^3$. Here $\mathcal{B}(V)$ is of finite type 9.

Assume (c) holds. Note that $s \neq -1$ since otherwise $r = -1$ and $q^2 r = 1$, a contradiction to $c_{12} = -3$. Now, $\mathcal{R}^1(V) \in b(q', r', s')$ with $(q', r', s') := (q, -s, qs^{-2})$. Here $r' \neq 1$. If $1 = r's' = -qs^{-1}$, then $q^3 r = -qr = rs = 1$. This is case (a). Furthermore, if $1 = -s' = q^3 s^{-2}$, then $s \in \mathbb{G}'_8$. Assume $\zeta \in \mathbb{G}'_8$ such that $s = \zeta^{-1}$, $r = \zeta$ and $q = s^{-2} = \zeta^2$. Here $\mathcal{B}(V)$ is of finite type 12. Otherwise, $c_{21}^{\mathcal{B}(\mathcal{R}^1(V))} < -1$ due to $p \neq 2$. Those cases have been discussed in Lemma 7.5.

Finally assume (d) holds. We can assume $1 \notin \{rs, q^3 r\}$ for those cases were discussed above. Moreover, $qr \neq 1$ due to $c_{12} = -3$. Hence $r \notin \mathbb{G}_4$. Then $\mathcal{R}^2(V) \in b(q', r', s')$ where $(q', r', s') = (q^{-1} r, r^{-1}, -1)$. Here $1 \notin \{r', q'r', q'^2 r', q', q'^2, q'^4\}$ due to the assumptions on r. If $1 = q'^3 r' = qr^2$, then $r \in \mathbb{G}'_8$ due to assumptions on r. Assume $\zeta \in \mathbb{G}'_8$ such that $r = -\zeta^{-1}$. Consequently, $q = r^6 = \zeta^2$. Here $\mathcal{B}(V)$ is of finite type 12. If $1 = q'^3 = qr^3$, then $r \in \mathbb{G}'_{12}$ due to assumptions on r. Assume $\zeta \in \mathbb{G}'_{12}$ such that $r = \zeta^{-1}$ and ,consequently, $q = r^{-3} = \zeta^3$. Here $\mathcal{B}(V)$ is of finite type 8. Otherwise, $c_{12}^{\mathcal{B}(\mathcal{R}^2(V))} < -3$. Those cases have been discussed in above lemmata. This completes the proof. □

Lemma 7.9 If $c_{12} \geq -2$ and $c_{21} = -1$, then $\operatorname{GKdim} \mathcal{B}(V) < \infty$ if and only if $\#\triangle_+ < \infty$.

Proof. First, assume $c_{12} = -2$. We consider the following cases:

(a) $q^2 r = rs = 1$;

(b) $q^2 r = 1$, $s = -1$, $r \neq -1$;

(c) $q \in \mathbb{G}'_3$, $rs = 1$;

(d) $q \in \mathbb{G}'_3$, $s = -1$.

(e) $q = 1$, $rs = 1$, $p = 3$.

(f) $q = 1$, $s = -1$, $p = 3$.

Note that in case (a) $\mathcal{B}(V)$ is of finite type 4 and in case (b) it is of finite type 5.

Assume $p = 2$ and (c) holds. If $qs \neq 1$, then $\mathcal{B}(V)$ is of finite type 6. Otherwise, $r = s^{-1} = q$ and, consequently, $q^2 r = 1$. This was considered in (a).

Next, assume (d) holds. Note that $r \notin \mathbb{G}'_3 = \{q, q^2\}$ for otherwise $qr = 1$ or this has been considered in case (b). Here $\mathcal{R}^2(V) \in b(q', r', s')$ with $(q', r', s') = (qr, r^{-1}, 1)$. Then $1 \notin \{r', q'r', q'^2 r', q', q'^2, q'^3\}$ due to the assumption on r. Thus, $c_{12}^{\mathcal{B}(\mathcal{R}^2(V))} < -2$. This has been discussed in above lemmata.

Now, assume $p \neq 2$. Assume (c) holds. If $qs \neq -1$, then $\mathcal{B}(V)$ is of finite type 6. Otherwise, $s = -q^{-1}$ and $r = -q$. Here $\mathcal{B}(V)$ is of finite type 6".

Next, suppose case (d). Again, $r \notin \mathbb{G}_2 \cup \mathbb{G}_3$ and $\mathcal{R}^2(V) \in b(q', r', s')$ with $(q', r', s') = (-qr, r^{-1}, -1)$. Here $1 \notin \{r', q'r', q'^2r'\}$ due to the assumption on r. If $1 = q' = -qr$, then $c_{12}^{\mathcal{B}(\mathcal{R}^2(V))} < -2$ due to $p \neq 3$. Hence this has been discussed in above lemmata. If $-1 = q' = -qr$, then $r \in \mathbb{G}'_3$ which has been excluded. Assume $1 = q'^3 = (-qr)^3 = -r^3$. By assumption on r this implies $r \in \mathbb{G}'_6$. If $q = r^2$, then $-qr = 1$ which we discussed above. Hence $r = -q$. Here $\mathcal{B}(V)$ is of finite type 7. Otherwise, $c_{12}^{\mathcal{B}(\mathcal{R}^2(V))} < -2$. This has been treated in above lemmata.

In case (e), if $s = -1$, then $\mathcal{B}(V)$ is of finite type 6"'. Otherwise it is of finite type 6'.

Now, assume (f) holds. Again, $r \notin \mathbb{G}_2$. Then $\mathcal{R}^2(V) \in b(q', r', s')$ with $(q', r', s') = (-r, r^{-1}, -1)$. Here $1 \notin \{r', q'r', q'^2r', q', q'^2, q'^3\}$ due to the assumption on r and $p = 3$. Thus, $c_{12}^{\mathcal{B}(\mathcal{R}^2(V))} < -2$. This has been discussed in above lemmata.

Finally if $c_{12} = c_{21} = -1$, then $\mathcal{B}(V)$ is of finite type 2 or 3.

This completes the proof. $\square$

This completes the proof of Theorem 7.1.

A | Finite Root Systems

The following table was published in [11, 18] and lists preconditions for q, r, s and p such that the corresponding root system is finite.

	$V \in b(q,r,s)$	parameters	char(K)
1	$(q,1,s)$	$q,s \in \mathbb{K}^\times$	
2	(q,q^{-1},q)	$q \in \mathbb{K}^\times \setminus \{1\}$	
3	$(q,q^{-1},-1),(-1,r,-1)$	$q,r \in \mathbb{K}^\times \setminus \mathbb{G}_2$	
4	(q,q^{-2},q^2)	$q \in \mathbb{K}^\times \setminus \mathbb{G}_2$	
5	$(q,q^{-2},-1),(-q^{-1},q^2,-1)$	$q \in \mathbb{K}^\times \setminus \mathbb{G}_4$	
6	$(\zeta,s^{-1},s),(\zeta,\zeta^{-1}s^{-1},\zeta s)$	$\zeta \in \mathbb{G}'_3, \zeta s \neq -1$ $s \in \mathbb{K}^\times \setminus \mathbb{G}_3$	$p \neq 3$
6′	$(1,s^{-1},s),(1,s^{-1},s)$	$s \in \mathbb{K}^\times \setminus \mathbb{G}_2$	$p = 3$
6″	$(\zeta,-\zeta,-\zeta^{-1})$	$\zeta \in \mathbb{G}'_3$	$p \neq 2,3$
6‴	$(1,-1,-1)$		$p = 3$
7	$(\zeta,-\zeta,-1),(\zeta^{-1},-\zeta^{-1},-1)$	$\zeta \in \mathbb{G}_3$	$p \neq 3$
8	$(-\zeta^2,\zeta,-1),(-\zeta^2,\zeta^3,-\zeta^{-2}),(-1,-\zeta^{-1},-\zeta^{-2}),$ $(-1,-\zeta,\zeta^3),(-1,\zeta^{-1},\zeta^3)$	$\zeta \in \mathbb{G}'_{12}$	$p \neq 2,3$
9	$(-\zeta^{-1},-\zeta^3,-1),(-\zeta^2,\zeta^3,-1),(-\zeta^2,\zeta,-\zeta^2)$	$\zeta \in \mathbb{G}'_{12}$	$p \neq 2,3$
9′	$(\zeta,\zeta,-1),(1,-\zeta,-1),(1,\zeta,1)$	$\zeta \in \mathbb{G}'_4$	$p = 3$
10	$(-\zeta^2,\zeta,-1),(\zeta^3,\zeta^{-1},-1),(\zeta^3,\zeta^{-2},-\zeta)$	$\zeta \in \mathbb{G}'_9$	$p \neq 3$
11	(q,q^{-3},q^3)	$q \in \mathbb{K}^\times \setminus (\mathbb{G}_2 \cup \mathbb{G}_3)$	
12	$(\zeta,-\zeta,-1),(\zeta^2,-\zeta^{-1},-1),(\zeta^2,\zeta,\zeta^{-1})$	$\zeta \in \mathbb{G}'_8$	$p \neq 2$
13	$(\zeta,\zeta^{-5},-1),(-\zeta^{-4},\zeta^5,-1),$ $(-\zeta^{-4},-\zeta^{-1},\zeta^6),(\zeta^{-1},\zeta,\zeta^6)$	$\zeta \in \mathbb{G}'_{24}$	$p \neq 2,3$
13′	$(-\zeta,\zeta^{-1},-1),(1,\zeta,-1),$ $(1,\zeta^{-1},-\zeta^2),(-\zeta^{-1},-\zeta,-\zeta^2)$	$\zeta \in \mathbb{G}'_8$	$p = 3$
14	$(\zeta,\zeta^2,-1),(-\zeta^{-2},\zeta^{-2},-1)$	$\zeta \in \mathbb{G}'_5$	$p \neq 5$
15	$(\zeta,\zeta^{-3},-1),(-\zeta^{-2},\zeta^3,-1),$ $(-\zeta^{-2},-\zeta^3,-1),(-\zeta,-\zeta^{-3},-1)$	$\zeta \in \mathbb{G}'_{20}$	$p \neq 2,5$
15′	$(\zeta,\zeta,-1),(1,-\zeta,-1),(1,\zeta,-1),(-\zeta,-\zeta,-1)$	$\zeta \in \mathbb{G}'_4$	$p = 5$
16	$(\zeta^5,-\zeta^{-3},-\zeta),(\zeta^5,-\zeta^{-2},-1),$ $(\zeta^3,-\zeta^2,-1),(\zeta^3,-\zeta^4,-\zeta^{-4})$	$\zeta \in \mathbb{G}'_{15}$	$p \neq 3,5$
16′	$(1,-\zeta^{-1},-\zeta^2),(1,-\zeta,-1),$ $(\zeta,-\zeta^{-1},-1),(\zeta,-\zeta^3,-\zeta^{-3})$	$\zeta \in \mathbb{G}'_5$	$p = 3$
16″	$(\zeta^{-1},-1,-\zeta),(\zeta^{-1},-\zeta,-1),$ $(1,-\zeta^{-1},-1),(1,-\zeta,-\zeta^{-1})$	$\zeta \in \mathbb{G}'_3$	$p = 5$
17	$(-\zeta,-\zeta^{-3},-1),(-\zeta^{-2},-\zeta^3,-1)$	$\zeta \in \mathbb{G}'_7$	$p \neq 7$
18	$(\zeta^{-1},-1,-\zeta),(\zeta^{-1},-\zeta,-1),(1,-\zeta^{-1},-1),$ $(1,-\zeta,-1),(\zeta,-\zeta^{-1},-1),(\zeta,-1,-\zeta^{-1})$	$\zeta \in \mathbb{G}'_3$	$p = 7$

B | Formulas for proving the existence of roots

In this section we want to collect some lengthy formulas which we use to prove results on roots of degree $m\alpha_1 + 3\alpha_2$. To improve readability of the other parts these were shifted to the appendix.

Lemma B.1 The following equations hold:

(i) $\partial_2([122]) = (2)_s(1-rs)u_1$.

(ii) $\partial_2(u_1^2) = b_1 q_{21} s u_2 + b_1 (2)_{qrs} u_1 x_1$.

(iii) $\partial_1\partial_2(u_1^3) = (3)_{qrs} b_1 u_1^2$.

(iv) $\partial_2^2(u_1^3) = (2)_s b_1^2 q_{21} \left(q_{21}^2 s^2 u_3 + (1+qrs+q^2rs) q_{21} s u_2 x_1 + (3)_{qrs} u_1 x_1^2\right)$.

(v) $\partial_2^3(u_1^3) = b_1^3 (3)_s^! q_{21}^3 x_1^3$.

(vi) If $qr^2s = -1$, then $\partial_1\partial_2([11212]) = b_1(1+qrs)(3)_q u_2$.

(vii) $\partial_2^2([11212]) = b_1 b_2 q_{21} (2)_s (1 - q^2r^2s) x_1^3$.

Proof. (i)

$$\begin{aligned}
\partial_2([122]) &= \partial_2(u_1x_2 - q_{12}sx_2u_1)\\
&= u_1 + ((1-r)s\cdot x_1x_2 - (1-r)q_{12}s\cdot x_2x_1) + q_{12}q_{21}s^2u_1\\
&= (1+s-rs-rs^2)u_1\\
&= (2)_s(1-rs)u_1 .
\end{aligned}$$

(ii)

$$\begin{aligned} & \partial_2(u_1^2) = b_1\left(u_1x_1 + q_{21}sx_1u_1\right) \\ = & \; b_1\left(u_1x_1 + q_{21}su_2 + qrsu_1x_1\right) \\ = & \; b_1q_{21}su_2 + b_1(2)_{qrs}u_1x_1\,. \end{aligned}$$

(iii)

$$\begin{aligned} \partial_1\partial_2(u_1^3) &= b_1\partial_1\left(u_1^2x_1 + q_{21}su_1x_1u_1 + q_{21}^2s^2x_1u_1^2\right) \\ &= (3)_{qrs}b_1u_1^2\,. \end{aligned}$$

(iv)

$$\begin{aligned} \partial_2^2(u_1^3) &= b_1\partial_2\left(u_1^2x_1 + q_{21}su_1x_1u_1 + q_{21}^2s^2x_1u_1^2\right) \\ &= b_1^2(2)_s\left(q_{21}u_1x_1^2 + q_{21}^2sx_1u_1x_1 + q_{21}^3s^2x_1^2u_1\right) \\ &= b_1^2(2)_s\big(q_{21}u_1x_1^2 + q_{21}^2su_2x_1 + q_{21}qrsu_1x_1^2 \\ &\quad +q_{21}^3s^2x_1u_2 + q_{21}^2qrs^2x_1u_1x_1\big) \\ &= b_1^2(2)_s\big(q_{21}u_1x_1^2 + q_{21}^2su_2x_1 + q_{21}qrsu_1x_1^2 \\ &\qquad\quad +q_{21}^3s^2u_3 + q_{21}^2q^2rs^2u_2x_1 + q_{21}^2qrs^2u_2x_1 + q_{21}(qrs)^2u_1x_1^2\big) \\ &= b_1^2(2)_sq_{21}\left(q_{21}^2s^2u_3 + (1+qrs+q^2rs)q_{21}su_2x_1 + (3)_{qrs}u_1x_1^2\right)\,. \end{aligned}$$

(v)

$$\begin{aligned} \partial_2^3(u_1^3) &= b_1^2(2)_sq_{21}\partial_2\left(q_{21}^2s^2u_3 + (1+qrs+q^2rs)q_{21}su_2x_1 + (3)_{qrs}u_1x_1^2\right) \\ &= b_1^2(2)_sq_{21}^3\left(s^2b_3 + (1+qrs+q^2rs)sb_2 + (3)_{qrs}b_1\right)x_1^3 \\ &= b_1^3(3)_s^!q_{21}^3x_1^3\,. \end{aligned}$$

(vi)

$$\begin{aligned} \partial_1\partial_2([11212]) &= \partial_1\partial_2(u_2u_1 - q^2rsq_{12}u_1u_2) \\ &= b_1u_2 + qrsb_2(2)_qx_1u_1 - b_2(2)_qq^2rsq_{12}u_1x_1 - b_1q^4r^3s^2u_2 \\ &= b_1u_2 + b_2(2)_qqrsu_2 + b_2(2)_qq^2rsq_{12}u_1x_1 \\ &\quad -b_2(2)_qq^2rsq_{12}u_1x_1 - b_1q^4r^3s^2u_2 \\ &= b_1(1+qrs(1-qr)(1+q) - q^4r^3s^2)u_2 = b_1(1+qrs)(3)_qu_2\,. \end{aligned}$$

(vii)

$$\begin{aligned}\partial_2^2([11212]) &= \partial_2^2(u_2u_1 - q^2rsq_{12}u_1u_2)\\ &= \partial_2\left(b_1u_2x_1 + b_2q_{21}sx_1^2u_1 - b_2q^2rsq_{12}u_1x_1^2 - q^2r^2s^2q_{21}b_1x_1u_2\right)\\ &= b_1b_2q_{21}(2)_s(1-q^2r^2s)x_1^3\,.\end{aligned}$$ □

Lemma B.2 The following equations hold:

(i) $\partial_1^2\partial_2(u_2^3) = b_2(2)_q(3)_{q^4r^2s}u_2^2\,.$

(ii) $\partial_2^3(u_2^3) = b_2^3(3)_s^!q_{21}^6x_1^6\,.$

(iii) $\partial_1^2\partial_2([11212]) = 0\,.$

(iv) $\partial_1^2\partial_2([111212]) = 0\,.$

(v) $\partial_2^2([11212]) = b_1b_2(2)_sq_{21}(1-q^2r^2s)x_1^3\,.$

(vi) $\partial_2^2([111212]) = b_1b_2(2)_sq_{21}(1-q^2r^2s)(1-q^3r^2)x_1^4\,.$

Proof.

(i)

$$\begin{aligned}\partial_1^2\partial_2(u_2^3) &= b_2\partial_1^2(u_2^2x_1^2 + q_{21}^2su_2x_1^2u_2 + q_{21}^4s^2x_1^2u_2^2)\\ &= b_2(2)_q(3)_{q^4r^2s}u_2^2\,.\end{aligned}$$

(ii)

$$\begin{aligned}\partial_2^3(u_2^3) &= b_2\partial_2^2(u_2^2x_1^2 + q_{21}^2su_2x_1^2u_2 + q_{21}^4s^2x_1^2u_2^2)\\ &= b_2^2(2)_s\partial_2(q_{21}^2u_2x_1^4 + q_{21}^4sx_1^2u_2x_1^2 + q_{21}^6s^2x_1^4u_2)\\ &= b_2^3(3)_s^!q_{21}^6x_1^6\,.\end{aligned}$$

(iii)

$$\begin{aligned}\partial_1^2\partial_2([11212]) &= \partial_1^2\partial_2(u_2u_1 - q^2rsq_{12}u_1u_2)\\ &= q^2rsq_{12}(\partial_1^2\partial_2(u_2)u_1 - u_1\partial_1^2\partial_2(u_2)))\\ &= 0\,.\end{aligned}$$

(iv) Follows from (iii) due to $[111212] = [x_1, [11212]]_c\,.$

(v)

$$\begin{aligned}\partial_2^2([11212]) &= \partial_2(b_1u_2x_1 + b_2q_{21}sx_1^2u_1 - b_2q^2rsq_{12}u_1x_1^2 - b_1q^2r^2s^2q_{21}x_1u_2)\\ &= b_1b_2(2)_sq_{21}(1-q^2r^2s)x_1^3\,.\end{aligned}$$

(vi)

$$\begin{aligned}\partial_2^2([111212]) &= \partial_2^2(x_1[11212] - q^3q_{12}^2[11212]x_1)\\ &= x_1\partial_2^2([11212]) - q^3r^2\partial_2^2([11212])x_1\\ &= b_1b_2(2)_sq_{21}(1-q^2r^2s)(1-q^3r^2)x_1^4\,.\end{aligned}$$ □

Note that $qr^2s+1=0$ implies that [1122] is not a root by Theorem 5.14. Hence there exists $\nu\in\mathbb{K}$ such that $[1122]=\nu u_1^2$ by Remark 5.16.

Lemma B.3 Assume $u_3\neq 0$, $[122]\neq 0$ and $qr^2s+1=0$.

(i) $\exists\mu\in\mathbb{K}$ such that $\partial_2([12122])=\mu u_1^2$.

(ii) $\exists\mu\in\mathbb{K}$ such that $\partial_1\partial_2([112122])=\mu q^2q_{12}^2b_1u_1^2$.

(iii) $\exists\mu\in\mathbb{K}$ such that

$$\begin{aligned}\partial_2^2([112122]) &= \mu b_1\big(q_{21}su_3 + (1+qrs+q^2rs-q^2r^3s)u_2x_1\\ &\qquad +(2)_{qrs}qq_{12}(1-qr^2)u_1x_1^2\big)\,.\end{aligned}$$

(iv) $\exists\mu\in\mathbb{K}$ such that

$$\begin{aligned}\partial_2^3([112122]) &= q_{21}\mu b_1\big(b_3s + b_2(1+qrs+q^2rs-q^2r^3s)\\ &\qquad +b_1(2)_{qrs}qr(1-qr^2)\big)\,x_1^3\,.\end{aligned}$$

Proof. (i)

$$\begin{aligned}
\partial_2([12122]) &= \partial_2(u_1[122] - qrs^2q_{12}[122]u_1)\\
&= (2)_s(1-rs)u_1^2 + q_{21}s^2b_1x_1[122]\\
&\quad -qrs^2q_{12}b_1[122]x_1 - qr^2s^3(2)_s(1-rs)u_1^2\\
&= (2)_s(1-rs)(1-qr^2s^3)u_1^2 + q_{21}s^2b_1(x_1[122] - qq_{12}^2[122]x_1)\\
&= (2)_s(1-rs)(1-qr^2s^3)u_1^2 + q_{21}s^2b_1[1122]\\
&= \underbrace{\left((2)_s(1-rs)(1-qr^2s^3) + q_{21}s^2b_1\nu\right)}_{=:\mu} u_1^2\,.
\end{aligned}$$

Note that $\mu \neq 0$ since otherwise $[12122] = 0$. But $[12122]$ is a root due to Theorem 5.14 if $[1222] \neq 0$ and otherwise due to Lemma 5.10 and

$$s_2(2\alpha_1 + \alpha_2) = 2\alpha_1 + 3\alpha_2\,.$$

(ii)

$$\begin{aligned}
&\partial_1\partial_2([112122])\\
=\;& \partial_1\partial_2(x_1[12122] - q^2q_{12}^3[12122]x_1)\\
=\;& \partial_1\left(x_1\partial_2([12122]) - q^2q_{12}^3q_{21}\partial_2([12122])x_1\right)\\
=\;& \mu\partial_1(x_1u_1^2 - q^2q_{12}^3q_{21}u_1^2x_1)\\
=\;& \mu b_1q^2q_{12}^2u_1^2\,.
\end{aligned}$$

(iii)

$$\begin{aligned}
&\partial_2^2([112122])\\
=\;& x_1\partial_2^2([12122]) - q^2q_{12}^3q_{21}^2\partial_2^2([12122])x_1\\
=\;& \mu\left(x_1\partial_2(u_1^2) - q^2q_{12}^3q_{21}^2\partial_2(u_1^2)x_1\right)\\
=\;& \mu b_1\left(q_{21}sx_1u_2 + (2)_{qrs}x_1u_1x_1 - q^2r^3su_2x_1 - q^2q_{12}^3q_{21}^2(2)_{qrs}u_1x_1^2\right)\\
=\;& \mu b_1\left(q_{21}su_3 + q^2rsu_2x_1 + (2)_{qrs}u_2x_1 + qq_{12}(2)_{qrs}u_1x_1^2\right.\\
&\left.-q^2r^3su_2x_1 - q^2q_{12}^3q_{21}^2(2)_{qrs}u_1x_1^2\right)\\
=\;& \mu b_1q_{21}su_3 + \mu b_1(1+qrs+q^2rs-q^2r^3s)u_2x_1\\
&+\mu b_1(1+qrs)qq_{12}(1-qr^2)u_1x_1^2\,.
\end{aligned}$$

(iv)

$$\begin{aligned}
& \partial_2^3([112122]) \\
= \; & \mu b_1 \partial_2 \left(q_{21} s u_3 + (1 + qrs + q^2 rs - q^2 r^3 s) u_2 x_1 + \right. \\
& \left. (1 + qrs) q q_{12} (1 - qr^2) u_1 x_1^2 \right) \\
= \; & q_{21} \mu b_1 \left(b_3 s + b_2 (1 + qrs + q^2 rs - q^2 r^3 s) \right. \\
& \left. + b_1 (2)_{qrs} qr (1 - qr^2) \right) x_1^3 \, . \qquad \square
\end{aligned}$$

Now, we apply the comultiplication to the superletters considered in Lemma 5.19, Lemma 5.22 and Lemma 5.23. Then using Lemma 6.12 those superletters can be transformed to homogeneous vectors primitive in $K_{>d}/K_{\geq d}$ with $r = 1$ or $r = 2$ under given preconditions.

Let $u_2' := [122]$ and recall the Shirshov-decomposition of 122 is $(12, 2)$. From (5.1) we deduce

$$\begin{aligned}
\Delta(u_1) &= u_1 \otimes 1 + b_1 x_1 \otimes x_2 + 1 \otimes u_1 \, , \\
\Delta(u_2) &= u_2 \otimes 1 + b_2 x_1^2 \otimes x_2 + (2)_q (1 - qr) x_1 \otimes u_1 + 1 \otimes u_2 \, , \\
\Delta(u_3) &= u_3 \otimes 1 + b_3 x_1^3 \otimes x_2 + \frac{b_3}{b_1} (3)_q x_1^2 \otimes u_1 + (1 - q^2 r)(3)_q x_1 \otimes u_2 + 1 \otimes u_3 \, ,
\end{aligned}$$

and, consequently,

$$\begin{aligned}
\Delta(u_2') &= \Delta(u_1)\Delta(x_2) - q_{12} s \Delta(x_2)\Delta(u_1) \\
&= u_1 \otimes x_2 + u_1 x_2 \otimes 1 + b_1 x_1 \otimes x_2^2 \\
&\quad + b_1 s x_1 x_2 \otimes x_2 + 1 \otimes u_1 x_2 + q_{12} s x_2 \otimes u_2 \\
&\quad - q_{12} s \left(q_{21} s u_1 \otimes x_2 + b_1 q_{21} x_1 \otimes x_2^2 + 1 \otimes x_2 u_1 \right. \\
&\quad \left. + x_2 u_1 \otimes 1 + b_1 x_2 x_1 \otimes x_2 + x_2 \otimes u_1 \right) \\
&= [122] \otimes 1 + 1 \otimes [122] + (1 - rs^2) u_1 \otimes x_2 + (1 - rs)(2)_s u_1 \otimes x_2 \, .
\end{aligned}$$

Lemma B.4 Assume $u_3 \neq 0$, $[122] \neq 0$, $(3)_{qr} s^! \neq 0$ and $qr^2 s + 1 = 0$. Then $[u_2, u_2']_c \neq 0$ and there are $\lambda, \lambda' \in \mathbb{K}$ such that

$$\begin{aligned}
\Delta([u_2, u_2']_c) \;\in\; & [u_2, u_2']_c \otimes 1 + 1 \otimes [u_2, u_2']_c \\
& + \lambda u_1^2 \otimes u_1 + \lambda' u_1 \otimes u_1^2 \\
& + K_{\geq 1}/K_{>1}.
\end{aligned}$$

Proof. First, by Lemma 5.19 [112122] is a root vector. Then $[u_2, u_2']_c \neq 0$ by Remark 4.29. Furthermore,

$$\begin{aligned}
\Delta([u_2, u_2']_c) &= \Delta(u_2)\Delta(u_2') - q^2 q_{12}^4 q_{21} s^2\, \Delta(u_2')\Delta(u_2) \\
&= u_2 u_2' \otimes 1 + 1 \otimes u_2 u_2' + (2)_q (1-qr) q r s^2 q_{12}\, x_1 u_2' \otimes u_1 \\
&\quad + (2)_s (1-rs) q^2 r s q_{12}\, u_1 \otimes u_2 x_2 \\
&\quad - q^2 r s^2 q_{12}^3\, (u_2' u_2 \otimes 1 + 1 \otimes u_2' u_2 \\
&\quad + (2)_q (1-qr)\, u_2' x_1 \otimes u_1 + (2)_s (1-rs)\, u_1 \otimes x_2 u2) \\
&\quad + \text{ terms } x \otimes y \text{ with } \deg(x) = a_1\alpha_1 + a_2\alpha_2,\ \frac{a_1}{a_2} > 1 \\
&= [u_2, u_2']_c \otimes 1 + 1 \otimes [u_2, u_2']_c \\
&\quad + (2)_q (1-qr) q r s^2 q_{12}\, [x_1, u_2']_c \otimes u_1 \\
&\quad + (2)_s (1-rs) q^2 r s q_{12}\, u_1 \otimes [u_2, x_2]_c \\
&\quad + \text{ terms } x \otimes y \text{ with } \deg(x) = a_1\alpha_1 + a_2\alpha_2,\ \frac{a_1}{a_2} > 1\,.
\end{aligned}$$

Now, [1122] is not a root vector by Theorem 5.14 since $qr^2 s = -1$. Thus,

$$[x_1, u_2']_c = [1122] = \mu u_1^2, \quad [u_2, x_2]_c \overset{\textit{Remark } 4.29}{=} [x_1, u_2']_c + \mu' u_1^2 = (\mu + \mu') u_1^2$$

for some $\mu, \mu' \in \mathbb{K}$. Thus, there are $\lambda, \lambda' \in \mathbb{K}$ such that

$$\begin{aligned}
\Delta([u_2, u_2']_c) &= [u_2, u_2']_c \otimes 1 + 1 \otimes [u_2, u_2']_c \\
&\quad + \lambda u_1^2 \otimes u_1 + \lambda' u_1 \otimes u_1^2 \\
&\quad + \text{ terms } x \otimes y \text{ with } \deg(x) = a_1\alpha_1 + a_2\alpha_2,\ \frac{a_1}{a_2} > 1\,. \qquad \square
\end{aligned}$$

Finally, we want to develop formulas for the comultiplication of [111212112] and [111211212] for certain braidings. These cases could not be treated for general p by the other tools we have seen so far. We will do this step by step on the corresponding Shirshov-decompositions.

First, the Shirshov-decomposition of 11212 is $(112, 12)$. Hence $[11212] = [u_2, u_1]_c$.

$$\begin{aligned}
\Delta([11212]) &= \Delta(u_2)\Delta(u_1) - q^2 r s q_{12} \Delta(u_1)\Delta(u_2) \\
&= u_2 u_1 \otimes 1 + b_1\, u_2 x_1 \otimes x_2 \\
&\quad + u_2 \otimes u_1 + b_2 q_{21} s\, x_1^2 u_1 \otimes x_2
\end{aligned}$$

$$
\begin{aligned}
& +b_1b_2q_{21}\; x_1^3\otimes x_2^2 + b_2\; x_1^2\otimes x_2u_1 \\
& +(2)_q(1-qr)qrs\; x_1u_1\otimes u_1 + b_2(2)_q qq_{21}\; x_1^2\otimes u_1x_2 \\
& +(2)_q(1-qr)\; x_1\otimes u_1^2 + q^2rsq_{12}\; u_1\otimes u_2 \\
& +b_1q^2q_{21}\; x_1\otimes u_2x_2 + 1\otimes u_2u_1 \\
& -q^2rsq_{12}\; u_1u_2\otimes 1 - b_2q^2rsq_{12}\; u_1x_1^2\otimes x_2 \\
& -(2)_q(1-qr)q^2rsq_{12}\; u_1x_1\otimes u_1 - q^2rsq_{12}\; u_1\otimes u_2 \\
& -b_1q^2r^2s^2q_{21}\; x_1u_2\otimes x_2 - b_1b_2q^2r^2sq_{21}\; x_1^3\otimes x_2^2 \\
& -b_2(2)_q q^2r^2s\; x_1^2\otimes x_2u_1 - b_1q^2rsq_{12}\; x_1\otimes x_2u_2 \\
& -q^4r^3s^2\; u_2\otimes u_1 - b_2q^4r^2sq_{21}\; x_1^2\otimes u_1x_2 \\
& -(2)_q(1-qr)q^3r^2s\; x_1\otimes u_1^2 - q^2rsq_{12}\; 1\otimes u_1u_2 \\
= \; & [11212]\otimes 1 + 1\otimes[11212] + b_1b_2q_{21}(1-q^2r^2s)\; x_1^3\otimes x_2^2 \\
& +b_1\big(u_2x_1 + (1-qr)q_{21}s\; x_1^2u_1 \\
& \qquad -(1-qr)q^2rsq_{12}\; u_1x_1^2 - q^2r^2s^2q_{21}\; x_1u_2\big)\otimes x_2 \\
& +\left(u_2 + (2)_q(1-qr)qrs\; x_1u_1\right. \\
& \qquad \left.-(1-qr)(2)_q q^2rsq_{12}\; u_1x_1 - q^4r^3s^2\; u_2\right)\otimes u_1 \\
& +b_2\; x_1^2\otimes\big(x_2u_1 + (2)_q qq_{21}\; u_1x_2 - (2)_q q^2r^2s\; x_2u_1 - q^4r^2sq_{21}\; u_1x_2\big) \\
& +x_1\otimes\big((2)_q(1-qr)\; u_1^2 + b_1q^2q_{21}\; u_2x_2 \\
& \qquad -b_1q^2rsq_{12}\; x_2u_2 - (2)_q(1-qr)\; q^3r^2su_1^2\big).
\end{aligned}
$$

Now, the Shirshov-decomposition of 111212 is $(1, 11212)$. Hence $[111212] = [x_1, [11212]]_c$. Using the above we conclude

$$
\begin{aligned}
\Delta([111212]) \;=\; & \Delta(x_1)\Delta([11212]) - q^3q_{12}^2\;\Delta([11212])\Delta(x_1) \\
=\; & x_1[11212]\otimes 1 + 1\otimes x_1[11212] \\
& -q^3q_{12}^2\;[11212]x_1\otimes 1 - q^3q_{12}^2\; 1\otimes x_1[11212] \\
& +q^2q_{12}\;\left(u_2 + (2)_q(1-qr)qrs\; x_1u_1\right. \\
& \qquad \left.-(1-qr)(2)_q q^2rsq_{12}\; u_1x_1 - q^4r^3s^2\; u_2\right)\otimes x_1u_1 \\
& -q^3q_{12}^2\;\left(u_2 + (2)_q(1-qr)qrs\; x_1u_1\right. \\
& \qquad \left.-(1-qr)(2)_q q^2rsq_{12}\; u_1x_1 - q^4r^3s^2\; u_2\right)\otimes u_1x_1 \\
& +\text{ terms } x\otimes y \text{ with } \deg(x) = a_1\alpha_1 + a_2\alpha_2,\; \frac{a_1}{a_2} > 2 \\
=\; & [111212]\otimes 1 + 1\otimes[111212]
\end{aligned}
$$

$$+q^2q_{12}\ \big(1+(2)_q(1-qr)qrs-q^4r^3s^2\big)\ u_2\otimes u_2$$
$$+\text{ terms } x\otimes y \text{ with } \deg(x)=a_1\alpha_1+a_2\alpha_2,\ \frac{a_1}{a_2}>2.$$

Lemma B.5 Assume $q=s\in\mathbb{G}'_{12}$, $r=q^8$. Then there are $\lambda,\lambda'\in\mathbb{K}$ such that

$$\begin{aligned}\Delta([111212112]) \in\ & [111212112]\otimes 1+1\otimes[111212112]\\ & +\lambda\ u_2^2\otimes u_2+\lambda'\ u_2\otimes u_2^2\\ & +\text{ terms } x\otimes y \text{ with } \deg(x)=a_1\alpha_1+a_2\alpha_2,\ \frac{a_1}{a_2}>2\,.\end{aligned}$$

Proof.

$$\begin{aligned}\Delta([111212112]) &= \Delta([111212])\Delta(u_2)-q^8r^4s^2\ \Delta(u_2)\Delta([111212])\\ &= [111212112]\otimes 1+1\otimes[111212112]+\lambda\ u_2^2\otimes u_2\\ &\quad +\lambda'\ u_2\otimes u_2^2+(1-(q^8r^4s^2)^2)\ [111212]\otimes u_2\\ &\quad +\text{ terms } x\otimes y \text{ with } \deg(x)=a_1\alpha_1+a_2\alpha_2,\ \frac{a_1}{a_2}>2\\ &= [111212112]\otimes 1+1\otimes[111212112]+\lambda\ u_2^2\otimes u_2+\lambda'\ u_2\otimes u_2^2\\ &\quad +\text{ terms } x\otimes y \text{ with } \deg(x)=a_1\alpha_1+a_2\alpha_2,\ \frac{a_1}{a_2}>2\end{aligned}$$

for some $\lambda,\lambda'\in\mathbb{K}$ due to $q^8r^4s^2=q^6=-1$. □

Lemma B.6 Assume $q\in\mathbb{G}'_{18}, r=q^{13}, s=q^5$ or $q\in\mathbb{G}'_9, r=q^4, s=q^5,\ p=2$. Then $[u_3,[11212]]_c\neq 0$ and there are $\lambda,\lambda'\in\mathbb{K}$ such that

$$\begin{aligned}\Delta([u_3,[11212]]_c) &= [u_3,[11212]]_c\otimes 1+1\otimes[u_3,[11212]]_c+\lambda\ u_2^2\otimes u_2+\lambda'\ u_2\otimes u_2^2\\ &\quad +\text{ terms } x\otimes y \text{ with } \deg(x)=a_1\alpha_1+a_2\alpha_2,\ \frac{a_1}{a_2}>2\,.\end{aligned}$$

Proof. First, by Lemma 5.23 [111211212] is a root vector. Then $[u_3,[11212]]_c\neq 0$ by Remark 4.29. Note that

$$1+(2)_q(1-qr)qrs-q^4r^3s^2=(3)_q(1+qrs)$$

due to $qr^2s=-1$ under given assumptions on q,r,s. Hence

$$\begin{aligned}\Delta([u_3,[11212]]_c) &= \Delta(u_3)\Delta([11212])-q^9r^3s^2q_{12}^3\ \Delta([11212])\Delta(u_3)\\ &= \big(u_3[11212]-q^9r^3s^2q_{12}^3\ [11212]u_3\big)\otimes 1\\ &\quad +1\otimes\big(u_3[11212]-q^9r^3s^2q_{12}^3\ [11212]u_3\big)\end{aligned}$$

$$
\begin{aligned}
& +(1-q^2r)(3)_q\ \big(q^6r^3s^2q_{12}\ x_1[11212] - q^9r^3s^2q_{12}^3\ [11212]x_1\big)\otimes u_2 \\
& +(3)_q(1-q^2r)\ u_2\otimes\big(q^6r^2sq_{12}\ u_3u_1 - q^9r^3s^2q_{12}^3\ u_1u_3\big) \\
& +\text{ terms } x\otimes y \text{ with } \deg(x)=a_1\alpha_1+a_2\alpha_2,\ \frac{a_1}{a_2}>2 \\
= \ & [u_3,[11212]]_c\otimes 1 + 1\otimes[u_3,[11212]]_c \\
& +(1-q^2r)(3)_q q^6r^3s^2q_{12}\ [111212]\otimes u_2 \\
& +(1-q^2r)(3)_q q^6r^2sq_{12}\ u_2\otimes[u_3,u_1]_c \\
& +\text{ terms } x\otimes y \text{ with } \deg(x)=a_1\alpha_1+a_2\alpha_2,\ \frac{a_1}{a_2}>2 \\
= \ & [u_3,[11212]]_c\otimes 1 + 1\otimes[u_3,[11212]]_c + \lambda\ u_2^2\otimes u_2 + \lambda'\ u_2\otimes u_2^2 \\
& +\text{ terms } x\otimes y \text{ with } \deg(x)=a_1\alpha_1+a_2\alpha_2,\ \frac{a_1}{a_2}>2
\end{aligned}
$$

for some $\lambda,\lambda'\in\mathbb{K}$ due to Remark 4.29 since [111212] is not a root due to $1,4\in\mathbb{J}$.$\square$

C | Zusammenfassung

Das Interesse an Nichols-Algebren ging vornehmlich aus der Theorie der punktierten Hopf-Algebren hervor. Umgekehrt ist die Klassifikation von Nichols-Algebren mit endlicher Dimension bzw. endlicher Gelfand-Kirillov-Dimension ein wichtiger Schritt für die Klassifizierung punktierter Hopf-Algebren von endlicher Dimension bzw. endlicher Gelfand-Kirillov Dimension unter gewissen Bedingungen [2].

Nichols-Algebren wurden zunächst als Bialgebren von Typ eins betrachtet [27]. Später wurden sie von verschiedenen Autoren charakterisiert [25, 26, 29, 30, 32, 33]. Als besonders zugänglich haben sich Nichols-Algebren von diagonalem Typ erwiesen, die als positiver Anteil von Quantengruppen auftreten [29]. Endlich-dimensionale Nichols-Algebren von diagonalem Typ sind in einer Reihe von Veröffentlichungen klassifiziert worden [12, 11, 14, 17, 18]. Wichtige Merkmale einer Nichols-Algebra $\mathcal{B}(V)$ von diagonalem Typ sind dabei ihr Wurzelsystem $\triangle$ und die zugehörige Basis von Produkten von Wurzelvektoren sowie der assoziierte Weyl-Gruppoid. In diesem Kontext wurden folgende Implikationen beobachtet:

$$\dim \mathcal{B}(V) < \infty \quad \overset{(1)}{\Rightarrow} \quad \#\triangle < \infty \quad \overset{(2)}{\Rightarrow} \quad \mathrm{GKdim}\,(\mathcal{B}(V)) < \infty\,.$$

Dabei ist bekannt, dass unter gewissen Bedingungen auch die Umkehrung von (1) zutrifft. Es wird vermutet, dass die Umkehrung von (2) im Allgemeinen auch gilt [4]. Diese Vermutung hat in den vergangenen Jahren zunehmend Aufmerksamkeit erhalten. Insbesondere wurde gezeigt, dass sie für Rang zwei Nichols-Algebren von diagonalem Typ über einem Körper der Charakteristik null [5] sowie darauf aufbauend für Nichols-Algebren über abelschen Gruppen zutrifft [4].

Ziel dieser Arbeit ist es, zu beweisen, dass diese Aussage für Rang zwei Nichols-Algebren von diagonalem Typ über einem beliebigen Körper gilt. Dabei ist zu beachten, dass es über beliebigen Körpern zusätzliche Beispiele von Nichols-Algebren mit endlichem Wurzelsystem gibt. Insbesondere existieren Beispiele mit einfachen Wurzeln α mit $\chi(\alpha, \alpha) = 1$, wobei χ den zugehörigen Bicharakter bezeichnet.

Solche Wurzeln implizieren unendliche Gelfand-Kirillov Dimension über Körpern von Charakteristik null [5]. Daher werden neue Hilfsmittel benötigt, um die Aussage für beliebige Körper zu zeigen. Das Resultat für Charakterstik null wird dabei neu bewiesen.

In Kapitel 2 zeigen wir, dass die Vektorraum-Basis aus geordneten Produkten von Monomen einer Algebra sich unter technischen Voraussetzungen umordnen lässt. Die Vektorraum-Basis einer Nichols-Algebra aus Produkten von Wurzelvektoren ist ein Beispiel, auf das sich dieses Resultat anwenden lässt.

Anschließend wird in Kapitel 3 die Gelfand-Kirillov-Dimension eingeführt und einige Aussagen zu algebraischen Konstruktionen werden gezeigt. Insbesondere beweisen wir eine Abschätzung der Gelfand-Kirillov Dimension einer Algebra gegen die von gewissen Sub-Quotienten.

Danach führen wir in Kapitel 4 in die Theorie der Nichols-Algebren ein. Dabei beschränken wir uns im Wesentlichen auf solche von diagonalem Typ. Bekannte Merkmale wie die PBW-basis und das Wurzelsystem werden diskutiert sowie der Weyl-Gruppoid skizziert. Für eine ausführlichere Einführung verweisen wir auf [2].

Um zu zeigen, dass eine Nichols-Algebra unendliche Gelfand-Kirillov-Dimension besitzt, benötigen wir einige Informationen über das Wurzelsystem. Dazu werden in Kapitel 5 die Ergebnisse aus [19, 34] zusammengefasst und um zusätzliche Resultate ergänzt. Insbesondere zeigen wir eine Methode, um die Existenz von unendlich vielen Wurzeln nachzuweisen.

Mit diesem Wissen werden in Kapitel 6 Bedingungen gesammelt, um Nichols-Algebren mit unendlicher Gelfand-Kirillov-Dimension zu identifizieren. Dabei folgen unsere Ansätze vor allem zwei Motivationen:

- ▷ Die Existenz von Wurzeln der Gestalt $k\alpha+\beta$ für alle $k \in \mathbb{N}$ zeigen, wobei $\alpha, \beta \in \mathbb{Z}^2$.
- ▷ Eine unendliche Kette von "enthaltenen" Nichols-Algebren mit strikt fallenden Gelfand-Kirillov-Dimension konstruieren.

Als Beispiel für den ersten Zugang wiederholen wir das Ergebnis zu Nichols-Algebren von affinem Cartan-Typ [5]. Der zweite Zugang wird genutzt, um unter stärkeren Bedingungen eine analoge Aussage zu beweisen, die für die Klassifizierung in Charakteristik null genutzt wurde.

Schließlich widmet sich Kapitel 7 dem schrittweisen Beweis des Hauptresultats:
Theorem 7.1 Sei $\mathbb{K}$ ein beliebiger Körper und $\mathcal{B}(V)$ eine Rang zwei Nichols-Algebra von diagonalem Typ über $\mathbb{K}$. Falls $\mathcal{B}(V)$ unendliche Gelfand-Kirillov-Dimension besitzt, so ist das zugehörige Wurzelsystem unendlich.

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