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# q-COHOMOLOGICALLY COMPLETE AND q-PSEUDOCONVEX DOMAINS

by

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A dissertation submitted for the degree of Doctor of Philosophy at the University of Warwick

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Dedicated to my daughter Anna

who deserves a better father.

TABLE OF CONTENTS

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It is a desperate task to express in words my gratitude to my friend Mike Eastwood, who also is my supervisor, so I will not even try: but the deep influence that his natural talent of mathematician and teacher had and has on my scientific education should not go unobserved; this thesis has been written under his constant and encouraging help and important parts were developed directly by him. I do not believe that a student can be helped more by his supervisor.

I am also grateful to Prof. David Epstein for showing me how to prove the result contained in the appendix.

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## SUMMARY

This dissertation is devoted to the study of the different viewpoints from which an open subset D of a Stein manifold M can be considered, as the geometric concepts of q-completeness and q-pseudoconvexity and the analytic ideas of vanishing of cohomology groups after a certain level and inextendibility of cohomology classes or holomorphic functions.

The idea is to generalize to any integer q the well known equivalence between 0-completeness and 0-cohomological completeness ( see theorem 2.3.1).

A step in this direction, namely that if D is q-complete then it is also q-cohomologically complete, was done in 1962 by Andreotti and Grauert, but the converse implication is still an open problem.

Using a rather indirect tool involving certain cohomology classes called "test classes" we can manage to prove that if D is cohomologically q-complete and has $C^2$ boundary then it is q-complete, and this is probably the most interesting result appearing in this thesis ( see theorem 3.3.1).

This method however can also be applied to answer certain natural questions about inextendibility of cohomology classes, analogous to inextendibility of holomorphic functions for domains of holomorphy, and the answer turns out to be not surprising if D has $C^2$ boundary (see theorem 4.1.8) but less intuitive in the general case and counterexamples illustrating this behaviour are discussed in chapter 3, section 4.

In particular we describe a particularly interesting application of the test classes that gives a lower bound on the number of analytic functions needed to define an analytic subvariety just touching $\overline{D}$ at a point x belonging to its boundary provided the behaviour of $\partial D$ near x is known (see theorem 4.2.3).

All these results canbe deduced without knowing the explicit expression for these cohomology classes, but such an expression in terms of Dolbeault cohomology and Cech cohomology is given in the last chapter: it can be observed that the test classes are related to the Bochner-Martinelli kernel.

# INTRODUCTION

E. E. Levi noticed in his paper [14] that a domain of holomorphy with $C^2$ boundary contained in a complex space $C^n$ is 0-pseudoconvex (for the definition of these concepts see chapter 2, section 2).

This observation introduced a challenging question for complex analists, which can be loosely stated as: is the converse true?

It was soon realized that the hypothesis on the boundary can be removed provided 0-pseudoconvexity is replaced by 0-completeness and that replacing $\mathbb{C}^n$ with any Stein manifold does not create any serious trouble; moreover 0-completeness makes sense even if the analytic domain that we are considering is not an open subset of a larger one and the Levi problem can be restated as:

If D is a 0-complete analytic manifold (space) is D Stein ?

The question was studied by mathematicians like Oka, Lelong, Bremermann and others; an interesting survey of the historical developement of this problem is contained in the introduction of [8], and in the same paper Grauert gives a positive answer to the above question for manifolds. In 1961 Narasimhan solved again in the affirmative the Levi problem for analytic spaces, in his article [15].

Meanwhile other powerful tools for the study of analytic spaces had been developed, specially cohomology theory, and a famous result known as Cartan's theorem B showed that a Stein space is always 0-cohomologically complete (see definition 2.2.3).

The situation was then very well settled because it is easy to prove the converse of this last statement, and so if D is an analytic space the following properties are equivalent:

(a) D is a Stein space (which is an analytic-function theoretic concept),

(b) D is 0-complete (geometric-analytic),

(c) D is cohomologically 0-complete (algebraic-analytic).

However an integer, zero, appears in the formulation of (b) and (c), so it is natural to ask whether the equivalence of (b) and (c) still holds when 0 is replaced by another integer q.

The first step in this direction is due to Andreotti and Grauert (see prop. [illegible].1.1); namely they proved that a q-complete analytic space is necessarily q-cohomologically complete.

The converse implication is, as far as we know, still unsolved nowadays (see [19] problem IV, p. 57), the difficulty consisting in the fact that it is not easy to define in a suitable way the concept of "q-Stein space".

In this dissertation I have been studying this problem in the particular but intuitively and historically important case when D is an open subset with $C^2$ boundary of a Stein manifold M, going back to a sort of "q-Levi problem" as originally stated.

The method used here is indirect but rather efficient and involves certain natural cohomology classes, called test classes, that have been studied already, though in a quite implicit way, by Andreotti and Norguet in [3] by means of the Bochner-Martinelli integral formula and in a more "cohomological" way by Eastwood in [5] and Laufer in [13].

The test classes provide a powerful tool which finally enables us to answer in the affirmative to the problem considered: namely we can prove that a domain with $C^2$ boundary contained in a Stein manifold is q-pseudoconvex if and only if it is q-cohomologically complete (see theorem [illegible].3.1).

This method is also useful to compare the concepts considered above (q-completeness, q-pseudoconvexity and q-cohomological completeness) with other rather classical ones like pseudoconvexity à la Grauert and Fritzsche (see def. 2.1.4).

Moreover we can investigate some intuitive questions about inextendibility of cohomology classes already studied by Andreotti and Norguet in [3] that arise naturally from the classical definition of domain of holomorphy and the observation that holomorphic functions can be considered as 0 cohomology classes (see theorem 4.1.8).

The general impression that arises from these results is that open subsets of Stein manifolds behave in a good, intuitive way analogous to domains of holomorphy if their boundary is $C^2$, but unexpected conclusions can be deduced if we drop this hypothesis: some examples of this odd behaviour are shown and discussed making use of complements of analytic varieties (example 3.4.2).

The methods used in this dissertation can also be applied to investigate the nature of germs of analytic varieties just "touching" at a point x the closure $\overline{D}$ of an open subset D of an analytic manifold provided we know the "bending" of the boundary of D at x, and a lower bound to the geometric codimension (=minimal number of germs of analytic functions necessary to define it) is given (see theorem 4.2.3).

Finally the cohomology classes used to deduce the above results are explicitely described in terms of Dolbeault and Leray cohomology: this is done in chapter 5.

A substantial part of this dissertation will be published in a joint paper, co-author Michael G. Eastwood, to appear with title "Pseudoconvex and Cohomologically Complete Domains".

## CHAPTER 1: The Projected q-Envelope of Holomorphy.

Let M be a n-dimensional Stein manifold, $n \geqslant 2$, an open connected subset $D \subseteq M$ will be called domain. This chapter is dedicated to the construction, for a domain $D \subseteq M$, of a sequence of sets

$$D \subseteq E_{n-2}(D) \subseteq E_{n-3}(D) \subseteq \ldots\ldots \subseteq E_0(D) \subseteq M,$$

the utility of which will be shown later; $E_q(D)$, $q=0,1,\ldots,n-2$, is called the projected q-envelope of holomorphy of D; some algebraic machinery is necessary to construct these sets and to show that they are well defined. We start with

### § 1: The Koszul Complex of a Point.

Let R be a commutative unitary ring and consider the graded algebra $\Lambda^{\cdot} R^n = \{\Lambda^p R^n\}_{p \in Z}$. We shall identify, as natural, $\Lambda^1 R^n$ with $R^n$ and with the R-module of column vectors of n elements of R. If $f_1, f_2, \ldots, f_n$ belong to R, they determine an element $\underline{f}$ in $\Lambda^1 R^n$, and so they can be used to define a R-homomorphism

$$d = d_{\underline{f}} : \Lambda^{\cdot} R^n \longrightarrow \Lambda^{\cdot} R^n \text{ given by } d_{\underline{f}}(w) = \underline{f} \wedge w, \forall w \in \Lambda^{\cdot} R^n.$$

It is immediate to check that $d \circ d = 0$, i,e, that d is a differential (of degree +1).

The resulting complex $(\Lambda^{\cdot} R^n, d_{\underline{f}})$ is called the Koszul complex of $\underline{f}$ and R and we shall denote it by $K^{\cdot}(\underline{f},R) = \{K^p(\underline{f},R)\}_{p \in Z}$.

If $\tilde{M}$ is any R-module we can define the Koszul complex $K^{\cdot}(\underline{f},\tilde{M})$ by $K^p(\underline{f},\tilde{M}) = K^p(\underline{f},R) \otimes_R \tilde{M}$ and differential $d_{\underline{f}} \otimes 1$.

Notice that $K^p(\underline{f},\tilde{M}) = 0$ for $p < 0$ and $p > n$, $K^0(\underline{f},\tilde{M}) = \tilde{M}$ and there is a natural identification $K^n(\underline{f},\tilde{M}) \overset{i}{\approx} \tilde{M}$.

As a matter of notation we indicate with $F_1, F_2, \ldots, F_n$ the formal symbols occurring in $K^{\cdot}(\underline{f},\tilde{M})$, so that $\underline{f} = \sum_{j=1}^{n} f_j F_j$ and

$i: K^n(\underline{f},\tilde{M}) \longrightarrow \tilde{M}$ is given by $i(m\, F_1 \wedge F_2 \wedge \ldots \wedge F_n) = m,\ \forall m \in \tilde{M}$. The cohomology of $K^{\cdot}(\underline{f},\tilde{M})$ is indicated by $H^{\cdot}(\underline{f},\tilde{M})$.

Lemma 1.1.1:- If for some $j = 1,2,\ldots,n$, $f_j$ is a unit of $R$ the complex $K^{\cdot}(\underline{f},\tilde{M})$ is acyclic, i.e. $H^p(\underline{f},\tilde{M}) = 0\ \forall p$ .

Proof:- Choose $g \in R$ s.t. $gf_j = 1$ and define a homomorphism $h : K^{p+1}(\underline{f},\tilde{M}) \longrightarrow K^p(\underline{f},\tilde{M}), \forall p$, as follows: an element $w \in K^{p+1}(\underline{f},\tilde{M})$ can be uniquely written as $w = F_j \wedge u + v$, where $u \in K^p(\underline{f},\tilde{M})$, $v \in K^{p+1}(\underline{f},\tilde{M})$ and $F_j$ does not appear in $u$ and $v$; define $h(w) = g\cdot u$. Then $(hd+dh)(w) = hd(F_j \wedge u) + hd(v) + dh(F_j \wedge u) + dh(v) = h(\sum_{i\neq j} f_i F_i \wedge F_j \wedge u) + h(\sum_{i=1}^{n} f_i F_i \wedge v) + d(g\cdot u) = -\sum_{i\neq j} g f_i F_i \wedge u + g f_j \cdot v + d(g\cdot u) = g\, f_j\, F_j \wedge u + g\, f_j\, v = F_j \wedge u + v = w$, i.e. $hd+dh=$Identity; therefore $K^{\cdot}(\underline{f},\tilde{M})$ is homotopy equivalent to the zero complex and so it is acyclic. □

Let now $\mathcal{R}$ be a sheaf of commutative unitary rings on a topological space $M$ and $\mathcal{M}$ be a sheaf of $\mathcal{R}$-modules, moreover suppose that $f_1,f_2,\ldots,f_n \in \Gamma(M,\mathcal{R})$; we shall denote by $\mathcal{K}^{\cdot}(\underline{f},\mathcal{M})$ the Koszul complex of sheaves given by the complex of complete presheaves

$$\mathcal{K}^{\cdot}(\underline{f},\mathcal{M})\ (U) = K^{\cdot}(\underline{f}_{|U}, \Gamma(U,\mathcal{M}))$$

for all open set $U \subseteq M$ (we remark that restriction maps clearly commute with $d_{\underline{f}}$, and so they are maps of complexes).

Throughout this dissertation, unless otherwise explicitly stated $M$ will be a n-dimensional, $n \geqslant 2$, Stein manifold and $\mathcal{R}$ its structure sheaf $\mathcal{O}$.

If $x \in M$ it is always possible (see [7] Satz 1 p. 91) to choose sections $f_1,f_2,\ldots,f_n \in \Gamma(M,\mathcal{O})$ s.t.
$\{x\} = V(f_1,f_2,\ldots\ldots,f_n) = \{ y \in M \text{ s.t. } f_1(y)=f_2(y)=\ldots=f_n(y)=0 \}$
If these functions also give local coordinates of $M$ at $x$, which can always be arranged, we shall denote them by $z_1,z_2,\ldots,z_n$.

The Koszul complex of the point $x \in M$ (with respect to the functions $f_1, f_2, \ldots, f_n$) is by definition the Koszul complex of sheaves $\mathcal{K}^{\cdot}(\underline{f}, \mathcal{O})$.

As a consequence of lemma 1.1.1 we have the following

Proposition 1.1.2 :- For every analytic sheaf $\mathcal{M}$ the complex of sheaves $\mathcal{K}^{\cdot}(\underline{f}, \mathcal{M})$ is acyclic on $M - \{x\}$ (i.e. $\forall y \neq x$ the complex of $\mathcal{O}_y$-modules $\mathcal{K}^{\cdot}(\underline{f}, \mathcal{M})_y$ is acyclic).

Proof:- $\forall y \neq x$ $\exists j = 1,2,\ldots,n$ s.t. $f_j(y) \neq 0$, i.e. the germ $(\tilde{f}_j)_y \in \mathcal{O}_y$ is a unit. Lemma 1.1.1 says precisely that $\mathcal{K}^{\cdot}(\underline{f}, \mathcal{M})_y$ is an acyclic complex of $\mathcal{O}_y$-modules. □

Therefore the sequence of sheaves

$$0 \longrightarrow \mathcal{K}^0(\underline{f}, \mathcal{M}) \xrightarrow{d_f} \mathcal{K}^1(\underline{f}, \mathcal{M}) \xrightarrow{d_f} \ldots\ldots \mathcal{K}^n(\underline{f}, \mathcal{M}) \longrightarrow 0$$

is exact on $M - \{x\}$ , and so it can be split into short exact sequences, i.e. there exist sheaves $\mathcal{L}_s(\underline{f}, \mathcal{M})$ , $s = 0,1,\ldots,n-2$ on $M - \{x\}$ s.t. the sequences

$$0 \longrightarrow \mathcal{L}_s(\underline{f}, \mathcal{M}) \longrightarrow \mathcal{K}^{n-s-1}(\underline{f}, \mathcal{M}) \longrightarrow \mathcal{L}_{s-1}(\underline{f}, \mathcal{M}) \longrightarrow 0$$

are exact, and $\mathcal{L}_{-1}(\underline{f}, \mathcal{M}) = \mathcal{K}^n(\underline{f}, \mathcal{M}) \stackrel{i}{=} \mathcal{M}$ , $\mathcal{L}_{n-2}(\underline{f}, \mathcal{M}) = \mathcal{K}^0(\underline{f}, \mathcal{M}) = \mathcal{M}$ .

As a matter of notation $\mathcal{L}_s(\underline{f}, \mathcal{O})$ and $\mathcal{K}^p(\underline{f}, \mathcal{O})$ will be indicated simply with $\mathcal{L}_s(\underline{f})$ and $\mathcal{K}^p(\underline{f})$; this is by far the most interesting case.

## § 2: The Test Classes.

As we have just seen, there are exact sequences of sheaves

$$0 \longrightarrow \mathcal{L}_s(\underline{f}) \longrightarrow \mathcal{K}^{n-s-1}(\underline{f}) \longrightarrow \mathcal{L}_{s-1}(\underline{f}) \longrightarrow 0$$

on $M - \{x\}$ , for $s = 0,1,\ldots,n-2$.

The connecting homomorphisms of the corresponding long exact sequences of cohomology

$$\ldots\ldots \longrightarrow H^s(M - \{x\}, \mathcal{L}_{s-1}(\underline{f})) \xrightarrow{\delta} H^{s+1}(M - \{x\}, \mathcal{L}_s(\underline{f})) \longrightarrow \ldots.$$

may be composed to give maps

$$\delta_s(\underline{f}) : \Gamma(M - \{x\}, \mathcal{L}_{-1}(\underline{f})) \longrightarrow H^{s+1}(M - \{x\}, \mathcal{L}_s(\underline{f})).$$

Now we observe that the restriction map $r : \Gamma(M, \mathcal{O}) \longrightarrow \Gamma(M - \{x\}, \mathcal{O})$ is an isomorphism by Riemann removable singularity theorem and we call again $\delta_s(\underline{f})$ the map given by the composition

$$\delta_s(\underline{f}) \circ i \circ r : \Gamma(M, \mathcal{O}) \longrightarrow H^{s+1}(M - \{x\}, \mathcal{L}_s(\underline{f})).$$

If g is an element of $\Gamma(M, \mathcal{O})$ we obtain <u>test</u> <u>classes</u>

$$\alpha_s(g, \underline{f}) =_{\text{def.}} \delta_s(\underline{f})(g) \in H^{s+1}(M - \{x\}, \mathcal{L}_s(\underline{f})), \quad s=0,1,\ldots,n-2.$$

We shall shortly see that the test classes $\alpha_s(g, \underline{f})$ are of particular interest when the germ $\tilde{g}_x \notin \mathcal{J}_x(\underline{f})$, where $\mathcal{J}_x(\underline{f})$ denotes, as classically, the stalk at x of the sheaf of ideals generated by $\underline{f}$: this is the case if $g = 1$, and $\alpha_s(1, \underline{f})$ will be written simply $\alpha_s(x, \underline{f})$.

The test classes say an important word about the envelope of holomorphy of a domain $D \subseteq M$.

## § 3: The Envelope of Holomorphy.

If D is a domain in a Stein manifold M there exists always a connected Stein manifold E(D), called the <u>envelope of holomorphy</u> of D, which contains D and is characterized by the fact that every holomorphic function on D extends uniquely to E(D). E(D) is not necessarily a subset of M because "sheeting" can occur (see [11] p.43), but in general E(D) is a Riemann domain over M with projection

$$\pi : E(D) \longrightarrow M$$

(the terminology Riemann domain just means that $\pi$ is a local isomorphism). This notation will be retained for the rest of this dissertation, and the symbol $\pi$ will always indicate the projection of a Riemann domain. This situation can be expressed by means of diagrams as follows:

(a) Firstly the diagram

(a)
$$\begin{array}{ccc} D & \hookrightarrow & E(D) \\ & \searrow \quad \swarrow_{\pi} & \\ & M & \end{array}$$
commutes and

(b) If we call $\mathcal{O}_E$ the structure sheaf of $E(D)$, the map $\pi$ induces a map $\pi^* : \mathcal{O} \longrightarrow \mathcal{O}_E$ given by $\pi^*(g)(y) =_{\text{def.}} g(\pi(y))$, $\forall y \in E(D)$: the diagram

(b)
$$\begin{array}{ccc} \Gamma(D,\mathcal{O}) & \xleftarrow{\ r\ } & \Gamma(E(D),\mathcal{O}_E) \\ & {}^{r}\nwarrow \quad \nearrow^{\pi^*} & \\ & \Gamma(M,\mathcal{O}) & \end{array}$$

where r indicates the restriction map, commutes and $r : \Gamma(E(D),\mathcal{O}_E) \longrightarrow \Gamma(D,\mathcal{O})$ is an isomorphism.

A good reference for all the above is [16].

A reason for calling the cohomology classes $\alpha_s(g,\underline{f})$ test classes is given by the following

Proposition 1.3.1:- Suppose $x \in M - D$ and $g \in \Gamma(M,\mathcal{O})$ is chosen in such a way that the germ $\tilde{g}_x \notin \mathcal{I}_x(\underline{f})$, e.g. $g(x) \neq 0$. Then

$$x \in \pi(E(D)) \quad \text{if and only if} \quad \alpha_0(g,\underline{f})|_D \neq 0$$

where $\alpha_0(g,\underline{f})|_D$ denotes the image of $\alpha_0(g,\underline{f})$ under the restriction

$$r : H^1(M - \{x\}, \mathcal{L}_0(\underline{f})) \longrightarrow H^1(D, \mathcal{L}_0(\underline{f})).$$

Proof:- (cfr. [5] Theorem 2.1). For $i=1,2,\ldots,n$ set $f_i^* = \pi^*(f_i)$, and construct the Koszul complex $\mathcal{K}^{\cdot}(\underline{f}^*)$ on $E(D)$; by proposition 1.1.2 $\mathcal{K}^{\cdot}(\underline{f}^*)$ is exact on $E(D) - \pi^{-1}(x)$, and we have an exact sequence of sheaves on $E(D) - \pi^{-1}(x)$

$$0 \longrightarrow \mathcal{L}_0 \longrightarrow \mathcal{K}^{n-1}(\underline{f}^*) \xrightarrow{d_{f^*}} \mathcal{K}^n(\underline{f}^*) \longrightarrow 0,$$

where $\mathcal{L}_0 = \operatorname{Ker} d_{f^*}$ and so is a coherent sheaf.

In the following diagram with exact rows

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathcal{L}_0(\underline{f}) & \longrightarrow & \mathcal{K}^{n-1}(\underline{f}) & \longrightarrow & \mathcal{K}^n(\underline{f}) & \longrightarrow & 0 \\
 & & \Big\downarrow{\scriptstyle \pi^*} & & \Big\downarrow{\scriptstyle \pi^*} & & \Big\downarrow{\scriptstyle \pi^*} & & \\
0 & \longrightarrow & \mathcal{L}_0 & \longrightarrow & \mathcal{K}^{n-1}(\underline{f}^*) & \longrightarrow & \mathcal{K}^n(\underline{f}^*) & \longrightarrow & 0
\end{array}$$

the square on the right hand side is commutative and so the map indicated with a dotted line that makes the first square commutative exists and is unique. By taking cohomology and using the above diagram (b) we obtain a commutative diagram

$$\begin{array}{ccc}
H^1(D, \mathcal{L}_0(\underline{f})) & \xleftarrow{\;r\;} & H^1(E(D) - \pi^{-1}(x), \mathcal{L}_0) \\
 & {\scriptstyle r}\nwarrow \quad \nearrow{\scriptstyle \pi^*} & \\
 & H^1(M - \{x\}, \mathcal{L}_0(\underline{f})). &
\end{array}$$

Now if $x \notin \pi(E(D))$, $E(D) - \pi^{-1}(x) = E(D)$ which is Stein and by Cartan's Theorem B $\pi^*(\alpha_0(g,\underline{f}) = 0$ and so also $\alpha_0(g,\underline{f})|_D = 0$.

To prove the converse we first remark that, since $\pi$ is a local isomorphism, it induces an isomorphism

$$\pi^* : \mathcal{I}_{\pi(y)}(\underline{f}) \longrightarrow \mathcal{I}_y(\underline{f}^*) \qquad \forall\, y \in E(D).$$

In particular $\pi^*(g) \notin \mathcal{I}_y(\underline{f}^*)$, $\forall y \in \pi^{-1}(x)$.

If $\alpha_0(g,\underline{f})|_D = 0$ there exist functions $h_1, h_2, \ldots, h_n$ in $\Gamma(D, \mathcal{O})$ s.t. $\sum_{i=1}^n h_i f_i = g$, but, since $r : \Gamma(E(D), \mathcal{O}_E) \longrightarrow \Gamma(D, \mathcal{O})$ is an isomorphism there are functions $h_i^* \in \Gamma(E(D), \mathcal{O}_E)$, with $r(h_i^*) = h_i$, s.t. $\sum_{i=1}^n h_i^* f_i^* = \pi^*(g)$. Now if $x \in \pi(E(D))$, $\exists\, y \in E(D)$, $\pi(y) = x$, so we find that $\pi^*(g) \in \mathcal{J}_y(\underline{f}^*)$, contradicting, by the above remark, our choice of g. The proposition is therefore proved. □

If $H^p(D,\mathcal{O}) = 0$ for p=1,2,....,n-1, then it is easy to show that $H^1(D,\mathcal{L}_0(\underline{f})) = 0$ (cfr. Prop. 3.1.3 ) for any $\underline{f}$ defining a point $x \in M - D$, and hence we can deduce that D is Stein. This suggests that we could use the test classes $\alpha_s(x,\underline{f})|_D$ for s=0,1,...,n-1 to measure how far D is from being Stein.

It follows from the above proposition that whether $\alpha_0(x,\underline{f})|_D$ vanishes or not is independent of choice of $\underline{f}$ and only depends on x and D. In the next section we shall prove that the same is true for $\alpha_s(x,\underline{f})|_D$ provided $f_1,f_2,....,f_n$ form a local coordinate system of M at x.

## § 4: Relationship between $\mathcal{K}^\cdot(\underline{f},\mathcal{O})$ and $\mathcal{K}^\cdot(\underline{z},\mathcal{O})$.

In what follows we shall suppose that $f_1,f_2,....,f_n$ is a general collection of global functions s.t. $V(f_1,f_2,....,f_n) = \{x\}$, and that $z_1,z_2,....,z_n$ are global functions that give local coordinates of M at x and s.t. $V(z_1,z_2,....,z_n) = \{x\}$. There is a relationship between $\underline{f}$ and $\underline{z}$: it is given by the following

Lemma 1.4.1 :- There exists a matrix valued holomorphic function $A: M \longrightarrow \mathbb{C}^{n\times n}$ s.t. $A\,\underline{z} = \underline{f}$.

Proof:- Let $\mathcal{J}$ denote the ideal sheaf of x. and $\mathcal{O}^{n\times n}$ the sheaf of germs of $n\times n$ matrices of holomorphic functions. As usual we shall indicate with $\mathcal{J}^n$ the sheaf of column vectors with entries in $\mathcal{J}$. Consider the exact sequence of sheaves

$$0 \longrightarrow \operatorname{Ker}\psi \longrightarrow \mathcal{O}^{n\times n} \xrightarrow{\psi} \mathcal{J}^n \longrightarrow 0$$

where $\psi(A) = A\,\underline{z}$ $\forall A \in \mathcal{O}^{n\times n}$ ( $\psi$ is surjective by definition of $z_1, z_2, \ldots\ldots, z_n$).

Since $\mathcal{O}^{n\times n}$ and $\mathcal{J}^n$ are coherent sheaves, also $\operatorname{Ker}\psi$ is coherent, and by Cartan's Theorem B, $H^1(M, \operatorname{Ker}\psi) = 0$.

The long exact sequence of cohomology associated to the above sequence of sheaves shows that the map

$$\psi : \Gamma(M, \mathcal{O}^{n\times n}) \longrightarrow \Gamma(M, \mathcal{J}^n)$$

is surjective. Moreover $\underline{f} \in \Gamma(M, \mathcal{J}^n)$ by definition and the conclusion follows. □

Lemma 1.4.2 ( corollary to Lemma 1.4.1):- There exists a map of complexes $A^{\cdot} : \mathcal{K}^{\cdot}(\underline{z}) \longrightarrow \mathcal{K}^{\cdot}(\underline{f})$ s.t. $A^0 : \mathcal{O} = \mathcal{K}^0(\underline{z}) \longrightarrow \mathcal{K}^0(\underline{f})$ is the identity.

Proof:- Identify, as usual $\mathcal{K}^1(\underline{z})$ and $\mathcal{K}^1(\underline{f})$ with $\mathcal{O}^n$ and define $A^1 : \mathcal{K}^1(\underline{z}) \longrightarrow \mathcal{K}^1(\underline{f})$ by means of the map A mentioned in the above Lemma: $A^{\cdot}$ is then automatically determined by the requirement for $A^0$ and by imposing it to be a map of graded algebras. One only needs to check that $A^{\cdot}$ is a map of complexes, and this is an immediate consequence of the fact that $A\,\underline{z} = \underline{f}$. □

We remark that $A^n : \mathcal{K}^n(\underline{z}) \longrightarrow \mathcal{K}^n(\underline{f})$ acts by multiplication with det A; we want now a map of complexes $B^{\cdot} : \mathcal{K}^{\cdot}(\underline{f}) \longrightarrow \mathcal{K}^{\cdot}(\underline{z})$ with suitable properties. To do this we need a purely algebraic

Lemma 1.4.3:- Let R be a commutative unitary ring, $\underline{z}, \underline{f} \in \Lambda^1 R^n$, and suppose that there exists a map of complexes

$$A^{\cdot} : K^{\cdot}(\underline{z}, R) \longrightarrow K^{\cdot}(\underline{f}, R)$$

s.t. $A^0 : K^0(z, R) \longrightarrow K^0(\underline{f}, R)$ is the identity;
Then there is also a map of complexes

$$B^{\cdot} : K^{\cdot}(\underline{f}, R) \longrightarrow K^{\cdot}(\underline{z}, R)$$

s.t. , after the natural identification of $K^n(\underline{f}, R)$ and $K^n(\underline{z}, R)$ with

R, $B^n: K^n(\underline{f},R) \longrightarrow K^n(\underline{z},R)$ is the identity.

<u>Proof</u>:- Let $R$-$\mathcal{M}od$ be the category of $R$-modules and consider the contravariant functor

$$\mathrm{Hom}(\ ,R) : R\text{-}\mathcal{M}od \longrightarrow R\text{-}\mathcal{M}od .$$

Using a classical notation, if $\tilde{M} \in R$-$\mathcal{M}od$, $\mathrm{Hom}(\tilde{M},R)$ will be indicated by $\tilde{M}^*$ (the dual of $\tilde{M}$), and if $h: \tilde{M} \longrightarrow \tilde{N}$ is a homomorphism of $R$-modules, $h^*$ will stand for $\mathrm{Hom}(h,R): \tilde{N}^* \longrightarrow \tilde{M}^*$.

By applying this functor to the complex $K^{\cdot}(\underline{z},R)$, (respectively $K^{\cdot}(\underline{f},R)$) we obtain a cocomplex $[K^{\cdot}(\underline{z},R)]^{\cdot}$ with differential $d^*_{\underline{z}}$ of degree -1 (resp. $[K^{\cdot}(\underline{f},R)]^{\cdot}$ with differential $d^*_{\underline{f}}$).

By functoriality of Hom ( ,R) the diagram

$$\text{(a)}\qquad \begin{array}{ccc} [K^{n-p}(\underline{f},R)]^* & \xrightarrow{\ [A^{n-p}]^*\ } & [K^{n-p}(\underline{z},R)]^* \\ \downarrow{\scriptstyle d^*_{\underline{f}}} & & \downarrow{\scriptstyle d^*_{\underline{z}}} \\ [K^{n-p-1}(\underline{f},R)]^* & \xrightarrow{\ [A^{n-p-1}]^*\ } & [K^{n-p-1}(\underline{z},R)]^* \end{array}$$

commutes $\forall p$.

Now define a $R$-homomorphism

$\Phi_p: K^p(\underline{f},R) \longrightarrow [K^{n-p}(\underline{f},R)]^*$, $\forall p$, as follows:

for all multiindexes $I$, $J$ with $|I| = p$, $|J| = n-p$, set

$$\langle \Phi_p(F_I), F_J \rangle = \begin{cases} 0 & \text{if } I \cup J \neq \{1,2,\dots,n\} \\ (-1)^{p(p-1)/2\ +\ \mathrm{sign}\,(I,J)} & \text{otherwise} \end{cases}$$

and extend it by $R$-linearity to all $K^p(\underline{f},R)$. It is immediate to check that $\Phi_p$ is an isomorphism.

Moreover we have the following identity:

$\forall I$, $\forall J$ with $|I| = p$, $|J| = n-p-1$ and $\forall i=1,2,\dots,n$

(b) $\langle \Phi_{p+1}(F_i \wedge F_I), F_J \rangle = \langle \Phi_p(F_I), F_i \wedge F_J \rangle$.

Indeed both sides vanish unless $I \cup J \cup \{i\} = \{1,2,\ldots\ldots,n\}$, and in this case we have:

$$p(p+1)/2 + \text{sign}\,(i,I,J) = p(p+1)/2 - p + \text{sign}\,(I,i,J) =$$
$$p(p-1)/2 + \text{sign}\,(I,i,J),$$

and the identity is proved.

The diagram

(c)
$$\begin{array}{ccc} K^p(\underline{f},R) & \xrightarrow{\;\Phi_p\;} & [K^{n-p}(\underline{f},R)]^* \\ \downarrow d_{\underline{f}} & & \downarrow d^*_{\underline{f}} \\ K^{p+1}(\underline{f},R) & \xrightarrow{\;\Phi_{p+1}\;} & [K^{n-p-1}(\underline{f},R)]^* \end{array}$$

commutes $\forall p$.

It is enough to prove that, $\forall I, \forall J$ with $|I| = p$, $|J| = n-p-1$,

$$\langle d^*_{\underline{f}} \circ \Phi_p(F_I), F_J \rangle = \langle \Phi_{p+1} \circ d_{\underline{f}}(F_I), F_J \rangle .$$

But

$$\langle d^*_{\underline{f}} \circ \Phi_p(F_I), F_J \rangle = \langle \Phi_p(F_I), d_{\underline{f}}(F_J) \rangle =$$
$$\langle \Phi_p(F_I), \textstyle\sum_{i=1}^n f_i\, F_i \wedge F_J \rangle = \textstyle\sum_{i=1}^n f_i \langle \Phi_p(F_I), F_i \wedge F_J \rangle$$

and, since the identity (b) holds, this is equal to

$$\textstyle\sum_{i=1}^n f_i \langle \Phi_{p+1}(F_i \wedge F_I), F_J \rangle = \langle \Phi_{p+1}( \textstyle\sum_{i=1}^n f_i\, F_i \wedge F_I), F_J \rangle =$$
$\langle \Phi_{p+1} \circ d_{\underline{f}}(F_I), F_J \rangle$, and the identity is proved.

Let us call $\Psi_p$ the homomorphism constructed in the analogous way that makes the diagram (d), appearing for typographical reasons in the following page, commutative $\forall p$.

Define the homomorphism $B^\cdot = \{B^p\} : K^\cdot(\underline{f},R) \longrightarrow K^\cdot(\underline{z},R)$, where $B^p = \Psi_p^{-1} \circ [A^{n-p}]^* \circ \Phi_p : K^p(\underline{f},R) \longrightarrow K^p(z,R)$.

$$
\text{(d)}\qquad
\begin{array}{ccc}
K^p(\underline{z},R) & \xrightarrow{\ \Psi_p\ } & [K^{n-p}(\underline{z},R)]^* \\
\Big\downarrow d_{\underline{z}} & & \Big\downarrow d^*_{\underline{z}} \\
K^{p+1}(\underline{z},R) & \xrightarrow{\ \Psi_{p+1}\ } & [K^{n-p-1}(\underline{z},R)]^*
\end{array}
$$

The fact that $B^{\cdot}$ is really a homomorphism of complexes follows from the commutativity of diagrams (a), (c), (d); moreover, since, after the natural identification of $K^n(\underline{f},R)$, $K^n(\underline{z},R)$, $[K^0(\underline{f},R)]^*$ and $[K^0(\underline{z},R)]^*$ with R, $\Phi_n = (-1)^{n(n-1)/2}$ id. $= \Psi_n$, we have $B^n$ = id. and the lemma is proved. □

The three lemmas contained in this section collected together give the following

<u>Proposition</u> 1.4.4:- There exists a homomorphism of complexes $B^{\cdot}: \mathcal{K}^{\cdot}(\underline{f}) \longrightarrow \mathcal{K}^{\cdot}(\underline{z})$ s.t. $B^n: \mathcal{K}^n(\underline{f}) \longrightarrow \mathcal{K}^n(\underline{z})$ is the identity. □

By using standard results of homological algebra and induction on $s=0,1,\ldots,n-2$, we can define sheaf homomorphisms $B: \mathcal{L}_s(\underline{f}) \longrightarrow \mathcal{L}_s(\underline{z})$ s.t. the following diagram with exact rows

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & \mathcal{L}_s(\underline{f}) & \longrightarrow & \mathcal{K}^{n-s-1}(\underline{f}) & \longrightarrow & \mathcal{L}_{s-1}(\underline{f}) & \longrightarrow & 0 \\
 & & \Big\downarrow B & & \Big\downarrow B & & \Big\downarrow B & & \\
0 & \longrightarrow & \mathcal{L}_s(\underline{z}) & \longrightarrow & \mathcal{K}^{n-s-1}(\underline{z}) & \longrightarrow & \mathcal{L}_{s-1}(\underline{z}) & \longrightarrow & 0
\end{array}
$$

commutes $\forall s$, the middle homomorphism being described in Prop. 1.4.4.

If $x \in M-D$, by taking the long exact sequences of cohomology we obtain a commutative diagram

$$
\begin{array}{ccc}
\Gamma(D,\mathcal{O}) & \xrightarrow{\ \delta_s(\underline{f})\ } & H^{s+1}(D,\mathcal{L}_s(\underline{f})) \\
\Big\downarrow B & & \Big\downarrow B \\
\Gamma(D,\mathcal{O}) & \xrightarrow{\ \delta_s(\underline{z})\ } & H^{s+1}(D,\mathcal{L}_s(\underline{z})) \,.
\end{array}
$$

But since $B: \Gamma(D,\mathcal{O}) \longrightarrow \Gamma(D,\mathcal{O})$ is the identity, $\forall g \in \Gamma(M,\mathcal{O})$

$$B(\alpha_s(g,\underline{f})|_D) = \alpha_s(g,\underline{z})|_D$$

and we have proved the following

Corollary 1.4.5:- (a) $\alpha_s(g,\underline{f})|_D = 0 \Longrightarrow \alpha_s(g,\underline{z})|_D = 0$,

(b) Whether $\alpha_s(g,\underline{z})|_D$ vanishes or not depends only on D, x and g and not on the particular choice of local coordinates $z_1,z_2,\ldots,z_n$ s.t. $V(z_1,z_2,\ldots,z_n) = \{x\}$ . □

Now suppose g = 1; this corollary allows us to simplify the notation and write $\alpha_s(x)$ instead of $\alpha_s(1,\underline{z})$ in the following definition and for the rest of the dissertation.

## §5:- The Projected q- Envelope of Holomorphy.

Definition 1.5.1:- (a) The projected q-envelope of holomorphy of D is the set

$$E_q(D) = D \cup \{x \in M - D \text{ s.t. } \alpha_q(x)|_D \neq 0\} \qquad q=0,1,\ldots,n-2.$$

(b) We say that D is $\alpha$-q-complete if $D = E_q(D)$.

Remark 1.5.2:- By construction $\alpha_q(x)|_D = 0 \Longrightarrow \alpha_{q+1}(x)|_D = 0$, so we have inclusion

$$D \subseteq E_{n-2}(D) \subseteq E_{n-3}(D) \subseteq \ldots\ldots \subseteq E_1(D) \subseteq E_0(D) = \pi E(D).$$

The last equality, which follows from Prop. 1.3.1, justifies the name given to $E_q(D)$.

Obviously D is $\alpha$-q-complete if and only if $\alpha_q(x)|_D = 0 \ \forall x \notin D$.

Unfortunately it is not clear that $E_q(D)$ is open, except in the case q = 0, although this does appear to be true in all cases I have checked.

$E_0(D)$ is Stein, and so $H^p(E_0(D),\mathcal{S}) = 0\ \forall p > 0\ \ \forall$ coherent sheaves $\mathcal{S}$. It would be desirable to have $H^p(E_q(D),\mathcal{S}) = 0\ \ \forall p > q$ $\forall \mathcal{S}$ coherent, at least when $E_q(D)$ is open; this is not the case, as it is shown in example 3.4.2.

In the next chapter we shall compare $\alpha$-c-completeness with other properties of open subsets of Stein manifolds, including cohomological completeness and pseudoconvexity, but before we want to give a first application of the test classes.

## § 6:- A Digression.

These few comments written here are motivated by purely aesthetics reasons and are inessential to the understanding of the rest of the dissertation, so that the reading of this section can be omitted; for this reason the style of the exposition will be less detailed.

Let $M$ be an analytic manifold and $x \in M$. Let $z_1, z_2, \ldots, z_n$ be local coordinates of $M$ at $x$, not necessarily globally defined, $U$ be an open neighbourhood of $x$ and $f_1, f_2, \ldots, f_n \in \Gamma(U,\mathcal{O})$ be s.t. $x$ is an isolated point of the variety $V(f_1, f_2, \ldots, f_n)$. Then there exists a germ of a $n \times n$ matrix valued holomorphic function $A$ s.t. $A\,\underline{\tilde{z}}_x = \underline{\tilde{f}}_x$.

Certainly it can be proved with a patient computation that $\det A \notin \mathcal{J}_x(\underline{f})$, but the following argument looks more elegant.

Since the problem is local we can suppose that $U$ is a polydisc in $\mathbb{C}^n$, that $A$ is defined and satisfies the equation $A\,\underline{z} = \underline{f}$ in all of $U$ and that $V(f_1, f_2, \ldots, f_n) \cap U = \{x\}$. Then we can construct the test classes

$$\alpha_s(x,\underline{z}) \in H^{s+1}(U - \{x\}, \mathcal{L}_s(\underline{z})) \quad \text{and} \quad \alpha_s(x,\underline{f}) \in H^{s+1}(U - \{x\}, \mathcal{L}_s(\underline{f}))$$

(cfr. ... ). Moreover the homomorphism B' in Prop. 1.4.4 has clearly the property that $B^0 : \mathfrak{K}^0(\underline{f}) \longrightarrow \mathfrak{K}^0(\underline{z})$ operates by multiplication with det A, so

$$\alpha_{n-2}(x,\underline{z}) = \det A \cdot \alpha_{n-2}(x,\underline{f}) = \alpha_{n-2}(\det A,\underline{f})$$

(the last equality follows from the fact that the connecting homomorphisms are $\mathcal{O}$-linear).

Using the fact that, if U' is any polydisc s.t. $x \in U' \subseteq U$, then $H^p(U' - \{x\},\mathcal{O}) = 0$ for $p=1,2,\dots,n-2$, (cfr.[10] Theorem 23), we claim that $\alpha_{n-2}(x,\underline{z})|_{U' - \{x\}} \neq 0$. Indeed if this is not the case, a simple reasoning leads to the conclusion $\alpha_0(x,\underline{z})|_{U' - \{x\}} = 0$ (cfr. Prop. 2.1.3). Thus, by Prop. 1.3.1, $x \notin S(U' - \{x\})$, which contradicts Riemann removable singularities theorem. The claim is therefore proved, and so also $\alpha_{n-2}(\det A,\underline{f})|_{U' - \{x\}} \neq 0$, which implies that $\alpha_0(\det A,\underline{f})|_{U' - \{x\}} \neq 0$, and so

$$\det A \notin (\mathrm{Im}\ d_{\underline{f}} : \Gamma(U',\mathfrak{K}^{n-1}) \longrightarrow \Gamma(U',\mathfrak{K}^n)) = \Gamma(U',\mathfrak{I}(\underline{f})),$$

and, since this is true for any U' as above, the conclusion follows.

CHAPTER 2: Pseudoconvexity and Completeness.

In this chapter we revise the classical concepts of completeness and pseudoconvexity and state a classical theorem comparing them with the property of $\mathbb{C}$-q-completeness described in chapter 1.

1: Hartogs' Figures and Completeness.

Let $\Delta$ denote the unit polydisc in $\mathbb{C}^n$ and consider the open sets

$$\Delta^q = \{z \in \Delta \text{ s.t. } |z_j| < \tfrac{1}{2} \text{ for } j > q\} \qquad q=1,2,\dots,n-1$$

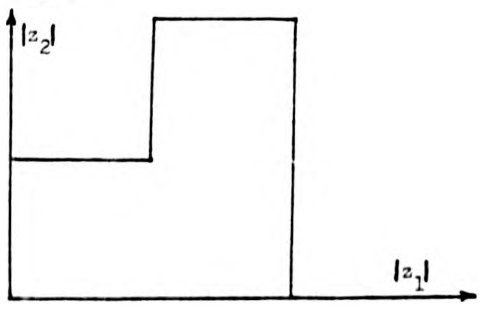
$$\Delta_k = \{z \in \Delta \text{ s.t. } \tfrac{1}{2} < |z_k| < 1\} \qquad k=1,2,\dots,n.$$

Definition 2.1.1:- For q=1,2,....,n-1 the q-Hartogs' figure is the open set

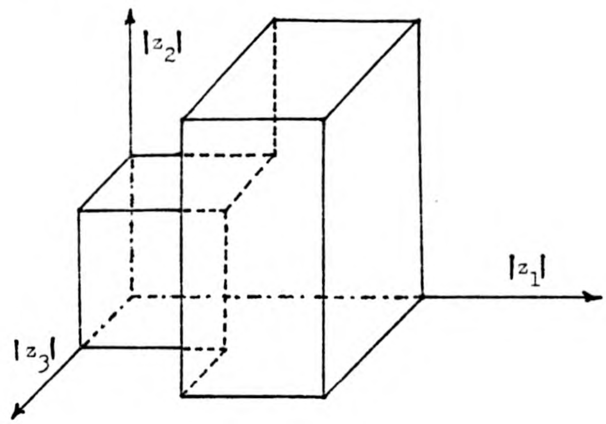
$$H_q = \Delta^q \cup \bigcup_{k=1}^{q} \Delta_k.$$

We give now the pictures of the Hartogs' figures in absolute space for n = 2 and for n = 3.

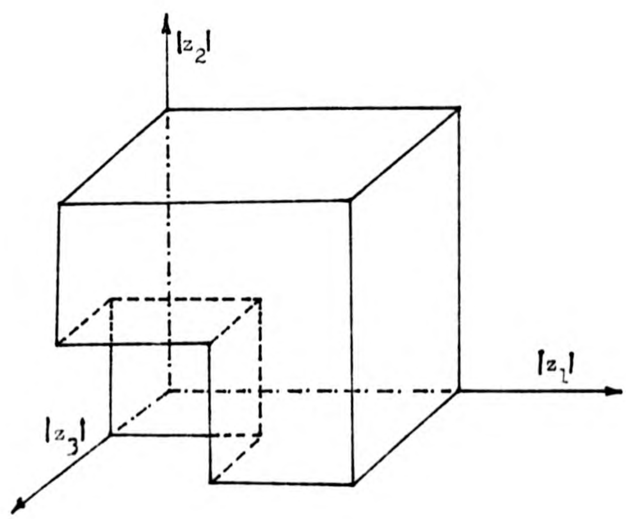
(a) Picture of $H_1 \subseteq \mathbb{C}^2$.



(b) Picture of $H_1 \subseteq \mathbb{C}^3$.




(c) Picture of $H_2 \subseteq \mathbb{C}^3$.



Lemma 2.1.2:- Every holomorphic fuction $g \in \Gamma(H_q, \mathcal{O})$ extends to a holomorphic function $h \in \Gamma(\Delta, \mathcal{O})$, i.e. $E_0(H_q) = \Delta$, $\forall q$.

Proof:- Set $B = \{z \in \Delta \text{ s.t. } |z_1| < \frac{3}{2}\}$ and define the holomorphic function $h_1 \in \Gamma(B, \mathcal{O})$ by

$$h_1(z_1, z_2, \ldots, z_n) = \frac{1}{2\pi i} \int_{|t| = \frac{3}{4}} \frac{g(t, z_2, \ldots, z_n)}{t - z_1}\, dt$$

( $h_1$ is well defined because $|t| = \frac{3}{4} \Rightarrow (t, z_2, \ldots, z_n) \in \Delta_1 \subseteq H_q$).

The functions $g$ and $h_1$ agree on the open set $\{z \text{ s.t. } |z_j| < \frac{1}{2}\ \forall j = 2, 3, \ldots, n\} \cap B$ by Cauchy integral formula and, by the uniqueness of analytic extension, they agree on $H_q \cap B$. Therefore they can be glued together to define a holomorphic function h on $H_q \cup B = \Delta$ that is the desired extension. □

Hartogs' figures enjoy interesting cohomological properties that can be deduced by using the open covering $\mathcal{U} = \{\Delta^q, \Delta_1, \ldots, \Delta_q\}$ of $H_q$. Since all sets in $\mathcal{U}$ are domains of holomorphy and therefore so are the intersections of any number of them, $\mathcal{U}$ is a Leray cover of $H_q$ and so, $\forall p$, $H^p(H_q, \mathcal{S}) = \check{H}^p(\mathcal{U}, \mathcal{S})$, for any coherent analytic sheaf $\mathcal{S}$ on $H_q$.

In particular, since $\mathcal{U}$ only has q+1 elements, $H^p(H_q, \mathcal{S}) = 0$, for $p > q$ and any coherent $\mathcal{S}$; a computation involving Laurent expansions and comparing of coefficients carried out with abundance of particulars in [2] p.218, shows that also $H^p(H_q, \mathcal{O}) = 0$ for $1 \leq p \leq q-1$.

We can now prove the following

Proposition 2.1.3:- $E_{q-1}(H_q) = \Delta$, yet $E_q(H_q) = H_q$.

Proof:- Take any point $x \in \mathbb{C}^n - H_q$, the sheaf $\mathcal{L}_q(\underline{z})$ is coherent on $H_q$, and so by the above comments $H^{q+1}(H_q, \mathcal{L}_q(\underline{z})) = 0$, therefore $\alpha_q(x)|_{H_q} = 0$ and thus $E_q(H_q) = H_q$ (i.e. $H_q$ is $\alpha$-q-complete).

On the other hand, consider the exact cohomology sequence

$$\cdots \longrightarrow H^s(H_q, \mathcal{R}^{n-s-1}(\underline{z})) \longrightarrow H^s(H_q, \mathcal{L}_{s-1}(\underline{z})) \xrightarrow{\delta} H^{s+1}(H_q, \mathcal{L}_s(\underline{z})).$$

$\mathcal{R}^{n-s-1}(\underline{z})$ is just the direct sum of some numbers of copies of $\mathcal{O}$, and so, by the above comment $H^s(H_q, \mathcal{R}^{n-s-1}(\underline{z})) = 0$ for $s=1,2,\ldots,q-1$, therefore $\delta$ is injective : then $\alpha_s(x)|_{H_q} = 0 \Longrightarrow \alpha_{s-1}(x)|_{H_q} = 0$, i.e. $E_s(H_q) = E_{s-1}(H_q)$ for $s=1,2,\ldots,q-1$: in particular we have $E_{q-1}(H_q) = E_0(H_q)$. But Lemma 2.1.2 says precisely that $E_0(H_q) = \Delta$ and the proposition is proved. □

In the above proof, $E_{q-1}(H_q) = E_0(H_q)$ was deduced using exclusively a cohomological property of $H_q$, namely from $H^p(H_q, \mathcal{O}) = 0$ for $1 \leq p < q$, so the same can be said for the sets $H_q$ appearing in the following

<u>Definition</u> 2.1.4:- (a) An open subset $H_q$ of a Stein manifold $M$ is said to be a q+1- <u>general</u> <u>Hartogs'</u> <u>figure</u> if $H^p(H_q, \mathcal{O}) = 0$ for $p=1,2,\ldots,q$.

(b) D is said to be <u>Hartogs'</u> <u>q-complete</u> if, for any general q+1 -Hartogs' figure $H_{q+1} \subseteq D$, also $\pi(E(H_{q+1})) = E_0(H_{q+1}) \subseteq D$.

The hypothesis that M is Stein is necessary to guarantee that $E(H_{q+1})$ exists or that $E_0(H_{q+1})$ can be defined.

## § 2:- q-Pseudoconvexity and q-Completeness.

We recall now the basic definitions of q-pseudoconvexity and q-completeness. In this section we only assume that M is an analytic manifold not necessarily Stein.

If $x \in \partial D$, the boundary of D, it is always possible to find a neighbourhood U of x and a <u>defining</u> <u>function</u> $\phi: U \longrightarrow \mathbb{R}$ of class $C^2$ such that

$$D \cap U = \{ y \in U \text{ s.t. } \Phi(y) < \Phi(x) \} .$$

(see [3] p. 202). If $\Phi$ can be chosen to be non singular at x, we say that D has $C^2$ boundary at x and if this is true at all points of $\partial D$ we say that D has $C^2$ boundary.

Let $\Phi$ be a defining function for D near $x \in \partial D$, and suppose $z_1, z_2, \ldots, z_n$ are local coordinates of M at x: the <u>complex Hessian</u> of $\Phi$ at x is the matrix

$$(\mathcal{H}\Phi)(x) = \left( \frac{\partial^2 \Phi(x)}{\partial z_i \partial \bar{z}_j} \right)_{i,j=1}^{n} .$$

$(\mathcal{H}\Phi)(x)$ is a Hermitian matrix and so there is a Hermitian form called again $(\mathcal{H}\Phi)(x)$ associated to it: $(\mathcal{H}\Phi)(x): T_x M \longrightarrow \mathbb{R}$ is given by

$$(\mathcal{H}\Phi)(x)\ (v) = \sum_{i,j=1}^{n} \frac{\partial^2 \Phi(x)}{\partial z_i \partial \bar{z}_j} v_i \bar{v}_j$$

$\forall v \in T_x M$, where $v_1, v_2, \ldots, v_n$ are the components of v with respect to the basis $\left( \frac{\partial}{\partial z_1}, \frac{\partial}{\partial z_2}, \ldots, \frac{\partial}{\partial z_n} \right)$ of the holomorphic tangent space $T_x M$ of M at x.

It is easy to check that the Hermitian form $(\mathcal{H}\Phi)(x)$ is an invariant under change of holomorphic coordinates (see [11] p.261), and so its signature does not depend on the particular choice of $(z_1, z_2, \ldots, z_n)$.

Now suppose that the defining function $\Phi$ is non singular at x, then it makes sense to consider the holomorphic tangent space $T_x \partial D$ of $\partial D$ at the regular point x:

$$T_x \partial D =_{\text{def.}} \left\{ v = \sum_{i=1}^{n} v_i \frac{\partial}{\partial z_i} \in T_x M \text{ s.t. } \sum_{i=1}^{n} \frac{\partial \Phi}{\partial z_i}(x)\, v_i = 0 \right\} .$$

(notice that $\frac{\partial \Phi}{\partial z_i}(x) \neq 0$ for at least one index i because $\Phi$ is non singular at x).

The restriction of $(\mathcal{H}\Phi)(x)$ to $T_x\partial D$ is called the <u>Levi form</u> of $\Phi$ at x and is indicated by $(\mathcal{L}\Phi)(x)$ ; it is easy to check that its signature does not depend even on the choice of the non singular defining function $\Phi$ for D at x. Therefore, provided $\Phi$ is non singular at x, the number of positive, negative and zero eigenvalues of $(\mathcal{L}\Phi)(x)$ are invariants depending only on x and D: they will be denoted respectively by $p(x,D)$, $n(x,D)$ and $z(x,D)$ or with $p(x)$, $n(x)$ and $z(x)$ when no confusion is possible.

<u>Definition</u> 2.2.1:- If D has $C^2$ boundary we say that D is (<u>weakly</u>) q-<u>pseudoconvex</u> if $n(x) \leq q \quad \forall x \in \partial D$. (cfr.[4]p. 209).

To give an interpretation of these three numbers we recall that if V is any complex linear subspace of $T_xM$ the Hermitian forms on $T_xM$ can be partially preordered as follows:
if A and B are two such forms we say that $A \geq B$ (resp. $A > B$) on V if, $\forall v \in V$, $A(v) \geq B(v)$ (resp. $\forall v \in V - \{0\}$ , $A(v) > B(v)$ ).

In particular if 0 denotes the zero Hermitian form and $A > 0$ (resp. $A < 0$, $A \geq 0$, $A \leq 0$) on V, we say that A is positive definite (resp. negative definite, positive semidefinite, negative semidefinite) on V.

Elementary facts in linear algebra say that the interpretation of $p(x)$, $n(x)$ and $z(x)$ is the following:
We can split $T_x\partial D$ in three linear subspaces $V_p \oplus V_n \oplus V_z = T_x\partial D$, s.t. $(\mathcal{H}\Phi)(x)$ (or $(\mathcal{L}\Phi)(x)$ ) is positive definite on $V_p$ and negative definite on $V_n$ , with $\dim_{\mathbb{C}} V_a = a(x)$ , $a=p,n,z$, and vanishes on $V_z$.

In order to define q-pseudoconvexity we need a $C^2$ boundary; to get rid of this assumption we introduce the concept of q-plurisubharmonic exhaustion function.

Here D only needs to be an analytic manifold not necessarily contained in a larger one.

Definition 2.2.2:- (a) A $C^2$ function $\Phi : D \longrightarrow \mathbb{R}$ is said to be an exhaustion function if, for all $c \in \mathbb{R}$, the sets

$$B_c = \{ x \in D \text{ s.t. } \Phi(x) < c \}$$

are compact.

(b) $\Phi$ is said to be a q-plurisubharmonic function if, $\forall x \in D$, the Hessian $(\mathcal{H}\Phi)(x)$ has at least n-q positive eigenvalues.

(c) We say that D is q-complete if there exists a q-plurisubharmonic exhaustion function $\Phi : D \longrightarrow \mathbb{R}$ .

Obviously $\Phi$ is q-plurisubharmonic if and only if, $\forall x \in D$, there exists a linear subspace V of dimension n-q of $T_x D$ s.t. the Hessian $(\mathcal{H}\Phi)(x)$ is positive definite on V.

For more informations about q-completeness see [17].

Definition 2.2.3:- An analytic manifold D is said to be cohomologically q-complete if, for all coherent analytic sheaf $\mathcal{S}$ on D and all $p > q$, $H^p(D, \mathcal{S}) = 0$ (see [17] p. 443).

## §3:- The Main Classical Theorem.

We shall collect now in a theorem the well known relations between the various concepts of completeness and pseudoconvexity defined so far.

Theorem 2.3.1 If D is a domain in a Stein manifold M the following conditions are equivalent:

(a) D is 0-complete,

(b) D is cohomologically 0-complete,

(b') $H^p(D,\mathcal{O}) = 0$ for $p > 0$,

(c) D is $\alpha$-0-complete,

(d) D is Hartogs' 0-complete,

(e) D is Stein (and in this case we say that D is a domain of holomorphy, as if $M = \mathbb{C}^n$),

and if D has $C^2$ boundary they are also equivalent to

(a') D is (weakly) 0-pseudoconvex.

<u>Proof</u>:- (a)$\Longleftrightarrow$(e) see [15].

(e)$\Longrightarrow$(b) is Cartan's theorem B [11] p.243 .

(b)$\Longrightarrow$(e) is easy and is proved in [11] p. 246.

(b)$\Longrightarrow$(b') trivially.

(b')$\Longrightarrow$(c) because if (b) is true then $\forall x \in M - D$, $H^1(D, \mathcal{L}_0(\underline{z})) = 0$ (cfr. Prop. 3.1.3 ).

(c)$\Longrightarrow$(e) by proposition 1.3.1.

(e)$\Longrightarrow$(a') if D has $C^2$boundary is proved in [11] p. 264, see also proposition ?.2.4 .

(d)$\Longleftrightarrow$(e) is trivial because D is itself a general 1-Hartogs' figure.

(a')$\Longrightarrow$(a) is done in case $M = \mathbb{C}^n$ in [12] p. 49, and the general case can be somehow reduced to this: this implication is proved in detail later in proposition 3.1.2. □

The principal aim of this dissertation is to see what happens if we replace 0 by q in the above statement. The reason why this theorem appears at this stage of the dissertation, in spite of the fact that some of the tools needed to prove the implications (b')$\Longrightarrow$(c) and (a')$\Longrightarrow$(a) have not yet been developed, is to emphasise the classical relevance of the problem. These two implications will not be used in what follows until a precise proof is given, so that the mathematical correctness of the dissertation is not affected.

CHAPTER 3:- A Generalisation of the Main Classical Theorem.

§1:- Relationships between q-Pseudoconvexity and Various Forms of q-Completeness.

Proposition 3.1.1 :- A q-complete analytic manifold is cohomologically q-complete.

Proof:- Apart from some irrelevant difference of notation this is the corollary on page 250 of [2]. □

Proposition 3.1.2 :- If D is a domain with $C^2$ boundary in a Stein manifold M and D is q-pseudoconvex then it is also q-complete.

Proof:- We shall divide the proof in several steps.

Step 1:- As there is always an analytic embedding of M into $\mathbb{C}^N$, for some large N (see [15] p.359) we can suppose at once that M is an analytic submanifold of $\mathbb{C}^N$. Choose a holomorphic tubular neighbourhood $p : V \longrightarrow M$ and set $\tilde{D} = p^{-1}(D)$ (cfr. [6] proof of lemma 1 , p. 131). We claim that, after shrinking V if necessary,

(a) $\forall x \in \partial\tilde{D}$, $\partial\tilde{D}$ is $C^2$ at x,

(b) If we consider $\tilde{D}$ as an open subset of $\mathbb{C}^N$ then $n(x,\tilde{D}) = n(p(x),D)$, for all $x \in \partial\tilde{D}$.

Indeed, since the problem is local we can suppose that local coordinates $z_1,z_2,\ldots,z_N$ have been chosen s.t., near x, $M = \{z \text{ s.t. } z_{N-n+1}=z_{N-n+2}=\ldots=z_N=0\}$ , $z_1,z_2,\ldots,z_n$ are local coordinates of M at x and $p(z_1,z_2,\ldots,z_N) = (z_1,z_2,\ldots,z_n,0,\ldots,0)$.

Let $\tilde{U}$ be a neighbourhood of x in $\mathbb{C}^N$ so small that $z_1,z_2,\ldots,z_N$ are defined in $\tilde{U}$ and that there exists a $C^2$ defining function $\Phi : U = \tilde{U} \cap M \longrightarrow \mathbb{R}$ for D with $d\Phi(x) \neq 0$ and $\tilde{U} \subseteq V$. By shrinking $\tilde{U}$ if necessary we can also suppose that $\tilde{U} \subseteq p^{-1}(U)$.

Define $\tilde{\Phi} : \tilde{U} \longrightarrow \mathbb{R}$ by $\tilde{\Phi} = \Phi \circ p$ i.e. $\tilde{\Phi}(z_1, z_2, \ldots, z_N) = \Phi(z_1, z_2, \ldots, z_n, 0, \ldots, 0)$. Then $\tilde{\Phi}$ is a defining function for $\tilde{D}$ at x. Moreover

$$T_x \partial\tilde{D} = \left\{ v \in T_x\mathbb{C}^N \text{ s.t. } \sum_{i=1}^N \frac{\partial\tilde{\Phi}}{\partial z_i}(x)\, v_i = 0 \right\} =$$

$$\left\{ v \in T_x\mathbb{C}^N \text{ s.t. } \sum_{i=1}^n \frac{\partial\Phi}{\partial z_i}(p(x))\, v_i = 0 \right\} \simeq T_{p(x)}\partial D$$

where as usual $v = \sum_{i=1}^N v_i \frac{\partial}{\partial z_i}$, and

$$\frac{\partial^2\tilde{\Phi}(x)}{\partial z_i \partial \bar{z}_j} = \begin{cases} \dfrac{\partial^2 \Phi(p(x))}{\partial z_i \partial \bar{z}_j} & \text{if } i,j \leq n \\ 0 & \text{otherwise.} \end{cases}$$

This proves the claim.

Step 2:- So, if we suppose that D is q-pseudoconvex we have that, $\forall x$ in $\partial\tilde{D}$, $\partial\tilde{D}$ is $C^2$ at x and $n(x, \tilde{D}) \leq q$.

Consider the function $\rho : \mathbb{C}^N \longrightarrow \mathbb{R}$ given by

$$\rho(y) = \begin{cases} \text{dist}(y, \partial\tilde{D}) & \text{if } y \in \bar{\tilde{D}} \\ -\text{dist}(y, \partial\tilde{D}) & \text{if } y \in \mathbb{C}^N - \tilde{D}, \end{cases}$$

Where dist denotes the Euclidean distance. Since $\forall x \in \partial\tilde{D}, \partial\tilde{D}$ is $C^2$ at x, we can conclude that there exists a neighbourhood $\tilde{U}'$ of $\partial\tilde{D}$ in $\mathbb{C}^N$ on which $\rho$ is $C^2$ (see appendix on page 61).

By shrinking $\tilde{U}'$ if necessary, we can also suppose that $\forall y$ in $\tilde{U}'$ there exists exactly one point $c(y) \in \partial\tilde{D} \cap \tilde{U}'$ which is the closest point to y under the Euclidean distance, that $d\rho(c(y)) \neq 0$ and that $n(c(y), \tilde{D}) \leq q$.

Let $\varphi : \tilde{U}' \cap \tilde{D} \longrightarrow \mathbb{R}$ be the function $\varphi = \log\rho$ : we claim that the Hessian $(\mathcal{H}\varphi)(y)$ has at most q positive eigenvalues $\forall y$.

Indeed suppose that this is false, i.e. there exists a point y in $\tilde{U}' \cap \tilde{D}$ s.t. $(\mathcal{H}\varphi)(y)$ has (at least) q+1 positive eigenvalues: according to the remark on page 25, there exist linear coordinates

$(t_1, t_2, \ldots, t_N)$ of $\mathbb{C}^N$ s.t. the Hermitian form given by the matrix

$$(c_{jk})_{j,k=1}^{q+1} = \left(\frac{\partial^2 \varphi(y)}{\partial t_j \partial \bar{t}_k}\right)_{j,k=1}^{q+1}$$

is positive definite on the linear subspace V of $T_y\mathbb{C}^N = \mathbb{C}^N$ spanned by $\left(\frac{\partial}{\partial t_1}, \frac{\partial}{\partial t_2}, \ldots, \frac{\partial}{\partial t_{q+1}}\right)$.

By Taylor's theorem we have

$$\varphi\left(y + \sum_{j=1}^{q+1} t_j \frac{\partial}{\partial t_j}\right) = \log \rho\left(y + \sum_{j=1}^{q+1} t_j \frac{\partial}{\partial t_j}\right) = \log \rho(y) +$$

$$\mathrm{Re}\left(\sum_{i=1}^{q+1} a_i t_i + \sum_{j,k=1}^{q+1} b_{jk} t_j t_k\right) + \sum_{j,k=1}^{q+1} c_{jk} t_j \bar{t}_k + O(|t|^2),$$

where $a_i = \frac{1}{2}\frac{\partial \varphi}{\partial t_i}(y)$ and $b_{jk} = \frac{\partial^2 \varphi(y)}{\partial t_j \partial t_k}$ are constants, and $O(|t|^2)$ has the property that $\lim_{t \to 0} \frac{O(|t|^2)}{|t|^2} = 0$ and so also

$$\lim_{t \to 0} \frac{O(|t|^2)}{\sum_{j,k}^{q+1} c_{jk} t_j \bar{t}_k} = 0 .$$

In order to simplify notation omit the limits of the summands and write $A(t) = y + \sum t_j \frac{\partial}{\partial t_j}$, $B(t) = \exp\left(\sum a_i t_i + \sum b_{jk} t_j t_k\right)$.

Then the above equality can be written as

$$\rho(A(t)) - \rho(y)|B(t)| = \left\{\exp\left(\sum c_{jk} t_j \bar{t}_k + O(|t|^2)\right) - 1\right\} \rho(y)|B(t)| =$$

$\left\{\sum c_{jk} t_j \bar{t}_k + O'(|t|^2)\right\} \rho(y)|B(t)|$, where the last equality is obtained by expanding in Taylor series the function exp and $O'(|t|^2)$ has the same properties as $O(|t|^2)$. Then one has

$$\lim_{t \to 0} \frac{\rho(A(t)) - \rho(y)|B(t)|}{\sum c_{jk} t_j \bar{t}_k} = \rho(y),$$

so we can choose $\varepsilon > 0$ small enough s.t. $\forall t$, $|t| < \varepsilon$, one has

(a) $A(t) \in \tilde{D} \cap \tilde{U}'$ and

(b) $\rho(A(t)) - \rho(y)|B(t)| > \frac{\rho(y)}{2} \cdot \sum c_{jk} t_j \bar{t}_k$.

Set $u = c(y) - y$ and define an analytic function T on the open ball $B_\varepsilon = \{t \in \mathbb{C}^{q+1} \text{ s.t. } |t| < \varepsilon\}$ :

$T: B_\varepsilon \longrightarrow \mathbb{C}^N$ is given by $T(t) = A(t) + u\, B(t)$.

We can also suppose that $\varepsilon$ is so small that $T(t) \in \tilde{U}'$ if $t \in B_\varepsilon$ . Then it is easy to check, and a picture shows how, that if $t \in B_\varepsilon$ one has

(c) $\rho(T(t)) \geqslant \rho(A(t)) - |u|\,|B(t)| \geqslant \frac{|u|}{2} \sum c_{jk}\, t_j \bar{t}_k \geqslant 0$,

This in particular proves that $T(t) \in \tilde{D}$ for all $t \in B_\varepsilon - \{0\}$ and, since $\rho(T(0)) = \rho(c(y)) = 0$, 0 is a minimum for the function $\rho \circ T : B_\varepsilon \longrightarrow \mathbb{R}$, and so, taking partial derivatives,

$\frac{\partial \rho \circ T(0)}{\partial t_j} = 0$ for all $j=1,2,\dots,q+1$.

Using the chain rule and the fact that $T$ is analytic we have:

(d) $\sum_{h=1}^{N} \frac{\partial \rho}{\partial z_h}(c(y))\, \frac{\partial T_h}{\partial t_j}(0) = 0$ for $j=1,2,\dots,q+1$.

In other words the vectors $\frac{\partial T}{\partial t_j}(0)$, $j=1,2,\dots,q+1$ , are in $T_{c(y)}\partial\tilde{D}$.

Moreover, $\forall t$ in $\mathbb{C}^{q+1}$, we have

(e) $\sum_{j,k=1}^{q+1} \frac{\partial^2 \rho \circ T(0)}{\partial t_j \partial \bar{t}_k}\, t_j \bar{t}_k \geqslant \frac{|u|}{4} \sum_{j,k=1}^{q+1} c_{jk}\, t_j \bar{t}_k$ .

To prove this we first observe that it is clearly enough to check it for small $|t|$.

From the above inequality (c), using Taylor series, we deduce

$$\mathrm{Re}\Big(\sum d_{jk}\, t_j t_k\Big) + \sum \frac{\partial^2 \rho \circ T(0)}{\partial t_j\, \partial \bar{t}_k}\, t_j \bar{t}_k + 0''(|t|^2) \geqslant \frac{|u|}{2} \sum c_{jk}\, t_j \bar{t}_k,$$

for all $t \in B_\varepsilon$ , where $d_{jk} = \frac{\partial^2 \rho \circ T(0)}{\partial t_j\, \partial t_k}$ are constants and $0''(|t|^2)$ has the same properties as $0(|t|^2)$.

Then, after reducing $\varepsilon$ if necessary, we have, $\forall t \in B_\varepsilon$ ,

$$\mathrm{Re}\Big(\sum d_{jk}\, t_j t_k\Big) + \sum \frac{\partial^2 \rho \circ T(0)}{\partial t_j\, \partial \bar{t}_k}\, t_j \bar{t}_k \geqslant \frac{|u|}{4} \sum c_{jk}\, t_j \bar{t}_k .$$

Let $t'_j = e^{i\theta} t_j$ for $0 \leq \theta \leq 2\pi$ ; writing t' in the above inequality and observing that the second and third term are unchanged under the substitution $t \longrightarrow t'$, we deduce , $\forall \theta$,

$$\mathrm{Re}\left( e^{i2\theta} \sum d_{jk}\, t_j t_k \right) + \sum \frac{\partial^2 \rho \bullet T(0)}{\partial t_j\, \partial \bar{t}_k}\, t_j \bar{t}_k \geq \frac{|u|}{4} \sum c_{jk}\, \bar{t}_j t_k \;,$$

and by choosing $\theta$ so that the first term is negative we prove the inequality (e).

Using again the chain rule and the fact that T is analytic we have that the Hermitian form

$$\left( \sum_{h,m=1}^{N} \frac{\partial^2 \rho(c(y))}{\partial z_h\, \partial \bar{z}_m} \cdot \frac{\partial T_h}{\partial t_j}(0) \left( \overline{\frac{\partial T_m}{\partial t_k}(0)} \right) \right)_{j,k=1}^{q+1}$$

is positive definite.

It follows easily that the Hermitian form $\left( \frac{\partial^2 \rho(c(y))}{\partial z_h\, \partial \bar{z}_m} \right)_{h,m=1}^{N}$ is positive definite on the linear subspace V of $T_{c(y)}\partial\tilde{D}$ spanned by the vectors $\frac{\partial T}{\partial t_j}(0)$ , j=1,2,.....,q+1 ; in particular it follows automatically that these vectors are linearly independent, so that $\dim_{\mathbb{C}} V = q+1$ ; but since $-\rho$ is a defining function for $\tilde{D}$ at c(y), we have that $n(c(y),\tilde{D}) \geq q+1$ and this contradicts our hypothesis, so that the claim is proved.

Step 3:- By restricting $\varphi$ to $\tilde{U}' \cap D$ we find a $C^2$ function, called again $\varphi : W = \tilde{U}' \cap D \longrightarrow \mathbb{R}$ s.t.

(a) $\lim_{y \longrightarrow \partial D} \varphi(y) = -\infty$ ,

(b) $(\mathcal{H}\varphi)(y)$ has at most q positive eigenvalues $\forall y$ in W.

Let F be a closed subset of M s.t. $D - W \subseteq \mathrm{int}\, F \subseteq F \subseteq D$ , and let $0 \leq \psi \leq 1$ be a $C^2$ bump function s.t. $\psi = 0$ on F, $\psi = 1$ in a neighbourhood of M − D , and suppose that F is chosen so that $\varphi(y) \leq 0$ for $y \notin F$.

By considering the function $\varphi' = \varphi \cdot \psi$ , we have that

(a) $\lim_{y \to \partial D} \varphi'(y) = -\infty$,

(b) $(\mathcal{H}\varphi')(y)$ has at most q positive eigenvalues $\forall y \in D - F$,

(c) $\varphi' \leq 0$.

Now we use the fact that M is Stein and so 0-complete (see theorem 2.?.1) i.e. there exists a 0-plurisubharmonic exhaustion function $\lambda : M \longrightarrow \mathbb{R}$.

$\forall n \in \mathbb{Z}$, the set $K_n = \{y \in M \text{ s.t. } \lambda(y) \leq n\}$ is compact, therefore so is $F \cap K_n$ and there exist constants $C_n$ s.t.

$C_n (\mathcal{H}\lambda)(y) - (\mathcal{H}\varphi')(y) > 0 \quad \forall y \in F \cap K_n$.

Now choose a $C^2$ function $f : \mathbb{R} \longrightarrow \mathbb{R}$ with the properties

(a) $f' > 0$, $f'' > 0$ always,

(b) $f'(r) > C_{E(r)+1}$, r, where $E(r)$ denotes the integral part of r.

(c) $f'(r) > C_0 \quad \forall r$, and consider the $C^2$ function

$\chi = f \circ \lambda - \varphi' : D \longrightarrow \mathbb{R}$.

First we notice that, $\forall c \in \mathbb{R}$, $B_c = \{y \in D \text{ s.t. } \chi(y) \leq c\}$ is contained, by the property (c) of $\varphi'$ in $\{y \in D \text{ s.t. } f \circ \lambda(y) \leq c\}$ which is compact by the assumptions on f and $\lambda$. Moreover $B_c$ is closed in D and, since $\lim_{y \to \partial D} \varphi'(y) = -\infty$, it is also closed in M. Thus $B_c$ is compact and $\chi$ is an exhaustion function.

For all $y \in D$ we have

$(\mathcal{H}\chi)(y) = f''(\lambda(y)) \cdot A(y) + f'(\lambda(y)) \cdot (\mathcal{H}\lambda)(y) - (\mathcal{H}\varphi')(y)$,

where $A(y) = \left(\frac{\partial \lambda}{\partial z_i}(y) \cdot \overline{\frac{\partial \lambda}{\partial z_j}}(y)\right)_{i,j=1}^{n}$ is a semipositive Hermitian form. If $y \in D - F$ then there exists a linear subspace V of $T_y D$, of dimension n-q where $-(\mathcal{H}\varphi')(y)$ is positive semidefinite. Therefore $(\mathcal{H}\chi)(y)$ is positive definite on V.

If $y \in F$ then either $y \in K_0 \cap F$ in which case

$$(\mathcal{H}\chi)(y) \geq f'(\lambda(y))\,(\mathcal{H}\lambda)(y) - (\mathcal{H}\varphi')(y) \geq$$
$$c_0\,(\mathcal{H}\lambda)(y) - (\mathcal{H}\varphi')(y) > 0 \quad ,$$

or $y \in (K_{n+1} - K_n) \cap F$ for some integer $n \geq 0$, in which case

$f'(\lambda(y)) > c_{n+1}$ and so

$$(\mathcal{H}\chi)(y) > c_{n+1}\,(\mathcal{H}\lambda)(y) - (\mathcal{H}\varphi')(y) > 0.$$

Therefore $\chi$ is also q-plurisubharmonic and we can finally say that the proposition is proved. □

<u>Proposition</u> 3.1.3:- If $H^p(D, \Theta) = 0$ for $p > q$, then for all $x \in M - D$, $H^s(D, \mathcal{L}_{s-1}(\underline{z})) = 0$ for $s > q$.

<u>Proof</u>:- Since $\mathcal{L}_{n-2}(\underline{z}) = \Theta$ the conclusion is vacuous unless $q \leq n-2$, and so assume that this is the case and, in particular that $H^{n-1}(D, \Theta) = 0$ .

The long exact cohomology sequence associated to

$$0 \longrightarrow \mathcal{L}_s(\underline{z}) \longrightarrow \mathcal{R}^{n-s-1}(\underline{z}) \longrightarrow \mathcal{L}_{s-1}(\underline{z}) \longrightarrow 0$$

together with the hypothesis shows that

$$H^s(D, \mathcal{L}_{s-1}(\underline{z})) \simeq H^{s+1}(D, \mathcal{L}_s(\underline{z})) \quad \text{for} \quad s > q.$$

Thus $H^s(D, \mathcal{L}_{s-1}(\underline{z})) \simeq H^{n-1}(D, \mathcal{L}_{n-2}(\underline{z})) = H^{n-1}(D, \Theta) = 0.$ □

<u>Proposition</u> 3.1.4:- If $H^p(D, \Theta) = 0$ for $p > q$, then D is $\alpha$-q-complete.

<u>Proof</u>:- By the above prop., $\forall x \in M - D$, $H^{q+1}(D, \mathcal{L}_q(\underline{z})) = 0$ and so $\alpha_q(x)|_D = 0$ . By remark 1.5.2 D is $\alpha$-q-complete. □

<u>Proposition</u> 3.1.5:- If D is $\alpha$-q-complete it is also Hartogs' q-complete.

<u>Proof</u>:- By what remarked just before def. 2.1.4 we know that, for any general $q+1$ - Hartogs' figure $H_{q+1}$ , $E_0(H_{q+1}) = E_q(H_{q+1})$.

So if $H_{q+1} \subseteq D$,

$$\pi(E(H_{q+1})) = E_0(H_{q+1}) = E_q(H_{q+1}) \subseteq E_q(D) = D,$$

where the last equality is by assumption. Therefore D is Hartogs' q-complete. □

To proceed further we need a result due to Andreotti and Grauert: we drop again the hypothesis that M is Stein.

<u>Proposition</u> 3.1.6:- Let M be a complex manifold, D an open subset of M, x a point in the boundary of D, U a neighbourhood of x and $\varphi: U \longrightarrow \mathbb{R}$ a $C^2$ defining function for D in U s.t. the complex Hessian $(\mathcal{H}\varphi)(y)$ has at least $k \geq 2$ negative eigenvalues for all y in U. Then there exist arbitrarily small open neighbourhoods Q of x with

(a) $H^p(D \cap Q, \mathcal{O}) = 0$ for $1 \leq p < k-1$ and

(b) The restriction map $\Gamma(Q, \mathcal{O}) \longrightarrow \Gamma(D \cap Q, \mathcal{O})$

surjective.

<u>Proof</u>: See [2], Proposition 12, p. 222. □

To use this result we come back immediately to the case when M is Stein.

<u>Proposition</u> 3.1.7:- If D is a domain with $C^2$ boundary in a Stein manifold M and D is Hartogs' q-complete then it is also q-pseudoconvex.

<u>Proof</u>:- Let x be any point in the boundary of D, U an open neighbourhood of x and $\psi: U \longrightarrow \mathbb{R}$ a $C^2$ defining function for D in U with $d\psi(x) \neq 0$.

If $f: \mathbb{R} \longrightarrow \mathbb{R}$ is any $C^2$ function then

$$(\mathcal{H}(f \circ \psi))(x) = f''(\psi(x)) \cdot A(x) + f'(\psi(x)) \cdot (\mathcal{H}\psi)(x)$$

where $A(x) = \left( \frac{\partial \Psi}{\partial z_i}(x) \overline{\frac{\partial \Psi}{\partial z_j}(x)} \right)_{i,j=1}^{n}$ is a positive semidefinite Hermitian form vanishing on $T_x \partial D$ and positive definite on the complex line spanned by the vector $v = \left( \frac{\partial \Psi}{\partial z_1}(x), \frac{\partial \Psi}{\partial z_2}(x), \ldots, \frac{\partial \Psi}{\partial z_n}(x) \right)$

In particular choose f given by $f(t) = -e^{-ct}$ where c is a constant in $\mathbb{R}$ s.t. $c > 0$ and

$$c > \frac{\sum_{i,j=1}^{n} \frac{\partial^2 \Psi(x)}{\partial z_i \partial \bar{z}_j} \frac{\partial \Psi}{\partial z_i}(x) \overline{\frac{\partial \Psi}{\partial z_j}(x)}}{\sum_{i,j=1}^{n} \left( \frac{\partial \Psi}{\partial z_i}(x) \right)^2 \left( \overline{\frac{\partial \Psi}{\partial z_j}(x)} \right)^2} = \frac{(\mathcal{H}\Psi)(x)(v)}{A(x)(v)} .$$

Then $f \circ \Psi = -e^{-c\Psi}$ is a defining function for D in U.

Let $V_n$ be a linear subspace of $T_x D$, with $\dim_{\mathbb{C}} V_n = n(x)$ (see remark on page 25), where $(\mathcal{H}\Psi)(x)$ is negative definite and let V be the linear subspace of $T_x M$ spanned by $V_n$ and v.

Then V has dimension $n(x)+1$ and if $w = u + \lambda v$, for some $u \in V_n$ and some $\lambda \in \mathbb{C}$, is a non zero vector in V then

$$(\mathcal{H} f \circ \Psi)(x)(w) = (\mathcal{H} -e^{-c\Psi})(x)(w) =$$

$$-c^2 e^{-c\Psi(x)} \cdot A(x)(w) + c\, e^{-c\Psi(x)} (\mathcal{H}\Psi)(x)(w) =$$

$$c\, e^{-c\Psi(x)} \left\{ -c\, A(x)(w) + (\mathcal{H}\Psi)(x)(w) \right\} =$$

$$c\, e^{-c\Psi(x)} \left\{ -c\, |\lambda|^2 \cdot A(x)(v) + (\mathcal{H}\Psi)(x)(u) + |\lambda|^2 (\mathcal{H}\Psi)(x)(v) \right\}.$$

Now either $\lambda = 0$ and $u \neq 0$, in which case we are left with the second summand that is negative, or $\lambda \neq 0$ and by our choice of c we get again a negative result. So we have proved that we can choose a defining function for D in U whose complex Hessian at x has $n(x)+1$ negative eigenvalues; by continuity, after perhaps shrinking U we find that $(\mathcal{H} -e^{-c\Psi})(y)$ has $n(x)+1$ negative eigenvalues for all $y \in U$.

So if D in not q-pseudoconvex, i.e. there exists at least one point $x \in \partial D$ with $n(x) \geq q+1$, we can find a defining function whose Hessian has at least q+2 negative eigenvalues in a neighbourhood of x . So we can apply proposition 3.1.6 with k=q+2 : if Q is as in the proposition, $D \cap Q$ is a q+1 general Hartogs' figure by (a) with $\pi(E(D \cap Q)) \nsubseteq D$ by (b) and so D is not Hartogs' q-complete and the proposition is proved.□

In the above proposition M needs to be Stein because otherwise the definition of Hartogs' completeness does not make sense (see remark after def.2.1.4), but since the problem is local the above proof applies actually to the following more general statement:

Let D be an open subset with $C^2$ boundary of an analytic manifold M; if $\forall x \in \partial D$ there exists a Stein neighbourhood U of x s.t. $U \cap D$ is Hartogs' q-complete (the definition makes sense now), then D is q-pseudoconvex.

## §2:- Inextendibility of Cohomology Classes.

In this section we discuss inextendibility questions analogous to those used in the classical definition of domains of holomorphy.

Again M does not need to be Stein , so in what follows D is an open subset of an analytic manifold M.

Following Andreotti and Norguet ([3] ,p. 199), we introduce, for any point $x \in \partial D$ and any analytic sheaf $\mathcal{S}$ on M, the $\mathcal{O}_x$-modules

$$H^p(D,x,\mathcal{S}) = \varinjlim H^p(D \cap U, \mathcal{S}),$$

$$H^p_+(D \cup \{x\}, \mathcal{S}) = \varinjlim H^p(D \cup U, \mathcal{S}) \text{ and}$$

$$H^p_x(\mathcal{S}) = \varinjlim H^p(U, \mathcal{S}).$$

where the direct limits are taken over all open neighbourhoods of x. Notice that

$$H^p_x(\mathcal{S}) = \begin{cases} \mathcal{S}_x & \text{if } p = 0 \\ 0 & \text{otherwise} \end{cases}$$

There are restriction maps:

$$\mu : H^p(D,\mathcal{S}) \longrightarrow H^p(D,x,\mathcal{S}),$$

$$\rho : H^p_+(D \cup \{x\},\mathcal{S}) \longrightarrow H^p(D,\mathcal{S}) \text{ and}$$

$$\lambda : H^p_x(\mathcal{S}) \longrightarrow H^p(D,x,\mathcal{S}).$$

By taking the direct limit of the usual Mayer-Vietoris sequence we obtain an exact sequence

$$\ldots\longrightarrow H^p_+(D \cup \{x\},\mathcal{S}) \longrightarrow H^p(D,\mathcal{S}) \oplus H^p_x(\mathcal{S}) \xrightarrow{\mu-\lambda} H^p(D,x,\mathcal{S}) \longrightarrow$$

$$\longrightarrow H^{p+1}_+(D \cup \{x\},\mathcal{S}) \longrightarrow \ldots\ldots$$

called again the Mayer-Vietoris sequence.

Definition 3.2.1:- A cohomology class $\xi \in H^p(D,\mathcal{S})$ is said to be extendible through x if $\xi \in \operatorname{im} \rho$.

From the Mayer-Vietoris sequence it follows immediately that, if $p > 0$, $\xi$ is extendible if and only if $\mu(\xi) = 0$.

Definition 3.2.2: We say that D is a q-domain of holomorphy if, for all $x \in \partial D$, there exists $p \leq q$ and a cohomology class $\xi$ in $H^p(D,\mathcal{O})$ which does not extend through x (cfr. [1] p. 138).

For $q = 0$ this is just the classical definition of domain of holomorphy, but for $q > 0$ there are many domains that are not q-domains of holomorphy for any q: we provide an easy

Example 3.2.3:- Let $\overline{\Delta}$ be the closed unit polydisc and set $D = \mathbb{C}^n - \overline{\Delta}$. Choose $x = (1,0,\ldots,0)$ ; if U is a neighbourhood of x small enough, $U \cap D$ is a domain of holomorphy, and therefore, $\forall\, p > 0$, $H^p(D,x,\mathcal{O}) = 0$ and it follows from the remark after def. 3.2.1 that every cohomology class in $H^p(D,\mathcal{O})$ extends through x; moreover every holomorphic function on D extends through x by Hartogs' theorem, and so D is not a q-domain of holomorphy for any q. This D does not have $C^2$ boundary, but the corners can clearly be smoothed without destroying the example.

Proposition 3.2.4:- If D is an open subset with $C^2$ boundary of an analytic manifold M and D is a q-domain of holomorphy then D is q-pseudoconvex (cfr. Prop. 4.1.4 and 4.1.5).

Proof:- The same argument used to prove prop. 3.1.7 with infinitesimal modifications shows that, for all $x \in \partial D$, the map

$$\lambda : H^p_x(\mathcal{O}) \longrightarrow H^p(D, x, \mathcal{O}) \text{ is surjective for } p \leq n(x)-1,$$

(cfr. also [3], théorème 1 , p. 20? ), and so if, for some point $x \in \partial D$, $n(x) \geq q+1$ , using the Mayer-Vietoris sequence, we deduce easily that

$$\rho : H^p_+(D \cup \{x\}, \mathcal{O}) \longrightarrow H^p(D, \mathcal{O}) \text{ is surjective for } p \leq q,$$

and so D is not a q-domain of holomorphy. □

## §3:- The Main Theorem.

In this section we collect together the results of this chapter to prove the following

Theorem 3.3.1:- Let D be a domain in a Stein manifold M, and consider the following statements:

(a) D is q-complete,

(a') D is q-pseudoconvex ( if D has $C^2$ boundary),

(b) D is q-cohomologically complete,

(b') $H^p(D,\mathcal{O}) = 0$ for $p > q$,

(c) D is $\mathbb{C}$-q-complete,

(d) D is Hartogs' q-complete,

(e) D is a q-domain of holomorphy.

Then (a) $\Longrightarrow$ (b) $\Longrightarrow$ (b') $\Longrightarrow$ (c) $\Longrightarrow$ (d) and, if D has $C^2$ boundary, (a), (a'), (b), (b'), (c) and (d) are all equivalent and follow from (e).

<u>Proof</u>:- (a) $\Longrightarrow$ (b) by prop. 3.1.1;

(b) $\Longrightarrow$ (b') trivially;

(b') $\Longrightarrow$ (c) by prop. 3.1.4;

(c) $\Longrightarrow$ (d) by prop. 3.1.5.

If D has $C^2$ boundary then

(d) $\Longrightarrow$ (a') by prop. 3.1.7;

(a') $\Longrightarrow$ (a) by prop. 3.1.2. and (e) $\Longrightarrow$ (a') by prop. 3.1.[illegible] □

§4:- Counterexamples.

Now we discuss some counterexamples to the missing implications in the above theorem when D does not have $C^2$ boundary, precisely we shall prove that the implications (c) $\Longrightarrow$ (b') and (e) $\Longrightarrow$ (b') are not true in general. The most interesting examples of such domains are complements of analytic subvarieties for which we have good tools to compute cohomology (see [10]).

<u>Proposition</u> 3.4.1:- Let V be an analytic subvariety of a Stein manifold M, and suppose $V = \bigcup V_j$ is a reduced decomposition of V in irreducible components. Moreover assume that for all $V_j$ and any Stein subset U of M, $H^p(U - V_j, \Theta) = 0$ unless $p = 0$ or $p = d_j - 1$, where $d_j$ is the codimension of $V_j$ (this hypothesis is not artificial since it is satisfied, for instance, if all $V_j$ are geometric comple-

te intersections, see [10], theorem 23 ).

Let $D = M - V$, then

$$E_q(D) = M - \bigcup_{d_j \leq q+1} V_j \ .$$

Proof:- If $x \in V_j$ and $d_j \leq q+1$ then $H^p(M - V_j, \mathcal{O}) = 0$ for $p > q$, by hypothesis, and so, by prop. 3.1.3, $H^{q+1}(M - V_j, \mathcal{L}_q(\underline{z})) = 0$; in particular $\alpha_q(x)|_{M - V_j} = 0$ and so also $\alpha_q(x)|_D = 0$. This proves that $E_q(D) \subseteq M - \bigcup_{d_j \leq q+1} V_j$ .

To show the reverse inclusion suppose $x \in V_k - \bigcup_{d_j \leq q+1} V_j$, for some k with $d_k > q+1$ . Choose a Stein neighbourhood U of x so small that $U \cap \bigcup_{d_j \leq q+1} V_j = \emptyset$. Then $H^p(U - V_k, \mathcal{O}) = 0$ for $1 \leq p < q+1$ and repeating exactly the same method as in proof of prop. 2.1.3 we may conclude that $E_q(U - V_k) = E_0(U - V_k)$.

However $E_0(U - V_k) = \pi E(U - V_k) = U$ by the Riemann removable singularities theorem since $d_k \geq 2$; so, as $D \supseteq U - V_k$,

$$x \in U = E_0(U - V_k) = E_q(U - V_k) \subseteq E_q(D)$$

and the proposition is proved. □

Example 3.4.2:- Let $M = \mathbb{C}^4$, $V = V_1 \cup V_2$, $D = M - V$, where

$V_1 = \{z \in \mathbb{C}^4 \text{ s. t. } z_1 = z_2 = 0\}$ and

$V_2 = \{z \in \mathbb{C}^4 \text{ s. t. } z_3 = z_4 = 0\}$ are two planes.

Then it is easy to see, with a simple application of the Mayer-Vietoris sequence of $M - V_1$ and $M - V_2$, that $H^2(D, \mathcal{O}) \neq 0$ (see [17] pp. 445-447). By prop. 3.4.1 however, we see that $E_1(D) = D$, so D is $\alpha$-1-complete (and thus Hartogs' 1-complete too). Also, since V may be defined by three analytic functions $(z_1z_3,\ z_2z_4,\ z_2z_3 + z_1z_4)$, it follows (see [17], Prop. 2.6 , p. 435) that D is 2-complete.

Thus D is $\alpha$-1-complete but nothing better than 2-complete and cohomologically 2-complete. In the notation of theorem 3.3.1 (c) $\not\Leftrightarrow$ (b').

This example also provides a negative answer to the following two natural questions:

(1) is it true that a q-domain of holomorphy is cohomologically q-complete?

(2) is it true that if $D$ is a q-domain of holomorphy then, for all $x \in \partial D$ and $\forall \xi \in H^{q}(D, \Theta)$, $\xi$ is extendible through $x$ ?

Here (2) is a question weaker than (1) and both have a positive answer if $q = 0$ (see theorem 2.3.1) or if $D$ has $C^2$ boundary ( see theorem 3.3.1).

However we claim that the domain $D$ constructed above is a 1-domain of holomorphy but there exists an element of $H^2(D, \Theta)$ which does not extend through the origin.

To see that $D$ is a 1-domain of holomorphy fix any $x$ in $V$ and consider the following commutative diagram with exact rows:

$$\begin{array}{ccccccc}
\longrightarrow & H^1(D, \mathcal{R}^2(\underline{z})) & \xrightarrow{\Phi} & H^1(D, \mathcal{L}_0(\underline{z})) & \xrightarrow{\delta} & H^2(D, \mathcal{L}_1(\underline{z})) \\
 & \downarrow \mu & & \downarrow \mu & & \downarrow \mu \\
\longrightarrow & H^1(D, x, \mathcal{R}^2(\underline{z})) & \longrightarrow & H^1(D, x, \mathcal{L}_0(\underline{z})) & \longrightarrow & H^2(D, x, \mathcal{L}_1(\underline{z})).
\end{array}$$

Now, since $E_1(D) = D$ and $E_0(D) = \mathbb{C}^4$ by the Riemann removable singularities theorem (or by prop. 3.1.4), we have that $\alpha_0(x)|_D \neq 0$ yet $\alpha_1(x)|_D = 0$. But $\alpha_1(x)|_D = \delta(\alpha_0(x)|_D)$ and so there exists an element $\xi \in H^1(D, \mathcal{R}^2(\underline{z}))$ such that $\Phi(\xi) = \alpha_0(x)|_D$.

If $\Delta$ is any polydisc neighbourhood of $x$, exactly the same argument can be repeated with $\Delta$ instead of $\mathbb{C}^4$. Therefore:
$\mu(\alpha_1(x)|_D) = 0$, $\mu(\alpha_0(x)|_D) \neq 0$ and so $\mu(\xi) \neq 0$.

By the remark just after def. 3.2.1 and identifying $\mathcal{R}^2$ with $\Theta^6$, we have that one of the components of $\xi$ does not extend through x, therefore D is a 1-domain of holomorphy. At this stage we can already say that the answer to question (1) is in general no, or, with the notation of theorem 3.3.1 , (e) $\not\Rightarrow$ (b').

However we can proceed further to answer question (2).

First we observe that an easy application of the Mayer-Vietoris sequence shows that

$$H^2(D,\mathcal{O}) \underset{\Psi}{\simeq} H^3(\mathbb{C}^4 - \{0\},\mathcal{O}).$$

Consider now the commutative diagram

$$\begin{array}{ccc} H^2(D,\mathcal{O}) & \xrightarrow{\ \Psi\ } & H^3(\mathbb{C}^4 - \{0\},\mathcal{O}) \\ \downarrow \mu & & \downarrow \mu \\ H^2(D,x,\mathcal{O}) & \longrightarrow & H^3(\mathbb{C}^4 - \{0\},x,\mathcal{O}) \end{array}$$

where x is the origin.

By prop. 3.4.1 $E_2(\mathbb{C}^4 - \{0\}) = \mathbb{C}^4$, so that $\alpha_2(x) \in H^3(\mathbb{C}^4 - \{0\},\mathcal{O}) = H^3(\mathbb{C}^4 - \{0\},\mathcal{L}_2(\underline{z}))$ is not zero. If $\Delta$ is any polydisc neighbourhood of the origin we can replace $\mathbb{C}^4$ with $\Delta$ in the above argument and we obtain the same conclusions. therefore $\mu(\alpha_2(x)) \neq 0$ and so also $\mu\circ\Psi^{-1}(\alpha_2(x)) \neq 0$ . As before, $\Psi^{-1}(\alpha_2(x))$ turns out to be an element of $H^2(D,\mathcal{O})$ which does not extend through the origin, and the answer to question (2) is again no.

## CHAPTER 4:- The Inextendibility Index.

We shall now discuss further the concept of inextendibility of cohomology classes.

### §1:- The Inextendibility Index.

In this section again we merely suppose that $D$ is an open subset of an analytic manifold $M$.

Let $\mathcal{S}$ be any analytic sheaf on $M$, $x \in \partial D$ and suppose that, for some $p \geq 0$, the map $\lambda : H^p_x(\mathcal{S}) \longrightarrow H^p(D,x,\mathcal{S})$ is not surjective: then we can introduce the following

<u>Definition</u> 4.1.1:- The <u>inextendibility index</u> of $\mathcal{S}$ at x is the non negative integer

$k(x,\mathcal{S}) = \min \{p$ s.t. $\lambda : H^p_x(\mathcal{S}) \longrightarrow H^p(D,x,\mathcal{S})$ is not surjective$\}$.

The hypothesis needed to define $k(x,\mathcal{S})$ is not always satisfied even for a coherent sheaf $\mathcal{S}$, for instance if $D = \mathbb{C}^n - \{x\}$, and $\mathcal{S}$ is the structure sheaf of the subvariety $\{x\}$ of $\mathbb{C}^n$, then $\forall p \geq 0$, $H^p(D,x,\mathcal{S}) = 0$, so $\lambda$ is always surjective and $k(x,\mathcal{S})$ cannot be defined: however if $\mathcal{S}$ is locally free near x, $k(x,\mathcal{S})$ is always well defined. Indeed, since the problem is local, we can suppose that $M$ is a polydisc $\Delta \ni x$ in $\mathbb{C}^n$ and that $\mathcal{S}$ is the structure sheaf $\mathcal{O}$. In these hypotheses the test class $\alpha_0(x) \in H^1(\Delta - \{x\}, \mathcal{L}_0(\underline{z}))$ can be defined and we have the following

<u>Lemma</u> 4.1.2:- If $0 = \mu(\alpha_0(x)|_{D\cap\Delta}) \in H^1(D\cap\Delta, x, \mathcal{L}_0(\underline{z}))$ then $\lambda : H^0_x(\mathcal{O}) \longrightarrow H^0(D,x,\mathcal{O})$ is not surjective.

*Proof*:- Let $U$ be an open neighbourhood of $x$ so small that $U \subseteq \Delta$ and $\alpha_0(x)|_{D \cap U} = 0$. Then there exist holomorphic functions $h_1, h_2, \ldots, h_n$ in $\Gamma(U \cap D, \mathcal{O})$ s.t. $\sum_{i=1}^n z_i h_i = 1$ and so also $\mu \sum_{i=1}^n z_i h_i = 1$. It follows that for at least one index $i$ $\mu(h_i) \notin \operatorname{im} \lambda$, because otherwise we get a contradiction evaluating both sides of the above equality at $x$. □

By (a local version of) prop.3.1.3 the hypotheses of the above lemma are in particular satisfied if $H^p(D,x,\mathcal{O}) = 0$ for $p \geq 1$, and so, recalling that $H_x^p(\mathcal{O}) = 0$ for $p \geq 1$, if $\lambda: H_x^p(\mathcal{O}) \longrightarrow H^p(D,x,\mathcal{O})$ is surjective for $p \geq 1$, then necessarily $k(x,\mathcal{O}) = 0$.

If for a point $x \in \partial D$ and for an analytic sheaf $\mathcal{S}$ coherent in a neighbourhood of $x$, $k(x,\mathcal{S})$ is well defined, there exists a relationship between $k(x,\mathcal{S})$ and $k(x,\mathcal{O})$, which we shall denote simply by $k(x)$: to make this precise we recall that if $\mathcal{S}$ is coherent in a neighbourhood $U$ of $x$ there exists a free resolution of $\mathcal{S}$ in $U$, after possibly shrinking $U$, of the type

$$0 \longrightarrow \mathcal{O}^{p_d} \longrightarrow \mathcal{O}^{p_{d-1}} \longrightarrow \ldots \longrightarrow \mathcal{O}^{p_0} \longrightarrow \mathcal{S} \longrightarrow 0$$

for some $d \leq n$ (Hilbert syzygy theorem, see [11] p. 74).

The smallest $d$ for which such a resolution exists is indicated by $d_x(\mathcal{S})$ and called the <u>homological</u> <u>codimension</u> of $\mathcal{S}_x$ (see [2] p. 197).

<u>Proposition</u> 4.1.3:- If $k(x,\mathcal{S})$ is well defined then $k(x,\mathcal{S}) \geq k(x) - d_x(\mathcal{S})$.

*Proof*:- Since the problem is local we can suppose that $M$ is a polydisc $\Delta \subseteq \mathbb{C}^n$ and that $\Delta$ has been chosen so small that there is a free resolution

$$0 \longrightarrow \mathcal{O}^{p_d} \longrightarrow \mathcal{O}^{p_{d-1}} \longrightarrow \ldots\ldots \longrightarrow \mathcal{O}^{p_0} \xrightarrow{\Psi} \mathcal{S} \longrightarrow 0$$

of $\mathcal{S}$ in $\Delta$ with $d = d_x(\mathcal{S})$. Also we suppose that $k(x) \geq d+1$, otherwise the problem is trivial.

The hypotheses are

(a) $\lambda : H^0_x(\Theta) \longrightarrow H^0(D,x,\Theta)$ is surjective

(b) $H^s(D,x,\Theta) = 0$ for $1 \leq s \leq k(x)-1$ ,

and we want to prove that

(a') $\lambda : H^0_x(\mathcal{S}) \longrightarrow H^0(D,x,\mathcal{S})$ is surjective and

(b') $H^s(D,x,\mathcal{S}) = 0$ for $1 \leq s \leq k(x) - d_x(\mathcal{S}) - 1$.

In order to do this we split the above exact sequence into short exact sequences

$$0 \longrightarrow \mathcal{K}_s \longrightarrow \Theta^{p_{s-1}} \longrightarrow \mathcal{K}_{s-1} \longrightarrow 0,$$

for $s=1,2,\ldots,d$, where $\mathcal{K}_d = \Theta^{p_d}$ and $\mathcal{K}_0 = \mathcal{S}$ .

We have a long exact sequence of (local) cohomology

$$\ldots\longrightarrow H^p(D,x,\Theta^{p_{s-1}}) \longrightarrow H^p(D,x,\mathcal{K}_{s-1}) \longrightarrow$$
$$\longrightarrow H^{p+1}(D,x,\mathcal{K}_s) \longrightarrow H^{p+1}(D,x,\Theta^{p_{s-1}}) \longrightarrow \ldots$$

from which we deduce, using the hypothesis (b), that

$$H^p(D,x,\mathcal{K}_{s-1}) \simeq H^{p+1}(D,x,\mathcal{K}_s) \quad \text{for } 1 \leq p \leq k(x)-2 \text{ and in}$$

particular for $1 \leq p \leq d-1$; so that

$$H^p(D,x,\mathcal{S}) = H^p(D,x,\mathcal{K}_0) \simeq H^{p+d}(D,x,\mathcal{K}_d) = H^{p+d}(D,x,\Theta^{p_d}).$$

But for $p \leq k(x) - d - 1$ , $H^{p+1}(D,x,\Theta^{p_d}) = 0$ by (b) and therefore (b') is proved.

Similarly we have

$$H^1(D,x,\mathcal{K}_1) \simeq H^d(D,x,\mathcal{K}_d) = H^d(D,x,\Theta^{p_d}) = 0 \text{ since } d \leq k(x)-1.$$

Therefore, by considering the commutative diagram with exact column wise the problem is trivial.

The hypotheses are

(a) $\lambda : H^0_x(\mathcal{O}) \longrightarrow H^0(D,x,\mathcal{O})$ is surjective

(b) $H^s(D,x,\mathcal{O}) = 0$ for $1 \leq s \leq k(x)-1$ ,

and we want to prove that

(a') $\lambda : H^0_x(\mathcal{S}) \longrightarrow H^0(D,x,\mathcal{S})$ is surjective and

(b') $H^s(D,x,\mathcal{S}) = 0$ for $1 \leq s \leq k(x) - d_x(\mathcal{S}) - 1$.

In order to do this we split the above exact sequence into short exact sequences

$$0 \longrightarrow \mathcal{R}_s \longrightarrow \mathcal{O}^{p_{s-1}} \longrightarrow \mathcal{R}_{s-1} \longrightarrow 0,$$

for $s=1,2,\dots,d$, where $\mathcal{R}_d = \mathcal{O}^{p_d}$ and $\mathcal{R}_0 = \mathcal{S}$ .

We have a long exact sequence of (local) cohomology

$$\dots \longrightarrow H^p(D,x,\mathcal{O}^{p_{s-1}}) \longrightarrow H^p(D,x,\mathcal{R}_{s-1}) \longrightarrow$$
$$\longrightarrow H^{p+1}(D,x,\mathcal{R}_s) \longrightarrow H^{p+1}(D,x,\mathcal{O}^{p_{s-1}}) \longrightarrow \dots$$

from which we deduce, using the hypothesis (b), that

$H^p(D,x,\mathcal{R}_{s-1}) \simeq H^{p+1}(D,x,\mathcal{R}_s)$ for $1 \leq p \leq k(x)-2$ and in particular for $1 \leq p \leq d-1$; so that

$$H^p(D,x,\mathcal{S}) = H^p(D,x,\mathcal{R}_0) \simeq H^{p+d}(D,x,\mathcal{R}_d) = H^{p+d}(D,x,\mathcal{O}^{p_d}).$$

But for $p \leq k(x) - d - 1$ , $H^{p+d}(D,x,\mathcal{O}^{p_d}) = 0$ by (b) and therefore (b') is proved.

Similarly we have

$$H^1(D,x,\mathcal{R}_1) \simeq H^d(D,x,\mathcal{R}_d) = H^d(D,x,\mathcal{O}^{p_d}) = 0 \text{ since } d \leq k(x)-1.$$

Therefore, by considering the commutative diagram with exact column

$$
\begin{array}{ccc}
H^0_x(\mathcal{O}^{p}{}_0) & \xrightarrow{\lambda} & H^0(D,x,\mathcal{O}^{p}{}_0) \\
\downarrow \Psi & & \downarrow \Psi \\
H^0_x(\mathcal{S}) & \xrightarrow{\lambda} & H^0(D,x,\mathcal{S}) \\
 & & \downarrow \\
 & & H^1(D,x,\mathcal{K}_1) = 0
\end{array}
$$

and the hypothesis (a) we see that also condition (b') is true. □

The first three lines of the proof of prop. 3.2.4 establish a relationship between $k(x)$ and the behaviour of $D$ near $x$, namely they prove the following

<u>Proposition</u> 4.1.4:- If $D$ has $C^2$ boundary at $x$, $n(x) \leq k(x)$. □

Using the Mayer-Vietoris sequence we deduce easily that

$\rho : H^p_+(D \cup \{x\}, \mathcal{O}) \longrightarrow H^p(D, \mathcal{O})$ is surjective for $p \leq k(x)-1$,

and so we have the following

<u>Proposition</u> 4.1.5:- If $D$ is a q-domain of holomorphy then, for all $x \in \partial D$, $k(x) \leq q$. □

Prop. 3.2.4 follows from these two propositions.

It is natural to ask if $n(x)$ is always equal to $k(x)$; we shall provide an example to show that this is not always the case.

It is clear that we must try an example with degenerate boundary (i.e. $z(x) \neq 0$) otherwise $n(y) = n(x)$, $\forall y \in U \cap D$, where $U$ is a small enough Stein neighbourhood of $x$, and so $U \cap D$ is $n(x)$-complete (see [17] Prop. 2.4, p. 4[illegible]4) and then $H^s(D,x,\mathcal{O}) = 0$ for $s > n(x)$, which implies $k(x) \leq n(x)$.

We also need some more material.

<u>Definition</u> 4.1.6:- An open subset $D \subsetneq \mathbb{C}^n$ is called a <u>tube</u> if there exists an open subset $\mathcal{B}(D) \subseteq \mathbb{R}^n$, called the <u>base</u> of $D$, s.t. $y \in D \Leftrightarrow \mathrm{Re}\, y \in \mathcal{B}(D)$.

If B is an open subset of $\mathbb{R}^n$ we indicate with $\mathcal{T}(B)$ the tube in $\mathbb{C}^n$ with base B. The <u>convex hull</u> of $B \subseteq \mathbb{R}^n$ is the smallest convex set containing B, and is denoted by ch(B).

We know (see [12] p. 41 ) that if D is a tube then $E(D) = \mathcal{T}(ch(\mathcal{B}(D)))$, and if we have an open neighbourhood U of the origin in $\mathbb{C}^n$ of the type $U = \{z \text{ s.t. } \operatorname{Re} z \in B, |\operatorname{Im} z| < \varepsilon$, for some open neighbourhood B of $0 \in \mathbb{R}^n$ and some $\varepsilon > 0\}$, then $E(U) = \{z \text{ s.t. } \operatorname{Re} z \in ch(B), |\operatorname{Im} z| < \varepsilon\}$.

We are now ready to produce the following

<u>Example</u> 4.1.7:- Let $\varphi: \mathbb{C}^2 \longrightarrow \mathbb{R}$ be defined by

$$\varphi(y_1, y_2) = (\operatorname{Re} y_1)^3 - \operatorname{Re} y_2$$

and set $D = \{y \text{ s.t. } \varphi(y) < 0\}$. An easy computation shows that

$n(y) = z(y) = 0$ , $p(y) = 1$ for $\operatorname{Re} y_1 > 0$,

$n(y) = 1$ , $z(y) = p(y) = 0$ for $\operatorname{Re} y_1 < 0$ and

$n(y) = p(y) = 0$ , $z(y) = 1$ for $\operatorname{Re} y_1 = 0$.

D is a tube and the above remarks show that $E(D) = \mathbb{C}^2$ , and that, for all open neighbourhoods W of the origin, $0 \in E(D \cap W)$, i.e. that $\lambda: H^0_x(\Theta) \longrightarrow H^0(D, x, \Theta)$ is surjective for x = the origin. This means that $k(0) \geqslant 1 > 0 = n(0)$.

Now suppose again that M is Stein: the following question has clearly a positive answer if q=0: Is it true that in the last non vanishing cohomology group $H^q(D, \Theta)$ there exists a cohomology class which does not extend through at least one point of $\partial D$?

We can give a positive answer and even something slightly better if D has $C^2$ boundary.

<u>Theorem</u> 4.1.8:- Let D be a domain in a Stein manifold M s.t. $H^p(D, \Theta) = 0$ for $p > q$ , but $H^q(D, \Theta) \neq 0$ ; suppose that for some

point $x \in \partial D, k(x) \geq q$ (this is always the case if $D$ has $C^2$ boundary by theorem 2.3.1 and prop. 4.1.4) ; then

$$\dim_{\mathbb{C}} \mu(H^q(D,\mathcal{O}) \bmod (\lambda(H^q_x(\mathcal{O}) \cap \mu(H^q(D,\mathcal{O}))) = \infty .$$

Proof:-To simplify the notation write $G = \lambda(H^q_x(\mathcal{O}) \cap \mu(H^q(D,\mathcal{O}))$. notice that $G = 0$ if $q \geq 1$.

Suppose that $\dim_{\mathbb{C}} \mu(H^q(D,\mathcal{O}) \bmod G = k < \infty$ . Then also $\dim_{\mathbb{C}} (\mu(H^q(D,\mathcal{O}) \bmod G)^N = k\,N = m < \infty$, for $N = \binom{n}{n-q-1}$ .

Take global sections $z_1, z_2, \ldots, z_n$ as done at the end of page 7, and set $f_1 = z_1^{m+1}$, $f_j = z_j$ for $j=2,\ldots,n$.

We can consider the test classes $\alpha_s(z_1^j, \underline{f})|_D \in H^{s+1}(D, \mathcal{L}_s(\underline{f}))$, for $j=0,1,\ldots,m$.

The hypothesis together with prop. 3.1.3 say that $H^{q+1}(D,\mathcal{L}_q(\underline{f}))$ vanishes and considering the exact sequence

$$\ldots \longrightarrow H^q(D, \mathcal{K}^{n-q-1}(\underline{f})) \xrightarrow{\Psi} H^q(D, \mathcal{L}_{q-1}(\underline{f})) \longrightarrow 0$$

we deduce that there exist elements $\eta_0, \eta_1, \ldots, \eta_m$ in $(H^q(D,\mathcal{O}))^N = H^q(D, \mathcal{K}^{n-q-1}(\underline{f}))$ s.t.

(a) $\Psi(\eta_j) = \alpha_{q-1}(z_1^j, \underline{f})|_D$ .

We claim that the vectors $\{\hat{\mu}(\eta_j) =_{\text{def}} \mu(\eta_j) \bmod G\}_{j=0}^m$ are linearly independent: this would contradict our choice of $m$.

So let $c_0, c_1, \ldots, c_m$ be complex numbers s.t.

(b) $\quad \sum_{j=0}^m c_j \hat{\mu}(\eta_j) = 0.$

We shall treat separately the cases $q = 0$ and $q > 0$.

Case (1):- $q = 0$. In this case $N = n$ and equation (b) means

$$\sum_{j=0}^m c_j \mu(\eta_j) = \lambda(\tilde{h}_1, \tilde{h}_2, \ldots, \tilde{h}_n) \text{ for some } \tilde{h}_i \in \mathcal{O}_x,\ i=1,\ldots,n,$$

i.e. if $\eta_j = (g_j^1, g_j^2, \ldots, g_j^n)$ we have

(c) $\sum_{j=0}^{m} c_j \mu(g_j^{\,i}) = \lambda(\tilde{h}_i)$ for $i=1,2,\ldots,n$.

Equation (a) means

$$z_1^{m+1} g_j^{\,1} + \sum_{i=2}^{n} z_i\, g_j^{\,i} = z_1^{\,j} \quad \text{for } j=0,1,\ldots,m.$$

(actually there might be some minus sign appearing in front of the $g_j^{\,i}$'s depending on the choice of the identification of $\Gamma(D,\mathcal{O})^n$ with $\Gamma(D,\mathcal{R}^{n-1}(\underline{f}))$ but this is clearly irrelevant).

From this and (c) we obtain

$$\mu(\sum_{j=0}^{m} c_j z_1^{\,j}) = \sum_{j=0}^{m} c_j \left[ z_1^{m+1} \mu(g_j^{\,1}) + \sum_{i=2}^{n} z_i \mu(g_j^{\,i}) \right] =$$

$$z_1^{m+1} \cdot \sum_{j=0}^{m} c_j \mu(g_j^{\,1}) + \sum_{i=2}^{n} \sum_{j=0}^{m} c_j z_i \mu(g_j^{\,i}) =$$

$$z_1^{m+1} \lambda(\tilde{h}_1) + \sum_{i=2}^{n} z_i \lambda(\tilde{h}_i) .$$

So we have the equality

(d) $\mu(\sum_{j=0}^{m} c_j z_1^{\,j}) = z_1^{m+1} \lambda(\tilde{h}_1) + \sum_{i=2}^{n} z_i \lambda(\tilde{h}_i)$.

By the uniqueness of analytic continuation we obtain that the equality

$$\sum_{j=0}^{m} c_j z_1^{\,j} = z_1^{m+1} h_1 + \sum_{i=2}^{n} z_i h_i$$

where $h_i$ is a representative of $\tilde{h}_i$, $\forall i$, holds in a neighbourhood of x; deriving both sides j times with respect to $z_1$ and evaluating at x we have $j!\, c_j = 0$ and the claim is proved.

Case (2):- $q > 0$. In this case (b) means

$\sum_{j=0}^{m} c_j \mu(\eta_j) = 0$ which implies that

$\sum_{j=0}^{m} c_j \mu(\alpha_{q-1}(z_1^{\,j}, \underline{f}|_D)) = 0$.

If $q \geq 2$, $k(x) \geq q$ implies that $H^s(D,x,\mathcal{O}) = 0$ for $s=1,2,\ldots,q-1$, and using the exact sequence

$$0 = H^s(D,x,\mathcal{R}^{n-s-1}(\underline{f})) \longrightarrow H^s(D,x,\mathcal{L}_{s-1}(\underline{f})) \longrightarrow H^{s+1}(D,x,\mathcal{L}_s(\underline{f}))$$

we obtain that

$$\text{(e)} \qquad \sum_{j=0}^{q} c_j \, \mu(\alpha_0(z_1^{\,j}, \underline{f})|_D) = 0,$$

and if q=1, (e) is the last inequality appearing above.

From (e) it follows that there exist elements $g_1, g_2, \ldots, g_n$ in $\Gamma(D, x, \Theta)$ s.t.

$$z_1^{\,q+1} g_1 + \sum_{i=2}^{n} z_i \, g_i = \mu\left(\sum_{j=0}^{q} c_j \, z_1^{\,j}\right),$$

(see the above remark in brackets), but, since $\Gamma(D, x, \Theta) = \lambda(\Theta)_x$, because $k(x) > 0$, we can find $\tilde{h}_1, \tilde{h}_2, \ldots, \tilde{h}_n$ in $\Theta_x$ s.t. $\lambda(\tilde{h}_i) = g_i$, $\forall i$, and we obtain again the equality (d) from which the conclusion follows in the same way, thus proving the claim and with it the theorem. □

## § 2:- The Analytic Touching Number.

We introduce another invariant motivated by the following

Proposition 4.2.1:- Suppose that D is an open subset of an analytic manifold M and that D has $C^2$ boundary at $x \in \partial D$. Then there exist an open neighbourhood U of x and s analytic functions $g_1, g_2, \ldots, g_s$ in $\Gamma(U, \Theta)$, for $s = n(x) + z(x) + 1$ $(= n - p(x))$ s.t., writing

$$V = \{ y \in U \text{ s.t. } g_1(y) = g_2(y) = \ldots = g_s(y) = 0 \},$$

we have $V \cap \bar{D} = \{x\}$, i.e. V touches $\bar{D}$ at x.

Proof:- See [4], Prop. 6, p. 209. □

Definition 4.2.2:- Without assumptions on the boundary, the analytic touching number a(x) of D at x is the least s for which we can find s germs of analytic functions $\tilde{g}_1, \tilde{g}_2, \ldots, \tilde{g}_s$ in $\Theta_x$ which define a subvariety V touching $\bar{D}$ at x as in the above proposition.

Prop. 4.2.1 shows that $a(x) \leq n(x) + z(x) + 1$. A lower bound is provided by the following

<u>Theorem</u> 4.2.3:- $a(x) \geq k(x) + 1$.

<u>Proof</u>:- Suppose that $\tilde{g}_1, \tilde{g}_2, \ldots, \tilde{g}_s$ are as in definition 4.2.2 and that $g_i$ is a representative for $\tilde{g}_i$ in a small Stein neighbourhood $U$ of $x$, $\forall i$, s.t. $\bar{D} \cap V = \{x\}$ and so $D \cap U \subseteq U - V$.

Then, by [10], theorem 23, p. 154, it follows that, for any Stein open set $U'$ with $x \in U' \subseteq U$,

$H^p(U' - V, \mathcal{O}) = 0$ for $p \geq s$, and so $H^p(U - V, x, \mathcal{O}) = 0$, $p \geq s$.

From prop. 4.1.3 we have that $H^s(U - V, x, \mathcal{L}_{s-1}(\underline{z})) = 0$ and in particular $\mu(\alpha_{s-1}(x)|_{U-V}) = 0$; by further restricting also $\mu(\alpha_{s-1}(x)|_D) = 0$ and hence there exists an integer $p \leq s - 1$ s.t. $\mu(\alpha_{p-1}(x)|_D) \neq 0$ and $\mu(\alpha_p(x)|_D) = 0$ for some $p \geq 1$ or else $\mu(\alpha_0(x)|_D) = 0$; in this second case lemma 4.1.2 says that $k(x) = 0 = p \leq s-1$ and the theorem is proved, and if $p \geq 1$ we can consider the exact sequence

$$\ldots \longrightarrow H^p(D, x, \mathcal{R}^{n-p-1}(\underline{z})) \xrightarrow{\Phi} H^p(D, x, \mathcal{L}_{p-1}(\underline{z})) \longrightarrow H^{p+1}(D, x, \mathcal{L}_p(\underline{z}))$$

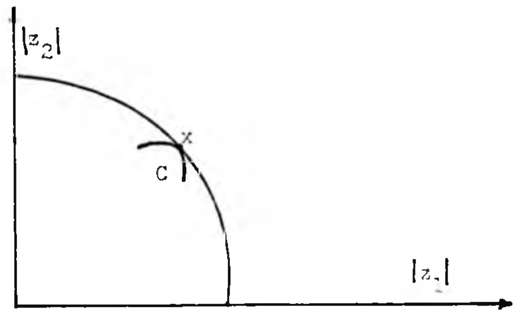
from which we deduce that there exists an element $\eta$ in $H^p(D, x, \mathcal{O})^N = H^p(D, x, \mathcal{R}^{n-p-1}(\underline{z}))$, $N = \binom{n}{n-p-1}$, s.t. $\Phi(\eta) = \mu(\alpha_{p-1}(x)|_D) \neq 0$. It follows that $\lambda: 0 = H^p_x(\mathcal{O}) \longrightarrow H^p(D, x, \mathcal{O})$ is not surjective and so $k(x) \leq p \leq s - 1$ and the theorem is proved.
(notice that in this proof, when restricting, $D \cap U$ has been indicated simply by $D$ to use a simpler notation). □

If $D$ has $C^2$ boundary at $x$, the inequalities proved in the last two sections can be summarized as

$$n(x) \leq k(x) \leq a(x) - 1 \leq n(x) + z(x).$$

The following particular case of the above theorem is particularly intuitive.

Let $\overline{B} = \{ z \in \mathbb{C}^2 \text{ s.t. } |z|^2 = z_1\bar{z}_1 + z_2\bar{z}_2 \leq 1 \}$ be the closed unit ball in $\mathbb{C}^2$, let $x \in \overline{B}$; does there exist a germ of a curve C s.t. $C \subsetneq \overline{B}$ and $C \cap \overline{B} = \{x\}$ ? (see picture in absolute space)




If such a germ exists it is of codimension 1 and so can be described by a single germ of analytic function, i.e. if we take $D = \mathbb{C}^2 - \overline{B}$, $a(x) = 1$, but an easy computation shows that $n(x) = 1$. So the existence of this germ of curve would contradict the theorem.

Loosely speaking this shows that one should not trust pictures when thinking in several complex variables.

CHAPTER 5:- Dolbeault and Leray Representatives for the Test Classes.

We shall give now an explicit form to the test classes $\alpha_s(g, \underline{f}) \in H^{s+1}(M - \{x\}, \mathcal{L}_s(\underline{f}))$.

## §1:- The Double Koszul Complex.

Let $(B^{\cdot}, \delta')$ be any complex of $\Gamma(M - \{x\}, \mathcal{O})$-modules s.t., $\forall p$, $H^p(B^{\cdot}, \delta') = H^p(M - \{x\}, \mathcal{O})$: the most relevant cases are the following:

(a) $B^q = C^q(\mathcal{U}, \mathcal{O})$, where $\mathcal{U}$ is a Leray covering of $M - \{x\}$, e.g. $\mathcal{U} = \{M - V(f_i)\}_{i=1}^{n}$, $\delta'$ = Cech cohomology differential.

(b) $B^q = \Gamma(M - \{x\}, \mathcal{E}^{0,q})$ where $\mathcal{E}^{0,q}$ is the sheaf of germs of differential forms of type $(0,q)$ and $\delta' = \bar{\partial}$ is the Dolbeault differential.

(c) $B^q = (M - \{x\}, \mathcal{A}^q)$ where $\mathcal{A}^{\cdot}$ is any analytic acyclic resolution of $\mathcal{O}$ on $M - \{x\}$ and $\delta'$ is induced from the differential of $\mathcal{A}^{\cdot}$.

We can construct the Koszul complex $K^{\cdot, q} =_{def} K^{\cdot}(\underline{f}, B^q)$ and, since $\delta'$ is a homomorphism of $\Gamma(M - \{x\}, \mathcal{O})$-modules the diagrams

$$\begin{array}{ccc} K^{p,q+1} & \xrightarrow{d} & K^{p+1,q+1} \\ \uparrow \delta' & & \uparrow \delta' \\ K^{p,q} & \xrightarrow{d} & K^{p+1,q} \end{array}$$

commute, so we can consider the double anticommutative complex $(K^{\cdot \cdot}; d, \delta)$ where $\delta = (-1)^q \delta'$ and it's total complex $(T^{\cdot}, \Delta)$ where $T^r = \bigoplus_{p+q=r} K^{p,q}$ and $\Delta = d + \delta$.

Definition 5.1.1:- The s-double Koszul complex, $s \in \mathbb{Z}$, is the double complex $K_s^{\cdot\cdot}$ given by

$$K_s^{p,q} \begin{cases} K^{p,q} & \text{if } p \leq n-s-2 \\ 0 & \text{otherwise} \end{cases}$$

and differential induced from that of $(K^{\cdot\cdot}; d, \delta)$.

The corresponding total complex is denoted by $(T_s^{\cdot}, \Delta_s)$.

The cohomology of $(T_s^{\cdot}, \Delta_s)$ does not depend on the particular choice of $B^{\cdot}$ because there is a spectral sequence

$$E_{p,q}^{\infty} \Longrightarrow H^{p+q}(T_s^{\cdot})$$

(see [18], Chapter IX) and the $2^{nd}$ spectral sequence, computed "vertically", shows that

$$E_{p,q}^{1} = H^q(M - \{x\}, \mathcal{K}_s^p(\underline{f})),$$

where $\mathcal{K}_s^p(\underline{f}) = \mathcal{K}^p(\underline{f})$ if $p \leq n-s-2$ and $\mathcal{K}_s^p(\underline{f}) = 0$ otherwise.

Lemma 5.1.2:- $H^p(T_{-2}^{\cdot}) = H^p(T^{\cdot}) = 0$ for all p.

Proof:- As we have just seen, we are free to choose $B^q = \Gamma(M - \{x\}, \mathcal{A}^q)$, with $\mathcal{A}^q$ as in (c) above; also write $M'$ instead of $M - \{x\}$.

Prop. 1.1.2 says that the sequence of sheaves

$$0 \longrightarrow \mathcal{K}^0(\underline{f}, \mathcal{A}^q) \longrightarrow \mathcal{K}^1(\underline{f}, \mathcal{A}^q) \longrightarrow \ldots\ldots \longrightarrow \mathcal{K}^n(\underline{f}, \mathcal{A}^q) \longrightarrow 0$$

is exact on $M'$ for all q, and so it can be split into short exact sequences

$$0 \longrightarrow \mathcal{L}_s(\underline{f}, \mathcal{A}^q) \longrightarrow \mathcal{K}^{n-s-1}(\underline{f}, \mathcal{A}^q) \longrightarrow \mathcal{L}_{s-1}(\underline{f}, \mathcal{A}^q) \longrightarrow 0$$

for $s = 0, 1, \ldots, n-2$, where $\mathcal{L}_{n-2}(\underline{f}, \mathcal{A}^q) = \mathcal{K}^0(\underline{f}, \mathcal{A}^q) = \mathcal{A}^q$ and $\mathcal{L}_{-1}(\underline{f}, \mathcal{A}^q) = \mathcal{K}^n(\underline{f}, \mathcal{A}^q) \simeq \mathcal{A}^q$ (see p. 8).

The hypothesis on $\mathcal{A}^{\cdot}$ says that $H^p(M', \mathcal{K}^t(\underline{f}, \mathcal{A}^q)) = 0$ for all q, all $p \geq 1$ and all t; it follows, using a computation rather standard in this dissertation (e.g. see proof of prop. 3.1.3), that

$$H^1(M', \mathcal{L}_{s-1}(\underline{f}, \mathfrak{a}^q)) \simeq H^{n-s}(M', \mathcal{L}_{n-2}(\underline{f}, \mathfrak{a}^q)) = H^{n-s}(M', \mathfrak{a}^q) = 0,$$

for all $s=0,1,\ldots,n-2$ and all $q$.

Therefore the sequence

$$0 \longrightarrow \Gamma(M', \mathcal{L}_s(\underline{f}, \mathfrak{a}^q)) \longrightarrow \Gamma(M', \mathfrak{K}^{n-s-1}(\underline{f}, \mathfrak{a}^q)) \longrightarrow \Gamma(M', \mathcal{L}_{s-1}(\underline{f}, \mathfrak{a}^q)) \longrightarrow 0$$

is exact $\forall s,q$. It follows that the long sequence

$$0 \longrightarrow \Gamma(M', \mathfrak{K}^0(\underline{f}, \mathfrak{a}^q)) \longrightarrow \Gamma(M', \mathfrak{K}^1(\underline{f}, \mathfrak{a}^q)) \longrightarrow \cdots \longrightarrow \Gamma(M', \mathfrak{K}^n(\underline{f}, \mathfrak{a}^q)) \longrightarrow 0$$

is exact for all $q$: in other words the double complex $K^{\cdot\cdot}$ has exact rows. The result is now easily deduced from the spectral sequence theory or, more simply, with a straightforward diagram chase. □

The cohomology of $T^{\cdot}_s$ is deeply related to that of $\mathcal{L}_s(\underline{f})$ as shown in the following

<u>Theorem</u> 5.1.3:- There are isomorphisms

$$\Psi_s : H^{n-1}(T^{\cdot}_s) \longrightarrow H^{s+1}(M', \mathcal{L}_s(\underline{f})) \quad \text{for } s=-1,0,\ldots,n-3$$

s.t. the diagrams

$$\begin{array}{ccc} H^{n-1}(T^{\cdot}_s) & \xrightarrow{\Psi_s} & H^{s+1}(M', \mathcal{L}_s(\underline{f})) \\ \Big\downarrow \rho & & \Big\downarrow \delta \\ H^{n-1}(T^{\cdot}_{s+1}) & \xrightarrow{\Psi_{s+1}} & H^{s+2}(M', \mathcal{L}_{s+1}(\underline{f})) \end{array}$$

commute, where $\rho$ is induced by the natural restriction

$$\rho : T^r_s = T^r_{s+1} \oplus K^{n-s-2, r-n+s+2} \longrightarrow T^r_{s+1}.$$

Moreover

(a) $\Psi_{-1} : H^{n-1}(T^{\cdot}_{-1}) \longrightarrow H^0(M', \mathcal{L}_{-1}(\underline{f})) = H^0(M', \mathfrak{K}^n(\underline{f}))$

is given by $\Psi_{-1}(\tau(\eta)) = d(\eta_{n-1,0})$

for all $\eta = \oplus_{p+q=n-1} \eta_{p,q} \in Z^{n-1}(T^{\cdot}_{-1}) = \{\eta \in T^{n-1}_{-1} \text{ s.t. } \Delta_{-1}(\eta) = 0\}$

where $\tau : Z^{n-1}(T_s^{\cdot}) \longrightarrow H^{n-1}(T_s^{\cdot})$ is the projection of cohomology.

Proof:- for all r, s we have a commutative diagram with split exact rows

$$\begin{array}{ccccccccc} 0 & \longrightarrow & K^{n-s-2,\, r-n+s+2} & \longrightarrow & T_s^r & \xrightarrow{\rho} & T_{s+1}^r & \longrightarrow & 0 \\ & & \big\downarrow \delta & & \big\downarrow \Delta_s & & \big\downarrow \Delta_{s+1} & & \\ 0 & \longrightarrow & K^{n-s-2,\, r-n+s+3} & \longrightarrow & T_s^{r+1} & \xrightarrow{\rho} & T_{s+1}^{r+1} & \longrightarrow & 0 , \end{array}$$

and so we can consider the exact sequence of complexes

$$0 \longrightarrow (K^{n-s-2,\,\cdot-n+s+2}, \delta) \longrightarrow (T_s^{\cdot}, \Delta_s) \xrightarrow{\rho} (T_{s+1}^{\cdot}, \Delta_{s+1}) \longrightarrow 0,$$

and, by taking cohomology, the exact sequence

$$(b_s) \qquad \cdots\cdots \longrightarrow H^{r-n+s+2}(M', \mathfrak{K}^{n-s-2}(\underline{f})) \longrightarrow H^r(T_s^{\cdot}) \longrightarrow$$

$$\xrightarrow{\rho} H^r(T_{s+1}^{\cdot}) \longrightarrow H^{r-n+s+3}(M', \mathfrak{K}^{n-s-2}(\underline{f})) \longrightarrow \cdots\cdots$$

The result will be deduced by setting r=n-1 and choosing s in an appropriate way. First, for s=-2, we have an exact sequence

$$(b_{-2}) \qquad H^{n-1}(T_{-2}^{\cdot}) \longrightarrow H^{n-1}(T_{-1}^{\cdot}) \xrightarrow{\Psi_{-1}} H^0(M', \mathfrak{K}^n(\underline{f})) \longrightarrow H^n(T_{-2}^{\cdot})$$

where $\Psi_{-1}$ is the connecting homomorphism; by lemma 3.1.2 $H^{n-1}(T_{-2}^{\cdot})$ and $H^n(T_{-2}^{\cdot})$ vanish, so that $\Psi_{-1}$ is an isomorphism; moreover an easy checking shows that $\Psi_{-1}$ satisfies the above equality (a).

Now suppose $n \geq 3$ otherwise the theorem is proved. We shall use the fact that $H^p(M', \Theta) = 0$ for $p=1,2,\ldots,n-2$ (see [10], theorem 23) and so $H^p(M', \mathfrak{K}^t(\underline{f})) = 0$ for $p=1,2,\ldots,n-2$ and all t.

It is clear that the first square in the following diagram with exact rows

$$\begin{array}{lccccccc} (b_{-1}) & H^0(M', \mathfrak{K}^{n-1}(\underline{f})) & \longrightarrow & H^{n-1}(T_{-1}^{\cdot}) & \xrightarrow{\rho} & H^{n-1}(T_0^{\cdot}) & \longrightarrow & 0 \\ & \big\downarrow \text{id.} & & \big\downarrow \Psi_{-1} & & \big\downarrow \Psi_0 & & \\ & H^0(M', \mathfrak{K}^{n-1}(\underline{f})) & \longrightarrow & H^0(M', \mathfrak{K}^n(\underline{f})) & \xrightarrow{\delta} & H^1(M', \mathcal{L}_0(\underline{f})) & \longrightarrow & 0 \end{array}$$

commutes, so that the isomorphism $\Psi_0$ indicated with dotted line exists and makes all the diagram commutative.

Now define $\Psi_s$ by induction for s=1,2,..,n-3 by imposing commutativity to the following diagram

$$
(b_{s-1})\qquad
\begin{array}{ccccccc}
0 \longrightarrow & H^{n-1}(T^{\cdot}_{s-1}) & \xrightarrow[\sim]{\rho} & H^{n-1}(T^{\cdot}_{s}) & \longrightarrow 0 \\
 & \big\downarrow \Psi_{s-1} & & \big\downarrow \Psi_s & \\
0 \longrightarrow & H^{s}(M', \mathcal{L}_{s-1}(\underline{f})) & \xrightarrow[\sim]{\delta} & H^{s+1}(M', \mathcal{L}_{s}(\underline{f})) & \longrightarrow 0
\end{array}
$$

The theorem is now proved. □

## §2:- Explicit Form for the Test Classes.

Let $g \in \Gamma(M, \Theta) \overset{r}{\simeq} \Gamma(M', \Theta) \overset{i}{\simeq} \Gamma(M', \mathcal{R}^n(\underline{f}))$ , see p. 9. Setting $\beta_s(g,\underline{f}) = \Psi_s^{-1}(\alpha_s(g,\underline{f}))$ for s=0,1,...,n-3 , the above theorem says that $\rho(\beta_{s-1}) = \beta_s$ , and so it is not a big abuse of language to write $\beta_{n-2} = \rho(\beta_{n-3})$, where $\beta_s$ is a shorter notation for $\beta_s(g,\underline{f})$.

Now we can find explicitly elements $\eta_s$ in $Z^{n-1}(T^{\cdot}_s)$ s.t. $\tau(\eta_s) = \beta_s$ in the two more relevant cases indicated with (a) and (b) at the beginning of this chapter.

According to theorem 5.1.3, $\Psi_{-1}^{-1}(i(g))$ is represented by an element $\eta_{-1} = \oplus_{p+q=n-1} \eta_{p,q} \in Z^{n-1}(T^{\cdot}_{-1})$ s.t. $d(\eta_{n-1,0}) = i(g)$, i.e. $\Delta_{-2}(\rho(\eta_{-1})) = (0,0,\ldots,0,i(g))$. Therefore $\eta_{-1}$ can be computed by diagram chasing and, again from theorem 5.1.3 we have $\eta_0 = \rho(\eta_{-1})$ and for s=1,2,...,n-2, $\eta_s = \rho(\eta_{s-1})$ so that if $\eta_{-1} = \oplus_{p+q=n-1} \eta_{p,q}$ we obtain that $\eta_s = \oplus_{\substack{p+q=n-1 \\ p \leq n-s-2}} \eta_{p,q}$ .

If $B^{\cdot} = \Gamma(M', \mathcal{E}^{0,\cdot})$ a diagram chase shows that

$$(h)\quad \eta_{p,q} = {\sum}'_{|I|=p} \omega_I F_I$$

where the symbol ' means that the sum is to be taken over increasing multiindices only and the coefficients $\omega_I \in \Gamma(M', \mathcal{C}^{0,q})$ are given by

$$\omega_I = s_{p,q}\, q!\cdot g \sum{}' \frac{(-1)^{\mathrm{sign}(i,J,I)}\ \bar{f}_i\ (\bar{\partial}\bar{f})_J}{(\sum_{k=1}^{n} f_k \bar{f}_k)^{q+1}}$$

where the sum is over increasing multiindices J s.t. $|J| = q$ and indices i so that $\{i\}\cup J \cup I = \{1,2,\dots,n\}$, and the term $s_{p,q}$ is as follows:

$$s_{p,q} = \begin{cases} 1 & \text{if } q = 0 \bmod 4 \\ (-1)^{p+1} & \text{if } q = 1 \bmod 4 \\ -1 & \text{if } q = 2 \bmod 4 \\ (-1)^{p} & \text{if } q = 3 \bmod 4 . \end{cases}$$

We observe that, apart from a coefficient $\pm 1$, the class $\eta_{0,n-1} \in \Gamma(M', \mathcal{C}^{0,n-1})$ is the Bochner-Martinelli kernel and that this class is used in [3] by means of the corresponding integral formula.

If $B' = \check{C}^{\cdot}(\mathcal{U}, \Theta)$ as in case (a) at the beginning of this chapter then

(a) $\eta_{p,q} = \sum'_{|I|=p} \omega_I\, F_I$

and in order to express the coefficients $\omega_I \in \check{C}^q(\mathcal{U}, \Theta) = \bigoplus'_{|K|=q+1} \Gamma(U_K, \Theta)$, where $U_K = U_{k_1} \cap U_{k_2} \cap \dots \cap U_{k_{q+1}}$ if $K = (k_1, k_2, \dots, k_{q+1})$, we write

$$\check{C}^q(\mathcal{U}, \Theta) = \bigoplus_{i=1}^{n} \bigoplus'_{\substack{(i,J)\\ |J|=q}} \Gamma(U_{(i,J)}, \Theta) ;$$

then it is easy to see that

where the symbol ' means that the sum is to be taken over increasing multiindices only and the coefficients $\omega_I \in \Gamma(M', \mathcal{E}^{0,q})$ are given by

$$\omega_I = s_{p,q}\, q! \cdot g \sum{}' \frac{(-1)^{\mathrm{sign}(i,J,I)}\, \bar{f}_i\, (\bar{\partial} \bar{f})_J}{\left(\sum_{k=1}^{n} f_k \bar{f}_k\right)^{q+1}}$$

where the sum is over increasing multiindices J s.t. $|J| = q$ and indices i so that $\{i\} \cup J \cup I = \{1,2,\ldots,n\}$, and the term $s_{p,q}$ is as follows:

$$s_{p,q} = \begin{cases} 1 & \text{if } q = 0 \bmod 4 \\ (-1)^{p+1} & \text{if } q = 1 \bmod 4 \\ -1 & \text{if } q = 2 \bmod 4 \\ (-1)^{p} & \text{if } q = 3 \bmod 4 . \end{cases}$$

We observe that, apart from a coefficient $\pm 1$, the class $\eta_{0,n-1} \in \Gamma(M', \mathcal{E}^{0,n-1})$ is the Bochner-Martinelli kernel and that this class is used in [3] by means of the corresponding integral formula.

If $B' = \check{C}^{\cdot}(\mathcal{U}, \Theta)$ as in case (a) at the beginning of this chapter then

(a) $\eta_{p,q} = \sum{}'_{|I|=p} \omega_I F_I$

and in order to express the coefficients $\omega_I \in \check{C}^q(\mathcal{U}, \Theta) = \bigoplus{}'_{|K|=q+1} \Gamma(U_K, \Theta)$, where $U_K = U_{k_1} \cap U_{k_2} \cap \ldots \cap U_{k_{q+1}}$ if $K = (k_1, k_2, \ldots, k_{q+1})$, we write

$$\check{C}^q(\mathcal{U}, \Theta) = \bigoplus_{i=1}^{n} \bigoplus{}'_{\substack{(i,J)\\ |J|=q}} \Gamma(U_{(i,J)}, \Theta) ;$$

then it is easy to see that

$$\omega_I = s_{p,q}\, q! \cdot g \cdot \sum{}' \frac{(-1)^{\mathrm{sign}(i,J,I)}}{f_{(i,J)}}$$

where the conventions for the summands are the same as above and $f_{(i,J)}$ stands for $f_i \cdot f_{j_1} \cdot f_{j_2} \cdot \ldots \cdot f_{j_q}$ if $J=(j_1, j_2, \ldots, j_q)$.

Appendix:- A Differential Topological Remark on the Distance Function.

In the proof of proposition 3.1.2 I claimed that the following statement is true:

Let x be a point in a open set $\tilde{U} \subseteq \mathbb{R}^N$ and let $\varphi : \tilde{U} \longrightarrow \mathbb{R}$ be a function of class $C^p$ with $p \geq 2$. Set $D = \{ y \in \tilde{U} \text{ s.t. } \varphi(y) < \varphi(x) \}$; moreover suppose that $d\varphi(x) \neq 0$. Then the function $\rho : \tilde{U} \longrightarrow \mathbb{R}$ defined by (dist indicates the Euclidean distance)

$$\rho(z) = \begin{cases} \text{dist}(z, \partial D) & \text{if } z \in \bar{D} \\ -\text{dist}(z, \partial D) & \text{if } z \notin D \end{cases} \quad \text{for all } z \in \tilde{U}$$

is again of class $C^p$ in a neigbourhood of x, and also $d\rho(x) \neq 0$.

Hormander ([12], p. 50) says that this is a consequence of the implicit function theorem, but the details seem to be rather mysterious. Prof. David Epstein suggested that the result can be proved as follows:

There is no loss of generality in assuming that $\varphi(x) = 0$ and that $d\varphi(y) \neq 0$ for all $y \in \tilde{U}$. An easy application of the Inverse Function Theorem (In.F.T.) shows that

$$F =_{\text{def}} \{ y \in U \text{ s.t. } \varphi(y) = 0 \} = \partial D.$$

Define the function $\nu : F \longrightarrow \mathbb{R}^N$ by $\nu(y) = \dfrac{d\varphi(y)}{|d\varphi(y)|}$, $\forall y \in U$: $\nu(y) \in S^{N-1}$, the unit sphere of centre 0 in $\mathbb{R}^N$ and $\nu(y)$ is a unitary vector orthogonal to F and pointing outside D (this is easily seen by checking that $\lim_{t \to 0} \dfrac{\varphi(y + t\nu(y))}{t} > 0$).

Consider the $C^{p-1}$ function $\Psi : F \times \mathbb{R} \longrightarrow \mathbb{R}^N$ given by $\Psi(y,t) = y + t\nu(y)$, for all $y \in F$ and $t \in \mathbb{R}$. $\Psi$ can be decomposed as follows:

$$\omega_I = a_{p,q}\, q! \cdot g \cdot \sum{}' \frac{(-1)^{\mathrm{sign}(i,J,I)}}{f_{(i,J)}}$$

wh re the conventions for the summands are the same as above and $f_{(i,J)}$ stands for $f_i \cdot f_{j_1} \cdot f_{j_2} \cdot \ldots\ldots \cdot f_{j_q}$ if $J=(j_1, j_2, \ldots, j_q)$.

$$F \times \mathbb{R} \xrightarrow{\alpha} F \times (S^{N-1} \times \mathbb{R}) \xrightarrow{\beta} F \times \mathbb{R}^N \xrightarrow{\gamma} \mathbb{R}^N, \qquad \Psi = \gamma \circ \beta \circ \alpha$$

where $\alpha(y,t) = (y,(\nu(y),t)),\ \forall y \in F,\ t \in \mathbb{R},$

$\beta(y,(u,t)) = (y,\ t \cdot u),\ \forall y \in F,\ t \in \mathbb{R},\ u \in S^{N-1}$ and

$\gamma(y,v) = y + v,\ \forall y \in F,\ v \in \mathbb{R}^N.$

The chain rule then shows that

$$d\Psi(y,t)(v_y,t') = v_y + \nu(y) \cdot t' + t\, d\nu(y)\,(v_y),$$

for all $y \in F$, $t \in \mathbb{R}$, $v_y \in T_yF$ and $t' \in T_t\mathbb{R} = \mathbb{R}$; in particular

$d\Psi(x,0)(v_x,t') = v_x + \nu(x)\, t'$ for all $v_x \in T_xF$ and $t' \in \mathbb{R}$ and, since $\nu(x) \notin T_xF$, $v_x$ and $\nu(x)$ are linearly independent so that $d\Psi(x,0)$ is injective and hence an isomorphism. From the In.F.T. it follows that we can find a neighbourhood $\tilde{U}'$ of $\Psi(x,0) = x$ in $\mathbb{R}^N$, a neighbourhood $U'$ of $x$ in $F$ and $\varepsilon > 0$ s.t.

$$\Psi : U' \times (-\varepsilon, \varepsilon) \longrightarrow \tilde{U}'$$

is a $C^{p-1}$ diffeomorphism. Let us call $\chi$ the inverse of $\Psi$.

$\chi$ can be written as $\chi = (c, \sigma)$ where $c : \tilde{U}' \longrightarrow F$ and $\sigma : \tilde{U}' \longrightarrow \mathbb{R}$ are $C^{p-1}$ functions.

From the definition of $\Psi$ it follows immediately that $c$ and $\sigma$ enjoy the following properties:

(a) $\sigma = -\rho$,

(b) $c(z) \in F$ is the closest point to $z$ in $F$, for all $z \in \tilde{U}'$.

(c) $\langle z - c(z),\ v_{c(z)} \rangle = 0$, where $\langle\ ,\ \rangle$ denotes the Euclidean inner product, for all $z \in \tilde{U}'$ and all $v_{c(z)} \in T_{c(z)}F$.

(d) $\sigma(z) = \langle z - c(z),\ \nu(c(z)) \rangle$ for all $z \in \tilde{U}'$.

We shall prove that $\sigma$ is actually a $C^p$ function with $d\sigma(x) \neq 0$, this is enough to show that $\rho$ has the same property by (a).

From (d), using the chain rule one obtains:

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