Double criticality and the two-way Boussinesq equation in stratified shallow water hydrodynamics

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Abstract. Double criticality and its nonlinear implications are considered for stratified $N-$layer shallow water flows with $N = 1, 2, 3$. Double criticality arises when the linearization of the steady problem about a uniform flow has a double zero eigenvalue. We find that there are two types of double criticality: non-semisimple (one eigenvector, and one generalized eigenvector), and semi-simple (two independent eigenvectors). Using a multiple scales argument, dictated by the type of singularity, it is shown that the weakly nonlinear problem near double criticality is governed by a two-way Boussinesq equation (non-semisimple case) and a coupled Korteweg-de Vries equation (semisimple case). Parameter values and reduced equations are constructed for the examples of two-layer and three-layer stratified shallow water hydrodynamics.

1 Introduction ....

Criticality is a central theme in stratified shallow water hydrodynamics. For one-layer flow with a free surface, criticality of a uniform flow corresponds to Froude number unity. In two-layer flow there are two Froude numbers and so the "Froude number unity" criterion fails, and a more general approach is needed. Hence, in $N-$layer flow of differing densities with $N \geq 2$ and continuously stratified flow, a zero eigenvalue in the linearization about a uniform flow is taken as the signature of criticality.

In this paper, we consider the problem of "double criticality", that is, when the linearization has two zero eigenvalues. There are two cases: non-semisimple (only one eigenvector but a second generalized eigenvector), or semi-simple (two linearly independent eigenvectors). Both cases appear in shallow water multi-layer flows. We show that double criticality is impossible in one layer, in two layers only non-semisimple double criticality is possible and arises when the velocities in the two layers are nonzero and of opposite sign. In three layers both types of double criticality are found for a large range of parameter values.

Once it is established that double criticality exists in simple shallow water stratified models, the next question is the implication for the nonlinear problem. For this question, we assume that the shallow water models are appended by dispersive terms. The most interesting case is non-semisimple double criticality, which we show leads to the emergence of a two-way Boussinesq equation of the form

$$q_{TT} + \left(\nu q_{XX} - \tfrac{1}{2}\kappa q^2\right)_{XX} = 0\,, \tag{1.1}$$

where T, X are slow time and space variables to be defined. The origin of $q(X,T)$, and definitions of κ and ν will emerge in the theory.

To focus on a class of systems, the starting point for the analysis in this paper is the general class of partial differential equations (PDEs)

$$\mathbf{U}_t + \mathbf{F}(\mathbf{U})_x = \mathbf{D}\mathbf{U}_{xxx}\,, \quad \mathbf{U} \in \mathbb{R}^n\,, \tag{1.2}$$

with $\mathbf{F} : \mathbb{R}^n \to \mathbb{R}^n$, the flux vector, a given smooth function, and $\mathbf{D}$ a given $n \times n$ constant matrix. This class of PDEs includes shallow water models for multi-layered flows of differing density with weak dispersion, and examples are given in §§5–8. Well-posedness of the initial-value problem for (1.2) is not required for the reduction theory. Indeed, the dispersionless shallow water equations for two or more layers are not in general well posed [1, 4].

Systems of the form (1.2) have $\mathbf{U} = \mathbf{U}_0$, with $\mathbf{U}_0 \in \mathbb{R}^n$ a constant vector, as an exact solution. These constant solutions represent *uniform flows* in shallow water hydrodynamics. The linearization of the flux vector about a constant state is the $n \times n$ matrix $\mathrm{D}\mathbf{F}(\mathbf{U}_0)$ defined by

$$\mathrm{D}\mathbf{F}(\mathbf{U}_0)\mathbf{V} := \frac{d}{ds}\mathbf{F}(\mathbf{U}_0 + s\mathbf{V})\bigg|_{s=0}\,, \quad \forall\, \mathbf{V} \in \mathbb{R}^n\,. \tag{1.3}$$

Therefore, the linearization of (1.2) about $\mathbf{U}_0$ takes the form

$$\mathbf{V}_t + \mathrm{D}\mathbf{F}(\mathbf{U}_0)\mathbf{V}_x = \mathbf{D}\mathbf{V}_{xxx}\,, \quad \mathbf{V} \in \mathbb{R}^n\,. \tag{1.4}$$

The eigenvalues of the $n \times n$ constant matrix $\mathrm{D}\mathbf{F}(\mathbf{U}_0)$ give the dispersionless characteristic speeds of the linearization. The characteristic polynomial of $\mathrm{D}\mathbf{F}(\mathbf{U}_0)$ is

$$\Delta(\lambda) = \det\big[\mathrm{D}\mathbf{F}(\mathbf{U}_0) - \lambda\mathbf{I}\big]\,. \tag{1.5}$$

These characteristic speeds are relative to a fixed frame of reference. A basic state $\mathbf{U}_0$ is said to be critical when

$$\Delta(0) = \det[\mathrm{D}\mathbf{F}(\mathbf{U}_0)] = 0 \quad \text{and} \quad \Delta'(0) \neq 0\,, \tag{1.6}$$

that is, when one of the dispersionless characteristic speeds vanishes. The state $\mathbf{U}_0$ is said to be doubly critical when

$$\Delta(0) = \Delta'(0) = 0 \quad \text{and} \quad \Delta''(0) \neq 0\,, \tag{1.7}$$

that is, when two of the dispersionless characteristic speeds vanish. A principal aim of this paper is to show how singularities of the flux vector, arising due to coalescence of zero eigenvalues, drive the reduction to weakly nonlinear model equations. Hence the precise form of the dispersion in (1.2) is of secondary importance although it is important that dispersion is included.

Using a multiple scales argument, with the scaling dictated by the singularity in the flux vector, the weakly nonlinear problem can be reduced to model equations: criticality (1.6) reduces (1.2) to the Korteweg-de Vries (KdV) equation, double criticality reduces (1.2) to either the two-way Boussinesq equation (non-semisimple case) or to coupled KdV equations (semisimple case). These reductions are summarized in Table 1.

Table 1: Criticality and emergent modulation equations

# zero Eigenvalues	dim Ker(D$\mathbf{F}$)	modulation equation	T	X
1: $\Delta(0) = 0$	1	$q_T + \kappa q q_X + \nu q_{XXX} = 0$	$\varepsilon^3 t$	εx
2: $\Delta(0) = 0$ $\Delta'(0) = 0$	1	$q_{TT} - (\kappa q q_X - \nu q_{XX})_{XX} = 0$	$\varepsilon^2 t$	εx
2: $\Delta(0) = 0$ $\Delta'(0) = 0$	2	$\mathbf{q}_T + \mathcal{K}(\mathbf{q})_X + \mathcal{V}\mathbf{q}_{XXX} = 0$ $\mathbf{q} = (q_1, q_2)$	$\varepsilon^3 t$	εx

In Table 1 the terms in the coupled KdV equations are

$$\mathcal{K}(\mathbf{q}) = \frac{1}{2}\begin{pmatrix} \mathcal{K}^1_{11}q_1^2 + 2\mathcal{K}^1_{12}q_1q_2 + \mathcal{K}^1_{22}q_2^2 \\ \mathcal{K}^2_{11}q_1^2 + 2\mathcal{K}^2_{12}q_1q_2 + \mathcal{K}^2_{22}q_2^2 \end{pmatrix} \quad \text{and} \quad \mathcal{V} = \begin{bmatrix} \nu_{11} & \nu_{12} \\ \nu_{21} & \nu_{22} \end{bmatrix} .$$

The coefficients κ, ν and the coefficients in $\mathcal{K}, \mathcal{V}$ appearing in Table 1 emerge from the reduction theory and are defined in terms of derivatives of the flux vector and a projection of the dispersion matrix.

The closest theory to double criticality in the literature is the theory of Grimshaw [17] for resonance. Grimshaw's theory considers resonance between two modes with the same (zero or nonzero) characteristic speeds and their unfolding, distinguishing two cases ("kissing" configuration and "bubbling" configuration) which correspond to the non-semisimple and semisimple cases in this paper and he shows that either the Boussinesq equation or coupled KdV arises in the weakly nonlinear problem. The principal new features in this paper are firstly the connection with the concept of criticality in fluid mechanics; secondly, the emphasis on systems of the form (1.2) where resonance and nonlinearity are both expressed in terms of properties of the flux vector; and thirdly a multiple scales reduction is used which confirms the formal asymptotic validity.

For the full Euler equations, a two-way Boussinesq equation has been derived by Hickernell [22, 23] for inviscid stratified shear flow. The basic state is a parallel flow rather than a uniform flow. Although neither resonance nor criticality is emphasized, it appears to be implicit in the derivation that a double non-semisimple zero eigenvalue appears in the linearization about the parallel flow. In the second paper [23] a study of the two-way Boussinesq equation and its implications are studied. In Helfrich & Pedlosky [21] a two-way Boussinesq equation is derived for a two-layer version of the quasi-geostrophic potential vorticity equation on the $\beta-$plane. They explicitly state that the Boussinesq equation arises due to the coalescence of two modes (see equation (2.18) in [21]), and it is implicit that the double root is non-semisimple. The two-way Boussinesq equation is given in equation (2.27) in [21]. This theory is extended to the case of a two-way Boussinesq equation with non-constant coefficients in [25]. Similarly, Mitsudera [29] works with the quasi-geostrophic potential vorticity equation and studies the linearization about parallel flow and cites resonance as the starting point for the derivation of both a coupled KdV equation (equation (2.16) and (2.25) in [29]) and a two-way Boussinesq equation (equation (4.3a) in [29]). Another setting where the two-way Boussinesq equation arises is in the weakly nonlinear problem when two characteristics coalesce in Whitham modulation theory [31].

The coupled KdV equation was first derived in the the fluid mechanics literature by Gear & Grimshaw [16]. It has since been found in a wide range of examples in stratified shear flow and stratified shallow water flow [19, 20, 28]. A recent review is given in [18]. The new features of this paper are firstly the connection between coupled KdV and criticality in fluid mechanics, secondly how the flux vector is used to define criticality and how its curvature generates the nonlinearity in the coupled KdV equation, and thirdly, an asymptotically valid multiple scales expansion is used in the reduction from (1.2) to the coupled KdV system.

The advantage of reduction of the full system, whether it be the Euler equations or the system (1.2), is that model equations like the two-way Boussinesq equation and the coupled KdV equation are easier to analyze and in this case both systems have a wide range of interesting solutions which give a clue to related, and possibly more complicated, solutions in the full system.

The strategy for showing the reduction from (1.2) to one of the model equations in Table 1 is to introduce an assumption about the form of the solution (ansatz) and then show it is satisfies (1.2) exactly up to some power of ε, where ε is a measure of the distance in parameter space from criticality. The reduction theory for the non-semisimple case is given in §4 and the reduction for the semisimple case is given in §7. The reduced equations are formally asymptotically valid, in the sense that the neglected terms vanish in the limit $\varepsilon \to 0$, but the question of convergence of the series in ε or validity of the reduction are not considered. However, in the case where $\mathbf{U}_{xxx}$ is replaced by $\mathbf{U}_{xx}$ in (1.2) the reduced equation in the semisimple case changes from KdV to Burgers, and a proof of validity and convergence in this case is given in [11].

The paper applies the theory to three examples: the one-layer, two-layer, and three-layer shallow water equations with a free surface. In one layer, criticality generates the KdV equation only. We find that it is impossible to reduce the one-layer shallow water equations to the 2-way Boussinesq equation, which contradicts longstanding results in the literature. However, we provide arguments from several viewpoints supporting this conclusion. In two-layer shallow water flow with a free surface, double criticality occurs and the reduction to (1.1) is derived in §6. The case of three layers has both types of double criticality, and the parameter values and reduced equations are derived in §8.

2 Criticality and the flux vector

Consider the linearization of the flux vector $\mathbf{F}$ about a family of constant states, $\mathbf{U}_0$, and suppose that the state $\mathbf{U}_0$ is *critical*, that is, it satisfies (1.6). For simplicity the same symbol $\mathbf{U}_0$ will be used for both the general constant state and the critical state (or doubly critical state) and which is intended will be clear from the context.

Associated with the simple zero eigenvalue at criticality are real right and left eigenvectors

$$\mathrm{D}\mathbf{F}(\mathbf{U}_0)\boldsymbol{\xi}_1 = 0 \quad \text{and} \quad \boldsymbol{\eta}_1^T \mathrm{D}\mathbf{F}(\mathbf{U}_0) = 0 \quad \text{with} \quad \langle \boldsymbol{\eta}_1, \boldsymbol{\xi}_1 \rangle = 1\,, \tag{2.1}$$

where here and throughout

$$\langle \mathbf{u}, \mathbf{v} \rangle = \mathbf{u}^T \mathbf{v}\,, \quad \mathbf{u}, \mathbf{v} \in \mathbb{R}^n\,, \tag{2.2}$$

is the standard inner product on $\mathbb{R}^n$.

Of interest in this paper is the case where the algebraic multiplicity of the eigenvalue zero is two, called "double criticality", occurring when the characteristic polynomial satisfies (1.7). A double zero eigenvalue has two cases: geometric multiplicity one or two. When the geometric multiplicity is one (the kernel of $\mathrm{D}\mathbf{F}(\mathbf{U}_0)$ has dimension one) a generalised eigenvector is needed,

$$\mathrm{D}\mathbf{F}(\mathbf{U}_0)\boldsymbol{\xi}_1 = 0 \quad \text{and} \quad \mathrm{D}\mathbf{F}(\mathbf{U}_0)\boldsymbol{\xi}_2 = \boldsymbol{\xi}_1 \,, \tag{2.3}$$

with left eigenvectors

$$\boldsymbol{\eta}_2^T \mathrm{D}\mathbf{F}(\mathbf{U}_0) = 0 \quad \text{and} \quad \boldsymbol{\eta}_1^T \mathrm{D}\mathbf{F}(\mathbf{U}_0) = \boldsymbol{\eta}_2^T \,. \tag{2.4}$$

The eigenvectors are numbered so that the natural normalization is in effect,

$$\langle \boldsymbol{\eta}_i, \boldsymbol{\xi}_j \rangle = \delta_{ij} \,, \quad i,j = 1,2 \,, \tag{2.5}$$

where δ_{ij} is the Kronecker delta. There are some subtleties with this normalization and the theory is recorded in Appendix A. In the case of double criticality with geometric multiplicity two there are two independent eigenvectors

$$\mathrm{D}\mathbf{F}(\mathbf{U}_0)\boldsymbol{\xi}_j = 0 \quad \text{and} \quad \boldsymbol{\eta}_j^T \mathrm{D}\mathbf{F}(\mathbf{U}_0) = 0 \,, \quad j = 1,2 \,, \tag{2.6}$$

with the normalization (2.5).

The theory is also valid if $\mathbf{U}_0$ is *near* a critical value. To be precise, let $\mathbf{U}_0(\alpha)$ where α can be interpreted as one of the components of $\mathbf{U}_0$, and re-phrase the condition (1.6) as

$$\det[\mathrm{D}\mathbf{F}(\mathbf{U}_0(\alpha))]\Big|_{\alpha=0} = 0 \,. \tag{2.7}$$

Then expand $\mathrm{D}\mathbf{F}$ in a Taylor series in α

$$\mathrm{D}\mathbf{F}(\mathbf{U}_0(\alpha)) = \mathrm{D}\mathbf{F}(\mathbf{U}_0(0)) + \alpha\mathbf{B} + \cdots \,, \tag{2.8}$$

with $\mathbf{B} = \partial_\alpha \mathrm{D}\mathbf{F}(\mathbf{U}_0(\alpha))\Big|_{\alpha=0}$. Then it will be clear in the theory that by taking $\alpha = \alpha_0 \varepsilon^p$ for some p then the theory will still go though, and will generate an unfolding term in the reduced equations. An example of this construction is given in §4.1.

In principle, one could carry the theory of multiple criticality to zeros of $\Delta(\lambda)$ of any order. However, each additional zero eigenvalue requires variation of additional parameters. Simple criticality arises from the variation of one parameter in the family of basic states, double criticality with one eigenvector requires varying two parameters, and double criticality with two eigenvectors requires varying three parameters. Hence these three cases will be the most common, with higher order criticality requiring at least four parameters.

3 From Center-manifold theory to the reduction of PDEs with dispersion

The reduction strategy here is a generalization of the center-manifold reduction in dynamical systems. Center-manifold theory [12] is used to reduce nonlinear ordinary differential equations (ODEs) to simpler ODEs when the linearization about an equilibrium is singular. Starting with a system of nonlinear ODEs,

$$\mathbf{U}_t + \mathbf{F}(\mathbf{U}) = 0\,, \tag{3.1}$$

and an equilibrium solution $\mathbf{U}_0 \in \mathbb{R}^n$, satisfying $\mathbf{F}(\mathbf{U}_0) = 0$, suppose that at some value of $\mathbf{U}_0$ the linearization of $\mathbf{F}$ about $\mathbf{U}_0$ is singular. Write $\mathbf{U}_0(\alpha)$, where α is a scalar and can be interpreted as one of the components of $\mathbf{U}_0$, and suppose that

$$\det\big[\mathrm{D}\mathbf{F}(\mathbf{U}_0(\alpha))\big] = 0 \quad \text{at} \quad \alpha = 0\,,$$

and the eigenvalue is simple with right $\boldsymbol{\xi}$ and left $\boldsymbol{\eta}$ eigenvectors. The strategy in center manifold theory is to split $\mathbb{R}^n$ into the kernel of $\mathrm{D}\mathbf{F}(\mathbf{U}_0(0))$ and its complement in $\mathbb{R}^n$ and then assume a solution of the form

$$U(t) = \mathbf{U}_0 + \varepsilon q(T)\boldsymbol{\xi} + \varepsilon^2 \mathbf{W}(T,\varepsilon)\,, \quad \text{with} \quad \langle \boldsymbol{\eta}, \mathbf{W}\rangle = 0\,, \tag{3.2}$$

and $T = \varepsilon t$. Substitution of the form of the solution (3.2) into (3.1) and expanding $\mathbf{F}(\mathbf{U})$ in a Taylor series gives

$$\varepsilon^2 q_T \boldsymbol{\xi} + \varepsilon^3 \mathbf{W}_T + \varepsilon q \mathrm{D}\mathbf{F}(\mathbf{U}_0(0))\boldsymbol{\xi} + \varepsilon\alpha q \mathbf{B}\boldsymbol{\xi} + \varepsilon^2 \mathrm{D}\mathbf{F}(\mathbf{U}_0(0))\mathbf{W} + \tfrac{1}{2}\varepsilon^2 q^2 \mathrm{D}^2\mathbf{F}(\mathbf{U}_0(0))(\boldsymbol{\xi},\boldsymbol{\xi}) + \cdots\,,$$

with $\mathbf{B} = \partial_\alpha \mathrm{D}\mathbf{F}(\mathbf{U}_0(\alpha))\big|_{\alpha=0}$. Taking the inner product with $\boldsymbol{\eta}$, and imposing the normalization $\langle \boldsymbol{\eta}, \boldsymbol{\xi}\rangle = 1$, letting $\alpha = \alpha_0\varepsilon$, and dividing by ε^2,

$$q_T + \mu q + \tfrac{1}{2}\kappa q^2 + \cdots = 0\,, \tag{3.3}$$

with

$$\mu = \alpha_0 \langle \boldsymbol{\eta}, \mathbf{B}\boldsymbol{\xi}\rangle \quad \text{and} \quad \kappa = \langle \boldsymbol{\eta}, \mathrm{D}^2\mathbf{F}(\mathbf{U}_0(0))(\boldsymbol{\xi},\boldsymbol{\xi})\rangle\,.$$

Neglecting the higher order terms, the normal form (3.3) is the standard normal form for the *saddle-node* bifurcation [12]. Its solutions then give information about the nonlinear behaviour in a neighborhood of the equilibrium solution $\mathbf{U}_0(0)$.

Now suppose that (3.1) is replaced by the left-hand side of (1.2), $\mathbf{U}_t + \big(\mathbf{F}(\mathbf{U})\big)_x = 0$, and change the scaling to $T = \varepsilon^3 t$, and $q \sim \varepsilon^2$, then the reduced equation (3.3) is modified to

$$q_T + \mu q_X + \kappa q q_X + \cdots = 0\,. \tag{3.4}$$

Now add in the dispersion term $\mathbf{D}\mathbf{U}_{xxx}$ to the governing equation. Since it is linear it just generates an additional term in (3.4) giving,

$$q_T + \mu q_X + \kappa q q_X - \nu q_{XXX} + \cdots = 0\,, \tag{3.5}$$

where $\nu = \langle \boldsymbol{\eta}, \mathbf{D}\boldsymbol{\xi}\rangle$. Neglecting the higher order terms then results in the KdV equation. It is this strategy, that has its origins in the center-manifold reduction, which will for the basis of the reduction theory here.

4 Double criticality and the 2-way Boussinesq equation

In this section the implications of nonlinearity on double criticality are considered for the case when the geometric multiplicity is one. With the appropriate scaling and an assumption about the form of the unknown solution (an ansatz) it is shown that the model in the weakly nonlinear problem is the two-way Boussinesq equation. The semisimple case requires a different scaling and solution ansatz and is considered in §7.

Starting with the general class of PDEs (1.2), suppose there exists a uniform flow $\mathbf{U}_0$ that is doubly critical (1.7) with one geometric eigenvector (2.3). Introduce the slow time and space scales

$$T = \varepsilon^2 t\,, \quad X = \varepsilon x\,, \tag{4.1}$$

and consider the following ansatz for $\mathbf{U}(x,t)$,

$$\mathbf{U}(x,t) = \mathbf{U}_0 + \varepsilon^2 q(X,T,\varepsilon)\boldsymbol{\xi}_1 + \varepsilon^3 p(X,T,\varepsilon)\boldsymbol{\xi}_2 + \varepsilon^4 \mathbf{W}(X,T,\varepsilon)\,, \tag{4.2}$$

with the conditions

$$\langle \boldsymbol{\eta}_j, \mathbf{W}(X,T,\varepsilon)\rangle = 0 \quad \text{for all } X,T,\varepsilon\,, \quad \text{and} \quad j = 1,2\,. \tag{4.3}$$

The key steps that make this ansatz effective are the scaling of the independent (4.1) and dependent variables ($q \sim \varepsilon^2$, $p \sim \varepsilon^3$, $\mathbf{W} \sim \varepsilon^4$), the importance of including the generalized eigenvector $\boldsymbol{\xi}_2$ in the ansatz, and requiring $\mathbf{W}$ to be orthogonal (4.3) to the generalized kernel of $\mathrm{D}\mathbf{F}(\mathbf{U}_0)$.

Substitute the ansatz (4.2) into (1.2) and then evaluate term by term in powers of ε. The terms in (1.2) are

$$\begin{aligned}
\mathbf{U}_t &= \varepsilon^4 q_T \boldsymbol{\xi}_1 + \varepsilon^5 p_T \boldsymbol{\xi}_2 + \varepsilon^6 \mathbf{W}_T\,,\\
\mathbf{F}(\mathbf{U})_x &= \varepsilon^4 p_X \boldsymbol{\xi}_1 + \varepsilon^5 \mathrm{D}\mathbf{F}(\mathbf{U}_0)\,\mathbf{W}_X\\
&\quad + \tfrac{1}{2}\varepsilon^5 \mathrm{D}^2\mathbf{F}(\mathbf{U}_0)\,(q\boldsymbol{\xi}_1 + \varepsilon p\boldsymbol{\xi}_2 + \varepsilon^2\mathbf{W}, q\boldsymbol{\xi}_1 + \varepsilon p\boldsymbol{\xi}_2 + \varepsilon^2\mathbf{W})_X + \cdots\\
\mathbf{D}\mathbf{U}_{xxx} &= \varepsilon^5 q_{XXX}\mathbf{D}\boldsymbol{\xi}_1 + \varepsilon^6 p_{XXX}\mathbf{D}\boldsymbol{\xi}_2 + \varepsilon^7 \mathbf{D}\mathbf{W}_{XXX}\,,
\end{aligned} \tag{4.4}$$

where

$$\mathrm{D}^2\mathbf{F}(\mathbf{U}_0)(\mathbf{V},\mathbf{W}) := \frac{\partial^2}{\partial s_1 \partial s_2}\mathbf{F}(\mathbf{U}_0 + s_1\mathbf{V} + s_2\mathbf{W})\bigg|_{s_1=s_2=0}\,.$$

Substitute the ε series for each term into (1.2) and split the equation into three parts in the directions $\boldsymbol{\xi}_1$, $\boldsymbol{\xi}_2$, and the complement. Let $\mathbf{P}$ represent the projection onto the complement of $\mathrm{span}\{\boldsymbol{\xi}_1, \boldsymbol{\xi}_2\}$. The three components of the splitting are then

$$\begin{aligned}
&\varepsilon^4 q_T + \varepsilon^4 p_X = \varepsilon^5 \mathbf{R}_1\\
&\varepsilon^5 p_T + \varepsilon^5 \kappa\, q q_X = \varepsilon^5 \nu q_{XXX} + \varepsilon^6 \mathbf{R}_2\\
&\varepsilon^6 \mathbf{W}_T + \varepsilon^5 \mathbf{P}\mathrm{D}\mathbf{F}(\mathbf{U}_0)\mathbf{W}_X + \varepsilon^5 \mathbf{P}\mathrm{D}^2\mathbf{F}(\mathbf{U}_0)(\boldsymbol{\xi}_1,\boldsymbol{\xi}_1) q q_X = \varepsilon^5 q_{XXX}\mathbf{P}\mathbf{D}\boldsymbol{\xi}_1 + \varepsilon^6 \mathbf{R}_3\,,
\end{aligned} \tag{4.5}$$

where

$$\kappa = \langle \boldsymbol{\eta}_2, \mathrm{D}^2\mathbf{F}(\mathbf{U}_0)(\boldsymbol{\xi}_1,\boldsymbol{\xi}_1)\rangle \quad \text{and} \quad \nu = \langle \boldsymbol{\eta}_2, \mathbf{D}\boldsymbol{\xi}_1\rangle\,, \tag{4.6}$$

and $\mathbf{R}_j$, $j = 1,2,3$, are remainder terms for which explicit expressions can be written down but are not needed. In the first two equations in (4.5) the remainder term is of higher order. Therefore dividing the first equation in (4.5) by ε^4, and the second equations by ε^5, reduces them to

$$\begin{aligned} q_T + p_X &= \varepsilon \mathbf{R}_1 \\ p_T + \kappa\, q q_X &= \nu q_{XXX} + \varepsilon \mathbf{R}_2 \,. \end{aligned} \tag{4.7}$$

Now, let $\widehat{\mathbf{R}}_3 = \mathbf{R}_3 - \mathbf{W}_T$, then the third equation in (4.5), after dividing by ε^5, reduces to

$$\frac{d}{dX}\left[\mathbf{P}\mathrm{D}\mathbf{F}(\mathbf{U}_0)\mathbf{W} + \tfrac{1}{2}\mathbf{P}\mathrm{D}^2\mathbf{F}(\mathbf{U}_0)(\boldsymbol{\xi}_1,\boldsymbol{\xi}_1)q^2 - q_{XX}\mathbf{P}\mathbf{D}\boldsymbol{\xi}_1\right] = \varepsilon\widehat{\mathbf{R}}_3 \,. \tag{4.8}$$

In this latter equation, the linear operator $\mathbf{P}\mathrm{D}\mathbf{F}(\mathbf{U}_0)$ is invertible since $\mathbf{P}$ is the projection onto the complement of the kernel of $\mathrm{D}\mathbf{F}(\mathbf{U}_0)$. Hence it is solvable for $\mathbf{W}(X,T,\varepsilon)$ as a power series in ε. Indeed, all three equations (4.7) and (4.8) can be solved order by order in ε. Expand the functions q, p, and $\mathbf{W}$ in a Taylor series in ε

$$\begin{aligned} q(X,T,\varepsilon) &= q_0(X,T) + \varepsilon q_1(X,T) + \mathcal{O}(\varepsilon^2)\,, \\ p(X,T,\varepsilon) &= p_0(X,T) + \varepsilon p_1(X,T) + \mathcal{O}(\varepsilon^2)\,, \\ \mathbf{W}(X,T,\varepsilon) &= \mathbf{W}_0(X,T) + \varepsilon \mathbf{W}_1(X,T) + \mathcal{O}(\varepsilon^2)\,. \end{aligned}$$

Substitution of these expansions into (4.7) and (4.8) generates a sequence of equations for $(q_j, p_j, \mathbf{W}_j)$, $j = 0,1,\ldots$. In principle this formal sequence can be solved order by order. However, only the leading order terms in q and p are needed and so the subscripts are dropped: $q_0(X,T) \equiv q(X,T,0)$ and $p_0(X,T) \equiv p(X,T,0)$. Let $\mathbf{B} = \mathbf{P}\mathrm{D}\mathbf{F}(\mathbf{U}_0)$, then, $\mathbf{B}$ is invertible. Therefore, to leading order,

$$\mathbf{W}_0 = -\tfrac{1}{2}\mathbf{B}^{-1}\mathbf{P}\mathrm{D}^2\mathbf{F}(\boldsymbol{\xi}_1,\boldsymbol{\xi}_1)q^2 + \mathbf{B}^{-1}\mathbf{P}\mathbf{D}\boldsymbol{\xi}_1\, q_{XX} + \mathbf{B}^{-1}f_0(T)\,.$$

where $f_0(T)$ is an arbitrary function of T, and q,p are the leading order terms in the q,p expansions.

To leading order the weakly nonlinear modulation equation associated with non-semisimple double criticality is

$$q_T + p_X = 0 \quad \text{and} \quad p_T + \kappa\, q q_X = \nu q_{XXX}\,. \tag{4.9}$$

Combining these two equations gives the scalar two-way Boussinesq equation

$$q_{TT} = -p_{XT} = -\frac{\partial}{\partial X}(p_T) = -\frac{\partial}{\partial X}\left(-\kappa\, q q_X + \nu q_{XXX}\right)\,,$$

or

$$q_{TT} + \left(\nu\, q_{XX} - \tfrac{1}{2}\kappa q^2\right)_{XX} = 0\,. \tag{4.10}$$

The coefficient of dispersion ν is defined in (4.6). When $\nu > 0$ ($\nu < 0$) we say that (4.10) is the "good" ("bad") Boussinesq equation. In the bad case, the initial value problem for the linear equation $q_{TT} + \nu q_{XXXX} = 0$ is not well posed, although the equation still has a significant range of interesting bounded solutions [24].

The more interesting coefficient is κ, defined in (4.6). This coefficient can be interpreted as a curvature as follows. Consider the scalar-valued function

$$\alpha(s) = \langle \boldsymbol{\eta}_2, \mathbf{F}(\mathbf{U}_0 + s\boldsymbol{\xi}_1)\rangle\,.$$

Then clearly $\alpha'(0) = \langle \boldsymbol{\eta}_2, \mathrm{D}\mathbf{F}(\mathrm{U}_0)\boldsymbol{\xi}_1 \rangle = 0$ at criticality and $\alpha''(0) = \kappa$; that is, κ is the classical scalar curvature of the one-parameter path through the flux vector, $\alpha(s)$, at $s = 0$.

The reduced equation for q in (4.10) is independent of the choice of scaling of $\boldsymbol{\xi}_1$ and $\boldsymbol{\xi}_2$ in the following sense. The normalization of the eigenvectors (2.5) still leaves freedom to scale $\boldsymbol{\xi}_1$ in (2.3) – as long as the same scaling is used on $\boldsymbol{\xi}_2$ (see Appendix A for details of the normalization theory for the eigenvectors in the non-semisimple case). However if $\boldsymbol{\xi}_1$ is multiplied by a non-zero real parameter, then $q(X, T, \varepsilon)$ in (4.2) should also be multiplied by the inverse of that real parameter in which case (4.2) is invariant, and the reduced equation (4.10) is then in terms of the scaled q.

4.1 Unfolding criticality and the second derivative

As discussed in §2 and shown in (2.8) the theory can be modified to include the case when the uniform flow is *near critical*. Take $\alpha = \alpha_0\varepsilon^2$ in (2.8) and add the fifth-order term $\alpha_0\varepsilon^5 q_X \mathbf{B}\boldsymbol{\xi}$ in the expansion for $\mathbf{F}(\mathbf{U})$ in (4.4). After projection, this term will modify (4.7) to

$$\begin{aligned} q_T + p_X &= \varepsilon \mathbf{R}_1 \\ p_T + \mu q_X + \kappa\, q q_X &= \nu q_{XXX} + \varepsilon \mathbf{R}_2 \,. \end{aligned} \tag{4.11}$$

with $\mu = \alpha_0 \langle \boldsymbol{\eta}, \mathbf{B}\boldsymbol{\xi} \rangle$. The Boussinesq equation (4.10) is then modified to

$$q_{TT} - \mu q_{XX} + \left(\nu\, q_{XX} - \tfrac{1}{2}\kappa q^2\right)_{XX} = 0\,. \tag{4.12}$$

In other words, the q_{XX} term does appear naturally in the emergent Boussinesq equation and it is associated with unfolding the uniform flow near criticality to order ε^2. Henceforth, the unfolding terms will not always be included, but they are easily added as needed, using the above argument.

5 Shallow water equations for one-layer flow

The most well known setting where the 2-way Boussinesq equation has appeared is in one-layer shallow water hydrodynamics. Indeed, this is the setting where the two-way Boussinesq equation was first derived [9]. In principle it should be a straightforward argument to use the theory of this paper to reduce a shallow-water Boussinesq equation for one layer to the two-way Boussinesq equation. However, this reduction fails, unless $h_0 = 0$.

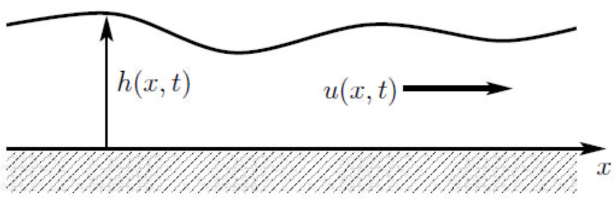

Figure 1: Schematic of the flow field for one layer shallow water hydrodynamics

Consider a typical example in the equivalence class of Boussinesq shallow water equations

$$\begin{aligned} h_t + uh_x + hu_x &= \tfrac{1}{6}h_0^3 u_{xxx} \\ u_t + uu_x + gh_x &= -\tfrac{1}{2}gh_0^2 h_{xxx}\,, \end{aligned} \tag{5.1}$$

for $h(x,t)$, the fluid depth, and $u(x,t)$, the horizontal fluid velocity, as shown schematically in Figure 1, where $g > 0$ is the gravity coefficient and h_0 a mean depth [14]. The system (5.1) can be written in the standard form (1.2) by taking

$$\mathbf{U} = \begin{pmatrix} h \\ u \end{pmatrix}, \quad \mathbf{F}(\mathbf{U}) = \begin{pmatrix} hu \\ \frac{1}{2}u^2 + gh \end{pmatrix}, \quad \mathbf{D} = \begin{bmatrix} 0 & \frac{1}{6}h_0^3 \\ -\frac{1}{2}gh_0^2 & 0 \end{bmatrix}. \tag{5.2}$$

Membership of (5.1) in the equivalence class of Boussinesq shallow water models is justified by the fact that the dispersion relation for the linearization of (5.1) is equivalent to that for the full water wave problem up to order $(kh_0)^2$. To see this, consider the linearization of (5.1) about $(h,u) = (h_0, u_0)$,

$$\begin{aligned} h_t + h_0 u_x &= \tfrac{1}{6}h_0^3 u_{xxx} \\ u_t + gh_x &= -\tfrac{1}{2}gh_0^2 h_{xxx}\,. \end{aligned} \tag{5.3}$$

It has dispersion relation

$$\frac{c^2}{gh_0} = \left(1 + \frac{1}{6}k^2h_0^2\right)\left(1 - \frac{1}{2}k^2h_0^2\right).$$

Expanding shows that it agrees with the exact dispersion relation, $\frac{c^2}{gh_0} = \frac{\tanh(kh_0)}{kh_0}$ up to second order. This Boussinesq equation is not well-posed. In the limit $kh_0 \to \infty$ the growth rate of the unstable linear modes tends to infinity generating an ill-posed initial value problem. However, the reduction to long-wave models (KdV, 2-way Boussinesq) goes through unchanged, independent of linear well-posedness.

The linearization of the flux vector of (5.1) about a constant state $\mathbf{U}_0 = (h_0, u_0)$ is

$$\mathrm{D}\mathbf{F}(\mathbf{U}_0) = \begin{bmatrix} u_0 & h_0 \\ g & u_0 \end{bmatrix},$$

with characteristic polynomial

$$\Delta(\lambda) := \det[\mathrm{D}\mathbf{F}(\mathbf{U}_0) - \lambda\mathbf{I}] = \lambda^2 - 2u_0\lambda + gh_0(F^2 - 1)\,, \quad F^2 = \frac{u_0^2}{gh_0}.$$

When $\Delta(0) = 0$ the classic Froude number unity condition for criticality is recovered, and it is straightforward to show, using the theory in §7 (see equations (7.5) and (7.6)), that the system (5.1) can be reduced to the KdV equation to leading order

$$q_T + \kappa q q_X - \nu q_{XXX} = 0$$

with $T = \varepsilon^3 t$, $X = \varepsilon x$ and

$$\nu = \frac{1}{6}\frac{gh_0^3}{u_0} \quad \text{and} \quad \kappa = \frac{3}{2}g\,.$$

This reduction recovers the usual velocity form of the KdV equation in shallow water (cf. equation (6.9c) on page 693 of DINGEMANS [14]).

For double criticality, and the emergence of a 2-way Boussinesq equation, a necessary condition is that $\mathbf{DF}(\mathbf{U}_0)$ should have a double nonsemisimple zero eigenvalue. But

$$\Delta(0) = \Delta'(0) = 0 \quad \Rightarrow \quad u_0 = h_0 = 0\,.$$

When criticality is satisfied, a double zero eigenvalue exists if and only if $h_0 = 0$. Hence, the theory of this paper suggests that the 2-way Boussinesq equation is not an asymptotically valid model for shallow water hydrodynamics in one layer. Another concept of validity is in the context of the initial-value problem where the flow due to initial data of the full water-wave problem is compared with the flow of the two-way Boussinesq equation with comparable initial data. This two concepts of validity will be considered in turn.

The 2-way Boussinesq equation for shallow water waves, originally derived by Boussinesq and quoted in the literature is

$$\eta_{tt} - gh_0\eta_{xx} = gh_0\left(\frac{3\eta^2}{2h_0} + \frac{h_0^2}{3}\eta_{xx}\right)_{xx}\,. \tag{5.4}$$

There are two natural scalings of this equation. Substitution of the Boussinesq scaling

$$T = \varepsilon^2 t\,, \quad X = \varepsilon x\,, \quad \eta = \varepsilon^2\widehat{\eta}\,,$$

gives

$$\widehat{\eta}_{TT} - \frac{gh_0}{\varepsilon^2}\widehat{\eta}_{XX} = gh_0\left(\frac{3\widehat{\eta}^2}{2h_0} + \frac{h_0^2}{3}\widehat{\eta}_{XX}\right)_{XX}\,. \tag{5.5}$$

There is clearly a problem in the limit as $\varepsilon \to 0$ with the second term unless $gh_0 \sim \varepsilon^2$. This problem is consistent with the requirements for the theory of double criticality in this paper. However, if $h_0 \sim \varepsilon^2$ then the dispersive term is of higher order putting it out of balance with the other terms in the equation, leaving a purely hyperbolic equation without dispersion. On the other hand, the limit $h_0 \to 0$ will bring in the importance of viscous terms, and so the inviscid model for shallow water hydrodynamics is no longer valid.

The other natural scaling is to take

$$T = \varepsilon t\,, \quad X = \varepsilon x\,, \quad \eta = \varepsilon^2\widehat{\eta}\,.$$

Substitution into (5.4) then gives

$$\widehat{\eta}_{TT} - gh_0\widehat{\eta}_{XX} = gh_0\varepsilon^2\left(\frac{3}{2h_0}\widehat{\eta}^2 + \frac{h_0^2}{3}\widehat{\eta}_{XX}\right)_{XX}\,. \tag{5.6}$$

However, in the limit as $\varepsilon \to 0$ the nonlinearity and dispersion are vanishingly small in comparison to the linear wave dynamics. One can try other scalings, but there does not appear to be any scaling that renders (5.4) homogeneous in ε.

There are other arguments in the literature for non-validity of (5.4). KEULEGAN & PATTERSON [26] argue that the derivation of the two-way Boussinesq equation requires the imposition of initial conditions so that the Boussinesq equation is restricted to be a one-way equation (effectively reducing it to KdV). Specifically they state between equations (102) and (103) that "*...restricting ourselves to waves propagated in the positive $x-$direction, we obtain ...*" Thereby nullifying the two-way property.

The rigorous validity of the initial value problem, comparing the flow of initial data for the full water wave problem with initial data for model equations has been considered by Schneider & Wayne [32, 33]. They argue that all solutions of the initial value problem settle into solutions of a left or right running KdV equation. These results are consistent with [26] and consistent with the fact that the two-way Boussinesq equation can be approximately decomposed into a left and right running KdV equations. Schneider & Wayne, in §5.2 of [33], give an explicit argument showing that the Boussinesq equation will not sustain a validity argument because of the presence of ε terms in the equation (as argued above), and they prove in Schneider & Wayne [32] that the only long-wave models which are asymptotically valid are the left and right running KdV equations.

The two-way Boussinesq equation can still be used as a model for shallow water hydrodynamics if limited to solutions travelling in one direction of the KdV type. Other more exotic solutions such as two-way standing waves are unlikely to accurately represent solutions of the full water-wave problem.

On the other hand, by stratifying the flow, by allowing two layers of different density in the shallow water model, conditions for the emergence of an asymptotically valid 2-way Boussinesq equation are found.

6 Double criticality for two-layer shallow water flow

The shallow water equations for two layers of differing density can be expressed in conservation law form $\mathbf{U}_t + \mathbf{F}(\mathbf{U})_x = 0$ with

$$\mathbf{U} = \begin{pmatrix} h_1 \\ h_2 \\ u_1 \\ u_2 \end{pmatrix} \quad \text{and} \quad \mathbf{F}(\mathbf{U}) = \begin{pmatrix} h_1 u_1 \\ h_2 u_2 \\ \frac{1}{2}u_1^2 + gh_1 + rgh_2 \\ \frac{1}{2}u_2^2 + gh_1 + gh_2 \end{pmatrix}, \tag{6.1}$$

where, as shown schematically in Figure 2, $h_1(x,t), h_2(x,t)$ are the layer depths and $u_1(x,t), u_2(x,t)$ are the horizontal velocities, and

$$r = \frac{\rho_2}{\rho_1} < 1\,. \tag{6.2}$$

These equations are derived from the Euler equations in [2, 3]. By adding in dispersion they take the form

$$\mathbf{U}_t + \mathbf{F}(\mathbf{U})_x = \mathbf{D}\mathbf{U}_{xxx}\,, \quad \mathbf{U} \in \mathbb{R}^4\,. \tag{6.3}$$

There are a range of derivations of dispersive Boussinesq-type equations for shallow-water two-layer fluids in the literature using various strategies and resulting in a range of forms (for example, [13, 30, 5, 15]), usually with mixed space and time derivatives in the dispersion. Here we use the Boussinesq equation derived in [15] because it is transformed so that the dispersion terms appear in the form (6.3) with all the dispersion in terms of space derivatives

$$\begin{aligned} \frac{\partial}{\partial t}(\rho_1 u_1) + \frac{\partial}{\partial x}(\tfrac{1}{2}\rho_1 u_1^2 + \rho_1 g h_1 + \rho_2 g h_2) &= a_{11}\frac{\partial^3 h_1}{\partial x^3} + a_{12}\frac{\partial^3 h_2}{\partial x^3} \\ \frac{\partial}{\partial t}(\rho_2 u_2) + \frac{\partial}{\partial x}(\tfrac{1}{2}\rho_2 u_2^2 + \rho_2 g h_1 + \rho_2 g h_2) &= a_{21}\frac{\partial^3 h_1}{\partial x^3} + a_{22}\frac{\partial^3 h_2}{\partial x^3}\,. \end{aligned} \tag{6.4}$$

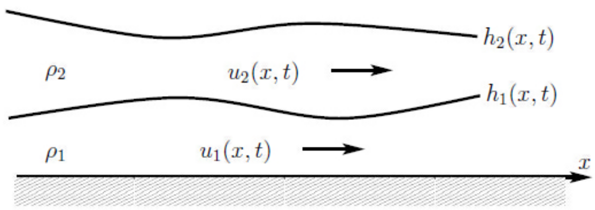

Figure 2: A schematic of two-layer stratified shallow water with a free surface.

with no change to the mass conservation equations. The coefficients are

$$
\begin{aligned}
a_{11} &= -\tfrac{1}{3}\rho_1 g h_1^2 - \rho_2 g h_1 h_2 - \tfrac{1}{2}\rho_2 g h_2^2 \\
a_{12} &= a_{21} = -\tfrac{1}{6}\rho_2 g h_1^2 - \tfrac{1}{4}\rho_2 g h_1 h_2 - \tfrac{1}{2}\tfrac{\rho_2^2}{\rho_1} g h_1 h_2 - \tfrac{5}{12}\rho_2 g h_2^2 \\
a_{22} &= -\tfrac{1}{2} r \rho_2 g h_1 h_2 - \tfrac{1}{3}\rho_2 g h_2^2 \,.
\end{aligned}
$$

The dispersive terms are derived in Chapter 2.5 of [15] including surface tension at the two interfaces. The dispersion is confirmed to be in the equivalence class of Boussinesq equations for this case by showing that the dispersion relation for the linearization of (6.4) agrees up to quadratic order with the exact dispersion relation for two-layer flow with a free surface.

The Donaldson-dispersion generates a $\mathbf{D}$ matrix of the form

$$
\mathbf{D} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ a_{11}/\rho_1 & a_{12}/\rho_1 & 0 & 0 \\ a_{12}/\rho_2 & a_{22}/\rho_2 & 0 & 0 \end{bmatrix} . \tag{6.5}
$$

Let $\mathbf{U}_0 \in \mathbb{R}^4$ be a uniform flow. In this section it will be shown that there are parameter values when this family of constant states undergoes double criticality, and the emergent two-way Boussinesq equation is constructed.

The linearization of the flux vector about the family of constant states is

$$
\mathrm{D}\mathbf{F}(\mathbf{U}_0) = \begin{pmatrix} u_1 & 0 & h_1 & 0 \\ 0 & u_2 & 0 & h_2 \\ g & rg & u_1 & 0 \\ g & g & 0 & u_2 \end{pmatrix} , \tag{6.6}
$$

where for simplicity the same symbol is used for the flow variable and the uniform flow. The eigenvalues of $\mathrm{D}\mathbf{F}(\mathbf{U}_0)$ satisfy,

$$
\Delta(\lambda) := \det[\lambda \mathbf{I} - \mathrm{D}\mathbf{F}(\mathbf{U}_0)] = \lambda^4 + a_3\lambda^3 + a_2\lambda^2 + a_1\lambda + a_0 = 0 \,,
$$

with

$$
\begin{aligned}
a_3 &= -2(u_1 + u_2) \\
a_2 &= 4u_1u_2 + u_1^2 + u_2^2 - gh_1 - gh_2 \\
a_1 &= 2gh_1u_2 + 2gh_2u_1 - 2u_1u_2(u_1 + u_2) \\
a_0 &= u_1^2u_2^2 + g(1-r)h_1gh_2 - gh_1u_2^2 - gh_2u_1^2
\end{aligned}
$$

These coefficients agree with the calculation in the Appendix of Lawrence [27]. The matrix $\mathbf{DF}(\mathbf{U}_0)$ has a simple zero eigenvalue (critical flow) if $a_0 = 0$ but $a_1 \neq 0$ and it has a double zero eigenvalue (double criticality) if $a_0 = a_1 = 0$ and $a_2 \neq 0$.

Introduce Froude numbers in each layer

$$F_1^2 = \frac{u_1^2}{gh_1} \quad \text{and} \quad F_2^2 = \frac{u_2^2}{gh_2}\,. \tag{6.7}$$

Then $a_0 = 0$ simplifies to

$$\big(F_1^2 - 1\big)\big(F_2^2 - 1\big) = r\,. \tag{6.8}$$

For fixed r, this condition generates a pair of hyperbolae in the (F_1^2, F_2^2) plane and is the well-known condition for critical flow of two layers with a free surface [6, 1, 27, 3]. A geometric view of the surface generated by (6.8) is given in [10].

For double criticality we need also $a_1 = 0$ which simplifies to

$$\frac{u_2}{gh_2}\big(F_1^2 - 1\big) + \frac{u_1}{gh_1}\big(F_2^2 - 1\big) = 0\,. \tag{6.9}$$

Combining (6.8) and (6.9) gives

$$(1 - F_1^2)^2 = -r\frac{h_2 u_1}{h_1 u_2}\,.$$

Hence, a necessary condition for double criticality is that *both* u_1 *and* u_2 *are nonzero and of opposite sign*

$$u_1 u_2 < 0\,. \tag{6.10}$$

For the algebraic multiplicity to be exactly two,

$$0 \neq \Delta''(0) = a_2 = 4u_1u_2 - gh_1(1 - F_1^2) - gh_2(1 - F_2^2)\,,$$

or

$$h_1(F_1^2 - 1) + h_2(F_2^2 - 1) \neq -\frac{4}{g}u_1 u_2\,. \tag{6.11}$$

When the three conditions (6.8), (6.9), and (6.11) are satisfied, there exists physically realizable values of F_1 and F_2 at which double criticality occurs. An example is

$$gh_1 = 27\,, \quad gh_2 = 16\,, \quad u_1 = 3\,, \quad u_2 = -2\,, \quad r = \tfrac{1}{2}\,. \tag{6.12}$$

With these parameter values,

$$F_1^2 = \frac{u_1^2}{gh_1} = \frac{1}{3} \quad \text{and} \quad F_2^2 = \frac{u_2^2}{gh_2} = \frac{1}{4}\,,$$

and the conditions $\Delta(0) = \Delta'(0) = 0$ and $\Delta''(0) \neq 0$ are satisfied.

6.1 Generalized eigenvectors

Assume the conditions (6.8) and (6.9) are satisfied. Then the algebraic multiplicity of the zero eigenvalue is two. By computing the eigenvectors, it will be shown that the geometric multiplicity is at most one.

An eigenvector $\boldsymbol{\xi}_1 \in \mathbb{R}^4$ satisfies $\mathrm{D}\mathbf{F}(\mathbf{U}_0)\boldsymbol{\xi}_1 = 0$ with $\mathrm{D}\mathbf{F}(\mathbf{U}_0)$ defined in (6.6). Writing this out gives

$$\begin{aligned} \xi_{13} &= -\tfrac{u_1}{h_1}\xi_{11} \\ \xi_{14} &= -\tfrac{u_2}{h_2}\xi_{12}\,, \end{aligned} \tag{6.13}$$

and

$$\begin{bmatrix} 1-F_1^2 & r \\ 1 & 1-F_2^2 \end{bmatrix} \begin{pmatrix} \xi_{11} \\ \xi_{12} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{6.14}$$

The matrix on the left is singular due to (6.8) and it has a simple zero eigenvalue. Existence of a second zero eigenvalue would require a zero trace but a zero trace violates (6.8). Note that the simple zero eigenvalue of the matrix in (6.14) does not contradict the fact that $\mathrm{D}\mathbf{F}(\mathbf{U}_0)$ has a double zero eigenvalue.

Combining (6.14) and (6.13), the sole geometric eigenvector at criticality is

$$\boldsymbol{\xi}_1 = a\widehat{\boldsymbol{\xi}}_1 \quad \text{with} \quad \widehat{\boldsymbol{\xi}}_1 = \begin{pmatrix} r \\ F_1^2 - 1 \\ -r\frac{u_1}{h_1} \\ -\frac{u_2}{h_2}(F_1^2-1) \end{pmatrix},$$

where a is an arbitrary nonzero real number.

Since the algebraic multiplicity is two, there must be a generalised eigenvector. See Appendix A for a summary of the theory of generalised eigenvectors needed. The generalized eigenvector $\boldsymbol{\xi}_2$ satisfies

$$\mathrm{D}\mathbf{F}(\mathbf{U}_0)\boldsymbol{\xi}_2 = \boldsymbol{\xi}_1\,. \tag{6.15}$$

Calculating gives

$$\begin{aligned} \xi_{23} &= a\tfrac{r}{h_1} - \tfrac{u_1}{h_1}\xi_{21} \\ \xi_{24} &= a\tfrac{(F_1^2-1)}{h_2} - \tfrac{u_2}{h_2}\xi_{22}\,, \end{aligned}$$

and

$$\begin{bmatrix} 1-F_1^2 & r \\ 1 & 1-F_2^2 \end{bmatrix} \begin{pmatrix} \xi_{21} \\ \xi_{22} \end{pmatrix} = -\frac{2a}{g} \begin{pmatrix} \frac{u_1}{h_1}r \\ \frac{u_2}{h_2}(F_1^2-1) \end{pmatrix}. \tag{6.16}$$

The solvability condition for this inhomogeneous problem is precisely (6.9). The generalized eigenvector is

$$\boldsymbol{\xi}_2 = a\widehat{\boldsymbol{\xi}}_2 + b\widehat{\boldsymbol{\xi}}_1, \quad \text{with} \quad \widehat{\boldsymbol{\xi}}_2 = \begin{pmatrix} 0 \\ -2\frac{u_1}{gh_1} \\ \frac{r}{h_1} \\ \frac{(F_1^2-1)}{h_2} + \frac{2}{g}\frac{u_1u_2}{h_1h_2} \end{pmatrix}, \tag{6.17}$$

where b is an arbitrary real number.

The adjoint eigenvectors satisfy

$$\mathrm{D}\mathbf{F}(\mathbf{U}_0)^T\boldsymbol{\eta}_2 = 0 \quad \text{and} \quad \mathrm{D}\mathbf{F}(\mathbf{U}_0)^T\boldsymbol{\eta}_1 = \boldsymbol{\eta}_2\,.$$

Computing as above gives,

$$\boldsymbol{\eta}_2 = c\widehat{\boldsymbol{\eta}}_2 \quad \text{with} \quad \widehat{\boldsymbol{\eta}}_2 = \begin{pmatrix} \frac{u_1}{h_1}(1-F_2^2) \\ -r\frac{u_2}{h_2} \\ -(1-F_2^2) \\ r \end{pmatrix},$$

where c is an arbitrary non-zero real number, and

$$\boldsymbol{\eta}_1 = c\widehat{\boldsymbol{\eta}}_1 + d\widehat{\boldsymbol{\eta}}_2 \quad \text{with} \quad \widehat{\boldsymbol{\eta}}_1 = \begin{pmatrix} -\frac{1}{h_1}(1-F_2^2) \\ -\frac{2}{g}\frac{u_1u_2}{h_1h_2}(1-F_2^2) + \frac{r}{h_2} \\ 0 \\ \frac{2}{g}\frac{u_1}{h_1}(1-F_2^2) \end{pmatrix},$$

where d is an arbitrary real number.

There are four free constants, but two can be fixed in order to ensure that the conditions (1.8) are satisfied. Using the strategy in Appendix A and computing,

$$\langle \widehat{\boldsymbol{\eta}}_1, \widehat{\boldsymbol{\xi}}_1 \rangle = \langle \widehat{\boldsymbol{\eta}}_2, \widehat{\boldsymbol{\xi}}_2 \rangle = r\left[\frac{4}{g}\frac{u_1u_2}{h_1h_2} + \frac{1}{h_1}(F_2^2-1) + \frac{1}{h_2}(F_1^2-1)\right],$$

and the term in brackets is nonzero due to the assumption (6.11). The parameters a and c are required to satisfy

$$ac = \langle \widehat{\boldsymbol{\eta}}_1, \widehat{\boldsymbol{\xi}}_1 \rangle^{-1}.$$

The inner product $\langle \widehat{\boldsymbol{\eta}}_2, \widehat{\boldsymbol{\xi}}_1 \rangle = 0$ by solvability of (6.15). It remains to compute the other inner product of cross terms,

$$\langle \widehat{\boldsymbol{\eta}}_1, \widehat{\boldsymbol{\xi}}_2 \rangle = \frac{4}{gh_1h_2}\left[2(1-F_2^2)u_2 - ru_1 - 2ru_2\right].$$

For normalization it is required that (1.9)

$$bc + ad = -(ac)^2\langle \widehat{\boldsymbol{\eta}}_1, \widehat{\boldsymbol{\xi}}_2 \rangle.$$

Therefore, we can take $a = 1$, $d = 0$, $b = -c\langle \widehat{\boldsymbol{\eta}}_1, \widehat{\boldsymbol{\xi}}_2 \rangle$, and

$$c = \frac{1}{r}\left[\frac{4}{g}\frac{u_1u_2}{h_1h_2} + \frac{1}{h_1}(F_2^2-1) + \frac{1}{h_2}(F_1^2-1)\right]^{-1}.$$

Hence the eigenvector set for double criticality is

$$\boldsymbol{\xi}_1 = \begin{pmatrix} r \\ F_1^2 - 1 \\ -r\frac{u_1}{h_1} \\ -\frac{u_2}{h_2}(F_1^2-1) \end{pmatrix}, \quad \boldsymbol{\xi}_2 = \begin{pmatrix} 0 \\ -2\frac{u_1}{gh_1} \\ \frac{r}{h_1} \\ \frac{(F_1^2-1)}{h_2} + \frac{2}{g}\frac{u_1u_2}{h_1h_2} \end{pmatrix} + b\widehat{\boldsymbol{\xi}}_1,$$

and

$$\boldsymbol{\eta}_1 = c\begin{pmatrix} -\frac{1}{h_1}(1-F_2^2) \\ -\frac{2}{g}\frac{u_1u_2}{h_1h_2}(1-F_2^2) + \frac{r}{h_2} \\ 0 \\ \frac{2}{g}\frac{u_1}{h_1}(1-F_2^2) \end{pmatrix}, \quad \text{and} \quad \boldsymbol{\eta}_2 = c\begin{pmatrix} \frac{u_1}{h_1}(1-F_2^2) \\ -r\frac{u_2}{h_2} \\ -(1-F_2^2) \\ r \end{pmatrix},$$

and they satisfy (1.8).

6.2 Emergent two-way Boussinesq equation

At double criticality, apply the theory of §4 to (6.3), to give the two-way Boussinesq equation at criticality

$$q_{TT} + \big(\nu\, q_{XX} - \tfrac{1}{2}\kappa q^2\big)_{XX} = 0\,. \tag{6.18}$$

Compute κ using the formula (4.6). The second derivative of the flux vector at $\mathbf{U}_0$ in the direction $\boldsymbol{\xi}_1$ is

$$\mathrm{D}^2\mathbf{F}(\mathbf{U}_0)(\boldsymbol{\xi}_1,\boldsymbol{\xi}_1) = \frac{d^2}{ds^2}\mathbf{F}(\mathbf{U}_0 + s\boldsymbol{\xi}_1)\bigg|_{s=0} = \begin{pmatrix} -2r^2\frac{u_1}{h_1} \\ -2\frac{u_2}{h_2}(1-F_1^2)^2 \\ r^2\frac{u_1^2}{h_1^2} \\ \frac{u_2^2}{h_2^2}(1-F_1^2)^2 \end{pmatrix},$$

and so

$$\kappa = \langle \boldsymbol{\eta}_2, \mathrm{D}^2\mathbf{F}(\mathbf{U}_0)(\boldsymbol{\xi}_1,\boldsymbol{\xi}_1)\rangle = -3cr^2\frac{u_1u_2}{h_1h_2}F_1^2\,, \tag{6.19}$$

using the conditions (6.8) and (6.9) to simplify. The coefficient κ in (6.19) is nonzero at double criticality and $\mathrm{sign}(\kappa) = \mathrm{sign}(c)$. To compute the coefficient of dispersion, use the definition of $\mathbf{D}$ in (6.5),

$$\nu = \langle \boldsymbol{\eta}_2, \mathbf{D}\boldsymbol{\xi}_1\rangle = \frac{cr}{\rho_1}\left(a_{11}(F_2^2-1) + 2a_{12} + \frac{a_{22}}{r}(F_1^2-1)\right)\,.$$

The expression for ν is quite complicated when the coefficients (6.4) are substituted. The coefficient can be explicitly computed for the numerical values (6.12) and we find that $\nu < 0$. The sign of ν determines whether the Boussinesq is good ($\nu > 0$) or bad ($\nu < 0$). Hence, for the numerical values (6.12) the two-way Boussinesq equation is linearly ill-posed. This ill-posedness is not surprising and may be capturing the reduction of the Kelvin-Helmholtz instability associated with the state (6.12). Even in the ill-posed case, the Boussinesq equation has a wide range of interesting solutions [7, 24, 23].

7 Semisimple double criticality and coupled KdV

In this section, modulation equations are derived for the weakly nonlinear problem in the case of double criticality with two linearly independent eigenvectors. Starting with the system (1.2) suppose that the conditions (1.6) and (1.7) are satisfied with two independent left and right eigenvectors

$$\mathrm{D}\mathbf{F}(\mathbf{U}_0)\boldsymbol{\xi}_j = 0 \quad \text{and} \quad \boldsymbol{\eta}_j^T\mathrm{D}\mathbf{F}(\mathbf{U}_0) = 0\,, \quad j = 1,2\,, \tag{7.1}$$

with normalization (1.8).

Introduce the functions

$$\mathcal{K}_{ij}^k = \langle \boldsymbol{\eta}_k, \mathrm{D}^2\mathbf{F}(\mathbf{U}_0)(\boldsymbol{\xi}_i,\boldsymbol{\xi}_j)\rangle \quad \text{and} \quad \nu_{ij} = \langle \boldsymbol{\eta}_i, \mathbf{D}\boldsymbol{\xi}_j\rangle\,, \quad i,j = 1,2\,. \tag{7.2}$$

We will show that the weakly nonlinear dynamics of (1.2) is governed by the coupled KdV equations

$$\begin{aligned}
\frac{\partial q_1}{\partial T} + \mathcal{K}^1_{11}\, q_1 \frac{\partial q_1}{\partial X} + \mathcal{K}^1_{12}\, \frac{\partial (q_1 q_2)}{\partial X} + \mathcal{K}^1_{22} q_2 \frac{\partial q_2}{\partial X} &= \nu_{11}\, \frac{\partial^3 q_1}{\partial X^3} + \nu_{12}\, \frac{\partial^3 q_2}{\partial X^3} \\
\frac{\partial q_2}{\partial T} + \mathcal{K}^2_{11}\, q_1 \frac{\partial q_1}{\partial X} + \mathcal{K}^2_{12}\, \frac{\partial (q_1 q_2)}{\partial X} + \mathcal{K}^2_{22} q_2 \frac{\partial q_2}{\partial X} &= \nu_{21}\, \frac{\partial^3 q_1}{\partial X^3} + \nu_{22}\, \frac{\partial^3 q_2}{\partial X^3}\,,
\end{aligned} \tag{7.3}$$

where the slow time and space scales correspond to the KdV scaling

$$T = \varepsilon^3 t\,, \quad X = \varepsilon x\,. \tag{7.4}$$

A special case of this result is when the zero eigenvalue is simple with right eigenvector $\boldsymbol{\xi}_1$ and left eigenvector $\boldsymbol{\eta}_1$, and the single KdV equation emerges

$$q_T + \kappa q q_X = \nu q_{XXX}\,, \tag{7.5}$$

with

$$\kappa = \langle \boldsymbol{\eta}_1, \mathrm{D}^2 F(\mathbf{U}_0)(\boldsymbol{\xi}_1, \boldsymbol{\xi}_1)\rangle \quad \text{and} \quad \nu = \langle \boldsymbol{\eta}_1, \mathbf{D}\boldsymbol{\xi}_1\rangle\,. \tag{7.6}$$

The strategy for deriving the modulation equation (7.3) is similar to §4. Introduce an ansatz for the solution of (1.2),

$$\mathbf{U}(x,t) = \mathbf{U}_0 + \varepsilon^2 q_1(X,T,\varepsilon)\boldsymbol{\xi}_1 + \varepsilon^2 q_2(X,T,\varepsilon)\boldsymbol{\xi}_2 + \varepsilon^4 \mathbf{W}(X,T,\varepsilon)\,, \tag{7.7}$$

with

$$\langle \boldsymbol{\eta}_j, \mathbf{W}(X,T,\varepsilon) = 0\,, \quad \text{for } j = 1,2\,, \quad \forall (X,T,\varepsilon)\,.$$

A key difference from the non-semisimple case is that the scaling is the same on both the $q_1\boldsymbol{\xi}_1$ and $q_2\boldsymbol{\xi}_2$ terms. Substitute (7.7) into (1.2)

$$\mathbf{U}_t = \varepsilon^5 (q_1)_T \boldsymbol{\xi}_1 + \varepsilon^5 (q_2)_T \boldsymbol{\xi}_2 + \varepsilon^7 \mathbf{W}_T\,.$$

$$\begin{aligned}
\mathbf{F}(\mathbf{U}) &= \mathbf{F}(\mathbf{U}_0) + \varepsilon^4 \mathrm{D}\mathbf{F}(\mathbf{U}_0)\mathbf{W} + \tfrac12 \varepsilon^4 \mathrm{D}^2 \mathbf{F}(\mathbf{U}_0)(q_1\boldsymbol{\xi}_1 + q_2\boldsymbol{\xi}_2 + \varepsilon^2 \mathbf{W}, q_1\boldsymbol{\xi}_1 + q_2\boldsymbol{\xi}_2 + \varepsilon^2\mathbf{W}) + \cdots \\
\mathbf{F}(\mathbf{U})_x &= \varepsilon^5 \mathrm{D}\mathbf{F}(\mathbf{U}_0)\mathbf{W}_X + \tfrac12 \varepsilon^5 D^2 \mathbf{F}(\mathbf{U}_0)(q_1\boldsymbol{\xi}_1 + q_2\boldsymbol{\xi}_2 + \varepsilon^2 \mathbf{W}, q_1\boldsymbol{\xi}_1 + q_2\boldsymbol{\xi}_2 + \varepsilon^2\mathbf{W})_X + \cdots \\
\mathbf{D}\mathbf{U}_{xxx} &= \varepsilon^5 \mathbf{D}((q_1)_{XXX}\boldsymbol{\xi}_1 + (q_2)_{XXX}\boldsymbol{\xi}_2 + \varepsilon^2 \mathbf{W}_{XXX})\,.
\end{aligned}$$

Split the equation into three parts according to the decomposition

$$\mathbb{R}^n = \operatorname{span}\{\boldsymbol{\xi}_1\} \oplus \operatorname{span}\{\boldsymbol{\xi}_2\} \oplus \mathbb{W}\,,$$

where $\mathbb{W}$ is the orthogonal complement with projection $\mathbf{P} : \mathbb{R}^n \to \mathbb{W}$ defined by

$$\mathbf{P} = \mathbf{I} - \boldsymbol{\xi}_1 \boldsymbol{\eta}_1^T - \boldsymbol{\xi}_2 \boldsymbol{\eta}_2^T\,.$$

The first two components, in the directions $\boldsymbol{\xi}_1$ and $\boldsymbol{\xi}_2$, are,

$$\begin{aligned}
\frac{\partial q_1}{\partial T} + \frac{\partial}{\partial X}\left(\tfrac12 \mathcal{K}^1_{11}\, q_1^2 + \mathcal{K}^1_{12}\, q_1 q_2 + \tfrac12 \mathcal{K}^1_{22}\, q_2^2\right) &= \nu_{11}\,(q_1)_{XXX} + \nu_{12}\,(q_2)_{XXX} + \varepsilon^2 \mathbf{R}_1 \\
\frac{\partial q_2}{\partial T} + \frac{\partial}{\partial X}\left(\tfrac12 \mathcal{K}^2_{11}\, q_1^2 + \mathcal{K}^2_{12}\, q_1 q_2 + \tfrac12 \mathcal{K}^2_{22}\, q_2^2\right) &= \nu_{21}\,(q_1)_{XXX} + \nu_{22}\,(q_2)_{XXX} + \varepsilon^2 \mathbf{R}_2\,,
\end{aligned}$$

where the coefficients are as defined in (7.2). The component in the $\mathbb{W}$ direction, after dividing by ε^5, is

$$\frac{d}{dX}\Big[\mathbf{P}\mathrm{D}\mathbf{F}(\mathbf{U}_0)\mathbf{W} + \tfrac{1}{2}\mathbf{P}\mathrm{D}^2\mathbf{F}(\mathbf{U}_0)(q_1\boldsymbol{\xi}_1 + q_2\boldsymbol{\xi}_2, q_1\boldsymbol{\xi}_1 + q_2\boldsymbol{\xi}_2)$$
$$-\mathbf{P}\mathbf{D}((q_1)_{XXX}\boldsymbol{\xi}_1 + (q_2)_{XXX}\boldsymbol{\xi}_2)\Big] = \varepsilon^2\mathbf{R}_3\,.$$

Since $\mathbf{P}\mathrm{D}\mathbf{F}(\mathbf{U}_0)$ is invertible, the $\mathbb{W}$ equation is solvable for $\mathbf{W}$ as a function of q_1 and q_2 to leading order. This asymptotic validity is confirmed by introducing a perturbation expansion for q_1, q_2 and $\mathbf{W}$,

$$\begin{aligned}
q_1(X,T,\varepsilon) &= q_1^0(X,T) + \varepsilon q_1^1(X,T) + \mathcal{O}(\varepsilon^2)\,,\\
q_2(X,T,\varepsilon) &= q_2^0(X,T) + \varepsilon q_2^1(X,T) + \mathcal{O}(\varepsilon^2)\,,\\
\mathbf{W}(X,T,\varepsilon) &= \mathbf{W}_0(X,T) + \varepsilon\mathbf{W}_1(X,T) + \mathcal{O}(\varepsilon^2)\,,
\end{aligned}$$

Substitution shows that q_1^0 and q_2^0 satisfy the coupled KdV equation and $\mathbf{W}_0$ can be determined as a function of q_1^0 and q_2^0. In principle the higher-order terms $(q_1^n, q_2^n, \mathbf{W}_n)$ can be solved term by term to any order with the above coupled KdV equation the asymptotically correct leading order term, although convergence of this asymptotic series is outside the scope of this paper. Replacing $q_1^0(X,T) \equiv q_1(X,T,0)$ and $q_2^0(X,T) \equiv q_2(X,T,0)$ gives the coupled KdV equation (7.3) as the appropriate modulation equation for the weakly nonlinear dynamics near double semisimple criticality.

The KdV equation and coupled KdV equations derived in (7.5) and (7.3) are relative to a frame of reference fixed in space.

8 Three-layer SWEs with a free surface

In this section the case of shallow water with three layers of differing density and stable stratification is considered. We find that there is a wide range of parameter values for which double criticality occurs.

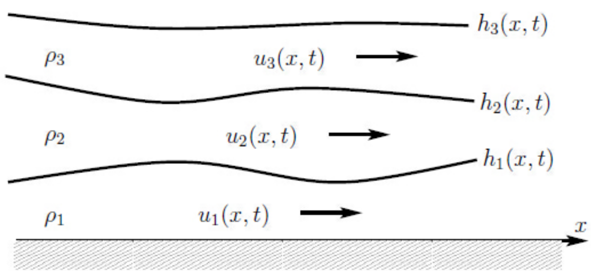

Figure 3: Schematic of the flowfield for three-layer shallow water hydrodynamics.

Let ρ_j, $h_j(x,t)$ and $u_j(x,t)$ be the density, depth and horizontal velocity in each layer, as shown in the schematic in Figure 3, with stable stratification

$$\rho_1 > \rho_2 > \rho_3 > 0\,. \tag{8.1}$$

The governing equations can be written in the form (1.2)

$$\mathbf{U}_t + \mathbf{F}(\mathbf{U})_x = \mathbf{D}\mathbf{U}_{xxx}\,, \quad \mathbf{U} \in \mathbb{R}^6\,, \tag{8.2}$$

with

$$\mathbf{U} = \begin{pmatrix} h_1 \\ h_2 \\ h_3 \\ u_1 \\ u_2 \\ u_3 \end{pmatrix} \quad \text{and} \quad \mathbf{F}(\mathbf{U}) = \begin{pmatrix} h_1 u_1 \\ h_2 u_2 \\ h_3 u_3 \\ \frac{1}{2}u_1^2 + gh_1 + r_2 gh_2 + r_3 gh_3 \\ \frac{1}{2}u_2^2 + gh_1 + gh_2 + \frac{r_3}{r_2}gh_3 \\ \frac{1}{2}u_3^2 + gh_1 + gh_2 + gh_3 \end{pmatrix}$$

where $r_2 = \rho_2/\rho_1$ and $r_3 = \rho_3/\rho_1$, and g is the gravitational constant. These equations are derived (for any number of layers) in Baines [2, 3].

A derivation of the dispersion matrix $\mathbf{D}$ for this case is outside the scope of this paper. Indeed, as far as we are aware there are no derivations in the literature of the dispersive terms for three-layer shallow water flow with a free surface. Here, it is assumed that $\mathbf{D}$ is a given constant matrix, and conditions for double criticality are derived as they are based on the flux vector only.

8.1 Parameter conditions for double criticality

Let $\mathbf{U}_0 = (h_1^0, h_2^0, h_3^0, u_1^0, u_2^0, u_3^0)$ be a constant uniform state, and look at the derivative of the flux vector evaluated on this state,

$$\mathrm{D}\mathbf{F}(\mathbf{U}_0) = \begin{bmatrix} u_1^0 & 0 & 0 & h_1^0 & 0 & 0 \\ 0 & u_2^0 & 0 & 0 & h_2^0 & 0 \\ 0 & 0 & u_3^0 & 0 & 0 & h_3^0 \\ g & gr_2 & gr_3 & u_1^0 & 0 & 0 \\ g & g & gr_3/r_2 & 0 & u_2^0 & 0 \\ g & g & g & 0 & 0 & u_3^0 \end{bmatrix}. \tag{8.3}$$

Here and henceforth the superscripts on h_j and u_j will be dropped to lighten notation, and moreover the same symbols h_j, u_j, $j = 1, 2, 3$, will be used for both the constant state and special values at which criticality and double criticality occur, with the distinction clear from the context. A constant state $\mathbf{U}_0$ is critical if (1.6) is satisfied with

$$\begin{aligned} \det(\mathrm{D}\mathbf{F}(\mathbf{U}_0)) \;\; = \;\; & g^3 h_1 h_2 h_3 \Bigg[(F_1^2 - 1)\left((F_2^2 - 1)(F_3^2 - 1) - \frac{r_3}{r_2} \right) \\ & \qquad - r_2 \left(F_3^2 - 1 + \frac{r_3}{r_2} \right) - r_3 F_2^2 \Bigg]\,, \end{aligned} \tag{8.4}$$

and so parameter values at simple criticality satisfy

$$(1 - F_1^2)\left((1 - F_2^2)(1 - F_3^2) - \frac{r_3}{r_2} \right) + r_2 \left(F_3^2 - 1 + \frac{r_3}{r_2} \right) + r_3 F_2^2 = 0\,, \tag{8.5}$$

where F_j corresponds to the Froude number in each layer,

$$F_j^2 = \frac{u_j^2}{gh_j}\,.$$

This condition for criticality agrees with that in Benton [6] (Benton's strategy is to first reduce the 6×6 matrix to a 3×3 matrix as in equation (8.12) below).

Parameter values satisfying (8.5) but $\Delta'(0) \neq 0$ correspond to simple criticality at which a single KdV equation can be expected to emerge. Here the interest is in double criticality. Computing $\Delta'(\lambda)$ at $\lambda = 0$ and dividing the expression by $g^3 h_1 h_2 h_3$, which is nonzero, the condition $\Delta'(0) = 0$ reduces to

$$\begin{aligned} \frac{2u_1}{gh_1}\left((1-F_2^2)(1-F_3^2) - \frac{r_3}{r_2}\right) & \\ + \frac{2u_2}{gh_2}\big((1-F_1^2)(1-F_3^2) - r_3\big) & \\ + \frac{2u_3}{gh_3}\big((1-F_1^2)(1-F_2^2) - r_2\big) &= 0\,. \end{aligned} \tag{8.6}$$

8.1.1 An example of double criticality

The range of parameter values satisfying the two conditions (8.5) and (8.6) is extensive. Here, an example of parameter values is computed.

First take the velocity in the middle layer to be zero: $F_2 = 0$. In this case two 2 layer flows are embedded in the 3 layer flow and double criticality is then an interaction between the upper layer and lower layer criticality.

With $F_2 = 0$, the first condition (8.5) factorizes into

$$\left(F_3^2 - 1 + \frac{r_3}{r_2}\right)\big(F_1^2 - 1 + r_2\big) = 0\,. \tag{8.7}$$

This equation is satisfied by either

$$F_3^2 = 1 - \frac{r_3}{r_2} \quad \text{or} \quad F_1^2 = 1 - r_2\,, \quad \text{(or both)}\,.$$

The second condition (8.6) reduces to

$$\frac{2u_1}{gh_1}\left(F_3^2 - 1 + \frac{r_3}{r_2}\right) + \frac{2u_3}{gh_3}\big(F_1^2 - 1 + r_2\big) = 0\,. \tag{8.8}$$

In fact the two equations (8.7) and (8.8) can not be both satisfied unless both factors in (8.7) are zero. Hence one solution for double criticality is

$$F_1^2 = 1 - r_2\,, \quad F_2^2 = 0\,, \quad F_3^2 = 1 - \frac{r_3}{r_2}\,. \tag{8.9}$$

However, the distinction between semisimple double criticality and non-semisimple double criticality can be made only by computing the eigenvectors.

8.2 Geometric eigenvector(s) at double criticality

To determine geometric multiplicity the kernel of $\mathrm{D}\mathbf{F}(\mathbf{U}_0)$ needs to be computed. Let $\boldsymbol{\xi} \in \mathbb{R}^6$ be an eigenvector of $\mathrm{D}\mathbf{F}(\mathbf{U}_0)$,

$$\mathrm{D}\mathbf{F}(\mathbf{U}_0)\boldsymbol{\xi} = 0\,, \tag{8.10}$$

with $\mathrm{D}\mathbf{F}(\mathbf{U}_0)$ given in (8.3). Since h_1, h_2 and h_3 are strictly positive the first three equations in (8.10) can be solved to give

$$\xi_{j+3} = -p_j \xi_j \,, \quad j = 1,2,3 \,, \quad p_j = \frac{u_j}{h_j} \,. \tag{8.11}$$

Substitution into the other three equations in (8.10) then gives a 3×3 system

$$\begin{bmatrix} 1-F_1^2 & r_2 & r_3 \\ 1 & 1-F_2^2 & \frac{r_3}{r_2} \\ 1 & 1 & 1-F_3^2 \end{bmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{pmatrix} = 0 \,. \tag{8.12}$$

Eliminate ξ_2 using the third equation,

$$\xi_2 = -\xi_1 - (1-F_3^2)\xi_3 \,, \tag{8.13}$$

and substitute into the first two equations, giving the 2×2 system

$$\begin{bmatrix} F_2^2 & \frac{r_3}{r_2} - (1-F_2^2)(1-F_3^2) \\ 1 - r_2 - F_1^2 & r_3 - r_2(1-F_3^2) \end{bmatrix} \begin{pmatrix} \xi_1 \\ \xi_3 \end{pmatrix} = 0 \,, \tag{8.14}$$

The geometric multiplicity is determined by the number of independent solutions of (8.14). When the conditions (8.9) are satisfied, the 2×2 matrix on the left-hand side of (8.14) vanishes giving two independent solutions, and hence the example (8.9) corresponds to the semisimple case. Before proceeding to computing the eigenvectors in this case, we first identify parameter values where the double criticality can be non-semisimple.

8.3 The non-semisimple case – an example

Since $F_2 = 0$ leads to semisimple double criticality, take $F_3 = 0$, simplifying the double criticality conditions (8.5) and (8.6) to

$$(1-F_1^2)\left((1-F_2^2) - \frac{r_3}{r_2}\right) + (r_3 - r_2) + r_3 F_2^2 = 0 \,, \tag{8.15}$$

and

$$\frac{2u_1}{gh_1}\left(1 - F_2^2 - \frac{r_3}{r_2}\right) + \frac{2u_2}{gh_2}\left(1 - F_1^2 - r_3\right) = 0 \,. \tag{8.16}$$

The criticality condition (8.15) can be factorized into

$$(F_1^2 - 1 + r_3)\left(F_2^2 - 1 + \frac{r_3}{r_2}\right) = r_2 + \frac{r_3^2}{r_2} - 2r_3 \,.$$

Plotting in the (F_1, F_2) plane gives two families of curves. Intersection with (8.16) then gives points of double criticality. For example, take

$$r_2 = \frac{1}{2}, \quad F_1^2 = r_3, \quad F_2^2 = \frac{1}{2} - r_3, \quad F_3 = 0 \,, \quad \text{and} \quad r_3 \neq \frac{1}{2} \,. \tag{8.17}$$

At these values, the conditions (8.15) and (8.16) are satisfied with the additional requirement

$$2\frac{u_2}{h_2} + \frac{u_1}{h_1} = 0\,. \tag{8.18}$$

With the parameter values (8.17) and the parameter constraint (8.18) the kernel of $\mathrm{D}\mathbf{F}(\mathbf{U}_0)$ is one dimensional, and the generalised eigenvectors are found to be

$$\widehat{\boldsymbol{\xi}}_1 = \begin{pmatrix} 1 \\ -2 \\ 1 \\ -p_1 \\ 2p_2 \\ 0 \end{pmatrix}, \quad \widehat{\boldsymbol{\xi}}_2 = \frac{1}{g(1-2r_3)} \begin{pmatrix} -2p_1 \\ 0 \\ 2p_1 \\ \frac{g(1-2r_3)}{h_1} + 2p_1^2 \\ -\frac{2g(1-2r_3)}{h_2} \\ \frac{g(1-2r_3)}{h_3} \end{pmatrix},$$

where

$$p_j = \frac{u_j}{h_j}, \quad j = 1, 2, 3\,.$$

The kernel of $\mathrm{D}\mathbf{F}(\mathbf{U}_0)^T$ is also one-dimensional with generalized eigenvectors

$$\widehat{\boldsymbol{\eta}}_1 = \begin{pmatrix} -\frac{2p_1^2}{g(1-2r_3)} - \frac{1}{h_1} \\ \frac{1}{h_2} \\ -\frac{r_3}{h_3} \\ \frac{2p_1}{g(1-2r_3)} \\ 0 \\ -\frac{2p_1 r_3}{g(1-2r_3)} \end{pmatrix}, \quad \widehat{\boldsymbol{\eta}}_2 = \begin{pmatrix} p_1 \\ -p_2 \\ 0 \\ -1 \\ 1 \\ -r_3 \end{pmatrix}.$$

Now use the algorithm in Appendix A to normalize these vectors. Without loss of generality take $a = 1$ and $d = 0$. Then the normalized generalized eigenvectors are

$$\boldsymbol{\xi}_1 = \widehat{\boldsymbol{\xi}}_1\,, \quad \boldsymbol{\xi}_2 = \widehat{\boldsymbol{\xi}}_2 + b\widehat{\boldsymbol{\xi}}_1\,, \quad \boldsymbol{\eta}_1 = c\widehat{\boldsymbol{\eta}}_1\,, \quad \boldsymbol{\eta}_2 = c\widehat{\boldsymbol{\eta}}_2\,.$$

The normalization (1.8) then requires

$$c = -\frac{gh_1h_2h_3(1-2r_3)}{4p_1^2h_1h_2h_3 + g(1-2r_3)(r_3h_1h_2 + 2h_1h_3 + h_2h_3)}\,, \tag{8.19}$$

and

$$b = -\frac{4cp_1}{g(1-2r_3)}\left[\frac{2r_3}{h_1(1-2r_3)} + \frac{1}{h_1} - \frac{r_3}{h_3}\right]. \tag{8.20}$$

This case, $F_3 = 0$, effectively embeds the two-layer example from §6 in the three-layer problem, and the two streams are required to go in opposite directions: the constraint (8.18) requires $u_1u_2 < 0$ and so the uniform flow is Kelvin-Helmholz unstable.

8.3.1 The form of the emergent Boussinesq equation

The nonlinear coefficient of the two-way Boussinesq equation can be calculated using the formula (4.6). The coefficient of the nonlinearity is found to be

$$\kappa := \tfrac{1}{2}\langle \boldsymbol{\eta}_2, \mathrm{D}^2\mathbf{F}(\mathbf{U}_0)(\boldsymbol{\xi}_1, \boldsymbol{\xi}_1)\rangle = \frac{c}{2p_1}(p_1^3 - 4p_2^3) = 3cp_2^2,$$

by using (8.18). The dispersion coefficient ν requires an explcit expression for the dispersion matrix $\mathbf{D}$. The important property of ν is its sign, and taking a clue from the two-layer case and the presence of Kelvin-Helmholz instability, it is expected that $\nu < 0$ in this case. Therefore the emergent Boussinesq model for three layered flow is

$$q_{TT}\big(\nu\, q_{XX} - 3cp_2^2 q^2\big)_{XX} = 0\,, \tag{8.21}$$

with the conjecture that $\nu < 0$. As in the two-layer case the emergent Boussinesq equations was a wide range of interesting solutions [7, 24, 23].

8.4 The semsimple case – an example

At the special case (8.9) it is clear that there are two independent solutions, giving the semisimple case. In this subsection the eigenvectors are calculated and the potential form of the emergent coupled KdV equation given. Fix the parameter values at

$$u_2 = 0\,, \quad F_1^2 = 1 - r_2\,, \quad F_3^2 = 1 - \frac{r_3}{r_2}\,. \tag{8.22}$$

It is now straightforward to calculate the eigenvectors

$$\boldsymbol{\xi}_1 = \begin{pmatrix} 1 \\ -1 \\ 0 \\ -u_1/h_1 \\ 0 \\ 0 \end{pmatrix}, \quad \boldsymbol{\xi}_2 = \begin{pmatrix} 0 \\ -r_3/r_2 \\ 1 \\ 0 \\ 0 \\ -u_3/h_3 \end{pmatrix},$$

and the adjoint eigenvectors, which satisfy

$$\mathrm{D}\mathbf{F}(\mathbf{U}_0)^T \boldsymbol{\eta} = 0\,,$$

are

$$\boldsymbol{\eta}_1 = \frac{h_1}{2u_1} \begin{pmatrix} u_1/h_1 \\ 0 \\ 0 \\ -1 \\ r_2 \\ 0 \end{pmatrix}, \quad \boldsymbol{\eta}_2 = \frac{h_3}{2u_3} \begin{pmatrix} 0 \\ 0 \\ u_3/h_3 \\ 0 \\ 1 \\ -1 \end{pmatrix}.$$

They have been normalised to satisfy (1.8).

8.4.1 The form of the emergent coupled KdV

The nonlinearity in the reduced system in the semisimple case is

$$\begin{aligned} &\frac{\partial}{\partial x}\left(\tfrac{1}{2}\mathcal{K}_{11}^1 q_1^2 + \mathcal{K}_{12}^1 q_1 q_2 + \tfrac{1}{2}\mathcal{K}_{22}^1 q_2^2\right) \\ &\frac{\partial}{\partial x}\left(\tfrac{1}{2}\mathcal{K}_{11}^2 q_1^2 + \mathcal{K}_{12}^2 q_1 q_2 + \tfrac{1}{2}\mathcal{K}_{22}^2 q_2^2\right) \end{aligned} \tag{8.23}$$

where $\mathcal{K}_{ij}^k$ is defined in (7.2). Only 2 out of 6 $\mathcal{K}_{ij}^k$ coefficients are nonzero in this case. Computing we find

$$\mathcal{K}_{11}^1 = -\frac{3}{2}\frac{u_1}{h_1}\,, \quad \mathcal{K}_{12}^1 = 0\,, \quad \mathcal{K}_{22}^1 = 0$$

$$\mathcal{K}_{11}^2 = 0\,, \quad \mathcal{K}_{12}^2 = 0\,, \quad \mathcal{K}_{22}^2 = -\frac{3}{2}\frac{u_3}{h_3}\,,$$

where u_1, u_3 and h_1, h_3 satisfy the constraints (8.22). Hence, the coupled KdV equations for three-layer shallow water at the parameter values (8.22) are

$$\begin{aligned} (q_1)_T - \tfrac{3u_1}{2h_1} q_1 (q_1)_x &= \nu_{11}(q_1)_{XXX} + \nu_{12}(q_2)_{XXX} \\ (q_2)_T - \tfrac{3u_3}{2h_3} q_2 (q_2)_x &= \nu_{21}(q_1)_{XXX} + \nu_{22}(q_2)_{XXX}\,. \end{aligned} \tag{8.24}$$

The coefficients of the nonlinearity agree with the results in Grimshaw [17]. In [17] coupled KdV equations for the case of three layers with a rigid lid are derived and in that case the coefficients of the nonlinearity are as in (8.24) but $\mathscr{K}_{22}^2$ has a positive sign but Roger Grimshaw (private communication) has confirmed that he also finds a negative sign for $\mathscr{K}_{22}^2$ when the upper layer is free. In [17] the coefficients of the dispersion terms in (8.24) are also calculated, as well as unfolding terms.

9 Concluding remarks

A theory has been presented that reduces the class of PDEs (1.2), in the neighborhood of uniform flows, to KdV, coupled KdV, or two-way Boussinesq depending on the structure of the zero eigenvalues of the Jacobian of the flux vector. The theory is relative a fixed frame of reference. The theory can be enriched by working relative to a moving frame. With the shift $x \mapsto x - ct$, the system of PDEs (1.2) becomes

$$\mathbf{U}_t + \big[\mathbf{F}(\mathbf{U}) - c\mathbf{U}\big]_x = \mathbf{D}\mathbf{U}_{xxx}\,, \quad \mathbf{U} \in \mathbb{R}^n\,. \tag{9.1}$$

The theory then goes through with $\mathbf{F}(\mathbf{U})$ replaced by $\widehat{\mathbf{F}}(\mathbf{U}, c) = \mathbf{F}(\mathbf{U}) - c\mathbf{U}$ with the additional parameter c. Criticality and double criticality are then relative to the moving frame.

The theory in this paper is based on the class of PDEs (1.2). However, the key features are (a) a class of uniform flows or parallel flows, (b) zero eigenvalues of the linearization about the basic state, (c) the algebraic multiplicity and geometric multiplicity of the zero eigenvalues, and (d) the form of the ansatz in (4.2) and (7.7). All these features should extend with appropriate modification to other systems of PDEs including the full Euler equations.

The theory can be extended to the case of PDEs of the type (1.2) in two space dimensions and time. The natural extension of (1.2) is

$$\mathbf{U}_t + (\mathbf{F}(\mathbf{U}))_x + (\mathbf{G}(\mathbf{U}))_y = \mathbf{D}_1\mathbf{U}_{xxx} + \mathbf{D}_2\mathbf{U}_{xxy} + \mathbf{D}_3\mathbf{U}_{xyy} + \mathbf{D}_4\mathbf{U}_{yyy}\,, \quad \mathbf{U} \in \mathbb{R}^n\,,$$

where $\mathbf{F}(\mathbf{U})$ and $\mathbf{G}(\mathbf{U})$ are the flux vectors in the x and y directions respectively. The constant vectors $\mathbf{U}_0 \in \mathbb{R}^n$ are still exact solutions. When either of the flux vectors is degenerate (either $\det[D\mathbf{F}(\mathbf{U}_0)] = 0$ or $\det[D\mathbf{G}(\mathbf{U}_0)] = 0$) then the strategy of this paper generalizes with appropriate choice of scaling and ansatz. The full details of this generalization will be considered elsewhere, but one can speculate that this strategy will generate the Kadomtsev-Petviashvili equation, the $2+1$ Boussinesq equation, and other potential coupled equations for higher order singularities.

The interest in the reduction of (1.2) or the full Euler equations to the two-way Boussinesq or coupled KdV is that the reduced equations are easier to analyze and their range of solutions give clues to related solutions in the full system. For example the 2-way Boussinesq equation has a vast range of interesting solutions: solitary wave solutions, has blow-up for some initial data and global existence for other initial data, and it is known to be completely integrable, and has a vast range of multi-pulse and multi-periodic solutions (e.g. [7, 8, 34, 24]). Similarly the coupled KdV equation is known to have a wide range of solitary wave solutions [19].

If $\mathrm{D}\mathbf{F}(\mathbf{U}_0)$ has a zero eigenvalue of multiplicity N with N linearly independent eigenvectors, then the theory of §7 can be generalized to show that N coupled KdV equations arise in the weakly nonlinear problem. However, the number of independent parameters required becomes large. However, the case of three coupled KdV equations are of interest and have been analyzed in the literature [35]. A more interesting problem is when the geometric multiplicity is less than the algebraic multiplicity. For example when the algebraic multiplicity is three and geometric multiplicity is two, then the theory here suggests that the weakly nonlinear problem will have a two-way Boussinesq equation coupled to a KdV equation.

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— Appendix —

A Normalization of generalized eigenvectors

There is a subtlety in the normalisation of generalised eigenvectors and so the necessary theory is recorded here. Consider a general (non-symmetric) $n\times n$ matrix $\mathbf{A}$ with characteristic polynomial

$$\Delta(\lambda) = \det\big[\mathbf{A} - \lambda\mathbf{I}\big] = 0\,.$$

Suppose zero is an eigenvalue of algebraic multiplicity two

$$\Delta(0) = \Delta'(0) = 0 \quad \text{and} \quad \Delta''(0) \neq 0\,, \tag{1.1}$$

but the geometric multiplicity (dimension of the kernel of $\mathbf{A}$) is one,

$$\mathbf{A}\boldsymbol{\xi}_1 = 0\,. \tag{1.2}$$

There exists a generalised eigenvector $\boldsymbol{\xi}_2$ satisfying

$$\mathbf{A}\boldsymbol{\xi}_2 = \boldsymbol{\xi}_1\,. \tag{1.3}$$

Since $\mathbf{A}$ is not symmetric, there exists adjoint eigenvectors $\boldsymbol{\eta}_1$ and $\boldsymbol{\eta}_2$ satisfying

$$\mathbf{A}^T\boldsymbol{\eta}_2 = 0 \quad \text{and} \quad \mathbf{A}^T\boldsymbol{\eta}_1 = \boldsymbol{\eta}_2\,. \tag{1.4}$$

Since $\boldsymbol{\xi}_2$ exists, the equation (1.3) is solvable and so

$$0 = \langle \mathbf{A}^T\boldsymbol{\eta}_2, \boldsymbol{\xi}_2\rangle = \langle \boldsymbol{\eta}_2, \mathbf{A}\boldsymbol{\xi}_2\rangle = \langle \boldsymbol{\eta}_2, \boldsymbol{\xi}_1\rangle \quad \Rightarrow \quad \langle \boldsymbol{\eta}_2, \boldsymbol{\xi}_1\rangle = 0\,. \tag{1.5}$$

There is one other identity that follows from the definitions of the eigenvectors and adjoint eigenvectors

$$\langle \boldsymbol{\eta}_1, \boldsymbol{\xi}_1\rangle = \langle \boldsymbol{\eta}_1, \mathbf{A}\boldsymbol{\xi}_2\rangle = \langle \mathbf{A}^T\boldsymbol{\eta}_1, \boldsymbol{\xi}_2\rangle = \langle \boldsymbol{\eta}_2, \boldsymbol{\xi}_2\rangle\,. \tag{1.6}$$

We are free to fix these two inner products, so take them to be unity,

$$\langle \boldsymbol{\eta}_1, \boldsymbol{\xi}_1\rangle = \langle \boldsymbol{\eta}_2, \boldsymbol{\xi}_2\rangle = 1\,. \tag{1.7}$$

The aim is to normalise the eigenvectors so that

$$\langle \boldsymbol{\eta}_i, \boldsymbol{\xi}_j\rangle = \delta_{i,j}\,, \quad i,j = 1,2\,. \tag{1.8}$$

The tricky case is the normalisation $\langle \boldsymbol{\eta}_1, \boldsymbol{\xi}_2\rangle$, since in general this inner product may not be zero.

To clarify the normalisation, it is necessary to make explicit the arbitrary constants. Let $\widehat{\boldsymbol{\xi}}_1$ be any fixed element in the kernel of $\mathbf{A}$, then

$$\boldsymbol{\xi}_1 = a\widehat{\boldsymbol{\xi}}_1\,,$$

with a an arbitrary non-zero real number. Substituting into (1.3) gives

$$\boldsymbol{\xi}_2 = a\widehat{\boldsymbol{\xi}}_2 + b\widehat{\boldsymbol{\xi}}_1\,,$$

where $\widehat{\boldsymbol{\xi}}_2$ is any fixed particular solution of $\mathbf{A}\widehat{\boldsymbol{\xi}}_2 = \widehat{\boldsymbol{\xi}}_1$, and b is an arbitrary (possibly zero) real number.

Introduce a similar construction for the adjoint eigenvectors

$$\boldsymbol{\eta}_2 = c\widehat{\boldsymbol{\eta}}_2\,,$$

where c is an arbitrary non-zero real number and $\widehat{\boldsymbol{\eta}}_2$ is any fixed element in the kernel of $\mathbf{A}^T$. Similarly

$$\boldsymbol{\eta}_1 = c\widehat{\boldsymbol{\eta}}_1 + d\widehat{\boldsymbol{\eta}}_2\,,$$

where $\widehat{\boldsymbol{\eta}}_1$ is any fixed particular solution of $\mathbf{A}^T\widehat{\boldsymbol{\eta}}_1 = \widehat{\boldsymbol{\eta}}_2$, and d is any arbitrary (possibly zero) real number.

At this point there are four free parameters in the construction of the eigenvectors and adjoint eigenvectors. However, two of the parameters can be fixed by the normalisation. Imposing (1.7) gives

$$\begin{aligned} 1 &= \langle \boldsymbol{\eta}_1, \boldsymbol{\xi}_1 \rangle = ac \langle \widehat{\boldsymbol{\eta}}_1, \widehat{\boldsymbol{\xi}}_1 \rangle \\ 1 &= \langle \boldsymbol{\eta}_2, \boldsymbol{\xi}_2 \rangle = ac \langle \widehat{\boldsymbol{\eta}}_2, \widehat{\boldsymbol{\xi}}_2 \rangle \,. \end{aligned}$$

These two equations give only one condition due to the equivalence (1.6). The condition $\langle \widehat{\boldsymbol{\eta}}_2, \widehat{\boldsymbol{\xi}}_1 \rangle = 0$ is satisfied independent of the choice of parameters, since

$$0 = \langle \boldsymbol{\eta}_2, \boldsymbol{\xi}_1 \rangle = ac \langle \widehat{\boldsymbol{\eta}}_2, \widehat{\boldsymbol{\xi}}_1 \rangle \,,$$

and $ac \neq 0$. The final normalisation is

$$\begin{aligned} 0 &= \langle \boldsymbol{\eta}_1, \boldsymbol{\xi}_2 \rangle \\ &= \langle c\widehat{\boldsymbol{\eta}}_1 + d\widehat{\boldsymbol{\eta}}_2, a\widehat{\boldsymbol{\xi}}_2 + b\widehat{\boldsymbol{\xi}}_1 \rangle \\ &= ac \langle \widehat{\boldsymbol{\eta}}_1, \widehat{\boldsymbol{\xi}}_2 \rangle + bc \langle \widehat{\boldsymbol{\eta}}_1, \widehat{\boldsymbol{\xi}}_1 \rangle + da \langle \widehat{\boldsymbol{\eta}}_2, \widehat{\boldsymbol{\xi}}_2 \rangle \\ &= ac \langle \widehat{\boldsymbol{\eta}}_1, \widehat{\boldsymbol{\xi}}_2 \rangle + \tfrac{b}{a} + \tfrac{d}{c} \end{aligned}$$

Hence, given $\langle \widehat{\boldsymbol{\eta}}_1, \widehat{\boldsymbol{\xi}}_2 \rangle$, choose b, d so that

$$bc + ad = -(ac)^2 \langle \widehat{\boldsymbol{\eta}}_1, \widehat{\boldsymbol{\xi}}_2 \rangle \,.$$

To summarise, once the four vectors $\widehat{\boldsymbol{\xi}}_1, \widehat{\boldsymbol{\xi}}_2, \widehat{\boldsymbol{\eta}}_1, \widehat{\boldsymbol{\eta}}_2$ are fixed, the four free parameters a, b, c, d are required to satisfy

$$ac = \langle \widehat{\boldsymbol{\eta}}_1, \widehat{\boldsymbol{\xi}}_1 \rangle^{-1} \quad \text{and} \quad bc + ad = -(ac)^2 \langle \widehat{\boldsymbol{\eta}}_1, \widehat{\boldsymbol{\xi}}_2 \rangle \,, \tag{1.9}$$

resulting in the normalisation (1.8).

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