International Journal of Engineering and Information Systems (IJEAIS)
ISSN: 2000-000X
Vol. 3 Issue 4, April – 2019, Pages: 40-49

The General Solution for a linear Second Order Homogenous Differential Equations with Variable Coefficients

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Abstract : *The main goal in this work to find the general solution for some kind of linear second order homogenous differential equations with variable coefficients which have the general form* $y'' + P(x)\,y' + Q(x)\,y = 0$ *, by using the substitution* $y = e^{\int Z(x)\,dx}$ *,which transform form the above equation to Riccati equation .*

Keyword :- Differential equations , Riccati equation , Bernoulli equation .

1. Introduction

Many researchers in this field of differential equations , may face a difficult in solving the linear second order differential equations by using known methods . Therefore ,they are trying to solve these equations by using the power series or the Frobenius method [1] .

Kathem [2] gave a method for solving the above equation ,this method depends to find a function $Z(x)$ such that $y = e^{\int Z(x)\,dx}$. Kathem [2] only gave examples which enable to find the general soluation by using this substitution .

2- Bernoulli Equation [3]

The general form that of **Bernoulli equation** has is written as

$$y' + p(x)\,y = q(x)\,y^{n} \quad n \neq 1$$

where p and q are functions of, x (or constants)

3- Riccati Equation [4]

The general form of Riccati equation is written as

$$y' = f(x) + g(x)\,y + h(x)y^{2} \qquad ...(1)$$

where f , g and h are given functions of x (or constants) . We can solve it , if one or more particular solutions of (1) can be found by inspection or otherwise . The general solution of (1) is easy to be obtained by the following conditions

i- If y_1 is a known particular solution , then the general solution can be obtained by the assumption :-

$$U = y - y_1 \ ,$$

then (1) transformed into Bernoulli equation

$$U\ (x) + (g + 2hy_1)U = hU^{2} \ ,$$

so , the general solution of (1) is given by

$$(y - y_1)\left[c - \int h(x)\chi(x)\,dx\right] = \chi(x) \ ;\chi(x) = e^{\int (g+2hy_1)dx}$$

ii-If y_1 and y_2 are two known particular solutions , then the general solution of (1) can be found by the assumption

$$U(y-y_2)=y-y_1 ,$$

then the general solution is given by :-

$$y-y_1=C(y-y_2)\ e^{\int h(x)\,(y_1-y_2)\,dx} ,$$

where C is an arbitrary constant .

iii- If y_1, y_2 and y_3 ,are three known particular solutions, say then the general solution of equation (1) is given as :

$$\frac{(y-y_1)(y_3-y_2)}{(y_3-y_1)(y-y_1)}=C ,$$

where C is an arbitrary constant

4- How Find The General Solution for the Linear Second Order Differential Equations

We can solve the equation

$$y''+P(x)y'+Q(x)y=0 \qquad ...(2)$$

by the following cases:

i- If $P(x)$ and $Q(x)$ are constants say $P(x)=a$ and $Q(x)=b$ then the equation (2) becomes

$$Z'(x)+Z^2(x)+aZ(x)+b=0 \qquad ...(3) ,$$

and the solution of (3) is given by :-

a) $$\boxed{y=e^{-\frac{a}{2}x}\left(C_1\cos\sqrt{b-\frac{a^2}{4}}x+C_2\sin\sqrt{b-\frac{a^2}{4}}\,x\right)} ,$$

if $b\neq\frac{a^2}{4}$,where α and β are arbitrary constants

b) $$\boxed{y=Ae^{-\frac{a}{2}x}(x+C)} ,$$

if $b=\frac{a^2}{4}$,where A and C are arbitrary constants .

proof :-

a) Since $Z'(x)+Z^2(x)+aZ(x)+b=0$, so

$$\frac{dZ}{\left(Z+\frac{a}{2}\right)^2+b-\frac{a^2}{4}}+dx=0\Rightarrow\int\frac{dZ}{\left(Z+\frac{a}{2}\right)^2+d^2}=-x+C;d^2=b-\frac{a^2}{4}$$

$$\Rightarrow\frac{1}{d}\tan^{-1}\left(\frac{Z+a/2}{d}\right)=-x+C\Rightarrow Z=d\tan(f-dx)-\frac{a}{2};f=dc$$

$$y=e^{\int\left(d\tan(\mathrm{f}-\mathrm{dx})-\frac{\mathrm{a}}{2}\right)dx}=e^{\ln\cos(f-dx)-\frac{a}{2}x+g}$$

$$y=Ae^{-\frac{a}{2}x}\cos(f-dx);A=e^g$$

$$y=e^{-\frac{a}{2}x}(C_1\cos\sqrt{b-\frac{a^2}{4}}\;x+C_2\sin\sqrt{b-\frac{a^2}{4}}\;x)\;;\;where\;\;C_1=A\cos f\;\;and\;\;C_2=A\sin f$$

4.1.Example:- For solving the differential equation

$$y''+2y'-3y=0\qquad,a=2\,and\,b=-3\;,$$

since $b\neq\frac{a^2}{4}$

Then , by using the above formula , we get the general solution which has the form

$$y=e^{-x}(C_1\cos\sqrt{-3-1}\;x+C_2\sin\sqrt{-3-1}\;x)\Rightarrow y=e^{-x}(C_1\cos 2ix+C_2\sin 2ix)$$

$$y=e^{-x}(C_1\frac{e^{-2x}+e^{2x}}{2}+C_2\frac{e^{-2x}-e^{2x}}{2i})=Ae^{-3x}+Be^{x}$$

$$,where\;A=(\frac{C_1}{2}+\frac{C_2}{2i})\;,B=(\frac{C_1}{2}-\frac{C_2}{2i})$$

b) If

$$b=\frac{a^2}{4}\Rightarrow\frac{dz}{\left(Z+\frac{a}{2}\right)^2}+dx=0\Rightarrow-\frac{1}{Z+\frac{a}{2}}=C_1-x\Rightarrow Z+\frac{a}{2}=\frac{1}{x+C}\quad;C=-C_1$$

Since

$$y=e^{\int Z(x)dx}\Rightarrow y=e^{\int\left(\frac{1}{x+C}-\frac{a}{2}\right)dx}\Rightarrow y=e^{\ln(x+C)-\frac{a}{2}x+C_2}$$

$$y=Ae^{-\frac{a}{2}x}(x+C)\quad ;A=e^{C_2}$$

4.2.Example:-For solving the differential equation

$$y''+4y'+4y=0 \qquad ;a=4,b=4$$

we will use the general form in the above formula and we get

$$y=e^{-\frac{a}{2}x}(Ax+B)\ ,B=AC\ \Rightarrow\ y=e^{-2x}(Ax+B)$$

ii-**If** $Q(x)=0$ **, then the general solution is given by :-**

$$\boxed{y=A\int e^{-\int pdx}dx+B}$$

proof:- Since

$$Z'(x)+Z^2(x)+PZ(x)+Q=0\Rightarrow Z'+Z^2+PZ=0\Rightarrow Z'+PZ=-Z^2$$

,

this is like Bernoulli equation , to solve it , let $Z^{-1}=t\Rightarrow t'-Pt=1$

.this equation is linear , and its integrating factor is given by:-

$$I.F=e^{-\int p(x)dx}\Rightarrow e^{-\int p(x)dx}dt-t\,p(x)e^{-\int p(x)dx}dx=e^{-\int p(x)dx}dx$$

$$\Rightarrow e^{-\int p(x)dx}t=\int e^{-\int p(x)dx}dx\Rightarrow Z=\frac{e^{-\int p(x)dx}}{\int e^{-\int p(x)dx}dx}$$

$$y=e^{\int\frac{e^{-\int p(x)dx}}{\int e^{-\int p(x)dx}dx}dx}$$

$$y=e^{\ln\int e^{-\int p(x)dx}dx+C_1}=A\int e^{-\int p(x)dx}dx+B \qquad ,A=e^{C_1}$$

4.3.Example :- For solving the differential equation

$$y''-\frac{2}{x}y'=0\ ,$$

we use the general form in the above formula and we get

$$y = A\int e^{-\int p(x)dx}dx + B \qquad \Rightarrow \qquad y = A\int e^{\int \frac{2}{x}dx}dx + B$$

$$y = A\int e^{2\ln x}dx + B \qquad \Rightarrow \qquad y = A_1 x^3 + B \qquad ; A_1 = \frac{A}{3}$$

iii-If $P(x) = 2\sqrt{Q(x)}$, then the equation (3) can be solved by the assumption $u = Z(x) + \sqrt{Q(x)}$,

since

$$Z' + Z^2 + PZ + Q = 0 \Rightarrow Z' + Z^2 + 2\sqrt{Q} + Q = 0 \Rightarrow Z' + \left(Z + \sqrt{Q}\right)^2 = 0 ,$$

to solve this equation , let

$$u = Z + \sqrt{Q} \Rightarrow Z' = u' - \frac{Q'}{2\sqrt{Q}} = u' - \frac{Q'}{P} \Rightarrow u' - \frac{Q'}{P} + u^2 = 0 \Rightarrow u' + u^2 = \frac{Q'}{P} ,$$

this is Riccati equation , with $f(x) = 1, g(x) = 0 \; and \; k(x) = \frac{Q'}{P}$

Now , there are many cases

1-If u_1 is a known solution to the last equation , then the general solution is given by

$$y = Ae^{\int\left(u_1 - \sqrt{Q(x)}\right)dx}\int e^{-2\int u_1 dx}dx$$

proof:-

The assumption $d = u - u_1$ transforms the equation to Bernoulli equation which has the form:-

$$d' + d^2 + 2u_1 d = 0 \Rightarrow d' + 2u_1 d = -d^2 ,$$

to solve it , we set

$d^{-1} = t \Rightarrow -d^{-2}d' = t' \Rightarrow d^{-2}d' = -t' \Rightarrow t' - 2u_1 t = 1$, this is linear equation , and its integrating factor is given by :-

$I.F = e^{-\int 2u_1 dx}$,so the general solution of the last equation is given by :-

$$e^{-\int 2u_1 dx} t = \int e^{-\int 2u_1 dx} dx \Rightarrow \frac{e^{-\int 2u_1 dx}}{d} = \int e^{-\int 2u_1 dx} dx$$

$$u - u_1 = \frac{e^{-\int 2u_1 dx}}{\int e^{-\int 2u_1 dx} dx} \Rightarrow u = \frac{e^{-\int 2u_1 dx}}{\int e^{-\int 2u_1 dx} dx} + u_1$$

$$Z + \sqrt{Q(x)} = \frac{e^{-\int 2u_1 dx}}{\int e^{-\int 2u_1 dx} dx} + u_1 \Rightarrow Z = \frac{e^{-\int 2u_1 dx}}{\int e^{-\int 2u_1 dx} dx} + u_1 - \sqrt{Q(x)}$$

$$y = e^{\int \left(\frac{e^{-\int 2u_1 dx}}{\int e^{-\int 2u_1 dx} dx} + u_1 - \sqrt{Q(x)} \right) dx}$$

$$y = e^{\ln \int e^{-\int 2u_1 dx} dx + \int \left(u_1 - \sqrt{Q(x)} \right) dx + C}$$

$$y = A\, e^{\int \left(u_1 - \sqrt{Q(x)} \right) dx} \int e^{-\int 2u_1 dx} dx \quad ; A = e^C$$

.

2-If u_1 and u_2 are two known solutions , then the general solution of the last equation is given by :-

$$y = e^{\int \left(\frac{u_1 - C u_2\, e^{\int (u_1 - u_2) dx}}{1 - C\, e^{\int (u_1 - u_2) dx}} - \sqrt{Q} \right) dx} \quad ; C = \text{constant}$$

proof:- From Riccati equation we get

$$u - u_1 = C(u - u_2)\, e^{\int (u_1 - u_2) dx} \quad ; C \text{ is any arbitrary constant}$$

so

$$u = \frac{u_1 - C u_2\, e^{\int (u_1 - u_2) dx}}{1 - C\, e^{\int (u_1 - u_2) dx}} \Rightarrow y = e^{\int \left(\frac{u_1 - C u_2\, e^{\int (u_1 - u_2) dx}}{1 - C\, e^{\int (u_1 - u_2) dx}} - \sqrt{Q} \right) dx}$$

3-if u_1, u_2 and u_3 are three known solutions , then the general solution of the last equation is given by :-

$$y = e^{\int\left(\frac{u_1 - CJ(x)u_2}{1 - CJ(x)} - \sqrt{Q(x)}\right)dx} \quad ; \; J(x) = \left(\frac{u_3 - u_1}{u_3 - u_2}\right) \quad ; \; C = cons\tan t$$

proof:- From Riccati equation we get :

$$\frac{u - u_1}{u - u_2} = C\left(\frac{u_3 - u_1}{u_3 - u_2}\right) \; ; C \text{ is any arbitrary constant}$$

$$\Rightarrow \frac{u - u_1}{u - u_2} = C\,J(x) \quad ; J(x) = \left(\frac{u_3 - u_1}{u_3 - u_2}\right)$$

$$\Rightarrow u - u_1 = C\,J(x)u - C\,J(x)u_2 \;\; \Rightarrow u = \frac{u_1 - CJ(x)u_2}{1 - CJ(x)}$$

$$\Rightarrow Z = \frac{u_1 - CJ(x)u_2}{1 - CJ(x)} - \sqrt{Q(x)}$$

$$y = e^{\int\left(\frac{u_1 - CJ(x)u_2}{1 - CJ(x)} - \sqrt{Q(x)}\right)dx} \quad ; \; J(x) = \left(\frac{u_3 - u_1}{u_3 - u_2}\right)$$

__Note__***:-*** Some of these equations can be transformed into variable separable equations and don't need the above formula to find the general solution

4.5. Example ***:-*** For solving the differential equation

$$y'' + 2xy' + x^2 y = 0 \quad ; \quad P(x) = 2x \;, Q(x) = x^2 \;,$$

by using the equation (3) we get

$$Z' + Z^2 + 2xZ + x^2 = 0 \;\Rightarrow Z' + (Z + x)^2 = 0 \quad ,$$

let

$$Z + x = t \Rightarrow Z' = t' - 1$$

$$\Rightarrow t' - 1 + t^2 = 0 \Rightarrow \frac{dt}{1 - t^2} - dx = 0 \Rightarrow \tanh^{-1} t = x + C \Rightarrow t = \tanh(x + C)$$

$$\Rightarrow Z = \tanh(x + C) - x \quad ,$$

since

$$y = e^{\int Z(x)dx} \Rightarrow y = e^{\int (\tanh(x+C) - x)dx}$$

$$y = e^{\ln\cosh(x+C) - \frac{1}{2}x^2 + a} \Rightarrow y = e^{-\frac{1}{2}x^2 + a}\cosh(x+C)$$

$$y = e^{-\frac{1}{2}x^2}\left(A\cosh x + B\sinh x\right) \qquad ; A = e^a\cosh C, B = e^a\sinh C$$

iv) If $P(x)$ and $Q(x)$ are not any one of the above cases , then the equation $Z' + Z^2 + P(x)Z + Q(x) = 0$ is like Riccati equation . As a result then there are three cases **:**

1-If Z_1 is a known solution to it, then the general solution of (1) is given by **:-**

$$y = A\, e^{\int Z_1 dx} \int e^{-\int (P + 2Z_1)dx} dx \qquad ; A = e^a$$

Proof :-

The assumption $Z = Z_1 + u$ transforms the equation to Bernoulli equation which has the form:-

$$u' + (P + 2Z_1)u + u^2 = 0 \quad ,$$

to solve it , we assume $u^{-1} = t$

$\Rightarrow t' - (P + 2Z_1)t = 1$ this is a linear equation , and its integrating factor (I.F) is given by :-

$$I.F = e^{-\int (P+2Z_1)dx} \Rightarrow t\,.\,e^{-\int (P+2Z_1)dx} = \int e^{-\int (P+2Z_1)dx} dx$$

$$\Rightarrow Z = \frac{e^{-\int (P+2Z_1)dx}}{\int e^{-\int (P+2Z_1)dx} dx} + Z_1 \Rightarrow y = e^{\int \left(\frac{e^{-\int (P+2Z_1)dx}}{\int e^{-\int (P+2Z_1)dx} dx} + Z_1 \right) dx}$$

$$\Rightarrow y = e^{\ln \int e^{-\int (P+2Z_1)dx} dx} \, e^{\int Z_1 dx + a}$$

$$\Rightarrow y = A \, e^{\int Z_1 dx} \int e^{-\int (P+2Z_1)dx} dx \qquad ; A = e^a$$

2- If Z_1 and Z_2 are two known solutions of it , then the general solution of this equation is given by :-

$$y = e^{\int \left(\frac{Z_1 - C Z_2 e^{\int (Z_1 - Z_2) dx}}{1 - C e^{\int (Z_1 - Z_2) dx}} \right) dx} \quad ; C = constant$$

proof:- From Riccati equation , we can write

$$Z \left(1 - C e^{\int (Z_1 - Z_2) dx} \right) = Z_1 - C Z_2 e^{\int (Z_1 - Z_2) dx} \Rightarrow Z = \frac{Z_1 - C Z_2 e^{\int (Z_1 - Z_2) dx}}{1 - C e^{\int (Z_1 - Z_2) dx}}$$

$$\Rightarrow y = e^{\int \left(\frac{Z_1 - C Z_2 e^{\int (Z_1 - Z_2) dx}}{1 - C e^{\int (Z_1 - Z_2) dx}} \right) dx}$$

3-If Z_1, Z_2 and Z_3 are three known solutions of it , then the general solution of this equation is given by :-

$$y = e^{\int \left(\frac{Z_1 - C J(x) Z_2}{1 - C J(x)} \right) dx} \quad ; C = constant \; and \; J(x) = \frac{Z_3 - Z_1}{Z_3 - Z_2}$$

proof:- From Riccati equation , we can write

$$\frac{Z - Z_1}{Z - Z_2} = C \left(\frac{Z_3 - Z_1}{Z_3 - Z_2} \right) \qquad ; C \text{ be any arbitrary constant}$$

$$\Rightarrow \frac{Z - Z_1}{Z - Z_2} = C \; J(x) \qquad ; J(x) = \frac{Z_3 - Z_1}{Z_3 - Z_2} \quad \Rightarrow Z - Z_1 = C \, J(x) \;\; Z - C \, J(x) \; Z_2$$

$$Z = \frac{Z_1 - C J(x) Z_2}{1 - C J(x)} \qquad \Rightarrow y = e^{\int \left(\frac{Z_1 - C J(x) Z_2}{1 - C J(x)} \right) dx}$$

4.6..Example :- For solving the differential equation

$$y'' + \frac{2}{x} y' - \frac{2}{x^2} y = 0 \; ,$$

we use the general form in the above formula , which is

$$y = A \, e^{\int Z_1 dx} \int e^{-\int (P + 2Z_1) dx} dx \qquad ; A = e^a \; ,$$

now, let $Z_1 = \frac{1}{x}$ (which is a particular solution of Riccati equation)

$$y = A\, e^{\int \frac{1}{x} dx} \int e^{-\int \frac{4}{x} dx} dx$$

$$y = A e^{\ln x} \int e^{-4 \ln x} dx = Ax\left(\frac{-x^{-3}}{3} + C_1 \right) = -\frac{A}{3} x^{-2} + Bx \qquad ; B = AC_1$$

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