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LARGE TIME BEHAVIOR OF EXCHANGE-DRIVEN GROWTH

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Abstract. Exchange-driven growth (EDG) is a model in which pairs of clusters interact by exchanging single unit with a rate given by a kernel $K(j,k)$. Despite EDG model's common use in the applied sciences, its rigorous mathematical treatment is very recent. In this article we study the large time behaviour of EDG equations. We show two sets of results depending on the properties of the kernel (i) $K(j,k) = b_j a_k$ and (ii) $K(j,k) = ja_k + b_j + \varepsilon\beta_j\alpha_k$. For type I kernels, under the detailed balance assumption, we show that the system admits unique equilibrium up to a critical mass ρ_s above which there is no equilibrium. We prove that if the system has an initial mass below ρ_s then the solutions converge to a unique equilibrium distribution strongly where if the initial mass is above ρ_s then the solutions converge to *cricital* equilibrium distribution in a weak sense. For type II kernels, we do not make any assumption of detailed balance and equilibrium is shown as a consequence of contraction properties of solutions. We provide two separate results depending on the monotonicity of the kernel or smallness of the total mass. For the first case we prove exponential convergence in the number of clusters norm and for the second we prove exponential convergence in the total mass norm.

1. **Introduction.** Exchange-driven growth (EDG) is a model for non-equilibrium cluster growth in which pairs of clusters interact by exchanging a single unit of mass (monomer) at a time. [1],[2]. In the recent years EDG has been used to model several natural and social phenomena such as migration [3], population dynamics [4] and wealth exchange [5]. EDG is also important mathematically for multiple reasons. Firstly, it is a model of intermediate complexity between the classical Becker-Doring (BD) model [6], [7], where the dynamics are well understood, and the Smoluchowski coagulation model, where the existing mathematical questions are much tougher. Secondly, EDG arises as the mean field limit of a class of interacting particle systems (IPS) that includes models of non-equilibrium statistical physics including zero-range processes [8], [9], [10], [11], [12], [13], [15], [16], [17], that have been intensively studied for a range of condensation phenomena that they exhibit. Despite its importance, rigorous results on the properties and behavior of the corresponding equations (existence, uniqueness, asymptotic behavior etc.) are few and have been obtained only very recently [18], [19]. It is the purpose of this

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article to continue the mathematical study of the EDG systems focusing on the large time asymptotic properties of solutions with explicit convergence rates where possible.

In EDG, the mathematical description of the mass exchange systems is given at the mesoscopic level and one studies the mean field rate equations (hereafter referred as EDG equations) ignoring fluctuations at the particle level. The main mathematical object of study is $c_j(t)$, the cluster size density, describing the volume fraction of the system which is occupied by clusters of size $j \geq 1$, where $j = 0$ corresponds to the empty (available) volume fraction not occupied by any particle. The rate of exchange from a $j-$cluster to a $k-$cluster is given by $K(j,k)$. Symbolically, the exchange process can be described in the following way. If $< j >$, $< k >$ denote the clusters of sizes $j > 0$, $k \geq 0$, then the rule of interaction is

$$< j > \oplus < k > \to < j-1 > \oplus < k+1 > \quad (j,k \text{ ordered})$$

The fact that the second index k can have the value zero breaks the symmetry in the interaction and is central to the paper. In general, $K(j,k)$ need not be a symmetric function (even for $j,k > 0$). This is another important difference between the EDG and coagulation (Smoluchowski) models. Mathematically, the infinite network of interactions are represented as a system of nonlinear ODEs

$$\dot{c}_0 = c_1 \sum_{k=0}^{\infty} K(1,k)c_k - c_0 \sum_{k=1}^{\infty} K(k,0)c_k, \tag{1.1}$$

$$\dot{c}_j = c_{j+1} \sum_{k=0}^{\infty} K(j+1,k)c_k - c_j \sum_{k=0}^{\infty} K(j,k)c_k \tag{1.2}$$

$$- c_j \sum_{k=1}^{\infty} K(k,j)c_k + c_{j-1} \sum_{k=1}^{\infty} K(k,j-1)c_k \ , \tag{1.3}$$

$$c_j(0) = c_{j,0} \qquad \{j = 0,1,2,...\}. \tag{1.4}$$

In [18] one of the authors provided the first mathematical investigation of EDG equations giving the fundamental properties such as global existence, uniqueness and non-existence. In particular, for general non-symmetric kernels whose growth is bounded as $K(j,k) \leq Cjk$ (for large j,k), unique classical solutions were shown to exist globally. Recently, these results for non-symmetric kernels were extended in [19], in particular, moment boundedness assumptions for the uniqueness were replaced with milder conditions. For symmetric kernels, it was shown in [18] that the existence result can be generalized to kernels whose growth rate is lying in the range $K(j,k) \leq C(j^{\mu}k^{v} + j^{\nu}k^{\mu})$, with $\mu,\nu \leq 2$, $\mu + v \leq 3$. Uniqueness of solutions was obtained under additional boundedness assumptions on the moments. On the other hand, for sufficiently fast growing kernels it was shown that no solution can exist provided that the initial distribution has a fat tail.

There exists a body of literature on the applications of EDG model in physical and social sciences. In these classical treatments exchange interactions are only defined among non-zero clusters and 0-clusters have no use or meaning. One of the key aspects of the current formulation of the EDG system given by (1.1)-(1.4) is the inclusion of the $0-$clusters (or available volume) representing the non-zero volume fraction accessible to particles. In this description volume or total number density, i.e., $\sum_{j\geq 0} c_j = \eta$ becomes a conserved quantity independently of the total mass density (denoted by ρ hereafter).

The presence of zero clusters influence the properties of the whole system most distinctly by allowing the particles to detach from non-zero clusters and re-occupy the available (free) volume. Effectively, this provides a "fresh" source of $1-$clusters to the system and is equivalent to $K(j,0) > 0$ in mathematical terms. This behavior was first demonstrated numerically in [20], where it was observed that the seemingly innocuous change in the kernel $(K(j,0) > 0)$ fundamentally alters the dynamical behavior, driving the system, towards a unique equilibrium (BD-like) instead of indefinite growth where the cluster densities eventually vanish (Smoluchowski-like when $K(j,0) = 0$). For a large class of kernels this observation was recently proven in [19].

In this article we study the large time behavior of the exchange-driven system concentrating on the cases where the exchange interaction rate (i.e., the kernel K) is separable and has either of the following forms

$$(I)\ K(j,k) = b_j a_k, \tag{1.5}$$

$$(II)\ K(j,k) = ja_k + b_j + \varepsilon\beta_j\alpha_k \tag{1.6}$$

where the b_j (and β_j) terms can be interpreted as "export" rate and a_j (and α_j) terms as the "import" rate of particles and $\varepsilon > 0$ is a small parameter.

For the type I separable kernels we show that, under a crucial balance assumption, the equilibrium cluster densities take the form $c_j = \frac{Q_j z(\rho,\eta)^j}{\sum_j Q_j z(\rho,\eta)^j}$ where $z(\rho,\eta)$ is a solution of a nonlinear equation and $Q_j = \prod_{k=1}^{k=j} \frac{a_{k-1}}{b_k}$ are combinatorial factors. The explicit form of the equilibria becomes useful in the analysis of behavior of solutions. In particular, the feature that the equilibrium densities are the minimizers of a certain (entropy) functional $V(c) = \sum c_j \ln(\frac{c_j}{Q_j}) - c_j$ on a chosen set

$$X_{\rho,\eta} = \{(c)_{j=1}^{\infty} : c_j \geq 0, \sum jc_j = \rho, \sum c_j = \eta\}$$

enables us to use the well developed entropy dissipation methods for the large time analysis. It is worth noting that, for this type of kernel, an equilibrium is possible only for a range of initial mass ρ_i satisfying $\rho_i \leq \rho_s$ where ρ_s is the critical mass. In this case (hereafter referred to as subcritical case) individual cluster densities can be explicitly obtained from a recursive relation. If $\rho_s < \infty$ and $\rho_i > \rho_s$, then, there will be no admissible equilibrium, indicating a phase transition. For type II separable kernels we do not make any assumption on the structure of equilibrium (no detailed balance assumption) and therefore no specific analysis of the forms of equilibrium will be made or needed except for its existence. That we do not impose any structural conditions on the equilibrium is one of the novelties in this paper.

The main goal of this article is to obtain rigorous results on the large time behavior of the EDG system. Below we give a brief outline of arguments and main findings. We provide two sets of results depending on the type of the kernel.

For type I kernels, we prove qualitative convergence results with mild assumptions on the kernel. In particular, we show that the time dependent system (1.1)-(1.4) goes strongly to equilibrium if the total mass is below a threshold value ρ_s. Above this critical value, a dynamic phase transition occurs and the excess initial mass $\rho_i - \rho_s$ forms larger and larger clusters while the rest of the system approaches to equilibrium weakly. This behavior is analogous to the simpler Becker-Doring system whose dynamics has been well studied [14], [21], [22], [23], [24], [25], [26], [27].

For the results, we first show that under the assumptions of [18] the system (1.1)-(1.4) form a semi-group. Then one naturally seeks a Lyapunov function which is decreasing in time and a suitable norm where the positive orbit is relatively compact and the Lyapunov function is continuous. Since mass is an invariant of the motion a first candidate for the suitable norm is the space $X = \{(c)_{j=1}^{\infty} : \sum jc_j < \infty\}$. The downside of this natural norm is that the positive orbit is not always compact. Quite similar to the classical case in BD equations using a weaker topology comes useful and the desired compactness result can be obtained even for the supercritical case. The remaining condition is then to satisfy the continuity of the Lyapunov function in the chosen metric. It turns out that the continuity does not generally hold for the "bare" form of the Lyapunov function but holds for the modified version

$$V_{z,y}(c) = V(c) - \ln z \sum jc_j - \ln y \sum c_j.$$

Here, the invariance of the total mass and volume is of crucial importance for preserving the monotonicity property of the new Lyapunov function. This naturally extends the approach taken in [14] where the only conserved quantity was total mass. With this modification we can show that $V_{z,y}$ is weakly (defined more precisely later) continuous at the special values $z = z_s$, $y = y_s$ (defined later) and the invariance principle can be applied to prove the weak convergence of solutions. For the subcritical case we enforce stronger conditions on the initial data to prove compactness and use the invariance principle to show the strong convergence.

Our second set of results with type II kernels on the large time behavior concern the convergence to equilibrium solutions without detailed balance. Both the existence of general equilibrium and the convergence to equilibrium are consequences of the key contraction properties (of solutions) arising from different assumptions on the kernels. These lead to two separate results of convergence. For each result we prove that solutions converge to the equilibrium exponentially fast in a suitable norm.

The proofs of rate of convergence rely on analyzing the evolution of two non-negative quantities which measure the distance of a solution from another solution (distribution) having the same mass. The task here is to show that, in each measure, this "distance" shrinks in time (contraction property). To show the first contraction property we assume the kernel satisfies certain monotonicity conditions. With this, one can show that solutions approach to equilibrium exponentially fast in the "number of clusters" norm. For the second contraction property, one can remove the monotonicity conditions on the kernel and impose a small mass condition on the system. The second approach is along the lines of [24]. Though more restrictive, with such an assumption one can show that solutions converge to equilibrium exponentially fast in the stronger "mass norm".

Part of the results of this paper, namely those in Section 3, overlap with some of the results in [19] which were independently obtained. Actually the results in [19] cover a class of kernels wider than those considered in Section 3 of this paper. Nevertheless, given that the proofs of the convergence results are simpler and give a clear intuition about properties of the kernels for the product kernels considered in Section 3 we decided to keep them (see the discussion about "export" and "import" tendencies). On the other hand, the analysis of the long time asymptotics for kernels of type II, for which detailed balance is not satisfied, has not been considered, to our knowledge, anywhere else. We consider this type of kernels in Section 4 of this paper. Besides providing the first explicit rates of convergence, the results in this article are also valuable as they illustrate that the EDG system shows structural similarities to the BD system and naturally generalizes it.

The organization of the rest of the paper is as follows. In Section 2 we recall some of the basic results on the well posedness of the EDG system and give important lemmas that will be used throughout. In Section 3, we study the form of the equilibria with type I kernels and define and analyze some important functions that will form the basis of arguments to prove the convergence to equilibrium (in weak and strong senses). In Section 4, we study the EDG system with type II kernels without the detailed balance assumption and prove exponential convergence to equilibrium in "weak" and strong senses with explicit rates.

2. **Fundamentals.** In this section we give the setting of the problem and provide some basic facts which will be used in the subsequent analysis. For the sequences of functions that we are interested the appropriate spaces are $X_\mu = \{x = (x_j)_{j=0}^\infty,\ x_j \in \mathbb{R}; \|x\|_\mu < \infty\}$. We equip the space with the norm $\|x\|_\mu = \sum_{j=1}^\infty j^\mu x_j$ where $\mu \geq 0$. Similarly, we define X_μ^+ the subspaces of non-negative sequences as $X_\mu^+ = \{x = (x_j),\ x_j \geq 0; \|x\|_\mu < \infty\}$.

Definition 1: We say the system (1.1)-(1.4) has a solution iff

(i) $c_j(t) : [0,\infty) \to [0,\infty)$ is continuous and $\sup_{t\in[0,\infty)} c_j(t) < \infty$

(ii) $\int_0^t \sum_{k=0}^\infty K(j,k)c_k ds < \infty$, $\int_0^t \sum_{k=1}^\infty K(k,j)c_k ds < \infty$ for all $t \in [0,T)$ $(T \leq \infty)$

(iii) $c_j(t) = c_j(0) + \int_0^t \left(c_{j+1}\sum_{k=0}^\infty K(j+1,k)c_k - c_j\sum_{k=0}^\infty K(j,k)c_k\right) ds$
$+ \int_0^t \left(-c_j\sum_{k=1}^\infty K(k,j)c_k + c_{j-1}\sum_{k=1}^\infty K(k,j-1)c_k\right) ds \qquad \{j \geq 1\}$

$c_0(t) = c_0(0) + \int_0^t c_1\sum_{k=0}^\infty K(1,k)c_k - \int_0^t c_0\sum_{k=1}^\infty K(k,0)c_k.$

In above and the rest of the paper, the cluster interaction kernel $K(\cdot,\cdot) : \mathbb{R}\times\mathbb{R} \to [0,\infty)$ is defined to be non-negative function. We also set $K(0,j) \equiv 0$ identically.

Definition 2: For a sequence $(c_j)_{j=1}^N$, we call the quantity $M_p^N(t) = \sum_{j=0}^N j^p c_j(t)$ as the $p^{th}-$moment of the sequence. If the sequence is infinite, then we denote the $p^{th}-$moment with $M_p(t) = \sum_{j=0}^\infty j^p c_j(t)$.

It is often useful to study the finite version of the infinite system where the equations are truncated at some order, say, $N < \infty$ as below

$$\dot{c}_0^N = c_1^N \sum_{k=0}^{N-1} K(1,k)c_k^N - c_0^N \sum_{k=1}^{N} K(k,0)c_k^N, \tag{2.1}$$

$$\begin{aligned}\dot{c}_j^N &= c_{j+1}^N \sum_{k=0}^{N-1} K(j+1,k)c_k^N - c_j^N \sum_{k=0}^{N-1} K(j,k)c_k^N \\ &- c_j^N \sum_{k=1}^{N} K(k,j)c_k^N + c_{j-1}^N \sum_{k=1}^{N} K(k,j-1)c_k^N,\ \{1 \leq j \leq N-1\}\end{aligned} \tag{2.2}$$

$$\dot{c}_N^N = -c_N^N \sum_{k=0}^{N-1} K(N,k)c_k^N + c_{N-1}^N \sum_{k=1}^{N} K(k,N-1)c_k^N, \tag{2.3}$$

with the initial conditions given by

$$c_j^N(0) = c_{j,0} \geq 0, \quad \{0 \leq j \leq N\}. \tag{2.4}$$

The fundamental properties of solutions are well known from the standard ODE theory. We also quote the following basic result from [18] whose proof we skip

Lemma 1. *Let g_j be a sequence of non-negative real numbers. Then,*

$$\sum_{j=0}^{N} g_j \frac{dc_j^N}{dt} = \sum_{j=1}^{N} (g_{j-1} - g_j) c_j^N \sum_{k=0}^{N-1} K(j,k) c_k^N + \sum_{j=0}^{N-1} (-g_j + g_{j+1}) c_j^N \sum_{k=1}^{N} K(k,j) c_k^N. \tag{2.5}$$

Two immediate results that one can draw from this lemma (by setting $g_j = 1$ and $g_j = j$) is the conservation of total number of clusters (volume) and total mass which also extends to the infinite system. The finite system will be useful and revisited when needed in order to gain further information on the original system.

Now, we state the some of the fundamental results on the solutions of the EDG system (1.1)-(1.4) with kernels allowing particles to hop on to the available volume ($K(j,0) > 0$), sometimes called as non-linear chipping. At this point, no assumptions are made on the kernel, but it is always assumed that the growth of the kernels (with respect to the entries) is sublinear (see [18] for well-posedness results for kernels growing faster than linear).

Theorem 1. *Let $K(j,k)$ be a general kernel satisfying $K(j,k) \leq Cjk$ for large enough j,k. Assume further that $M_p(0) < \infty$ for some $p > 1$. Then the infinite system (1.1)-(1.4) has a global solution $c \in X_1$ where $c_j(t)$ is continuously differentiable. Moreover for any $t < \infty$ $M_p(t) < \infty$ and*

$$\sum_{0}^{\infty} c_j(t) = \sum_{0}^{\infty} c_j(0), \tag{2.6}$$

$$\sum_{0}^{\infty} jc_j(t) = \sum_{0}^{\infty} jc_j(0). \tag{2.7}$$

It can be shown that the global existence and conservation laws still hold if one replaces the moment assumption ($M_p(0) < \infty$) with a slower growth assumption on the kernels.

Theorem 2. *Let K satisfy $K(j,k) \leq Cb_j a_k$ (with $a_j, b_j = o(j)$) and $M_1(0) < \infty$.Then the infinite system (1.1)-(1.4) has a global solution $(c_j) \in X_1$ where $c_j(t)$ is continuously differentiable.*

While Theorem 1 shows that individual cluster size densities are continuous in time, when studying the asymptotics we will need to treat the cluster size distribution as an element in the space X_1. The following result is an immediate consequence continuity of cluster densities and Dini's uniform convergence theorem.

Proposition 1. *Let c be the solution of (1.1)-(1.4). Then $c : [0,T) \to X_1$ is continuous and the series $\sum_{j=1}^{\infty} jc_j(t)$ is uniformly convergent on compact intervals of $[0,T)$.*

When discussing the convergence to equilibrium, in addition to strong convergence (in the X_1 norm) we will also make use of weak$*$ convergence which has also been frequently used in the analysis of the Becker-Doring equations.

Definition 3: We say that a sequence $\{x^i\}$ in X_1 converges weak$*$ to $x \in X_1$ ($\rightharpoonup^*$symbolically) if the following holds

(i) $\sup_{i\geq 1}\left\|x^i\right\|_1 < \infty$,
(ii) $x^i_j \to x_j$ as $i \to \infty$ for each $j = 1, 2, ...$

The virtue behind using this concept of convergence is two-fold. First, as briefly mentioned in the introduction, the positive orbit of the flow generated by EDG equations are not generally compact in X_1. In those cases it will be convenient to consider a finite ball $B_\rho = \{x \in X_1, \|x\| < \rho\}$ induced with the metric

$$dist(x, y) = \sum_{j=0}^{\infty} |x_j - y_j|$$

where the B_ρ is compact and the weak∗ convergence is equivalent to convergence in this new metric. A second benefit of studying the weak∗ convergence is that one can easily characterize the cases where weak∗ convergence becomes equivalent to strong convergence in X_1 thanks to the following lemma [14].

Lemma 2. *If $x^j \rightharpoonup^* x$ in X_1 and $\left\|x^j\right\| \to \|x\|$, then it follows that $x^j \to x$.*

In this new topology we frequently use the following definition of continuity.

Definition 4: Let $S \subset X_1$. A function $f : S \to R$ is said to be weak∗ continuous iff $x^j \rightharpoonup^* x$ implies $f(x^j) \to f(x)$ as $j \to \infty$.

A typical example of weak∗ continuous function in X_1 is the function $W(x) = \sum_{j=0}^{\infty} g_j x_j$. This function is weak∗ continuous if and only if the coefficients satisfy $g_j = o(j)$ near infinity.

As the last item of this section we establish the link between the solutions generated by the EDG equations (under the setting of this paper) and the concept of generalized flow introduced in [14] which is defined as below.

Definition 5: A generalized flow G on a metric space Y is a family of continuous mappings $\phi : [0, \infty) \to Y$ with the properties

(i) if $\phi \in G$ and $t \geq 0$ then ϕ_t defined by $\phi_t(s) = \phi(t + s)$ belongs to G.

(ii) if $y \in Y$ there exists at least one $\phi \in G$ with $\phi(0) = y$

(iii) if $\phi^i \in G$ and $\phi^i(0)$ converges to y in Y, then there exists a subsequence $\phi^{i(k)}$ and an element of $\phi \in G$ such that $\phi^{i(k)}(t) \to \phi(t)$ uniformly on compact intervals of $[0, \infty)$ (with $\phi(0) = y$).

The generalized flow is related to semigroup in the following way.

Definition 6. We say that a generalized flow is a semigroup if for each $y \in Y$, there is a unique $\phi(t)$ with $\phi(0) = y$ and the flow is given by a map $T(t) : Y \to Y$ with T satisfying the properties

(i) $T(0) = identity$

(ii) $T(s + t) = T(s)T(t)$

(iii) the mapping $(t, \phi(0)) \to T(t)\phi(0)$ is continuous from $[0, \infty) \times Y \to Y$.

The next results show that, depending on the growth properties of the kernel, the EDG system generates a generalized flow in the strong or weak sense (of convergence).

Proposition 2. *Let the conditions in Theorem 1 hold ($a_j, b_j = O(j)$). Then the system (1.1)-(1.4) generates a generalized flow on X_1^+.*

Proof. Properties (i) and (ii) (in Definition 5) are clear from the definition of a solution of (1.1)-(1.4). The continuity of $\phi : [0, \infty) \to X_1^+$ is due to Proposition 1. For property (iii), consider the sequence $\phi^i(0) \to \phi(0)$ in X_1. For each j consider the family $\{\phi^i_j(t)\}_{i=1}^{\infty}$, $\{\dot{\phi}^i_j(t)\}_{i=1}^{\infty}$ which are uniformly bounded since$K(j, k) \leq Cjk$ and $\sum_{j\geq 1} j(\phi^i_j)(t) \leq C$. Then by Arzela-Ascoli theorem, for each j there is a subsequence

$i(k)$ such that $(\phi_j^{i(k)})(t) \to \phi_j(t)$. We need to show that ϕ is the limit of ϕ^i in X_1. For this, we use the conservation of mass from Theorem 1

$$\lim_{i(k)\to\infty} \sum_{j\geq 1} j\phi_j^{i(k)}(t) = \lim_{i(k)\to\infty} \sum_{j\geq 1} j\phi_j^{i(k)}(0) = \sum j\phi_j(0) = \sum j\phi_j(t)$$

which implies $\left\|\phi^{i(k)}(t)\right\|_1 \to \|\phi(t)\|_1$. Then, by Lemma 2 we get $\phi^{i(k)}(t) \to \phi(t)$ in X_1, proving the Proposition. □

Proposition 3. *Assume the conditions of Theorem 2 hold ($a_j, b_j = o(j)$). Then the system (1.1)-(1.4) generates a generalized flow on B_ρ^+.*

Proof. Consider $d(\phi^i(0), \phi(0)) \to 0$ in B_ρ^+. Then $\phi_j^i(0) \to \phi_j^i(0)$ in particular. By Theorem 2 and following arguments similar to the previous proposition one can construct a subsequence $\phi_j^{i(k)}(t)$ which converges uniformly to some $\phi_j(t)$ for each j and satisfies $\sum_{j\geq 1} j\phi_j^{i(k)}(t) \leq C$ uniformly in i. But this implies $(\phi^i)^{N_i} \to^* \phi^i$ which is equivalent to $d(\phi^{N_i(k)}(t), \phi(t)) \to 0$. □

Since one of the requirements for the generalized flow to be a semigroup is the uniqueness we need the following uniqueness result from [18] for the EDG system.

Theorem 3. *Let the conditions of Theorem 1 be satisfied with $M_p(0) < \infty$ for some $p > 2$. Then the ODE system (1.1)-(1.4) has a unique solution in X_1.*

With the theorem above and the arguments used in proof of the main existence theorem one can show that the infinite system (1.1)-(1.4) actually forms a semigroup.

Theorem 4. *Let the conditions of Theorem 1 be satisfied with $M_p(0) < \infty$ for some $p > 2$. Then the ODE system (1.1)-(1.4) forms a semigroup.*

Proof. Properties (i) and (ii) follow from the definition of solution and Theorem 1. Under the conditions of the theorem the uniqueness follows from Theorem 3. Property (iii) is a consequence of Proposition 1. □

Finally, we end the subsection with the following result from [18] which is a straightforward computation.

Lemma 3. *Let $c_j(t)$ be a solution of the EDG system (1.1)-(1.4). Then one has the following identities*

$$\sum_{j=m}^{\infty} c_j(t) - \sum_{j=m}^{\infty} c_j(0) = \int_0^t I_{m-1}(c(s))ds,$$

$$\sum_{j=m}^{\infty} jc_j(t) - \sum_{j=m}^{\infty} jc_j(0) = \int_0^t \sum_{j=m}^{\infty} I_j(c(s))ds + m\int_0^t I_{m-1}(c(s))ds.$$

3. Convergence to equilibrium with detailed balance.

3.1. **Equilibria and minimizers.** We say that c_j is an equilibrium solution if $\dot{c}_j(t) = I_{j-1} - I_j = 0$ for all $j \geq 0$ where

$$I_j = \sum_{k\geq 1} K(k,j)c_k c_j - \sum_{k\geq 0} K(j+1,k)c_{j+1}c_k \tag{3.1}$$

is the density current. We set $I_{-1} = 0$. This implies $I_j = 0$ for all j. Furthermore, throughout this section we assume $K(j,k) = b_j a_k$ (type I kernel). This gives the following recursive relationship between the cluster densities

$$c_{j+1} = \frac{a_j}{b_{j+1}} \frac{B}{A} c_j = \frac{a_j ... a_0}{b_{j+1} ... b_1} \left(\frac{B}{A}\right)^{j+1} c_0, \tag{3.2}$$

where $B = \sum_{j=1}^{\infty} b_j c_j$ and $A = \sum_{k=0}^{\infty} a_j c_j$. Thanks to the separability assumption of the kernel one can see that the detailed balance condition is satisfied, i.e., for each j,k the forward and backward rates in the exchange reaction

$$< j+1 > \oplus < k > \to < j > \oplus < k+1 >$$

are equal. Indeed, using the separability and the first equality in (3.2), one easily verifies that

$$K(j+1,k) c_{j+1} c_k = K(k+1,j) c_j c_{k+1}. \tag{3.3}$$

In order for $(c)_{j=0}^{\infty}$ be a true equilibrium distribution, the set of equations for c_j, A, B must be solved simultaneously. We show this by finding a unique distribution for a given the total number (density) η and total mass (density) ρ of clusters.

Let $\frac{Q_j}{Qj-1} := \frac{a_{j-1}}{b_j}$ and $Q_0 = 1$. Consider the forms for mass and number density, i.e., $\rho = \sum_{j=1}^{\infty} j Q_j \left(\frac{B}{A}\right)^j c_0$ and $\eta = \sum_{j=0}^{\infty} Q_j \left(\frac{B}{A}\right)^j c_0$. For the consistency of solutions, we need to show there is a unique $z(\rho,\eta)$ such that $\frac{\sum_{j=1}^{\infty} jQ_k z^j}{\sum_{j=0}^{\infty} Q_j z^j} = \frac{\rho}{\eta}$. Let z_s be the radius of convergence of for the series $\sum_{j=0}^{\infty} jQ_j z^j$ which is given by

$$z_s^{-1} = \lim_{j\to\infty} (Q_j)^{1/j}. \tag{3.4}$$

Define the function F

$$F(z) = \frac{\sum_{j=0}^{\infty} jQ_j z^j}{\sum_{j=0}^{\infty} Q_j z^j}.$$

Proposition 4. *The function $F(z)$ is strictly increasing on $0 \le z < z_s$.*

Proof. For $z < z_s$ the series $\sum_{j=0}^{\infty} Q_j z^j$ and $\sum_{j=0}^{\infty} jQ_j z^j$ can be differentiated term by term.

$$\frac{dF(z)}{dz} = \frac{\sum_{j=0}^{\infty} j^2 Q_j z^{j-1} \sum_{k=0}^{\infty} Q_k z^k - \sum_{j=0}^{\infty} jQ_j z^j \sum_{k=0}^{\infty} kQ_k z^{k-1}}{\left(\sum_{j=0}^{\infty} Q_j z^j\right)^2}.$$

Using the symmetry of the sum in the first term of the numerator one has

$$\frac{dF(z)}{dz} = \frac{\sum_{j=0}^{\infty} \sum_{k=0}^{\infty} Q_j Q_k z^{j+k-1}((j^2+k^2)/2 - jk)}{\left(\sum_{j=0}^{\infty} Q_j z^j\right)^2}.$$

Since $(j^2+k^2)/2 \ge jk$ holds for any $j,k \ge 0$ the numerator is positive and hence $\frac{dF(z)}{dz} > 0$, proving the proposition. □

Now, we define the critical mass density ρ_s as

$$\rho_s = \eta \sup_{z<z_s} F(z).$$

Then, for a given $\rho < \rho_s$ there is a unique value of $z(\rho,\eta)$ satisfying the equality $F(z(\rho,\eta)) = \frac{\rho}{\eta}$. This therefore uniquely determines $\frac{B}{A}$ and c_0^e as

$$\frac{B}{A} = z(\rho,\eta), \qquad c_0^e = y(\rho,\eta) = \frac{\eta}{\sum_{j=0}^{\infty} Q_j z(\rho,\eta)^j}.$$

Also, for $\rho = \rho_s$, we denote $y_s = \frac{\eta}{\sum_{j=0}^{\infty} Q_j z_s^j}$. Hence, we have proved

Proposition 5. *Let $\rho, \eta < \infty$ be given. Then, if $\rho < \rho_s$ the EDG system admits a unique equilibrium distribution c^e given by*

$$c_j^e(\rho,\eta) = Q_j z(\rho,\eta)^j y(\rho,\eta).$$

If $\rho > \rho_s$, then there is no equilibrium state with density ρ.

Next, we define some functions which will be useful in the analysis. Consider the function $G(c) = \sum_{j=0}^{\infty} c_j(\ln(c_j) - 1)$ which has the form of entropy. We first state an elementary result whose proof follows easily from the points made after Definition 4 (see [14]).

Lemma 4. *The function $G(c) = \sum_{j=0}^{\infty} c_j(\ln(c_j) - 1)$ is finite and weak$*$ continuous on X_1^+.*

From the lemma, clearly $G(c)$ is also bounded on the ball $B_\rho = \{x \in X : \|x\|_1 < \rho\}$. Now we define the relative entropy

$$V(c) = G(c) - \sum_{j=1}^{\infty} jc_j \ln(Q_j)^{1/j} = \sum_{j=0}^{\infty} c_j(\ln(\frac{c_j}{Q_j}) - 1).$$

It is assumed throughout the paper that $z_s > 0$ which is equivalent to $\lim_{j\to\infty}(Q_j)^{1/j} < \infty$. Hence $V(c)$ is bounded from below. If we further assume $\liminf (Q_j)^{1/j} > 0$, then $V(c)$ becomes bounded from above also. Next, we define the *modified* relative entropy

$$V_{z,y}(c) = V(c) - \ln z \sum_{j=1}^{\infty} jc_j - \ln y \sum_{j=0}^{\infty} c_j$$

and the set

$$X_1^{+,\rho,\eta} = \{x \in X^+ : \sum_{j=1}^{\infty} jx_j = \rho, \sum_{j=1}^{\infty} x_j = \eta\}.$$

The next theorem shows the relationship between the equilibrium solutions and the minimizers of the relative entropy and modified relative entropy functions.

Theorem 5. *Assume that $z_s(\rho,\eta) < \infty$ and $\rho < \infty$. Then,*

(i) If $0 \le \rho \le \rho_s$, then $c^e(\rho,\eta)$ is the unique minimizer of $V_{z(\rho,\eta),y(\rho,\eta)}$ on X_1^+ and of $V(c)$ on $X_1^{+,\rho,\eta}$. Furthermore, every minimizing sequence c^i of V on $X_1^{+,\rho,\eta}$ converges strongly to $c^e(\rho,\eta)$ in X_1.

(ii) If $\rho_s < \rho < \infty$, then every minimizing sequence c^i of $V_{z_s,y_s}(c)$ on $X_1^{+,\rho,\eta}$ converges weakly to $c^e(\rho_s,\eta)$ but not strongly in X_1 and

$$\inf_{c \in X_1^+} V_{z_s,y_s}(c) = V_{z_s,y_s}(c_s).$$

Proof. For (i), one first observes that the function $c_j \to c_j\left(\ln(\frac{c_j}{Q_j z(\rho,\eta)^j y(\rho,\eta)}) - 1\right)$ has the unique minimum at $c_j = Q_j z(\rho,\eta)^j y(\rho,\eta)$. Hence the function $V_{z(\rho,\eta),y(\rho,\eta)}(c)$ is minimized (over X_1) exactly at the equilibrium distribution $c_j^e(\rho,\eta)$. Clearly,

$c_j^e(\rho,\eta)$ is also the minimizer of $V(c)$ on the set $X_1^{+,\rho,\eta}$. Now, because c_j^i is bounded on X_1^+ and because $c_j^i \to c_j^e$ for each j, one has $c^i \rightharpoonup c^e$. Since, mass of the sequence is constant on the set $X_1^{\rho,\eta}$, then by Lemma 2, one gets $c^i \to c^e$ in X_1.

For the proof of (ii), we take $\rho > \rho_s$. By the first part of the theorem $c^e(\rho_s,\eta)$ is the minimizer of $V_{z_s,y_s}(c)$ in X_1 and hence $V_{z_s,y_s}(c) \geq V_{z_s,y_s}(c^e(\rho_s,\eta))$. Now consider a special sequence $c^i \in X_1^{\rho,\eta}$ defined by

$$c_j^i = c_j^e(\rho_s,\eta) + \delta_{ij}(\frac{\rho-\rho_s}{i}).$$

It is clear that $c^i \rightharpoonup^* c^e(\rho_s,\eta)$. Also, it can be shown by a straightforward computation that $V_{z_s,y_s}(c^i) \to V_{z_s,y_s}(c^e(\rho_s,\eta))$. However the convergence cannot be strong as $\left\|c^i\right\|_1 = \rho > \rho_s = \|c^e(\rho_s,\eta)\|_1$. □

In the sequel, it will be important to know the continuity property of $V_{z,y}(c)$. We have the following.

Proposition 6. *$V_{z,y}(c)$ is weak$*$ continuous on X_1^+ if $\lim_{j\to\infty}(Q_j)^{1/j}$ exists and $z = z_s$.*

Proof. Recall that a function $W(c) = \sum_{j=1}^\infty g_j c_j$ is weak$*$ continuous if and only if $g_j = o(j)$. Recall also that $V_{z,y}(c) = G(c) - \sum_{j=1}^\infty \ln(Q_j)c_j - \ln z \sum_{j=1}^\infty jc_j - \ln y \sum_{j=0}^\infty c_j$. Since $G(c)$ is weak$*$ continuous by Lemma 4, for $V_{z,y}(c)$ to be weak$*$ continuous, one needs $\ln Q_j + j\ln(z) + \ln(y) = o(j)$ or equivalently $\lim_{j\to\infty}\frac{\ln(Q_j z^j y)}{j} = 0$. But this follows if and only if $\lim_{j\to\infty}(Q_j)^{\frac{1}{j}} z = 1$, that is, $z = z_s$. □

Remark: Recall from the earlier discussions that z_s is the radius of convergence of the series $\sum_{k=0}^\infty Q_k z^k$. A more direct way to compute the radius of convergence is the ratio test which gives $z_s = \lim_{k\to\infty}(Q_k/Q_{k+1}) = \lim \frac{b_{k+1}}{a_k}$. So, the behavior of the equilibria (and the conditions for the dynamic phase transition as shown in the next section) is decided by the competition in the tendency of exchange favoring "export" against "import" of monomers ($K(j,k) = b_j a_k$). This leads to following scenarios

(i) $\lim \frac{b_{k+1}}{a_k} = \infty$, (*exporting particle wins over importing and the cluster growth is impeded)*: In this case $z_s = \infty$. Hence, for any initial mass the system can support equilibrium.

(ii) $\lim \frac{b_{k+1}}{a_k} = \alpha > 0$ (*exporting and importing are comparable*): In this case $z_s = \alpha$ and whether or not the system can support an equilibrium depends on the comparison of ρ and $\rho_s = \frac{\sum_{k=1}^\infty kQ_k z_s^k}{\sum_{k=0}^\infty Q_k z_s^k}$. If $\rho > \rho_s$ then there will be no equilibrium.

(iii) $\lim \frac{b_{k+1}}{a_k} = 0$ (*importing particle wins over exporting and clusters grow in time*): In this case $z_s = 0$ and hence there is no equilibrium irrespective of the initial mass.

3.2. **Lyapunov functions and asymptotic behavior.** In this section we show the convergence of solutions to equilibrium in the strong or weak$*$ senses. The approach is similar to [14]. The main instrument is the relative entropy $V(c)$ whose minimization was discussed in the previous section. We anticipate that, evolving in time, $c(t)$ becomes the minimizing sequence for V. It is therefore important to know how V will behave in time.

We first quote a preliminary result from [18] that guarantees the positivity of the cluster densities.

Proposition 7. *Let c^N solve the truncated EDG system (2.1)-(2.4) and $c_j^N(0) > 0$ for some j. Then $c_j^N(t) > 0$ for any $t > 0$.*

Note that the same result holds for the solution $c(t)$ of the original infinite system (1.1)-(1.4). Next we need need the following lemma which will be needed to show that the relative entropy is non-increasing.

Lemma 5. *Let a_j, b_j, c_j be a sequence of non-negative numbers with $j \geq 0$. Let, for a given integer $N \geq 1$, $A^{N+1} = \sum_{j=0}^{N} a_j c_j$ and $B^{N+1} = \sum_{j=1}^{N+1} b_j c_j$. Define $I_j^{N+1} = a_j c_j B^{N+1} - b_{j+1} c_{j+1} A^{N+1}$ for $0 \leq j \leq N$ and zero otherwise. Then one has the inequality*

$$D^N(c) := -\sum_{j=0}^{N+1} (I_{j-1}^{N+1} - I_j^{N+1}) \ln(\frac{c_j}{Q_j}) \geq 0.$$

Proof. We prove this by recursively summing the terms. Let $I_{j-1}^{N+1} - I_j^{N+1} = R_j^{N+1}$. From the definitions, we can relate R_j^N and R_j^{N+1}. For the "lower boundary" term $(j = 0)$,

$$R_0^{N+1} = R_0^N + 0 - (a_0 c_0 b_{N+1} c_{N+1} - b_1 c_1 a_N c_N) \ln(\frac{c_0}{Q_0}), \quad j = 0. \tag{3.5}$$

The middle terms are related by

$$R_j^{N+1} = R_j^N + [(a_{j-1} c_{j-1} b_{N+1} c_{N+1} - b_j c_j a_N c_N) - (a_j c_j b_{N+1} c_{N+1} - b_{j+1} c_{j+1} a_N c_N) \ln(\frac{c_j}{Q_j}). \tag{3.6}$$

The "upper boundary" $j = N, N+1$ are then related by

$$R_N^{N+1} = R_N^N + \left[(a_{N-1} c_{N-1} b_{N+1} c_{N+1} - b_N c_N a_N c_N) - (a_N c_N B^{N+1} - b_{N+1} c_{N+1} A^{N+1})\right] \ln(\frac{c_N}{Q_N}), \tag{3.7}$$

$$R_{N+1}^{N+1} = (a_N c_N B^{N+1} - b_{N+1} c_{N+1} A^{N+1}) \ln(\frac{c_{N+1}}{Q_{N+1}}). \tag{3.8}$$

Now, for adjacent indices $j, j+1$ we combine the second term (in bracket) of j^{th} equation with the first term $(j+1)^{th}$ equation which gives

$$(a_j c_j b_{N+1} c_{N+1} - b_{j+1} c_{j+1} a_N c_N) \ln(\frac{c_{j+1}}{Q_{j+1}} \frac{Q_j}{c_j}). \tag{3.9}$$

Next, we expand the A^{N+1}, B^{N+1} terms in equations (3.7), (3.8) noting that $a_N c_N B^{N+1} - b_{N+1} c_{N+1} A^{N+1} = a_N c_N B^N - b_{N+1} c_{N+1} A^N$. Combining the $(j+1)^{th}$ and terms in (3.7), (3.8) (inside the bracket) we get

$$(a_N c_N b_{j+1} c_{j+1} - b_{N+1} c_{N+1} a_j c_j) \ln(\frac{c_{N+1}}{Q_{N+1}} \frac{Q_N}{c_N}). \tag{3.10}$$

Now, summing over the index j and recalling $\frac{Q_{j+1}}{Q_j} = \frac{a_j}{b_{J+1}}$ the desired sum in the statement of the lemma can be written as

$$\begin{aligned}\sum_{j=0}^{N+1} R_j^{N+1} \ln(\frac{c_j}{Q_j}) &= \sum_{j=0}^{N} R_j^N \ln(\frac{c_j}{Q_j}) - \sum_{j=0}^{N+1} (a_j c_j b_{N+1} c_{N+1} - b_{j+1} c_{j+1} a_N c_N) \ln(\frac{c_{N+1} b_{N+1} a_j c_j}{c_N c_N b_{j+1} c_{j+1}}) \\ &\leq \sum_{j=0}^{N} R_j^N \ln(\frac{c_j}{Q_j}).\end{aligned}$$

The second line followed since $(x-y)\ln(\frac{x}{y})>0$ for any real number pairs $x,y\geq 0$. Repeating the arguments for $j\leq N$ and reducing the index number we find

$$\sum_{j=0}^{N+1} R_j^{N+1}\ln(\frac{c_j}{Q_j})\leq (a_0c_0-b_1c_1)\ln(\frac{c_1}{c_0}\frac{b_1}{a_0})\leq 0.$$

which completes the proof. ☐

Theorem 6. *Let $a_j, b_j = O(j/\ln j)$ and $c_j(t)$ be the solution of (1.1)-(1.4). Assume that $c_j(0)>0$ for some j and $0<\lim_{j\to\infty}(Q_j)^{1/j}<\infty$ holds. Then*

$$V(c(t)) = V(c(0)) - \int_0^t D(c(s))ds, \tag{3.11}$$

where $D(c)\geq 0$ and is given by

$$D(c) := \sum_0^{\infty}(a_jc_jB - b_{j+1}c_{j+1}A)\ln(\frac{a_jc_j}{b_{j+1}c_{j+1}}). \tag{3.12}$$

Proof. Consider the truncated sum

$$V^N(c) = \sum_{j=0}^{N} c_j(\ln(\frac{c_j}{Q_j})-1).$$

Let $I_N(c) = a_Nc_NB(c) - b_{N+1}c_{N+1}A(c)$. Differentiating $V^N(c)$ we get

$$\dot{V}^N(c) = \sum_{j=0}^{N}\dot{c}_j\ln(\frac{c_j}{Q_j}) = -D^{N-1}(c) - I_N(c)\ln(\frac{c_N}{Q_N}) \tag{3.13}$$

$$= -D^N(c) - I_N(c)\ln(\frac{c_{N+1}}{Q_{N+1}}), \tag{3.14}$$

where $D^{N-1}(c) = \sum_{j=0}^{N-1}(a_jc_jB - b_{j+1}c_{j+1}A)\ln(\frac{a_jc_j}{c_{j+1}b_{j+1}})$. Since $-I_N\ln c_N \leq -Ba_Nc_N\ln(c_N)$ and $-I_N\ln c_{N+1}\geq Ab_{N+1}c_{N+1}\ln(c_{N+1})$ rearranging (3.13) gives

$$-\dot{V}^N(c)\geq D^{N-1}(c) + Ba_Nc_N\ln(c_N) - I_N(c)\ln(Q_N) \tag{3.15}$$

$$-\dot{V}^N(c)\leq D^N(c) - Ab_{N+1}c_{N+1}\ln(c_{N+1}) - I_N(c)\ln(Q_{N+1}).$$

We first observe that, since $a_j\leq C\frac{j}{\ln(j)}$, then $a_Nc_N\ln(c_N)$ and $b_{N+1}c_{N+1}\ln(c_{N+1})$ go to zero uniformly as $N\to\infty$. Indeed, one has

$$|a_Nc_N\ln(c_N)|\leq |a_Nc_N\ln(Nc_N)| + |a_Nc_N\ln(N)|$$

and the first tern on the right hand side above goes to zero since $Nc_N\to 0$ for $N\to\infty$. The second term goes to zero by the assumption in the theorem (the bound on a_N). Similarly one finds $\lim_{N\to\infty} b_{N+1}c_{N+1}\ln(c_{N+1}) = 0$. Now, integrating the first inequality in (3.15) we have

$$\int_0^t D^{N-1}(c)ds + \int_0^t Ba_Nc_N\ln(c_N)ds - \int_0^t I_N(c)\ln(Q_N)ds \leq V^N(c(0)) - V^N(c(t)) \tag{3.16}$$

Now, we already showed that the second term on the left hand side of 3.16 goes to zero for large N. For the third term, using the Lemma 3 and noting $0<\lim_{N\to\infty}(Q_N)^{1/N}\leq C$, one has

$$\int_0^t I_N(c)\ln(Q_N)ds\to 0 \quad \text{as } N\to\infty.$$

Repeating these arguments for the term after the second inequality in (3.15) we find

$$\int_0^t D^{N-1}(c)ds + o(1) \le V^N(c(0)) - V^N(c(t)) \le \int_0^t D^N(c)ds + o(1).$$

Finally, using monotone convergence theorem the result follows. □

Remark: By adding and subtracting the term $AB\ln(\frac{B}{A})$ to $D(c)$ one can put $D(c)$ in a more convenient

$$D(c) := \sum_0^\infty (a_j c_j B - b_{j+1}c_{j+1}A)\ln(\frac{a_j c_j B}{b_{j+1}c_{j+1}A})$$

where each term in the summation is non-negative.

For the integral equality (3.11) the bounds on the export and import rates $a_j, b_j = O(j/\ln j)$ were needed while they are not essential for the well posedness as discussed in Section 2. It would be nice, therefore, to have a similar result for $V(c)$ in the more general case when $a_j, b_j = O(j)$. The following corollary provides that.

Corollary 1. *Let $a_j, b_j = O(j)$. Let $c_j(t)$ be a solution of (1.1)-(1.4) as in Theorem 1. Assume that $c_j(0) > 0$ and $0 < \lim_{j\to\infty}(Q_j)^{1/j} < \infty$ holds. Then*

$$V(c(t)) \le V(c(0)) - \int_0^t D(c(s))ds. \tag{3.17}$$

Proof. Take the truncated system (2.1)-(2.4) and the approximation V^N

$$V^N(c^N(t)) = V(c^N(0)) - \int \sum_{j=0}^{N-1}(a_j c_j^N B^N(c^N) - b_{j+1}c_{j+1}^N A^N(c^N))\ln(\frac{c_{j+1}^N b_{j+1}}{a_j c_j^N}). \tag{3.18}$$

Fix $n \in \mathbb{N}$ and consider the subsequence $N(k) > n$ which converges to the solution of the original EDG system. By Lemma 5 $D^{N(k)-1}(c^{N(k)}) \ge D^n(c^{N(k)})$. Then, since n is finite one has

$$\liminf \int D^{N(k)-1}(c^{N(k)}) \ge \int D^n(c).$$

Also, by Lemma 4 and condition $0 < \lim_{j\to\infty}(Q_j)^{1/j} < \infty$ one has

$$V(c) \le \lim_{N(k)\to\infty} \inf V(c^{N(k)}(t)).$$

Lastly, one has $V(c^{N(k)}(0)) \to V(c(0))$ and we arrive at

$$\begin{aligned} V(c(t)) &\le \lim_{N(k)\to\infty} \inf V(c^{N(k)}(t)) \\ &= \lim_{N(k)\to\infty} V(c^{N(k)}(0)) - \lim_{N(k)\to\infty} \inf \int D^{N(k)-1}(c^{N(k)}(s))ds \\ &\le V(c(0)) - \int_0^t D^n(c(s))ds. \end{aligned}$$

Passing to the limit $n \to \infty$ yields the result. □

For the asymptotic behavior we study the positive orbit of the flow $O^+(\phi) = \cup_{t\ge 0}\phi(t)$ where $\phi(t) = T(t)c(0)$. We define the $\omega-$limit set by $\omega(\phi) = \{x \in X : \phi(t_j) \to x$ for some sequence $t_j\}$. Also, we say that the set $S \subset O^+(\phi)$ is quasi-invariant iff for $\phi(0) \in S$, $\phi(t) \in S$ for every $t \ge 0$. We quote the following result from the general theory which is standard.

Proposition 8. *Suppose that $O^+(\phi)$ is relatively compact. Then $\omega(\phi)$ is non empty, quasi-invariant and* $\lim_{t\to\infty} dist(\phi(t), \omega(\phi)) = 0$.

We can now prove the main theorems of this section. In the sequel let c^ρ denote $c^e(\rho,\eta)$ for brevity. The first theorem below shows the weak$*$ convergence under fairly general conditions.

Theorem 7. *Consider the system (1.1)-(1.4) with $K(j,k) = b_j a_k$. Let $a_j, b_j = O(j/\ln j)$ for large j. Let the initial density be given $\rho_0 = \sum_{k=1}^\infty kc_k(0) < \infty$ and assume also that $\lim_{j\to\infty} \frac{b_{j+1}}{a_j} = z_s$ $(0 < z_s < \infty)$. Then $c(t) \rightharpoonup^* c^\rho$ for some ρ with $0 \le \rho \le \min(\rho_0, \rho_s)$.*

Proof. The EDG system under the conditions of the theorem generates a generalized flow on B^+_ρ. Consider the function $V_{z_s,y_s}(c)$. From Proposition 6 it is continuous on $B^+_{\rho_0}$. Also since total mass density $\sum_{k=1}^\infty kc_k(t)$ and total number densities $\sum_{k=1}^\infty c_k(0)$ are conserved by Theorem 6 we have

$$V_{z_s,y_s}(c(t)) = V_{z_s,y_s}(c(0)) - \int_0^t D(c(s))ds$$

Boundedness of $\sum_{k=1}^\infty kc_k(t)$ also implies that $O^+(c)$ is relatively compact in B_{ρ_0}. By the invariance principle $\omega(c)$ is non empty and consists of points $V_{z_s,y_s}(c) = const$ which implies that, for any element in $\bar{c} \in \omega(c)$, $D(\bar{c}) = 0$ and hence $\bar{c}$ has the form $\bar{c}_r = Q_j \left(\frac{B(\bar{c})}{A(\bar{c})}\right)^j \bar{c}_0(t)$ for some $\bar{c}(t) \in \omega(c)$. But, this is exactly the form of equilibrium solutions. Since the mass density cannot increase and an equilibrium is admissible, at most, up to the critical mass ρ_s, it follows that $\omega(c)$ consists of equilibria $c^{\rho,\eta}$ with $0 < \rho \le \min(\rho_0, \rho_s)$. Actually, ω consists of a single point since $V_{z_s,y_s}(c^\rho) = \sum_{j\ge1} c_j^\rho \ln(\frac{c_j^\rho}{Q_j z_s^j y_s}) - \sum_{j\ge1} c_j^\rho$is strictly decreasing in ρ as can be seen from direct computation

$$\frac{dV_{z_s,y_s}(c^\rho)}{d\rho} = \frac{d}{d\rho}\left(\rho \ln(\frac{z(\rho)}{z_s})\right) + \frac{d\ln(y(\rho))}{d\rho} = \ln(\frac{z(\rho)}{z_s})$$

where we used conservation of mass and volume multiple times. Then by Proposition 8 $dist(c(t), c^\rho) \to 0$ as $t \to \infty$, completing the proof. □

We can strengthen the theorem for the subcritical case by making further assumptions on the strength of "export" tendency over the "import" in the system. More precisely, let

$$(H1)\ \lim_{j\to\infty} \frac{a_j}{b_{j+1}} = 0 \tag{3.19}$$

hold. Then we can prove the following strong convergence result.

Theorem 8. *Let $c_j(t)$ solve the system (1.1)-(1.4) as in Theorem 3 with an initial mass $\rho_0 = \sum_{j=0}^\infty jc_j(0)$. Assume that $a_j, b_j = O(j)$ and (3.19) (Hypothesis H1) holds. Then $c(t) \to c^{\rho_0}$ strongly in X_1.*

Proof. $H1$ implies that the radius of convergence of the series $z_s = \infty$ which is equivalent to $\lim_{j\to\infty}(Q_j)^{1/j} = 0$. By the monotonicity of $V(c)$ one has $V(c(t)) \le V(c(0))$. Also, by Proposition 4 $\sum_{j=0}^\infty c_j(t)\ln(c_j) < \infty$. Hence we have

$$-\sum_{j=0}^\infty jc_j(t)\ln(Q_j)^{1/j} \le C.$$

Since $-\ln\left((Q_j)^{1/j}\right) \to \infty$ by $H1$, it follows that $O^+(c)$ is relatively compact in X_1. By the invariance principle the limit set has the form $\bar{c}_j(t) = Q_j\left(\frac{B(\bar{c})}{A(\bar{c})}\right)^j \bar{c}_0(t)$ where $\bar{c}_j(0) = \lim_{t_i\to\infty} c_j(t_i)$ for some sequence t_i. Hence $\bar{c}_j(t)$ has the form of equilibrium solutions. By the conservation of number and mass density in time, i.e., $\sum_{j=0}^{\infty}\bar{c}_j(t) = \eta$, $\sum_{j=1}^{\infty} j\bar{c}_j(t) = \rho_0$ and the uniqueness of equilibrium solutions, one concludes that $\omega(c)$ consists of a single point, that is, the equilibrium solutions that correspond to the pair (ρ_0, η). By Proposition 8 $c(t)$ converges strongly to c^{ρ_0}. □

If the *exporting* and *importing* tendencies are comparable as in Remark 1 Case (ii), then the above argument does not work and we need extra conditions to secure the strong convergence. We will need to control the moments of the initial distribution and crucially make use of a uniform comparison of b_j, a_j which will replace $(H1)$ i.e.,

$$(H2)\quad \frac{b_j}{a_j} \geq z_s \quad \text{for } j \geq 1. \tag{3.20}$$

Theorem 9. *Let $c_j(t)$ solve the system (1.1)-(1.4) and $\rho_0 \leq \rho_s$. Let $b_j \geq Cj^{\lambda}$ $(-1 \leq \lambda < 1)$ and $\sum_{j=0}^{\infty} j^p c_j(0) < \infty$ for some $p > 2-\lambda$. Assume further that (3.20) (Hypothesis H2) holds and $a_j, b_j = O(j/\ln j)$. Then $c(t) \to c^{\rho}$ strongly in X_1.*

Proof. The main line of argument, as in the previous theorem, is to show that $O^+(c)$ is relatively compact in X_1. This will follow by showing that $M_m(t) < C$ for some $m > 1$. Consider the p^{th} moment of the system $M_p := \sum_{j=0}^{\infty} j^p c_j(t)$ with $(2-\lambda < p \leq 2)$. By Theorem 1, $M_p(t) < \infty$ for any $t < \infty$. Now, choose $m < p$ such that $m > 2-\lambda$ still holds. By Lemma 1, one has

$$\dot{M}_m = \sum_{j\geq 1}((j-1)^m - j^m)b_j c_j A + \sum_{j\geq 0}((j+1)^m - j^m)a_j c_j B.$$

Taylor expanding the $(j-1)^m$ and $(j+1)^m$ terms up to second order we find

$$\dot{M}_m \leq -\sum_{j\geq 1} m j^{m-1} b_j c_j A + \sum_{j\geq 1} m j^{m-1} a_j c_j B + CAB.$$

Note that A, B depend on time. By Theorem 7 $d(c(t), c^e) \to 0$, or in particular $c(t) \to^* c^e$ as $t \to \infty$. Then one has $\lim_{t\to\infty}\sum_{j\geq 1} g_j c_j(t) \to \sum_{j\geq 1} g_j c_j^e$ for any $g_j = o(j)$. Therefore, since $a_j, b_j = o(j)$, it follows that $\frac{B(c(t))}{A(c(t))} \to \frac{B(c^e)}{A(c^e)} = z(\rho) < z_s$ as $t \to \infty$.

Now, since $\frac{b_j}{a_j} \geq z_s$ by the assumption in the theorem, there is a t_* and $\delta > 0$ such that $-\frac{A(c(t))}{B(c(t))} + \frac{a_j}{b_j} \leq -\delta$ for $t > t_*$ and

$$\dot{M}_m \leq C + m\sum_{j\geq 1}(-b_j c_j A + a_j c_j B) j^{m-1} \leq C + m\sum_{j\geq 1}(-\frac{A}{B} + \frac{a_j}{b_j}) j^{m-1} B b_j c_j.$$

By the fact that $B(c(t)) > \varepsilon$ for some $\varepsilon > 0$ for t_* large enough (since $\lim_{t\to\infty} B(c(t) = B(c^e) > 0$) and the condition $b_j \geq Cj^{\lambda}$ $(-1 \leq \lambda < 1)$ we find

$$\dot{M}_m \leq C - C\delta\varepsilon\sum_{j\geq 1} j^{m-1+\lambda} c_j.$$

Integrating both sides and noting $M_{m-1+\lambda} \leq M_m$ we get

$$M_{m-1+\lambda}(t) \leq C(t-t_*) + M_m(t_*) - C\int_{t_*}^{t} M_{m-1+\lambda}(s)ds.$$

Comparing this to the solution of $x(t) = C(t-t_*) + M_m(t_*) - C\int_{t_*}^{t} x(s)ds$ we find $M_{m-1+\lambda}(t) \leq C$ for all $t > t_*$. Since $m > 2-\lambda$ by our choice, it follows that the tail of the distribution jc_j uniformly approaches to zero giving the relative compactness of the orbit in X_1. Arguing similarly as in Theorem 8 one sees that $c(t) \to c^{\rho_0}$ strongly. □

Remark: Without essentially changing the proof, the hypothesis $(H2)$ could be replaced with $\frac{b_j}{a_j} < z_s$ for finitely many j values.

4. Convergence to equilibrium without detailed balance.

In this section we extend the study of convergence of time dependent solutions to equilibrium without imposing a structure condition on the equilibria. Our goal is to obtain explicit convergence rates to equilibrium. We assume, throughout this section, the following

$$(H3) \quad K(j,k) = ja_k + b_j + \varepsilon\beta_j\alpha_k \text{ with } a_j \geq \tilde{a} \text{ for some } \tilde{a} > 0. \tag{4.1}$$

Depending on the type of assumptions for the system we obtain two different convergence results. Each result relies on a key contraction property of the time dependent solution. The first contraction property is a consequence of the monotonicity of the a_j, b_j functions which leads to exponentially fast convergence in the "weak" metric ($dist(c,d) = \|c-d\|_0 = \sum_{j\geq 0}|c_j - d_j|$). The second contraction property follows from the total mass of the system being sufficiently small and is used to show exponentially fast convergence in the "strong" metric ($\|c-d\|_1 = \sum_{j\geq 1} j\,|c_j - d_j|$). Such a contraction property was first shown to hold for the coagulation-fragmentation systems under a similar small mass assumption [24].

4.1. Exponentially fast weak convergence to equilibrium.

Here our approach is partly motivated by that, in the EDG equations, a_j represents the import rate of particles (and hence causes growth of clusters) and b_j represents the export rate (and hence causes breakdown of clusters). In this interpretation one would expect that for monotonically increasing b_j (in j) and monotonically decreasing a_j the dynamics favor the approach to equilibrium which would be manifested in the convergence rates. The following theorem supports this interpretation.

Theorem 10. *Consider the EDG system (1.1)-(1.4). Let the hypotheses of Theorem 1 be satisfied with $M_1(0) = \rho$ and $M_p(0) < \infty$ (for some $p > 2$). Let the kernel have the form in (4.1) (Hypothesis H3) with a_j non-increasing, b_j non-decreasing, α_j, β_j bounded and $\varepsilon > 0$ small. Then the solutions of (1.1)-(1.4) converge to a unique equilibrium in the sense that*

$$\sum_{j\geq 1} \left|c_j(t) - c_j^e\right| \leq 4\rho e^{-\gamma t}, \tag{4.2}$$

where $\gamma > 0$ can be computed explicitly (and depends on $\tilde{a}, \varepsilon$ and the bounds of α_j, β_j).

The main idea of the theorem (covered in the next lemma) is based on defining an appropriate time dependent quantity which is positive and measures the distance between two solutions (of the same mass) and showing that this distance contracts in time, i.e., the two solutions approach to each other. It will then be shown that the limit solution is actually an equilibrium.

To prove the contraction, one focuses on the evolution of the tail of the distributions defined by $C_j(t) = \sum_{k\geq j} c_k(t)$. This approach proved useful in studying Becker-Doring systems [14],[23],[26] and were also recently adopted to prove some of the key properties of the EDG system such as nonexistence of solutions [18] and uniqueness without additional moment assumptions [19].

Lemma 6. *Under the conditions of Theorem 1 consider two solutions c_j, d_j of the system (1.1)-(1.4) with the same initial mass and the same initial volume. Let $M_p(0) < \infty$ (for some $p > 2$) and (4.1) hold (Hypothesis $H3$). Assume further that a_j is non-increasing, b_j non-decreasing, α_j, β_j are bounded and $\varepsilon > 0$ is small enough. Then the solutions approach to each other exponentially fast as*

$$\sum_{j\geq 1} |c_j(t) - d_j(t)| \leq 4\rho e^{-\gamma t}.$$

Proof. We first consider the dynamics for C_j, the tail of $(c_j)_{j=1}^{\infty}$. By direct computation the evolution equation for C_j is

$$\begin{aligned}\dot{C}_j &= \sum_{k\geq 1} K(k, j-1)c_k c_{j-1} - \sum_{k\geq 0} K(j,k)c_k c_j \\ &= \sum_{k\geq 1}(ka_{j-1} + b_k)c_k c_{j-1} - \sum_{k\geq 0}(ja_k + b_j)c_k c_j \\ &+ \varepsilon \sum_{k\geq 1} \beta_k \alpha_{j-1} c_k c_{j-1} - \varepsilon \sum_{k\geq 0} \beta_j \alpha_k c_k c_j.\end{aligned}$$

Taking the sum over "k" and denoting, as before, $A(c) = \sum_{j\geq 0} a_j c_j$, $B(c) = \sum_{j\geq 1} b_j c_j$ and defining $\tilde{A}(c) = \sum_{j=0}^{\infty} \alpha_j c_j$, $\tilde{B}(c) = \sum_{j=1}^{\infty} \beta_j c_j$ one gets

$$\dot{C}_j = \rho a_{j-1} c_{j-1} + B(c)c_{j-1} - jc_j A(c) - b_j c_j + \varepsilon \alpha_{j-1} c_{j-1} \tilde{B}(c) - \varepsilon \beta_j c_j \tilde{A}(c),$$

where we used $\rho = \sum_{j\geq 1} jc_j$ and $1 = \sum_{j\geq 0} c_j$. Similarly, for the other solution d_j, one has

$$\dot{D}_j = \rho a_{j-1} d_{j-1} + B(d)d_{j-1} - jd_j A(d) - b_j d_j + \varepsilon \alpha_{j-1} d_{j-1} \tilde{B}(d) - \varepsilon \beta_j d_j \tilde{A}(d).$$

Now define the difference of terms $e_j = c_j - d_j$ and $E_j = C_j - D_j$. One can write

$$\begin{aligned}\dot{E}_j &= \rho a_{j-1} e_{j-1} + (B(c)c_{j-1} - B(d)d_{j-1}) - (jc_j A(c) - jd_j A(d)) - b_j e_j \\ &+ \varepsilon \alpha_{j-1}(\tilde{B}(c)c_{j-1} - \tilde{B}(d)d_{j-1}) - \varepsilon \beta_j (c_j \tilde{A}(c) - d_j \tilde{A}(d)).\end{aligned}$$

Then, since $e_j = E_j - E_{j+1}$, for the difference terms in the parenthesis, one can write

$$c_j A(c) - d_j A(d) = e_j A(c) + d_j(A(c) - A(d)) = (E_j - E_{j+1})A(c) + d_j(A(c) - A(d)),$$

$$B(c)c_{j-1} - B(d)d_{j-1} = e_{j-1}B(c) + d_{j-1}(B(c) - B(d)) = (E_{j-1} - E_j)B(c) + d_{j-1}(B(c) - B(d)).$$

Denoting, for brevity, $A(c) - A(d) = A(e)$ and $B(c) - B(d) = B(e)$ (similarly for $\tilde{A}, \tilde{B}$) we find that the tail of the difference of solutions evolves according to

$$\begin{aligned}\dot{E}_j &= \rho a_{j-1}(E_{j-1} - E_j) + (E_{j-1} - E_j)B(c) + d_{j-1}B(e) \\ &\quad - j(E_j - E_{j+1})A(c) - jd_jA(e) - b_j(E_j - E_{j+1}) \\ &\quad + \varepsilon\alpha_{j-1}\left(\tilde{B}(c)(E_{j-1} - E_j) + d_{j-1}\tilde{B}(e)\right) - \varepsilon\beta_j\left(\tilde{A}(c)(E_j - E_{j+1}) + d_j\tilde{A}(e)\right).\end{aligned}$$

Now we show that the tail of the difference, E_j, goes to zero. For this purpose, consider the absolute value of the tail density $|E_j|$. Taking the time derivative we get

$$\frac{d\,|E_j|}{dt} = sgn(E_j)\dot{E}_j$$

$$\begin{aligned}&= sgn(E_j)\left(\rho a_{j-1}(E_{j-1} - E_j) + (E_{j-1} - E_j)B(c) + d_{j-1}B(e)\right) \\ &+ sgn(E_j)\left(-j(E_j - E_{j+1})A(c) - jd_jA(e) - b_j(E_j - E_{j+1})\right) \\ &+ \varepsilon sgn(E_j)\left(\alpha_{j-1}\left(\tilde{B}(c)(E_{j-1} - E_j) + d_{j-1}\tilde{B}(e)\right) - \beta_j\left(\tilde{A}(c)(E_j - E_{j+1}) + d_j\tilde{A}(e)\right)\right).\end{aligned}$$

Since $sgn(E_j)E_j = |E_j|$ and $E_{j\pm1} \leq |E_{j\pm1}|$, summing over j in both sides gives

$$\sum_{j=1}^{\infty}\frac{d\,|E_j|}{dt} \leq \sum_{j=1}^{\infty}\left(\rho a_{j-1}(|E_{j-1}| - |E_j|) + (|E_{j-1}| - |E_j|)B(c) + d_{j-1}\,|B(e)|\right) \tag{4.3}$$

$$+ \sum_{j=1}^{\infty}\left(j(|E_{j+1}| - |E_j|)A(c) + jd_j\,|A(e)| + b_j(|E_{j+1}| - |E_j|)\right) \tag{4.4}$$

$$+ \varepsilon\alpha_{j-1}\left(\tilde{B}(c)(|E_{j-1}| - |E_j|) + d_{j-1}\tilde{B}(e)\right) + \varepsilon\beta_j\left(\tilde{A}(c)(|E_{j+1}| - |E_j|) + d_j\tilde{A}(e)\right). \tag{4.5}$$

Now, let S_1, S_2, S_3 denote the sum of the three sums on the right hand side of (4.3), S_4, S_5, S_6 denote the three sums in (4.4) and S_7, S_8, S_9, S_{10} denote the four terms in (4.5). We treat each S_j separately. For the first term, we have

$$S_1 = \rho\sum_{j=1}^{\infty}a_{j-1}(|E_{j-1}| - |E_j|)\rho = \rho a_0\,|E_0| + \rho\sum_{j=1}^{\infty}(a_j - a_{j-1})\,|E_j|$$

where the term $a_0\,|E_0|$ is zero by the conservation of total volume, that is, $E_0 = \sum_{j=0}^{\infty}c_j - \sum_{j=0}^{\infty}d_j = 0$. For the second term S_2 we find

$$S_2 = B(c)\sum_{j=1}^{\infty}(|E_{j-1}| - |E_j|) = 0.$$

For S_3 we first observe, since $\sum_{j=1}^{\infty}d_{j-1} = 1$ (total volume),

$$S_3 = \sum_{j=1}^{\infty}d_{j-1}\,|B(e)| = |B(e)|\,,$$

while $|B(e)|$ can be written as

$$\begin{aligned}|B(e)| &= \left|\sum_{j=1}^{\infty}b_je_j\right| = \left|\sum_{j=1}^{\infty}b_j(E_j - E_{j+1})\right| \leq b_1\,|E_1| + \left|\sum_{j=2}^{\infty}(b_j - b_{j-1})E_j\right| \\ &\leq b_1\,|E_1| + \sum_{j=2}^{\infty}|b_j - b_{j-1}|\,|E_j|\,.\end{aligned}$$

Next we compute the terms in (4.4), S_4, S_5, S_6. For the S_4 term we find

$$S_4 = \sum_{j=1}^{\infty} j(|E_{j+1}| - |E_j|)A(c) = -A(c)\,|E_1| - A(c)\sum_{j=2}^{\infty}(j-1-j)\,|E_j| = -A(c)\sum_{j=1}^{\infty}|E_j|\,.$$

The S_5 term reads

$$\sum_{j=1}^{\infty} jd_j\,|A(e)| = \rho\,|A(e)|$$

where the $|A(e)|$ term can be, using $E_0 = 0$, written as

$$|A(c) - A(d)| = \left|\sum_{j=0}^{\infty} a_j e_j\right| = \left|\sum_{j=0}^{\infty} a_j(E_j - E_{j+1})\right| \le a_0\,|E_0| + \left|\sum_{j=1}^{\infty}(a_j - a_{j-1})E_j\right|$$
$$\le \sum_{j=1}^{\infty}|a_j - a_{j-1}|\,|E_j|\,.$$

Next, shifting the indices, the S_6 term can be written as

$$\sum_{j=1}^{\infty} b_j(-|E_j| + |E_{j+1}|) = -b_1\,|E_1| + \sum_{j=2}^{\infty}(b_{j-1} - b_j)\,|E_j|\,.$$

Now, we notice that, by the non-increasing property of a_j, $a_j - a_{j-1} = -|a_j - a_{j-1}|$ and hence S_1 and S_5 are opposite of each other and cancel out. Similarly, by the non-decreasing property of b_j, $b_{j-1} - b_j = -|b_j - b_{j-1}|$ and therefore S_3 and S_6 also cancel each other in the sum. Then, since $S_2 = 0$ by computation, we are left with the following

$$\sum_{j=1}^{\infty}\frac{d\,|E_j|}{dt} \le -A(c)\sum_{j=1}^{\infty}|E_j| + S_7 + S_8 + S_9 + S_{10}. \tag{4.6}$$

Finally, we treat the $S_7, ..., S_{10}$ terms. Setting $\beta_0 = 0$ and repeating the manipulations done for $S_1, ..., S_6$ we find

$$S_7 + S_8 \le \varepsilon\tilde{B}(c)\sum_{j=1}^{\infty}|\alpha_j - \alpha_{j-1}|\,|E_j| + \varepsilon\tilde{A}(d)\sum_{j=1}^{\infty}|\beta_j - \beta_{j-1}|\,|E_j|\,,$$

$$S_9 + S_{10} \le \varepsilon\tilde{A}(c)\sum_{j=1}^{\infty}|\beta_j - \beta_{j-1}|\,|E_j| + \varepsilon\tilde{B}(d)\sum_{j=1}^{\infty}|\alpha_j - \alpha_{j-1}|\,|E_j|\,.$$

Now, since $|\alpha_j - \alpha_{j-1}|\,, |\beta_j - \beta_{j-1}| \le 2L$ for some $L > 0$, we have

$$S_7 + S_8 + S_9 + S_{10} \le 8\varepsilon L^2\sum_{j=1}^{\infty}|E_j|\,.$$

Adding all terms in (4.6) and using $A(c) = \sum_{j=0}^{\infty} a_j c_j \ge \sum_{j=0}^{\infty}\tilde{a}c_j = \tilde{a}$ gives

$$\sum_{j=1}^{\infty}\frac{d\,|E_j|}{dt} \le -\tilde{a}\sum_{j=1}^{\infty}|E_j| + \varepsilon\left(8L^2\sum_{j=1}^{\infty}|E_j|\right) \le -(\tilde{a} - 8L^2\varepsilon)\sum_{j=1}^{\infty}|E_j|\,.$$

By Lemma 3 it can be seen that $\sum_{j=1}^{\infty}\frac{d|E_j|}{dt} = \frac{d}{dt}\left(\sum_{j=1}^{\infty}|E_j|\right)$ from which we get $\sum_{j=1}^{\infty}|E_j(t)| \le \sum_{j=1}^{\infty}|E_j(0)|\,e^{-(\tilde{a}-\varepsilon 8L^2)t}$. To finish the proof we observe

$$|e_j| \le |E_j| + |E_{j+1}|\,,$$

and then taking the sum we arrive at

$$\sum_{j=0}^{\infty} |e_j| \leq 2 \sum_{j=1}^{\infty} |E_j| \leq 2 \left(\sum_{j=1}^{\infty} |E_j(0)| \right) e^{-(\tilde{a}-8\varepsilon L^2)t}. \tag{4.7}$$

Finally, we recall that $E_j(0) = \sum_{k \geq j} e_k(0)$ and the sum $\sum_{j=1}^{\infty} |E_j(0)|$ can be bounded as

$$\begin{aligned} \sum_{j=1}^{\infty} |E_j(0)| &\leq \sum_{j=1}^{\infty} \sum_{k \geq j} |e_k(0)| = \sum_{k=1}^{\infty} \sum_{j=1}^{k} |e_k(0)| = \sum_{k=1}^{\infty} k\, |e_k(0)| \\ &\leq \sum_{k=1}^{\infty} k(c_k + d_k) \leq 2\rho, \end{aligned}$$

where in the first line we changed the order of summation. Using this in (4.7) completes the proof. □

Although the result is obtained only for non-decreasing b_j and non-increasing a_j, the involvement of the monotonicity gives a clear sign that the result should generalize (see the Conclusion section).

As a consequence of this lemma, all solutions having the same mass go to the equilibrium solution exponentially fast which is embedded in Proposition *9*. Next, we show the following lemma which uniformly bounds the moments.

Lemma 7. *Let the conditions of Theorem 1 be satisfied with $M_p(0) < \infty$ ($p > 2$). Then for any solution of (1.1)-(1.4) one has $M_p(t) < \infty$ for $t \geq 0$.*

Proof. We make the proof for $n = 2$ and the general proof is inferred by induction. Consider the truncated system *(2.1)-(2.4)*. Using Lemma 1 we have

$$\begin{aligned} \dot{M}_2^N(t) = &\sum_{j=0}^{N-1} \sum_{k=1}^{N} ((j+1)^2 - j^2)(ka_j + b_k + \varepsilon \beta_k a_j) c_j^N c_k^N \\ &+ \sum_{j=1}^{N} \sum_{k=0}^{N-1} ((j-1)^2 - j^2)(ja_k + b_j + \varepsilon \beta_j a_k) c_j^N c_k^N. \end{aligned}$$

Expanding the terms in the parenthesis we find

$$\begin{aligned} \dot{M}_2^N(t) \leq &\sum_{j=0}^{N-1} (2j+1) a_j c_j^N \rho^N + \sum_{j=0}^{N-1} (2j+1) c_j^N B^N + \varepsilon \sum_{j=0}^{N-1} (2j+1) \alpha_j \tilde{B}^N c_j^N \\ &+ \sum_{j=1}^{N} (1-2j) j A^N c_j^N + \sum_{j=1}^{N} (1-2j) b_j c_j^N + \varepsilon \sum_{j=1}^{N} (1-2j) \beta_j \tilde{A}^N c_j^N, \end{aligned}$$

where $\rho^N = \sum_{j=1}^{N} j c_j^N$, $A^N = \sum_{j=0}^{N-1} a_j c_j^N$, $B^N = \sum_{j=1}^{N} b_j c_j^N$ (and similarly for $\tilde{A}^N, \tilde{B}^N$). Using $0 < \tilde{a} \leq a_j \leq a_0$, $b_1 \leq b_j \leq \bar{b} j$ and the bound $\alpha_j, \beta_j \leq L$ we get the inequality

$$\begin{aligned}\dot{M}_2^N(t) &\le (2\rho+1)a_0\rho + \bar{b}(2\rho+1)\rho + \varepsilon L^2(2\rho^N+1) \\ &\quad + \rho^N a_0 - 2\tilde{a}(\sum_{j=1}^{N} c_k^N(0))M_2^N(t) + (\bar{b} - 2b_1\rho^N) + \varepsilon(L^2 - 2\beta_{\min}\alpha_{\min}(\sum_{j=1}^{N} c_k^N(0))\rho^N) \\ &\le C(\rho, a_0, b_1, \bar{b}, \beta_{\min}, L, \varepsilon) - 2\tilde{a}(\sum_{j=1}^{N} c_k^N(0))2M_2^N(t).\end{aligned} \tag{4.8}$$

where, in the second line, we used the conservation of volume for the truncated system $\sum_{j=1}^{N} c_k^N(t) = \sum_{j=1}^{N} c_k(0)$ and chose $N > N^*$ large enough such that $\sum_{j=1}^{N^*} c_k^N(0) > 0$. By Gronwall inequality we see that $M_2^N(t)$ is uniformly bounded. Hence we can pass to the limit $N \to \infty$ and hence

$$M_2(\infty) \le C(\rho, a_0, b_{\min}, \bar{b}, \beta_{\min}, L, \varepsilon)/(2\tilde{a}\sum_{j=1}^{N^*} c_k(0)).$$

By induction and following similar steps of computations it can easily be shown that $M_n(t)$ is finite for any $n > 2$. □

Now we can show the existence of equilibrium solutions.

Proposition 9. *Let the hypotheses of Theorem 1 be satisfied with $M_p(0) < \infty$ ($p > 2$) and (4.1) (hypothesis H3) hold. Assume further that a_j is non-increasing, b_j is non-decreasing and α_j, β_j are bounded and $\varepsilon > 0$ is small. Then for any solution satisfying (1.1)-(1.4) one has*

$$\lim_{t\to\infty} |\dot{c}_j(t)| = 0 \quad (\textit{for } j \ge 0).$$

Proof. Let $d(t) = c(t+\delta)$. Then, by the contraction property (Lemma 6) we have

$$\sum_{j=0}^{\infty} \frac{|c_j(t+\delta) - c_j(t)|}{\delta} \le 4\rho e^{-\gamma t} \sum_{j=0}^{\infty} \frac{|c_j(\delta) - c_j(0)|}{\delta}. \tag{4.9}$$

To take the limit $\delta \to 0$ we first observe that the term in the right hand side can be written as

$$\frac{|c_j(\delta) - c_j(0)|}{\delta} = |\dot{c}_j(\delta_j)| \tag{4.10}$$

by the mean value theorem. Now, since $c_j \in C^1$ and $K(j,k) \le Cjk$ (for j,k large) we have

$$\begin{aligned}\sum_{j=0}^{\infty} |\dot{c}_j(\delta_j)| &\le \sum_{j=0}^{\infty}\sum_{k=0}^{\infty} K(j+1,k)c_k(\delta_j)c_{j+1}(\delta_j) + \sum_{j=0}^{\infty}\sum_{k=0}^{\infty} K(j,k)c_k(\delta_j)c_j(\delta_j) \\ &\quad + \sum_{j=1}^{\infty}\sum_{k=1}^{\infty} K(k,j)c_k(\delta_j)c_j(\delta_j) + \sum_{j=1}^{\infty}\sum_{k=1}^{\infty} K(k,j-1)c_k(\delta_j)c_{j-1}(\delta_j) \\ &\le C\sum_{j=0}^{\infty}(j+1)c_j(\delta_j) + C\sum_{j=1}^{\infty} jc_j(\delta_j)\end{aligned}$$

where we used $c_k \le C/k^p$ by the previous lemma (uniform in time). Then, it follows that $\sum_{j=0}^{\infty} |\dot{c}_j(\delta_j)| \le C$, showing that the sum on the right hand side of (4.9) is bounded. Therefore we can pass to the limit $\delta \to 0$ in (4.9). Finally, we let $t \to \infty$ to finish the proof of lemma. □

After all the preparatory lemmas, the proof of Theorem 10 now becomes clear.

Proof. (of Theorem 10) By Proposition 9, for any solution of (1.1)-(1.4) with the same mass, $\dot{c}_j(t)$ approaches to zero exponentially fast. This implies that $c_j(t)$ has a limit. Indeed, consider for arbitrary t_1, t and $t > t_1$ the difference $|(c_j(t) - c_j(t_1)|$. By Proposition 9 (and by Lemma 6), one has $|(c_j(t) - c_j(t_1)| \leq Ce^{-\gamma t_1}$ implying that the $c_j(t)$ values are bounded and get closer (uniformly in time). Hence the infinite time limit exists and by Proposition 9 the limit is an equilibrium (denoted by c_j^e). Now, this equilibrium is also a trivial solution of (1.1)-(1.4) and have the same mass with original time dependent solution (by Lemma 7). Hence, $c(t)$ converges to c^e exponentially fast as in (*4.2*). Finally, we argue that the equilibrium is unique. This is because if there was any other equilibrium d^e, going through the algebra of Lemma 6 for the nonlinear equations C^e and D^e, we would obtain

$$0 \leq -(\tilde{a} - 8L^2\varepsilon) \sum_{j=1}^{\infty} \left| C_j^e - D_j^e \right|$$

from which we would conclude $C_j^e = D_j^e$ implying $c_j^e = d_j^e$. □

Remark: *The exponential convergence in the* $\|\cdot\|_0$ *norm in Theorem 10 shows, in particular, that* $c(t) \rightharpoonup^* c^e$. *By Lemma 7 we also have* $\|c(t)\| \to \|c^e\|$. *Then, by Lemma 2 one actually has* $c(t) \to c^e$ in X_1, *though we do not know how fast this convergence is in the strong mass norm.*

Remark: It can be seen from the proof of Lemma *6* that the boundedness assumption on α_j, β_j can be replaced with milder conditions such as $|\alpha_j - \alpha_{j-1}| \leq C$, $|\beta_j - \beta_{j-1}| \leq C$ which includes unbounded kernels.

4.2. **Exponentially fast strong convergence to equilibrium.** Theorem 10 relied heavily on the monotonicity properties of a_j, b_j functions. It is desirable to relax these conditions. In our next result, we show that when the total mass is sufficiently small, the monotonicity assumption can be dropped and exponential convergence to equilibrium is achieved in the mass norm. More precisely we prove the following.

Theorem 11. *Consider the (1.1)-(1.4) system. Let the hypothesis of Theorem 1 be satisfied with* $M_1(0) = \rho$ *and* $M_2(0) < \infty$. *Let the Hypotheses H3 (4.1) and H4 (4.11) hold. Assume further that the mass of the system is sufficiently small. Then the solutions of (1.1)-(1.4) converge to a unique equilibrium in the sense that*

$$\sum_{j \geq 1} j \left| c_j(t) - c_j^e \right| \leq 2\rho e^{-\gamma t}$$

for some $\gamma > 0$.

The growth conditions on the kernels stated in the theorem are as follows.

$$(H4) \qquad \begin{array}{c} 0 < a_{\min} \leq a_j \leq \bar{a}j \text{ and } 0 < b_{\min} \leq b_j \leq \bar{b}j \text{ for } j \geq 1 \\ 0 < \alpha_{\min} \leq \alpha_j \leq \bar{\alpha}j \text{ and } 0 < \beta_{\min} \leq \beta_j \leq \bar{\beta}j \text{ for } j \geq 1 \end{array} \tag{4.11}$$

We first need a lemma showing the boundedness of the moments of solutions. As in Section 4.1, we do not assume detailed balance. But, differently from the previous subsection, due to the faster growth rate in the a_j functions, we cannot, in general, show finiteness of all moments for small mass uniformly. However, with a modification of Lemma 7, we can show that the second moment is bounded.

Lemma 8. *Let, for the system (1.1)-(1.4), the conditions of Theorem 1 be satisfied with $M_2(0) < \infty$ and assume that the Hypotheses H3 (4.1) and H4 (4.11). Then for small enough mass ρ the system has bounded second moment.*

Proof. We show this by formal computations which can be made rigorous by truncated solutions in just the same way as in Section 4.1. Setting $g_j = j^2$ in Lemma 1 we get

$$\sum_{j\geq 1} j^2 \dot{c}_j = \sum_{j\geq 0} \left((j+1)^2 - j^2\right)(a_j\rho + B(c) + \varepsilon\alpha_j \tilde{B}(c))c_j + \sum_{j\geq 1} \left((j-1)^2 - j^2\right)(jA(c) + b_j + \varepsilon\beta_j \tilde{A}(c))c_j$$

$$= \sum_{j\geq 0}(2j+1)(a_j\rho + B(c) + \varepsilon\alpha_j\tilde{B}(c))c_j + \sum_{j\geq 1}(-2j+1)(jA(c) + b_j + \varepsilon\beta_j\tilde{A}(c))c_j$$
$$\leq 2\bar{a}\rho\sum_{j\geq 0} j^2 c_j + \rho(a_0 + \bar{a}\rho) + 2\rho B(c) + B(c) + 2\varepsilon\bar{\alpha}\bar{\beta}\rho\sum_{j\geq 0} j^2 c_j + \varepsilon(\alpha_0 + \bar{\alpha}\rho)\bar{\beta}\rho$$
$$- 2\tilde{a}\sum_{j\geq 1} j^2 c_j - 2\sum_{j\geq 1} j b_{\min} c_j - 2\varepsilon\sum_{j\geq 1} j\beta_{\min} c_j \tilde{A}(c) + \rho A(c) + B(c) + \varepsilon\bar{\beta}\rho\tilde{A}(c),$$

where, in the fourth and fifth lines, we used $A(c) = \sum_{j\geq 0} a_j c_j \leq a_0 + \sum_{j\geq 1} a_j c_j \leq a_0 + \bar{a}\rho$ and $B(c) = \sum_{j\geq 1} b_j c_j \leq \bar{b}\rho$ (similarly for $\tilde{A}(c)$ and $\tilde{B}(c)$). After rearranging the terms we have

$$\sum_{j\geq 1} j^2 \dot{c}_j \leq 2(\bar{a}\rho + 2\varepsilon\bar{\alpha}\bar{\beta}\rho - \tilde{a})\sum_{j\geq 1} j^2 c_j + \rho\left(2a_0 + 2\bar{a}\rho + 2\bar{b}\rho + 2\bar{b} + 2\varepsilon\bar{\beta}(\alpha_0 + \bar{\alpha}\rho) - 2b_{\min}\right).$$

If $\rho < \frac{\tilde{a}}{\bar{a} + 2\varepsilon\bar{\alpha}\bar{\beta}}$ then the differential inequality yields that the second moment is bounded and in particular

$$M_2(\infty) \leq \rho\frac{\left(2a_0 + 2\bar{a}\rho + 2\bar{b}\rho + 2\bar{b} + 2\varepsilon\bar{\beta}(\alpha_0 + \bar{\alpha}\rho) - b_{\min}\right)}{2(\tilde{a} - \bar{a}\rho - 2\varepsilon\bar{\alpha}\bar{\beta}\rho)}.$$

□

Remark. It is worth saying that the result is not specific to M_2.The proof can be extended to higher moments so long as the total mass ρ is small enough, that is, for any given $p > 0$, i.e., $M_p(\infty) < C$. However, the smallness requirement will depend on the value of p.

Next, as in Section 4.1 we show the contraction property of solutions.

Lemma 9. *Let the conditions of Theorem 1 be satisfied and c_j and d_j be two solutions of the system (1.1)-(1.4) with the same initial mass and same initial volume. Assume that $M_2(0) < \infty$ and the Hypotheses H3 (4.1) and H4 (4.11) hold with the total mass (density) ρ and ε small enough. Then the two solutions approach to each other in the sense that*

$$\sum_{j\geq 1} j\,|c_j(t) - d_j(t)| \leq 2\rho e^{-\gamma t}.$$

The general idea of proof is similar to the contraction result in Section 4.1. However, in this case it is more convenient to use difference of individual cluster densities (not the tail) to measure the difference of time dependent solutions, i.e.,

$\sum_{j\geq 1} j\,|c_j(t)-d_j(t)|$. The goal is to show that its derivative satisfies a differential inequality which yields the result.

Proof. Let c_j and d_j and be the time dependent and equilibrium solutions and $e_j = c_j - d_j$ be the difference. Setting $g_j = jsgn(e_j)$ in Lemma 1 (with $N\to\infty$) one gets

$$\sum_{j\geq 1} jsgn(e_j)\dot{c}_j = \sum_{j\geq 0}\left((j+1)sgn(e_{j+1}) - jsgn(e_j)\right)(a_j\rho + B(c) + \varepsilon\alpha_j\tilde{B}(c))c_j$$
$$+ \sum_{j\geq 1}\left((j-1)sgn(e_{j-1}) - jsgn(e_j)\right)(jA(c) + b_j + \varepsilon\beta_j\tilde{A}(c))c_j,$$

Subtracting from above the equation for $\sum_{j\geq 1} jsgn(e_j)\dot{d}_j$ and noting $A(c)c_j - A(d)d_j = e_jA(c) + d_jA(e)$ and $B(c)c_j - B(d_j)d_j = e_jB(c) + d_jB(e)$ (and using similar notations for $\tilde{A}$ and $\tilde{B}$) we get

$$\sum_{j\geq 1} j\,|\dot{e}_j| = \sum_{j\geq 0}\left((j+1)sgn(e_{j+1}) - jsgn(e_j)\right)(\rho a_je_j + e_jB(c) + d_jB(e)) \quad (4.12)$$
$$+ \sum_{j\geq 1}\left((j-1)sgn(e_{j-1}) - jsgn(e_j)\right)(je_jA(c) + jd_jA(e) + b_je_j) \quad (4.13)$$
$$+ \varepsilon\sum_{j\geq 0}\left((j+1)sgn(e_{j+1}) - jsgn(e_j)\right)\alpha_j(e_j\tilde{B}(c) + d_j\tilde{B}(e)), \quad (4.14)$$
$$+ \varepsilon\sum_{j\geq 1}\left((j-1)sgn(e_{j-1}) - jsgn(e_j)\right)\beta_j(e_j\tilde{A}(c) + d_j\tilde{A}(e)), \quad (4.15)$$

where in (4.12) we implicitly used $\rho = \sum_{j\geq 0} jc_j = \sum_{j\geq 0} jd_j$ and $1 = \sum_{j\geq 0} c_j = \sum_{j\geq 0} d_j$. Upon distributing the $(j\pm 1)sgn(e_{j\pm 1}) - jsgn(e_j)$ over the terms inside the parenthesis in each line on the right hand side of (4.12), we produce a total of 10 terms which we denote by $S_1, ..., S_{10}$ respectively. For each S_i term we obtain an inequality.

For S_1, using $|sgn(e_{j+1})| \leq 1$ and $sgn(e_j)e_j = |e_j|$, we write

$$S_1 = \sum_{j\geq 0}\left((j+1)sgn(e_{j+1}) - jsgn(e_j)\right)\rho a_je_j \leq \rho\sum_{j\geq 0} a_j\,|e_j|$$
$$\leq \rho\sum_{j\geq 1}(a_j + a_0)\,|e_j|\,,$$

where in the second line we used $|e_0| \leq \sum_{j\geq 1}|e_j|$ which follows from $\sum_{j\geq 0} e_j = 0$ (conservation of volume). Similarly, for S_2, one has

$$S_2 = \sum_{j\geq 0}\left((j+1)sgn(e_{j+1}) - jsgn(e_j)\right)e_jB(c) \leq 2B(c)\sum_{j\geq 1}|e_j|\,.$$

For S_3 we observe $|B(e)| \leq \sum_{j\geq 1} b_j\,|e_j|$ and obtain

$$S_3 = \sum_{j\geq 0}\left((j+1)sgn(e_{j+1}) - jsgn(e_j)\right)d_jB(e)$$
$$\leq \sum_{j\geq 0}(2j+1)d_j\sum_{k\geq 1} b_k\,|e_k|\,.$$

For S_4 term we again use $|sgn(e_{j-1})| \leq 1$ and find

$$\begin{aligned} S_4 &= \sum_{j\geq 1} \left((j-1)sgn(e_{j-1}) - jsgn(e_j)\right) je_j A(c) \\ &\leq -\sum_{j\geq 1} j\left|e_j\right| A(c). \end{aligned}$$

The S_5 term, using $|A(e)| \leq \sum_{j\geq 0} a_j |e_j| \leq \sum_{j\geq 1}(a_j + a_0)|e_j|$, gives

$$\begin{aligned} S_5 &= \sum_{j\geq 1} \left((j-1)sgn(e_{j-1}) - jsgn(e_j)\right) jd_j A(e) \\ &\leq \sum_{j\geq 1}(2j-1)jd_j \sum_{k\geq 1}(a_k + a_0)\left|e_k\right|. \end{aligned}$$

And, S_6 reads

$$S_6 = \sum_{j\geq 1} \left((j-1)sgn(e_{j-1}) - jsgn(e_j)\right) b_j e_j \leq -\sum_{j\geq 1} b_j \left|e_j\right|.$$

Looking at the terms one notices that S_6 cancels part of the term on the right hand side of S_3 since $\sum_{j\geq 0} d_j = 1$ which leaves $\sum_{j\geq 0}(2j)d_j \sum_{k\geq 1} b_k |e_k|$. Similarly, S_1 cancels the negative part on the right hand side of S_5 since $\sum_{j\geq 1} jd_j = \rho$. Combining with the rest of the terms in (4.12) we get

$$\begin{aligned} \sum_{j\geq 1} j\left|\dot{e}_j\right| &\leq 2B(c)\sum_{j\geq 1}\left|e_j\right| + \sum_{j\geq 0}(2j)d_j \sum_{k\geq 1} b_k \left|e_k\right| \qquad (4.16) \\ &\quad - \sum_{j\geq 1} j\left|e_j\right| A(c) + \sum_{j\geq 1}(2j)jd_j \sum_{k\geq 1}(a_k + a_0)\left|e_k\right| \\ &\quad + S_7 + S_8 + S_9 + S_{10}. \qquad (4.17) \end{aligned}$$

We now estimate the perturbation terms $S_7, ..., S_{10}$ in a similar fashion. S_7 and S_8 are given by

$$S_7 = \varepsilon \sum_{j\geq 0} \left((j+1)sgn(e_{j+1}) - jsgn(e_j)\right) \alpha_j e_j \tilde{B}(c) \leq \varepsilon \sum_{j\geq 1}(\alpha_0 + \bar{\alpha}j)\left|e_j\right| \tilde{B}(c),$$

$$\begin{aligned} S_8 &= \varepsilon \sum_{j\geq 0} \left((j+1)sgn(e_{j+1}) - jsgn(e_j)\right) \alpha_j d_j \tilde{B}(e) \leq \varepsilon \sum_{j\geq 0} (2j+1)\,\alpha_j d_j \left|\tilde{B}(e)\right| \\ &\leq \varepsilon \sum_{j\geq 0}(2j^2\bar{\alpha} + \bar{\alpha}j + \alpha_0)d_j \sum_{k\geq 1} \beta_k \left|e_k\right|, \end{aligned}$$

where we used $\sum_{j\geq 0} \alpha_j |e_j| \leq \sum_{j\geq 1}(\alpha_j + \alpha_0)|e_j|$ (since $|e_0| \leq \sum_{j\geq 1}|e_j|$). Similarly, $S9$ and $S10$ are given by

$$S_9 = \varepsilon \sum_{j\geq 1} \left((j-1)sgn(e_{j-1}) - jsgn(e_j)\right) \beta_j e_j \tilde{A}(c) \leq -\varepsilon \sum_{j\geq 1} \beta_{\min} \left|e_j\right| \tilde{A}(c),$$

$$S_{10} = \varepsilon \sum_{j\geq 1} \left((j-1)sgn(e_{j-1}) - jsgn(e_j)\right) \beta_j d_j \tilde{A}(e) \leq \varepsilon \sum_{j\geq 1}(2j-1)d_j\beta_j \sum_{k\geq 0} \alpha_k \left|e_k\right|.$$

By the bounds given in the theorem $B(c) \leq \sum_{j\geq 1} \bar{b}jc_j \leq \bar{b}\rho$ and $A(c) \geq \sum_{j\geq 1} \tilde{a}c_j \geq \tilde{a}$. Also, $\sum_{j\geq 1}|e_j| \leq \sum_{j\geq 1} j|e_j|$ and $\sum_{k\geq 1}(a_k + a_0)|e_k| \leq (a_0 + \bar{a})\sum_{k\geq 1} k|e_k|$. Then using $|e_0| \le \sum_{j\ge1} |e_j|$ several times (4.16) reduces to

$$\sum_{j\ge1} j\,|\dot e_j| \le 2\bar b\rho \sum_{j\ge1} j\,|e_j| + 2\rho\bar b \sum_{j\ge1} j\,|e_j| - a_{\min}\sum_{j\ge1} j\,|e_j| + 2M_2(a_0+\bar a)\sum_{k\ge1} k\,|e_k|$$
$$+\,\varepsilon\left((\alpha_0+\bar\alpha)\rho\bar\beta + 2\bar\alpha\bar\beta M_2 + \alpha_0\bar\beta d_0 + \bar\alpha\bar\beta\rho\right)\sum_{k\ge1} k\,|e_k| + \varepsilon 2\bar\beta M_2(\alpha_0+\bar\alpha)\sum_{k\ge1} k\,|e_k|\,.$$

From Lemma 8 we know

$$M_2 = \sum_{j\ge1} j^2 c_j \le P(\rho) := \frac{\rho\left(2a_0 + 2\bar a\rho + 2\bar b\rho + 2\bar b + 2\varepsilon\bar\beta(\alpha_0+\bar\alpha\rho) - b_{\min}\right)}{2(\tilde a - \bar a\rho - 2\varepsilon\bar\alpha\bar\beta\rho)}.$$

Hence one gets the differential inequality

$$\sum_{j\ge1} j\,|\dot e_j| \le \left(4\bar b\rho + 2(a_0+\bar a)P(\rho) - a_{\min}\right)\sum_{j\ge1} j\,|e_j| \tag{4.18}$$
$$+\,\varepsilon\left((\alpha_0+2\bar\alpha)\rho + 2(2\bar\alpha+\alpha_0)P(\rho) + \alpha_0 d_0 + \bar\alpha\rho\right)\bar\beta\sum_{j\ge1} j\,|e_j|\,. \tag{4.19}$$

It is then clear that, for ρ and ε small enough, the parenthesis on the right hand side of (4.18) has a negative value (say $-\gamma < 0$) giving

$$\sum_{j\ge1} j\,|\dot e_j| \le \sum_{j\ge1} j\,|e_j(0)|\,e^{-\gamma t} \le \sum_{j\ge1} j(c_j(0)+d_j(0))e^{-\gamma t} \le 2\rho e^{-\gamma t}$$

which proves the lemma. □

As the last ingredient for the theorem, we have the existence of the equilibrium solutions which is analogous to Proposition 9. The proof follows similar steps to Proposition 9, hence we skip it.

Proposition 10. *Assume the conditions of Theorem 1 with $M_2(0) < \infty$. Let Hypotheses H3 (4.1) and H4 (4.11) hold and the total mass be small enough. Then for any solution satisfying (1.1)-(1.4) one has*

$$\lim_{t\to\infty} |\dot c_j(t)| = 0 \quad (\textit{for } j \ge 0).$$

Collecting all of the results we can now prove the main theorem of this subsection.

Proof. (of Theorem 11). For mass sufficiently small, by Lemma 9 any two solutions of (1.1)-(1.4) with the same mass approach to each other exponentially fast. In particular the infinite time limit exists and by Lemma 8 and Proposition 10 this limit is an equilibrium and has the same mass. It can also be argued, as in Theorem 10, that the equilibrium is unique. Hence all time dependent solutions with equal mass converge to the unique equilibrium solution. □

5. **Conclusion.** In this article, we studied the large time behavior of the EDG system, particularly the convergence of solutions to equilibrium with explicit convergence rates where possible. Due to the complexities arising in a fully general kernel form we focussed on two special but fairly general classes of separable kernels (in product and sum forms).

For the first class of kernels ($K(j,k) = b_j a_k$) that we considered, we showed the existence of equilibria under the assumption of detailed balance. The crucial finding is that not all initial mass values can support equilibrium solutions. Much like in the Becker-Doring system, above a critical mass ρ_c, the EDG system undergoes a dynamic phase transition. By employing a well known entropy method we proved the strong convergence of solutions to unique equilibrium distribution for initial masses below the critical mass and weak convergence of solutions to the critical equilibrium distribution for initial masses above the critical mass. The question of how fast these convergences occur in each case is left for future investigations.

For the second class of kernels given by $K(j,k) = ja_k + b_j + \varepsilon\beta_j\alpha_k$, we proved, as a by-product of a contraction property, the existence of a unique equilibrium and convergence of solutions to this equilibrium in the ("weak") number of cluster norm. The property followed from the monotonicity of b_j, a_j an assumption motivated by the heuristic interpretations that a_j,b_j represent the *import/growth* and *export/fragmentation*. While these analogies (between the a_j, b_j of BD systems and EDG systems) are appealing and acceptable to a certain extent, one should bear in mind that, in the exchange systems the dynamics is so intertwined that a_j, b_j should not be regarded too simplistically or being mere copies of coagulation and fragmentation rates as in the BD system. Nevertheless, the arguments suggest that the result should generalize which we state as a conjecture

Conjecture. *Consider the EDG system (1.1)-(1.4) system. Let the conditions of Theorem 1 be satisfied. Assume for the kernel $K(j,k)$ that it is non-decreasing in the first component and non-increasing in the second component. Then the solutions of (1.1)-(1.4) converge to equilibrium exponentially fast in the sense of Theorem 10.*

For second class of kernels it was shown that the monotonicity assumption can be replaced with a bound condition on the total mass of the system. With this alternative condition we proved exponential convergence of solutions to unique equilibrium in the mass ("strong") norm. We do not know, if this condition is only a technical assumption or an intrinsic requirement.

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