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Krein signature and Whitham modulation theory: the sign of characteristics and the "sign characteristic"

Thomas J. Bridges[1] *& Daniel J. Ratliff*[2]

1. *Department of Mathematics, University of Surrey, Guildford GU2 7XH, UK*

2. *Department of Mathematical Sciences, Loughborough University, Loughborough, Leicestershire LE11 3TU, UK*

Dedicated to Roger H. J. Grimshaw on the occasion of his 80th birthday

Abstract. In classical Whitham modulation theory, the transition of the dispersionless Whitham equations from hyperbolic to elliptic is associated with a pair of nonzero purely imaginary eigenvalues coalescing and becoming a complex quartet, suggesting that a Krein signature is operational. However, there is no natural symplectic structure. Instead, we find that the operational signature is the "sign characteristic" of real eigenvalues of Hermitian matrix pencils. Its role in classical Whitham single-phase theory is elaborated for illustration. However, the main setting where the sign characteristic becomes important is in multiphase modulation. It is shown that a necessary condition for two coalescing characteristics to become unstable (the generalization of the hyperbolic to elliptic transition) is that the characteristics have opposite sign characteristic. For example the theory is applied to multiphase modulation of the two-phase travelling wave solutions of coupled nonlinear Schrödinger equations.

Keywords: multiphase modulation, wave action, theory of characteristics, coupled NLS, sign characteristic, quadratic matrix pencil.

1 Introduction

In dispersionless one-phase Whitham modulation theory the governing equations are the pair of equations

$$q_T = \Omega_X \quad \text{and} \quad \frac{\partial}{\partial T}\mathscr{A}(\omega+\Omega, k+q) + \frac{\partial}{\partial X}\mathscr{B}(\omega+\Omega, k+q) = 0\,, \tag{1.1}$$

for the unknown modulation frequency and wavenumber, $\Omega(X,T)$ and $q(X,T)$. Here (ω, k) are parameters representative of the wavetrain from which the Whitham modulation equations

[1]Corresponding author email: T.Bridges@surrey.ac.uk

are obtained, and $X = \varepsilon x$ and $T = \varepsilon t$ are slow time and space scales. The first equation is called *conservation of waves* and the second is called *conservation of wave action* [24]. The functions $\mathscr{A}$ and $\mathscr{B}$ are deduced, using classical Whitham theory, from a Lagrangian averaged over a periodic travelling wave with frequency ω and wavenumber k,

$$\mathscr{A} = \mathscr{L}_\omega \,, \quad \mathscr{B} = \mathscr{L}_k \,. \tag{1.2}$$

The Whitham modulation equations (WMEs) in (1.1) are a closed nonlinear first order set of PDEs for the functions Ω and q. The linearization of these equations about the basic state, represented by (ω, k), is

$$q_T = \Omega_X \quad \text{and} \quad \mathscr{A}_\omega \Omega_T + \mathscr{A}_k q_T + \mathscr{B}_\omega \Omega_X + \mathscr{B}_k q_X = 0 \,, \tag{1.3}$$

or, with the assumption $\mathscr{A}_\omega \neq 0$, they can be written in the standard form,

$$\begin{pmatrix} q \\ \Omega \end{pmatrix}_T + \mathbf{T}(\omega, k) \begin{pmatrix} q \\ \Omega \end{pmatrix}_X = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \,, \tag{1.4}$$

where

$$\mathbf{T}(\omega, k) = \frac{1}{\mathscr{A}_\omega} \begin{bmatrix} 0 & -\mathscr{A}_\omega \\ \mathscr{B}_k & \mathscr{A}_k + \mathscr{B}_\omega \end{bmatrix} \,. \tag{1.5}$$

Here, $\mathscr{A}$ and $\mathscr{B}$ are evaluated at $\Omega = q = 0$. The characteristics are

$$c^\pm = \frac{\mathscr{A}_k + \mathscr{B}_\omega}{2\mathscr{A}_\omega} \pm \frac{1}{\mathscr{A}_\omega} \sqrt{-\Delta_L} \,, \quad \Delta_L = \det \begin{bmatrix} \mathscr{A}_\omega & \mathscr{A}_k \\ \mathscr{B}_\omega & \mathscr{B}_k \end{bmatrix} \,, \tag{1.6}$$

using the identities (1.2). The modulation instability is recovered by letting

$$\begin{pmatrix} q(X,T) \\ \Omega(X,T) \end{pmatrix} = \mathrm{Re} \left\{ \begin{pmatrix} \widehat{q} \\ \widehat{\Omega} \end{pmatrix} \mathrm{e}^{\lambda T + \mathrm{i}\nu X} \right\} \,,$$

and substituting into (1.3) giving

$$\lambda = \mathrm{i} c^\pm \nu \,,$$

and so an unstable exponent (positive real part of λ) with modulation wave number ν exists precisely when $\Delta_L > 0$. As Δ_L changes sign the eigenvalues change from purely imaginary to a complex quartet as shown schematically in Figure 1. This type of stability transition is familiar from the theory of linear Hamiltonian systems and in that setting the collision and instability occurs since the eigenvalues have opposite Krein signature [11], and the nonlinear theory associated with that transition is called the Hamiltonian Hopf bifurcation. However, in the present case there is no obvious symplectic structure.

On the other hand there is a sign invariant that plays a similar role. It is related to Krein signature but more general in that even zero eigenvalues can have a signature. It is *the sign characteristic of Hermitian matrix pencils* [9]. The Hermitian pencil structure of (1.3) is evoked by multiplying the conservation of waves by $\mathscr{A}_\omega$, assuming $\mathscr{A}_\omega \neq 0$, and combining the two equations in (1.3) as

$$\begin{bmatrix} 0 & \mathscr{A}_\omega \\ \mathscr{A}_\omega & \mathscr{A}_k + \mathscr{B}_\omega \end{bmatrix} \begin{pmatrix} \Omega \\ q \end{pmatrix}_T + \begin{bmatrix} -\mathscr{A}_\omega & 0 \\ 0 & \mathscr{B}_k \end{bmatrix} \begin{pmatrix} \Omega \\ q \end{pmatrix}_X = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \,. \tag{1.7}$$

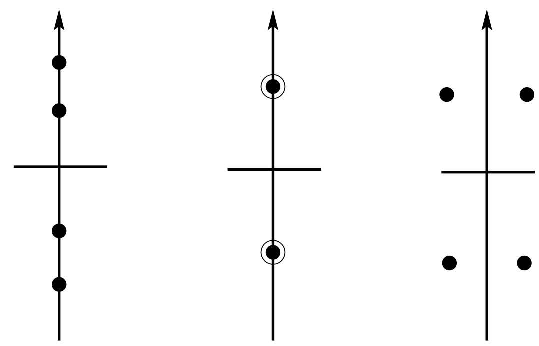

Figure 1: Collision of purely imaginary eigenvalues in the Whitham equations.

The two coefficient matrices are symmetric. Now the normal mode ansatz

$$\begin{pmatrix} \Omega \\ q \end{pmatrix} = \begin{pmatrix} \widehat{\Omega} \\ \widehat{q} \end{pmatrix} \mathrm{e}^{\mathrm{i}\alpha(x-ct)},$$

generates the following Hermitian matrix eigenvalue problem

$$\left(\begin{bmatrix} -\mathscr{A}_\omega & 0 \\ 0 & \mathscr{B}_k \end{bmatrix} + c \begin{bmatrix} 0 & \mathscr{A}_\omega \\ \mathscr{A}_\omega & \mathscr{A}_k + \mathscr{B}_\omega \end{bmatrix} \right) \begin{pmatrix} \widehat{\Omega} \\ \widehat{q} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \tag{1.8}$$

The generalization of this structure to the case of wavetrains with multiple phases (or multiphase wavetrains) is obtained by just replacing each entry with a matrix Jacobian,

$$\left[\begin{bmatrix} -\mathrm{D}_{\boldsymbol{\omega}}\mathbf{A} & 0 \\ 0 & \mathrm{D}_{\mathbf{k}}\mathbf{B} \end{bmatrix} + c \begin{bmatrix} 0 & \mathrm{D}_{\boldsymbol{\omega}}\mathbf{A} \\ \mathrm{D}_{\boldsymbol{\omega}}\mathbf{A} & \mathrm{D}_{\mathbf{k}}\mathbf{A} + \mathrm{D}_{\boldsymbol{\omega}}\mathbf{B} \end{bmatrix} \right] \begin{pmatrix} \widehat{\boldsymbol{\Omega}} \\ \widehat{\mathbf{q}} \end{pmatrix} = \begin{pmatrix} \mathbf{0} \\ \mathbf{0} \end{pmatrix}, \tag{1.9}$$

assuming that $\mathrm{D}_{\boldsymbol{\omega}}\mathbf{A}$ is invertible. For $N-$phase modulation the matrix entries are $N \times N$ and the full symmetric matrices are $2N \times 2N$. The derivation of the multiphase system is given in §4.

One of our main observations is that the linearization of the Whitham modulation equations, for any $N-$phase modulation, $N \geq 1$, leads to a Hermitian matrix pencil in the following form with the characteristics, c, as eigenvalues

$$[\mathbf{F} + c\mathbf{G}]\mathbf{v} = 0, \tag{1.10}$$

where $\mathbf{F}$ and $\mathbf{G}$ are $2N \times 2N$ matrices with $\mathbf{G}$ in general indefinite but invertible. This system is in standard form for a Hermitian matrix pencil relative to the indefinite metric $\mathbf{G}$ [9]. Eigenvalues (characteristics) are roots of the characteristic polynomial

$$\Delta(c) = \det[\mathbf{F} + c\mathbf{G}] = 0, \tag{1.11}$$

and every simple real eigenvalue $c = c_1$ has a sign characteristic

$$S(c_1) = \mathrm{sign}\big(\langle \mathbf{v}_1, \mathbf{G}\mathbf{v}_1 \rangle\big),$$

where $\mathbf{v}_1$ is the associated eigenvector, $[\mathbf{F} + c_1\mathbf{G}]\mathbf{v}_1 = 0$, and $\langle\cdot,\cdot\rangle$ is the standard inner product on $\mathbb{R}^{2N}$ The sign characteristic is invariant under congruence [9]. Although the naming "sign characteristic" emerged in the linear algebra literature (a history appears in [14]), the eigenvalues here are characteristics and so the name is quite appropriate!

One of the main observations of this paper is that every simple characteristic in the linearized Whitham modulation equations has a sign characteristic. This observation is connected to the theory of Hermitian matrix pencils to show that a necessary condition for two characteristics to coalesce and transition from real to complex is that they have opposite sign characteristic.

As in the case of Krein signature, the sign characteristic can be obtained graphically. When $\mathbf{G} = \mathtt{i}\mathbf{J}$ where $\mathtt{i}^2 = -1$ and $\mathbf{J}$ is a symplectic operator, the "graphical Krein signature" is defined by (cf. Kollar & Miller [13]) (see also Binding & Volkmer [2]). Here, taking $\mathbf{G}$ to be an arbitrary symmetric operator, the *graphical sign characteristic* is defined by embedding (1.10) in a classical symmetric eigenvalue problem

$$[\mathbf{F} + c\mathbf{G}]\mathbf{v} = \mu(c)\mathbf{v}\,, \tag{1.12}$$

with c now treated as a parameter. The idea is to draw a graph of the branches of $\mu(c)$ by solving

$$\det[\mathbf{F} + c\mathbf{G} - \mu(c)\mathbf{I}] = 0\,.$$

A schematic showing two branches of typical $\mu-$curves is shown in Figure 2. Real eigenvalues (characteristics) are obtained by intersection of a graph with the $c-$axis. At simple crossings the sign of the slope is the sign characteristic. Differentiating $\mu(c)$ at a simple zero eigenvalue, $c = c_1$, gives

$$\operatorname{sign}(\mu'(c_1)) = \operatorname{sign}\big(\langle\mathbf{v}_1, \mathbf{G}\mathbf{v}_1\rangle\big) = S(c_1)\,. \tag{1.13}$$

At double roots there is a new sign which is based on the sign of the curvature of the graph

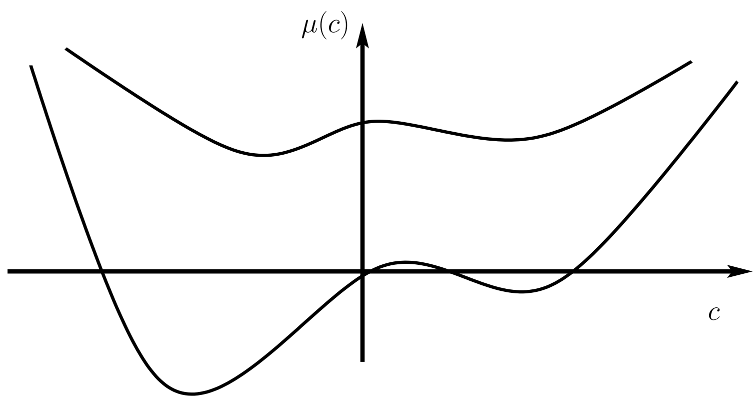

Figure 2: Schematic of two branches of graphical sign characteristic.

at that point, and this sign will appear in the normal form at coalescence (see §3).

The graphical sign characteristic [13] will be important in this paper in two ways. Firstly, in examples it is the most efficient way to identify the sign of each simple characteristic, and secondly at coalescing characteristics it provides a geometric characterization of the sign of the coalescence, expressed in terms of the curvature of the graph.

Another approach to studying the Hermitian matrix pencil that arises in Whitham modulation theory (1.9) is to convert it to a quadratic Hermitian matrix pencil,

$$\left[\mathrm{D}_{\boldsymbol{\omega}}\mathbf{A}\,c^2 + c(\mathrm{D}_{\mathbf{k}}\mathbf{A} + \mathrm{D}_{\boldsymbol{\omega}}\mathbf{B}) + \mathrm{D}_{\mathbf{k}}\mathbf{B}\right]\widehat{\mathbf{q}} = \mathbf{0}\,. \tag{1.14}$$

A parallel theory can then be developed for sign characteristic of quadratic Hermitian matrix pencils [5, 8, 12, 15, 17, 18]. The quadratic formulation turns out to be most efficient in applications and is used for the example in §5.

The authors interest in the classification and coalescence problem for characteristics stems from an interest in developing a generalization of Whitham modulation theory in the neighborhood of coalescing characteristics. The one-phase case is developed in [3] and in order to develop an analogue of this theory for the multiphase case, a theory for how and when coalescing characteristics arise and become unstable is required.

An outline of the paper is as follows. First the linear one-phase Whitham theory is given in detail as it is the paradigm of what follows even in the multi-phase case. An abstract normal form at coalescence is developed in §3 including the role of curvature of the graphical representation of the sign characteristic. The main setting is multiphase modulation in Whitham modulation theory. An abstract theory for group invariant Lagrangians is first developed, which leads to robust dispersionless vector-valued Whitham modulation equations. Linearizing these equations gives a family of Hermitian matrix pencils with the dimension dependent on the number of phases which can be arbitrary but finite. The theory is applied to coupled nonlinear Schrödinger (NLS) equations and graphical sign characteristic is shown to be the most efficient way to identify the sign characteristic of characteristics and identify coalescence points. Generalizations and extensions are discussed in the concluding remarks section.

2 Linear one-phase Whitham theory

The one-phase case is elementary, resulting in $\mathbf{F}$ and $\mathbf{G}$ being 2×2 symmetric matrices, but it provides a setting where the key features of coalescing characteristics can be seen explicitly. Thus, it is helpful to illustrate the theory in this setting before generalizing it to the $N-$ phase case.

The characteristics of linearized one-phase Whitham equations are eigenvalues of the Hermitian matrix pencil (1.10) with

$$\mathbf{F} = \begin{bmatrix} -\mathscr{A}_\omega & 0 \\ 0 & \mathscr{B}_k \end{bmatrix} \quad \text{and} \quad \mathbf{G} = \begin{bmatrix} 0 & \mathscr{A}_\omega \\ \mathscr{A}_\omega & \mathscr{A}_k + \mathscr{B}_\omega \end{bmatrix}. \tag{2.1}$$

The characteristics are invariant under *congruence transformation*. Two square matrices $\widetilde{\mathbf{F}}$ and $\mathbf{F}$ are said to be congruent if there exists a nonsingular matrix $\mathbf{T}$ such that

$$\widetilde{\mathbf{F}} = \mathbf{T}^T\mathbf{F}\mathbf{T}\,.$$

Congruence transformations preserve the *sign* of the eigenvalues of $\mathbf{F}$ but not the *value* of the eigenvalues. However, they preserve the eigenvalues of a Hermitian matrix pencil when the congruence transformation is applied to both $\mathbf{F}$ and $\mathbf{G}$ (cf. §4 of [14]).

Transform (1.10) and (2.1) using the congruence transformation

$$\mathbf{T} = \begin{pmatrix} \mathscr{A}_\omega^{-1} & -c_g \\ 0 & 1 \end{pmatrix}, \quad \text{with } c_g = \frac{\mathscr{A}_k}{\mathscr{A}_\omega},$$

assuming $\mathscr{A}_\omega \neq 0$. The matrices $\mathbf{F}$ and $\mathbf{G}$ become

$$\widetilde{\mathbf{F}} = \mathbf{T}^T\mathbf{F}\mathbf{T} = \begin{bmatrix} -\mathscr{A}_\omega^{-1} & c_g \\ c_g & \mathscr{A}_\omega^{-1}\Delta_L \end{bmatrix} \quad \text{and} \quad \widetilde{\mathbf{G}} = \mathbf{T}^T\mathbf{G}\mathbf{T} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \tag{2.2}$$

where Δ_L is the Lighthill determinant (1.6). Let $\mathbf{v} = \mathbf{T}\widetilde{\mathbf{v}}$, then the eigenvalue problem (1.10) with (2.1) is transformed to

$$\left(\begin{bmatrix} -\mathscr{A}_\omega^{-1} & c_g \\ c_g & \mathscr{A}_\omega^{-1}\Delta_L \end{bmatrix} + c \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \right) \begin{pmatrix} \tilde{v}_1 \\ \tilde{v}_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \tag{2.3}$$

with characteristics

$$c = -c_g \pm \mathscr{A}_\omega^{-1}\sqrt{-\Delta_L}\,,$$

when $\Delta_L \neq 0$, and eigenvectors

$$\widetilde{\mathbf{v}}^{\pm} = \begin{pmatrix} \mp\sqrt{-\Delta_L} \\ -1 \end{pmatrix}.$$

In the hyperbolic region ($\Delta_L < 0$) the sign characteristics are

$$S(c^{\pm}) = \operatorname{sign}\left(\langle \tilde{\mathbf{v}}^{\pm}, \tilde{\mathbf{G}}\tilde{\mathbf{v}}^{\pm} \rangle \right) = \pm 1\,.$$

It is precisely these two characteristics of opposite sign that coalesce at $\Delta_L = 0$ and become complex, generating instability and changing the type of the Whitham equations.

There is one other feature of interest. When the two characteristics coalesce the double characteristic has algebraic multiplicity two but geometric multiplicity one, even though both $\mathbf{F}$ and $\mathbf{G}$ are symmetric. This is another anomaly of the case of indefinite $\mathbf{G}$.

Firstly, the algebraic multiplicity two follows since

$$\Delta(c) = -\mathscr{A}_\omega^{-2}\Delta_L - (c + c_g)^2\,,$$

and so when $\Delta_L = 0$ and $c = -c_g$,

$$\Delta(-c_g) = \Delta'(-c_g) = 0 \quad \text{and} \quad \Delta''(-c_g) = -2 \neq 0\,.$$

Evaluating the Hermitian matrix pencil at the double characteristic,

$$\widetilde{\mathbf{F}} - c_g\widetilde{\mathbf{G}} = \begin{bmatrix} -\mathscr{A}_\omega^{-1} & 0 \\ 0 & 0 \end{bmatrix},$$

yielding only one geometric eigenvector

$$\widetilde{\mathbf{v}}^{(1)} = \begin{pmatrix} 0 \\ 1 \end{pmatrix},$$

and the second eigenvector is generalized

$$\left[\widetilde{\mathbf{F}} - c_g \widetilde{\mathbf{G}}\right] \widetilde{\mathbf{v}}^{(2)} = -\widetilde{\mathbf{G}} \widetilde{\mathbf{v}}^{(1)} = \begin{pmatrix} -1 \\ 0 \end{pmatrix} , \tag{2.4}$$

giving

$$\widetilde{\mathbf{v}}^{(2)} = \begin{pmatrix} \mathscr{A}_\omega \\ 0 \end{pmatrix} + a \, \widetilde{\mathbf{v}}^{(1)} , \quad a \in \mathbb{R} ,$$

with a arbitrary.

Although the discussion in this section was elementary, it contains the essential features of the general case of coalescence of two characteristics, and these are elaborated on the next section.

3 Canonical form near coalescence

Consider the general symmetric matrix pencil,

$$\left[\mathbf{F} + c \, \mathbf{G}\right] \mathbf{v} = 0 , \quad \mathbf{v} \in \mathbb{R}^n , \tag{3.5}$$

where $\mathbf{F}, \mathbf{G}$ are arbitrary real symmetric $n \times n$ matrices with $\mathbf{G}$ invertible. Associated with this pencil is the characteristic polynomial, $\Delta(c) = 0$, with

$$\Delta(c) = \det\left[\mathbf{F} + c\mathbf{G}\right] . \tag{3.6}$$

The focus of this section is on the local behaviour when $\mathbf{G}$ is indefinite and a pair of real eigenvalues c_1 and c_2 coalesce and become complex, clarifying the role of the sign characteristic and identifying the normal form at coalescence. Here attention will be restricted to the real symmetric case, although the theory is essentially the same in the complex Hermitian case.

A simple eigenvalue c_1, satisfying,

$$\Delta(c_1) = 0 \quad \text{but} \quad \Delta'(c_1) \neq 0 ,$$

with eigenvector $\mathbf{v}_1$,

$$\left[\mathbf{F} + c_1 \mathbf{G}\right] \mathbf{v}_1 = 0 , \tag{3.7}$$

has a sign characteristic

$$S(c_1) := \mathrm{sign}\big(\langle \mathbf{v}_1, \mathbf{G}\mathbf{v}_1 \rangle\big) .$$

When $\mathbf{G}$ is positive definite all of the signs are positive, and all the eigenvalues are real. To see the latter, let $\mathbf{G}^{1/2}$ be the positive square root of $\mathbf{G}$ and let $\mathbf{v} = \mathbf{G}^{1/2}\mathbf{w}$ in (3.5). Then it can be reduced to the standard symmetric eigenvalue problem

$$\left[\mathbf{G}^{-1/2}\mathbf{F}\mathbf{G}^{-1/2} + c \, \mathbf{I}\right] \mathbf{w} = 0 , \quad \mathbf{w} \in \mathbb{R}^n , \tag{3.8}$$

showing that all the eigenvalues are real (this argument is due to [22]).

Hence a necessary condition for complex eigenvalues to exist is indefinite $\mathbf{G}$. To see that indefinite $\mathbf{G}$ is sufficient, consider the canonical example

$$\mathbf{G} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \quad \text{and} \quad \mathbf{F} = \begin{bmatrix} a & b \\ b & \alpha \end{bmatrix}, \tag{3.9}$$

where a, b, α are arbitrary real numbers. Then the eigenvalues satisfying $\Delta(c) = 0$ are

$$c_1 = -b + \sqrt{a\alpha} \quad \text{and} \quad c_2 = -b - \sqrt{a\alpha}\,,$$

with $S(c_1) = -1$ and $S(c_2) = +1$ as long as $a\alpha > 0$. Fixing $a > 0$ and letting α pass through zero, from positive to negative values, it is clear that two eigenvalues with opposite sign characteristic collide and become complex.

Now look more closely at the structure of the matrix when two characteristics are at exact coalescence. Here a normal form is derived at the coalescence. It is a special case of the abstract theory presented in Theorem 9.2 of Lancaster & Rodman [14]. The difference here is that the normal form is derived explicitly, highlighting the secondary sign and the connection with the canonical form obtained in the one-phase case in §2.

Starting with the general form (3.5), suppose two eigenvalues with opposite sign coalesce at $c = c_* \neq 0$. The algebraic conditions on the characteristic polynomial (3.6) are

$$\Delta(c_*) = \Delta'(c_*) = 0 \quad \text{and} \quad \Delta''(c_*) \neq 0\,. \tag{3.10}$$

The algebraic multiplicity is two, but the geometric multiplicity is one, generating a Jordan chain

$$\begin{aligned} \big[\mathbf{F} + c_*\mathbf{G}\big]\mathbf{v}^{(1)} &= 0 \\ \big[\mathbf{F} + c_*\mathbf{G}\big]\mathbf{v}^{(2)} &= -\mathbf{G}\mathbf{v}^{(1)}\,. \end{aligned} \tag{3.11}$$

The chain terminates at two when

$$\big[\mathbf{F} + c_*\mathbf{G}\big]\mathbf{v}^{(3)} = -\mathbf{G}\mathbf{v}^{(2)}\,,$$

is not solvable; that is

$$\sigma := \langle \mathbf{v}^{(1)}, \mathbf{G}\mathbf{v}^{(2)} \rangle \neq 0\,.$$

For a system of the form (3.5) with such a double eigenvalue we have the following normal form theorem.

Theorem. *Suppose the system (3.5) has a double non-zero eigenvalue of algebraic multiplicity two and geometric multiplicity one. Then, projection onto the* $\mathrm{span}\{\mathbf{v}^{(1)}, \mathbf{v}^{(2)}\}$ *takes the matrix pencil* $\mathbf{F} + c\mathbf{G}$ *to the canonical form*

$$\left[\begin{pmatrix} s & c_* \\ c_* & 0 \end{pmatrix} - c\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\right], \tag{3.12}$$

where $s = -\mathrm{sign}(\sigma)$.

Remarks.

- The normal form (3.12) appears as part of the general theory in Theorem 9.2 of [14]. In the second row of equation (9.4) in [14] the normal form (3.12) is represented as the term $\eta_j\big((\lambda+\alpha_j)F_{\ell_j}+G_{\ell_j}\big)$ where η_j is the sign characteristic and α_j represents c_*.
- The sign s is invariant under congruent transformations that preserve the form of the matrix in (3.12).

Proof. First establish the orthogonality properties of the basis. Solvability of the second equation of (3.11) requires $\mathbf{G}\mathbf{v}^{(1)}$ to be in the range of $[\mathbf{F}+c_*\mathbf{G}]$ giving the condition

$$\langle \mathbf{v}^{(1)}, \mathbf{G}\mathbf{v}^{(1)}\rangle = 0\,, \tag{3.13}$$

and as a consequence from (3.11),

$$\langle \mathbf{v}^{(1)}, \mathbf{F}\mathbf{v}^{(1)}\rangle = 0\,. \tag{3.14}$$

Let $\widetilde{\mathbf{v}}^{(2)}$ be any fixed solution of the second equation in (3.11) and define

$$\mathbf{v}^{(2)} = \widetilde{\mathbf{v}}^{(2)} + b\mathbf{v}^{(1)}\,.$$

Then choose b (e.g. $b = -\frac{1}{2}\langle\widetilde{\mathbf{v}}^{(2)}, \mathbf{G}\widetilde{\mathbf{v}}^{(2)}\rangle/\langle\mathbf{v}^{(1)}, \mathbf{G}\widetilde{\mathbf{v}}^{(2)}\rangle$) so that the generalized eigenvector satisfies,

$$\langle \mathbf{v}^{(2)}, \mathbf{G}\mathbf{v}^{(2)}\rangle = 0\,. \tag{3.15}$$

This consequentially gives, from (3.11), that

$$\langle \mathbf{v}^{(2)}, \mathbf{F}\mathbf{v}^{(2)}\rangle = -\langle \mathbf{v}^{(2)}, \mathbf{G}\mathbf{v}^{(1)}\rangle = -\sigma\,. \tag{3.16}$$

The strategy now is just to project onto $\mathrm{span}\{\mathbf{v}^{(1)}, \mathbf{v}^{(2)}\}$. The form obtained is not unique, and here the basis is scaled so that the form in (3.12) is obtained which contains all the essential features in a tidy way. It is slightly different from the normal form in [14] and is constructed to resonate with the canonical form (2.3) of the one-phase construction in Whitham modulation theory.

Consider the transformation matrix

$$\mathbf{V} = \big[h_1\mathbf{v}^{(1)}\,\big|\,h_2\mathbf{v}^{(2)}\big] \in \mathbb{R}^{n\times 2}\,.$$

The simplest case is to just take $h_1 = h_2 = 1$ but we will tweak them to find an optimal normal form. Acting on $\mathbf{F}, \mathbf{G}$ on the left and right gives

$$\begin{aligned}
\mathbf{V}^T\mathbf{F}\mathbf{V} &= \begin{bmatrix} h_1^2\langle \mathbf{v}^{(1)}, \mathbf{F}\mathbf{v}^{(1)}\rangle & h_1h_2\langle \mathbf{v}^{(1)}, \mathbf{F}\mathbf{v}^{(2)}\rangle \\ h_2h_1\langle \mathbf{v}^{(2)}, \mathbf{F}\mathbf{v}^{(1)}\rangle & h_2^2\langle \mathbf{v}^{(2)}, \mathbf{F}\mathbf{v}^{(2)}\rangle \end{bmatrix} \\
\mathbf{V}^T\mathbf{G}\mathbf{V} &= \begin{bmatrix} h_1^2\langle \mathbf{v}^{(1)}, \mathbf{G}\mathbf{v}^{(1)}\rangle & h_1h_2\langle \mathbf{v}^{(1)}, \mathbf{G}\mathbf{v}^{(2)}\rangle \\ h_2h_1\langle \mathbf{v}^{(2)}, \mathbf{G}\mathbf{v}^{(1)}\rangle & h_2^2\langle \mathbf{v}^{(2)}, \mathbf{G}\mathbf{v}^{(2)}\rangle \end{bmatrix}\,.
\end{aligned}$$

With the orthogonality properties (3.13)-(3.15) the latter expression reduces to

$$\mathbf{V}^T\mathbf{G}\mathbf{V} = h_1h_2\sigma\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\,.$$

Now use the equations (3.14) and (3.16) to simplify the first expression,

$$\mathbf{V}^T\mathbf{F}\mathbf{V} = \begin{bmatrix} 0 & -c_* h_1 h_2 \sigma \\ -c_* h_1 h_2 \sigma & -\sigma h_2^2 \end{bmatrix},$$

where c_* is the distinguished value of c introduced in (3.10). Taking $h_1 h_2 \sigma = -1$ and $|\sigma| h_2^2 = 1$ reduces it to the canonical form

$$\left[\begin{pmatrix} 0 & c_* \\ c_* & s \end{pmatrix} - c\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\right].$$

The form (3.12) is obtained by permuting the rows and columns, which is a congruence transformation. ■

It is reasonable to conjecture that the versal unfolding of the normal form (3.12) in the space of matrices that preserve the canonical $\mathbf{G}$ in (3.12) is

$$\left[\begin{pmatrix} s & c_* \\ c_* & \varepsilon \end{pmatrix} - c\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\right].$$

modulo a shift in the value of c_*, where $\varepsilon \in \mathbb{R}$ is a small parameter, giving a characteristic polynomial of

$$c = c_* \pm \sqrt{s\varepsilon}.$$

However a proof of the unfolding requires the theory of versal deformation of matrices (e.g. [7, 10, 16]) and is outside the scope of this paper.

3.1 The sign s and curvature

In terms of graphical sign characteristic the secondary sign s at the coalescence is the curvature of the graph. To establish this, start with the eigenvalue problem for the graph $\mu(c)$ in (1.12), and differentiate with respect to c

$$\big[\mathbf{F} + c\mathbf{G}\big]\dot{\mathbf{v}} + \mathbf{G}\mathbf{v} = \dot{\mu}\mathbf{v} + \mu\dot{\mathbf{v}}, \tag{3.17}$$

with the dot representing differentiation with respect to c here. Evaluating (3.17) at a simple eigenvalue c_1 (for then $\mu(c_1) = 0$) with eigenvector $\mathbf{v}^{(1)}$,

$$\big[\mathbf{F} + c_1\mathbf{G}\big]\dot{\mathbf{v}} + \mathbf{G}\mathbf{v}^{(1)} = \dot{\mu}\mathbf{v}^{(1)},$$

and solvability gives

$$\dot{\mu} = \frac{\langle \mathbf{G}\mathbf{v}^{(1)}, \mathbf{v}^{(1)} \rangle}{\|\mathbf{v}^{(1)}\|^2}; \tag{3.18}$$

that is, the sign of $\dot{\mu}$ is the sign characteristic, confirming (1.13) in the introduction.

Now suppose the eigenvalue is double with algebraic multiplicity two and geometric multiplicity one as above. Then $\dot{\mu} = 0$ and (3.17) reduces to

$$\big[\mathbf{F} + c_*\mathbf{G}\big]\dot{\mathbf{v}} + \mathbf{G}\mathbf{v}^{(1)} = 0,$$

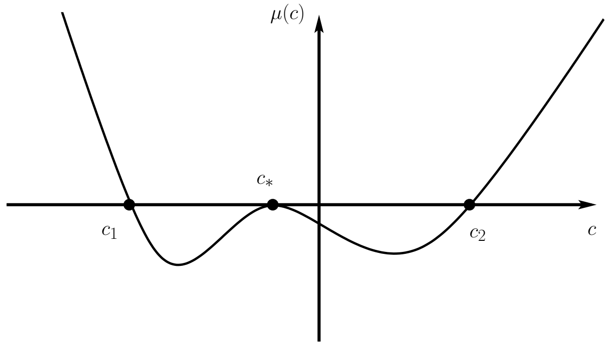

Figure 3: Schematic of graphical sign characteristic showing generic points (c_1 and c_2) and a point of coalescence (c_*), illustrating the sign of the curvature of the graph at coalescence.

giving a solution (using (3.11)) of $\dot{\mathbf{v}} = \mathbf{v}^{(2)} + \ell\mathbf{v}^{(1)}$ where $\ell \in \mathbb{R}$ is arbitrary.

Differentiate (3.17) again

$$\big[\mathbf{F} + c\mathbf{G}\big]\ddot{\mathbf{v}} + 2\mathbf{G}\dot{\mathbf{v}} = \ddot{\mu}\mathbf{v} + 2\dot{\mu}\dot{\mathbf{v}} + \mu\ddot{\mathbf{v}}\,. \tag{3.19}$$

Evaluate at $c = c_*$ noting that $\mu = \dot{\mu} = 0$,

$$\big[\mathbf{F} + c_*\mathbf{G}\big]\ddot{\mathbf{v}} = -2\mathbf{G}\dot{\mathbf{v}} + \ddot{\mu}\mathbf{v}^{(1)} = -2\mathbf{G}\big(\mathbf{v}^{(2)} + \ell\mathbf{v}^{(1)}\big) + \ddot{\mu}\mathbf{v}^{(1)}\,. \tag{3.20}$$

Applying solvability,

$$\ddot{\mu} = 2\frac{\langle \mathbf{G}\mathbf{v}^{(2)}, \mathbf{v}^{(1)}\rangle}{\|\mathbf{v}^{(1)}\|^2}\,, \tag{3.21}$$

giving that the sign of $\ddot{\mu}$ is $-s$ in the normal form, hence a kind of secondary characteristic sign. In terms of the graph $(c, \mu(c))$ it is the sign of the curvature at the coalescence. To see this recall the formula for plane curvature of a graph

$$\kappa := \frac{\ddot{\mu}(c)}{(1 + \dot{\mu}^2)^{3/2}}\,.$$

But at the point of coalescence $\dot{\mu}(c) = 0$ and so (3.21) is precisely the plane curvature of the graph $(c, \mu(c))$ at the point of coalescence.

A schematic of the graphical sign characteristic is shown in Figure 3. At simple crossings (giving simple roots of $\Delta(c) = 0$) the sign of the slope is the sign characteristic, and at double roots the sign s gives minus the sign of the curvature of the graph at that point.

Another anomaly of the case where $\mathbf{G}$ is indefinite is that the codimension for coalescence is one (only one parameter needs to be varied), and the generic collision leads to complex eigenvalues. This is to be contrasted with the case of positive definite $\mathbf{G}$ where the codimension for coalescence is two and the eigenvalues after collision remain real. This latter behaviour can be illustrated by taking $\mathbf{F}$ to be in the form (3.9) but with $\mathbf{G}$ the identity. Then the eigenvalues are

$$2c = -\big(a + \alpha\big) \pm \sqrt{(a - \alpha)^2 + 4b^2}\,,$$

and so *two* conditions, $a = \alpha$ and $b = 0$, are required for coalescence, and the double eigenvalue is semisimple.

4 Modulation of multiphase wave trains

In this section the Whitham modulation theory for multiphase wavetrains is set up in such a way that the linearization of the equations results in a Hermitian matrix pencil of the form (1.9). The Whitham modulation equations for multiphase wavetrains were first derived and studied by Ablowitz & Benney [1], although if potential variables are included as additional phases, Whitham [23] uses multiphase modulation for water waves; see also [24, §14.7], where he refers to the additional phases as "pseudo-phases". Ablowitz and Benney derived the conservation of wave action for scalar fields with two phases in detail, and showed how the theory generalized to $N-$phases. The theory in [1] shows that in general one should expect small divisors, but weakly nonlinear solutions could still be obtained. However for integrable systems, multiphase averaging and the Whitham equations are robust and rigorous, and a general theory can be obtained (e.g. Flashka et al. [6]). On the other hand if the system is not integrable, but there is an $N-$fold symmetry, then again a theory for conservation of wave action can be developed without small divisors and smoothly varying $N-$phase wavetrains (e.g. Ratliff [19, 20]). In essence the conservation of wave action is replaced by the conservation law generated by the symmetry.

In this section a general class of toral-invariant Lagrangians is introduced where the multiphase wavetrains are relative equilibria. Modulation of these wavetrains then leads to a class of multiphase Whitham modulation equations. So for $(x,t) \in [x_1,x_2] \times [t_1,t_2]$, let $\mathbf{u}(x,t)$ be an $\mathbb{R}^n-$valued field governed by the Euler-Lagrange equation associated with the abstract Lagrangian

$$\mathcal{L}(\mathbf{u}) = \int_{t_1}^{t_2} \int_{x_1}^{x_2} L(\mathbf{u}_t, \mathbf{u}_x, \mathbf{u}) \, \mathrm{d}x\mathrm{d}t \,,$$

with Euler-Lagrange equation $\mathcal{E}(\mathbf{u}) = 0$ where

$$\mathcal{E}(\mathbf{u}) := \frac{\partial}{\partial t}\left(\frac{\delta L}{\delta \mathbf{u}_t}\right) + \frac{\partial}{\partial x}\left(\frac{\delta L}{\delta \mathbf{u}_x}\right) - \frac{\delta L}{\delta \mathbf{u}} \,.$$

Now suppose that the Lagrangian is invariant with respect to the action of a compact Lie group. For simplicity take this Lie group to be the abelian group $\mathbb{T}^N := S^1 \times \cdots \times S^1$, the $N-$torus, with matrix representation G_θ (an $n \times n$ orthogonal matrix) with $\theta = (\theta_1, \ldots, \theta_N)$, and infinitesimal generators g_j,

$$\mathrm{g}_j(\mathbf{u}) := \frac{\partial}{\partial \theta_j} G_\theta \mathbf{u}\bigg|_{\theta=0} \,, \quad j = 1, \ldots, N \,.$$

Invariance of the Lagrangian requires

$$L(G_\theta \mathbf{u}_t, G_\theta \mathbf{u}_x, G_\theta \mathbf{u}) = L(\mathbf{u}_t, \mathbf{u}_x, \mathbf{u}) \quad \forall\, \theta \in \mathbb{T}^N \,. \tag{4.1}$$

Because of the simple nature of the group action a proof of Noether's theorem is elementary. Differentiate (4.1) with respect to θ_j, set $\theta = 0$, integrate over (x,t), and use $\mathcal{E}(\mathbf{u}) = 0$, giving

$$\frac{\partial}{\partial t} A_j(\mathbf{u}) + \frac{\partial}{\partial x} B_j(\mathbf{u}) = 0 \,, \quad j = 1, \ldots, k \,,$$

with components

$$A_j(\mathbf{u}) := \left\langle \frac{\delta L}{\delta \mathbf{u}_t}, \mathrm{g}_j(\mathbf{u}) \right\rangle , \quad B_j(\mathbf{u}) := \left\langle \frac{\delta L}{\delta \mathbf{u}_x}, \mathrm{g}_j(\mathbf{u}) \right\rangle = 0 , \quad j = 1, \ldots, N . \tag{4.2}$$

It is these components when evaluated on a basic state and averaged which will appear in the Whitham modulation theory. The inner product in (4.2) is the standard inner product on $\mathbb{R}^n$ without integration as the averaging trivializes in the case where the multiphase solutions are associated with a group. The definition of A_j is actually

$$A_j(\mathbf{u}) := \overline{\left\langle G_\theta \frac{\delta L}{\delta \mathbf{u}_t}, G_\theta \mathrm{g}_j(\mathbf{u}) \right\rangle}^{\mathbb{T}^N} , \quad j = 1, \ldots, N ,$$

but $\mathbb{T}^N-$invariance of the inner product reduces this expression to the standard inner product on $\mathbb{R}^n$ (noting that the fields $\mathbf{u}(x,t)$ lie in $\mathbb{R}^n$ whereas the symmetry group is $\mathbb{T}^N$). Similar argument for B_j, $j = 1, \ldots, N$.

Systems of the above form have a general class of solutions – relative equilibria – which are aligned with the group orbit. These have the form

$$\mathbf{u}(x,t) = \widehat{\mathbf{u}}(\theta, \boldsymbol{\omega}, \mathbf{k}) := G_{\theta(x,t)} \widehat{\mathbf{a}}(\boldsymbol{\omega}, \mathbf{k}) ,$$

with $\theta_j = k_j x + \omega_j t + \theta_j^{(0)}$, $j = 1, \ldots, N$. A relation between $\boldsymbol{\omega}, \mathbf{k}$ and amplitude $\widehat{\mathbf{a}}$ is obtained by substituting into $\mathcal{E}(\widehat{\mathbf{u}}) = 0$. An example is given in the next section. Substitution of this family of relative equilibria into the components of the conservation laws and averaging over θ gives

$$\mathbf{A}(\boldsymbol{\omega}, \mathbf{k}) = \begin{pmatrix} \mathscr{A}_1(\boldsymbol{\omega}, \mathbf{k}) \\ \vdots \\ \mathscr{A}_N(\boldsymbol{\omega}, \mathbf{k}) \end{pmatrix} , \quad \mathbf{B}(\boldsymbol{\omega}, \mathbf{k}) = \begin{pmatrix} \mathscr{B}_1(\boldsymbol{\omega}, \mathbf{k}) \\ \vdots \\ \mathscr{B}_N(\boldsymbol{\omega}, \mathbf{k}) \end{pmatrix} .$$

Roman letters are used for the components of the conservation laws A_j, B_j when they are considered as functions of (x,t). After averaging, they are expressed in terms of bold vectors $\mathbf{A}, \mathbf{B}$ with calligraphic entries.

Now introduce *phase modulation*: postulate a solution of $\mathcal{E}(\mathbf{u}) = 0$ with $\mathbf{u}$ of the form

$$\mathbf{u}(x,t) = \widehat{\mathbf{u}}(\boldsymbol{\theta} + \varepsilon^{-1}\boldsymbol{\phi}, \boldsymbol{\omega} + \boldsymbol{\Omega}, \mathbf{k} + \mathbf{q}) + \varepsilon \boldsymbol{\omega}(\boldsymbol{\theta} + \varepsilon^{-1}\boldsymbol{\phi}, X, T, \varepsilon) ,$$

with $X = \varepsilon x$ and $T = \varepsilon t$, with $\boldsymbol{\phi}$, $\boldsymbol{\Omega}$, and $\mathbf{q}$ functions of X, T, ε. Substitution of this ansatz into $\mathcal{E}(\mathbf{u}) = 0$ and setting to zero the terms of order ε^1 gives the WMEs. This can be done by direct calculation (e.g.[4, Chapter 6] for the case without symmetry and scalar-valued $\mathbf{u}$), but a simpler and more elegant approach is to first multisymplectify the Lagrangian density and use multisymplectic Noether theory (see §3 of Ratliff [20]).

Either way, at first order the solvability condition requires that $\boldsymbol{\Omega}$ and $\mathbf{q}$ satisfy the $N-$phase Whitham modulation equations,

$$\mathbf{q}_T = \boldsymbol{\Omega}_X \quad \text{and} \quad \frac{\partial}{\partial T}\mathbf{A}(\boldsymbol{\omega} + \boldsymbol{\Omega}, \mathbf{k} + \mathbf{q}) + \frac{\partial}{\partial X}\mathbf{B}(\boldsymbol{\omega} + \boldsymbol{\Omega}, \mathbf{k} + \mathbf{q}) = 0 , \tag{4.3}$$

where $\boldsymbol{\omega} \in \mathbb{R}^N$ and $\mathbf{k} \in \mathbb{R}^N$ are given constants representative of the basic state. The mappings $\mathbf{A}$ and $\mathbf{B}$ are variational,

$$\mathbf{A}(\boldsymbol{\omega} + \boldsymbol{\Omega}, \mathbf{k} + \mathbf{q}) = \mathrm{D}_{\boldsymbol{\omega}} \mathscr{L}(\boldsymbol{\omega} + \boldsymbol{\Omega}, \mathbf{k} + \mathbf{q})\,,$$

and

$$\mathbf{B}(\boldsymbol{\omega} + \boldsymbol{\Omega}, \mathbf{k} + \mathbf{q}) = \mathrm{D}_{\mathbf{k}} \mathscr{L}(\boldsymbol{\omega} + \boldsymbol{\Omega}, \mathbf{k} + \mathbf{q})\,.$$

Given a smooth function $\mathscr{L}$, the pair of equations (4.3) is a closed first-order system of PDEs for $\boldsymbol{\Omega}$ and $\mathbf{q}$. An example is given in the next section.

The interest here is in the characteristics of the system (4.3), linearized about the basic state represented by $(\boldsymbol{\omega}, \mathbf{k})$,

$$\mathbf{q}_T = \boldsymbol{\Omega}_X \quad \text{and} \quad \mathrm{D}_{\boldsymbol{\omega}}\mathbf{A}\,\Omega_T + \mathrm{D}_{\mathbf{k}}\mathbf{A}\,\mathbf{q}_T + \mathrm{D}_{\boldsymbol{\omega}}\mathbf{B}\,\Omega_X + \mathrm{D}_{\mathbf{k}}\mathbf{B}\,\mathbf{q}_X = 0\,, \tag{4.4}$$

where

$$\mathrm{D}_{\boldsymbol{\omega}}\mathbf{A}\,\boldsymbol{\xi} := \frac{d}{dr}\mathbf{A}(\boldsymbol{\omega} + r\boldsymbol{\xi}, \mathbf{k})\bigg|_{r=0}\,, \quad \text{for any } \boldsymbol{\xi} \in \mathbb{R}^N\,,$$

with similar expressions for the other terms. Multiplying the first equation of (4.4) by $\mathrm{D}_{\boldsymbol{\omega}}\mathbf{A}$, the two equations can be combined into

$$\begin{bmatrix} -\mathrm{D}_{\boldsymbol{\omega}}\mathbf{A} & 0 \\ 0 & \mathrm{D}_{\mathbf{k}}\mathbf{B} \end{bmatrix} \begin{pmatrix} \boldsymbol{\Omega} \\ \mathbf{q} \end{pmatrix}_T + \begin{bmatrix} 0 & \mathrm{D}_{\boldsymbol{\omega}}\mathbf{A} \\ \mathrm{D}_{\boldsymbol{\omega}}\mathbf{A} & \mathrm{D}_{\mathbf{k}}\mathbf{A} + \mathrm{D}_{\boldsymbol{\omega}}\mathbf{B} \end{bmatrix} \begin{pmatrix} \boldsymbol{\Omega} \\ \mathbf{q} \end{pmatrix}_X = \begin{pmatrix} \mathbf{0} \\ \mathbf{0} \end{pmatrix}\,. \tag{4.5}$$

With the normal mode ansatz

$$\boldsymbol{\Omega} = \widehat{\boldsymbol{\Omega}}\mathrm{e}^{\mathrm{i}k(x+ct)} \quad \text{and} \quad \mathbf{q} = \widehat{\mathbf{q}}\mathrm{e}^{\mathrm{i}k(x+ct)}\,,$$

the result is the Hermitian matrix pencil on $\mathbb{R}^{2N}$,

$$\left(\begin{bmatrix} -\mathrm{D}_{\boldsymbol{\omega}}\mathbf{A} & 0 \\ 0 & \mathrm{D}_{\mathbf{k}}\mathbf{B} \end{bmatrix} + c \begin{bmatrix} 0 & \mathrm{D}_{\boldsymbol{\omega}}\mathbf{A} \\ \mathrm{D}_{\boldsymbol{\omega}}\mathbf{A} & \mathrm{D}_{\mathbf{k}}\mathbf{A} + \mathrm{D}_{\boldsymbol{\omega}}\mathbf{B} \end{bmatrix} \right) \begin{pmatrix} \widehat{\boldsymbol{\Omega}} \\ \widehat{\mathbf{q}} \end{pmatrix} = \begin{pmatrix} \mathbf{0} \\ \mathbf{0} \end{pmatrix}\,. \tag{4.6}$$

This Hermitian matrix pencil can be analyzed directly on $\mathbb{R}^{2N}$, or it can be reduced to a quadratic formulation. The first equation of (4.6) gives

$$\widehat{\boldsymbol{\Omega}} = c\widehat{\mathbf{q}}\,,$$

and substitution into the second equation reduces it to the following quadratic Hermitian matrix pencil on $\mathbb{R}^N$,

$$\mathbf{E}(c)\widehat{\mathbf{q}} = 0\,, \tag{4.7}$$

with

$$\mathbf{E}(c) := \mathrm{D}_{\boldsymbol{\omega}}\mathbf{A}\,c^2 + \big(\mathrm{D}_{\mathbf{k}}\mathbf{A} + \mathrm{D}_{\boldsymbol{\omega}}\mathbf{B}\big)\,c + \mathrm{D}_{\mathbf{k}}\mathbf{B}\,. \tag{4.8}$$

The characteristic polynomial is

$$\Delta(c) = \det\left[\mathbf{E}(c)\right]\,. \tag{4.9}$$

Now suppose c_1 is a simple real characteristic root

$$\Delta(c_1) = 0 \quad \text{and} \quad \Delta'(c_1) \neq 0\,.$$

The sign characteristic of c_1 is defined by [8, 15, 17],

$$S(c_1) = \mathrm{sign}\left(\langle \mathbf{v}_1, \mathbf{E}'(c_1)\mathbf{v}_1\rangle\right) \,,$$

where $\mathbf{v}_1 \in \mathbb{R}^N$ is the associated eigenvector, $\mathbf{E}(c_1)\mathbf{v}_1 = 0$. It is proved in [15] that the sign characteristic of roots of (4.9) based on the quadratic representation (4.8) is equal to the definition obtained by transforming (4.8) to a linear matrix pencil as in (4.6) when $\mathrm{D}_{\boldsymbol{\omega}}\mathbf{A}$ is nonsingular.

We are now in a position to discuss the sign characteristic for multiphase wavetrains. The main idea is to once again introduce a new parameter μ and embed the characteristic problem into

$$\det\left[\mathbf{E}(c) - \mu(c)\mathbf{I}\right] = 0\,.$$

This form is slightly different from (1.12) in that $\mathbf{E}(c)$ is a quadratic Hermitian matrix pencil, but the theory of graphical sign characteristic goes through as in the case where $\mathbf{E}(c)$ depends linearly on c. The determinant of this problem leads to a characteristic problem in μ which may be solved, with coalescing characteristics occurring whenever there is a double root in μ. In the simplest multiphase case, where there are only two phases present, $\mathbf{E}(c)$ is quadratic in c and a 2×2 matrix, writing out the determinant gives a polynomial that is quadratic in μ and quartic in c,

$$\mu^2 - \mathrm{Tr}(\mathbf{E}(c))\mu + \det[\mathbf{E}(c)] = 0\,, \tag{4.10}$$

where $\det[\mathbf{E}(c)]$ is given in (5.9). The solution set consists of two curves in the $(c, \mu(c))$ plane. A schematic is shown in Figure 2 in the introduction.

5 Coalescing characteristics in coupled NLS equations

Coupled NLS (CNLS) equations are a canonical example of a PDE generated by a Lagrangian with a toral symmetry. CNLS will be used here to show a simple but nontrivial example of coalescing characteristics in Whitham modulation theory.

Any number of NLS equations can be coupled together, but here the case of two is considered where the symmetry group is $\mathbb{T}^2 = S^1 \times S^1$,

$$\begin{array}{rcl} \mathrm{i}\dfrac{\partial \Psi_1}{\partial t} + \alpha_1 \dfrac{\partial^2 \Psi_1}{\partial x^2} + (\beta_{11}|\Psi_1|^2 + \beta_{12}|\Psi_2|^2)\Psi_1 &=& 0 \\ \mathrm{i}\dfrac{\partial \Psi_2}{\partial t} + \alpha_2 \dfrac{\partial^2 \Psi_2}{\partial x^2} + (\beta_{21}|\Psi_1|^2 + \beta_{22}|\Psi_2|^2)\Psi_2 &=& 0\,, \end{array} \tag{5.1}$$

where the coefficients α_j, β_{ij}, $i, j = 1, 2$, are given real constants, with $\beta_{21} = \beta_{12}$. This system is the Euler-Lagrange equation for

$$\mathcal{L}(\Psi) = \int_{t_1}^{t_2}\int_{x_1}^{x_2} L(\Psi_t, \Psi_x, \Psi)\,\mathrm{d}x\mathrm{d}t\,,$$

with $\Psi := (\Psi_1, \Psi_2)$ and

$$\begin{array}{rcl} L &=& \dfrac{\mathrm{i}}{2}\left(\overline{\Psi}_1(\Psi_1)_t - \Psi_1(\overline{\Psi}_1)_t\right) + \dfrac{\mathrm{i}}{2}\left(\overline{\Psi}_2(\Psi_2)_t - \Psi_2(\overline{\Psi}_2)_t\right) \\ && -\alpha_1\left|(\Psi_1)_x\right|^2 - \alpha_2\left|(\Psi_2)_x\right|^2 + \tfrac{1}{2}\beta_{11}|\Psi_1|^4 + \beta_{12}|\Psi_1|^2|\Psi_2|^2 + \tfrac{1}{2}\beta_{22}|\Psi_2|^4\,, \end{array}$$

with the overline indicating complex conjugate. The toral symmetry follows from the fact that $(\mathrm{e}^{\mathrm{i}\theta_1}\Psi_1, \mathrm{e}^{\mathrm{i}\theta_2}\Psi_2)$ is a solution of (5.1), for any $(\theta_1, \theta_2) \in S^1 \times S^1$, when (Ψ_1, Ψ_2) is a solution. The complex coordinates can be converted to real coordinates, generating a standard action of $\mathbb{T}^2$ but will not be needed as the necessary calculations can be done in the complex setting.

Noether's theorem gives the conservation laws

$$(A_j)_t + (B_j)_x = 0\,, \quad j = 1, 2\,, \tag{5.2}$$

with

$$A_j = \tfrac{1}{2}|\Psi_j|^2 \quad \text{and} \quad B_j = \alpha_1 \mathrm{Im}(\overline{\Psi}_1(\Psi_1)_x)\,, \quad j = 1, 2\,. \tag{5.3}$$

The basic state, a family of relative equilibria associated with the $\mathbb{T}^2$ symmetry, has the form

$$\Psi_j(x,t) = \Psi_j^0(\boldsymbol{\omega}, \mathbf{k})\mathrm{e}^{\mathrm{i}\theta_j(x,t)}\,, \quad \theta_j(x,t) = k_j x + \omega_j t + \theta_j^0\,, \quad j = 1, 2\,.$$

Substitution into the governing equations (5.1) generates the required relationship between the amplitudes and the frequencies and wavenumbers,

$$\begin{bmatrix} \beta_{11} & \beta_{12} \\ \beta_{12} & \beta_{22} \end{bmatrix} \begin{pmatrix} |\Psi_1^0|^2 \\ |\Psi_2^0|^2 \end{pmatrix} = \begin{pmatrix} \omega_1 + \alpha_1 k_1^2 \\ \omega_2 + \alpha_2 k_2^2 \end{pmatrix}\,.$$

or

$$\begin{aligned} |\Psi_1^0|^2 &= \frac{1}{\beta}\Big(\beta_{22}(\omega_1 + \alpha_1 k_1^2) - \beta_{12}(\omega_2 + \alpha_2 k_2^2)\Big) \\ |\Psi_2^0|^2 &= \frac{1}{\beta}\Big(\beta_{11}(\omega_2 + \alpha_2 k_2^2) - \beta_{21}(\omega_1 + \alpha_1 k_1^2)\Big)\,, \end{aligned} \tag{5.4}$$

with $\beta = \beta_{11}\beta_{22} - \beta_{12}^2 \neq 0$.

The key vectors $\mathbf{A}(\boldsymbol{\omega}, \mathbf{k})$ and $\mathbf{B}(\boldsymbol{\omega}, \mathbf{k})$ needed for analysis of the linearization, are obtained by substituting (5.4) into the components of the conservation law (5.3). They are

$$\mathbf{A}(\boldsymbol{\omega}, \mathbf{k}) = \begin{pmatrix} \mathscr{A}_1(\boldsymbol{\omega}, \mathbf{k}) \\ \mathscr{A}_2(\boldsymbol{\omega}, \mathbf{k}) \end{pmatrix} \quad \text{and} \quad \mathbf{B}(\boldsymbol{\omega}, \mathbf{k}) = \begin{pmatrix} \mathscr{B}_1(\boldsymbol{\omega}, \mathbf{k}) \\ \mathscr{B}_2(\boldsymbol{\omega}, \mathbf{k}) \end{pmatrix}\,,$$

with

$$\begin{aligned} \mathscr{A}_1(\boldsymbol{\omega}, \mathbf{k}) &= \frac{1}{2\beta}\Big(\beta_{22}(\omega_1 + \alpha_1 k_1^2) - \beta_{12}(\omega_2 + \alpha_2 k_2^2)\Big) \\ \mathscr{A}_2(\boldsymbol{\omega}, \mathbf{k}) &= \frac{1}{2\beta}\Big(\beta_{11}(\omega_2 + \alpha_2 k_2^2) - \beta_{21}(\omega_1 + \alpha_1 k_1^2)\Big)\,, \end{aligned} \tag{5.5}$$

and

$$\begin{aligned} \mathscr{B}_1(\boldsymbol{\omega}, \mathbf{k}) &= \frac{\alpha_1 k_1}{\beta}\Big(\beta_{22}(\omega_1 + \alpha_1 k_1^2) - \beta_{12}(\omega_2 + \alpha_2 k_2^2)\Big) \\ \mathscr{B}_2(\boldsymbol{\omega}, \mathbf{k}) &= \frac{\alpha_1 k_1}{\beta}\Big(\beta_{11}(\omega_2 + \alpha_2 k_2^2) - \beta_{21}(\omega_1 + \alpha_1 k_1^2)\Big)\,, \end{aligned} \tag{5.6}$$

Differentiation then generates the key matrices in $\mathbf{E}(c)$ in (4.8),

$$\mathrm{D}_{\boldsymbol{\omega}}\mathbf{A} = \frac{1}{2\beta}\begin{pmatrix} \beta_{22} & -\beta_{12} \\ -\beta_{12} & \beta_{11} \end{pmatrix}\,,$$

and

$$\mathrm{D}_{\mathbf{k}}\mathbf{A} = \frac{1}{\beta}\begin{pmatrix} \alpha_1\beta_{22}k_1 & -\alpha_2\beta_{12}k_2 \\ -\alpha_1\beta_{12}k_1 & \alpha_2\beta_{11}k_2 \end{pmatrix} = \mathrm{D}_{\boldsymbol{\omega}}\mathbf{B}^T ,$$

and

$$\mathrm{D}_{\mathbf{k}}\mathbf{B} \;=\; \tfrac{1}{\beta}\begin{pmatrix} \alpha_1\beta|\Psi_1^0|^2 + 2\beta_{22}\alpha_1^2k_1^2 & -2\beta_{12}\alpha_1\alpha_2k_1k_2 \\ -2\beta_{12}\alpha_1\alpha_2k_1k_2 & \alpha_2\beta|\Psi_2^0|^2 + 2\alpha_2^2\beta_{11}k_2^2 \end{pmatrix} . \tag{5.7}$$

Despite the simplicity of these wavetrain solutions, we will show that the modulation associated with them still provides rich and nontrivial dynamics.

The strategy will be to compute the quadratic polynomical (4.10) and then use the graphical sign characteristic to identify coalescing characteristics. The polynomial for $\mu(c)$ is

$$\mu^2 - \mathrm{Tr}(\mathbf{E}(c))\mu + \det[\mathbf{E}(c)] = 0 , \tag{5.8}$$

The coefficients are

$$\Delta(c) := \det[\mathbf{E}(c)] = a_0c^4 + a_1c^3 + a_2c^2 + a_3c + a_4 , \tag{5.9}$$

with

$$\begin{aligned}
a_0 &= \tfrac{1}{4}\beta^{-1} ,\\
a_1 &= \beta^{-1}(\alpha_1k_1 + \alpha_2k_2) ,\\
a_2 &= \tfrac{1}{2}\beta^{-1}\big[\alpha_2(\beta_{11}|\Psi_1^0|^2 + 2\alpha_1k_1^2) + \alpha_1(\beta_{22}|\Psi_2^0|^2 + 2\alpha_2k_2^2) + 8\alpha_1\alpha_2k_1k_2\big] ,\\
a_3 &= 2\alpha_1\alpha_2\beta^{-1}\big(k_1(\beta_{22}|\Psi_2^0|^2 + 2\alpha_2k_2^2) + k_2(\beta_{11}|\Psi_1^0|^2 + 2\alpha_1k_1^2)\big)\\
a_4 &= \alpha_1\alpha_2\beta^{-1}\big((\beta_{11}|\Psi_1^0|^2 + 2\alpha_1k_1^2)(\beta_{22}|\Psi_2^0|^2 + 2\alpha_2k_2^2) - |\Psi_1^0|^2|\Psi_2^0|^2\beta_{12}^2\big) .
\end{aligned}$$

The trace is

$$\begin{aligned}
\mathrm{Tr}(\mathbf{E}(c)) \;=\; \frac{1}{\beta}\,&\big[\tfrac{1}{2}c^2(\beta_{11} + \beta_{22}) + 2c(\alpha_1k_1\beta_{22} + \alpha_2k_2\beta_{11})\\
&+\alpha_1\beta|\Psi_1^0|^2 + \alpha_2\beta|\Psi_2^0|^2 + 2\beta_{22}\alpha_1^2k_1^2 + 2\alpha_2^2\beta_{11}k_2^2\big] .
\end{aligned}$$

The strategy is to fix parameter values and plot the graph $(c, \mu(c))$ and then the formula (3.18) can be used to read off the sign characteristics from the graph.

5.1 Coalescing characteristics for "standing waves"

Consider first the case of "standing waves". The term is in quotes as these waves are not conventional CNLS standing waves. They are called standing waves here because one phase reduces to $\theta_1 = \omega t + kx$ and other phase is $\theta_2 = \omega t - kx$; that is, two plane waves travelling in opposite directions, and they have equal amplitudes, reminiscent of the classical concept of standing waves in physics. With this simplification the analysis of parameter values for coalescence can be carried out analytically. Standing waves in this sense are constructed by choosing the following parameter values

$$\beta_{11} = \beta_{22} = \gamma,\ \alpha_1 = \alpha_2 = \alpha,\ k = k_1 = -k_2,\ \omega_1 = \omega_2 = \omega , \tag{5.10}$$

so that $|\Psi_1^0|^2 = |\Psi_2^0|^2 = \Delta$. The choice of parameter values results in $a_1 = a_3 = 0$ and

$$\begin{aligned} a_0 &= \tfrac{1}{4}\beta^{-1}\,, \\ a_2 &= \beta^{-1}\big[\alpha\Delta(\gamma+\beta E_2^2) - 4\alpha^2k^2\big] = \beta^{-1}\alpha\big[(\gamma\Delta+2\alpha k^2) - 4\alpha k^2\big]\,, \\ a_4 &= \alpha^2\beta^{-1}\Delta^2\big((\gamma+\beta E_1^2)^2 - \beta_{12}^2\big) = \alpha^2\beta^{-1}\big((\gamma\Delta+2\alpha k^2)^2 - \Delta^2\beta_{12}^2\big) \end{aligned}$$

Hence there are four characteristics and they satisfy the biquadratic equation

$$a_0c^4 + a_2c^2 + a_4 = 0\,,$$

giving

$$c^2 = \frac{1}{2a_0}\left(-a_2 \pm \sqrt{a_2^2 - 4a_0a_4}\right)\,.$$

In such a case, coalescing characteristics occur precisely when

$$\big[\alpha\Delta(\gamma+\beta E_1^2) - 4\alpha^2k^2\big]^2 - \alpha^2\Delta^2\big((\gamma+\beta E_1^2)^2 - \beta_{12}^2\big) = 0\,,$$

which is equivalent to

$$(\gamma-\beta E_1^2)^2 - (\gamma+\beta E_1^2)^2 + \beta_{12}^2 = 0\,.$$

The solution to this is

$$\gamma\beta E_1^2 = \frac{1}{4}\beta_{12}^2\,, \quad \text{or} \quad k^2 = \frac{\beta_{12}^2\Delta}{8\alpha\gamma}\,.$$

Thus, it is clear from this that one must have $\alpha\gamma > 0$ for real solutions. This calculation of coalescing characteristics is direct and does not include the sign characteristics. It is the graphical sign characteristic that brings out the sign characteristic of the coalescing roots.

5.2 NLS characteristics from a graphical sign characteristic perspective

For general values of parameters, the characteristic polynomial (5.9) is a full quartic and so an analytic expression for the emergence of coalescing characteristics is less tractable. Instead, we resort to a graphical argument to show the existence of coalescing eigenvalues, achieved using graphical sign characteristic.

For illustrative purposes, first consider the standing wave case from §5.1. Choose parameter values from the analysis in (5.1) where coalescence occurs. An example is shown in Figure 4, with parameter values given in the caption. In the figure both $\mu-$curves are shown. With the lower curve beginning at the points where $|\mu'(c)| = \infty$. Due to symmetry two points of coalescence appear simultaneously at $\pm c_*$. At the points where $|\mu'(c)| = \infty$ the curves do not vanish but instead become complex. The curve appearing in Figure 4 is that for purely real μ, and the points where $|\mu'(c)| = \infty$ is where the function $\mu(c)$ satisfying $\det\big[\mathbf{E} - \mu(c)\mathbf{I}\big] = 0$ becomes complex valued. Such μ are never zero for any given $c \in \mathbb{R}/(c_-, c_+)$, where $c_\pm \approx \pm 4$ are the values of vertical tangency, and thus are not in need of consideration.

Now consider the general case where the symmetry conditions (5.10) no longer hold. While analytical study is more cumbersome, the graphical approach is straightforward. For example,

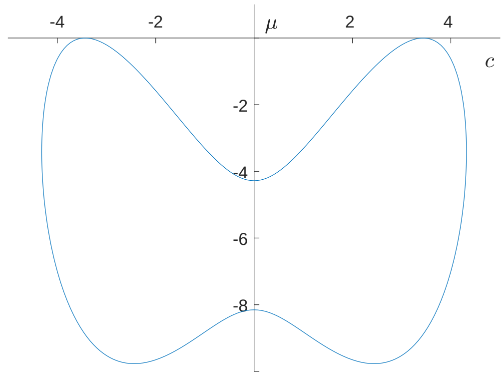

Figure 4: An example of the graphical sign characteristic μ in the case of coalescing characteristics for the standing wave case, with $\alpha_1 = \alpha_2 = 1$, $\gamma = \frac{3}{10}$, $\beta_{12} = \frac{4}{3}$, and $\Delta = 5$. The critical values of c are $c^* \approx \pm 3.4373$.

using parameter values motivated by Salman & Berloff [21] one can show the presence of coalescing characteristics, as illustrated in Figure 5. The parameter values are given in the caption. The coalescence is generic as the parameters are varied (it does not degenerate into a triple root) and the curvature is positive. Hence the normal form (3.12) is operational with $s = -1$ and $c_* \approx 3.9165$. It is apparent from the graph that the coalescing characteristics have opposite slope (and hence by (3.18) opposite sign characteristic).

A similar strategy can be used to identify other parameter choices where μ has a double root and thus the presence of a coalescing characteristics. However, the main aims of this section have been accomplished: coalescing characteristics are easy to find even in simple examples, the sign characteristic is essential for identifying potential instabilities, and the graphical sign characteristic is an efficient way to identify the sign characteristic and the curvature at coalescence.

6 Concluding remarks

By coupling an arbitrary but finite number of NLS equations, with toral symmetry, one can obtain Hermitian matrix pencils of arbitrary but finite size. The limit to infinity of the number of phases is intriguing and would lead to the study of Hermitian operator pencils.

The difference between the sign characteristic and Krein signature is in the choice of $\mathbf{G}$. The sign characteristic is defined for any Hermitian $\mathbf{G}$ whereas in the symplectic case it has the special form

$$\mathbf{G} = \mathtt{i}\mathbf{J}\,, \tag{6.1}$$

where $\mathbf{J}$ is a standard unit symplectic operator. In this case a sign is not defined for zero eigenvalues. The reason is that at a zero eigenvalue the eigenvector can be chosen to be real in which case the bilinear form with $\mathbf{G}$ as in (6.1) vanishes. On the other hand, with general Hermitian $\mathbf{G}$ the sign of zero eigenvalues is in general well-defined and non-zero.

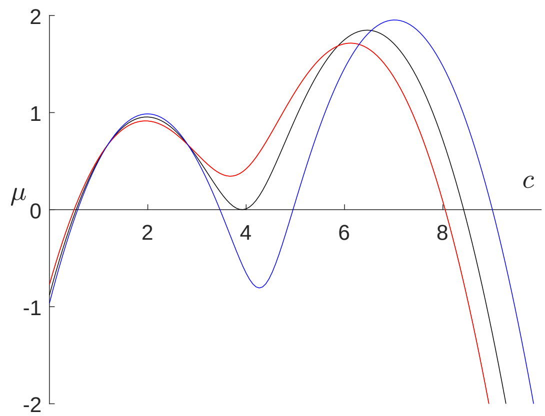

Figure 5: An illustration of how the graphical sign characteristic in the non-symmetric case leads to a coalescing characteristic emerging for the linearization about a two-phase wavetrain in the coupled NLS system (5.1). In the figure, the parameter values are chosen to be $\alpha_1 = \alpha_2 = 1$, $\beta_{11} = \beta_{22} = -1$, $|\Psi_1^0|^2 = 1$, $|\Psi_2^0|^2 = 2$, $k_1 = -1$ for all curves, and $\beta_{12} = 0.4$, $k_2 = -3.5$ (blue), $\beta_{12} = 0.6733$ (accurate to 4 decimal places), $k_2 = -3.2$ (black) and $\beta_{12} = 0.9$, $k_2 = -3$ (red). The critical value of c is $c^* \approx 3.9165$.

The motivation for this study is to feed into the development of a nonlinear theory near coalescing characteristics. Such a theory has been developed for the single phase modulation in [3], and the aim is to extend it to multphase case. Multiphase modulation for the breakdown of Whitham modulation equations at zero characteristics, leading to emergence of KdV, has been developed in [19, 20], and the aim is to adapt this theory to the case where the Whitham equations breakdown at nonzero coalescing characteristics.

An intriguing direction for the analysis of characteristics and their sign is extension to the $2+1$ case where the basic state depends on a vector-valued frequency and two vector valued wavenumbers, $(\boldsymbol{\omega}, \mathbf{k}, \ell)$. Extending the dispersionless Whitham modulation equations (4.3) to $2+1$, linearizing, and introducing the normal mode ansatz,

$$\mathbf{q} = \widehat{\mathbf{q}} \mathrm{e}^{\mathrm{i}\alpha(x+ct+\beta y+x_0)} ,$$

where β here is the ratio of the $y-$direction wavenumber to the $x-$direction wavenumber, leads to a quadratic Hermitian matrix pencil of the form

$$\mathbf{E}(c,\beta)\widehat{\mathbf{q}} = 0 ,$$

with

$$\begin{aligned} \mathbf{E}(c,\beta) \;\; &= \;\; \mathrm{D}_{\boldsymbol{\omega}}\mathbf{A}c^2 + \big(\mathrm{D}_{\mathbf{k}}\mathbf{A} + \mathrm{D}_{\boldsymbol{\omega}}\mathbf{B}\big)c \\ &\qquad + \mathrm{D}_{\mathbf{k}}\mathbf{B} + (\mathrm{D}_{\mathbf{k}}\mathbf{C} + \mathrm{D}_{\ell}\mathbf{B}\big)\beta + \mathrm{D}_{\ell}\mathbf{C}\beta^2 . \end{aligned}$$

For each fixed β this is a quadratic Hermitian matrix pencil in c, and for each fixed c it is a quadratic Hermitian matrix pencil in β. Either way, or even if it can be treated as a two-parameter quadratic Hermitian matrix pencil, new features are expected to arise.

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