# MODULAR REPRESENTATIONS OF FINITE GROUPS WITH UNSATURATED SPLIT (B,N)-PAIRS

N. B. Tinberg

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Department of Mathematics

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I thank Elaine Shiels for her kindness and help and special thanks go to my friend John D. Jarratt for his encouragement, companionship, and sense of humour.

# DECLARATION

The proof of (A) I 2.7 is due to J.A. Green who proved that E is Frobenius in the case of saturated split (B,N)-pairs. I thank him for permitting me to include it in my thesis.

## SUMMARY


Let $k$ be an algebraically closed field of characteristic $p > 0$. Let $G = (G,B,N,R,U)$ be a finite group which satisfies all conditions of a split (B,N)-pair except that of saturation; we allow $C = \bigcap_{n \in N} U^n > 1$. Let $Y \cong \mathrm{Ind}_U^G(k_U)$ $E = \mathrm{End}_{kG}(Y)$ where $k_U$ is the trivial U-module $k$. In part (A) we discuss $E$ and the set of isomorphism classes of (finite dimensional) right E-modules and recover most of the work of Curtis, Richen and Sawada on the modular representations of split (B,N)-pairs by using a recent result of Green. By this method we are able to discard the saturation condition from the general theory. The main results of (A) are:

(1) E is Frobenius .

(2) Every simple right E-module is one-dimensional and is thus given by a multiplicative character $\Psi : E \to k$ .

(3) Each such $\Psi$ is determined by a vector $(\chi, \mu_1, \ldots, \mu_n)$ where $\chi$ is a linear character of $B$ and $\mu_i \in k$.

Using a result of Kantor and Seitz on 2-transitive permutation groups we show that if $p$ is odd then $C \trianglelefteq G$ for all unsaturated split (B,N)-pairs and give an example when $p = 2$ and $C \ntrianglelefteq G$.

Results of (A) are applied to the parabolic subgroups $G_J$ $(J \subseteq R)$ of $G$ and to $Y_J \cong \mathrm{Ind}_U^{G_J}(k_U)$ in order to study the indecomposable components of Y. In part (B) we determine:

(1) a formula which describes how $\mathrm{Ind}_{G_J}^G(V)$ breaks up as a direct sum of indecomposable components of $Y$ for any indecomposable $kG_J$-module $V$ which is a component of $Y_J$;

(2) the dimensions of the indecomposable components of $Y$ and find an irreducible character of $G$ corresponding to the Steinberg character;

(3) the vertices of the indecomposable components of Y;

(4) a permutation on the set of indecomposable components of $Y$ taking each to its dual;

(5) a set of generators for the indecomposable components of $E$ (and Y) based on Bromich's work.

We also extend Green's work on G-algebras with permutation base to those with monomial base.

## CONVENTION

This thesis has two major divisions, (A) and (B), each containing its own reference list. Each such division contains chapters (designated by Roman numerals) and each chapter contains various sections (designated by Arabic numerals). The convention adopted for referring to results within the thesis can best be illustrated by the following example: Assume (A) II 2.12 is the result to which we wish to refer. If we are in (B) we refer to it as (A) II 2.12; if we are in (A) III we refer to it as II 2.12 and if we are in (A) II we refer to it simply as 2.12.

## STANDARD NOTATIONS AND ABBREVIATIONS

| | |
|---|---|
| $\|X\|$ | the cardinality of $X$ |
| $X \subseteq T$ | $X$ is a subset of $T$ |
| $T \setminus X$ | the complement of $X$ in $T$ |
| If $G$ is a group, | |
| $H \leq G$ | $H$ is a subgroup of $G$ |
| $H \trianglelefteq G$ | $H$ is a normal subgroup of $G$ |
| $\langle S_1, \ldots, S_t \rangle$ | the subgroup of $G$ generated by $S_1, \ldots, S_t \subseteq G$ |

If $k$ is any field and $M$ is a $kG$-module, $M|H$ denotes the restriction of $M$ to $H \leq G$ (we sometimes write $\rho|H$ if $\rho$ is the character afforded by $M$).

| | |
|---|---|
| hcf | highest common factor |
| dim | dimension |

Throughout this thesis all vector spaces are assumed to be finite dimensional.

# (A) Modular representations of finite groups with split (B,N)-pairs.

## I. Unsaturated (B,N)-pairs.

Assume $p$ is a prime number. Let $G = (G,B,N,R,U)$ be a finite group which satisfies the following conditions:

(i) $G$ has a (B,N)-pair (according to [3, Definition 2.1, p. B- 8] ) where $H = B \cap N$ and the Weyl group $W = N/H$ is generated by the set $R = \{w_1,\dots,w_n\}$ of special generators.

(ii) There exists a p-subgroup $U$ of $G$ such that $B = UH$ is a semi-direct product, $U$ is normal in $B$ and $H$ is abelian with order prime to $p$.

Then $G$ satisfies all axioms of a split (B,N)-pair ( [3, Definition 3.1, p. B-12] ) except that of saturation; we allow the intersection of the N-conjugates of $B$ to be larger than $H$. We say $G$ has an <u>unsaturated split (B,N)-pair of characteristic $p$ and rank $n$.</u> The term unsaturated means 'not necessarily saturated.' We assume unless otherwise stated that $k$ is an algebraically closed field of characteristic $p$. Let $Y \cong \mathrm{Ind}_U^G(k_U)$ and $E = \mathrm{End}_{kG}(Y)$ where $k_U$ is the trivial U-module $k$. Sawada ([8]) was the first to examine $Y$ and $E$ for groups with split (B,N)-pairs and established a bijective correspondence between the set of isomorphism classes of irreducible left kG-modules and the set of isomorphism classes of irreducible right E-modules. In doing so he relied on work done by Curtis ([3]) and Richen ([7]) on irreducible kG-modules. We will start by discussing the the E-modules directly and be able to recover most of the results of Curtis, Richen and Sawada by using a recent theorem of Green ([5]). By this method we will be able to discard the saturation condition.

Notations. Since $H$ is abelian, $U$ a p-group, all modular representations of $B$ are linear and we let $\hat{B} = \mathrm{Hom}(B,k^*)$ where $k^* = k\setminus\{1\}$. If $x,g \in G$ then $x^g = g^{-1}xg$. For any subset $T$ of $G$, $[T] = \sum_{t\in T} t \in kG$ and $T^g = g^{-1}Tg$ (similarly for $J^w$ where $J \subseteq R$, $w \in W$). Let $w \in W$, $(w) \in N$ with $(w)H = w$. For $X$ any subgroup of $G$ containing $H$ we write $Xw$ for $X(w)$ (similarly for $wX$, $XwX$). If $A$ is any subgroup of $G$ normalised by $H$, then $A^{(w)} = A^{h(w)}$ any $h \in H$ so we write $A^w$. Since $H$ is abelian the Weyl group $W$ acts on the elements of $H$ by $h^w = h^{(w)}$.

Let $\nu : N \to W$ be the natural epimorphism and the length of $w \in W$ as a minimal product of generators is denoted $l(w)$. The unique element of maximal length in $W$ is written $w_o$.

Let $y \in Y$ correspond to $1_{kG} \otimes_{kU} 1_k$. If $\{g_i \mid i \in I\}$ is a left transversal for the cosets of $U$ in $G$ then $Y = kGy$ has k-basis $\{g_i y \mid i \in I\}$.

We assume that $\{(w) \mid w \in W\}$ is a fixed but arbitrary set of coset representatives of $H$ in $N$.

The reader will notice that the proofs of certain facts in I have been deferred to (A) II where the specific rank one case is discussed.

1. Preliminaries. In this section we state results which, though proven in [3] and [7] under the assumption of saturation, do not actually depend on that condition. For example, statements in [7, Chapter II] which do not involve $H = B \cap N$ will be true in the unsaturated case. We also make adjustments to other results when necessary to suit our unsaturated

hypothesis.

Notation. Let $w \in W$. Then ${}_wB^+ = B \cap B^w$; ${}_wU^+ = U \cap U^w$; ${}_wB^- = B \cap B^{w_0w}$; and ${}_wU^- = U \cap U^{w_0w}$.

Remark 1. Notice that ${}_wB^+ = {}_wU^+H$, ${}_wB^- = {}_wU^-H$ (see [7, proof of Theorem 3.3(b), p.444] ) and that $H$ normalises ${}_wU^+$, ${}_wU^-$ for any $w \in W$.

1.1 Lemma. The intersection of the N-conjugates of B is $B \cap B^{w_0}$. Also $\bigcap_{n \in N} U^n = \bigcap_{w \in W} U^w = U \cap U^{w_0}$.

Proof. We need only show that $B \cap B^{w_0} \subseteq {}_wB^+$ for all $w \in W$. A proof of this fact can be found in [7, proof of Lemma 2.4, p.441] . The second statement follows from the remark above.

Remark 2. Let $C = {}_{w_0}U^+$. Then $C^w = C$ for all $w \in W$ by 1.1 .

1.2 Lemma. Let $w, v \in W$ satisfy $l(vw) = l(v) + l(w)$. Then ${}_{vw}U^- = {}_wU^- ({}_vU^-)^w$ and ${}_wU^- \cap ({}_vU^-)^w = C$.

Proof. The first part follows by an easy induction on $l(w)$ from [7, proof of Theorem 3.3(a), p.444] .

By 1.1, $C \subseteq {}_wU^- \cap ({}_vU^-)^w$ and

$$
\begin{aligned}
{}_wU^- \cap ({}_vU^-)^w &= U \cap U^{w_0w} \cap U^{w_0vw} \cap U^w \\
&\subseteq U^{w_0w} \cap U^w \\
&= (U^{w_0} \cap U)^w \\
&= C \quad \text{by remark 2.}
\end{aligned}
$$

1.3 Corollary. Let $w \in W$. Then $U = {}_wU^+ \, {}_wU^-$ and ${}_wU^+ \cap {}_wU^- = C$. Hence $|U| \, c = |{}_wU^+| \, |{}_wU^-|$ where $c = |C|$.

Proof. Let $v = w_0w^{-1}$ and apply 1.2 .

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By 1.1, $C \subseteq {}_wU^- \cap ({}_vU^-)^w$ and

$$\begin{aligned} {}_wU^- \cap ({}_vU^-)^w &= U \cap U^{w_0w} \cap U^{w_0vw} \cap U^w \\ &\subseteq U^{w_0w} \cap U^w \\ &= (U^{w_0} \cap U)^w \\ &= C \text{ by remark 2.} \end{aligned}$$

1.3 Corollary. Let $w \in W$. Then $U = {}_wU^+\,{}_wU^-$ and ${}_wU^+ \cap {}_wU^- = C$. Hence $|U|\,c = |{}_wU^+|\,|{}_wU^-|$ where $c = |C|$.

Proof. Let $v = w_0w^{-1}$ and apply 1.2 .

1.4 Let $w \in W$. Let $\Omega_w$ be a left transversal (containing 1) of ${}_{w^{-1}}U^-$ by C. Then $\Omega_w$ is automatically a transversal of U by ${}_{w^{-1}}U^+$ by 1.3 and $|\Omega_w| = |{}_{w^{-1}}U^-| / c$.

Also $BwB = UwB = \Omega_w wB$.

<u>Notation</u>. For $w_i \in R$, write $\Omega_i$ for $\Omega_{w_i}$, $B_i$ for ${}_{w_i}B^-$ and $U_i$ for ${}_{w_i}U^-$.

The following short lemmas are consequences of results proven in the rank one case (see Chapter II, 1.1-1.4) and the Bruhat Decomposition Theorem (see [1, Theorem 1, p.25]).

1.5 <u>Lemma</u>. Let $w \in W$. Then $\Omega_w{}^{(w)} \cap B = 1$.

1.6 <u>Lemma</u>. Let $w, w_1, w_2 \in W$, $u_1, u_2 \in U$, $h_1, h_2 \in H$.
Then

$$u_1h_1(w_1)U = u_2h_2(w_2)U \Leftrightarrow w_1 = w_2,\ u_2{}^{-1}u_1 \in {}_{w^{-1}}U^+,\ h_1 = h_2.$$

The set $\Gamma = \{u_w h(w) \mid h \in H, u_w \in \Omega_w, w \in W\}$ is a transversal for the left cosets of U in G.

1.7 <u>Lemma</u>. Every element of G can be uniquely expressed as $g = u(w)hu'$ where $w \in W$, $u \in \Omega_w$, $h \in H$ and $u' \in U$.

The next lemma is a consequence of 1.6 and 1.7.

1.8 <u>Lemma</u>. The elements of N form a transversal for the U-U double cosets of G.

## <u>2. The endomorphism algebra E.</u>

In this section we characterise the simple right E-modules.

By 1.8 E has k-basis $\{A_n \mid n \in N\}$ where $A_n(y) = p_n y$ and $p_n$ is the sum of those $\gamma \in \Gamma$ which lie in UnU (see, for example [8, p.32]) The elements $A_n$ $(n \in N)$ are clearly independent of the choice of transversal of the cosets of $U$ in $G$. Therefore, using 1.6

2.1 $$A_n(y) = [\Omega_w]\, ny$$

$$p_n = [\Omega_w]\, n \quad \text{where} \quad \nu(n) = w \,.$$

Clearly $p_h = h$ all $h \in H$. Multiplication in $E$ is given by the formulae

2.2 $$A_m A_n = \sum_{t \in N} c_{mnt} A_t \qquad (m, n \in N)$$

where $c_{mnt} = z_{mnt} \cdot 1_k$ and $z_{mnt} \in \mathbb{Z}$ is the number of pairs $(\gamma, \xi) \in \Gamma \times \Gamma$ such that $\gamma \in UnU$, $\xi \in UmU$ and $\gamma\xi \in tU$ since $A_t(y)$ is the sum of all the distinct U-translates of $ty$ and $gy = g'y \Leftrightarrow gU = g'U$ any $g, g' \in G$. The following lemma is immediate:

2.3 <u>Lemma</u>. If $t, m, n \in N$ are such that $UtU \not\subseteq UnUmU$, then the coefficient of $A_t$ in $A_m A_n$ is zero.

2.4 <u>Lemma</u>. Let $n, m \in N$ with $\nu(n) = v$, $\nu(m) = w$ be such that $l(vw) = l(v) + l(w)$. Then $A_m A_n = A_{nm}$.

Proof. We know $$A_m A_n(y) = [\Omega_v]\, n\, [\Omega_w] my$$
$$= [\Omega_v]\, n\, [\Omega_w] n^{-1} nmy \,.$$

By 1.2 $_{w^{-1}v^{-1}}U^- = {}_{v^{-1}}U^- ({}_{w^{-1}}U^-)^{v^{-1}}$ and $|_{w^{-1}v^{-1}}U^-|\, c = |_{v^{-1}}U^-||_{w^{-1}}U^-|$. We see that $A_m A_n(y)$ is the sum of $|\Omega_v||\Omega_w|$ U-translates of $nmy$ by our choice of transversals (1.4). Therefore $A_m A_n = \lambda A_{nm}$ where $\lambda$ is the integer $|\Omega_v||\Omega_w| / |\Omega_{vw}|$. By 1.4

$$\lambda = \frac{|_{v^{-1}}U^-||_{w^{-1}}U^-|}{c^2} \cdot \frac{c}{|_{w^{-1}v^{-1}}U^-|} \quad \text{so that}$$

$\lambda = 1$ as required.

2.5 <u>Corollary</u>. Let $h \in H$, $n \in N$. Then

$$A_n A_h = A_{hn} = A_{n^{-1}hn} A_n \,.$$

2.6 Corollary. The set $\{A_h, A_{(w_i)} \mid h \in H, w_i \in R\}$ k-algebra generates E.

We can now state and prove one of the main results of this paper. The proof is due to Green who proved it for the saturated case. Notice that the proof relies only on 2.4 and is therefore true for any field.

2.7 Proposition. Let G be a finite group with an unsaturated split (B,N)-pair of characteristic p and rank n. Let k be any field. Then E is a Frobenius algebra.

Proof. Let $q \in N$ satisfy $\nu(q) = w_o$, the unique element of maximal length in W. Let $f: E \times E \to k$ be given as follows: for $\alpha, \beta \in E$, $f(\alpha, \beta)$ is to be the coefficient of $A_q$ in the expression of $\alpha\beta$ as a linear combination of the basis elements $\{A_n \mid n \in N\}$. Certainly f is bilinear and associative and we need only show that f is non-degenerate. Let $\{Z_n \mid n \in N\}$ be the basis of E given by $Z_n = A_{n^{-1}q}$.

2.8 Let $n, n' \in N$, $\nu(n) = w$, $\nu(n') = w'$. Then $f(Z_n, A_{n'})$ is zero if either (i) $l(w) > l(w')$ or (ii) $l(w) = l(w')$ but $w \neq w'$. In the case $w = w'$, $f(Z_n, A_{n'}) = \delta_{n,n'}$ (that is, 1 for $n = n'$ and 0 otherwise).

Proof of 2.8 By 2.3 the coefficient of $A_q$ in $Z_nA_{n'} = A_{n^{-1}q}A_{n'}$ is 0 if $UqU \not\subseteq Un'Un^{-1}qU$. So $f(Z_n, A_{n'})$ is certainly 0 if

$$Bw_oB \not\subseteq Bw'Bw^{-1}w_oB \qquad (*)$$

Since $l(w'w^{-1}w_o) \leq l(w') + l(w^{-1}w_o) = l(w') + l(w_o) - l(w)$ (*) holds in (i) or (ii)

If $w = w'$, we see that $A_{n^{-1}q}A_{n'} = A_{n'n^{-1}q}$ by 2.4

since $l(w'w^{-1}w_o) = l(w_o) = l(w') + l(w^{-1}w_o)$. Hence

$f(Z_n, A_{n'})$ is 0 or 1 depending upon whether $n \neq n'$ or $n = n'$ and 2.8 is proved.

Now the elements of $N$ can be totally ordered so that $l(\nu(n)) < l(\nu(n')) \Rightarrow n < n'$. So if for $n, n' \in N$ we have $n \geqslant n'$ then we must have $l(\nu(n)) \geqslant l(\nu(n')$ By 2.8 $f(A_n, A_{n'}) = \delta_{n,n'}$ and we see that the matrix $(f(Z_n, A_{n'}))_{n,n' \in N}$ is unitriangular and hence non-singular. We have shown that $f$ is non-degenerate and the proof of Proposition 2.7 is completed.

Definition. Let $w_i \in R$. Define $G_i = \langle U, U_i^{w_i} \rangle$,

$H_i = G_i \cap H$.

2.9 Lemma. (see [3, Proposition 3.7, p.B-15]) Let $w_i \in R$. We can arrange that $(w_i) \in G_i$. In this case $G_i = UH_i \cup \Omega_i H_i (w_i) U$.

Proof. Consider $P_i = B \cup Bw_iB$ and any representative $(w_i)'$ of $w_i$. Let $1 \neq u \in \Omega_i$. Then $u^{(w_i)'} \in P_i$ and if $u^{(w_i)'} \in B$ then $u = 1$ by 1.5. Therefore $u^{(w_i)'} \in Bw_iB = \Omega_i w_i B$. Hence there exists a representative $(w_i) \in U\Omega_i^{(w_i)'} U$. The subgroup $\langle U, \Omega_i^{(w_i)'} \rangle$ does not depend on $(w_i)'$ since $U_i^{(w_i)'} = U_i^{w_i} = \Omega_i^{(w_i)'c^{(w_i)'}} = \Omega_i^{(w_i)'c}$ and $\langle U, \Omega_i^{(w_i)'} \rangle = \langle U, \Omega_i^{(w_i)'c} \rangle = \langle U, U_i^{w_i} \rangle$.

The subgroup $G_i$ has the required form since $G_i \subset P_i$.

We assume from now on that $(w_i) \in G_i$, for every $w_i \in R$.

The proofs of the following two lemmas can be found in Chapter II, 1.6 and 2.4.

2.10 <u>Structural Equations in $G$</u>. Let $w_i \in R$, $\Omega_i^* = \Omega_i \setminus \{1\}$.

There exist functions $f_i : \Omega_i^* \to \Omega_i^*$, $g_i : \Omega_i^* \to U$,

$h_i : \Omega_i^* \to H$ where $f_i$ is a bijection, such that for every $u \in \Omega_i^*$

$$(w_i)u(w_i) = f_i(u)h_i(u)(w_i)g_i(u) .$$

Since $(w_i) \in G_i$, $h_i(u) \in H_i$ for all $u \in \Omega_i^*$.

2.11 <u>Lemma</u>. Let $w_i \in R$. Then

$$A_{(w_i)}^2 = A_{(w_i)} \sum_{s=1}^{b(i)} A_{h_i(u_{i_s})} \quad \text{where } b(i) = |\Omega_i^*|$$

and $u_{i_1}, \ldots, u_{i_{b(i)}}$ are certain elements of $\Omega_i^*$ (not necessarily distinct).

The following formulae were first determined by Sawada ([8, Proposition 2.6, p. 34]) for the saturated case.

2.12 <u>Formulae</u>. Let $n \in N$, $\nu(n) = w$.

(i) If $l(w_i w) = l(w)+1$, then $A_n A_{(w_i)} = A_{(w_i)n}$ .

(ii) If $l(w_i w) = l(w)-1$, then $A_n A_{(w_i)} = A_n \sum_{s=1}^{b(i)} A_{h_i(u_{i_s})}$ .

(iii) If $l(w w_i) = l(w)+1$, then $A_{(w_i)} A_n = A_{n(w_i)}$ .

(iv) If $l(w w_i) = l(w)-1$, then $A_{(w_i)} A_n = \sum_{s=1}^{b(i)} A_{(w_i)^{-1} h_i(u_{i_s})(w_i)} A_n$

<u>Proof</u>. Parts (i) and (iii) follow from 2.4 . For (ii) let $w = w_i v$ with $l(v) = l(w)-1$. Then $(w_i)^{-1}n = m \in N$, $\nu(m) = v$ and $A_n = A_{(w_i)m} = A_m A_{(w_i)}$ by 2.4 . Therefore

$$A_n A_{(w_i)} = A_m A_{(w_i)}^2$$

$$= A_m A_{(w_i)} \sum_{s=1}^{b(i)} A_{h_i}(u_{i_s}) \quad \text{by 2.11}$$

$$= A_n \sum_{s=1}^{b(i)} A_{h_i}(u_{i_s}) \quad \text{by 2.4 .}$$

Part (iv) is proved similarly using Lemma 2.5 .

Definition. Let $\chi \in \hat{B}$, $w \in W$. Then ${}^w\chi \in \hat{B}$ where ${}^w\chi(hu) = \chi(h^w u)$ for $h \in H$, $u \in U$.

The proof of the following lemma is based on [3, proof of Theorem 4.3a, p.B-20] .

2.13 Lemma. Every irreducible right E-module $X$ is one-dimensional and if $X = kx$ there exists a character $\chi \in \hat{B}$ uniquely defined by $x A_h = \chi(h) x$ for all $h \in H$.

Proof. Every one-dimensional right E-module will uniquely determine a character of $B$ since by 2.4 $A_h A_{h'} = A_{h'h} = A_{h'} A_h$ $(h, h' \in H)$.

Let $\chi \in B$, $E_\chi = \frac{1}{|H|} \sum_{h \in H} \chi(h^{-1}) A_h$ . Then $E_\chi A_h = \chi(h) E_\chi$ all $h \in H$ and $1_E = \sum_{\chi \in \hat{B}}^{\oplus} E_\chi$. Since $X = \sum_{\chi \in \hat{B}}^{\oplus} X E_\chi$ there exists $\chi \in \hat{B}$ with $X E_\chi \neq 0$ .

For $0 \neq z \in X$, such that $z E_\chi \neq 0$ and let $t = z E_\chi$ . Then $t A_h = \chi(h)\, t$ all $h \in H$ .

Choose $w \in W$ of maximal length so that $x = t A_{(w)} \neq 0$. Then $x$ affords the character ${}^w\chi$ , that is

$$x A_h = {}^w\chi(h)\, x \quad \text{since} \quad x A_h = t A_{(w)} A_h$$

$$= t A_{(w)^{-1}h(w)} A_{(w)} \quad \text{by 2.5}$$

$$= {}^w\chi(h)\, t A_{(w)} .$$

We now consider $x A_{(w_i)}$ for $w_i \in R$.

Case 1. $l(w_i w) > l(w)$

Then $x A_{(w_i)} = t A_{(w)} A_{(w_i)}$

$= t A_{(w_i)(w)}$ by 2.12 (i)

$= t A_{(w_i w)h}$ some $h \in H$ since $\nu((w_i)(w)) = \nu((w_i w))$

$= t A_h A_{(w_i w)}$ by 2.4

$= \chi(h)\, t A_{(w_i w)}$

$= 0$ by choice of $w$.

Case 2. $l(w_i w) < l(w)$

Then $x A_{(w_i)} = t A_{(w)} A_{(w_i)}$

$= t A_{(w)} \sum_{s=1}^{b(i)} A_{h_i(u_{i_s})}$ by 2.12 (ii)

$= x \sum_{s=1}^{b(i)} A_{h_i(u_{i_s})}$

$= \sum_{s=1}^{b(i)} {}^{w}\chi(h_i(u_{i_s}))\, x$ .

Therefore $x$ generates a one-dimensional right $E$-submodule of $X$ by 2.6. But $X$ irreducible $\Rightarrow X = kx$ .

We are able to formulate more results based on the rank one case, the first being the following crucial lemma.

2.14 Lemma. Fix $\chi \in \hat{B}$, $w_i \in R$. Let $d_i = \sum_{s=1}^{b(i)} \chi(h_i(u_{i_s}))$ . If $d_i \neq 0$ then $\chi | H_i = 1$. Hence $d_i = -1$ .

Proof. By Theorem 3.2 of Chapter II there exists a one-dimensional $P_i = B \cup B w_i B$ - module $M$ such that if $M$ affords $\xi : P_i \to k^*$ then $\xi | H = \chi | H$. Now $G_i$ is

We now consider $x A_{(w_i)}$ for $w_i \in R$.

Case 1. $l(w_i w) > l(w)$

Then $x A_{(w_i)} = t A_{(w)} A_{(w_i)}$

$= t A_{(w_i)(w)}$ by 2.12 (i)

$= t A_{(w_i w)h}$ some $h \in H$ since $\nu((w_i)(w)) = \nu((w_i w))$

$= t A_h A_{(w_i w)}$ by 2.4

$= \chi(h)\ t A_{(w_i w)}$

$= 0$ by choice of $w$.

Case 2. $l(w_i w) < l(w)$

Then $x A_{(w_i)} = t A_{(w)} A_{(w_i)}$

$$= t A_{(w)} \sum_{s=1}^{b(i)} A_{h_i(u_{i_s})} \quad \text{by 2.12 (ii)}$$

$$= x \sum_{s=1}^{b(i)} A_{h_i(u_{i_s})}$$

$$= \sum_{s=1}^{b(i)} {}^{w}\chi(h_i(u_{i_s}))\, x \quad .$$

Therefore $x$ generates a one-dimensional right $E$-submodule of $X$ by 2.6. But $X$ irreducible $\Rightarrow X = kx$ .

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If $d_i \neq 0$ then $\chi | H_i = 1$. Hence $d_i = -1$ .

Proof. By Theorem 3.2 of Chapter II there exists a one-dimensional $P_i = B \cup Bw_iB$ - module $M$ such that if $M$ affords $\xi : P_i \to k^*$ then $\xi | H = \chi | H$. Now $G_i$ is generated by p-groups so that $\zeta|G_i = 1$ and $\zeta|H_i = 1$. Therefore $\chi|H_i = 1$ and since $h_i(u_{i_s}) \in H_i$ $(s=1,\ldots,b(i))$ (by 2.10) and $b(i) = |\Omega_i| - 1$, the result follows since $1 < |\Omega_i|$ is a power of $p$.

2.15 Lemma. Let $\psi$ be any multiplicative character $\psi: E \to k$. Then there exist $\chi \in \hat{B}$, $\mu_1,\ldots,\mu_n \in k$ such that

$$\left.\begin{array}{lll} \text{(i)}\ \psi(A_h) = \chi(h) & \text{all} & h \in H \\ \text{(ii)}\ \psi(A_{(w_i)}) = \mu_i & & (1 \leq i \leq n) \end{array}\right\} \quad (*)$$

Moreover, $\mu_i = 0$ or $-1$ and $\mu_i \neq 0$ implies $\chi|H_i = 1$.

Proof. Part (i) follows from 2.13 and (ii) follows from 2.11 and 2.14.

We might call the sequence $(\chi, \mu_1, \ldots, \mu_n)$ the 'weight of $\psi$' to correspond with Curtis' terminology.

Definition. Let $J \subseteq R$. Then $W_J = \langle w_i \mid w_i \in J \rangle$.

2.16 Lemma. Let $\chi \in \hat{B}$, $J \subseteq R$. Suppose $\chi|H_i = 1$ for every $w_i \in J$. Then ${}^w\chi = \chi$ all $w \in W_J$.

Proof. It is sufficient to show ${}^{w_i}\chi = \chi$ for all $w_i \in J$. Since $\chi|H_i = 1$, $d_i = \sum_{s=1}^{b(i)} \chi(h_i(u_{i_s})) \neq 0$ every $w_i \in J$ and the result follows by Lemma 3.1 of Chapter II.

The above lemma is also proved in [3, Lemma 5.4, p.B-26] and [7, Corollary 3.22, p.453] under the saturation condition.

We wish to prove the converse of 2.15; that is, given any sequence $(\chi, \mu_1, \ldots, \mu_n)$ where $\chi \in \hat{B}$, $\mu_i \in k$ $(1 \leq i \leq n)$ and where $\mu_i = 0$ or $-1$ with $\mu_i \neq 0$ implying $\chi|H_i = 1$, then there exists a multiplicative character $\psi: E \to k$ with properties (*). In order to do this we place additional restrictions on the choice of coset representatives $\{(w_i) \mid w_i \in R\}$.

The following lemma is due to Tits. A proof can be found in [4, (1G), p.5] .

2.17 <u>Lemma</u>. Let $w_i \in R$. Then $B_i \cup B_i w_i B_i$ is a subgroup of $G$.

<u>Remark.</u> Notice that the above lemma does not depend on a saturated condition since $B_i = U_i H$, $U \cap U^{w_o}$ is normalised by $H$ and $U \cap U^{w_o} \subset U_i$ $(w_i \in R)$.

2.18 <u>Lemma</u>. Let $w_i \in R$. Then coset representative $(w_i)$ can be chosen in $\langle U_i, U_i^{w_i} \rangle$.

<u>Proof</u>. Clearly $\langle U_i, U_i^{w_i} \rangle \subset B_i \cup B_i w_i B_i = U_i H \cup U_i H w_i U_i$ .

If $U_i^{w_i} \subset U_i H$ then $U_i^{w_i} = U_i$ so that

$$B^{w_i} = U_i^{w_i} ({}_{w_i}U^+)^{w_i} H$$

$$= U_i \, {}_{w_i}U^+ H$$

$$= B, \text{ contrary to the } (B,N)\text{-pair axioms.}$$

Hence $U_i^{w_i} \cap U_i H w_i U_i$ is non-empty and there exists a coset representative $n_i$ and $u_1, u_2, u_3 \in U_i$ such that

$$u_1^{w_i} = u_2 \, n_i \, u_3 \, .$$

2.19 The coset representative $(w_i)$ can be chosen in $U_i U_i^{w_i} U_i$ and the proof of 2.18 is completed.

<u>Remark</u>. Statement 2.19 is important since we are able to choose the coset representatives $\{(w_i) \mid w_i \in R\}$ in the same way whether the (B,N)-pair is saturated or not (see [2, Lemma 2.2, p.351] or [3, Definition 3.9, p.B-16] ).

We assume from now on that coset representatives $\{(w_i) \mid w_i \in R\}$ are chosen according to 2.19.

The next lemma, proved by Richen in [7, Lemma 3.28, p.456] holds in the unsaturated case.

2.20 Lemma. Let $J \subseteq R$. Coset representatives $\{(w) \mid w \in W_J\}$ can be chosen so that if $w, w' \in W_J$ then

$$(w)(w')(ww')^{-1} \in H_J = \langle H_i^{w} \mid w \in W_J, w_i \in J \rangle .$$

Definition. For any $\chi \in \hat{B}$, let $e(\chi) = \sum_{h\in H} \chi(h^{-1})A_h$ .

2.21 Theorem. (see [8, Proposition 3.1, p.36] ) Let $J \subseteq R$ and let coset representatives $\{(w) \mid w \in W_J\}$ be chosen according to 2.20. Let $\chi \in \hat{B}$ and suppose $\chi|H_i = 1$ all $w_i \in J$. Let

$$z(J,\chi) = e({}^{w_o}\chi) \sum_{w\in W_J} A_{(w)(w_o)} .$$

Then $z = z(J,\chi)$ generates a one-dimensional right E-module (right ideal of E) with the following properties:

$$\text{(i)}\quad z A_h = \chi(h)\, z \qquad (h \in H)$$

$$\text{(ii)}\quad z A_{(w_i)} = \begin{cases} 0 & w_i \in J \text{ or } \chi|H_i \neq 1 \\ -z & w_i \notin J \text{ and } \chi|H_i = 1 \end{cases}$$

Proof. By 2.6, we need only verify properties (i) and (ii). Take $h \in H$, $w \in W_J$. Then

$$\begin{aligned} e({}^{w_o}\chi)A_{(w)(w_o)}A_h &= e({}^{w_o}\chi)A_{h(w)(w_o)} \quad \text{by } 2.4 \\ &= e({}^{w_o}\chi)A_{(w)(w_o)(w_o)^{-1}(w)^{-1}h(w)(w_o)} \\ &= e({}^{w_o}\chi)A_{(w_o)^{-1}(w)^{-1}h(w)(w_o)}A_{(w)(w_o)} \\ &\qquad \text{by } 2.4 \end{aligned}$$

$= e({}^{w_o}\chi)\,{}^{w_o}\chi((w_o)^{-1}(w)^{-1}h(w)(w_o))A_{(w)(w ...}$

$= e({}^{w_o}\chi)\ \chi(h)\ A_{(w)(w_o)}$ by 2.16

so that $z\,A_h = \chi(h)\,z$ any $h \in H$.

(i) Take $w_i \notin J$. Then $l(w_i w w_o) < l(w w_o)$ (see [3, proof of Lemma 5.5, p.B-27] ) for all $w \in W_J$. And

$$e({}^{w_o}\chi)A_{(w)(w_o)}A_{(w_i)} = e({}^{w_o}\chi)A_{(w)(w_o)}\sum_{s=1}^{b(i)} A_{h_i(u_{i_s})} \quad \text{by 2.12(ii)}$$

$$= \sum_{s=1}^{b(i)} \chi(h_i(u_{i_s}))\ e({}^{w_o}\chi)\ A_{(w)(w_o)}$$

so that by 2.14

$$z\,A_{(w_i)} = \begin{cases} 0 & \chi|H_i \neq 1 \\ -z & \chi|H_i = 1 \end{cases}$$

(ii) Now suppose $w_i \in J$. We take a decomposition of $W_J$ into cosets $\{w, w_i w\}$ with respect to the subgroup $\langle w_i \rangle$. We show that terms in $z\,A_{(w_i)}$ corresponding to $w$ and $w_i w$ cancel each other. Without loss of generality we may assume $l(w_i w w_o) = l(w w_o)+1$ (as in [3, proof of Lemma 5.5, p.B-27] ).

The term corresponding to $w$ in $z\,A_{(w_i)}$ is (by 2.12(i))

$$e({}^{w_o}\chi)A_{(w)(w_o)}A_{(w_i)} = e({}^{w_o}\chi)A_{(w_i)(w)(w_o)} \qquad (1)$$

Since $l(w_i w w_o)-1 = l(w w_o)$ the term corresponding to $w_i w$ is

$$e({}^{w_o}\chi)A_{(w_i w)(w_o)}A_{(w_i)} = e({}^{w_o}\chi)A_{(w_i w)(w_o)}\sum_{s=1}^{b(i)} A_{h_i(u_{i_s})}$$

by 2.12(ii)

$$= e({}^{w_o}\chi)\sum_{s=1}^{b(i)} A_{h_i(u_{i_s})(w_i w)(w_o)}$$

by 2.4

By 2.5 this last term is

$$\sum_{s=1}^{b(i)} e({}^{w_o}\chi)A_{(w_o)^{-1}(w_iw)^{-1}h_i(u_{i_s})(w_iw)(w_o)}A_{(w_iw)(w_o)}$$

$$= \sum_{s=1}^{b(i)} e({}^{w_o}\chi)\,{}^{w_o}\chi((w_o)^{-1}(w_iw)^{-1}h_i(u_{i_s})(w_iw)(w_o))A_{(w_iw)(w_o)}$$

$$= \sum_{s=1}^{b(i)} \chi(h_i(u_{i_s}))e({}^{w_o}\chi)A_{(w_iw)(w_o)} \quad \text{by } 2.16$$

$$= -e({}^{w_o}\chi)A_{(w_iw)(w_o)} \quad \text{since } w_i \in J,\ \chi|H_i = 1$$

$$= -e({}^{w_o}\chi)A_{(w_i)(w)h(w_o)} \quad \text{some } h \in H_J \text{ by } 2.20$$

$$= -e({}^{w_o}\chi)A_{(w_o)^{-1}h(w_o)}A_{(w_i)(w)(w_o)} \quad \text{by } 2.4$$

$$= -e({}^{w_o}\chi)\,{}^{w_o}\chi((w_o)^{-1}h(w_o))A_{(w_i)(w)(w_o)}$$

$$= -e({}^{w_o}\chi)A_{(w_i)(w)(w_o)} \quad \text{since } \chi|H_J = 1 \text{ by } 2.16.$$

The term above cancels with (1) and the proof is completed.

Remarks. (1) As in the saturated case we can show that for every $w_i \in R$, $H_i = H \cap \langle U_i, U_i^{w_i}\rangle$ (see for example [8, Lemma 3.6, p. 38]) using lemmas 2.9, 2.17, and 2.18. It then follows that

a) $H_i^{w_i} = H_i$ all $w_i \in R$

b) $H_{w_ow_iw_o}^{w_ow_i} = H_i$ all $w_i \in R$

Therefore for $\chi \in \hat{B}$

(2) $M(\chi) = w_oM({}^{w_o}\chi)w_o$ since

$$\chi|H_i = 1 \Leftrightarrow {}^{w_o}\chi\,|H_i^{w_o} = 1$$

$$\Leftrightarrow {}^{w_o}\chi|H_{w_ow_iw_o}^{w_ow_iw_o} = 1 \text{ by (b)}$$

$$\Leftrightarrow {}^{w_o}\chi|H_{w_ow_iw_o} = 1 \text{ by (a)}.$$

The following two remarks were proved by Sawada ([8]) for the saturated case and remain true for unsaturated pairs:

(3) The map $(J,\chi) \to (J^{w_o}, {}^{w_o}\chi)$ is a bijection of the set of admissible G-pairs where $J^{w_o} = w_oJw_o$.

Let $z = z(J,\chi)$ be as in 2.21 and let $z$ afford the E-character $\varphi(J,\chi)$. Then since $z(J^{w_o},{}^{w_o}\chi) = e(\chi)\sum_{w\in W_J} A_{(w_o)(w)}$ it follows from the proof of 2.21 that

(4) $z(J^{w_o},{}^{w_o}\chi)$ generates a left E-module which affords the E-character $\varphi(J,\chi)$; that is

$$A_h z(J^{w_o},{}^{w_o}\chi) = \chi(h)z(J^{w_o},{}^{w_o}\chi) \quad \text{all } h \in H$$

$$A_{(w_i)}z(J^{w_o},{}^{w_o}\chi) = (\varphi(J,\chi)A_{(w_i)})z(J^{w_o},{}^{w_o}\chi) \quad \text{all } w_i \in R.$$

We will use this fact later.

We can now prove the converse of 2.15, one of the main results of this chapter. We might call the sequence $(\chi,\mu_1,\ldots,\mu_n)$ an 'admissible vector' if $\chi \in \hat{B}$, all $\mu_i \in \{0,-1\}$ and $\mu_i \neq 0$ implies $\chi|H_i = 1$.

2.22 Theorem. Let G be a finite group with an unsaturated split (B,N)-pair of characteristic p and rank n, and let k be an algebraically closed field of the same characteristic. Given any sequence $(\chi,\mu_1,\ldots,\mu_n)$ where $\chi: B \to k^*$ is a homomorphism, $\mu_i \in k$ $(1\leq i\leq n)$ such that $\mu_i = 0$ or $-1$, there exists a multiplicative character $\psi: E \to k$ given by $\psi(A_h) = \chi(h)$ all $h \in H$ and $\psi(A_{(w_i)}) = \mu_i$ $(1\leq i\leq n)$ if and only if for any $i \in \{1,\ldots,n\}$ with $\mu_i \neq 0$ we have $\chi|H_i = 1$.

Proof. ($\Rightarrow$) Follows by 2.15 .

($\Leftarrow$) Let $J = \{w_i \in R \mid \mu_i = 0 \text{ and } \chi|H_i = 1\}$ .

Let $z(J,\chi)$ be as in Theorem 2.21 and the result follows.

Remark. We have shown that $(\chi,\mu_1,\ldots,\mu_n)$ is the weight of some multiplicative character $\psi : E \to k$ if and only if it is an admissible vector.

Definition. Let $\chi \in \hat{B}$, $J \subseteq M(\chi) = \{w_i \in R \mid \chi|H_i = 1\}$. Then $(J,\chi)$ is called an admissible pair.

By 2.21 each admissible pair $(J,\chi)$ determines an admissible vector $(\chi,\mu_1,\ldots,\mu_n)$ where $\mu_i = 0$ (for $w_i \in J$ or $\chi|H_i \neq 1$) or $\mu_i = -1$ (for $w_i \notin J$ and $\chi|H_i = 1$). If for each admissible vector $(\chi,\mu_1,\ldots,\mu_n)$ we let $J = \{w_i \in R \mid \mu_i = 0 \text{ and } \chi|H_i = 1\}$ we see by 2.22 that the correspondence

$$(J,\chi) \leftrightarrow (\chi,\mu_1,\ldots,\mu_n)$$

described above is a bijective one between the set of all admissible pairs and the set of all admissible vectors. We now show how such weights and vectors correspond to Curtis' weights (see [3, Definition 4.2, p.B-17,B-18] ) and find a full set of irreducible left kG-modules in Y.

Definition. Let M be any finite dimensional left kG-module. Let $F(M) = \{m \in M \mid um = m \text{ , all } u \in U\}$.

Green ( [5, 1.3] ) describes how $F(M)$ may be regarded as a right E-module. In fact if $m \in F(M)$ and $\alpha \in E$

$$m\alpha = p_\alpha m \quad \text{where} \quad \alpha(y) = p_\alpha(y) \quad (p_\alpha \in kG) .$$

In particular (by 2.1)

2.23
$$m\, A_{(w_i)} = [\Omega_i](w_i)m \qquad (w_i \in R)$$
$$m\, A_h = hm \qquad (h \in H)$$

all $m \in F(M)$.

Green proves ([5, Theorem 2]) that the correspondence $M \to F(M)$ induces a bijection between the set of isomorphism classes of irreducible left kG-modules and the set of isomorphism classes of simple right E-modules. Since we have shown that all simple right E-modules are one-dimensional (2.13), $F(M)$ is one dimensional if $M$ is an irreducible kG-module and $F(M)$ is associated with an admissible vector $(\chi, \mu_1, \ldots, \mu_n)$ by 2.22. By 2.23 this vector coincides with the Curtis-Richen weight of $M$ and any non-zero $m \in F(M)$ is called a 'weight element' of weight $(\chi, \mu_1, \ldots, \mu_n)$. In other words $F(M)$ is precisely the set of all weight elements in M and M irreducible implies M has a unique U (hence B) line .

The following theorem was first proved by Sawada ([8]) using Curtis-Richen results ([3], [7]) and therefore relies on the saturation hypothesis.

2.24 <u>Theorem</u>. Let $G$ be a finite group with an unsaturated split $(B,N)$-pair of characteristic $p$ and rank $n$. Let $k$ be an algebraically closed field of the same characteristic. There exist bijective correspondences between the following:

(i) the set of admissible vectors,

(ii) the set of admissible pairs,

(iii) the set of isomorphism classes of simple right E-modules, and

(iv) the set of isomorphism classes of irreducible left kG-modules.

These correspondences are given by:

$(\chi, \mu_1, \ldots, \mu_n) \leftrightarrow (J, \chi) \leftrightarrow kz(J, \chi) \leftrightarrow kGz(J, \chi)(y)$

Proof. We need only verify the correspondence between (iii) and (iv). Green ([5, 1.3c]) proves that the map $E \to F(Y)$ given by $\beta \to \beta(y)$ $(\beta \in E)$ is a right E-isomorphism. Let $(J, \chi)$ be an admissible pair. Since $z(J, \chi)$ generates a one-dimensional right ideal of E (2.21), $kz(J, \chi)(y)$ is a one-dimensional right E-submodule of F(Y). Therefore by [5, 2.6a], $kGz(J, \chi)(y)$ is an irreducible left kG-module and $F(kGz(J, \chi)(y)) = kz(J, \chi)(y)$. If M is any irreducible left kG-module, there exists an admissible pair $(J, \chi)$ with $F(M) \cong kz(J, \chi) \cong kz(J, \chi)(y)$ as right E-modules. But M irreducible implies $M \cong kGz(J, \chi)(y)$. Therefore $\{kGz(J, \chi)(y) \mid (J, \chi) \text{ admissible}\}$ is a full set of irreducible left kG-modules. (Curtis also determines such a set in [3, Corollary 6.12, p.B-37].)

## II. The rank one case.

Assume k is any algebraically closed field of characteristic p. If G is a finite group with an unsaturated split (B,N)-pair (G,B,N,R,U) then for any $w_i \in R$ the parabolic subgroup $P_i = B \cup Bw_iB$ has an unsaturated split (B,N)-pair $(P_i, B, N_i, \{w_i\}, U)$ of rank one where $N_i = H \cup w_i H$. Let $(w_i) \in N$ satisfy $(w_i)H = w_i$. We show in section 2 that the set $\{A'_h, A'_{(w_i)} \mid h \in H\}$ k-algebra generates $E_i = \mathrm{End}_{kP_i}(Y_i)$ where $Y_i \cong \mathrm{Ind}_U^{P_i}(k_U)$. By Corollary 2.6 of Chapter I there exists an injective k-linear algebra homomorphism

$$\Phi : E_i \to E \quad \text{given by}$$

$$A'_h \to A_h \quad (h \in H)$$

$$A'_{(w_i)} \to A_{(w_i)}$$

since the set $\{h, h(w_i) \mid h \in H\}$ forms part of a transversal for the U-U double cosets in G (see Chapter I, 2.2). Therefore results proved for the rank one case can be extended to G.

It becomes necessary in section 3 to examine $d = \sum_{s=1}^{b} \chi(h(u_s))$ where $\chi \in \hat{B}$ is fixed and the $h(u_s)$ $(s=1,\ldots,b)$ are certain elements of H determined by $(w_i)$ and Richen's 'structural equations.' Since these equations exist for every $w_i \in R$, we refer in Chapter I to $d_i = \sum_{s=1}^{b(i)} \chi(h(u_{i_s}))$.

Therefore we now assume G has an unsaturated split (B,N) pair (G,B,N,R,U) of rank one. Let $W = N/H = \{1, w\}$. The subgroup $U \cap U^w$ is denoted by ${}_wU^+$. As in Chapter I, $Y \cong \mathrm{Ind}_U^G(k_U)$, y corresponds to $1_{kG} \otimes_{kU} 1_k$ so that $Y = kGy$. Let $E = \mathrm{End}_{kG}(Y)$. Let $(w) \in N$ satisfy $(w)H = w$.

1. Cosets of $G$ by $U$.

1.1 Let $\Omega$ be any left transversal (containing 1) of $U$ by ${}_wU^+$. Then $\Omega^{(w)} \cap B = 1$.

Proof. Since $\Omega \cap B^w \subsetneq B \cap B^w = (U \cap U^w)H$, the result follows.

Remark. Note that $|\Omega| > 1$, for otherwise $U = {}_wU^+$, $wBw = B$, contrary to the $(B,N)$-pair axioms.

1.2 Cosets of the form $gU$ $(g \in G)$ contained in $BwB = BwU$ are of the form $uh(w)U$ for some $u \in U$, $h \in H$. Moreover, if $u_1, u_2 \in U$ and $h_1, h_2 \in H$ then

$$u_1h_1(w)U = u_2h_2(w)U \iff u_2^{-1}u_1 \in {}_wU^+ \text{ and } h_1 = h_2 .$$

Proof. Clearly $u_1h_1(w)U = u_2h_1(w)U$ if $u_1 = u_2u$ for some $u \in {}_wU^+$ since $H$ normalises $U$ and ${}_wU^+$.

Say $u_1h_1(w) = u_2h_2(w)u$ $(u \in U)$.

Then $u_2^{-1}u_1 = h_2(w)u(w)^{-1}h_1^{-1}$

$= (w)h_2^wu(h_1^{-1})^w(w)^{-1}$ so that $u_2^{-1}u_1 \in B^w \cap B = {}_wU^+H$.

Therefore $u_2^{-1}u_1 \in {}_wU^+$ since it is an element whose order is a power of $p$. Therefore $h_2^wu(h_1^{-1})^w \in {}_wU^+ \subset U$ so that $(h_2^wu(h_2^{-1})^w)(h_2^w(h_1^{-1})^w) \in U$ . Therefore $h_2^w(h_1^{-1})^w \in U$ and it follows $h_2 = h_1$ .

1.3 Let $\Gamma = \{h, u(w)h \mid h \in H, u \in \Omega\}$. Then $\Gamma$ is a set of representatives of left cosets of $U$ in $G$.

Proof. We know that for $h', h \in H$

(i) $UhU = Uh'U \iff h = h'$

(ii) $UhU \neq Uh'(w)U$ (for otherwise $(w) \in B$)

(iii) $Uh(w)U = Uh'(w)U \iff h = h'$ (by 1.2)

1.4 Every element $g$ of $G$ can be uniquely expressed as $g = u_1h$ or $g = u(w)hu_2$ with $u_1,u_2 \in U$, $u \in \Omega$ , $h \in H$.

Proof. The result follows by 1.2, the fact that $B$ is the semidirect product of $U$ and $H$ and that $BwB = \Omega wB$.

1.5 The elements of $N$ form a transversal for the $U$-$U$ double cosets in $G$.

Proof. By 1.3 and 1.4.

Richen determines 'structural equations' in the saturated case and we adapt his proof in [7, p.445] to suit our hypothesis.

1.6 Structural equations in $G$. Let $\Omega^* = \Omega \setminus \{1\}$. For any $u \in \Omega^*$ there exist functions $f: \Omega^* \to \Omega^*$, $g: \Omega^* \to U$, $h: \Omega^* \to H$ where $f$ is a bijection and

$$(w)u(w) = f(u)h(u)(w)g(u) .$$

Proof. Let $u \in \Omega^*$. If $u^{(w)} \in B$, then $u = 1$ by 1.1. Therefore $u^{(w)} \in BwB$ and $(w)u(w) \in BwB = \Omega wHU$ and the existence of $f: \Omega^* \to \Omega$, $g: \Omega^* \to U$, and $h: \Omega^* \to H$ is established by 1.4. Say there exists $u \in \Omega^*$ for which $f(u) = 1$. Then $(w)u(w) = h(u)(w)g(u)$ so that $u(w) \in B$, $(w) \in B$, contradiction.

Now say there exist $u, u_1 \in \Omega^*$ with $f(u) = f(u_1)$. Then $(w)u^{-1}(w)^{-1} = (w)^2 g(u)^{-1}(w)^{-1}h(u)^{-1}f(u)^{-1}$ so that

$$(w)u^{-1}u_1(w)^{-1} = (w)u^{-1}(w)^{-1}(w)u_1(w)(w)^{-2}$$

$$= (w)^2 g(u)^{-1}(w)^{-1}h(u)^{-1}h(u_1)(w)g(u_1)(w)^{-2} \in B .$$

Therefore $(w)u^{-1}u_1(w)^{-1} \in U^w \cap B \subseteq {}_wU^+H$. Then $(w)u^{-1}u_1(w)^{-1} \in {}_wU^+$ since it is an element in $B$ whose order is a power of $p$. Finally $u^{-1}u_1 \in {}_wU^+$ so that $u_1 = u$ and $f$ is bijective.

## 2. The endomorphism algebra E.

As in Chapter I section 2 the set $\{A'_n \mid n \in N\}$ is a k-basis for E where for $h \in H$

2.1 $$A'_h(y) = hy$$

$$A'_{h(w)}(y) = [\Omega]^h(w)y .$$

It is easy to see that

2.2 $A'_h A'_{(w)} = A'_{(w)h}$ and $A'_{(w)} A'_h = A'_{h(w)}$ for any $h \in E$.

Therefore

2.3 The set $\{A'_h, A'_{(w)} \mid h \in H\}$ k-algebra generates E.

2.4 Lemma. There exist elements $u_1, \ldots, u_b$ (not necessarily distinct) belonging to $\Omega^*$ such that

$$A'^{\,2}_{(w)} = A'_{(w)} \sum_{s=1}^{b} A'_{h(u_s)} \quad \text{where}$$

$b = |\Omega| - 1$.

Proof. We can write $A'^{\,2}_{(w)} = \sum_{h \in H} \lambda_h A'_h + \sum_{h \in H} \lambda_{h(w)} A'_{h(w)}$

where $\lambda_h, \lambda_{h(w)} \in k$ all $h \in H$. Fix $h \in H$. We show

(i) if $\lambda_h \neq 0$ then $h = (w)^2$ and $\lambda_{(w)^2} = |\Omega| 1_k$

(ii) if $\lambda_{h(w)} \neq 0$ then $h = h(u)$ some $u \in \Omega^*$

Proof of (i): By Chapter I (2.2) there exist $u_1, u_2 \in \Omega$ such that $u_1(w)u_2(w) \in hU$. We must have $u_2 = 1$ for otherwise $(w)^{-1}u_2(w) \in (w)^{-2}hU \subset B$ contradicting 1.1. Now $u_1(w)^2 \in hU \Leftrightarrow (w)^2 = h$. It follows that $\lambda_{(w)^2} = |\Omega| 1_k$.

Proof of (ii): If $\lambda_{h(w)} \neq 0$ there exist $u_1, u_2 \in \Omega$ such that $u_1(w)u_2(w) \in h(w)U$. Therefore by 1.6

$Uh(u_2)(w)U = Uh(w)U$ so that $h = h(u_2)$ by 1.3 (iii).

We know that $A'_{(w)}{}^2(y)$ is a sum of $|\Omega|^2$ U-translates of y; that is $|\Omega|^2$ terms of the form $gy = \gamma y$ ($\gamma \in \Gamma$, $g \in \gamma U$). If the term $\gamma y$ appears so will each of its distinct U-translates of which there are $|\Omega|$ in number. If we call $\gamma y$ and its set of distinct U-translates an 'orbit' then by (i) and because $|\Omega|^2 - |\Omega| = |\Omega|(|\Omega| - 1)$, we see that there are $|\Omega| - 1$ such orbits in $\sum_{h \in H} \lambda_{h(w)} A'_{h(w)}$. By (ii) $A'_{(w)}{}^2$ has the required form since $1 < |\Omega|$ is a power of p.

<u>Definition</u>. For $\chi \in \hat{B}$, let $e(\chi) = \sum_{h \in H} \chi(h^{-1}) A'_h$.

Notice that $A'_h \, e(\chi) = e(\chi) \, A'_h = \chi(h) \, e(\chi)$ for any $h \in H$.

We now fix $\chi \in \hat{B}$.

3. <u>Examination of</u> $d = \sum_{s=1}^{b} \chi(h(u_s))$.

Remember that ${}^w\chi \in \hat{B}$ where ${}^w\chi(hu) = \chi(h^w u)$ any $h \in H$, $u \in U$.

3.1 <u>Lemma</u>. Assume $d \neq 0$. Then ${}^w\chi = \chi$.

<u>Proof</u>. Let $v = e({}^w\chi) A'_{(w)}$. By 2.3 v generates a one-dimensional right E-module since

$$
\begin{aligned}
\text{(i) } v A'_h &= e({}^w\chi) A'_{(w)} A'_h \\
&= e({}^w\chi) A'_{h(w)} \quad \text{by 2.2} \\
&= e({}^w\chi) A'_{(w)(w)^{-1}h(w)} \\
&= e({}^w\chi) A'_{(w)^{-1}h(w)} A'_{(w)} \quad \text{by 2.2} \\
&= \chi(h) \, v \quad \text{for all } h \in H, \text{ and}
\end{aligned}
$$

$$\begin{aligned}
\text{(ii)}\ v A'_{(w)} &= e({}^{w}\chi)A'^{\,2}_{(w)} \\
&= e({}^{w}\chi)A'_{(w)} \sum_{s=1}^{b} A'_{h(u_s)} \quad \text{by 2.4} \\
&= \sum_{s=1}^{b} \chi(h(u_s))\, v \quad \text{by part (i)} \\
&= d\, v .
\end{aligned}$$

Therefore there exists a multiplicative character $\phi : E \to k$ such that $\phi(A'_{(w)}) = d$ and $\phi(A'_h) = \chi(h)$ all $h \in H$.

But $\phi(A'_{(w)}A'_h) = \phi(A'_{(w)^{-1}h(w)}A'_{(w)})$ any $h \in H$ by 2.2

so that $\phi(A'_{(w)})\phi(A'_h) = \phi(A'_{(w)^{-1}h(w)})\phi(A'_{(w)})$ any $h \in H$

and so $d\chi(h) = {}^{w}\chi(h)\, d$ all $h \in H$ and the result follows.

3.2 <u>Theorem</u>. Assume $d \neq 0$. Then there exists a one-dimensional kG-module $M_o$ affording the character $\xi : G \to k^*$ with $\xi|H = \chi|H$.

<u>Proof</u>. By 3.1 $A'_{(w)}$ commutes with $e(\chi)$. Hence $e(\chi)$ is in the centre of $E$ and

$$e(\chi)E = e(\chi)E\, e(\chi) = k\, e(\chi) \oplus k\, e(\chi)A'_{(w)}$$

is an algebra which has basis $e = e(\chi)$ and $t = e(\chi)A'_{(w)}$. Now $e^2 = e$, $et = te = t$, $t^2 = dt$ and $e = e_o + e_1$ is a decomposition of $e$ into primitive idempotents in $e(\chi)E$ where $e_o = (1/d)(de - t)$ and $e_1 = (1/d)t$. Let $Y_\chi = e(\chi)Y$. Then $Y_\chi$ is a kG-module of dimension $|G : B| = |\Omega| + 1$ since $Y_\chi \cong \mathrm{Ind}_B^G(L_\chi)$ where $L_\chi$ is a kB-module affording the character $\chi$. Let $M_o = e_o(Y)$ and $M_1 = e_1(Y)$. Then $Y_\chi = M_o \oplus M_1$ where

$M_o$ and $M_1$ are indecomposable left kG-modules. We show that the dimension of $M_o$ is one by showing the dimension of $M_1$ is $|\Omega|$. Let $x_1 = e_1(y)$. Then $x_1$ is U-invariant and

$$\begin{aligned}
[\Omega](w)x_1 &= [\Omega](w)e_1(y) \\
&= e_1([\Omega](w)y) \\
&= e_1 A'_{(w)}(y) \\
&= 1/d\ e(\chi)\ A'_{(w)}{}^2(y) \\
&= 1/d\ e(\chi)\ A'_{(w)} \sum_{s=1}^{b} A'_{h(u_s)}(y) \quad \text{by 2.4} \\
&= 1/d\ e(\chi)\ A'_{(w)} \sum_{s=1}^{b} \chi(h(u_s))y \quad \text{since } e(\chi) \text{ and } A'_{(w)} \text{ commute} \\
&= d\ e_1(y) \\
&= d\ x_1 \neq 0 \quad \text{as } d \neq 0 .
\end{aligned}$$

Therefore $M_1$ contains an element $x = (w)\ x_1$ such that $[\Omega]x \neq 0$ and $x$ is stabilised by ${}_wU^+$. Let $L = \mathrm{Ind}_T^U(k_T)$ where $T = {}_wU^+$. Then there exists a surjective kU-map $\theta : L \to kUx$ given by $\theta(z) = x$ where $z = 1 \otimes 1$. Hence $\theta\left( \sum_{\omega\in\Omega} \omega z \right) = \sum_{\omega\in\Omega} \omega x \neq 0$.

Since U is a p-group, socle(L) is its space of U-invariants which is clearly $[\Omega]z$. Therefore $\theta$ is a bijection and the k-space kU x has dimension $|\Omega|$. But $kUx \subset M_1$ and dimension $M_1$ = dimension $Y_\chi$ - dimension $M_o \leq |\Omega|$ so that dimension of $M_1$ is $|\Omega|$.

Assume $M_o$ affords the character $\xi : G \to k^*$ and let $v = e_o(y)$. Then $M_o = kv$ and if $h \in H$

$$hv = h(1/d(de - t)(y))$$

$$= 1/d(dhe - ht)(y)$$

$$= 1/d(d\,\chi(h)\,e(y) - h\,e\,A'_{(w)}(y))$$

$$= 1/d(d\,\chi(h)e(y) - h\,A'_{(w)}e(y))$$ since $e$ and $A'_{(w)}$ commute

$$= 1/d(d\,\chi(h)e(y) - A'_{(w)}(h\,e(y))$$

$$= 1/d(d\,\chi(h)e(y) - A'_{(w)}\,\chi(h)e(y))$$

$$= \chi(h)\,v\,.$$ Therefore $\xi|H = \chi|H$.

## III. Normality of C - A counterexample.

In this short chapter we examine the subgroup $C = U \cap U^{w_o}$. Clearly $C = 1$ if and only if $G$ has a saturated split (B,N)-pair. If $C$ is normal in $G$ there is a bijection between the set of isomorphism classes of irreducible kG-modules and the set of isomorphism classes of irreducible k(G/C)-modules since $C$ is a p-group; and since $G/C$ has a saturated split (B,N)-pair $(G/C, B/C, N, R, U/C)$ the results of (A) I could have been deduced from the 'saturated' theory. Since $C$ is normalised by $H$ and $N$ (see (A) I Remark 2 of section 1) $C \trianglelefteq G$ if and only if $C \trianglelefteq U$. We show that if $C_i = U \cap U^{w_i} \trianglelefteq U$ all $w_i \in R$ then $C \trianglelefteq U$; that is, $C \trianglelefteq G$ if this condition is satisfied for all rank 1 parabolic subgroups of $G$. Using a theorem of Kantor and Seitz [6] on doubly-transitive permutation groups we show that $C \trianglelefteq G$ if $p$ is odd and we give an example of a rank 1 (B,N)-pair when $p = 2$ and $C \ntrianglelefteq G$.

1.1 Lemma. $U = \langle (U_i)^{w^{-1}} \mid w \in W \text{ and } l(ww_i) = l(w)+1 \rangle$.

Proof. Let $w = w_{i_1} \ldots w_{i_t}$ be a reduced expression for $w \in W$. It follows from (A) I 1.2 that

$$_{w}U^{-} = (U_{i_t})(U_{i_{t-1}})^{w_{i_t}} \ldots (U_{i_1})^{w_{i_2} \ldots w_{i_t}}$$

since $l(w_{i_t} \ldots w_{i_s} w_{i_{s-1}}) = l(w_{i_t} \ldots w_{i_s}) + 1$ any $2 \leqslant s \leqslant t-1$.

Since $U = {}_{w_o}U^{-}$ we have $U \subseteq \langle (U_i)^{w^{-1}} \mid w \in W, l(ww_i) = l(w) + 1 \rangle$

Also if $l(ww_i) = l(w) + 1$ then $(U_i)^{w^{-1}} \subseteq U$ using [7, Lemma 2.8, p.441] which doesn't depend on saturation.

Since $C^w = C$ all $w \in W$

1.2 <u>Lemma</u>. $C \trianglelefteq U$ if and only if $C \trianglelefteq U_i$ all $w_i \in R$.

1.3 <u>Lemma</u>. Let $w_i \in R$. Assume $U \cap U^{w_i} \trianglelefteq U$. Then $C \trianglelefteq U_i$.

<u>Proof</u>. We have $C = U \cap U^{w_o} \cap U^{w_i w_o}$

$$= U \cap (U \cap U^{w_i})^{w_i w_o} .$$

By assumption $(U \cap U^{w_i})^{w_i w_o} \trianglelefteq U^{w_i w_o}$ so that $C \trianglelefteq U \cap U^{w_i w_o}$ and $C = C^{w_o w_i} \trianglelefteq U^{w_o w_i} \cap U = U_i$ .

The next lemma is immediate by 1.3 and 1.2.

1.4 <u>Lemma</u>. Say $C_i = U \cap U^{w_i} \trianglelefteq P_i = B \cup Bw_iB$ all $w_i \in R$. Then $C \trianglelefteq G$; that is $C \trianglelefteq G$ if this condition is satisfied by all the rank 1 parabolic subgroups of $G$.

Lemma 1.4 tells us that we can restrict our attention to the rank 1 case so suppose then that

$$G = B \cup BwB \quad \text{where}$$

$(G,B,N,\{w\},U)$ is an unsaturated split $(B,N)$-pair of rank 1. Then

a) $G$ acts 2-transitively on $\Omega = G/B$, the space of cosets $gB$ $(g \in G)$ and

b) $\hat{G} = G/Z$ acts faithfully and 2-transitively on $\Omega$ where

$$Z = \bigcap_{g \in G} B^g .$$

Let $\alpha, \beta \in \Omega$ where $\alpha = B$, $\beta = wB$. Notice $|\Omega| = |G/B| = 1 + p^l$ where $2 < |U/C| = p^l$ and

$$(\hat{G})_\alpha = B/Z, \text{ the stabiliser in } \hat{G} \text{ of } \alpha .$$

Since $U$ is a p-group, $U \trianglelefteq B$, $B/Z$ contains a normal nilpotent subgroup $Q = UZ/Z$ which is transitive on $\Omega \setminus \{\alpha\}$ since $BwB = UwB$.

Since $C^w = C$ all $w \in W$

1.2 Lemma. $C \trianglelefteq U$ if and only if $C \trianglelefteq U_i$ all $w_i \in R$.

1.3 Lemma. Let $w_i \in R$. Assume $U \cap U^{w_i} \trianglelefteq U$. Then $C \trianglelefteq U_i$.

Proof. We have $C = U \cap U^{w_o} \cap U^{w_i w_o}$

$$= U \cap (U \cap U^{w_i})^{w_i w_o}.$$

By assumption $(U \cap U^{w_i})^{w_i w_o} \trianglelefteq U^{w_i w_o}$ so that $C \trianglelefteq U \cap U^{w_i w_o}$ and $C = C^{w_o w_i} \trianglelefteq U^{w_o w_i} \cap U = U_i$.

The next lemma is immediate by 1.3 and 1.2.

1.4 Lemma. Say $C_i = U \cap U^{w_i} \trianglelefteq P_i = B \cup Bw_iB$ all $w_i \in R$. Then $C \trianglelefteq G$; that is $C \trianglelefteq G$ if this condition is satisfied by all the rank 1 parabolic subgroups of $G$.

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$$Z = \bigcap_{g \in G} B^g .$$

Let $\alpha, \beta \in \Omega$ where $\alpha = B$, $\beta = wB$. Notice $|\Omega| = |G/B| = 1 + p^l$ where $2 \leq |U/C| = p^l$ and

$$(\hat{G})_\alpha = B/Z, \text{ the stabiliser in } \hat{G} \text{ of } \alpha .$$

Since $U$ is a p-group, $U \trianglelefteq B$, $B/Z$ contains a normal nilpotent subgroup $Q = UZ/Z$ which is transitive on $\Omega \setminus \{\alpha\}$ since $BwB = UwB$.

By a result of Kantor and Seitz [6, Theorem C', p. 131] <u>either</u>

(i) $Q$ is regular on $\Omega \setminus \{\alpha\}$ which implies in particular that $Q_\beta = 1$ and

$$Q_\beta = \{uZ \mid u \in U,\ u(wB) = wB\} = \{uZ \mid u \in U,\ u^w \in B\}$$

$$= (U \cap U^w)Z/Z$$

$$= CZ/Z\ .$$

Therefore $CZ/Z = 1$ implies $C \leq Z$

so that $C \leq Z \cap U \leq U \cap B^w \leq U \cap U^w = C$

so that $C = Z \cap U$. But $Z \trianglelefteq G$ will then give $C \trianglelefteq G$.

<u>or</u>

(ii) $G$ contains a regular normal subgroup of order $q^2$ where $q$ is a Mersenne prime ($q = 2^r - 1$, $r$ prime). Therefore $|\Omega| = q^2$ is an odd integer and

$|\Omega \setminus \{\alpha\}|$ is even which implies $p^l$ is even

and $p = 2$.

We have therefore proved the following theorem:

1.5 <u>Theorem</u>. If $p$ is odd, $C \trianglelefteq G$ for all unsaturated split (B,N)-pairs.

The argument in [6, proof of Corollary 1, p. 139] leads to the following example of a rank 1 unsaturated split (B,N)-pair where $p = 2$, $C \ntrianglelefteq G$.

Let $x_0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, $x_1 = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \in GL(2,3)$. Let $U = \langle x_0, x_1 \rangle$ with defining relations $x_1^8 = x_0^2 = 1$; $x_0^{-1} x_1 x_0 = x_1^3$.

Then $U \in Syl_2(GL(2,3))$ and the elements of $U$ are:

$$1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad x_0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$x_1 = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \qquad x_0 x_1 = \begin{pmatrix} 1 & -1 \\ -1 & -1 \end{pmatrix}$$

$$x_1^2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \qquad x_0 x_1^2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$x_1^3 = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \qquad x_0 x_1^3 = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

$$x_1^4 = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \qquad x_0 x_1^4 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$x_1^5 = \begin{pmatrix} -1 & 1 \\ -1 & -1 \end{pmatrix} \qquad x_0 x_1^5 = \begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix}$$

$$x_1^6 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \qquad x_0 x_1^6 = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$$

$$x_1^7 = \begin{pmatrix} -1 & -1 \\ 1 & -1 \end{pmatrix} \qquad x_0 x_1^7 = \begin{pmatrix} -1 & -1 \\ -1 & 1 \end{pmatrix}$$

Let $M = V(2,3)$, the space of 2-dimensional column vectors over $GF(3)$. We have a map

$$\tau : U \to Aut(M) \text{ given by}$$

$$x \to \tau_x : m \to xm \qquad (x \in U,\ m \in M) .$$

Let $G = \{(m,x) \mid m \in M,\ x \in U\}$ be the semi-direct product of $M$ and $U$. Multiplication in $G$ is given by

$$(m,x)(m',x') = (m + xm', xx') \quad \text{for } m,m' \in M,\ x,x' \in U.$$

The identity of $G$ is $(0,1)$ and

$$M_1 = \{(m,1) \mid m \in M\} \cong M \quad \text{and} \quad M_1 \trianglelefteq G$$

$$U_1 = \{(0,x) \mid x \in U\} \cong U .$$

Let $\Omega = V(2,3)$ $(= M)$. Then $G$ acts on $\Omega$ by

$$(0,x)(v) = xv$$

$$(m,1)(v) = m + v \quad \text{all } m \in M,\ x \in U,\ v \in \Omega .$$

The following are easily verified:

a) $G_{\binom{0}{0}} = U_1$

b) $U_{1\binom{1}{0}} = \{(0,1), (0,x_0)\}$ .

c) $U_1$ is transitive on $\Omega\setminus\left\{\binom{0}{0}\right\}$ and since $(m,1)\binom{0}{0} = m$ any $m \in V(2,3)$,

d) $G$ is transitive on $\Omega$ . Hence $G$ is 2-transitive on $\Omega$ and

e) $G = G_a \cup G_a g G_a$ any $g \in G\setminus G_a$, $a \in \Omega$ .

We now show that <u>$G$ has an unsaturated split $(B,N)$-pair of rank 1</u>:

Let $w = \left(\binom{1}{0}, \begin{pmatrix}-1 & 0\\ 0 & -1\end{pmatrix}\right) \in G$. Then $w^2 = 1$, $w \notin U_1$ and

$$w\left(\binom{1}{0}\right) = \binom{0}{0} \quad ; \quad w\left(\binom{0}{0}\right) = \binom{1}{0} .$$ Let $N = \{1,w\}$, $B = U_1$ and $H = 1$. Then $G = \langle U_1, N\rangle$ by a) and e) taking $a = \binom{0}{0}$, $g = w$. Also $wU_1,\ U_1w,\ wU_1w \subseteq U_1 \cup U_1wU_1 = G$.

The Weyl group is $N$ and $R = \{w\}$. Lastly $wU_1w \neq U_1$ for otherwise there exists $u_1 \in U_1$ with

$$wu_1w = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} .$$

Applying $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ to both sides gives

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \text{, contradiction.}$$

Now $C = U_1 \cap U_1^w$ and $U_1 \cap U_1^w \subseteq U_{1\left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right)}$ since $w$ exchanges $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$. Using b) it follows that $U_{1\left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right)} \subseteq wU_1w$. Therefore $C = \{(0,1), (0,x_0)\}$ but $C$ is not normalised by the element $(0,x_1)$ for example since $x_1^{-1}x_0x_1 = x_0x_1^6$ .

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## (B) Some indecomposable modules of groups with split $(B,N)$-pairs.

## I. Determination of irreducible modular representations of parabolic subgroups.

Assume $G = (G,B,N,R,U)$ is a finite group with an unsaturated split $(B,N)$-pair of characteristic $p$ and rank $n$ and $k$ is an algebraically closed field of the same characteristic. Let $J$ be any fixed subset of $R$. Then $G_J = (G_J, B, N_J, J, U)$ is an unsaturated split $(B,N)$-pair (see [1, Proposition 1, p. 28] ) of characteristic $p$ and rank $|J|$ where $W_J = \langle w_i \mid w_i \in J \rangle$ , $N_J = \nu^{-1}(W_J)$ and $G_J = \bigcup_{w \in W_J} BwB$. Notice that $H = B \cap N = B \cap N_J$.

Assume $\{(w) \mid w \in W_J\}$ is a fixed but arbitrary set of representatives of the cosets of $N_J$ by $H$.

1.1 Notations and Definitions. We denote by $w_J$ the unique element of maximal length in $W_J$. Let $w \in W_J$. Define

$${}_wU^{-J} = U \cap U^{w_J w}$$

$$C_J = U \cap U^{w_J} .$$

We write ${}_{w_i}U^{-J}$ as $U_i^J$ for $w_i \in J$, ${}_wU^{-R}$ as ${}_wU^-$, $C_R$ as $C$ and $w_R$ as $w_o$. Clearly $C \subseteq C_J$ and $C_J^w = C_J$ any $w \in W_J$ (see (A) I 1.1). The reader is reminded that ${}_wU^+ = U \cap U^w$ any $w \in W$. For convenience we write ${}_{w^{-1}}U^{-J}$ as $U_w^{-J}$ and ${}_{w^{-1}}U^+$ as $U_w^+$ .

Take $w \in W$. Let $\Omega_w$ $(1 \in \Omega_w)$ be a left transversal of $U_w^-$ by $C$. (Write $\Omega_i$ for $\Omega_{w_i}$ any $w_i \in R$.)

1.2 Lemma. Let $w \in W_J$. Then $\Omega_w$ is also a left transversal of $U_w^{-J}$ by $C_J$.

Proof. We show $U_w^{-J} = U_w^- C_J$ and $U_w^- \cap C_J = C$ . (*)
Let $a = w_o w_J$, $b = w^{-1}$. Then $l(w_J w^{-1}) = l(w_J) - l(w^{-1})$ since $w^{-1} \in W_J$. Therefore

$$l(w_o w_J w^{-1}) = l(w_o) - l(w_J w^{-1}) = l(w_o w_J) + l(w^{-1})$$ and

we apply (A) I 1.2. We then have

$${}_{ab}U^- = {}_bU^-({}_aU^-)^b \text{ and } {}_bU^- \cap ({}_aU^-)^b = C.$$

But ${}_aU^- = U \cap U^{w_J} = C_J$ so that $({}_aU^-)^b = C_J^{w^{-1}} = C_J$ and the lemma is proved.

Remarks. 1) Since $U = U_w^- U_w^+$ any $w \in W$ ((A), I.1.3) it follows that $U = U_w^- C_J U_w^+ = U_w^{-J} U_w^+$ any $w \in W_J$ by (*). Moreover just as ${}_wU^+ \cap {}_wU^- = C$ any $w \in W$ we have ${}_wU^+ \cap {}_wU^{-J} = C_J$ for any $w \in W_J$. Of course $\Omega_w$ remains a transversal for $U$ modulo $U_w^+$.

2) In (A)I 2.9 we chose $(w_i) \in \langle U, (U_i)^{w_i} \rangle$ for any $w_i \in R$. Notice that by (*) if $w_i \in J$ that

$$\begin{aligned} \langle U, (U_i)^{w_i} \rangle &= \langle U, (U_i)^{w_i} C_J \rangle \\ &= \langle U, (U_i)^{w_i} C_J^{w_i} \rangle \\ &= \langle U, (U_i C_J)^{w_i} \rangle \\ &= \langle U, (U_i^J)^{w_i} \rangle . \end{aligned}$$

Most of the results below follow from the work in (A) and the proofs are omitted.

1.3 Lemma. The set $\Gamma_J = \{u_w h(w) \mid h \in H, u_w \in \Omega_w, w \in W_J\}$ is a left transversal of $G_J$ by $U$.

Notice that $\Gamma_J$ can be taken to be a subset of $\Gamma = \Gamma_R$.

1.4 Lemma. The elements of $N_J$ form a transversal for the U-U double cosets in $G_J$.

Let $Y_J \cong \mathrm{Ind}_U^{G_J}(k_U)$ and $y_J$ correspond to $1_{kG_J} \otimes_{kU} 1_k$ so that $Y_J = kG_J y_J$. Denote by $E_J$ the endomorphism algebra $\mathrm{End}_{kG_J}(Y_J)$. We let $Y_R = Y$, $y_R = y$ and $E_R = E$.

We've shown ((A),I 2.1) that $E$ has k-basis $\{A_n \mid n \in N\}$ where $A_n(y) = [\Omega_w]ny$ for $nH = w$. In fact:

1.5 Lemma. The k-algebra $E_J$ has k-basis $\{A'_n \mid n \in N_J\}$ where $A'_n(y_J) = [\Omega_w]ny_J$ where $nH = w \in W_J$.

The k-linear map $\theta : E_J \to E$ given by $A'_n \to A_n$ for $n \in N_J$ is an injective algebra homomorphism by Lemma 1.4 and using (A) I 2.2. Therefore any right E-module $X$ can be made into a right $E_J$-module by restriction; that is, if $x \in X$

$$x A'_n = x A_n \quad \text{for} \quad n \in N_J .$$

It is via the 'structual equations' of (A) I 2.10 for $w_i \in J$ and the map $\theta$ above that results from (A) can be applied to our present case.

1.6 Lemma. The set $\{A'_h, A'_{(w_i)} \mid h \in H, w_i \in J\}$ k-algebra generates $E_J$.

1.7 Theorem.(i) The algebra $E_J$ is Frobenius.

(ii) All simple (right) $E_J$-modules are one-dimensional.

Remember that for any $w_i \in R$, $H_i = H \cap \langle U, (U_i)^{w_i} \rangle$ and if $\chi \in \hat{B} = \mathrm{Hom}(B,k^*)$ then $M(\chi) = \{w_i \in R \mid \chi|H_i = 1\}$. If $w \in W$, ${}^w\chi \in \hat{B}$ where ${}^w\chi(hu) = \chi(h^w u)$ any $h \in H$, $u \in U$. By remark 2 above the work in (A) can be applied to both the $E$ and $E_J$ ($kG$ and $kG_J$) irreducible modules.

Definition. Let $\chi \in \hat{B}$, $S \subseteq M(\chi) \cap J = M_J(\chi)$. We call $(S, \chi)$ an admissible $G_J$-pair. Let $P_J$ be the set of all such pairs.

1.8 Theorem. The multiplicative characters of $E_J$ and the set of isomorphism classes of irreducible $kG_J$-modules are in a one-to-one correspondence with the elements of $P_J$. In particular if the character $\varphi$ corresponds to the admissible $G_J$-pair $(S, \chi)$ then $\varphi = \varphi_J(S, \chi)$ is given by

$$\varphi(A'_h) = \chi(h) \quad \text{any} \quad h \in H$$

$$\varphi(A'_{(w_i)}) = \begin{cases} 0 & w_i \in S \text{ or } \chi|H_i \neq 1 \\ -1 & w_i \notin S \text{ and } \chi|H_i = 1 \end{cases} .$$

Moreover if $kz_J(S, \chi)$ is the right $E_J$-module affording $\varphi_J(S, \chi)$ (see (A) I 2.21 ) and $M_J(S, \chi) = kG_J z_J(S, \chi)(y_J)$ then $\{M_J(S, \chi) \mid (S, \chi) \in P_J\}$ is a full set of irreducible left $kG_J$-modules.

Sawada proved the following lemma in the case $J = R$ when $G$ has a saturated $(B,N)$-pair ([9, Corollary 3.5 (ii), p. 37]).

1.9 Lemma. The indecomposable components of $Y_J$ have simple head and simple socle and are in a one-to-one correspondence with the elements of $P_J$.

Proof. See [6, Theorem 1 (i), (2.3), (2.6) Remark 2] and 1.7 (ii).

The following lemma is most useful. Curtis ([4, Theorem 6.15, p. B-38]) first proved it for the case $J = R$ under the saturation condition. We adapt his proof.

1.10 Lemma. Let $M_J(S, \chi)$ be as above. Then $M_J(S, \chi)$ has a unique B-line and this line affords the character $\chi$. Moreover the parabolic subgroup $G_S$ is the full stabiliser in $G_J$ of that line.

<u>Proof</u>. We know by [6, Theorem 2] that $F(M_J(S,\chi))$, the k-space of U-invariant elements in $M_J(S,\chi)$, is an irreducible $E_J$-module. By 1.7 (ii) and [6, 2.6a] $F(M_J(S,\chi)) = kz_J(S,\chi)(y_J)$ is the unique U-line and hence the unique B-line in $M_J(S,\chi)$. Let $z = z_J(S,\chi)(y_J)$.

The map $\zeta : \beta \to \beta(y_J)$ is a right $E_J$-isomorphism of $E_J \to F(Y_J)$ with $z$ the image of $z_J(S,\chi)$. Since $z_J(S,\chi)A'_h = \chi(h)z_J(S,\chi)$ any $h \in H$ by Theorem 1.8, we get by applying $\zeta$ to both sides that

$$z\,A'_h = \chi(h)\,z \quad \text{any } h \in H.$$

Since $hz = zA'_h$ (see for example (A) I 2.23) for all $h \in H$, $kz$ affords $\chi$ as required.

Now let $w_i \in S$, $X_i = \Omega_i^* = \Omega_i\setminus\{1\}$. Then by 1.8

$$0 = z\,A'_{(w_i)} = [\Omega_i](w_i)z \quad \text{(see (A), I 2.23)}$$

$$= \Big[(w_i)^2 + \sum_{u\in X_i}(w_i)u(w_i)\Big]z$$

$$= (w_i)^2 z + \sum_{u\in X_i} f_i(u)h_i(u)(w_i)g_i(u)z$$

(by (A) I 2.10 ) where $f_i(u) \in X_i$, $h_i(u) \in H_i$, $g_i(u) \in U$ any $u \in X_i$

$$= ((w_i)^2 z) + \sum_{u\in X_i} f_i(u)h_i(u)(w_i)z \quad \text{since}$$

$z$ is U-invariant

$$= z + [\Omega_i](w_i)z - (w_i)z \quad \text{since } (w_i)^2 \in H_i,$$

$\chi|H_i = 1$ and ${}^{w_i}\chi = \chi$ by (A) I 2.16

$$= z - (w_i)z\ .$$

It follows that $(w)z = z$ any $w \in W_S$ by adapting (A) I 2.20 to our present case.

Conversely, say $w_i \in J$ satisfies $(w_i)z = \lambda z$ some $\lambda \in k$. Then there exists $\mu \in k$ such that

$$z A'_{(w_i)} = \mu z = [\Omega_i](w_i)z = \lambda |\Omega_i| z = 0$$

(since $1 > |\Omega_i|$ is a p-power) so that $\mu = 0$. Now $z$ is stabilised by $U$ so it is stabilised by $U_i^J$. Therefore $z$ is stabilised by $H_i$ so that $\chi | H_i = 1$. By 1.8 $w_i$ must belong to $S$ and the lemma is proved.

## 2. Restriction and Induction Formulae.

In this section we discuss the relationship between (1) simple $E$ modules and simple $E_J$ modules, and (2) the indecomposable components of $Y$ and those of $Y_J$.

2.1 Lemma. Let $\varphi : E_J \to k$ be any multiplicative character. Then there exists a multiplicative character $\psi : E \to k$ such that $\varphi = \psi | E_J$. In fact if $\varphi$ is determined by the admissible $G_J$-pair $(S, \chi)$ and $\psi$ is determined by the admissible G-pair $(K, \chi')$ then

$$\varphi = \psi | E_J \Leftrightarrow \chi = \chi' \quad \text{and} \quad S = K \cap J.$$

Proof. We prove the second statement and the first will follow with $K = S$. Notice that by 1.8

$$S = \{w_i \in J \mid \varphi(A'_{(w_i)}) = 0 \text{ and } \chi | H_i = 1\}$$

$$K = \{w_i \in R \mid \psi(A_{(w_i)}) = 0 \text{ and } \chi | H_i = 1\}.$$

($\Rightarrow$) Let $\varphi = \psi | E_J$. Clearly $\chi = \chi'$ since $\varphi(A'_h) = \psi(A_h)$ all $h \in H$. Also $S \subseteq K \cap M_J(\chi) = K \cap J$. If there exists $w_i \in K \cap J$ but $w_i \notin S$ then we must have $\varphi(A'_{(w_i)}) = -1$

and $\psi(A_{(w_i)}) = 0$, contradiction. So $K \cap J = S$.

($\Leftarrow$) Say $\chi = \chi'$ and $S = K \cap J$. Then $\varphi(A'_h) = \psi(A_h) = \chi(h)$ all $h \in H$ and by 1.6 we need only show $\varphi(A'_{(w_i)}) = \psi(A_{(w_i)})$ all $w_i \in J$. We consider the following cases:

a) $w_i \in S$. Then $w_i \in K$ and $\varphi(A'_{(w_i)}) = \psi(A_{(w_i)}) = 0$.

b) $w_i \notin S$, $w_i \in M(\chi)$. Then $w_i \notin K$ and so

$\varphi(A'_{(w_i)}) = \psi(A_{(w_i)}) = -1$.

c) $w_i \notin S$, $w_i \notin M(\chi)$. Then $\varphi(A'_{(w_i)}) = \psi(A_{(w_i)}) = 0$.

Now take any decomposition of $Y_J$ as a direct sum of indecomposable $kG_J$-modules. By 1.9 given any admissible $G_J$-pair $(S,\chi)$ exactly one such summand has head isomorphic to $M_J(S,\chi)$.

2.2 Lemma. Let $Y_J(S,\chi)$ be the component of $Y_J$ whose head is isomorphic to $M_J(S,\chi)$. Then $Y_J(S,\chi)$ is unique up to isomorphism by the Krull-Schmidt Theorem and

(1) $$Y_J = \sum_{(S,\chi)\in P_J}^{\oplus} Y_J(S,\chi) \quad \text{is a decomposition}$$

of $Y_J$ into indecomposable $kG_J$-submodules. The socle of $Y_J(S,\chi)$ is isomorphic to $M_J(S^{w_J}, {}^{w_J}\chi)$ where $S^{w_J} = w_J S w_J$.

Proof. Let $\pi_J(S,\chi) \in E_J$ be the projection of $Y_J$ onto $Y_J(S,\chi)$. Then $1_Y = \sum_{(S,\chi)\in P_J} \pi_J(S,\chi)$ is an orthogonal decomposition of the identity $1_Y$ of $E$ into primitive idempotents in $E$.

We have arranged that

$$\varphi_J(S,\chi)\,(\pi_J(K,\chi')) = \begin{cases} 1 & S = K,\ \chi = \chi' \\ 0 & \text{otherwise} \end{cases}$$

Since $\pi_J(S,\chi)z_J(S^{w_J}, {}^{w_J}\chi) = z_J(S^{w_J}, {}^{w_J}\chi)$ for any $(S,\chi) \in P_J$ (see remark 4 following (A) I 2.21) we have (as in [9, Theorem 3.10 p. 40]) that $\pi_J(S,\chi)Y_J \supseteq M_J(S^{w_J}, {}^{w_J}\chi)$ and the result follows.

and $\psi(A_{(w_i)}) = 0$, contradiction. So $K \cap J = S$.

($\Leftarrow$) Say $\chi = \chi'$ and $S = K \cap J$. Then $\varphi(A'_h) = \psi(A_h) = \chi(h)$ all $h \in H$ and by 1.6 we need only show $\varphi(A'_{(w_i)}) = \psi(A_{(w_i)})$ all $w_i \in J$. We consider the following cases:

a) $w_i \in S$. Then $w_i \in K$ and $\varphi(A'_{(w_i)}) = \psi(A_{(w_i)}) = 0$.

b) $w_i \notin S$, $w_i \in M(\chi)$. Then $w_i \notin K$ and so $\varphi(A'_{(w_i)}) = \psi(A_{(w_i)}) = -1$.

c) $w_i \notin S$, $w_i \notin M(\chi)$. Then $\varphi(A'_{(w_i)}) = \psi(A_{(w_i)}) = 0$.

Now take any decomposition of $Y_J$ as a direct sum of indecomposable $kG_J$-modules. By 1.9 given any admissible $G_J$-pair $(S,\chi)$ exactly one such summand has head isomorphic to $M_J(S,\chi)$.

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(1) $$Y_J = \sum^{\oplus}_{(S,\chi)\in P_J} Y_J(S,\chi) \quad \text{is a decomposition}$$

of $Y_J$ into indecomposable $kG_J$-submodules. The socle of $Y_J(S,\chi)$ is isomorphic to $M_J(S^{w_J}, {}^{w_J}\chi)$ where $S^{w_J} = w_J S w_J$.

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Since $\pi_J(S,\chi) z_J(S^{w_J}, {}^{w_J}\chi) = z_J(S^{w_J}, {}^{w_J}\chi)$ for any $(S,\chi) \in P_J$ (see remark 4 following (A) I 2.21) we have (as in [9, Theorem 3.10 p. 40]) that $\pi_J(S,\chi)Y_J \supseteq M_J(S^{w_J}, {}^{w_J}\chi)$ and the result follows.

Notations. Let $D, F$ be subgroups of $G_J$, $D \subseteq F$. If $L$, $L'$ are $kD$-modules then $(L, L')_{kD}$ denotes the k-space $Hom_{kD}(L, L')$ of all D-maps $\theta : L \to L'$. (Similarly we write $(Z, Z')_{E_J}$ for $E_J$-modules $Z$ and $Z'$). We sometimes write $L^F$ for $Ind_D^F(L)$.

Let $\{L_\chi \mid \chi \in \hat{B}\}$ be a full set of irreducible left $kB$ (or $kH$)-modules. Then each $L_\chi$ is one-dimensional and it is easy to see that

$$(k_U)^B \cong \sum_{\chi \in \hat{B}}^{\oplus} L_\chi .$$

Hence

$$(2) \qquad Y_J \cong \sum_{\chi \in \hat{B}}^{\oplus} L_\chi^{G_J} .$$

2.3 Lemma. Let $\chi \in \hat{B}$. Then $L_\chi^{G_J} \cong \sum_{S \subseteq M_J(\chi)}^{\oplus} Y_J(S, \chi)$ .

Proof. By (1), (2) and the Krull-Schmidt Theorem, it is enough to show that $Y_J(S, \chi')$ is a component of $L_\chi^{G_J}$ only if $\chi = \chi'$.

Now $Y_J(S, \chi')$ is a component of $L_\chi^{G_J}$

$\Rightarrow (L_\chi^{G_J}, M_J(S, \chi'))_{kG_J} \neq 0$ by 2.2

$\Rightarrow (L_\chi, M_J(S, \chi'))_{kB} \neq 0$ (Frobenius Reciprocity)

$\Rightarrow \chi = \chi'$ and $S \subseteq M_J(\chi)$ since $M_J(S, \chi')$ has unique B-line affording $\chi'$ by 1.10.

Since $Ind_{G_J}^G(L_\chi^{G_J}) \cong L_\chi^G$ and $Ind_{G_J}^G(Y_J) \cong Y$ we can prove the following lemma: (Write $Y_R(K, \chi)$ as $Y(K, \chi)$ for any admissible G-pair $(K, \chi)$)

2.4 <u>Lemma</u>. Let $(S, \chi) \in P_J$. Then

$$Y_J(S,\chi)^G \cong \sum_{\substack{K \subseteq M(\chi) \\ S = K \cap J}}^{\oplus} Y(K,\chi) \ .$$

<u>Proof</u>. By Lemma 2.3 and the same lemma applied to the case $J = R$

$$\sum_{S \subseteq M_J(\chi)}^{\oplus} Y_J(S,\chi)^G \cong \sum_{K \subseteq M(\chi)}^{\oplus} Y(K,\chi) \ .$$

By the Krull-Schmidt Theorem, $Y_J(S,\chi)^G \cong \sum_Q^{\oplus} Y(K,\chi)$ the sum over some set $Q$ of admissible G-pairs. This implies that head $Y_J(S,\chi)^G \cong \sum_Q^{\oplus} M(K,\chi)$ summed over the same set $Q$ by 2.2 .

Now $(Y_J(S,\chi)^G, M(J',\chi))_{kG} \neq 0$

$\Leftrightarrow$ $(Y_J(S,\chi), M(J',\chi)_{kG_J})_{kG_J} \neq 0$ (Frobenius Reciprocity)

$\Leftrightarrow$ $(F(Y_J(S,\chi)), F(M(J',\chi)_{kG_J}))_{E_J} \neq 0$ (+)

by [6, 2.1a].

By [6, Theorem 1(iii)] and Lemma 2.2 head $F(Y_J(S,\chi))$ affords the character $\varphi_J(S,\chi)$ of $E_J$. But $F(M(J',\chi)) = F(M(J',\chi)_{kG_J})$ is a one-dimensional space and affords the E-character $\varphi_R(J',\chi)$. Therefore statement (+) is equivalent to

$$\varphi_J(S,\chi) = \varphi_R(J',\chi)|E_J$$

$\Leftrightarrow$ $S = J' \cap J$ by 2.1 and the result follows.

Using the same methods we can show:

2.4 Lemma. Let $(S,\chi) \in P_J$. Then

$$Y_J(S,\chi)^G \cong \sum_{\substack{K \subseteq M(\chi) \\ S = K \cap J}}^{\oplus} Y(K,\chi) .$$

Proof. By Lemma 2.3 and the same lemma applied to the case $J = R$

$$\sum_{S \subseteq M_J(\chi)}^{\oplus} Y_J(S,\chi)^G \cong \sum_{K \subseteq M(\chi)}^{\oplus} Y(K,\chi) .$$

By the Krull-Schmidt Theorem, $Y_J(S,\chi)^G \cong \sum_Q^{\oplus} Y(K,\chi)$ the sum over some set $Q$ of admissible G-pairs. This implies that head $Y_J(S,\chi)^G \cong \sum_Q^{\oplus} M(K,\chi)$ summed over the same set $Q$ by 2.2 .

Now $(Y_J(S,\chi)^G, M(J',\chi))_{kG} \neq 0$

$\Leftrightarrow (Y_J(S,\chi), M(J',\chi)_{kG_J})_{kG_J} \neq 0$ (Frobenius Reciprocity)

$\Leftrightarrow (F(Y_J(S,\chi)), F(M(J',\chi)_{kG_J}))_{E_J} \neq 0$ (+)

by [6, 2.1a].

By [6, Theorem 1(iii)] and Lemma 2.2 head $F(Y_J(S,\chi))$ affords the character $\varphi_J(S,\chi)$ of $E_J$. But $F(M(J',\chi)) = F(M(J',\chi)_{kG_J})$ is a one-dimensional space and affords the E-character $\varphi_R(J',\chi)$. Therefore statement (+) is equivalent to

$$\varphi_J(S,\chi) = \varphi_R(J',\chi)|E_J$$

$\Leftrightarrow S = J' \cap J$ by 2.1 and the result follows.

Using the same methods we can show:

2.5 Corollary. Let $J \subseteq K \subseteq R$ and let $(S,\chi)$ be an admissible $G_J$-pair. Then

$$Y_J(S,\chi)^{G_K} \cong \sum^{\oplus}_{\substack{Q \subseteq M_K(\chi) \\ S = Q \cap J}} Y_K(Q,\chi) .$$

2.6 Corollary. Let $(K,\chi)$ be an admissible $G$-pair. Then $Y(K,\chi)$ is induced from the parabolic subgroup $G_{M(\chi)}$.

Proof. Take $J = M(\chi)$ in 2.4 to get

$$Y_{M(\chi)}(K,\chi)^G \cong Y(K,\chi).$$

We now restate Lemma 2.4 using 2.1 but first we introduce some new notation.

Notation. Write $Y_J(S,\chi)$ as $Y_J(\varphi)$ if $\varphi : E_J \to k$ is determined by the admissible $G_J$-pair $(S,\chi)$. Similarly we write $Y(K,\chi')$ as $Y(\psi)$ if $\psi : E \to k$ is determined by the admissible $G$-pair $(K,\chi')$.

2.7 Lemma. Let $\varphi : E_J \to k$ be any multiplicative character of $E_J$. Then

$$\mathrm{Ind}_{G_J}^{G}(Y_J(\varphi)) \cong \sum^{\oplus}_{\substack{\psi : E \to k \\ \psi | E_J = \varphi}} Y(\psi) .$$

We now consider an arbitrary subgroup $G_1$ of $G$ which contains $U$ and discuss the relationship between $Y_1 \cong \mathrm{Ind}_{G_1}^{G}(k_{G_1})$ and $Y$.

2.8 Lemma. Let $G_1$ be as above. Then $Y_1$ is a component of $Y$; that is, there exists a $kG$-module $X$ such that $Y \cong X \oplus Y_1$.

Proof. Let $t = 1_{kG} \otimes_{kG_1} 1_k$ . Then $Y_1 = kGt$. Let $\{x_i \mid i \in I\}$ be a left transversal of $G$ by $G_1$, $\{v_j \mid j \in J\}$ be a

2.5 Corollary. Let $J \subseteq K \subseteq R$ and let $(S, \chi)$ be an admissible $G_J$-pair. Then

$$Y_J(S,\chi)^{G_K} \cong \sum^{\oplus}_{\substack{Q \subseteq M_K(\chi) \\ S = Q \cap J}} Y_K(Q,\chi) \ .$$

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We now consider an arbitrary subgroup $G_1$ of $G$ which contains $U$ and discuss the relationship between $Y_1 \cong \mathrm{Ind}_{G_1}^{G}(k_{G_1})$ and $Y$.

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Proof. Let $t = 1_{kG} \otimes_{kG_1} 1_k$. Then $Y_1 = kGt$. Let $\{x_i \mid i \in I\}$ be a left transversal of $G$ by $G_1$, $\{v_j \mid j \in J\}$ be a left transversal of $G_1$ by $U$. The map $\xi : Y \to Y_1$ given by $x_i v_j y \to x_i v_j t = x_i t$ $(i \in I,\ j \in J)$ is surjective. Since $U$ is a Sylow p-subgroup of $G$, p does not divide $|G_1 : U|$ and the map $\theta : Y_1 \to Y$ given by $t \to 1/|G_1 : U| \sum_{j \in J} v_j y$ satisfies $\xi\theta = 1_{Y_1}$ and the result follows.

Let $G_1$ be as above. Set $B_1 = B \cap G_1$, $N_1 = N \cap G_1$. Since $U \subseteq G_1$, $HB_1 = B$ and $G_1 = B_1 N_1 B_1$. There exists a subset $S \subseteq R$ such that $HG_1 = G_1 H = G_S$ and $(G_1, B_1, N_1)$ is a (B,N)-pair whose Weyl group is isomorphic to $W_S$ (see [14, Proposition 2.5, p. 317]). Clearly $G_S$ is the unique minimal parabolic subgroup containing $G_1$. In a recent paper ([10]) Sawada describes all such $G_1$ and in particular shows that $G_1$ contains $\langle U_i, (U_i)^{w_i} \rangle$ all $w_i \in S$ (see [10, proof of Theorem 1.6(ii)]). Therefore $\{(w_i) \mid w_i \in S\}$ can be taken to be in $G_1$ (see (A) I 2.18) We use these facts to prove the following useful lemma:

2.9 Lemma. Let $U \subseteq G_1$ be a subgroup of $G$. Let $G_S$ be the unique minimal parabolic subgroup containing $G_1$. Then

$$Y_1 \cong \sum_{\substack{S \subseteq J \\ \chi|B \cap G_1 = 1}}^{\oplus} Y(J, \chi) .$$

Proof. By 2.8 $Y_1$ is a component of $Y$. Therefore, by the Krull-Schmidt Theorem, $Y_1 \cong \sum_Q^{\oplus} Y(J, \chi)$; this sum over some set $Q$ of admissible G-pairs $(J, \chi)$. By 2.2 head $Y_1 \cong \sum_{(J,\chi) \in Q}^{\oplus} M(J, \chi)$.

Now $(Y_1, M(J,\chi))_{kG} \neq 0 \Rightarrow (k_{G_1}, M(J,\chi)_{kG_1})_{kG_1} \neq 0$

(Frobenius Reciprocity)

$\Rightarrow M(J,\chi)$ contains a trivial $G_1$-line

$\Rightarrow$ the unique B-line of $M(J,\chi)$, say km, is also a trivial $G_1$-line

$\Rightarrow \chi | B \cap G_1 = 1$ and km is also a $G_S$-line (since $HG_1 = G_S$)

$\Rightarrow \chi | B \cap G_1 = 1$ and $S \subseteq J$ (by 1.10).

On the other hand, say $\chi | B \cap G_1 = 1$ and $S \subseteq J$. Then the unique B-line km of $M(J,\chi)$ is also a $G_S$-line

$\Rightarrow$ km is a trivial $G_1$-line (since $G_1 = B_1 N_1 B_1$, $\{(w_i) \mid w_i \in S\} \subset N_1$ and we can arrange that $(w)m = m$ all $w \in W_S$ as in the proof of 1.10)

$\Rightarrow (k_{G_1}, M(J,\chi)_{kG_1})_{kG_1} \neq 0$

$\Rightarrow (Y_1, M(J,\chi))_{kG} \neq 0$ using Frobenius Reciprocity.

We apply this lemma to parabolic subgroups of G.

2.10 <u>Corollary</u>. Let $S \subseteq R$. Then

$$\mathrm{Ind}_{G_S}^G (k_{G_S}) \cong \sum_{S \subseteq J \subseteq R}^{\oplus} Y(J, 1_B) \quad \text{where}$$

$1_B$ is the trivial character of B. In particular

$$\mathrm{Ind}_B^G (k_B) \cong \sum_{J \subseteq R}^{\oplus} Y(J, 1_B) \quad .$$

## II. The dimensions of the indecomposable components of Y.

Let $(K, \chi)$ be an admissible G-pair; that is, $\chi \in \hat{B}$ and $K \subseteq M(\chi)$. We aim to calculate the dimension of $Y(K, \chi)$.

The Weyl group of a (B,N)-pair is isomorphic to the Weyl group of a root system in Euclidean space (see [7, p. 459] ) in such a way that R corresponds to the set of fundamental reflections. We therefore define

$\Delta = \{a_1, \ldots, a_n \mid w_i \in R\}$ to be the set of fundamental or simple roots of this root system. If $J \subseteq R$ let

$$\Delta_J = \{a_i \mid w_i \in J\} \quad .$$

1. Distinguished coset representatives.

The following sets were first defined by Solomon in [13] for arbitrary Coxeter groups:

1.1 Definitions. For each subset J of R define

$$X_J = \{w \in W \mid w(\Delta_J) > 0\}$$

$$V_J = \{w \in W \mid w(\Delta_J) > 0,\ w(\Delta_{\hat{J}}) < 0\} \quad \text{where } \hat{J} = R \setminus J.$$

The next lemma follows from the definitions.

1.2 Lemma. Let $J \subseteq R$. Then $X_J = \bigcup_{J \subseteq K \subseteq R} V_K$ and this is a partition of $X_J$.

1.3 Lemma. For any $J \subseteq R$ the set $X_J$ is a set of left coset representatives for W modulo $W_J$. If $w \in W$ and $w = xv$ with $x \in X_J$, $v \in W_J$ then $l(w) = l(x) + l(v)$.

Proof. See [13, Lemma 8, p.227].

1.4 Lemma. If $v \in V_J$ then $vw_{\hat{J}} \in X_{\hat{J}}$ and $l(v) = l(vw_{\hat{J}}) + l(w_{\hat{J}})$.

Proof. See [13, Lemma 9, p. 228].

1.5 <u>Corollary</u>. Let $v \in V_J$. Then $v = ww_{\hat{J}}$ with $l(v) = l(w) + l(w_{\hat{J}})$ and $w \in X_{\hat{J}}$.

1.6 <u>Corollary</u>. For any $J \subseteq R$, $w_{\hat{J}}$ is the unique element of minimal length in $V_J$.

1.7 <u>Lemma</u>. Let $J \subseteq R$. Then $G = \bigcup_{w \in X_J} BwG_J$, a disjoint union.

<u>Proof</u>. The result follows from the Bruhat decomposition of $G$, 1.3 and the fact that for any $w, w' \in X_J$, $BwG_J = Bw'G_J \Rightarrow wW_J = w'W_J$ (see [1, Proposition 2, p. 28] ).

<u>Notation</u>. Let $w \in W$. Set $q^w = |B:B_w^+| = |U:U_w^+| = |U_w^-:C| = |\Omega_w|$.

1.8 <u>Lemma</u>. (i) $q^{w_i} > 1$ any $w_i \in R$

(ii) $q^w = q^{w^{-1}}$ any $w \in W$

(iii) Let $w = w_{i_1} \cdots w_{i_t}$ be a reduced expression for $w$. Then $q^w = q^{w_{i_1}} \cdots q^{w_{i_t}}$.

<u>Proof</u>. (i) $|\Omega_{w_i}| > 1$ since otherwise $w_i B w_i = B$, contradicting the axioms for a $(B,N)$-pair; (ii) Trivial; (iii) Follows from an easy induction on $l(w)$ from (A) I 1.2 .

The following is a generalisation of Solomon's result ([12, p. 387]) for Chevalley groups:

1.9 <u>Lemma</u>. Let $J \subseteq R$. Then $|G:G_J| = \sum_{w \in X_J} q^w$.

<u>Proof</u>. The lemma follows from 1.7 and the fact that $B \cap wG_Jw^{-1} = B \cap wBw^{-1}$ any $w \in X_J$.

2. <u>Dimensions and Brauer characters</u>.

Let $(K, \chi)$ be an admissible $G$-pair. By I 2.6

$$Y_{M(\chi)}(K, \chi)^G \cong Y(K, \chi)$$

so that $\dim Y(K,\chi) = |G:G_{M(\chi)}| \dim Y_{M(\chi)}(K,\chi)$ .
In order to determine the dimension of $Y(K,\chi)$ we need only calculate the dimension of $Y_{M(\chi)}(K,\chi)$ since we can calculate $|G:G_S|$ for any $S \subseteq R$ by 1.9.

Therefore we replace $G$ by $G_{M(\chi)}$; that is <u>we assume that $R = M(\chi)$ and that the sets $X_J$, $V_J$ have been defined for $J \subseteq M(\chi)$, $W = W_{M(\chi)}$ in 1.1</u> . Let $d_J = \dim Y(J,\chi)$ any $J \subseteq R$. By I Lemma 2.4

$$(1) \qquad Y_J(J,\chi)^G \cong \sum_{J \subseteq K \subseteq R}^{\oplus} Y(K,\chi) \qquad (J \subseteq R)$$

2.1 <u>Lemma</u>. For any $J \subseteq R$, the dimension of $Y_J(J,\chi)$ is 1.

<u>Proof</u>. Firstly, $M_J(J,\chi)$ has a unique B-line, say $km$, and this line is $G_J$-stable by I 1.10. Since $M_J(J,\chi)$ is an irreducible $kG_J$-module, $M_J(J,\chi) = M$ must be equal to $km$ and have dimension one. Therefore $M|_U$ is U-trivial, that is $M|_U \cong k_U$. Since $U$ is a Sylow p-subgroup of $G$, $|G_J:U|$ is prime to $p$ so that $M$ is a component of $(M|_U)^{G_J} \cong (k_U)^{G_J} \cong Y_J$. Therefore $M$ is isomorphic to $Y_J(S,\chi')$ for some admissible $G_J$-pair $(S,\chi')$. Hence head $Y_J(S,\chi') = M = M_J(J,\chi)$ so that $(S,\chi') = (J,\chi)$ by I 2.2 and $M \cong Y_J(J,\chi)$ is one-dimensional.

We then have from (1) that:

$$(2) \qquad x_J = \sum_{J \subseteq K} d_K \quad \text{where } |G:G_J| = x_J \quad (J \subseteq R)$$

Write $v_S = \sum_{w \in V_S} q^w$ for any $S \subseteq R$. By 1.2 and 1.9 we also have:

(3) $$x_J = \sum_{J \subseteq K} v_K \qquad (J \subseteq R) .$$

2.2 Lemma. The dimension of $Y(J,\chi)$ is $\sum_{w \in V_J} q^w$ .

Proof. We prove by decreasing induction on $|J|$ that $d_J = v_J$. Firstly $d_R = \dim Y_R(R,\chi) = 1$ by 2.1 and $v_R = \sum_{w \in V_R} q^w = 1$ since $V_R = \{1\}$. Now suppose $|J| \leqslant |R|$ and that $d_K = v_K$ all $K \subseteq R$ with $|K| > |J|$. The result follows using (2) and (3).

2.3 Lemma. The indecomposable component $Y(\Phi, \chi)$ is irreducible for the empty set $\Phi$ .

Proof. By 2.2 $\dim Y(\Phi,\chi) = q^{w_o} = |U:C|$. By I 2.2 $Y(\Phi,\chi)$ has socle isomorphic to $M(\Phi,\chi) = kGm$ where $m = e(\chi)A_{(w_o)}(y)$ (see (A) I Theorem 2.21 ). Here $e(\chi) = \sum_{h \in H} \chi(h^{-1})A_h$ . Let $x = (w_o)m$. Then

$$\begin{aligned} [\Omega_{w_o}]x &= [\Omega_{w_o}](w_o)e(\chi)A_{(w_o)}(y) \\ &= e(\chi)A_{(w_o)}([\Omega_{w_o}](w_o)\, y) \\ &= e(\chi)A_{(w_o)}{}^2(y) \quad \text{(see (A) I 2.23 )} \\ &= (-1)^{l(w_o)}\chi(h')e(\chi)A_{(w_o)}(y) \quad \text{using (A) I 2.12} \\ &\quad \text{for some } h' \in H \\ &= (-1)^{l(w_o)}\chi(h')m \neq 0 . \end{aligned}$$

Consider the kU-epimorphism $\alpha:(k_C)^U \to kUx$ given by $z \to x$ where $z = 1\otimes 1$. Since U is a p-group, the socle of $(k_C)^U$ is its space of U-invariants which is clearly $[\Omega_{w_o}]z$. Because $\alpha([\Omega_{w_o}]z) \neq 0$ $\alpha$ must be a bijection and kUx has dimension $q^{w_o}$. But $kUx \subseteq M(\Phi, \chi)$ implies that $M(\Phi, \chi) \cong Y(\Phi, \chi)$.

We can apply results of this section to the case $\chi = 1_B$, the trivial character of B for in this case $M(\chi) = R$. From the proofs of Lemmas I 1.10 and II 2.1 see that for any subset $J \subseteq R$

$$Y_J(J, 1_B) \cong k_{G_J} \quad .$$

By I 2.4

$$(k_{G_J})^G \cong \sum_{J\subseteq K\subseteq R}^{\oplus} Y(K, 1_B) \; .$$

Let $\eta_K$ be the Brauer character of $Y(K, 1_B)$ any $K \supseteq J$. Then

$$1_{G_J}^G = \sum_{J\subseteq K\subseteq R} \eta_K \qquad (\text{all } J \subseteq R)$$

Solving these equations for $\eta_J$ (see [1, Exercise 25, p. 44-45]) we see that

$$\eta_J = \sum_{J\subseteq K\subseteq R} (-1)^{|K\setminus J|} 1_{G_K}^G \quad .$$

And specifically, we get the Steinberg character when $J = \Phi$ :

$$\eta_\Phi = \sum_{K\subseteq R} (-1)^{|K|} 1_{G_K}^G \quad .$$

Let $\gamma = \sum_{K \subseteq R} (-1)^{|K|} 1_{G_K}^G$ be the ordinary character corresponding to the Steinberg character; that is $1_{G_K}^G$ is interpreted as being an ordinary character of G for all $K \subseteq R$. Curtis has shown ([3]) that $\gamma$ is irreducible for an arbitrary finite group G with a (B,N)-pair. Let p-reg. = $\{x \in G \mid$ p does not divide the order of $x\}$. We conclude this section with the following lemma:

2.4 Lemma. If G is a finite group with an unsaturated split (B,N)-pair

$$\gamma|_{p\text{-reg.}} = \eta\Phi,$$

that is, $\gamma$ remains irreducible as a Brauer character.

Remark Bromich determined the $\eta_J$ in [2, (7.1.12)].

## 3. Examples.

1. Consider the group $G$ given in (A) III. We have $G = \{(m,x) \mid m \in M,\ x \in U\}$ where $M = V(2,3)$ and $U \in Syl_2(GL(2,3))$. Let $k$ be an algebraically closed field of characteristic 2. Since $H = 1$, $B = U_1 = \{(0,x) \mid x \in U\}$, the only character of $B$ is the identity character $1_B$. Therefore $Y \cong (k_{U_1})^G$ is a direct sum of two indecomposable kG-modules corresponding to the admissible pairs $(\Phi, 1_B)$ and $(\{w\}, 1_B)$. Both these components are in fact irreducible (Lemmas 2.1 and 2.3). The dimension of $Y(\Phi, 1_B) = |U_1|/|C| = 8$ and the dimension of $Y(\{w\}, 1_B)$ is 1. Since the dimension of $Y = |G:U_1| = |M| = 9$, the dimensions concur.

2. Let $G = GL(3,p)$ where $k$ is an algebraically closed field of characteristic $p$. Then $G$ has a split (B,N)-pair with $B = \{\text{upper triangular matrices}\}$, $N = \{\text{monomial matrices}\}$, $U = \{\text{uni-upper triangular matrices}\}$ and $H = \{\text{diagonal matrices}\}$. The Weyl group $W \cong S_3 = \langle w_1, w_2 \rangle = \langle R \rangle$. We can take $n_1, n_2 \in N$ where $n_1 H = w_1$ and $n_2 H = w_2$ and

$$n_1 = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad n_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix} \quad \text{since}$$

$$G_{a_1} = \left(\begin{array}{c|c} SL(2,p) & 0 \\ \hline 0 & 1 \end{array}\right) \quad \text{and} \quad G_{a_2} = \left(\begin{array}{c|c} 1 & 0 \\ \hline 0 & SL(2,p) \end{array}\right). \quad \text{Hence}$$

$$H_1 = H_{a_1} = \left\{ \begin{pmatrix} t & 0 & 0 \\ 0 & t^{-1} & 0 \\ 0 & 0 & 1 \end{pmatrix} \;\middle|\; t \in GF(p) \right\} \quad \text{and}$$

$$H_2 = H_{a_2} = \left\{ \begin{pmatrix} 1 & 0 & 0 \\ 0 & t & 0 \\ 0 & 0 & t^{-1} \end{pmatrix} \middle| \ t \in GF(p) \right\} .$$

$W = \{1, w_1, w_2, w_1w_2, w_2w_1, w_1w_2w_1\}$ . The longest element of $W$ is therefore $w_1w_2w_1 = w_2w_1w_2$. For any $J \subseteq R$, $X_J = \{w \in W \mid w(\Delta_J) > 0\}$.

a). $J = \{w_1\}$. Then $X_J = \{1, w_2, w_1w_2\}$ and

$G_J$ is of the form $\left\{ \begin{pmatrix} * & * & * \\ * & * & * \\ 0 & 0 & * \end{pmatrix} \in GL(3,p) \right\}$.

$G_J$ has a split (B,N)-pair which is not saturated as

$$C_J = \left\{ \begin{pmatrix} 1 & 0 & \alpha \\ 0 & 1 & \beta \\ 0 & 0 & 1 \end{pmatrix} \middle| \alpha,\beta \in GF(p) \right\} .$$ But $C_J \trianglelefteq U$, $C_J \ntrianglelefteq G$.

Now $|U:C_J| = |\Omega_{w_1}| = p$ and $|G:G_J| = \sum_{w \in X_J} |\Omega_w|$ .

Then

$$U^+_{w_1w_2} = \left\{ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & \alpha \\ 0 & 0 & 1 \end{pmatrix} \middle| \ \alpha \in GF(p) \right\}$$ and

$$U^+_{w_2w_1} = \left\{ \begin{pmatrix} 1 & \alpha & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \middle| \ \alpha \in GF(p) \right\}$$ so that

$|\Omega_{w_1w_2}| = |\Omega_{w_2w_1}| = p^2$ and $|G:G_J| = 1 + p + p^2$ .

b). $K = \{w_2\}$. Then $X_K = \{1, w_1, w_2w_1\}$ and

$$G_K = \left\{ \begin{pmatrix} * & * & * \\ 0 & * & * \\ 0 & * & * \end{pmatrix} \in GL(3,p) \right\}$$ with $$C_K = \left\{ \begin{pmatrix} 1 & \alpha & \beta \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \middle| \alpha,\beta \in GF(p) \right\}$$

As above in a) $|G:G_K| = 1 + p + p^2$

Let $0 \leq \sigma_1, \sigma_2, \sigma_3 \leq p-2$. Define $\chi_{\sigma_1,\sigma_2,\sigma_3} : H \to k^*$ by

$$h = \begin{pmatrix} h_1 & & 0 \\ & h_2 & \\ 0 & & h_3 \end{pmatrix} \to h_1^{\sigma_1} h_2^{\sigma_2} h_3^{\sigma_3} \qquad (h \in H)$$

These are all the characters of $H$ and
$\hat{B} = \{\chi_{\sigma_1,\sigma_2,\sigma_3} \mid 0 \leq \sigma_1, \sigma_2, \sigma_3 \leq p-2\}$. Notice that $\chi_{\sigma_1,\sigma_2,\sigma_3} = (\chi_{1,0,0})^{\sigma_1}(\chi_{0,1,0})^{\sigma_2}(\chi_{0,0,1})^{\sigma_3}$.

Therefore

c). $$M(\chi_{\sigma_1,\sigma_2,\sigma_3}) = \begin{cases} \Phi & \text{if } \sigma_1 \neq \sigma_2;\ \sigma_2 \neq \sigma_3 \\ w_1 & \text{if } \sigma_1 = \sigma_2;\ \sigma_2 \neq \sigma_3 \\ w_2 & \text{if } \sigma_1 \neq \sigma_2;\ \sigma_2 = \sigma_3 \\ w_1, w_2 & \text{if } \sigma_1 = \sigma_2 = \sigma_3 \end{cases}$$

In each case below we find all $S \subseteq M(\chi)$ for a fixed $\chi = \chi_{\sigma_1,\sigma_2,\sigma_3}$ and give the dimension of $Y(S, \chi)$. Notice in each case that we must have $\sum_{S \subseteq M(\chi)} \dim Y(S,\chi) = |G:B| = 1 + 2p + 2p^2 + p^3$ (see I 2.3 case $J = R$).

<u>Case 1</u>: $\chi = \chi_{\sigma_1,\sigma_2,\sigma_3}$ and all $\sigma_i$ distinct $i = 1,2,3$. The only admissible G-pair is $(\Phi, \chi)$ since $M(\chi) = \Phi$ and $Y_\Phi(\Phi, \chi)$ has dimension 1. Therefore $Y(\Phi, \chi)$ has dimension $|G:B| = 1 + 2p + 2p^2 + p^3$.

<u>Case 2:</u> $\chi = \chi_{\sigma,\tau,\sigma}$ $\tau \neq \sigma$. As in case 1 $M(\chi) = \Phi$ and $Y(\Phi, \chi)$ has dimension $1 + 2p + 2p^2 + p^3$.

<u>Case 3:</u> $\chi = \chi_{\sigma,\sigma,\tau}$ $\tau \neq \sigma$. Then $M(\chi) = \{w_1\}$ and we have two admissible G-pairs $(\Phi, \chi)$ and $(\{w_1\}, \chi)$.

Let $0 \leq \sigma_1, \sigma_2, \sigma_3 \leq p-2$. Define $\chi_{\sigma_1,\sigma_2,\sigma_3} : H \to k^*$ by

$$h = \begin{pmatrix} h_1 & & 0 \\ & h_2 & \\ 0 & & h_3 \end{pmatrix} \to h_1^{\sigma_1} h_2^{\sigma_2} h_3^{\sigma_3} \qquad (h \in H)$$

These are all the characters of H and $\hat{B} = \{\chi_{\sigma_1,\sigma_2,\sigma_3} \mid 0 \leq \sigma_1, \sigma_2, \sigma_3 \leq p-2\}$. Notice that $\chi_{\sigma_1,\sigma_2,\sigma_3} = (\chi_{1,0,0})^{\sigma_1}(\chi_{0,1,0})^{\sigma_2}(\chi_{0,0,1})^{\sigma_3}$.

Therefore

c). $$M(\chi_{\sigma_1,\sigma_2,\sigma_3}) = \begin{cases} \Phi & \text{if } \sigma_1 \neq \sigma_2;\ \sigma_2 \neq \sigma_3 \\ w_1 & \text{if } \sigma_1 = \sigma_2;\ \sigma_2 \neq \sigma_3 \\ w_2 & \text{if } \sigma_1 \neq \sigma_2;\ \sigma_2 = \sigma_3 \\ w_1, w_2 & \text{if } \sigma_1 = \sigma_2 = \sigma_3 \end{cases}$$

In each case below we find all $S \subseteq M(\chi)$ for a fixed $\chi = \chi_{\sigma_1,\sigma_2,\sigma_3}$ and give the dimension of $Y(S,\chi)$. Notice in each case that we must have $\sum_{S \subseteq M(\chi)} \dim Y(S,\chi) = |G:B| = 1 + 2p + 2p^2 + p^3$ (see I 2.3 case J = R).

<u>Case 1</u>: $\chi = \chi_{\sigma_1,\sigma_2,\sigma_3}$ and all $\sigma_i$ distinct $i = 1,2,3$. The only admissible G-pair is $(\Phi,\chi)$ since $M(\chi) = \Phi$ and $Y_\Phi(\Phi,\chi)$ has dimension 1. Therefore $Y(\Phi,\chi)$ has dimension $|G:B| = 1 + 2p + 2p^2 + p^3$.

<u>Case 2:</u> $\chi = \chi_{\sigma,\tau,\sigma}$ $\tau \neq \sigma$. As in case 1 $M(\chi) = \Phi$ and $Y(\Phi,\chi)$ has dimension $1 + 2p + 2p^2 + p^3$.

<u>Case 3:</u> $\chi = \chi_{\sigma,\sigma,\tau}$ $\tau \neq \sigma$. Then $M(\chi) = \{w_1\}$ and we have two admissible G-pairs $(\Phi,\chi)$ and $(\{w_1\},\chi)$.

(i) $(\Phi, \chi)$. Now $Y_J(\Phi, \chi)$ has dimension $|U:C_J| = p$ by a) above so that $Y(\Phi, \chi)$ has dimension $|G:G_J|p = p + p^2 + p^3$.

(ii) $(\{w_1\}, \chi)$. Now $Y_J(\{w_1\}, \chi)$ has dimension 1 so that $Y(\{w_1\}, \chi)$ has dimension $|G:G_J| = 1 + p + p^2$.

Case 4: $\chi = \chi_{\sigma,\tau,\tau}$ $\tau \neq \sigma$. Then $M(\chi) = \{w_2\}$ and as in Case 3 $Y(\Phi, \chi)$ has dimension $p + p^2 + p^3$ and $Y(\{w_2\}, \chi)$ has dimension $1 + p + p^2$.

Case 5: $\chi = \chi_{\sigma,\sigma,\sigma}$. Then $M(\chi) = R$ and there are four admissible G-pairs:

(i) $(\Phi, \chi)$. Then $Y(\Phi, \chi)$ has dimension $|U/C| = |U| = p^3$. This is the Steinberg character.

(ii) $(\{w_1\}, \chi)$. Now $V_{\{w_1\}} = \{w_2, w_1w_2\}$ so that $Y(\{w_1\}, \chi)$ has dimension $\sum_{v \in V_{\{w_1\}}} |\Omega_w| = p + p^2$.

(iii) $(\{w_2\}, \chi)$. Then $V_{\{w_2\}} = \{w_1, w_2w_1\}$ so that $Y(\{w_2\}, \chi)$ has dimension $p + p^2$ as in (ii).

(iv) $(\{w_1, w_2\}, \chi)$. The dimension of $Y(\{w_1, w_2\}, \chi)$ is 1.

We summarise these results in the following table:

| Type of $\chi$ | Number of such $\chi$ | Number of $J$ with $J \subseteq M(\chi)$ | Dimensions o[f] $Y(J, \chi)$. |
|---|---|---|---|
| $\chi_{\sigma_1,\sigma_2,\sigma_3}$ | (p-1)(p-2)(p-3) | 1 | $1+2p+2p^2+p^3$ |
| $\chi_{\sigma,\tau,\sigma}$ | (p-1)(p-2) | 1 | $1+2p+2p^2+p^3$ |
| $\chi_{\sigma,\sigma,\tau}$ | (p-1)(p-2) | 2 | $p+p^2+p^3$<br>$1+p+p^2$ |
| $\chi_{\sigma,\tau,\tau}$ | (p-1)(p-2) | 2 | $p+p^2+p^3$<br>$1+p+p^2$ |
| $\chi_{\sigma,\sigma,\sigma}$ | (p-1) | 4 | $p^3$<br>$p+p^2$<br>$p+p^2$<br>1 |

Therefore the total number of $\chi$ is

$$(p-1)\{p^2 - 5p + 6 + 3p - 6 + 1\} = (p-1)(p^2 - 2p + 1)$$

$$= (p-1)^3 = |H|.$$

The total number of components of $Y$ is

$$(p-1)(p-2)(p-3) + (p-1)(p-2) + 2(p-1)(p-2) + 2(p-1)(p-2) + 4(p-1)$$

$$= (p-1)(p^2 - 5p + 6 + 5p - 10 + 4)$$

$$= (p-1)p^2$$

= the number of isomorphism classes of irreducible kG-modules

## 4. Generators for the indecomposable summands of E and Y.

In this section we generalise some of Bromich's work ([2]) on the decomposition of the algebra $\mathrm{End}_{kG}(L)$ where $L \cong \mathrm{Ind}_B^G(k_B)$ for G with a (B,N)-pair and k an algebraically closed field of characteristic p. For each $\chi \in \hat{B}$, let $E(\chi) = \frac{1}{|H|} \sum_{h \in H} \chi(h^{-1})A_h$. Since $E = \sum^{\oplus}_{\chi \in \hat{B}} E(\chi)E$ is a decomposition of $E = \mathrm{End}_{kG}(Y)$ into right ideals we decompose $E(\chi)E$ for a fixed $\chi \in \hat{B}$. We need only make slight adjustments to Bromich's definitions and proofs as her proofs will apply in the case $M(\chi) = R$.

The following is a generalisation of [2, section 4.4]: Fix $\chi \in \hat{B}$, $J \subseteq M(\chi)$. Assume coset representatives $\{(w) \mid w \in W_J\}$ have been chosen according to (A) I 2.19, 2.20.

4.1 Notations (see section 1).

$$X_J = X_{J,\chi} = \{w \in W_{M(\chi)} \mid w(\Delta_J) > 0\}$$

$$V_J = V_{J,\chi} = \{w \in W_{M(\chi)} \mid w(\Delta_J) > 0,\ w(\Delta_{\hat{J}}) < 0\}$$
$$\text{where } \hat{J} = M(\chi)\setminus J$$

$$T = T_\chi = \{w \in W \mid w(\Delta_{M(\chi)}) > 0\}.$$

Notice if $M(\chi) = R$, then $T = \{1\}$.

As in section 1:

4.2 Lemma. (i) T is a set of left coset representatives of W modulo $W_{M(\chi)}$ and if $t \in T$, $w \in W_{M(\chi)}$, $l(tw) = l(t) + l(w)$.

(ii) $X_J$ is a set of left coset representatives of $W_{M(\chi)}$ modulo $W_J$ and if $x \in X_J$, $w \in W_J$ then $l(xw) = l(x) + l(w)$.

(iii) $W_{M(\chi)} = \bigcup_{S \subseteq M(\chi)} V_S$ , a disjoint union.

4.3 Definitions. Let $e_J = e_{J,\chi} = E(\chi) \sum_{w \in W_J} A_{(w)}$

$$o_J = o_{J,\chi} = E(\chi)(-1)^{l(w_J)} A_{(w_J)}$$

where $w_J$ is the unique element of maximal length in $W_J$.

4.4 Lemma. (i) $e_J A_h = A_h e_J = \chi(h) e_J$ all $h \in H$.

(ii) $o_J A_h = A_h o_J = \chi(h) o_J$ all $h \in H$.

(iii) $e_J A_{(w_i)} = A_{(w_i)} e_J = 0$ all $w_i \in J$.

(iv) $o_J A_{(w_i)} = A_{(w_i)} o_J = -o_J$ all $w_i \in J$.

Proof. Most of (i) and (ii) follows since $A_h E(\chi) = E(\chi) A_h = \chi(h) A$ any $h \in H$. But $J \subseteq M(\chi)$ is important:

$$e_J A_h = E(\chi) \sum_{w \in W_J} A_{(w)} A_h$$

$$= E(\chi) \sum_{w \in W_J} A_{(w)^{-1} h (w)} A_{(w)} \qquad ((A)\ I\ 2.5)$$

$$= E(\chi) \sum_{w \in W_J} {}^{w}\chi(h) A_{(w)}$$

$$= E(\chi) \sum_{w \in W_J} \chi(h) A_{(w)} \qquad ((A)\ I\ 2.16)$$

$$= \chi(h) e_J.$$

Also $o_J A_h = E(\chi)(-1)^{l(w_J)} A_{(w_J)^{-1} h (w_J)} A_{(w_J)}$ ((A) I 2.5)

$$= E(\chi)(-1)^{l(w_J)}\ {}^{w_J}\chi(h) A_{(w_J)}$$

$$= \chi(h) o_J \text{ (using ((A) I 2.16)}.$$

For (iii) take a decomposition of $W_J$ into cosets $\{w, w_i w\}$ with respect to $\langle w_i \rangle$. We show terms corresponding to $w$ and $w_i w$ cancel for any $w \in W_J$. Fix $w \in W_J$. Without loss of generality assume $l(w_i w) = l(w) + 1$. Then

$$A_{(w)}A_{(w_i)} = A_{(w_i)(w)} \quad \text{by (A) I 2.12 (i)}$$

$$= A_h A_{(w_i w)} \quad \text{for some } h \in H \text{ with } \chi(h) = 1$$

by choice of representatives

so that $E(\chi)A_{(w)}A_{(w_i)} = E(\chi)A_{(w_i w)}$ . $\qquad$ (+)

But $A_{(w_i w)}A_{(w_i)} = A_{(w_i w)} \sum_{s=1}^{b(i)} A_{h_i(u_{i_s})}$ by (A) I 2.12 (ii)

$$= \sum_{s=1}^{b(i)} A_{(w_i w)^{-1} h_i(u_{i_s})(w_i w)} A_{(w_i w)}$$

so that $E(\chi)A_{(w_i w)}A_{(w_i)} = E(\chi) \sum_s {}^{w_i w}\chi(h_i(u_{i_s}))\ A_{(w_i w)}$

$$= -E(\chi)A_{(w_i w)} \quad \text{by (A) I 2.16} \quad (++)$$

Therefore $e_J A_{(w_i)} = E(\chi) \sum_{w \in W_J} A_{(w)}A_{(w_i)} = 0$ by (+) and (++).

Similarly for $A_{(w_i)}e_J$.

For (iv) we know that $l(w_i w_J) = l(w_J w_i) = l(w_J) - 1$ any $w_i \in J$ so that

$$o_J A_{(w_i)} = E(\chi)(-1)^{l(w_J)} A_{(w_J)} A_{(w_i)}$$

$$= E(\chi)(-1)^{l(w_J)} A_{(w_J)} \sum_{s=1}^{b(i)} A_{h_i(u_{i_s})} \quad \text{((A) I 2.12 (ii)}$$

$$= E(\chi)(-1)^{l(w_J)} \sum_{s=1}^{b(i)} A_{(w_J)^{-1} h_i(u_{i_s})(w_J)} A_{(w_J)}$$

$$= E(\chi) \sum_{s=1}^{b(i)} {}^{w_J}\chi(h_i(u_{i_s}))\ (-1)^{l(w_J)} A_{(w_J)}$$

$= -o_J$ by (A) I 2.16. Similarly for $A_{(w_i)}o_J$.

4.5 Lemma. The elements $o_J$ and $e_J$ are idempotents in E.

Proof. First notice that $E(\chi)$ is an idempotent. By 4.4 (iii) $e_J A_{(w)} = 0$ any $1 \neq w \in W_J$. Therefore $e_J^2 = e_J A_{(1)} = e_J$. Also $o_J^2 = o_J E(\chi)(-1)^{l(w_J)} A_{(w_{i_t})} \cdots A_{(w_{i_1})}$ where $w_J = w_{i_1} \cdots w_{i_t}$ is a reduced expression. All $w_{i_m}$ $(1 \leq m \leq t)$ belong to J so that by 4.4 (iv)

$$o_J^2 = (-1)^{2l(w_J)} E(\chi)\ (-1)^{l(w_J)} A_{(w_J)} = o_J .$$

4.6 Lemma. ([2, (4.4.4)]) Let $v \in V_J$, $x \in T$. Then

(i) $o_{\hat{J}}\, A_{(v)} = E(\chi) A_{(v)}$

(ii) $e_J o_{\hat{J}}\, A_{(v)} = E(\chi) \sum_{w \in W_J} A_{(v)(w)}$

(iii) $e_J o_{\hat{J}}\, A_{(v)} A_{(x)} = E(\chi) \sum_{w \in W_J} A_{(x)(v)(w)}$

Proof. (i) $o_{\hat{J}}\, A_{(v)} = (-1)^{l(w_{\hat{J}})} E(\chi) A_{(w_{\hat{J}})} A_{(v)}$. Let $w_{\hat{J}} = w_{i_1} \cdots w_{i_t}$ be a reduced expression so that $A_{(w_{\hat{J}})} = A_h A_{(w_{i_t})} \cdots A_{(w_{i_1})}$ some $h \in H$ with $\chi(h) = 1$ since $J \subseteq M(\chi)$. But $v \in V_J$ implies $l(vw_i) = l(v) - 1$ any $w_i \in \hat{J}$ so that $o_{\hat{J}}\, A_{(v)} = (-1)^{2l(w_{\hat{J}})} E(\chi) A_{(v)}$ by (A) I 2.16 and repeated applications of (A) I 2.12 (iv).
(ii) $e_J o_{\hat{J}}\, A_{(v)} = e_J\, A_{(v)} = E(\chi) \sum_{w \in W_J} A_{(w)} A_{(v)}$ . But $V_J \subseteq X_J$ and the result follows by 4.2 (ii). Part (iii) follows by (ii) and 4.2 (i) since $vw \in W_{M(\chi)}$ any $w \in W_J$.

Notation. Let $v \in W_{M(\chi)}$, $w \in T$. If we define

$$\sigma(v,w) = e_J o_{\hat{J}}\, A_{(v)} A_{(w)} \quad \text{where } v \in V_J, \text{ the value}$$

of J is uniquely determined (by 4.1 (iii)).

4.7 Lemma. ([2, (4.4.7)]) Let $\mathcal{J}_\chi = \{\sigma(v,w) \mid v \in V_J,\ w \in T\}$. Then $\mathcal{J}_\chi$ is a set of linearly independent elements of $E(\chi)E$.

Proof. Say there exist $\lambda_{v,w} \in k$ with

$$0 = \sum_{v\in V_J} \sum_{w\in T} \lambda_{v,w}\, \sigma(v,w)\ . \qquad (+)$$

Let $S_n = \sum_{\substack{v,w \\ l(w)+l(v)\geq n}} \lambda_{v,w}\, \sigma(v,w)$ . We show that if $S_n = 0$

then $\lambda_{v,w} = 0$ all $w,v$ with $l(w) + l(v) = n$ which will imply $S_{n+1} = 0$. Since expression (+) is equivalent to $S_0 = 0$ we will have proved by induction on $n$ that all $\lambda_{v,w}$ are zero.

Let $v_1,\ldots,v_t$ be all elements in $W_{M(\chi)}$ which satisfy the following condition: For each $v_i$ $(1\leq i\leq t)$ there exists (at least one) $w \in T$ with $l(v_i) + l(w) = n$. Then $v_i \in V_{J(i)}$ some unique subset $J(i) \subseteq R$ by 4.2 (iii). Let $w \in T$. Then

$$\sigma(v_i,w) = e_{J(i)}{}^{\circ}{}_{\widehat{J(i)}}\ A_{(v_i)}A_{(w)}$$

$$= E(\chi) \sum_{w''\in W_{J(i)}} A_{(w)(v_i)(w'')} \qquad \text{by 4.6 (ii)}$$

$$= E(\chi)A_{(w)(v_i)} + E(\chi)\left\{\text{sum of terms } A_{(w')} \text{ with } l(w') > l(w) + l(v_i)\right\} \quad \text{by 4.2 (i)}\ .$$

Hence $$S_n = \sum_{i=1}^{t} \sum_{\substack{w\in T \\ \text{where} \\ l(w)+l(v_i)=n}} \lambda_{v_i,w}\, E(\chi)A_{(w)(v_i)}$$

$$+\ E(\chi)\left\{\begin{array}{l}\text{linear combination of terms } A_{(w')} \\ \text{with } l(w') > n\end{array}\right\}$$

$$= \sum_{h\in H} \sum_{i=1}^{t} \sum_{\substack{w\in T \\ l(w)+l(v_i)=n}} \frac{1}{|H|} \chi(h^{-1}) \lambda_{v_i,w} A(w)(v_i)h$$

$$+ \left\{ \begin{array}{l} \text{linear combination of terms } A_{(w')}A_h \\ \text{where } h \in H,\ l(w') > n \end{array} \right\} .$$

But the elements $\{(w)(v)h \mid w \in T,\ v \in W_{M(\chi)},\ h \in H\}$ are all distinct (use 4.2 (i) and (iii)) and are in fact all the elements of $N$. Since $\{A_n \mid n \in N\}$ is a k-basis for $E$ this implies that if $S_n = 0$ then

$$\chi(h^{-1})\lambda_{v_i,w} = 0 \quad \text{all } h \in H,\ w \in T \text{ with } l(v_i) + l(w) = n \text{ any } v_i\ (1\leqslant i\leqslant t)$$

$$\Rightarrow \lambda_{v_i,w} = 0 \quad \text{all } w \in T, \text{ with } l(v_i) + l(w) = n \text{ any } v_i\ (1\leqslant i\leqslant t)$$

$$\Rightarrow S_{n+1} = 0 \quad \text{as required.}$$

4.8 Lemma. ([2, (4.4.9)]) Let $\theta \in E$. Then there exist $\zeta_{v,w} \in k$ such that

$$e_J o_J^{*} \theta = \sum_{v\in V_J,\, w\in T} \zeta_{v,w}\, \sigma(v,w) .$$

Proof. We know $\theta = \sum_{h\in H, w\in W} \lambda_{h,w} A_h A_{(w)} \qquad (\lambda_{h,w} \in k)$

and by 4.4 (ii) we need only show that for all $w \in W$ $e_J o_J^{*} A_{(w)}$ has the required form. We do this by induction on $l(w)$. If $l(w) = 0$ then $w = 1$ and

$$e_J o_J^{*} = e_J o_J^{2} = (-1)^{l(w_J^{*})} e_J o_J^{*} A_{(w_J^{*})} = (-1)^{l(w_J^{*})} \sigma(w_J^{*},1).$$

Assume $l(w) \geqslant 1$. Let $w = w_i w'$ with $l(w) = l(w') + 1$.
By induction

$$e_J \circ \hat{J} A_{(w')} = \sum_{\substack{v \in V_J \\ x \in T}} \beta_{v,x}\, \sigma(v,x) \qquad (\beta_{v,w} \in k).$$

But $A_{(w)} = A_h A_{(w')} A_{(w_i)}$ some $h \in H$ by (A) I 2.12 (i)

so we need only show that $\sigma(v,x)A_{(w_i)}$ has the required form for any $v \in V_J$, $x \in T$, $w_i \in R$. Fix $v \in V_J$, $x \in T$, $w_i \in R$.

We know $A_{(v)}A_{(x)} = A_{(x)(v)}$ by 4.2 (i).

<u>Case I</u>. $l(w_i xv) = l(xv) - 1$; $v^{-1}x^{-1}(a_i) < 0$.

Then $\sigma(v,x)A_{(w_i)} = e_J \circ \hat{J}\, A_{(x)(v)} A_{(w_i)}$

$$= e_J \circ \hat{J}\, A_{(x)(v)} \sum_{s=1}^{b(i)} A_{h_i(u_{i_s})} \quad \text{by (A) I 2.12 (ii)}$$

$$= e_J \circ \hat{J} \sum_{s=1}^{b(i)} A_{(v)^{-1}(x)^{-1}h_i(u_{i_s})(x)(v)}\, A_{(x)(v)}$$

by (A) I 2.5

$$= \sum_{s=1}^{b(i)} \chi((v)^{-1}(x)^{-1}h_i(u_{i_s})(x)(v))\, \sigma(v,x)$$

by 4.4 (ii)

<u>Case II</u>. $l(w_i xv) = l(xv) + 1$; $v^{-1}x^{-1}(a_i) > 0$.

(i) $x^{-1}(a_i) > 0$; $l(w_i x) = l(x) + 1$

a) $x^{-1}(a_i) \neq a_t$ any $a_t \in \Delta_{M(\chi)}$. Then

$w_i x \in T$ and $\sigma(v,x)A_{(w_i)} = e_J \circ \hat{J}\, A_{(v)}A_{(w_i)(x)}$

by (A) I 2.12 (i)

$= e_J \circ \hat{J} A_h A_{(v)} A_{(w_i x)}$ some $h \in H$

$= \chi(h)\, \sigma(v, w_i x)$ .

b) $x^{-1}(a_i) = a_t$ some $a_t \in \Delta_{H(\chi)}$. Then

$x^{-1}w_i x = w_t$ and $v^{-1}(a_t) > 0$ so that either

1) $v^{-1}(a_t) \neq a_s$ any $a_s \in \Delta_J$ or

2) $v^{-1}(a_t) = a_s$ some $a_s \in \Delta_J$ .

In 1) $w_t v \in V_J$ and

$$\sigma(v,x)A_{(w_i)} = e_J o \hat{J} A_{(w_i)(x)(v)} \quad \text{by 4.2 (i)}$$

$$= \chi(h) e_J o \hat{J} A_{(x)(w_t)(v)} \qquad (+)$$

for some $h \in H$

$$= \chi(h') e_J o \hat{J} A_{(w_t v)} A_{(x)} \quad \text{some } h' \in H$$

$$= \chi(h') \sigma(w_t v, x).$$

In 2) we have $v^{-1}w_t v = w_s \in J$ and

$$\sigma(v,x)A_{(w_i)} = e_J A_{(v)}A_{(x)}A_{(w_i)} \quad \text{by 4.6 (i)}$$

$$= \chi(h) e_J A_{(x)(w_t)(v)} \quad \text{as in (+)}$$

above for some $h \in H$

$$= \chi(h') e_J A_{(x)(v)(w_s)} \quad \text{some } h' \in H$$

$$= \chi(h'') e_J A_{(w_s)} A_{(x)(v)} \quad \text{some } h'' \in H$$

$$= 0 \quad \text{by 4.4 (iii).}$$

(ii) $x^{-1}(a_i) < 0$. Then $l(w_i x) = l(x) - 1$ and

$$\sigma(v,x)A_{(w_i)} = e_J o \hat{J} A_{(v)}A_{(x)} \sum_{s=1}^{b(i)} A_{h_i(u_{i_s})} \quad \text{by (A) I 2.12 (ii).}$$

This expression will have the required form as in Case I.

4.9 Lemma. Let $\mathcal{S} = \{ e_{J,\chi} \circ_{\hat{J},\chi} A_{(v)} A_{(w)} \mid \chi \in \hat{B},\ w \in T_{\chi},\ J \subseteq M(\chi),\ v \in V_{J,\chi} \}$

Then $\mathcal{S}$ is a set of linearly independent elements of $E$ and form a basis for $E$.

Proof. The set $\mathcal{S}$ is linearly independent by 4.7 and the direct sum decomposition $E = \sum^{\oplus}_{\chi \in \hat{B}} E(\chi)E$. The elements of $\mathcal{S}$ must form a k-basis since

$$|\mathcal{S}| = \sum_{\chi \in \hat{B}} |\mathcal{S}_{\chi}| = \sum_{\chi \in \hat{B}} \sum_{J \subseteq M(\chi)} |V_{J,\chi}|\,|T_{\chi}|$$

$$= \sum_{\chi \in \hat{B}} |W_{M(\chi)}|\ |W:W_{M(\chi)}| \qquad \text{by 4.2}$$

$$= \sum_{\chi \in \hat{B}} |W|$$

$$= |H|\ |W| = |N| .$$

4.10 Corollary. a) For each $\chi \in \hat{B}$, $\mathcal{S}_{\chi}$ is a k-basis for $E(\chi)E$.

b) For a fixed admissible G-pair $(J, \chi)$ $e_J \circ_{\hat{J}} E$ has dimension $|V_{J,\chi}|\,|T_{\chi}|$ and the set $\{ e_J \circ_{\hat{J}} A_{(v)} A_{(w)} \mid v \in V_{J,\chi},\ w \in T_{\chi} \}$ is a basis .

Proof. Part a) follows by 4.7 and the proof of 4.9. Part b) follows by 4.7 and 4.8.

4.11 Corollary. For each admissible G-pair $(J, \chi)$ $e_{J,\chi} \circ_{\hat{J},\chi} E$ is an indecomposable right E-module (ideal) and

(1) $$E = \sum^{\oplus}_{(J,\chi) \in P} e_{J,\chi} \circ_{\hat{J},\chi} E .$$

Proof. Decomposition (1) follows from 4.8 and 4.9. Each $e_{J,\chi} \circ \hat{J},\chi E$ must therefore be an indecomposable right E-module since $e_{J,\chi} \circ \hat{J},\chi E \neq 0$ and there is a bijective correspondence between the set of indecomposable components of E, the set of indecomposable components of Y and the set P of admissible G-pairs.

From decomposition (1) we can write

$$1_Y = \sum_{(J,\chi)\in P} p_{J,\chi} \quad (p_{J,\chi} \in e_{J,\chi} \circ \hat{J},\chi E).$$

Then the $p_{J,\chi}$ are mutually orthogonal primitive idempotents in E and

(2) $$p_{J,\chi} E = e_{J,\chi} \circ \hat{J},\chi E .$$

Since $E(Y) = Y$ we get by applying both sides of (2) to Y that

(3) $$p_{J,\chi}(Y) = e_{J,\chi} \circ \hat{J},\chi (Y) \text{ and}$$

4.12 Corollary. $Y = \sum^{\oplus}_{(J,\chi)\in P} e_{J,\chi} \circ \hat{J},\chi (Y)$ is a decomposition of Y into indecomposable kG-submodules.

4.13 Corollary. For any admissible G-pair $(J,\chi)$

$$e_{J,\chi} \circ \hat{J},\chi (Y) \cong Y(J,\chi) .$$

Proof. In order to identify $e_{J,\chi} \circ \hat{J},\chi (Y)$ we need only show that $\varphi_R(J,\chi)(p_{J,\chi}) = 1$ (see proof of I 2.2). This will follow if we show that $\varphi_R(J,\chi)(e_{J,\chi} \circ \hat{J},\chi) \neq 0$ by (3).

Let $w \in W$. Then by (A) I 2.21

$$\varphi_R(J,\chi)A_{(w)} = \begin{cases} 0 & w \notin W_{\hat{J}} \\ (-1)^{l(w)} & w \in W_{\hat{J}} \end{cases}$$

It is easily seen that $\varphi_R(J,\chi)e_{J,\chi} = \varphi_R(J,\chi)o_{\hat{J},\chi} = 1.$

We now consider $Y_J$ as a subspace of $Y$ for any $J \subseteq R$ and we take $y_J = y$ (see I 1.3) so that $Y_J = kG_J y$. We can consider $e_{J,\chi}$ and $o_{J,\chi}$ as elements of $E_J$ via the injective algebra homomorphism $\theta : E_J \to E$ given in (B) I section 1. As such, $e_{J,\chi}$ affords the $E_J$ character $\varphi_J(J,\chi)$ and $o_{J,\chi}$ affords the $E_J$ character $\varphi_J(\Phi,\chi)$.

Therefore

4.14 $\quad e_{J,\chi}(Y_J) \cong M_J(J,\chi)$ and

$o_{\hat{J},\chi}(Y_{\hat{J}}) \cong M_{\hat{J}}(\Phi,\chi)$ for any admissible G-pair $(J,\chi)$.

By 2.1 and 2.3 $e_{J,\chi}(Y_J) \cong Y_J(J,\chi)$ and $o_{\hat{J},\chi}(Y_{\hat{J}}) \cong Y_{\hat{J}}(\Phi,\chi)$

Since $Y = \sum^{\oplus}_{\alpha \in \mathcal{A}} \alpha Y_J$ where $\mathcal{A}$ is a set of representatives of left cosets of $G$ by $G_J$

$$e_{J,\chi}(Y) = \sum_{\alpha\in\mathcal{A}} e_{J,\chi}(\alpha Y_J)$$

$$= \sum_{\alpha\in\mathcal{A}} \alpha\, e_{J,\chi}(Y_J)$$

$$= \sum^{\oplus}_{\alpha\in\mathcal{A}} \alpha\, e_{J,\chi}(Y_J) \quad \text{since } e_{J,\chi}(Y_J) \subseteq Y_J.$$

Therefore $\left[e_{J,\chi}(Y_J)\right]^G \cong e_{J,\chi}(Y)$ and similarly $\left[o_{\hat{J},\chi}(Y_J)\right]^G \cong o_{\hat{J},\chi}(Y)$. Therefore by 2.4

4.15 <u>Lemma.</u> $\quad e_{J,\chi}(Y) \cong Y_J(J,\chi)^G \cong \sum^{\oplus}_{K \supseteq J} Y(K,\chi)$

$o_{\hat{J},\chi}(Y) \cong Y_{\hat{J}}(\Phi,\chi)^G \cong \sum^{\oplus}_{S \subseteq J} Y(S,\chi)$

and $Y(J,\chi)$ is the unique common indecomposable component of $Y_J(J,\chi)^G$ and $Y_{\hat{J}}(\Phi,\chi)^G$.

## III. The vertices of the indecomposable components of Y.

In this chapter we calculate a vertex for $Y(J,\chi)$ for any admissible G-pair $(J,\chi)$. We refer to Green's work on G-algebras and generalise the notion of G-algebras with permutation base to those with monomial base. The author realises that the vertices of the components of Y can be calculated by appealing only to L.L.Scott's work [11] on permutation modules (see 3.7 Remark (ii)). We include the work on the monomial case for general interest.

### 1. Preliminaries on G-Algebras.

We begin by recalling some definitions and results from [5, p. 138-141]. We assume that G is any finite group and k is any commutative ring with identity.

Definition. A G-algebra over k is a k-algebra A with identity element on which G acts as a group of k-algebra automorphisms; that is, $g \in G$ acts on $a \in A$ to give $a^g \in A$ making A into a right G-module and

$$(ab)^g = a^g b^g \quad \text{all } a,b \in A,\ g \in G.$$

Notice $1^g = 1$ for all $g \in G$ where 1 is the identity of A.

Definition. Let A be a G-algebra over k. For each subgroup H of G, define

$$A_H = \{ a \in A \mid a^h = a \ \text{ all } h \in H \}.$$

1.1 Lemma. Let H be a subgroup of G. Then $A_H$ is a subalgebra of A and if H and K are both subgroups of G,

$$H \leq K \Rightarrow A_K \subseteq A_H .$$

1.2 Definition. If $H$ and $K$ are both subgroups of $G$ and $H \leq K$, we define the k-linear map $T_{H,K}: A_H \to A_K$ by

$$T_{H,K}(a) = \sum_{v \in V} a^v \qquad (a \in A_H)$$

where $V$ is a set of representatives of the cosets $Hv$ in $K$. Since $a \in A_H$, $T_{H,K}(a)$ does not depend on the choice of $V$. Moreover $T_{H,K}(a)^x = T_{H,K}(a)$ any $x \in K$ since $Vx$ is an H-transversal of $K$ if $V$ is.

Definition. If $H$ and $K$ are subgroups of $G$ with $H \leq K$ define

$$A_{H,K} = \text{Image } T_{H,K} = T_{H,K}(A_H).$$

1.3 Lemma. Let $A$ be an G-algebra and let $D,H,K$ be subgroups of $G$ with $D \leq H \leq K$. Let $a \in A_H$, $b \in A_K$, $g \in G$. Then

(i) $T_{H,K}(ab) = T_{H,K}(a)\, b$

(ii) $T_{H,K}(ba) = b\, T_{H,K}(a)$

(iii) $T_{H,K}T_{D,H} = T_{D,K}$ (Transitivity Law)

(iv) $(A_H)^g = A_{H^g}$

(v) $T_{H,K}(a)^g = T_{H^g,K^g}(a^g)$

1.4 Lemma. If $H$ and $K$ are subgroups of $G$ with $H \leq K$ then $A_{H,K}$ is an ideal of $A_K$.

Proof. By 1.3 (i) and (ii).

Notation. If $D$ and $H$ are subgroups of $G$, then $D \leq_G H$ means that $D$ is conjugate in $G$ to a subgroup of $H$, $D =_G H$ means $D$ is conjugate in $G$ to $H$.

Assume $k$ is a field of characteristic $p > 0$. We must include the notion of defect groups in G-algebras.

1.5 <u>Theorem</u>. Let $A$ be a G-algebra over $k$ and let $e$ be a primitive idempotent in the algebra $A_G$. Then there exists a subgroup $D$ of $G$ such that

(i) $e \in A_{D,G}$ and

(ii) if $e \in A_{H,G}$ for any subgroup $H$ of $G$, then $D \leq_G H$.

Thus $D$ is determined up to conjugacy in $G$ and we call $D$ <u>a defect group of $e$</u> in the G-algebra A.

1.6 <u>Lemma</u>. Let $D,H,K$ be subgroups of $G$ with $D \leq H \leq K$ and $\operatorname{hcf}(p, |H:D|) = 1$. Then $A_{D,K} = A_{H,K}$.

<u>Proof</u>. We always have

$$A_{H,K} = T_{H,K}(A_H) \supseteq T_{H,K}(A_{D,H}) = A_{D,K} \quad \text{by 1.3 (iii).}$$

However when $p$ does not divide $|H:D|$ the map

$T_{D,H}: A_D \to A_H$ is surjective for we have $A_H \subseteq A_D$

and if $\beta \in A_H$ then

$$T_{D,H}(\beta) = \sum_{v \in V} \beta^v \qquad \text{where } V \text{ is a set of representatives}$$

of cosets $Dv$ in $H$

$$= |H:D|\,\beta \neq 0 .$$

Therefore $A_{D,H} = A_H$ and $A_{D,K} = T_{H,K}(A_{D,H}) = T_{H,K}(A_H) = A_{H,K}$.

1.7 <u>Lemma</u>. The defect group D defined above is a p-subgroup of G.

Let $M$ be a left kG-module. The k-algebra $\mathcal{E} = \mathcal{E}(M) = \operatorname{End}_k (M)$ can be made into a G-algebra by defining

$$\theta^g(m) = g^{-1}\theta(gm) \quad \text{for} \quad \theta \in \mathcal{E}, \; g \in G, \; m \in M.$$

For any subgroup H of G, $\mathcal{E}_H$ is the algebra of kH-endomorphisms of M.

We conclude this section with the following lemma:

1.8 Lemma. Let M be any left kG-module and let $\varepsilon$ be an idempotent in $\mathcal{E}_G$. Then

$\varepsilon$ is primitive in $\mathcal{E}_G \Leftrightarrow \varepsilon M$ is an indecomposable kG-module.

The defect group of $\varepsilon$ in $\mathcal{E}_G$, $\varepsilon M$ indecomposable, coincides with the vertex of $\varepsilon M$.

## 2. G-algebras with monomial base.

For this discussion we assume k is an integral domain and that A is a G-algebra over k. We generalise the discussion of [5, p. 141-142].

2.1 Definitions. A line L is a free 1-dimensional k-submodule of A.

A is said to have a monomial base if there exists a finite set of lines $\Lambda$ such that

(i) $A = \sum\limits_{L \in \Lambda}^{\oplus} L$

(ii) $\Lambda$ is permuted by G; that is if $L \in \Lambda$, $g \in G$, then $L^g \in \Lambda$.

Given any line L there exists at least one free generator $\omega_L$, so that each element of L can be written uniquely as $\xi\omega_L$ ($\xi \in k$). The set $\{\omega_L \mid L \in \Lambda\}$ is then a free-basis for A by 2.1 (i) which affords a monomial representation of G by 2.1 (ii) of dimension equal to the cardinality of $\Lambda$.

Let $H$ be a subgroup of $G$ and let $\{\Lambda_i \mid i \in I\}$ be the set of H-orbits of $\Lambda$. Let $\{L_i \mid L_i \in \Lambda_i, i \in I\}$ be a set of representatives of these H-orbits. For each $i \in I$ choose $\omega_i = \omega_{L_i}$, a free k-generator of $L_i$.

<u>Notation</u>. For $L \in \Lambda$ let $H(L) = \{h \in H \mid L^h = L\}$, the stabiliser of $L$ in $H$. Also denote by $L^H$ the sum in $A$ of the elements in the H-orbit of $L$. Ofcourse there are $|H:H(L)|$ elements in the H-orbit of $L$.

2.2 <u>Definition</u>. Let $L = k\omega_L \in \Lambda$. The <u>character</u> $\varphi_L$ of $H(L)$ is given by

$$\varphi_L : H(L) \to k^* \quad \text{where}$$

$$\omega_L{}^h = \varphi_L(h)\omega_L \quad \text{for} \quad h \in H(L). \quad \text{Here}$$

$k^*$ is the group of units of $k$.

The character $\varphi_L$ is easily seen to be independent of the choice of free generator for if $\omega'_L$ is another free generator of $L$ and $\omega'^h_L = \varphi'_L(h)\,\omega'_L$ then

$$\begin{aligned} \omega'_L = \xi\omega_L \text{ some } \xi \in k \Rightarrow \quad \omega'^h_L &= \xi\omega_L{}^h \\ &= \xi\varphi_L(h)\,\omega_L \\ &= \varphi_L(h)\omega'_L \quad \text{any } h \in H(L \end{aligned}$$

Returning to the H-orbit representative $L_i$, let $\varphi_i = \varphi_{L_i} : H(L_i) \to k^*$ and let $X_i$ be a set of representatives of cosets $H(L_i)x$ in $H$. Then $L_i{}^x$ gives all elements in the H-orbit of $L_i$ without repetition as $x$ ranges over $X_i$. Hence

2.3 The elements $\{ \omega_i^x \mid x \in X_i \}$ are independent and $\mathcal{L} = \bigcup_{i \in I} \{ \omega_i^x \mid x \in X_i \}$ is a k-basis for $A$ since $\mathcal{L}$ contains a free generator from each $L \in \Lambda$.

The set $\mathcal{L}$ is called an H-standardised basis of A.

To determine the action of $h \in H$ on $\omega_i^x$ write $xh = fv$, $f \in H(L_i)$, $v \in X_i$. Then

2.4 $$(\omega_i^x)^h = \omega_i^{xh} = \omega_i^{fv} = \varphi_i(f)\, \omega_i^v .$$

This resembles the usual procedure for giving an induced representation in explicit matrix form.

We show that not all H-orbits make a contribution to the subalgebra $A_H$. We make the following definition:

2.5 Definition. A line $L = k\omega$ is called H-special if $\omega^f = \omega$ all $f \in H(L)$.

Clearly $L$ is H-special $\Leftrightarrow \varphi_L(h) = 1$ all $h \in H(L)$

$$\Leftrightarrow \omega \in A_{H(L)}.$$

2.6 Remarks. (i) This property is invariant to the choice of $\omega$, the free generator and (ii) L is H-special if and only if $L^h$ is H-special for all $h \in H$. (We write $L^h = L^x$ for some representative x of cosets $H(L)x$ in H and $L^x = k\omega^x$ if $L = k\omega$. We must show $(\omega^x)^f = \omega^x$ all $f \in H(L^x) = x^{-1}H(L)x$. Then $\omega^{xf} = \omega^{(xfx^{-1})x} = \omega^x$ if L is H-special.)

Definition. Let H be a subgroup of G. Let $L = k\omega \in \Lambda$ be H-special. Then define

$$\omega^H = T_{H(L),H}(\omega) = \sum_{x \in X} \omega^x \quad \text{where } X$$

is a set of representatives of cosets $H(L)x$ in $H$.
Ofcourse $\omega^H$ is invariant to the choice of $X$.

We can say that the H-orbit $\Lambda_i$ is H-special if any line in $\Lambda_i$ is H-special (by Remark 2.6(ii)).
The following is a generalisation of [5, Lemma 5a, p. 141].

2.7 Lemma. Let $\mathcal{J} = \{ \omega_i^H \mid \Lambda_i$ is H-special, $i \in I\}$. Then $\mathcal{J}$ is a k-basis for $A_H$.

Proof. Let $a \in A$. By 2.3

$$a = \sum_{i\in I} \sum_{x \in X_i} \xi_{i,x}\, \omega_i^{\,x} \qquad (\xi_{i,x} \in k)$$

and

$$a \in A_H \Leftrightarrow \sum_{i,x} \xi_{i,x}\, \omega_i^{\,x} = \sum_{i,x} \xi_{i,x} (\omega_i^{\,x})^h \quad \text{all } h \in H$$

$$\Leftrightarrow \sum_{i,x} \xi_{i,x}\, \omega_i^{\,x} = \sum_{i,x} \xi_{i,x}\, \varphi_i(f)\, \omega_i^{\,v}$$

where for every $h \in H$, $xh = fv$ $(f \in H(L_i),\ v \in X_i)$ by 2.4

$\Leftrightarrow$ for all $i \in I$, $x \in X_i$

$\xi_{i,v} = \xi_{i,x}\, \varphi_i(f)$ where $xh = fv$ any $h \in H$

$\Leftrightarrow$ for all $i \in I$, $x \in X_i$, $\xi_{i,v} = \xi_{i,x}\, \varphi_i(f)$ for all $f \in H(L_i)$, all $v \in X_i$, since as $h$ runs through $H$, $xh$ also runs through $H$ so that $f$ ranges over all the elements of $H(L_i)$ and $v$ ranges over all those of $X_i$.

$\Leftrightarrow$ for all $i \in I$, $x \in X_i$ either $L_i$ is H-special and $\xi_{i,v} = \xi_{i,x}$ all $v \in X_i$ or $L_i$ is not H-special in which case there exists $f' \in H(L_i)$ such that $\varphi_i(f') \neq 1$. Therefore since $\xi_{i,x} = \xi_{i,x}\varphi_i(f')$ we must have $\xi_{i,x} = 0$ all $x \in X_i$.

We continue in the spirit of [5, section 5].

2.8 Lemma. Let $H$ and $D$ be subgroups of $G$ with $D \leq H$. Let $L = k\omega \in \Lambda$ be D-special. Then

$$T_{D,H}(\omega^D) = \begin{cases} |H(L):D(L)|\,\omega^H & \text{if } L \text{ is H-special} \\ 0 & \text{otherwise} \end{cases}$$

Proof. If $L$ is not H-special $T_{D,H}(\omega^D) = 0$ by 2.7.

$$\begin{aligned} \text{Now}\quad T_{D,H}(\omega^D) &= T_{D,H}T_{D(L),D}(\omega) \\ &= T_{D(L),H}(\omega) \quad \text{by 1.3 (iii)} \\ &= T_{H(L),H}T_{D(L),H(L)}(\omega) \quad \text{by 1.3 (iii)} \\ &= T_{H(L),H}|H(L):D(L)|(\omega) \quad \text{if } L \text{ is H-special} \\ &= |H(L):D(L)|\,\omega^H. \end{aligned}$$

Definition. Let $D,H$ be as above and $L = k\omega$. Define

$$N(\omega;D,H) = \operatorname*{hcf}_{h\in H}\{|H(L):D^h \cap H(L)|\} \quad \text{where}$$

$D^h = h^{-1}Dh$.

2.9 Lemma. ([5, Lemma 5d, p. 141]) The set

$$\mathcal{N} = \{ N(\omega_i;D,H)\,\omega_i^H \mid \Lambda_i \text{ is H-special, } i \in I \}$$

is a k-basis for $A_{D,H}$.

Proof. By 2.7 $A_D$ is k-generated by $\{\omega^D \mid L = k\omega,\ L \text{ is D-special}$
By 2.8 $\{|H(L):D(L)|\,\omega^H \mid L = k\omega,\ L \text{ is H-special}\}$ k-generates $A_{D,H}$. Now $\omega^H = \omega_i^H$ for exactly one $i \in I$ and $\omega_i^H = \omega^H$ if and only if there exists $h \in H$ such that $\omega = \omega_i^h$. Therefore $A_{D,H}$ has k-basis $\{\lambda_i\,\omega_i^H \mid i \in I,$ $\Lambda_i$ is H-special$\}$ where $\lambda_i$ is the highest common factor

of the set of integers $\{|H(L_i^h):D(L_i^h)|,\ h \in H\}$.

Since $G(L^g) = G(L)^g$ any line $L$ and any $g \in G$ $H(L^g) = H \cap G(L^g) = (H^{g^{-1}} \cap G(L))^g$. Therefore for any $h \in H$

$$|H(L_i^h):D(L_i^h)| = |H^{h^{-1}} \cap G(L_i):D^{h^{-1}} \cap G(L_i)|$$

$$= |H(L_i):D^{h^{-1}} \cap H(L_i)|.$$

Let $t$ be any field of characteristic zero with discrete valuation $\vee$ such that $\vee(p) = 1$ for some prime $p$. Let $R$ be the valuation ring of $\vee$ and let $F = R/P$ where $P$ is the unique maximal ideal of $R$ so that characteristic of $F$ is $p$. If $k = R$ or $F$ we can replace each integer $N(\omega_i;D,H)$ by the highest power of $p$ dividing it for if $N = p^r N'$ with $(p, N') = 1$ then $N'$ is a unit in $k$ and $p^r k = Nk$.

2.10 <u>Lemma</u>. ([5, Lemma 5d, p. 142]). Let $k = R$ or $F$. The set $\mathcal{N} = \{p^{n(\omega_i;D,H)}\ \omega_i{}^H \mid \Lambda_i$ is H-special, $i \in I\}$ is a k-basis for $A_{D,H}$ where

$$n(\omega_i;D,H) = \min_{h \in H} \{ \vee |H(L_i):D^h \cap H(L_i)| \} .$$

L.L. Scott ([11, p. 104]) defines the notion of a 'defect group' of a basis element in $End_{kG}(M)$ where $M$ is a permutation module. We generalise his definition for the monomial case. Assume $k = F$.

<u>Definition</u>. Let $\Lambda_i$ be an H-special H-orbit and $S$ be a p-subgroup of $H$. Then $S$ is called a <u>defect group of $\Lambda_i$</u> if $S$ is a Sylow p-subgroup of $H(L_i)$ some $L_i \in \Lambda_i$.

Since $H(L_i{}^h) = H(L_i)^h$ any $h \in H$, a defect group is determined up to conjugacy in $H$. We choose a fixed defect group of an H-special H-orbit $\Lambda_i$ and denote it by $\Delta(\Lambda_i)$.

We conclude this section with the following lemma:

2.11 Lemma. Let $k$ be a field of characteristic $p$. Let $D$ and $H$ be subgroups of $G$ with $D \leq H$. The ideal $A_{D,H}$ has k-basis consisting of those H-special $\omega_i{}^H$ where $\Delta(\Lambda_i) \underset{H}{\leq} D$.

Proof. By Lemma 2.10, $A_{D,H}$ has k-basis consisting of all H-special $\omega_i{}^H$ for which $n(\omega_i;D,H) = 0$ since we are in a field of characteristic $p$.

Now $n(\omega_i;D,H) = 0 \Leftrightarrow \nu|H(L_i):D^h \cap H(L_i)| = 0$ for some $h \in H$

$\Leftrightarrow (p, |H(L_i):D^h \cap H(L_i)|) = 1$ for some $h \in H$

$\Leftrightarrow D^h$ contains a Sylow p-subgroup of $H(L_i)$ for some $h \in H$

$\Leftrightarrow \Delta(\Lambda_i) \underset{H}{\leq} D$

Remark. By 1.6 we need only consider $A_{D,H}$ for p-subgroups $D$.

3. An example.

Let $k$ be any field of characteristic $p$ and $U$ any subgroup of $G$. We consider the special case when

$$M \cong \mathrm{Ind}_U^G(S)$$

where $S$ is a one-dimensional kU-module. Let $\lambda : U \to k^*$

be the character of U afforded by $S = ks$.

We have already seen that $\mathcal{E} = \mathrm{End}_k(M)$ is a G-algebra. Let $\{x_i \mid i \in I\}$ be a set of representatives of cosets $x_iU$ in G. Let $x_1 = 1$.

The group G acts transitively on I by the action $(g, i) \to gi$ given by $g(x_iU) = x_{gi}U$ (any $g \in G$, $i \in I$). Since $gx_i = x_{gi}(x_{gi}^{-1}gx_i)$ we have that $x_{gi}^{-1}gx_i$ lies in U and

3.1 $$g(x_i \otimes s) = \lambda(x_{gi}^{-1}gx_i)x_{gi} \otimes s \quad \text{for all } g \in G,\ i \in I.$$

Therefore M has monomial base since $M = \sum_{i \in I}^{\oplus} x_i \otimes S$ as k-spaces and G acts on the set of lines $\{x_i \otimes S \mid i \in I\}$ by

$$(x_i \otimes S)^g = g^{-1}(x_i \otimes S) = x_{g^{-1}i} \otimes S.$$

For each pair $(i,j) \in I \times I$ let $\theta_{i,j} \in \mathcal{E}$ be defined by

3.2 $$\theta_{i,j}(x_{i'} \otimes s) = \begin{cases} x_j \otimes s & i = i' \\ 0 & \text{otherwise.} \end{cases}$$

Then $\{\theta_{i,j} \mid (i,j) \in I \times I\}$ is a k-basis for $\mathcal{E}$ and

$$\mathcal{E} = \sum_{i,j}^{\oplus} M_{i,j} \quad \text{where } M_{i,j} = k\,\theta_{i,j}.$$ Let

$\Lambda$ denote the set of lines $\{M_{i,j} \mid (i,j) \in I \times I\}$. We need to calculate the precise action of $g \in G$ on an arbitrary line $M_{i,j}$:

Let $g \in G$. Then

$$\theta_{i,j}{}^{g}(x_{i'}\otimes s) = g^{-1}\,\theta_{i,j}(g(x_{i'}\otimes s))$$

$$= g^{-1}\,\theta_{i,j}(x_{gi'}\otimes \lambda(x_{gi'}^{-1}gx_{i'})s) \quad \text{by } 3.1$$

$$= \begin{cases} g^{-1}\,\lambda(x_i{}^{-1}gx_{g^{-1}i})x_j\otimes s & gi' = i \\ 0 & gi' \neq i \end{cases}$$

$$= \begin{cases} \lambda(x_i{}^{-1}gx_{g^{-1}i})\,\lambda(x_{g^{-1}j}^{-1}g^{-1}x_j)x_{g^{-1}j}\otimes s & gi' = i \\ 0 & gi' \neq i \end{cases}$$

by 3.1. Hence

3.3 $$\theta_{i,j}{}^{g} = \lambda(x_i{}^{-1}gx_{g^{-1}i})\,\lambda(x_j{}^{-1}gx_{g^{-1}j})^{-1}\theta_{g^{-1}i,g^{-1}j}$$

and $M_{i,j}{}^{g} = M_{g^{-1}i,g^{-1}j}$ for any $g \in G$, $(i,j) \in I \times I$.

Therefore $G(M_{i,j}) = x_iUx_i{}^{-1} \cap x_jUx_j{}^{-1}$.

Statements 3.2 and 3.3 combine to show

3.4 $\mathcal{E}$ is a G-algebra with monomial base $\Lambda = \{M_{i,j} \mid (i,j)\in I \times I\}$.

In order to calculate $E = \mathcal{E}_G$, the algebra of all kG-endomorphisms of M we must find the G-orbits on the set of lines $\Lambda$. We examine the action of G on I x I given by $g(i,j) \rightarrow (gi,gj)$.

Each G-orbit of I x I must contain at least one element of the form $(1,j)$ some $j \in I$. Clearly $(1,j)$ and $(1,s)$ are in the same G-orbit if and only if j and s are in the same U-orbit.

3.5 Let $1 = j_1, j_2, \ldots, j_r$ be representatives of the U-orbits on I. Then $\{M_{1,1}; M_{1,j_2}; \ldots; M_{1,j_r}\}$ is a set of representatives of the G-orbits on $\Lambda$. Assume $M_{1,j} = k\,\theta_{1,j}$ is from the orbit $\Lambda_j$.

We concern ourselves with G-special G-orbits which are described by the following lemma.

3.6 Lemma. The G-orbit $\Lambda_j$ is G-special if and only if $\lambda(t) = \lambda(x_j^{-1}tx_j)$ all $t \in G(M_{1,j})$.

Proof. The orbit $\Lambda_j$ is G-special

$$\Leftrightarrow \theta_{1,j}{}^t = \theta_{1,j} \quad \text{all} \quad t \in G(M_{1,j})$$

$$\Leftrightarrow \theta_{1,j} = \lambda(tx_{t^{-1}1})\,\lambda(x_j^{-1}tx_{t^{-1}j})^{-1}\,\theta_{t^{-1}1,t^{-1}j}$$

all $t \in G(M_{1,j})$ by 3.3

$$\Leftrightarrow \lambda(tx_1) = \lambda(t) = \lambda(x_j^{-1}tx_j) \quad \text{all} \quad t \in G(M_{1,j}).$$

3.7 Remarks. (i) The orbit $\Lambda_1$ is always G-special.

(ii) If $\lambda$ is the trivial character (identity character) on S all orbits are G-special and we are in the permutation case.

By 2.10 the defect group D of a primitive idempotent $\varepsilon \in \mathcal{E}_G$ will contain (up to conjugacy in G) some of the $\Delta(\Lambda_j)$'s. The following lemma is more precise.

3.8 Lemma. Let $\varepsilon$ be a primitive idempotent in $\mathcal{E}_G$ and let D be the defect group of $\varepsilon$. Then $\varepsilon$ has a unique decomposition

$$(1) \qquad \varepsilon = \sum_{\substack{\Lambda_j \text{ G-special} \\ \Delta(\Lambda_j) \leq_G D}} \xi_j\,\theta_{1,j}{}^G \qquad (\xi_j \in k^*)$$

and D is actually equal to one of the $\Delta(\Lambda_j)$ some $j \in I$.

More precisely, D coincides with any maximal (with respect to order) one of the $\Delta(\Lambda_j)$ given in (1).

Proof. Decomposition (1) follows from 2.1Φ. For any $j \in I$

$$\theta_{1,j}{}^{G} = \frac{1}{|G(M_{1,j}) : \Delta(\Lambda_j)|} \; T_{\Delta(\Lambda_j),G}(\theta_{1,j})$$

since

$$T_{\Delta(\Lambda_j),G}(\theta_{1,j}) = T_{G(M_{1,j}),G} T_{\Delta(\Lambda_j),G(M_{1,j})}(\theta_{1,j})$$

by 1.3 (iii)

$$= |G(M_{1,j}):\Delta(\Lambda_j)| \, T_{G(M_{1,j}),G}(\theta_{1,j})$$

$$= |G(M_{1,j}):\Delta(\Lambda_j)| \; \theta_{1,j}{}^{G} .$$

(2) Therefore $\varepsilon \in \sum_{\substack{\Lambda_j \text{ G-special} \\ \Delta(\Lambda_j) \underset{G}{\leq} D}} \mathcal{E}_{\Delta(\Lambda_j),G}$ .

By Rosenberg's lemma, $\varepsilon \in \mathcal{E}_{\Delta(\Lambda_t),G}$ some $t \in I$ given in (2) so that $D \underset{G}{\leq} \Delta(\Lambda_t)$ by 1.5. Therefore $D \underset{G}{=} \Delta(\Lambda_t)$.

If $\Delta(\Lambda_q)$ is a maximal one among the subgroups given in (1) then

$$|\Delta(\Lambda_q)| \geq |\Delta(\Lambda_t)| = |D| \geq |\Delta(\Lambda_q)|$$

so that $D \underset{G}{=} \Delta(\Lambda_q)$.

Remark. L.L. Scott gives a statment analogous to the above lemma in [11, p. 105] in which he defines a defect group of a primitive idempotent in $\mathcal{E}_G$ in the permutation case by the properties in the lemma.

In [11, Proposition 3(2), p. 106] Scott gives a characterisation of defect groups using certain modular characters of $\mathcal{E}_G$. We give a similar lemma based on [8, Lemma 3.1, p. 211] .

3.9 Lemma. Say $\tau$ is a k-algebra epimorphism of $\mathcal{E}_G$ onto a simple algebra S. Let f be a primitive idempotent of $\mathcal{E}_G$. Assume $\tau(f) \neq 0$. For any subgroup $D \leq G$

$$f \in \mathcal{E}_{D,G} \Leftrightarrow \tau(\mathcal{E}_{D,G}) \neq 0.$$

Proof. Clearly $f \in \mathcal{E}_{D,G}$ implies $\tau(\mathcal{E}_{D,G}) \neq 0$. Say $\tau(\mathcal{E}_{D,G}) \neq 0$. Since $\tau(\mathcal{E}_{D,G})$ is an ideal of $\tau(\mathcal{E}_G) = S$ we must have $\tau(\mathcal{E}_{D,G}) = S$ since S is simple. Therefore there exists $a \in \mathcal{E}_{D,G}$ such that $\tau(a) = \tau(f)$ so that

$$f \in \mathcal{E}_{D,G} + \text{kernel}\,\tau \quad (\text{since } f = a + (f - a)).$$

Therefore $f \in \mathcal{E}_{D,G}$ by Rosenberg's lemma.

## 4. Vertices of $Y(J,\chi)$.

We now assume that $G = (G,B,N,R,U)$ is an (unsaturated) split (B,N)-pair of characteristic p and k is an algebraically closed field of the same characteristic. Let $\{x_i \mid i \in I\}$ be a set of representatives of the left cosets of U in G and all notations are as in the preceeding sections 1,2, and 3. We take $M \cong Y$, $S \cong k_U$.

We have shown that E has k-basis $\{A_n\}$ where

$A_n(y) = [\Omega_w]ny$ where $nH = w$. By 2.7, 3.5 and 3.7(ii) $\{\vartheta_{1,1}{}^G, \vartheta_{1,j_2}{}^G, \ldots, \vartheta_{1,j_{|N|}}{}^G \mid 1, j_2, \ldots, j_{|N|}$ are representatives of the U-orbits of I$\}$ is also a k-basis of E where

4.1 $$\vartheta_{1,j}{}'(x_i y) = \begin{cases} x_j y & x_i \in U \\ 0 & \text{otherwise} \end{cases}$$

any $i, j \in I$.

Relating the bases $\{A_n\}$ and $\{\vartheta_{1,j}{}^G\}$ we see that

4.2 Lemma. $\vartheta_{1,j}{}^G = A_n \Leftrightarrow Ux_jU = UnU$.

Proof. Notice that given any $j \in I$ there exists a unique $n \in N$ for which $UnU = Ux_jU$ and if $Ux_jU = Ux_sU$ any $s, j \in \{1, \ldots, |N|\}$ then there is $u \in U$ such that $s = uj$ so that $s$ and $j$ belong to the same U-orbit; that is $j = s$.

Now $\vartheta_{1,j}{}^G = \sum_{z \in Z} \vartheta_{1,j}{}^z$ where $Z$ is a set of representatives of cosets $(U \cap x_jUx_j^{-1})z$ in $G$ (by 3.3).

Let $z = tx$ where $X$ is a set of representatives of cosets $Ux$ in $G$ and $T$ is a set of representatives of cosets $(U \cap x_jUx_j^{-1})t$ in $U$ and $t \in T$, $x \in X$.

Then $$\vartheta_{1,j}{}^G(y) = \sum_{\substack{t \in T \\ x \in X}} \vartheta_{1,j}{}^{tx}(y)$$

$$= \sum_{t,x} \vartheta_{x^{-1}t^{-1}1, x^{-1}t^{-1}j}(y) \quad \text{by 3.3}$$

$$= \sum_{t \in T} \vartheta_{t^{-1}1, t^{-1}j}(y)$$ (since the contribution of $x$ is 0 unless $x \in U$)

$$= \sum_{t \in T} x_{t^{-1}j}\, y$$

$$= \sum_{t\in T} t^{-1}x_j y \quad .$$

If $Ux_jU = UnU = \Omega_w nU$, $x_j$ has (unique) decomposition $x_j = u_1nu_2$ where $u_1 \in \Omega_w$, $u_2 \in U$ (see (A) I 1.7). Then $x_jUx_j^{-1} = u_1(nUn^{-1})u_1^{-1} = u_1(wUw^{-1})u_1^{-1}$ for $nH = w$ (1) and the set $T$ can be taken to be $u_1\Omega_w^{-1}u_1^{-1}$ so that

$$\begin{aligned}\theta_{1,j}^G(y) &= \sum_{t\in T} t^{-1}x_j y \\ &= u_1[\Omega_w]u_1^{-1}x_j y \\ &= u_1[\Omega_w]ny \\ &= [\Omega_w]ny \\ &= A_n(y) \quad \text{and} \quad \theta_{1,j}^G = A_n.\end{aligned}$$

Conversely, say $Ux_jU = Un'U$ for $n' \in N$, $n'H = w'$. Then $\theta_{1,j}^G = A_n$ implies $[\Omega_w]ny = [\Omega_{w'}]n'y$ by the work above so that $A_n = A_{n'}$ and $n = n'$.

The following lemma is immediate by (1):

4.3 Lemma. Let $Ux_jU = UnU$ for some $n \in N$. Then $\Delta(\Lambda_j) \underset{U}{=} {}_wU^+$ where $nH = w^{-1}$.

We have shown in I section 2 that we can use the following notation:

$$Y = \sum_{(J,\chi)\in P}{}^{\oplus} Y(J,\chi) \quad \text{is a decomposition}$$

of Y into indecomposable kG-modules summed over the set P of admissible G-pairs and

$$1_Y = \sum_{(J,\chi)\in P} \pi(J,\chi) \quad \text{is an orthogonal}$$

decomposition of $1_Y$ into primitive idempotents where

$Y(J,\chi) = \pi(J,\chi)Y$ and we arrange (see I 1.8) that

$$\rho_R(J,\chi)\pi(S,\chi') = \begin{cases} 1 & J = S, \quad \chi = \chi' \\ 0 & \text{otherwise} \end{cases}$$

We can now calculate the vertex of $Y(J,\chi)$ for any admissible G-pair $(J,\chi)$.

4.4 <u>Theorem</u>. Let $(J,\chi)$ be an admissible G-pair. Then ${}_{w_{\hat{J}}}U^+$ is a vertex for the indecomposable component $Y(J,\chi)$ where $\hat{J} = M(\chi)\setminus J$.

<u>Proof</u>. Let $\tau = \rho_R(J,\chi)$. Then $\tau(\pi(J,\chi)) \neq 0$ and we can apply Lemma 3.9. Let $w \in W$. Then there exists $h \in H$ for which

(1) $$\tau(A_{(w)}) = \begin{cases} (-1)^{l(w)}\chi(h) & w \in W_{\hat{J}} \\ 0 & w \notin W_{\hat{J}} \end{cases}$$

by I 1.8. Now let $P$ be any p-subgroup of $G$. Then

$\tau(\mathcal{E}_{P,G}) \neq 0 \Leftrightarrow$ there exists $j \in I$ such that $\tau(\theta_{1,j})^G \neq 0$ with $\Delta(\Lambda_j) \leq_G P$ by 2.11

$\Leftrightarrow$ there exists $n \in N$ such that $\tau(A_n) \neq 0$ and $U_w^+ \leq_G P$ where $nH = w$ by 4.2 and 4.3 .

By 3.9 and (1) we see that the vertex is ${}_wU^+$ for some $w \in W_{\hat{J}}$ . For any $w \in W$ with reduced expression $w = w_{i_1} \ldots w_{i_t}$ we have

(2) $$|{}_wU^+| = \frac{|U|}{q^{w_{i_1}} \ldots q^{w_{i_t}}} \quad \text{by II 1.8 (iii)}$$

If $w \in W_{\hat{J}}$, $w \neq w_{\hat{J}}$, then there exists $v \in W_{\hat{J}}$ with $w_{\hat{J}} = wv$ with $l(v) \geqslant 1$ and $l(w) + l(v) = l(w_{\hat{J}})$ since $w_{\hat{J}}$ is the unique element of maximal length in $W_{\hat{J}}$. Now $\tau(A_{(w_{\hat{J}})}) \neq 0$ and $|{}_w U^+| > |{}_{w_{\hat{J}}} U^+|$ any $w \in W_{\hat{J}}$, $w \neq w_{\hat{J}}$,

by (2) and II 1.8 (i). Hence ${}_{w_{\hat{J}}} U^+$ must be a vertex of $Y(J, \chi)$ by the minimality of its order.

Remark. This theorem shows the importance of 3.9 which allows us to calculate the vertex of $Y(J, \chi)$ with little information about the idempotent $\pi(J, \chi)$.

4.5 Lemma. Let $(J, \chi)$ be an admissible G-pair. Then $Y(J, \chi)$ is projective if and only if $M(\chi) = R$, $J = \Phi$ and $C = 1$.

Proof. $Y(J, \chi)$ is projective $\Leftrightarrow$ vertex of $Y(J, \chi)$ is 1

$$\Leftrightarrow |U : {}_{w_{\hat{J}}} U^+| = |U|$$

$$\Leftrightarrow q^{w_{\hat{J}}} = |C|\, q^{w_o}$$

$$\Leftrightarrow C = 1,\ w_{\hat{J}} = w_o \quad \text{(using II 1.8)}$$

$$\Leftrightarrow C = 1,\ \hat{J} = R$$

$$\Leftrightarrow C = 1,\ M(\chi) = R,\ J = \Phi .$$

Our last lemma of this section uses the main result of II section 2.

4.6 Lemma. Let $\chi \in \hat{B}$ be such that $M(\chi) = R$. Since ${}_{w_{\hat{J}}} U^+$ is a vertex for $Y(J, \chi)$ the dimension of $Y(J, \chi)$ is divisible by $q^{w_{\hat{J}}}$. In fact $q^{w_{\hat{J}}}$ is the highest power of $p$ dividing the dimension.

*Proof.* The first statement follows since the dimension of $Y(J,\chi)$ is divisible by $|U: \text{vertex } Y(J,\chi)|$ since $U$ is a Sylow $p$-subgroup of $G$. By II 1.6 $w_{\hat{J}}$ is the unique element of minimal length in $V_J$ so that if $w \in V_J$ $q^{w_{\hat{J}}}$ divides $q^w$ by II 1.8(iii). Since by II 2.2

$$\dim Y(J,\chi) = \sum_{w \in V_J} q^w$$

$$= q^{w_{\hat{J}}} (1 + d)$$

where $d$ is divisible by $p$, the result follows.

## 5. The duality of Y.

The module Y is self-dual, that is $Y \cong Y^* = \mathrm{Hom}_k(Y,k)$ since $((k_U)^G)^* \cong (k_U^*)^G \cong k_U^G$. Therefore there exists a permutation $(J,\chi) \to (J',\chi')$ of the set of admissible G-pairs such that $Y(J,\chi)^* \cong Y(J',\chi')$. (Notice this implies that all $Y(J,\chi)$ have simple socle if and only if all $Y(J,\chi)$ have simple head.) We determine this permutation in this section.

As an alternative to the classification of irreducible modules of groups with split (B,N)-pairs by weights (or equivalently by admissible G-pairs), Curtis shows that each such irreducible module is completely determined by its unique B-line and the parabolic subgroup which is the full stabiliser of that line (see [4, Theorem 6.15, p. 3-38]). We showed in I 1.10 that this remains true in the unsaturated case and it is using this point of view that we compute our result.

5.1 Lemma. Let $J \subseteq R$ and let $\rho : G_J \to k^*$ be a homomorphism afforded by the $kG_J$-module $L_\rho$. Let $(S,\chi)$ be an admissible G-pair. Then there exists a kG-monomorphism

$$\rho^*: M(S,\chi) \to L_\rho^{\,G}$$

if and only if $G_J$ stabilises the unique B-stable line of $M(S,\chi)^*$.

Proof. There exists an injective homomorphism

$$\rho^*: M(S,\chi) \to L_\rho^{\,G}$$

$$(M(S,\chi), L_\rho^{\,G})_{kG} \neq 0$$

$$(M(S,\chi)_{kG_J}, L_\rho)_{kG_J} \neq 0 \quad \text{(Frobenius Reciprocity)}$$

$\Leftrightarrow$ there exists $0 \neq f\colon M(G,\chi) \to k$ such that $f$ is a homomorphism and $f(gx) = \rho(g)f(x)$ for all $x \in M(G,\chi)$ and all $g \in G_J$.

$\Leftrightarrow$ there exists $0 \neq f \in M(G,\chi)^*$ such that $gf = \rho(g^{-1})f$ all $g \in G_J$.

$\Leftrightarrow$ $G_J$ stabilises the unique B-stable line $kf$ of $M(G,\chi)^*$.

<u>Definition</u>. If $\chi \in \hat{B}$, define $\chi^*\colon B \to k^*$ by $\chi^*(b) = \chi(b^{-1})$ all $b \in B$. Then $\chi^* \in \hat{B}$ and $M(\chi) = M(\chi^*)$.

5.2 <u>Lemma</u>. Let $(J,\chi)$ be an admissible G-pair. Then

$$Y(J,\chi)^* \cong Y(J,\chi^*) .$$

<u>Proof</u>. By I 2.2 we need only determine which admissible G-pair $(J',\chi')$ satisfies $M(J',\chi') \cong M(J,\chi)^*$. Let $M = M(J,\chi)$ have unique B-line $km$ affording $\chi$. Then $kU(w_o)m = kG(w_o)m = M$ (since Proposition 3.3 (v) and Theorem 4.3 (b) of [4] hold for unsaturated split pairs and we have the structural equations of (A) I 2.10 ). Therefore as in the proof of [4, Theorem 6.6, p. E-32]:

$$M = k(w_o)m \oplus \text{rad}\,(kU)(w_o)m \qquad \text{and}$$

$k(w_o)m$ affords ${}^{w_o}\chi$. Let $\lambda \in M^*$ be given as follows: If $m' \in M$ then $\lambda(m')$ is the coefficient of $(w_o)m$ in the decomposition above, that is

$$m' = \lambda(m')(w_o)m + x_1 \quad \text{where} \quad x_1 \in \text{rad}\,(kU)(w_o)m.$$

Then $k\lambda$ is the unique U-line in $M^*$ since for all $u \in U$

$$u^{-1}m' = \lambda(m')u^{-1}(w_o)m + u^{-1}x_1$$

$$= \lambda(m')((u^{-1} - 1)(w_o)m + (w_o)m) + x_2$$

where $x_2 \in \operatorname{rad}(kU)(w_o)m$

$$= \lambda(m')(w_o)m + x_3 \quad \text{where} \quad x_3 \in \operatorname{rad}(kU)(w_o)m$$

so that $u\lambda = \lambda$. Furthermore if $h \in H$ then

$$h^{-1}m' = \lambda(m')h^{-1}(w_o)m + h^{-1}x_1$$

$$= \lambda(m')\,{}^{w_o}\chi(h^{-1})(w_o)m + x_4 \quad \text{where}$$

$x_4 \in \operatorname{rad}(kU)(w_o)m$ since $H$ normalises $U$. Therefore $k\lambda$ affords the character $({}^{w_o}\chi)^*$.

The parabolic subgroup $G_{J^{w_o}}$ is contained in the full stabiliser of $k\lambda$ since for all $w_i \in J$ we have $(w_i)m = m$ (see I 1.10) and

(i) $(w_o)(w_i)(w_o)^{-1}(w_o)m = (w_o)m$ and

(ii) $(w_o)(w_i)(w_o)^{-1} \operatorname{rad}(kU)(w_o)m \subseteq \operatorname{rad}(kU)(w_o)m$.

The second statement follows as in [4, proof of Theorem 6.6, p. B-33] using [4, Corollary 3.6, p. B-14] which holds in the unsaturated case since $C_J^w = C_J$ all $w \in W_J$ (see I 1.1).

Let the full stabiliser of $k\lambda$ be $G_T$ with $T \supseteq J^{w_o}$. Then $M^* \cong M(T,({}^{w_o}\chi)^*)$ and $Y(J,\chi)^* \cong Y(T^{w_o},\chi^*)$ by I 2.2 since $({}^{w_o}\chi)^* = {}^{w_o}(\chi^*)$. We show $T = J^{w_o}$.

By results II 2.2 and III 4.6

$$d = \text{dimension } Y(T^{w_o},\chi^*) = |G:G_{M(\chi^*)}| \sum_{w \in V_{T^{w_o}}} q^w$$

where $V_{T^{w_o}}$ is a certain subset of $W_{M(\chi^*)} = W_{M(\chi)}$

and $d = |G:G_{M(\chi)}|\, q^{w_{(\widehat{T^{w_o}})}} (1 + t)$ where $w_{(\widehat{T^{w_o}})}$ is the unique element of maximal length in $M(\chi)\setminus T^{w_o}$ and $t$ is an integer divisible by $p$. But also

$d = \dim Y(J,\chi) = |G:G_{M(\chi)}|\, q^{w_{\hat{J}}} (1 + t')$ where $t'$ is divisible by $p$ and $\hat{J} = M(\chi)\setminus J$.

Hence $q^{w_{(\widehat{T^{w_o}})}} = q^{w_{\hat{J}}}$ (*)

If $J \subset T^{w_o}$ then $(\widehat{T^{w_o}}) \subset \hat{J}$ and $w_{\hat{J}} = w_{(\widehat{T^{w_o}})}\, v$ for some $v$ with $l(v) \geq 1$ and $l(w_{\hat{J}}) = l(w_{(\widehat{T^{w_o}})}) + l(v)$.

By II 1.8 we must have

$q^{w_{\hat{J}}} = q^{w_{(\widehat{T^{w_o}})}}\, q^{v}$. But $q^{v} > 1$ all $v \neq 1$

gives a contradiction to (*). Hence $J = T^{w_o}$.

5.3 <u>Corollary</u>. Let $(J,\chi)$ be an admissible G-pair. Then $M(J,\chi)^* \cong M(J^{w_o}, ({}^{w_o}\chi)^*)$.

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