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ON THE ELLIPTIC-HYPERBOLIC TRANSITION IN WHITHAM MODULATION THEORY

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Abstract. The dispersionless Whitham modulation equations in one space dimension and time are generically hyperbolic or elliptic, and breakdown at the transition, which is a curve in the frequency-wavenumber plane. In this paper, the modulation theory is reformulated with a slow phase and different scalings resulting in a phase modulation equation near the singular curves which is a geometric form of the two-way Boussinesq equation. This equation is universal in the same sense as Whitham theory. Moreover, it is dispersive, and it has a wide range of interesting multiperiodic, quasiperiodic and multi-pulse localized solutions. This theory shows that the elliptic-hyperbolic transition is a rich source of complex behaviour in nonlinear wave fields. There are several examples of these transition curves in the literature to which the theory applies. For illustration the theory is applied to the complex nonlinear Klein-Gordon equation which has two singular curves in the manifold of periodic travelling waves.

Key words. nonlinear waves, modulation, Lagrangian, multisymplectic, traveling waves

AMS subject classifications. 74J30,70S05,76B15

1. Introduction.

Modulational instability is one of the key ways that periodic travelling waves become unstable. The wavelength of the perturbation is slightly longer than the wavelength of the underlying periodic wave. In conservative systems this instability, in the weakly nonlinear case, is most closely associated with the *Benjamin-Feir instability* [4], and in non-conservative systems with the *Eckhaus instability* [15]. For weakly nonlinear periodic travelling waves, the simplest way to analyze modulational instability is to derive a nonlinear Schrödinger equation or complex Ginzburg Landau equation [35]. A history of the beginnings of modulation instability is given in [40].

For *finite-amplitude* periodic travelling waves in conservative systems modulation instability is captured by the Whitham modulation theory. For a nonlinear periodic travelling wave of frequency ω and wavenumber k, modulation of the form

$$k \mapsto k + q(X,T,\varepsilon) \quad \text{and} \quad \omega \mapsto \omega + \Omega(X,T,\varepsilon)\,, \tag{1}$$

where $X = \varepsilon x$, $T = \varepsilon t$, in the Whitham theory, results in

$$q_T = \Omega_X \quad \text{and} \quad \mathscr{A}_T + \mathscr{B}_X = 0\,, \tag{2}$$

to leading order in ε, where $\mathscr{A}(\omega+\Omega, k+q)$ and $\mathscr{B}(\omega+\Omega, k+q)$ are the *wave action* and *wave action flux* respectively, evaluated on the family of periodic travelling waves [37, 38]. The Whitham modulation equations (WMEs) in (2) are a closed nonlinear first order set of PDEs for the functions Ω and q. Generically, the WMEs are either hyperbolic or elliptic. The linearization of these equations about the basic state, represented by ω and k, is

$$q_T = \Omega_X \quad \text{and} \quad \mathscr{A}_\omega \Omega_T + \mathscr{A}_k q_T + \mathscr{B}_\omega \Omega_X + \mathscr{B}_k q_X = 0\,, \tag{3}$$

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or, with the assumption $\mathscr{A}_\omega \neq 0$, they can be written in the standard form,

$$\begin{pmatrix} q \\ \Omega \end{pmatrix}_T + \mathbf{A}(\omega,k)\begin{pmatrix} q \\ \Omega \end{pmatrix}_X = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \tag{4}$$

where

$$\mathbf{A}(\omega,k) = \frac{1}{\mathscr{A}_\omega}\begin{bmatrix} 0 & -\mathscr{A}_\omega \\ \mathscr{B}_k & \mathscr{A}_k + \mathscr{B}_\omega \end{bmatrix}. \tag{5}$$

Here, $\mathscr{A}$ and $\mathscr{B}$ are evaluated at $\Omega = q = 0$. The characteristics are

$$c^\pm = \frac{\mathscr{A}_\omega + \mathscr{B}_k}{2\mathscr{A}_\omega} \pm \frac{1}{\mathscr{A}_\omega}\sqrt{-\Delta_L}\,, \quad \Delta_L = \det\begin{bmatrix} \mathscr{A}_\omega & \mathscr{A}_k \\ \mathscr{B}_\omega & \mathscr{B}_k \end{bmatrix} = \det\begin{bmatrix} \mathscr{L}_{\omega\omega} & \mathscr{L}_{\omega k} \\ \mathscr{L}_{k\omega} & \mathscr{L}_{kk} \end{bmatrix}, \tag{6}$$

using the identities, $\mathscr{A} = \mathscr{L}_\omega$ and $\mathscr{B} = \mathscr{L}_k$, from Whitham theory, where $\mathscr{L}$ is the averaged Lagrangian. The modulation instability is recovered by letting

$$\begin{pmatrix} q(X,T) \\ \Omega(X,T) \end{pmatrix} = \mathrm{Re}\left\{ \begin{pmatrix} \widehat{q} \\ \widehat{\Omega} \end{pmatrix} \mathrm{e}^{\lambda T + \mathrm{i}\nu X} \right\},$$

and substituting into (3) giving

$$\lambda = \mathrm{i}c^\pm \nu\,,$$

and so an unstable exponent (positive real part of λ) with modulation wave number ν exists precisely when $\Delta_L > 0$. While ν is of order one, $\nu X = \varepsilon\nu x$, and so the modulation wave number can be interpreted as being of order ε in the original coordinates. Since the WMEs are dispersionless, there is no wavenumber cutoff of the modulation instability.

In terms of characteristics, this modulation instability highlights the Lighthill condition [22]: when $\Delta_L > 0$ the linearized WMEs are elliptic and when $\Delta_L < 0$ they are hyperbolic. This criterion, and other features of Whitham modulation theory have been widely studied and there is a vast literature; recent examples are the book [20], the review articles [29, 11], and the special issue on Whitham theory [5].

In this paper the interest is in the case when the Lighthill determinant is singular

$$\Delta_L := \det\begin{bmatrix} \mathscr{A}_\omega & \mathscr{A}_k \\ \mathscr{B}_\omega & \mathscr{B}_k \end{bmatrix} = 0 \quad \text{but} \quad \mathscr{A}_\omega \neq 0 \quad \text{and} \quad \mathscr{A}_k \neq 0\,. \tag{7}$$

The condition $\Delta_L = 0$ defines a curve in the (ω,k) plane locally separating stable and unstable states. The set $\Delta_L^{-1}(0)$, which is not necessarily connected, will be denoted by

$$\Sigma^1 = \Delta_L^{-1}(0) = \left\{ (\omega,k) \in U \subset \mathbb{R}^2 \;:\; \Delta_L = 0 \right\}, \tag{8}$$

where U is the open subset of $\mathbb{R}^2$ for which periodic travelling waves exist. This notation comes from singularity theory and is elaborated further in §2, as the geometry of Σ^1 appears in the phase modulation theory. A typical Σ^1 curve is shown in Figure 1.

As far as we are aware, a modulation theory near an elliptic-hyperbolic transition curve, generalizing Whitham modulation theory, has not been attempted heretofore. One strategy for deriving a new modulation equation near a Σ^1 curve is to take the

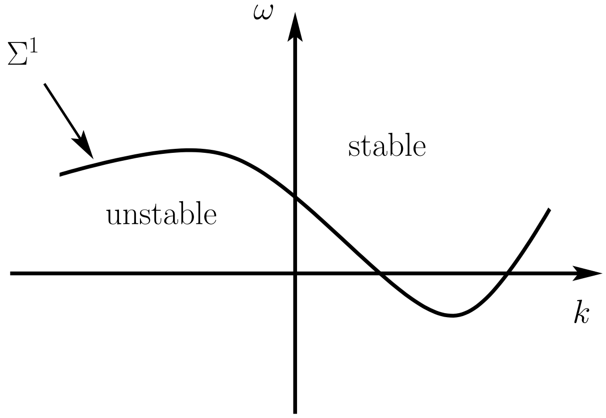

Fig. 1. *A typical curve defined by $\Delta_L = 0$ in the (ω, k) plane.*

Whitham theory to higher order. Luke [25] has given a theory and algorithm for deducing higher-order Whitham equations. However, the theory is quite complicated after the first order, and a clear closed system does not immediately emerge.

Another strategy is to change the time scale. The breakdown of the WMEs can be interpreted as a signal that a change in time scale, from $T = \varepsilon t$ to $T = \varepsilon^2 t$, is appropriate. Another feature of points on Σ^1 curves with $\mathscr{A}_k \neq 0$ is that the linearized WMEs have a double characteristic with nonzero speed, suggesting a moving frame is appropriate. Since $\mathscr{A}_k = \mathscr{B}_\omega$, the speed at the double characteristic is

$$c_g = \frac{\mathscr{A}_k}{\mathscr{A}_\omega}\,. \tag{9}$$

The symbol c_g is used as this velocity is a form of nonlinear group velocity. It is interpretable as (minus) the derivative of the frequency with respect to the wavenumber with wave action fixed. There are various generalizations of group velocity to the nonlinear regime in Whitham theory (e.g. [17, 30]). The definition (9) is preferred here as it is the velocity at the double characteristic, and arises naturally in the nonlinear modulation theory.

Our strategy for developing a nonlinear modulation theory near Σ^1 curves is to slow down the time scale, go into a c_g-boosted moving frame, and slow down the phase, wavenumber and frequency modulation. The modulation mapping (1) is replaced by

$$k \mapsto k + \varepsilon^2 q(X, T, \varepsilon)\,, \tag{10}$$

and

$$\omega \mapsto \omega - c_g \varepsilon^2 q(X, T, \varepsilon) + \varepsilon^3 \Omega(X, T, \varepsilon)\,, \tag{11}$$

with

$$X = \varepsilon(x - c_g t) \quad \text{and} \quad T = \varepsilon^2 t\,.$$

Substitution into the governing equations, which are the Euler-Lagrange equations based on a general abstract Lagrangian, then leads at fifth order in ε via a solvability condition to the new modulation equations replacing (2) and (3),

$$q_T = \Omega_X \quad \text{and} \quad \mathscr{A}_\omega \Omega_T + \kappa q q_X + \mathscr{K} q_{XXX} = 0\,. \tag{12}$$

Differentiating the second equation with respect to X and using the first equation shows that it is a variant of the two-way Boussinesq equation, but with coefficents that are universal in the same sense that the Whitham equations are universal (that they follow from the abstract properties of the Lagrangian). The importance of $\mathscr{A}_\omega \neq 0$ shows up in the first coefficient. The second coefficient, κ is the second derivative of the mapping $(\omega, k) \mapsto (\mathscr{A}(\omega, k), \mathscr{B}(\omega, k))$ evaluated on the kernel of the first derivative, and the coefficient of dispersion $\mathscr{K}$ is determined by a Jordan chain argument. The details of the derivation leading to (12) are given in §4.

A two-way Boussinesq equation is derived via phase modulation in [34], but in that case the Whitham theory has a double *zero* characteristic, and the phase modulation is *relative to a stationary frame of reference*. Moreover, that theory requires two parameters and is not associated directly with a stability-instability transition. The theory in this paper addresses the stability-instability transition directly, and will be more prevalent in applications as it only requires the variation of a single parameter.

There are several interesting consequences due to the modulation equation (12) near Σ^1 curves: dispersion is generated, thereby admitting coherent structures (e.g. solitary waves), and a wide range of complex solutions are generated (multi-pulse solitary waves [18], breathers [13], blowup [7, 36], integrable structures [6]) and it has its own elliptic-hyperbolic dichotomy. The two-way Boussinesq equation is said to be elliptic ("bad") if it is linearly ill-posed (corresponding in this case to $\mathscr{A}_\omega \mathscr{K} < 0$) and hyperbolic ("good") for the reverse sign. The good Boussinesq equation moderates the modulational instability, whereas the bad Boussinesq equation enhances the instability. In either case, dispersion identifies a cut-off wavenumber for the modulation instability which is absent in the dispersionless WMEs.

There are two familiar examples in the literature where Σ^1 curves arise. The first is stabilization of the Benjamin-Feir instability, for water waves on infinite depth, at large amplitude [24, 28, 39]. This case is interpreted in terms of the theory here in §6. The second is stabilization of the Benjamin-Feir instability when the depth parameter is below a critical threshold, $kh_0 \approx 1.363$ [3, 16]. This latter case occurs in the weakly nonlinear regime, and a theory for this case is developed by Johnson [19] near the threshold by extending the nonlinear Schrödinger equation to higher order. A new example has recently been discovered by Maiden & Hoefer [26] where an elliptic-hyperbolic transition has been discovered in modulation of viscous fluid conduit waves. However, in the latter two examples the modulation is *multiphase* and so the theory of this paper does not directly apply (see comments in §8). Here an example, based on modulation of a one-phase periodic travelling wave solution of a nonlinear complex Klein-Gordon equation, is presented where all the details can be worked out explicitly and it illustrates the key features induced by the elliptic-hyperbolic transition.

There is an interesting geometry associated with the mapping

$$(\omega, k) \mapsto (\mathscr{A}(\omega, k), \mathscr{B}(\omega, k))\,,$$

and it is developed in §2. The condition $\Delta_L = 0$ defines a curve in the (ω, k) plane which locally separates stable and unstable regions. The image defines a curve in $(\mathscr{A}, \mathscr{B})$ space. The geometry of these curves appears in the modulation theory. The modulation theory is developed for general conservative PDEs generated by a Lagrangian, and the background for this is developed in §3. The details of the modulation theory are presented in §4. In §5 features of the emergent two-way Boussinesq equation are discussed. Two examples of the application of the theory are presented: §6 applies the theory to the instability-stability transition of the Benjamin-Feir instability of Stokes waves in deep water, and §7 computes Σ^1 curves, and the reduced Boussinesq equation for periodic travelling waves of a nonlinear complex Klein-Gordon equation.

2. The frequency-wavenumber mapping. The geometry of the *frequency-wavenumber map*

$$(\omega, k) \mapsto \begin{pmatrix} \mathscr{A}(\omega, k) \\ \mathscr{B}(\omega, k) \end{pmatrix} := \mathbf{F}(\omega, k)\,, \tag{13}$$

appears centrally within the modulation theory. The Jacobian of this mapping,

$$\mathrm{D}\mathbf{F}(\omega, k) := \begin{bmatrix} \mathscr{A}_\omega & \mathscr{A}_k \\ \mathscr{B}_\omega & \mathscr{B}_k \end{bmatrix}\,,$$

is degenerate on the Σ^1 curves (7). With the assumptions (7), the trace of $\mathrm{D}\mathbf{F}$ is nonzero and so the zero eigenvalue of $\mathrm{D}\mathbf{F}$ is simple with geometric eigenvector

$$\mathrm{D}\mathbf{F}(\omega, k)\mathbf{n} = 0\,.$$

Since $\mathrm{D}\mathbf{F}$ is symmetric, $\mathbf{n}$ is both a left and right eigenvector. In terms of c_g

$$\mathbf{n} = \begin{pmatrix} -c_g \\ 1 \end{pmatrix}\,, \tag{14}$$

modulo a nonzero multiplicative constant. Although this eigenvector is not unique the choice (14) is canonical in that it will be shown to be relevant in the modulation theory.

The symbol $\mathbf{n}$ is used for the eigenvector in (14) because it is a normal vector. However, it is not the normal vector to the curve Σ^1, it is the normal vector to the image of this curve in the $(\mathscr{A}, \mathscr{B})$ plane. To see this first look at the geometry of the curve defined by Σ^1. To lighten the notation define

$$f(\omega, k) := \Delta_L(\omega, k)\,.$$

Then the normal vector to the curve $\Delta_L = 0$ is proportional to ∇f. A schematic is shown on the left in Figure 2. Now parameterize the curve $\Delta_L = 0$ by $(\omega(s), k(s))$. Then a tangent vector on the image of the mapping $\mathbf{F}(\omega(s), k(s))$ is

$$\begin{bmatrix} \mathscr{A}_\omega & \mathscr{A}_k \\ \mathscr{B}_\omega & \mathscr{B}_k \end{bmatrix} \begin{pmatrix} \dot{\omega} \\ \dot{k} \end{pmatrix}\,.$$

The left eigenvector $\mathbf{n}$ of $\mathrm{D}\mathbf{F}$ is orthogonal to this direction, giving a normal vector on the image curve in $(\mathscr{A}, \mathscr{B})-$space. A schematic is shown on the right in Figure 2.

The geometry of mappings from a plane to a plane is a fundamental problem in singularity theory and the basic results can be found in the first few chapters of

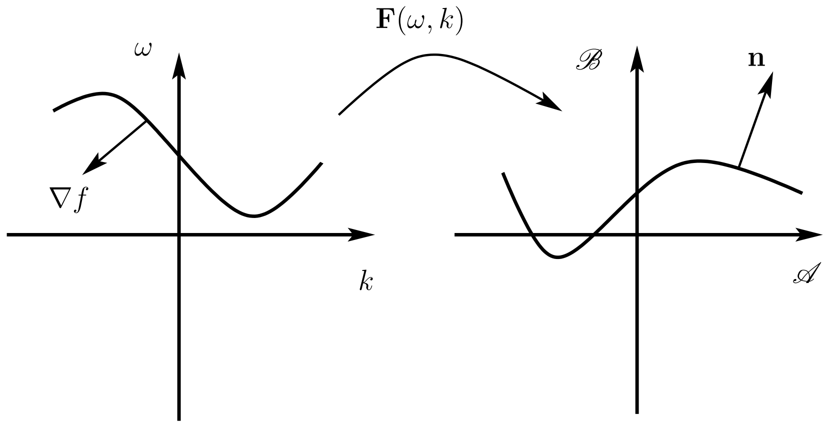

FIG. 2. *The singular curve defined by $\Delta_L = 0$ on the left, and its image curve under the mapping $\mathbf{F}$ is on the right.*

Arnold et al. [1]. For a mapping from the plane to the plane with a Σ^1 singularity, there are generically two types of curves: either

$$\begin{aligned} T_p\Sigma^1 \oplus \mathrm{Ker}(\mathrm{D}\mathbf{F}) &= \mathbb{R}^2 \qquad \text{(fold)} \\ T_p\Sigma^1 &= \mathrm{Ker}(\mathrm{D}\mathbf{F}) \qquad \text{(cusp)}\,, \end{aligned}$$

where $p = (\omega, k) \in \Sigma^1$. Since $T_p\Sigma^1 = \mathrm{Ker}(\nabla f)$, the fold condition is

$$\langle \nabla f, \mathbf{n} \rangle \neq 0\,, \tag{15}$$

where $\langle \cdot, \cdot \rangle$ is the standard inner product on $\mathbb{R}^2$. The cusp condition is simply

$$\langle \nabla f, \mathbf{n} \rangle = 0\,. \tag{16}$$

All other potential singularities of mappings from the plane to the plane are not stable under perturbation, a result known as Whitney's theorem [1], although one can potentially have many cusps [21]. This geometry plays a central role in the modulation theory, as it turns out that $\kappa \neq 0$ in (12) is precisely related to (15).

Define

$$\kappa = \langle \mathbf{n}, \mathrm{D}^2\mathbf{F}(\omega, k)(\mathbf{n}, \mathbf{n}) \rangle\,, \quad (\omega, k) \in \Sigma^1\,, \tag{17}$$

with $\mathbf{n}$ in the canonical form (14). The expression on the right is the *intrinsic second derivative* [31, 1]. It is the ordinary second derivative of the mapping $\mathbf{F}$ but evaluated on the kernel of the first derivative. It is widely used in singularity theory (cf. Chapter 3 of [1]).

The connection between κ in (17) and the fold condition (15) is the following

$$\langle \nabla f, \mathbf{n} \rangle = \left(\frac{\mathscr{A}_\omega + \mathscr{B}_k}{\|\mathbf{n}\|^2} \right) \langle \mathbf{n}, \mathrm{D}^2\mathbf{F}(\omega, k)(\mathbf{n}, \mathbf{n}) \rangle\,. \tag{18}$$

The coefficient on the right is nonzero since the zero eigenvalue of $\mathrm{D}\mathbf{F}$ is simple. The formula (18) is proved as follows. The function f can be characterized as

$$f(\omega, k) = \det[\mathrm{D}\mathbf{F}(\omega, k)]\,,$$

and so, using the formula for the derivative of a determinant,

$$f_\omega = \mathrm{Trace}\left(\mathrm{D}\mathbf{F}^{\#}\mathrm{D}\mathbf{F}_\omega\right) \quad \text{and} \quad f_k = \mathrm{Trace}\left(\mathrm{D}\mathbf{F}^{\#}\mathrm{D}\mathbf{F}_k\right)\,,$$

where $\mathrm{D}\mathbf{F}^{\#}$ is the adjugate of $\mathrm{D}\mathbf{F}$. Combining

$$\langle \nabla f, \mathbf{n}\rangle = n_1 f_\omega + n_2 f_k = \mathrm{Trace}\left(\mathrm{D}\mathbf{F}^{\#}(n_1\mathrm{D}\mathbf{F}_\omega + n_2\mathrm{D}\mathbf{F}_k)\right)\,.$$

Now note that the adjugate of a 2×2 matrix of rank 1 is proportional to $\mathbf{n}\mathbf{n}^T$, and in this case it is exactly

$$\mathrm{D}\mathbf{F}^{\#} = \frac{\mathrm{Tr}(\mathrm{D}\mathbf{F})}{\|\mathbf{n}\|^2}\mathbf{n}\mathbf{n}^T\,,$$

a formula which can be confirmed by direct calculation. Since $\mathrm{Tr}(\mathbf{n}\mathbf{n}^T\mathbf{A}) = \langle \mathbf{n}, \mathbf{A}\mathbf{n}\rangle$ for any 2×2 matrix $\mathbf{A}$, the formula (18) follows.

Writing out (17) using the canonical form for $\mathbf{n}$ in (14),

$$\kappa = (\mathscr{B}_{kk} - c_g\mathscr{A}_{kk}) - 2c_g(\mathscr{B}_{\omega k} - c_g\mathscr{A}_{\omega k}) + c_g^2(\mathscr{B}_{\omega\omega} - c_g\mathscr{A}_{\omega\omega})\,. \tag{19}$$

It is this form of the intrinsic second derivative κ that shows up in the modulation theory as the coefficient of nonlinearity in the modulation equation (12).

It is important to note that the "intrinsic" nature of the second derivative does not mean that the value of κ is independent of the choice of $\mathbf{n}$. As an eigenvector $\mathbf{n}$ is not unique and multiplication of $\mathbf{n}$ by a nonzero constant multiplies κ by that constant cubed, and so it can even change the sign of κ. The intrinsic label signifies that the affine part of the second derivative is removed, and the trilinear form of the second derivative remains the same. See [31, 1] for further detail on intrinsic derivatives.

3. Lagrangian setup and basic state.

The starting point for the modulation theory is a general class of PDEs generated by an abstract Lagrangian,

$$\mathscr{L}(U) = \int_{t_1}^{t_2}\int_{x_1}^{x_2} \mathcal{L}(U, U_x, U_t)\,\mathrm{d}x\mathrm{d}t\,, \tag{20}$$

where $U(x,t)$ is a vector-valued field on the rectangle $[x_1,x_2]\times[t_1,t_2]\subset\mathbb{R}^2$. It is advantageous to first transform the Lagrangian density to multisymplectic form,

$$\mathscr{L}(Z) = \int_{t_1}^{t_2}\int_{x_1}^{x_2} \left[\tfrac{1}{2}\langle Z, \mathbf{M}Z_t\rangle + \tfrac{1}{2}\langle Z, \mathbf{J}Z_x\rangle - S(Z)\right]\,\mathrm{d}x\mathrm{d}t\,, \tag{21}$$

where now $Z\in\mathbb{R}^n$ for each (x,t) and n is assumed to be even. The Lagrangian density is the same in going from (20) to (21) but the representation (21) has more structure. The operators $\mathbf{M}$ and $\mathbf{J}$ are constant skew-symmetric $n\times n$ matrices and $S:\mathbb{R}^n\to\mathbb{R}$ is a given smooth function. The transformation from (20) to (21), effectively a double Legendre transform, is discussed in previous papers [10, 8, 33, 34]. The Euler-Lagrange equation deduced from the Lagrangian (21) takes the concise form

$$\mathbf{M}Z_t + \mathbf{J}Z_x = \nabla S(Z)\,,\quad Z\in\mathbb{R}^n\,. \tag{22}$$

The theory could be developed directly on the primitive abstract Lagrangian (20) but partitioning the Lagrangian density as in (21) gives added structure that greatly simplifies the theory.

The basic state is a periodic travelling wave solution of wavelength $2\pi/k$ and period $2\pi/\omega$ of the form

$$Z(x,t) = \widehat{Z}(\theta,\omega,k)\,, \quad \widehat{Z}(\theta+2\pi,\cdot) = \widehat{Z}(\theta,\cdot)\,, \quad \theta = kx+\omega t+\theta_0\,, \tag{23}$$

with arbitrary phase shift θ_0. There is the usual assumption on existence and smoothness of this solution so that the necessary differentiation in θ, k, and ω is meaningful. The basic state satisfies

$$\omega \mathbf{M}\widehat{Z}_\theta + k\mathbf{J}\widehat{Z}_\theta = \nabla S(\widehat{Z})\,. \tag{24}$$

An important property of the structure is multisymplectic Noether theory [10] associated with conservation of wave action, that is,

$$\nabla A(\widehat{Z}) = \mathbf{M}\widehat{Z}_\theta \quad \text{and} \quad \nabla B(\widehat{Z}) = \mathbf{J}\widehat{Z}_\theta\,, \tag{25}$$

where A, B are the components of the action conservation law, $\widehat{Z}(\theta,\omega,k)$ is the basic state, and the gradient is defined with respect to the inner product including averaging over θ,

$$\langle\!\langle U,V\rangle\!\rangle := \frac{1}{2\pi}\int_0^{2\pi} \langle U,V\rangle\,\mathrm{d}\theta\,, \tag{26}$$

where $\langle\cdot,\cdot\rangle$ is the standard inner product on $\mathbb{R}^n$.

To get the components of the conservation law for wave action, average the Lagrangian, evaluated on the family of travelling waves, over θ,

$$\mathscr{L}(\omega,k) = \frac{1}{2\pi}\int_0^{2\pi}\left[\frac{\omega}{2}\langle \mathbf{M}\widehat{Z}_\theta,\widehat{Z}\rangle + \frac{k}{2}\langle \mathbf{J}\widehat{Z}_\theta,\widehat{Z}\rangle - S(\widehat{Z})\right]\mathrm{d}\theta\,,$$

and differentiate with respect to ω and k, giving

$$\begin{aligned} \mathscr{L}_\omega &:= \mathscr{A}(\omega,k) = \tfrac{1}{2}\langle\!\langle \mathbf{M}\widehat{Z}_\theta,\widehat{Z}\rangle\!\rangle \\ \mathscr{L}_k &:= \mathscr{B}(\omega,k) = \tfrac{1}{2}\langle\!\langle \mathbf{J}\widehat{Z}_\theta,\widehat{Z}\rangle\!\rangle\,. \end{aligned} \tag{27}$$

The key feature here is that the wave action and wave action flux, evaluated on the family of periodic travelling waves, are related to the tangent vectors of the waves via the structure matrices $\mathbf{M}$ and $\mathbf{J}$. This is multisymplectic Noether theory in action.

The first derivatives needed for $\mathrm{D}\mathbf{F}$ and Δ_L are

$$\begin{aligned} \mathscr{A}_\omega &= \langle\!\langle \mathbf{M}\widehat{Z}_\theta,\widehat{Z}_\omega\rangle\!\rangle\,, \quad \mathscr{A}_k = \langle\!\langle \mathbf{M}\widehat{Z}_\theta,\widehat{Z}_k\rangle\!\rangle\,, \\ \mathscr{B}_\omega &= \langle\!\langle \mathbf{J}\widehat{Z}_\theta,\widehat{Z}_\omega\rangle\!\rangle\,, \quad \mathscr{B}_k = \langle\!\langle \mathbf{J}\widehat{Z}_\theta,\widehat{Z}_k\rangle\!\rangle\,. \end{aligned} \tag{28}$$

The second derivatives needed in the construction of κ can be simplified by using a boosted symplectic structure. Define

$$\mathbf{K} := \mathbf{J} - c_g\mathbf{M}\,. \tag{29}$$

Then differentiating (28) and combining gives

$$\begin{aligned} \mathscr{B}_{\omega\omega} - c_g\mathscr{A}_{\omega\omega} &= \langle\!\langle \mathbf{K}\widehat{Z}_{\theta\omega},\widehat{Z}_\omega\rangle\!\rangle + \langle\!\langle \mathbf{K}\widehat{Z}_\theta,\widehat{Z}_{\omega\omega}\rangle\!\rangle \\ \mathscr{B}_{\omega k} - c_g\mathscr{A}_{\omega k} &= \langle\!\langle \mathbf{K}\widehat{Z}_{\theta k},\widehat{Z}_\omega\rangle\!\rangle + \langle\!\langle \mathbf{K}\widehat{Z}_\theta,\widehat{Z}_{\omega k}\rangle\!\rangle \\ \mathscr{B}_{kk} - c_g\mathscr{A}_{kk} &= \langle\!\langle \mathbf{K}\widehat{Z}_{\theta k},\widehat{Z}_k\rangle\!\rangle + \langle\!\langle \mathbf{K}\widehat{Z}_\theta,\widehat{Z}_{kk}\rangle\!\rangle\,. \end{aligned} \tag{30}$$

3.1. Linearization about the periodic basic state. Define the linear operator

$$\mathbf{L}W = \left[\mathrm{D}^2 S(\widehat{Z}) - k\mathbf{J}\frac{d}{d\theta} - \omega\mathbf{M}\frac{d}{d\theta}\right] W\,, \tag{31}$$

obtained by linearizing (24). Then differentiating (24) with respect to θ, k and ω gives,

$$\begin{aligned}
\mathrm{D}^2 S(\widehat{Z})\widehat{Z}_\theta &= k\mathbf{J}\widehat{Z}_{\theta\theta} + \omega\mathbf{M}\widehat{Z}_{\theta\theta}\,,\\
\mathrm{D}^2 S(\widehat{Z})\widehat{Z}_k &= k\mathbf{J}\widehat{Z}_{\theta k} + \omega\mathbf{M}\widehat{Z}_{\theta k} + \mathbf{J}\widehat{Z}_\theta\\
\mathrm{D}^2 S(\widehat{Z})\widehat{Z}_\omega &= k\mathbf{J}\widehat{Z}_{\theta\omega} + \omega\mathbf{M}\widehat{Z}_{\theta\omega} + \mathbf{M}\widehat{Z}_\theta\,,
\end{aligned}$$

or

$$\mathbf{L}\widehat{Z}_\theta = 0\,, \quad \mathbf{L}\widehat{Z}_k = \mathbf{J}\widehat{Z}_\theta\,, \quad \text{and} \quad \mathbf{L}\widehat{Z}_\omega = \mathbf{M}\widehat{Z}_\theta\,, \tag{32}$$

with other derivatives following a similar pattern. The first equation of (32) shows that $\widehat{Z}_\theta$ is in the kernel of $\mathbf{L}$, and it is natural to assume that the kernel is no larger. Hence assume

$$\mathrm{Kernel}(\mathbf{L}) = \mathrm{span}\{\widehat{Z}_\theta\}\,. \tag{33}$$

For inhomogeneous equations that arise in the modulation theory and the Jordan chain theory, a solvability condition will be needed. With the assumption (33) and the symmetry of $\mathbf{L}$, the solvability condition for the inhomogeneous equation $\mathbf{L}W = F$ is

$$\mathbf{L}W = F \quad \text{is solvable if and only if} \quad \langle\!\langle \widehat{Z}_\theta, F\rangle\!\rangle = 0\,. \tag{34}$$

3.2. A twisted symplectic Jordan chain. The second and third equation of (32) show that there are potentially two non-trivial Jordan chains associated with the zero eigenvalue of $\mathbf{L}$ with geometric eigenvector $\widehat{Z}_\theta$. In previous work [8, 33, 34], the phase modulation theory required a longer Jordan chain formed from *either* a $\mathbf{J}-$chain or an $\mathbf{M}-$chain. Here the intertwining of these two chains will be required in the phase modulation theory. Then, using (32),

$$\mathbf{L}\big(\widehat{Z}_k - c_g\widehat{Z}_\omega\big) = \big(\mathbf{J} - c_g\mathbf{M}\big)\widehat{Z}_\theta = \mathbf{K}\widehat{Z}_\theta\,, \tag{35}$$

using the boosted symplectic structure $\mathbf{K}$ (29) in the last equality. Therefore, define

$$\xi_1 = \widehat{Z}_\theta \quad \text{and} \quad \xi_2 = \widehat{Z}_k - c_g\widehat{Z}_\omega\,. \tag{36}$$

Then a mixed $\mathbf{K}-$Jordan chain of length two is formed

$$\mathbf{L}\xi_1 = 0 \quad \text{and} \quad \mathbf{L}\xi_2 = \mathbf{K}\xi_1\,. \tag{37}$$

It is the extension of this chain and its connection with the singularity (7) that will appear in the modulation theory. Since the symplectic structure assures that the chain length is even, a proposed longer chain is

$$\begin{aligned}
\mathbf{L}\xi_1 &= 0\\
\mathbf{L}\xi_2 &= \mathbf{K}\xi_1\\
\mathbf{L}\xi_3 &= \mathbf{K}\xi_2\\
\mathbf{L}\xi_4 &= \mathbf{K}\xi_3\,.
\end{aligned} \tag{38}$$

In this chain it is either assumed that $\mathbf{K}$ is invertible or $\mathbf{K}\xi_j \neq 0$ for $j = 1, 2, 3$.

The second equation in (38) is solvable due to (32), and the third equation is solvable since

$$
\begin{aligned}
\langle\!\langle \widehat{Z}_\theta, \mathbf{K}\xi_2 \rangle\!\rangle &= \langle\!\langle \widehat{Z}_\theta, \mathbf{K}(\widehat{Z}_k - c_g \widehat{Z}_\omega) \rangle\!\rangle \\
&= -\langle\!\langle \mathbf{K}\widehat{Z}_\theta, (\widehat{Z}_k - c_g \widehat{Z}_\omega) \rangle\!\rangle \\
&= -\langle\!\langle \mathbf{J}\widehat{Z}_\theta - c_g \mathbf{M}\widehat{Z}_\theta, (\widehat{Z}_k - c_g \widehat{Z}_\omega) \rangle\!\rangle \\
&= -\mathscr{B}_k + c_g \mathscr{B}_\omega + c_g \mathscr{A}_k - c_g^2 \mathscr{A}_\omega \\
&= -\tfrac{1}{\mathscr{A}_\omega} \Delta_L \,,
\end{aligned} \tag{39}
$$

using (28), and $\Delta_L = 0$ on Σ^1 curves. The fourth equation in (38) is solvable due to even-ness of the Jordan chain, but it can be confirmed explicitly,

$$
\begin{aligned}
\langle\!\langle \widehat{Z}_\theta, \mathbf{K}\xi_3 \rangle\!\rangle &= -\langle\!\langle \mathbf{K}\xi_1, \xi_3 \rangle\!\rangle \\
&= -\langle\!\langle \mathbf{L}\xi_2, \xi_3 \rangle\!\rangle \\
&= -\langle\!\langle \xi_2, \mathbf{L}\xi_3 \rangle\!\rangle \\
&= -\langle\!\langle \xi_2, \mathbf{K}\xi_2 \rangle\!\rangle \\
&= 0 \,,
\end{aligned}
$$

with the last line following from skew-symmetry of $\mathbf{K}$. This Jordan chain terminates at four if the next equation

$$\mathbf{L}\xi_5 = \mathbf{K}\xi_4 \,,$$

is not solvable; that is, when

$$\langle\!\langle \widehat{Z}_\theta, \mathbf{K}\xi_4 \rangle\!\rangle := -\mathscr{K} \neq 0 \,. \tag{40}$$

It is this coefficient $\mathscr{K}$ that shows up as the coefficient of dispersion in the modulation equation (12).

To summarize: *for $(\omega, k) \in \Sigma^1$, with the assumption (33), the algebraic multiplicitity of the zero eigenvalue of $\mathbf{L}$ is at least four and is exactly four when $\mathscr{K} \neq 0$.*

4. Modulation ansatz.

Given the family of basic states, $\widehat{Z}(\theta, \omega, k)$, the classical Whitham modulation equations (2) are obtained using the modulation ansatz

$$Z(x,t) = \widehat{Z}\left(\theta + \frac{1}{\varepsilon}\phi, \omega + \Omega, k + q\right) + \varepsilon W\left(\theta + \frac{1}{\varepsilon}\phi, X, T, \varepsilon\right) \,, \tag{41}$$

with ϕ dependent on (X, T, ε),

$$q = \phi_X \,, \quad \Omega = \phi_T \,, \quad X = \varepsilon x \,, \quad T = \varepsilon t \,.$$

Substitution of the ansatz (41) into the Euler-Lagrange equation (22) leads, via a solvability condtion at order ε^1, to the dispersionless conservation of wave action in (2). This modulation ansatz is valid away from a Σ^1 curve.

For $(\omega, k) \in \Sigma^1$ the ansatz needs to be modified. *A posteriori* it is confirmed that the appropriate modification of (41) is

$$Z(x,t) = \widehat{Z}(\theta + \varepsilon\phi, \omega - c_g \varepsilon^2 q + \varepsilon^3 \Omega, k + \varepsilon^2 q) + \varepsilon^3 W(\theta, X, T, \varepsilon) \,. \tag{42}$$

The conservation of waves is still operational

$$q = \phi_X\,, \quad \Omega = \phi_T\,, \quad \text{and} \quad q_T = \Omega_X\,, \tag{43}$$

but the scaling of the independent variables is changed to

$$X = \varepsilon\big(x - c_g t\big) \quad \text{and} \quad T = \varepsilon^2 t\,, \quad \text{with} \quad c_g := \frac{\mathscr{L}_{\omega k}}{\mathscr{L}_{\omega\omega}} = \frac{\mathscr{A}_k}{\mathscr{A}_\omega}\,. \tag{44}$$

The strategy is then to substitute the ansatz (42) into the Euler-Lagrange equation (22), expand everything in powers of ε, and solve order by order in ε. While the ansatz (42) is new, particularly in how the speed c_g affects the modulation, the machinations of the expansions is similar to previous work [8, 33, 34], and so only a summary is given. The zeroth, first, and second order equations in ε reproduce the equation for the basic state, the linearization, and conservation of waves (43). At third order the resulting equation is

$$\begin{aligned} \mathbf{L}W_3 &= q_X\left[\mathbf{J}\widehat{Z}_k - c_g\mathbf{M}\widehat{Z}_k + c_g^2\mathbf{M}\widehat{Z}_\omega - c_g\mathbf{J}\widehat{Z}_\omega\right] \\ &= q_X\mathbf{K}\big(\widehat{Z}_k - c_g\widehat{Z}_\omega\big) = \mathbf{K}\xi_2\,, \end{aligned} \tag{45}$$

using (29) and (36). Here, W_3 is obtained from the Taylor expansion of W,

$$\varepsilon^3 W(\theta, X, T, \varepsilon) = \varepsilon^3 W_3(\theta, X, T) + \varepsilon^4 W_4(\theta, X, T) + \varepsilon^5 W_5(\theta, X, T) + \cdots\,.$$

The equation (45) is solvable for $(\omega, k) \in \Sigma^1$ due to (39). Hence

$$W_3 = q_X\xi_3 + \alpha\xi_1\,, \tag{46}$$

where at this point $\alpha(X, T)$ is an arbitrary function.

4.1. Fourth order equation.

The fourth order equation simplifies to

$$\begin{aligned} &\mathbf{L}\big(W_4 - q_{XX}\xi_4 - \alpha_X\xi_2 - \phi q_X(\xi_3)_\theta - \alpha\phi\widehat{Z}_{\theta\theta}\big) \\ &\qquad = q_T\left(\mathbf{M}\widehat{Z}_k - c_g\mathbf{M}\widehat{Z}_\omega\right) + \Omega_X\left(\mathbf{J}\widehat{Z}_\omega - c_g\mathbf{M}\widehat{Z}_\omega\right)\,. \end{aligned}$$

A curiosity in the theory is that the q_T and Ω_X terms are exactly solvable for $(\omega, k) \in \Sigma^1$ since

$$q_T\langle\!\langle \widehat{Z}_\theta, (\mathbf{M}\widehat{Z}_k - c_g\mathbf{M}\widehat{Z}_\omega)\rangle\!\rangle = q_T(-\mathscr{A}_k + c_g\mathscr{A}_\omega) = 0\,,$$

and

$$\Omega_X\langle\!\langle \widehat{Z}_\theta(\mathbf{J}\widehat{Z}_\omega - c_g\mathbf{M}\widehat{Z}_\omega)\rangle\!\rangle = \Omega_X(\mathscr{B}_\omega - c_g\mathscr{A}_\omega) = 0\,,$$

using the definition of c_g and the cross-derivatives $\mathscr{A}_k = \mathscr{B}_\omega$.

The complete solution for W_4 is therefore

$$W_4 = q_T\eta + q_{XX}\xi_4 + \alpha_X\xi_2 + \phi q_X(\xi_3)_\theta + \alpha\phi\widehat{Z}_{\theta\theta} + \beta\xi_1\,, \tag{47}$$

where $\beta(X, T)$ is arbitrary at this point, and η is a particular solution of

$$\mathbf{L}\eta = \mathbf{M}\widehat{Z}_k - 2c_g\mathbf{M}\widehat{Z}_\omega + \mathbf{J}\widehat{Z}_\omega\,. \tag{48}$$

The solution η of this equation will not be needed explicitly in the theory, only its abstract definition in (48).

4.2. Solvability at fifth order. After some simplification, the fifth order terms reduce to

$$
\mathbf{L}\widetilde{W}_5 = \Omega_T \mathbf{M}\widehat{Z}_\omega + qq_X \Big[\mathbf{K}\Upsilon + \mathbf{K}(\xi_3)_\theta - \mathrm{D}^3 S(\widehat{Z})(\xi_2,\xi_3)\Big] + q_{XXX}\mathbf{K}\xi_4 + \Omega_{XX}\big(\mathbf{M}\xi_3 + \mathbf{K}\eta\big)\,, \tag{49}
$$

where $\widetilde{W}_5$ incorporates all terms that are exactly solvable and,

$$
\Upsilon := \widehat{Z}_{kk} - 2c_g \widehat{Z}_{\omega k} + c_g^2 \widehat{Z}_{\omega\omega}\,.
$$

An explicit expression for $\widetilde{W}_5$ can be constructed but is not needed as solvability delivers the modulation equation (12).

The awkward term in (49) is the Ω_{XX} term which would make the resulting modulation equation non-conservative. However, it too is in the range of $\mathbf{L}$, and it is the abstract definition of the function η in (48) that is used to show that this term is removable,

$$
\begin{aligned}
\langle\!\langle \widehat{Z}_\theta, \mathbf{M}\xi_3 + \mathbf{K}\eta \rangle\!\rangle &= \langle\!\langle \widehat{Z}_\theta, \mathbf{M}\xi_3 \rangle\!\rangle - \langle\!\langle \mathbf{K}\widehat{Z}_\theta, \eta \rangle\!\rangle \\
&= \langle\!\langle \widehat{Z}_\theta, \mathbf{M}\xi_3 \rangle\!\rangle - \langle\!\langle \mathbf{L}\xi_2, \eta \rangle\!\rangle \\
&= -\langle\!\langle \mathbf{M}\widehat{Z}_\theta, \xi_3 \rangle\!\rangle - \langle\!\langle \xi_2, \mathbf{L}\eta \rangle\!\rangle \\
&= -\langle\!\langle \mathbf{L}\widehat{Z}_\omega, \xi_3 \rangle\!\rangle - \langle\!\langle \xi_2, \mathbf{M}\widehat{Z}_k - c_g\mathbf{M}\widehat{Z}_\omega + \mathbf{K}\widehat{Z}_\omega \rangle\!\rangle \\
&= -\langle\!\langle \widehat{Z}_\omega, \mathbf{L}\xi_3 \rangle\!\rangle - \langle\!\langle \xi_2, \mathbf{M}\widehat{Z}_k - c_g\mathbf{M}\widehat{Z}_\omega + \mathbf{K}\widehat{Z}_\omega \rangle\!\rangle \\
&= -\langle\!\langle \widehat{Z}_\omega, \mathbf{K}\xi_2 \rangle\!\rangle - \langle\!\langle \xi_2, \mathbf{M}\widehat{Z}_k - c_g\mathbf{M}\widehat{Z}_\omega + \mathbf{K}\widehat{Z}_\omega \rangle\!\rangle \\
&= -\langle\!\langle \xi_2, \mathbf{M}(\widehat{Z}_k - c_g\widehat{Z}_\omega) \rangle\!\rangle \\
&= -\langle\!\langle \xi_2, \mathbf{M}\xi_2 \rangle\!\rangle \\
&= 0\,,
\end{aligned}
$$

using skew-symmetry of $\mathbf{M}$ and $\mathbf{K}$, symmetry of $\mathbf{L}$, the Jordan chain, and the function η (48). Therefore there exists a function δ such that

$$
\mathbf{L}\delta = \mathbf{M}\xi_3 + \mathbf{K}\eta\,.
$$

This simplifies the fifth order equation to

$$
\mathbf{L}\big(\widetilde{W}_5 - \Omega_{XX}\delta\big) = \Omega_T\mathbf{M}\widehat{Z}_\omega + q_{XXX}\mathbf{K}\xi_4 + qq_X\Big[\mathbf{K}\Upsilon + \mathbf{K}(\xi_3)_\theta - \mathrm{D}^3 S(\widehat{Z}^o)(\xi_2,\xi_3)\Big]\,.
$$

This equation is solvable if and only if the right hand side is orthogonal to $\widehat{Z}_\theta$, giving

$$
a_1\Omega_T + a_2 qq_X + a_3 q_{XXX} = 0\,, \tag{50}
$$

with

$$
\begin{aligned}
a_1 &= \langle\!\langle \widehat{Z}_\theta, \mathbf{M}\widehat{Z}_\omega \rangle\!\rangle \\
a_2 &= \Big\langle\!\!\Big\langle \widehat{Z}_\theta, \Big[\mathbf{K}\Upsilon + \mathbf{K}(\xi_3)_\theta - \mathrm{D}^3 S(\widehat{Z})(\xi_2,\xi_3)\Big] \Big\rangle\!\!\Big\rangle \\
a_3 &= \langle\!\langle \widehat{Z}_\theta, \mathbf{K}\xi_4 \rangle\!\rangle\,.
\end{aligned} \tag{51}
$$

Now

$$a_1 = \langle\!\langle \widehat{Z}_\theta, \mathbf{M}\widehat{Z}_\omega \rangle\!\rangle = -\langle\!\langle \mathbf{M}\widehat{Z}_\theta, \widehat{Z}_\omega \rangle\!\rangle = -\mathscr{A}_\omega \,,$$

using (28), and appeal to (40) shows that

$$a_3 = -\langle\!\langle \mathbf{K}\xi_1, \xi_4 \rangle\!\rangle = -\mathscr{K} \,.$$

Using the geometry of the frequency-wavenumber map it is shown below that $a_2 = -\kappa$, where κ is defined in (17), giving the final form of the modulation equation as

$$\mathscr{A}_\omega \Omega_T + \kappa q q_X + \mathscr{K} q_{XXX} = 0 \quad \text{and} \quad q_T = \Omega_X \,, \tag{52}$$

and for non-degeneracy of this equation it is assumed that

$$\mathscr{A}_\omega \neq 0 \,, \quad \kappa \neq 0 \,, \quad \text{and} \quad \mathscr{K} \neq 0 \,. \tag{53}$$

4.3. The geometry of the frequency-wavenumber map and a_2. It is not at all obvious that the intrinsic second derivative that arises from the geometry of the frequency-wavenumber map (17) should be related to the above coefficient a_2 that appears from the modulation analysis as the coefficient of nonlinearity. However, with the canonical choice of normal vector $\mathbf{n}$, they are exactly equal and this is proved as follows. The expression that arises in the modulation analysis and solvability condition is

$$a_2 = \left\langle\!\!\left\langle \widehat{Z}_\theta, \left[\mathbf{K}\Upsilon + \mathbf{K}(\xi_3)_\theta - \mathrm{D}^3 S(\widehat{Z})(\xi_2, \xi_3) \right] \right\rangle\!\!\right\rangle \,. \tag{54}$$

Differentiating $\mathbf{L}\xi_3 = \mathbf{K}\xi_2$ with respect to θ,

$$\mathbf{L}(\xi_3)_\theta + \mathrm{D}^3 S(\widehat{Z})(\xi_1, \xi_3) = \mathbf{K}(\xi_2)_\theta \,,$$

Hence, with $\widehat{Z}_\theta = \xi_1$, the second term in (54) is

$$\begin{aligned}
\langle\!\langle \widehat{Z}_\theta, \mathbf{K}(\xi_3)_\theta \rangle\!\rangle &= -\langle\!\langle \mathbf{K}\xi_1, (\xi_3)_\theta \rangle\!\rangle \\
&= -\langle\!\langle \mathbf{L}\xi_2, (\xi_3)_\theta \rangle\!\rangle \\
&= -\langle\!\langle \xi_2, \mathbf{L}(\xi_3)_\theta \rangle\!\rangle \\
&= -\langle\!\langle \xi_2, \mathbf{K}(\xi_2)_\theta - \mathrm{D}^3 S(\widehat{Z})(\xi_1, \xi_3) \rangle\!\rangle \\
&= -\langle\!\langle \xi_2, \mathbf{K}(\xi_2)_\theta \rangle\!\rangle + \langle\!\langle \xi_2, \mathrm{D}^3 S(\widehat{Z})(\xi_1, \xi_3) \rangle\!\rangle \\
&= -\langle\!\langle \xi_2, \mathbf{K}(\xi_2)_\theta \rangle\!\rangle + \langle\!\langle \xi_1, \mathrm{D}^3 S(\widehat{Z})(\xi_2, \xi_3) \rangle\!\rangle \,,
\end{aligned}$$

using $\mathbf{L}\xi_2 = \mathbf{K}\xi_1$, skew-symmetry of $\mathbf{K}$, symmetry of $\mathbf{L}$, and permutation of the trilinear form in the last line. Substitution of the expression for $\langle\!\langle \widehat{Z}_\theta, \mathbf{K}(\xi_3)_\theta \rangle\!\rangle$ into a_2 reduces it to

$$a_2 = \langle\!\langle \mathbf{K}\xi_2, (\xi_2)_\theta \rangle\!\rangle + \langle\!\langle \widehat{Z}_\theta, \mathbf{K}\Upsilon \rangle\!\rangle \,. \tag{55}$$

Start with this expression for a_2, and substitute for Υ and ξ_2,

$$
\begin{aligned}
-a_2 &= -\langle\!\langle \mathbf{K}\xi_2, (\xi_2)_\theta \rangle\!\rangle - \langle\!\langle \widehat{Z}_\theta, \mathbf{K}\Upsilon \rangle\!\rangle \\
&= \langle\!\langle \mathbf{K}(\widehat{Z}_{k\theta} - c_g \widehat{Z}_{\omega\theta}), \widehat{Z}_k - c_g \widehat{Z}_\omega \rangle\!\rangle + \langle\!\langle \mathbf{K}\widehat{Z}_\theta, \Upsilon \rangle\!\rangle \\
&= \langle\!\langle \mathbf{K}\widehat{Z}_{\theta k}, \widehat{Z}_k \rangle\!\rangle - c_g \langle\!\langle \mathbf{K}\widehat{Z}_{\theta k}, \widehat{Z}_\omega \rangle\!\rangle - c_g \langle\!\langle \mathbf{K}\widehat{Z}_{\theta\omega}, \widehat{Z}_k \rangle\!\rangle \\
&\quad + c_g^2 \langle\!\langle \mathbf{K}\widehat{Z}_{\theta\omega}, \widehat{Z}_\omega \rangle\!\rangle + \langle\!\langle \mathbf{K}\widehat{Z}_\theta, \Upsilon \rangle\!\rangle \\
&= \langle\!\langle \mathbf{K}\widehat{Z}_{\theta k}, \widehat{Z}_k \rangle\!\rangle - c_g \langle\!\langle \mathbf{K}\widehat{Z}_{\theta k}, \widehat{Z}_\omega \rangle\!\rangle - c_g \langle\!\langle \mathbf{K}\widehat{Z}_{\theta\omega}, \widehat{Z}_k \rangle\!\rangle + c_g^2 \langle\!\langle \mathbf{K}\widehat{Z}_{\theta\omega}, \widehat{Z}_\omega \rangle\!\rangle \\
&\quad + \langle\!\langle \mathbf{K}\widehat{Z}_\theta, \widehat{Z}_{kk} - 2c_g \widehat{Z}_{\omega k} + c_g^2 \widehat{Z}_{\omega\omega} \rangle\!\rangle \\
&= \big(\langle\!\langle \mathbf{K}\widehat{Z}_{\theta k}, \widehat{Z}_k \rangle\!\rangle + \langle\!\langle \mathbf{K}\widehat{Z}_\theta, \widehat{Z}_{kk} \rangle\!\rangle \big) + c_g^2 \big(\langle\!\langle \mathbf{K}\widehat{Z}_{\theta\omega}, \widehat{Z}_\omega \rangle\!\rangle + \langle\!\langle \mathbf{K}\widehat{Z}_\theta, \widehat{Z}_{\omega\omega} \rangle\!\rangle \big) \\
&\quad - c_g \big(\langle\!\langle \mathbf{K}\widehat{Z}_{\theta k}, \widehat{Z}_\omega \rangle\!\rangle + \langle\!\langle \mathbf{K}\widehat{Z}_{\theta\omega}, \widehat{Z}_k \rangle\!\rangle + 2\langle\!\langle \mathbf{K}\widehat{Z}_\theta, \widehat{Z}_{\omega k} \rangle\!\rangle \big) \\
&= \big(\mathscr{B}_{kk} - c_g \mathscr{A}_{kk} \big) - 2c_g \big(\mathscr{B}_{\omega k} - c_g \mathscr{A}_{\omega k} \big) + c_g^2 \big(\mathscr{B}_{\omega\omega} - c_g \mathscr{A}_{\omega\omega} \big) \\
&= \langle \mathbf{n}, \mathrm{D}^2\mathbf{F}(\omega, k)(\mathbf{n}, \mathbf{n}) \rangle \\
&= \kappa \,,
\end{aligned}
$$

when $\mathbf{n}$ is in canonical form (14). The third to last step follows from the substitution of the identities (30). This completes the derivation of the phase modulation equations (52) on Σ^1 curves.

4.4. Invariance under coordinate change. Since the modulation equation (52) relies on two eigenvector choices there is a potential non-uniqueness in the final form. The first potential non-uniqueness is the choice of geometric eigenvector ξ_1 of the zero eigenvalue of $\mathbf{L}$,

$$\mathbf{L}\xi_1 = 0 \quad \Rightarrow \quad \xi_1 = b\widehat{Z}_\theta \,,$$

where b is an arbitrary multiplicative constant. This constant is then multiplied by each element in the Jordan chains. Hence a_1 and a_3 in (50) would be multiplied by b^2. However, the signs of a_1 and a_3 would not change and the factor b^2 can be removed by scaling. The other eigenvector choice is $\mathbf{n}$ and

$$\mathrm{D}\mathbf{F}(\omega, k)\mathbf{n} = 0 \quad \Rightarrow \quad \mathbf{n} = b \begin{pmatrix} -c_g \\ 1 \end{pmatrix} ,$$

for some nonzero constant b. In this case the only change would be a scale factor on κ, $\kappa \mapsto b^3\kappa$. Since κ multiplies a nonlinearity, scaling q (or ϕ) using b^3 would eliminate this scale factor in κ. A change in sign of κ is eliminated by a change in sign of q. Hence, with the canonical choices $\xi_1 = \widehat{Z}_\theta$ and $\mathbf{n}$ as in (14), and the modulation ansatz (42), the modulation equation (52) is uniquely defined.

4.5. Unfolding from Σ^1 curves. Instead of taking Δ_L to be identically zero, it can be taken to be of order ε^2 giving an unfolding of the two-way Boussinesq equation

$$\mathscr{A}_\omega \Omega_T + \mu q_X + \kappa q q_X + \mathscr{K} q_{XXX} = 0 \quad \text{and} \quad q_T = \Omega_X \,, \tag{56}$$

where $\mathrm{sign}(\mu) = \mathrm{sign}(\mathscr{A}_\omega \Delta_L)$. In this case, the combined equation is the classical two-way Boussinesq equation with a second derivative in X term

$$\mathscr{A}_\omega q_{TT} + \mu q_{XX} + \left(\tfrac{1}{2}\kappa q^2\right)_{XX} + \mathscr{K} q_{XXXX} = 0 \,. \tag{57}$$

This unfolded version allows one to extend the discussion from solely along the Σ^1 curves to the neighbourhood around them, characterised by the small parameter ε.

5. The two-way Boussinesq equation. Once the modulation equation (52) is derived in a specific context, analysis of the solutions follows the standard strategy. Assuming all the coefficients are non-zero, the dependent and independent variables can be scaled so that the coefficients are ± 1, simplifying the form of the equation.

Starting with (57), scale X, T, and q and let

$$s_1 = \mathrm{sign}(\Delta_L) \quad \text{and} \quad s_2 = \mathrm{sign}\left(\mathscr{A}_\omega \mathscr{K}\right) .$$

Denote the scaled space and time variables by ξ and τ, and the scaled q by $u(\xi,\tau)$. Then the two-way Boussinesq equation is reduced to the standard form

$$u_{\tau\tau} + s_1 u_{\xi\xi} + \left(\tfrac{1}{2}u^2\right)_{\xi\xi} + s_2 u_{\xi\xi\xi\xi} = 0\,, \quad s_1, s_2 = \pm 1\,. \tag{58}$$

The set Σ^1 locally separates the subset of the (ω, k) for which travelling waves exist into two regions: elliptic ($s_1 = +1$) and hyperbolic ($s_1 = -1$). The sign s_2 indicates whether the resulting two-way Boussinesq equation is good ($s_2 = +1$) or bad ($s_2 = -1$). In the latter case, the initial value problem for the linearized system is ill posed.

Consider the linearization of (58) about the trivial solution and consider a normal mode solution of the form $\mathrm{e}^{\mathrm{i}(\hat{k}\xi + \hat{\omega}\tau)}$, then the dispersion relation is of the form

$$\hat{\omega}^2 = -s_1 \hat{k}^2 + s_2 \hat{k}^4\,.$$

There are four cases depending on the signs s_1 and s_2, and they are shown in Figure 3. The figure plots $\hat{\omega}^2$ against $\hat{k}^2$ and so $\hat{\omega}^2 < 0$ indicates linear instability of the trivial solution which in turn reflects linear instability of the basic travelling wave.

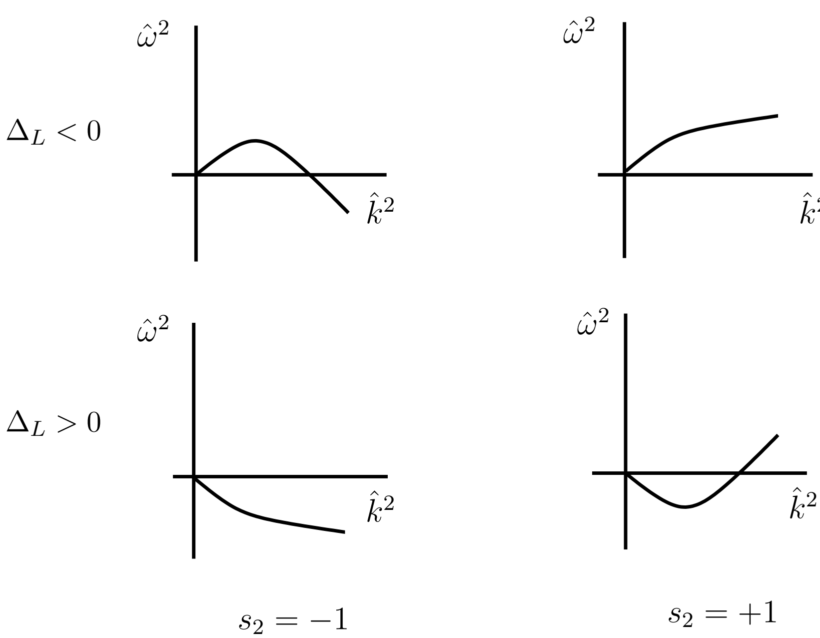

Fig. 3. *The four cases determined by the signs $s_1 = \mathrm{sign}(\Delta_L)$ and $s_2 = \mathrm{sign}(\mathscr{A}_\omega\mathscr{K})$ in the two-way Boussinesq equation near a Σ^1 curve.*

When $s_1 < 0$ (the upper two cases in Figure 3) then either an unstable band emerges at finite $\hat{k}$ when $s_2 = -1$ or the Boussinesq equation is also hyperbolic ($s_2 = +1$). When $s_1 > 0$ (lower two cases in Figure 3) then either a cutoff wave number emerges with re-stabilization at finite $\hat{k}$ (as in the lower right diagram with $s_2 = +1$), or instability is further enhanced ($s_1 = +1$ and $s_2 = -1$).

The simplest class of nonlinear solutions of (58) are travelling solitary wave solutions, for example,

$$u(\xi,\tau) = \widehat{u}(\xi + \gamma\tau)\,,$$

which satisfies the ODE

$$\left(\gamma^2\widehat{u} + s_1\widehat{u} + \tfrac{1}{2}\widehat{u}^2 + s_2\widehat{u}''\right)'' = 0\,.$$

Integrating and taking the function of integration to be constant

$$s_2\widehat{u}'' + (s_1 + \gamma^2)\widehat{u} + \tfrac{1}{2}\widehat{u}^2 = h\,.$$

The constant of integration h is fixed by initial data or the value of $\widehat{u}$ at infinity. For appropriate parameter values, this planar ODE has a family of periodic solutions and a homoclinic orbit which represent periodic travelling waves and a solitary travelling wave solution of (58). The implication of these solutions is that the transition from elliptic to hyperbolic of a periodic travelling wave of the original system generates a coherent structure in the transition, which is represented by the above solitary wave. However, there is much more complexity generated at the transition. Hirota [18] shows that there is a large family of $N-$soliton solutions to (58) as well. Further details especially in the case $N = 2$ are given in [18]. Numerical simulations of the case $N = 2$ are presented in [27].

The two-way Boussinesq equation is also generated by a Lagrangian, and has both a Hamiltonian and multisymplectic structure (e.g. [6], §10 of [9], and [12]).

6. Example: finite-amplitude stabilization of Stokes waves. The four scenarios in Figure 3 can be used to identify the type of stability-instability transition in the water wave problem at finite-amplitude, linearized about Stokes waves on deep water. It was first shown by Longuet-Higgins [24] that the Benjamin-Feir instability of Stokes travelling waves in deep water stabilizes at finite amplitude. This stabilization can be seen most clearly in the numerics of McLean [28]. Linear stability exponents for finite-amplitude Stokes waves in deep water are computed, and in Figure 2 of [28] stability regions are plotted as a function of the modulation wavenumbers, for a sequence of amplitudes. Three-dimensional instabilities (two modulation wavenumbers) are plotted but only the two-dimensional (one modulation wavenumber) instabilities are of interest here. At low amplitude the Benjamin-Feir instability is operational and it persists as the amplitude increases, until a wave steepness of $h/\lambda \approx 0.108$ is reached, where h is crest to trough distance and λ the wavelength. At this value, the region of modulation instability in wavenumber space detaches from the orgin (see the transition in going from Figure 2(c) to 2(d) in [28]).

Independently, in the same year, Whitham [39] showed that the stabilization point was precisely a transition point associated with $\Delta_L = 0$. Whitham first transforms the averaged Lagrangian into a functional $\mathscr{H}$ based on the energy,

$$\mathscr{L}(\omega,k,I) = \omega I - \mathscr{H}(k,I)\,,$$

where I is the value of the wave action. The amplitude of the wave is parameterized in terms of wave action (see [17] for discussion of Whitham modulation theory in terms of $\mathscr{H}(k, I)$). In terms of $\mathscr{H}(k, I)$ the Lighthill determinant is

$$\det \begin{bmatrix} \mathscr{A}_\omega & \mathscr{A}_k \\ \mathscr{B}_\omega & \mathscr{B}_k \end{bmatrix} = \frac{\mathscr{H}_{kk}}{\mathscr{H}_{II}} \,.$$

An explicit transformation from $\mathscr{L}$ to $\mathscr{H}$ is given in the introduction and Appendix A of [17]. The sign here differs from [17] and [39] as they define wave-action flux with the opposite sign.

Whitham [39] then argues (see §10 in [39]) that the energy takes a self similar form

$$\mathscr{H}(k, I) = \frac{g}{k^2} W(\zeta) \quad \text{with} \quad \zeta := \frac{k^3 I}{\sqrt{gk}} \,.$$

He then appeals to the tabulated values of the energy in Longuet-Higgins [23] to show that $\mathscr{H}_{II} > 0$ and does not change sign along a branch of Stokes waves, but shows that $\mathscr{H}_{kk} = 0$ precisely at $h/\lambda = 0.109$ which agrees, to numerical accuracy, with the change of stability found in [24] and [28].

With this association between the stability-instability transition point and vanishing of the Lighthill determinant, the theory of this paper can be used to deduce that the two-way Boussinesq equation is generated at the transition.

Going by the transition in Figure 2 of [28], the appropriate Boussinesq model is the bad Boussinesq with $s_2 = -1$, and Δ_L goes from positive to negative as the amplitude increases, corresponding to the two left graphs in Figure 3. Since the sign of the coefficient of the nonlinearity in (52) is not important, and generically it is nonzero, the appropriate Boussinesq model for water waves near the instability-stability transition of Stokes waves is

$$u_{\tau\tau} + s_1 u_{\xi\xi} \pm \left(\tfrac{1}{2}u^2\right)_{\xi\xi} - u_{\xi\xi\xi\xi} = 0 \,, \quad s_1 = \pm 1 \,, \tag{59}$$

with $s_1 = +1$ below the amplitude threshold and $s_1 = -1$ above.

This example is not of much interest physically since the numerics of [28] show that the above threshold point is surrounded by unstable Stokes waves. Below the threshold the waves are modulationally unstable, and above the threshold other finite-wavenumber instabilities and multidimensional (two modulation wavenumbers) take over. However, it is of theoretical interest in that it shows how limited qualitative information, obtained numerically, is sufficient to predict the nature of the modulation equation near the transition point.

7. Example: Σ^1 curves and explicit reduction for a nonlinear wave equation.

Consider the nonlinear wave equation, a complex Klein-Gordon (CKG) equation,

$$\Psi_{tt} = \Psi_{xx} - \Psi + |\Psi|^2 \Psi \,, \tag{60}$$

for the complex-valued function $\Psi(x, t)$, which is a model for the nonlinear dynamics near the Kelvin-Helmholtz instability [2]. The CKG equation is generated by the Lagrangian

$$\mathcal{L}(\Psi, \overline{\Psi}) = \frac{1}{2} \int_{t_1}^{t_2} \int_{x_1}^{x_2} \left[-|\Psi_t|^2 + |\Psi_x|^2 + |\Psi|^2 - \tfrac{1}{2}|\Psi|^4 \right] \mathrm{d}x \mathrm{d}t \,, \tag{61}$$

on the set $[x_1, x_2] \times [t_1, t_2] \subset \mathbb{R}^2$. The variation $\delta\mathcal{L}/\delta\overline{\Psi} = 0$, with fixed endpoints, generates (60), and $\delta\mathcal{L}/\delta\overline{\Psi} = 0$ generates the conjugate of (60). Multisymplectification of CKG will be introduced below when required for the calculation of the dispersion coefficient $\mathscr{K}$.

7.1. Periodic travelling waves. The CKG equation (60) has a family of exact periodic travelling wave solutions

$$\Psi(x,t) = \Psi_0 \mathrm{e}^{\mathrm{i}\theta}\,, \quad \theta = kx + \omega t + \theta_0\,, \tag{62}$$

and substitution into (60) gives the nonlinear dispersion relation, relating amplitude to the frequency and wavenumber

$$|\Psi_0|^2 = 1 - \omega^2 + k^2\,. \tag{63}$$

This solution set consists of a hyperboloid of one sheet in the three dimensional space (ω, k, r) with $r = |\Psi_0| > 0$. The projection of this hyperboloid onto the (ω, k) plane is shown in Figure 4. The unshaded region is the set where solutions of (63) exist and it consists of

$$U = \left\{(\omega, k) \in \mathbb{R}^2 \ : \ \omega^2 < 1 + k^2\,,\ k \neq 0\right\}. \tag{64}$$

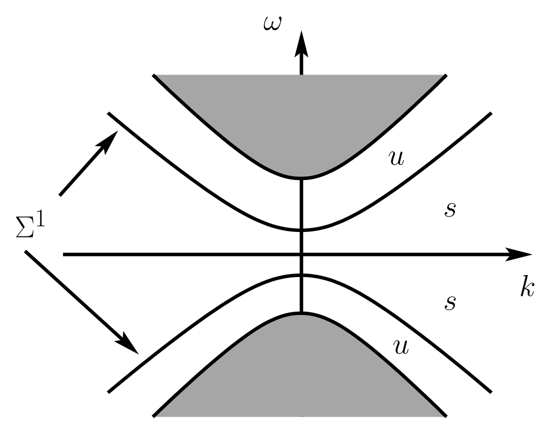

FIG. 4. *Regions of existence and Σ^1 curve for the family of periodic travelling wave solutions of CKG. The symbols s (u) denote regions where the periodic travelling wave is stable (unstable).*

7.2. Conservation law and Σ^1 curves. The conservation law which represents conservation of wave action is due to an S^1-symmetry: $\mathrm{e}^{\mathrm{i}s}\Psi$ is a solution of CKG whenever Ψ is a solution for any $s \in \mathbb{R}$. The conservation law is

$$A_t + B_x = 0\,, \quad \text{with} \quad A = -\mathrm{Im}(\overline{\Psi}\Psi_t)\,, \quad B = \mathrm{Im}(\overline{\Psi}\Psi_x)\,.$$

Evaluate the components of the conservation law on the family of periodic travelling waves

$$\begin{aligned} \mathscr{A}(\omega,k) &= -\omega|\Psi_0|^2 = -\omega(1+k^2-\omega^2) \\ \mathscr{B}(\omega,k) &= k|\Psi_0|^2 = k(1+k^2-\omega^2)\,. \end{aligned} \tag{65}$$

They can also be obtained by substituting (62) into (61), averaging, and differentiating with respect to ω and k. The matrix in the Lighthill determinant is

$$\begin{bmatrix} \mathscr{A}_\omega & \mathscr{A}_k \\ \mathscr{B}_\omega & \mathscr{B}_k \end{bmatrix} = \begin{bmatrix} -1-k^2+3\omega^2 & -2\omega k \\ -2\omega k & 1+3k^2-\omega^2 \end{bmatrix}.$$

Setting the determinant to zero gives

$$\begin{aligned} \Delta_L &= \mathscr{A}_\omega \mathscr{B}_k - \mathscr{A}_k \mathscr{B}_\omega \\ &= (-1-k^2+3\omega^2)(1+3k^2-\omega^2) - 4\omega^2 k^2 \\ &= -(1-\omega^2+k^2)(1-3\omega^2+3k^2) \\ &= -|\Psi_0|^2(1-3\omega^2+3k^2)\,. \end{aligned}$$

Hence the only non-trivial points in U where $\Delta_L = 0$ are when the second factor vanishes

$$\Sigma^1 = \left\{ (\omega,k) \in U \;:\; \omega^2 - k^2 = \tfrac{1}{3} \right\}, \tag{66}$$

with U defined in (64). The singular set Σ^1 consists of two curves and they are labelled in Figure 4, and the stable (unstable) regions in the $(\omega,k)-$plane are labelled with s (u). The image of Σ^1 in the $(\mathscr{A},\mathscr{B})$ plane consists of the two curves

$$\mathscr{A}^2 - \mathscr{B}^2 = \frac{4}{27}\,.$$

All the points in Σ^1 are fold points. There are no cusp points in this example, and so $\kappa \neq 0$. Explicitly,

$$\kappa = (\mathscr{B}_{kk} - c_g \mathscr{A}_{kk}) - 2c_g(\mathscr{B}_{\omega k} - c_g \mathscr{A}_{\omega k}) + c_g^2(\mathscr{B}_{\omega\omega} - c_g \mathscr{A}_{\omega\omega})\,.$$

Computing

$$c_g = \left.\frac{\mathscr{A}_k}{\mathscr{A}_\omega}\right|_{\Sigma^1} = -\frac{\omega}{k}\,,$$

and

$$\begin{aligned} [\mathscr{B}_{\omega\omega} - c_g \mathscr{A}_{\omega\omega}]\big|_{\Sigma^1} &= 4k + \frac{2}{k} \\ [\mathscr{B}_{\omega k} - c_g \mathscr{A}_{\omega k}]\big|_{\Sigma^1} &= -4\omega \\ [\mathscr{B}_{kk} - c_g \mathscr{A}_{kk}]\big|_{\Sigma^1} &= 4k - \frac{3}{2k}\,. \end{aligned}$$

Combining gives

$$\kappa = \frac{2}{3k^3}\,.$$

Since $\mathscr{A}_\omega\big|_{\Sigma^1} = 2k^2$, the emergent two-way Boussinesq equation is

$$2k^2 q_{TT} + \frac{2}{3k^3}(qq_X)_X + \mathscr{K} q_{XXXX} = 0\,.$$

It remains to compute the coefficient of dispersion. It can be computed in this case by deriving the dispersion relation for the linearization of (60) about the periodic travelling wave, but the Jordan chain strategy is used instead to illustrate it in an example, and because it is the most general strategy for more complex problems.

7.3. Multisymplectification, linearization and $\mathscr{K}$. A Legendre transform can be used to develop the multisympletic formulation of CKG, but it is simple enough to write down directly. Let

$$\mathbf{a} = \begin{pmatrix} \mathrm{Re}(\Psi) \\ \mathrm{Im}(\Psi) \end{pmatrix}, \quad \mathbf{b} = \mathbf{a}_t\,, \quad \text{and} \quad \mathbf{c} = \mathbf{a}_x\,.$$

Then CKG has the multisymplectic formulation

$$\begin{bmatrix} 0 & -\mathbf{I}_2 & 0 \\ \mathbf{I}_2 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{pmatrix} \mathbf{a} \\ \mathbf{b} \\ \mathbf{c} \end{pmatrix}_t + \begin{bmatrix} 0 & 0 & \mathbf{I}_2 \\ 0 & 0 & 0 \\ -\mathbf{I}_2 & 0 & 0 \end{bmatrix} \begin{pmatrix} \mathbf{a} \\ \mathbf{b} \\ \mathbf{c} \end{pmatrix}_x = \begin{pmatrix} \mathbf{a} - \|\mathbf{a}\|^2\mathbf{a} \\ \mathbf{b} \\ -\mathbf{c} \end{pmatrix},$$

where $\mathbf{I}_2$ is the 2×2 identity matrix, or

$$\mathbf{M}Z_t + \mathbf{J}Z_x = \nabla S(Z)\,,$$

with

$$\mathbf{K} = \mathbf{J} - c_g\mathbf{M} = \begin{bmatrix} 0 & c_g\mathbf{I}_2 & \mathbf{I}_2 \\ -c_g\mathbf{I}_2 & 0 & 0 \\ -\mathbf{I}_2 & 0 & 0 \end{bmatrix}, \quad Z = \begin{pmatrix} \mathbf{a} \\ \mathbf{b} \\ \mathbf{c} \end{pmatrix} \in \mathbb{R}^6\,,$$

and

$$S(Z) = \tfrac{1}{2}\|\mathbf{b}\|^2 - \tfrac{1}{2}\|\mathbf{c}\|^2 + \tfrac{1}{2}\|\mathbf{a}\|^2 - \tfrac{1}{4}\|\mathbf{a}\|^4\,.$$

In these coordinates the basic state is

$$\widehat{Z}(\theta,\omega,k) = \mathrm{G}_\theta \begin{pmatrix} \widehat{\mathbf{a}} \\ \widehat{\mathbf{b}} \\ \widehat{\mathbf{c}} \end{pmatrix}, \quad \mathbf{b} = \omega\mathbf{J}_2\widehat{\mathbf{a}}\,, \quad \mathbf{c} = k\mathbf{J}_2\widehat{\mathbf{a}}\,,$$

with $\|\widehat{\mathbf{a}}\|^2 = 1 - \omega^2 + k^2$,

$$\mathrm{G}_\theta = R_\theta \oplus R_\theta \oplus R_\theta\,, \quad R_\theta = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}, \quad \text{and} \quad \mathbf{J}_2 = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}.$$

The linearized operator $\mathbf{L}$ is

$$\mathbf{L} = \begin{bmatrix} (1-\|\hat{\mathbf{a}}\|^2)\mathbf{I} - 2\hat{\mathbf{a}}\hat{\mathbf{a}}^T & \omega\mathbf{J}_2 & -k\mathbf{J}_2 \\ -\omega\mathbf{J}_2 & \mathbf{I}_2 & 0 \\ k\mathbf{J}_2 & 0 & -\mathbf{I}_2 \end{bmatrix}, \tag{67}$$

and the Jordan chain satisfies $\mathbf{L}\xi_j = \mathbf{K}\xi_{j-1}$, $j = 1,2,3,4$ with $\xi_0 = 0$. Computing

$$\xi_1 = \widehat{Z}_\theta = \mathrm{G}_\theta \begin{pmatrix} \mathbf{J}_2\widehat{\mathbf{a}} \\ -\omega\widehat{\mathbf{a}} \\ -k\widehat{\mathbf{a}} \end{pmatrix},$$

and, with $\gamma = (k + \omega c_g)\|\widehat{\mathbf{a}}\|^{-2}$,

$$\xi_2 = \mathrm{G}_\theta \begin{pmatrix} \gamma\hat{\mathbf{a}} \\ -(c_g - \omega\gamma)\mathbf{J}_2\hat{\mathbf{a}} \\ (1+k\gamma)\mathbf{J}_2\hat{\mathbf{a}} \end{pmatrix} + \mathbb{R}\xi_1\,, \quad \xi_3 = \mathrm{G}_\theta \begin{pmatrix} 0 \\ -\gamma c_g\hat{\mathbf{a}} \\ \gamma\hat{\mathbf{a}} \end{pmatrix} + \mathbb{R}\xi_1\,,$$

where $\mathbb{R}\xi_1$ represents the arbitrary amount of homogeneous solution. The first three terms in the Jordan chain will be sufficient for computing $\mathscr{K}$ since

$$\mathscr{K} := \langle\!\langle \mathbf{K}\xi_1, \xi_4 \rangle\!\rangle = \langle\!\langle \mathbf{L}\xi_2, \xi_4 \rangle\!\rangle = \langle\!\langle \xi_2, \mathbf{L}\xi_4 \rangle\!\rangle = \langle\!\langle \xi_2, \mathbf{K}\xi_3 \rangle\!\rangle \,.$$

Hence

$$\mathscr{K} = \langle\!\langle \xi_2, \mathbf{K}\xi_3 \rangle\!\rangle = \gamma^2(1-c_g^2)\|\widehat{\mathbf{a}}\|^2 \,.$$

Now using the restrictions

$$c_g = -\frac{\omega}{k}\,, \quad \|\widehat{\mathbf{a}}\|^2 = \frac{2}{3} \quad \text{and} \quad \gamma = -\frac{1}{2k} \quad \text{when} \quad (\omega,k) \in \Sigma^1 \,,$$

it follows that

$$\mathscr{K} = -\frac{1}{18k^4} \,.$$

7.4. CKG to Boussinesq reduction. The Boussinesq model for $(\omega,k) \in \Sigma^1$ is therefore

$$2k^2 q_{TT} + \frac{2}{3k^3}(qq_X)_X - \frac{1}{18k^4} q_{XXXX} = 0 \,. \tag{68}$$

The importance of the assumption $k \neq 0$ in U (64) is evident here. The resulting Boussinesq equation is the linearly ill-posed version since $\mathscr{A}_\omega \mathscr{K} < 0$. Unfolding and scaling leads to the following canonical form

$$u_{\tau\tau} + s_1 u_{\xi\xi} + (\tfrac{1}{2}u^2)_{\xi\xi} - u_{\xi\xi\xi\xi} = 0 \,, \quad s_1 = \pm 1 \,,$$

where $s_1 = -1$ ($s_1 = +1$) on the stable (unstable) side of the Σ^1 curve (66).

To summarize, the CKG equation (60) has a family of exact periodic travelling waves. Modulation of these travelling waves in the neighbourhood of the Σ^1 curves (66) leads to a reduction to the two-way Boussinesq equation (68). The reduced equation contains a range of bounded periodic, quasiperiodic and localized solutions, but it also portends more dramatic behaviour in the original CKG equation in that it is linearly ill-posed and so general initial data may be dramatically unstable.

8. Coalescing characteristics and multiphase wavetrains. The theory in this paper is for basic states with one phase. However there are many examples in the literature where at least two phases are present. Examples are modulation of the cnoidal wave solutions of the KdV equation (§16.14 of [38]), modulation of Stokes waves in finite depth coupled to mean flow (§16.6-16.11 in [38]), and modulation of viscous fluid conduit periodic waves (Maiden & Hoefer [26]). In the latter two examples there is an elliptic-hyperbolic transition. However the theory of this paper does not apply directly and needs to be generalized to multiphase wavetrains. A theory for bifurcation of multiphase wavetrains near a zero characteristic has recently been developed by Ratliff & Bridges [32]. Hence there is some optimism that the theory of this paper can be generalized to the elliptic-hyperbolic transition in multiphase wavetrains, but is outside the scope of this paper.

9. Concluding remarks. The modulation equations derived here

$$q_T = \Omega_X \quad \text{and} \quad \mathscr{A}_\omega \Omega_T + \kappa q q_X + \mathscr{K} q_{XXX} = 0 \,, \tag{69}$$

are *asymptotically* valid in that the modulation ansatz (42) satisfies the governing equation (22) exactly with an error of order ε^6. However, this theory gives no indication of convergence to all orders in ε.

Rigorous validity of the theory presented here is an open question, and outside the scope of this paper. Rigorous validity is generally done in three steps: show that the original equation has a well-defined existence theory, show that the reduced equation has a well-defined existence theory, and then show that the difference between the exact and approximate solution stays close for a time interval of order ε^{-p}, for some $p > 0$.

Even considering validity of the CKG reduction to Boussinesq as an example, rather than reduction from an abstract Lagrangian, there is still a difficulty with the fact that the reduced equation (69) may not be well posed in general, particularly in the case where $\mathscr{A}_\omega\mathscr{K} < 0$, which arises in the CKG example. Hence methodology based on Cauchy-Kowalevskaya in a space of functions which are complex analytic in a strip would be required. This approach was successfully used by Düll & Schneider [14] in their proof of the validity of *elliptic* Whitham modulation equations in a reduction from the nonlinear Schrödinger equation.

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