High-order metaphysics as high-order abstractions and choice in set theory:
Or how to distinguish the good high-order metaphysics from their "bad company"

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The thesis is:

The link between the high-order metaphysics and abstractions, on the one hand, and choice in the foundation of set theory, on the other hand, can distinguish unambiguously the “good” principles of abstraction from the “bad” ones and thus resolve the “bad company problem” as to set theory. Thus it implies correspondingly a more precise definition of the relation between the axiom of choice and “all company” of axioms in set theory concerning directly or indirectly abstraction: the principle of abstraction, axiom of comprehension, axiom scheme of specification, axiom scheme of separation, subset axiom scheme, axiom scheme of replacement, axiom of unrestricted comprehension, axiom of extensionality, etc.

The base of that link is: The good abstraction is always a choice in the sense of set theory; or in other words, that abstraction, to which a choice does not correspond, is a “bad abstraction” implying contradictions. The concept of choice in set theory is fundamental (like that of point in geometry) and cannot be defined rigorously otherwise than contextually and indirectly by the axiom of choice as an unary operation on a set, the result of which is an exactly determined element belonging to the set. Any definition of the axiom concerns this operation. Whatever definition of the axiom of choice is granted, it implies that an element of the set of all subsets of any set can be chosen under its conditions whatever they are. If and only if that element is selective, i.e. it can be chosen under the corresponding conditions, under which the axiom of choice is accepted as valid, that set can be defined unambiguously by some combinations of predicates according to the conditions of the given axiom or axiom scheme concerning abstraction in set theory.

Often, the axiom of choice or its weaker versions are additionally appended to the corpus of all rest axioms of set theory. Thus, the necessity of choice for any axiom of abstraction in set theory is hidden. Choice is implicitly added by some kind of restriction as to the corresponding axiom concerning abstraction, e.g. as the requirement for the abstracted set “A” to be true subset of some other set “B”. Indeed the abstracted set can be implicitly chosen as an element of the set of all subsets of “B”. However, any edition of the axiom of choice is not necessary in that case and can be added or not to the rest axioms.

If the axiom of choice is involved explicitly, therefor Hume’s principle (Boolos 1987: 5) can serve as a criterion for distinguishing the “good” principles of abstractions from the “bad” ones as to set theory as follows:

The axiom of choice allows of existing of a well-ordered model “M” for any set “S” under the conditions of the axiom of choice so that a one-to-one mapping between “M” and “S” exists. If Hume’s principle should be restricted only to “M”, the axiom of choice allows it to be expanded to any “S” with no generalization. In other words, the axiom of choice is the condition for all possible principle of abstraction (Linnebo 2009a: 372) to be able to be reducible to Hume’s principle as to set theory. This allows of investigating some analogs of the axiom of choice as to well-founding (Linnebo 2009a: 374-375), e.g. some principles of individuation (Linnebo 2009b: 221-223) in a well-ordered series. Thus individuation can be discussed as an analog of choice in set theory.

Both choice and individuation determine the concept of abstraction in the ontology of mathematics.

They exclude as “bad” those abstractions (or principles), which does not content (or require) choice or individuation. For example, the set of all sets is a bad abstraction for it does not admit any choice or individuation of that set among another. In other words, an abstraction makes sense if and only if it excludes at least one element or individual from its extension. This means that “COLLAPSE” (Linnebo 2010) can be interpreted as ‘choice’ in the sense of set theory.

One can continue further to Popper’s “asymmetry between verifiability and falsifiability; an asymmetry which results from the logical form of universal statements. For these are never derivable from singular statements, but can be contradicted by singular statement” (Popper 1935: 13) as the next generalization in the same sequence. Indeed a scientific theory as a form of rather extended abstraction should demonstrate at least one possible counterexample unlike any “metaphysics” using the distinction in his book (Popper 1935: 9).

So, the abstraction possesses two definitive properties: generalization and distinction. The "bad" abstractions only generalize, but do not distinguish from anything. The property of distinction is formalized in set theory implicitly by the concept of choice and explicitly by the axiom of choice. An element from any finite set can be chosen without the help of the axiom of choice. The property of generalization is formalized in set theory by means of some axiom or axiom scheme of abstraction, e.g. that of specification. That abstraction is restricted by certain relevant condition within the corresponding axiom according to the general requirement for the independence of the axioms for that restriction is much weaker than the most editions of the axiom of choice.

If the axiom of choice is granted in some edition of it, a corresponding infinite series of embedded, stronger and stronger (i.e. more and more distinctive) abstractions will be well-ordered and thus well-founded.

The first element of that series will be a "bad" abstraction, though, if it is not obtained under the restricting condition within the corresponding axiom or axiom scheme about abstraction in set theory and thus independently of the axiom of choice. An alternative is to be utilized those editions of the axiom of choice, which allow a "pure" choice or a choice from an empty set. If that is the case, no abstraction in set theory would be "bad" for e.g. certain distinguishing choice is postulated even in relation to the set of all sets because its complement is the empty set admitting a choice under that edition of the axiom of choice, too. Such an approach is analogous to that solving of the "bad company problem" (Linnebo 2009a), according to which no abstraction is "bad" under certain condition.

Of course, the leave of a "pure" choice or a choice from an empty set does not seem intuitively enough justified. Then, what about the corresponding case in Hume's principle? It will be about the case

"$\boldsymbol{Fs} = \boldsymbol{Gs} = \boldsymbol{0}$", which is out of its formulation. So this case can serve as grounding though rather doubtful is not explicit as the principle starts directly from "$\boldsymbol{Fs} = \boldsymbol{Gs} = \boldsymbol{1}$". That kind of beginning is analogous to the restriction in the corresponding axiom or axiom scheme about abstraction in set theory unlike e.g. the axiom of unrestricted comprehension in "naïve" set theory.

Furthermore, if Hume's problem can serve as the criterion to distinguish the "bad" from the "good" abstractions, this in turn implies the axiom of choice as to set theory. Indeed if the abstractions constitute a well-founded series always, the corresponding sets obtained by those abstractions will in turn constitute a well-ordered series always. This implies well-ordering theorem, which is equivalent to the axiom of choice.

One can designate as the "name" or "natural name" of a set that logical function, which is equivalent to it according to the corresponding axiom (or axiom scheme) of abstraction in set theory. Then, what is the relation between the name and the choice of one and the same set? Can a set be chosen without having any name? Or vice versa: can a set be named without being chosen? This paper suggests the equivalence of the name and the choice of one and the same set for it seems intuitively justified. However, I do not know whether this can be proved in set theory or would require an additional axiom.

Furthermore, "This set has this name" should be a decidable proposition. However, the so-called Gödel first incompleteness theorem, "Satz VI" (Gödel 1931: 187) implies that there are such sets and such names, about which that proposition is not decidable if the conditions of the validity of the theorem are satisfied. This implies for the name of any set to be imposed suitable restrictions, which should exclude the application of Gödel's theorem. I can choose as a name any proposition, which is not decidable in thus.

One believes that this can be avoided by the restriction in the corresponding postulate in set theory for the names to be finite or to consist of a finite set of free variables. However, what about the sets having no finite name, but possessing an infinite name? Is there at least one set of that kind? Obviously, yes, there is: e.g. any transcendental number without any special designation like "π", "e", etc. One need an actual infinite set, e.g. that of its digits, in order to construct its name. However the restriction of name in the corresponding axiom scheme in set theory about abstraction should exclude it in thus saving the theory from the Gödel undecidable propositions as names of sets.

Consequently, there is another form of the "bad company problem" in set theory as to the admissible names used as the names of sets in abstractions: For example, the Gödel propositions and the names of transcendental numbers constitute such a "bad company". However, the axiom of choice is able to

distinguish unambiguously even between them: Indeed, the transcendental number being single can be chosen while any set specified by some undecidable proposition cannot be chosen for it is always doubled by the empty set as the proposition can be interpreted both as true and as false. Not one, but two possible and different choices correspond to that kind of proposition.

What about the analog of these difficulties in terms of Hume's principle? This would be the case "$\boldsymbol{Fs} = \boldsymbol{Gs} = \infty$", where "$\infty$" should interpret as actual infinity. That case is also out of the formulation of the principle for it is specific to the concept of abstraction in set theory.

If the abstraction is understood as both generalization and choice as above, the question about the relation between generalization and choice appears. It is a kind of anti-symmetry: Indeed the choice indicates an element from a set while the generalization indicates a set, to which a given element belongs. That anti-symmetry might be visualized as reversal direction. As generalization is choice in reverse direction, as choice is generalization in reverse direction. Any given abstraction might be thought as a point of a segment between the end points of choice and generalization. Both extremes of that segment can generate contradictions and should be cut.

The concept (and even the quantity) of information can be also utilized in order to describe abstraction as the relation (and even the ratio) between choice and generalization. Indeed information can be discussed as an order reached by a series of successive choices and the quantity of information is the minimal amount of elementary choices necessary for this order to be created. The unit of the quantity of information is that elementary choice defined as the choice between two equally probable alternatives: one bit of information.

However, that concept of information is not applicable to infinite series or sets, which are the interesting area in set theory. The notion of quantum information involved by quantum mechanics can be considered as a relevant generalization as to infinity. The unit of quantum information, one qubit, is a generalization of bit as a choice among a continuum of alternatives. Furthermore Hilbert space, in which quantum information is definable, can be introduced as a generalization of the positive integers, after which any positive integer is replaced by a corresponding cell of a qubit.

Hume's principle can be relevantly and rather heuristically generalized, too: In the quantum principle of Hume "$\boldsymbol{Gs}$" should be interpreted as some "many" and "$\boldsymbol{Fs}$" as some "much" of one and the same abstraction. Indeed the axiom scheme in set theory about abstraction can be interpreted as a scheme of tautologies, in which each name designates a set as a whole, i.e. as a "much", while the collection of elements designates as a "many" consisting of separated individuals.

That quantum principle of Hume is quite meaningful and exceptionally well interpretable in terms of quantum mechanics and the theory of quantum information. Furthermore, it can serve as a reference frame to the problems about the foundation of set theory just as the original principle is such a one about arithmetic and thus as a link.

Conclusions:

1. The logical acceptability of high-order metaphysics can be equivalently reduced to that of high-order abstractions, that is to an unlimited series of abstractions from abstractions
2. Thus one needs a relevant and general criterion for which cases of an abstraction from other abstractions are acceptable
3. Abstraction is both generalization and choice
4. Abstraction in set theory needs the axiom of choice
5. The concept and quantity of information can serve to describe abstraction
6. The description of abstraction in set theory needs the concept and quantity of quantum information
7. Hume's principle can be fruitfully generalized in terms of quantum information

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