

# Triangular Acceleration Methods of Second Kind for Improving the Values of Integrals Numerically

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**Abstract:** The aims of this study are to introduce acceleration methods that are called triangular acceleration methods, which come within the series of several acceleration methods that generally known as Al-Tememe's acceleration methods of the second kind which are discovered by (Ali Hassan Mohammed). These methods are useful in improving the results of determining numerical integrals of continuous integrands where the main error is of the forth order with respect to accuracy, partial intervals and the fasting of calculating the results specifically to accelerate results come out by Simpson's method. Also, it is possible to make use of it to improve the results of solving differential equations numerically of the main error of the forth order

There are numerical methods for calculating single integrals that are bounded in their integration intervals.

1. Trapezoidal Rule
2. Midpoint Rule
3. Simpson's Rule

It is called Newton–Cotes formulas.

This study will introduce Simpson's method to find approximate values of single integrals of continuous integrands through using triangular acceleration methods, which fall within Al-Tememe's acceleration series- the second kind. We will compare these methods in respect to accuracy and the fasting of approaching these values to the real value (analytical) of those integrals.

Let's assume the integration J:

$$J = \int_{x_0}^{x_{2n}} f(x) dx \quad (1)$$

Such that  $f(x)$  is a continuous integrand Lies above  $x$  - axis in the interval  $[x_0, x_{2n}]$ . Generally, Newton–Cotes formula for integration can be written in the following form:

$$\int_{x_0}^{x_{2n}} f(x) dx = G(h) + E_G(h) + R_G$$

Such that  $G(h)$  represents (Lagrangian – Approximation) of integration value  $J$ ,  $G$  refers to the type of the rule,  $E_G(h)$  is the correction terms that can be added to  $G(h)$  and  $R_G$  is the remainder .

the Simpson's rule value  $G(h)$  will referred by  $S(h)$  and it is givenby:

$$S(h) = \frac{h}{3} [f(a) + 4f(a+h) + 2f(a+2h) + 4f(a+3h) + \dots + 2f(a+(2n-2)h) + 4f(a+(2n-1)h) + f(b)]; h = \frac{b-a}{2n} \quad \text{and } n=1,2,\dots$$

And the general formula for  $E_G(h)$  is given by :

$$(E_S(h) = E_G(h)) \quad (f_{x_{2n}}^{(3)} - f_{x_0}^{(3)}) + \dots + (f_{x_{2n}}^{(3)} - f_{x_0}^{(5)}) + \frac{1}{1512} (E_S(h) = \frac{1}{180}$$

Fox [3]

So, when integrands of integration is a continuous function and their derivatives are in each point of integration intervals  $[x_0, x_{2n}]$ , it is possible to write error formula as:

$$E = J - s(h) = A_1 h^4 + A_2 h^6 + A_3 h^8 + \dots$$

Such that  $A_1, A_2, A_3, \dots$  are constants that their values do not depend on  $h$  but on the values of the derivatives of integrand in the end of the integration interval.

## 2. Al-Tememe's triangular acceleration of second kind

We will introduce six rules of Al-Tememe's triangular acceleration, which come within Al-Tememe's acceleration series of the second type.

It is mentioned above that the error in Simpson's rule can be written as the following:

$$E = A_1 h^4 + A_2 h^6 + \dots = h^3 (A_1 h + A_2 h^3 + \dots) \quad (3)$$

$$\cong h^3 \sin h \quad \text{since } (\sinh = h - \frac{1}{6} h^3 + \frac{1}{120} h^5 - \frac{1}{5040} h^7 + \dots) \quad [2]$$

If we assume that J value calculated numerically based on Simpson's rule as  $S_1(h_1)$  when  $h=h_1$  and  $S_2(h_2)$ , when  $h=h_2$ , so:

$$J-S_1(h_1) \cong h_1^3 \sin h \tag{4}$$

$$J-S_2(h_2) \cong h_2^3 \sin h \tag{5}$$

From the equations (4) and (5), we get:

$$A^S_{\text{Sin}} \cong \frac{(h_1^3 \sin h_1) S_2(h_2) - (h_2^3 \sin h_2) S_1(h_1)}{h_1^3 \sin h_1 - h_2^3 \sin h_2} \tag{6}$$

The formula (6) is called Al-Tememe's sine triangular acceleration of the second kind that referred to by ( $A^S_{\text{Sin}}$ )

Similarly, the second triangular acceleration rule can be written. The error E can be written as :

$$E = h^4(A_1h + A_2h^2 + A_3h^4 + \dots) \cong h^4 \cosh$$

Based on the same sine method mentioned above, we get the following:

$$A^S_{\text{Cos}} \cong \frac{(h_1^4 \cos h_1) S_2(h_2) - (h_2^4 \cos h_2) S_1(h_1)}{h_1^4 \cos h_1 - h_2^4 \cos h_2} \tag{7}$$

$$\text{Since } (\cosh = 1 - \frac{1}{2}h^2 + \frac{1}{24}h^4 - \frac{1}{720}h^6 - \dots) \tag{2}$$

We call the formula (7) as Al-Tememe's cosine triangular acceleration of the second kind that referred to by ( $A^S_{\text{Cos}}$ ).

Similarly, we find the third triangular acceleration rule that we will call Al-Tememe's tangent triangular acceleration of the second kind, which is referred to by ( $A^S_{\text{Tan}}$ ) and Al-Tememe's fourth triangular acceleration rule of the second kind that we will call it Al-Tememe's secant triangular acceleration of the second kind, which is referred to by ( $A^S_{\text{Sec}}$ ). These laws are:

$$A^S_{\text{Tan}} \cong \frac{(h_1^3 \tan h_1) S_2(h_2) - (h_2^3 \tan h_2) S_1(h_1)}{h_1^3 \tan h_1 - h_2^3 \tan h_2} \tag{8}$$

$$\text{Since } (\tanh = h + \frac{1}{3}h^3 + \frac{2}{15}h^5 + \frac{17}{315}h^7 + \frac{62}{2835}h^9 + \dots) \tag{2}$$

$$A^S_{\text{Sec}} \cong \frac{(h_1^4 \sec h_1) S_2(h_2) - (h_2^4 \sec h_2) S_1(h_1)}{h_1^4 \sec h_1 - h_2^4 \sec h_2} \tag{9}$$

$$\text{Since } (\text{sech} = 1 + \frac{1}{2}h^2 + \frac{5}{24}h^4 + \frac{61}{720}h^6 + \frac{277}{8064}h^8 + \dots) \tag{2}$$

Now, we will derive the fifth triangular acceleration : since the error is

$$E(h) = A_1h^4 + A_2h^6 + \dots = h^2(A_1h^2 + A_2h^4 + \dots) \tag{10}$$

$$\cong h^2 (\cosh - 1) \tag{11}$$

$$= -2 \sin^2\left(\frac{h}{2}\right) \tag{12}$$

So, the fifth triangular acceleration rule will be:

$$A^S_{\text{Sin}^2} \cong \frac{\left(h_2^2 \sin^2\left(\frac{h_2}{2}\right)\right) S_2(h_2) - \left(h_1^2 \sin^2\left(\frac{h_1}{2}\right)\right) S_1(h_1)}{h_2^2 \sin^2\left(\frac{h_2}{2}\right) - h_1^2 \sin^2\left(\frac{h_1}{2}\right)} \tag{13}$$

Formula (13) is called the triangular acceleration square sine rule. To derive the sixth triangular acceleration law,

Also the error it can be written as  $E = 2\cos^2\frac{h}{2}$ . Similarly, the sixth triangular acceleration law of the second kind is given by

$$A^S_{\text{Cos}^2} \cong \frac{\left(h_2^2 \cos^2\left(\frac{h_2}{2}\right)\right) S_2(h_2) - \left(h_1^2 \cos^2\left(\frac{h_1}{2}\right)\right) S_1(h_1)}{h_2^2 \cos^2\left(\frac{h_2}{2}\right) - h_1^2 \cos^2\left(\frac{h_1}{2}\right)} \tag{14}$$

The formula (14) is called the triangular acceleration square cosine rule.

### 3.Examples:

We will review some integrals that have continuous integrands on the interval of integration using triangular acceleration methods of Al-Tememe to improve the results numerically:

**3.1:**  $I = \int_1^2 \ln(x) dx$  and its analytical value is 0.3862943611989 which is rounded for 14 decimal.

**3.2:**  $I = \int_0^1 e^x dx$  and its analytical value is 0.71828182845904 which is rounded to 14 decimal.

**3.3:**  $I = \int_1^2 \sqrt{x} dx$  and its analytical value is 1.21895141649746 which is rounded to 14 decimal.

**3.4:**  $I = \int_1^2 \frac{1}{x} dx$  and its analytical value is 0.69314718055995 which is rounded to 14 decimal.

### 4.the Results

The integrand of integration  $I = \int_1^2 \ln(x) dx$  is continuous in the integration interval [1,2], and the formula of correction terms of Simpson's rule as above mentioned (equation 3).

We put  $EPS=10^{-12}$  (represents the absolute error of the subsequent value-previous value). We get the results shown in table (1).

We get correct value through accelerating  $A^S_{\cos}$  and other accelerations to 11 decimal when  $n=30,32,34,36,38,40$  while by using Simpson's method without acceleration was correct to 7 decimal when  $n=40$  while acceleration of  $A^S_{\cos^2}$ , we get the same accuracy when  $n=30,32,34,36,38$ .

the integrand of integration  $I = \int_0^1 e^x(x) dx$  is continuous in the integration interval [0,1], and the formula of correction terms of Simpson's rule as above mentioned (equation 3).

We put  $EPS=10^{-12}$  (represents the absolute error of the subsequent value- previous value). We get the results in table (2). We get correct value through accelerating  $A^S_{\cos}$  to 11 decimal when  $n=22,24,26,28$  and also for the other accelerations. While the value by using Simpson's method without acceleration was correct to 7 decimal except  $A^S_{\sin}$  acceleration, we get the same accuracy when  $n=18,20,22,24$ .

the integrand of integration  $I = \int_1^2 \sqrt{x} dx$  is continuous in the integration interval [1,2], and the formula of correction terms of Simpson's rule as above mentioned (equation 3).

We put  $EPS=10^{-12}$  (represents the absolute error of the subsequent value- previous value). We get the results in table (3). We get correct value through accelerating  $A^S_{\cos}$  to 11 decimal when  $n=22,24,26,28,30$  and also for all other accelerations. While the value by using Simpson's method without acceleration was correct to 8 decimal.

To find the values integrand of integration  $I = \int_1^2 \frac{1}{x} dx$  is continuous in the integration interval [1,2], and the formula of correction terms of Simpson's rule as above mentioned (equation 3).

We put  $EPS=10^{-12}$  (represents the absolute error of the subsequent value- previous value). We get the results in table (4). We get correct value through accelerating  $A^S_{\cos}$  to 11 decimal when  $n=42,44,46,48,50$  and also for all other accelerations. While the value by using Simpson's method without acceleration was correct to 7 decimal except  $A^S_{\cos^2}$  acceleration, we get the same accuracy when  $n=42,44,46,48$ .

**5.Conclusion:**

We conclude from the mentioned tables that these acceleration methods have the same efficiency and give high accuracy of results during limited number of partial intervals.

n	Values of Simpson's rule	$A^S_{\cos(h)}$	$A^S_{\sin(h)}$	$A^S_{\tan(h)}$	$A^S_{\sec(h)}$	$A^S_{\sin^2(\frac{h}{2})}$	$A^S_{\cos^2(\frac{h}{2})}$
2	0.38583460216543						
4	0.38625956281457	0.38628789438754	0.38628789381218	0.38628789294917	0.38628789266151	0.38628837116822	0.38628936824786
6	0.38628716327880	0.38629395728400	0.38629395725415	0.38629395720938	0.38629395719446	0.38629398180820	0.38629403149164
8	0.38629204346631	0.38629430230208	0.38629430229800	0.38629430229188	0.38629430228985	0.38629430564686	0.38629431238295
10	0.38629340380481	0.38629434756405	0.38629434756313	0.38629434756176	0.38629434756131	0.38629434831256	0.38629434981677
12	0.38629389730141	0.38629435696707	0.38629435696680	0.38629435696638	0.38629435696625	0.38629435719284	0.38629435764608
14	0.38629411005202	0.38629435957687	0.38629435957677	0.38629435957661	0.38629435957656	0.38629435965999	0.38629435982676
16	0.38629421367579	0.38629436046092	0.38629436046088	0.38629436046081	0.38629436046079	0.38629436049621	0.38629436056699
18	0.38629426895381	0.38629436080728	0.38629436080726	0.38629436080723	0.38629436080722	0.38629436082396	0.38629436085740
20	0.38629430059436	0.38629436095891	0.38629436095890	0.38629436095888	0.38629436095888	0.38629436096747	0.38629436098464
22	0.38629431975207	0.38629436103135	0.38629436103134	0.38629436103133	0.38629436103133	0.38629436103605	0.38629436104547
24	0.38629433189634	0.38629436106849	0.38629436106849	0.38629436106848	0.38629436106848	0.38629436107121	0.38629436107667
26	0.38629433989431	0.38629436108867	0.38629436108867	0.38629436108867	0.38629436108867	0.38629436109033	0.38629436109364
28	0.38629434533433	0.38629436110019	0.38629436110019	0.38629436110019	0.38629436110019	0.38629436110124	0.38629436110332
30	0.38629434913817	0.38629436110705	0.38629436110705	0.38629436110705	0.38629436110705	0.38629436110773	0.38629436110909
32	0.38629435186233	0.38629436111128	0.38629436111128	0.38629436111128	0.38629436111128	0.38629436111173	0.38629436111264
34	0.38629435385453	0.38629436111397	0.38629436111397	0.38629436111397	0.38629436111397	0.38629436111428	0.38629436111491
36	0.38629435533858	0.38629436111573	0.38629436111573	0.38629436111573	0.38629436111573	0.38629436111595	0.38629436111639
38	0.38629435646235	0.38629436111691	0.38629436111691	0.38629436111691	0.38629436111691	0.38629436111706	0.38629436111738
40	0.38629435732589	0.38629436111771	0.38629436111771	0.38629436111771	0.38629436111771	0.38629436111783	

Table no.(1) to calculate integration  $I = \int_1^2 \ln(x) dx = 0.38629436111989$  by simpson's rule with the trianglnation methods of Al-tememe

n	Values of simpson's rule	$A^S_{\cos(h)}$	$A^S_{\sin(h)}$	$A^S_{\tan(h)}$	$A^S_{\sec(h)}$	$A^S_{\sin^2(\frac{h}{2})}$	$A^S_{\cos^2(\frac{h}{2})}$
2	1.71886115187659						

4	1.71831884192175	1.71828268682341	1.71828268755765	1.71828268865897	1.71828268902607	1.71828207838376	1.71828080596898
6	1.71828916992083	1.71828186599555	1.71828186602764	1.71828186607577	1.71828186609182	1.71828183963070	1.71828178621831
8	1.71828415469990	1.71828183336278	1.71828183336697	1.71828183337325	1.71828183337535	1.71828182992545	1.71828182300297
10	1.71828278192482	1.71828182953748	1.71828182953840	1.71828182953979	1.71828182954025	1.71828182878213	1.71828182726417
12	1.71828228843802	1.71828182878149	1.71828182878177	1.71828182878218	1.71828182878232	1.71828182855573	1.71828182810250
14	1.71828207679867	1.71828182857717	1.71828182857727	1.71828182857742	1.71828182857747	1.71828182849448	1.71828182832858
16	1.71828197405189	1.71828182850904	1.71828182850908	1.71828182850915	1.71828182850917	1.71828182847405	1.71828182840386
18	1.71828191936081	1.71828182848262	1.71828182848264	1.71828182848267	1.71828182848268	1.71828182846612	1.71828182843304
20	1.71828188810386	1.71828182847114	1.71828182847115	1.71828182847116	1.71828182847117	1.71828182846268	1.71828182844572
22	1.71828186919936	1.71828182846567	1.71828182846568	1.71828182846569	1.71828182846569	1.71828182846104	1.71828182845174
24	1.71828185722555	1.71828182846288	1.71828182846289	1.71828182846289	1.71828182846289	1.71828182846020	1.71828182845481
26	1.71828184934489	1.71828182846137	1.71828182846138	1.71828182846138	1.71828182846138		1.71828182845648
28	1.71828184398734	1.71828182846051	1.71828182846051	1.71828182846052	1.71828182846052		1.71828182845743

Table no.(2) to calculate integration  $I = \int_0^1 e^x dx = 1.71828182845904$  by simpson's rule with the triangulation methods of Al-tememe

n	Values of simpson's rule	$A^S_{\cos(h)}$	$A^S_{\sin(h)}$	$A^S_{\tan(h)}$	$A^S_{\sec(h)}$	$A^S_{\sin^2(\frac{h}{2})}$	$A^S_{\cos^2(\frac{h}{2})}$
2	1.21886550798991						
4	1.21894515685709	1.21895046694332	1.21895046683548	1.21895046667373	1.21895046661982	1.21895055630463	1.21895074318373
6	1.21895013467771	1.21895135999548	1.21895135999009	1.21895135998202	1.21895135997933	1.21895136441848	1.21895137337903
8	1.21895100538783	1.21895140840331	1.21895140840259	1.21895140840149	1.21895140840113	1.21895140900008	1.21895141020192
10	1.21895124701366	1.21895141464592	1.21895141464575	1.21895141464551	1.21895141464543	1.21895141477887	1.21895141504605
12	1.21895133447083	1.21895141593250	1.21895141593250	1.21895141593237	1.21895141593235	1.21895141597251	1.21895141605283
14	1.21895137212513	1.21895141628804	1.21895141628802	1.21895141628799	1.21895141628798	1.21895141630275	1.21895141633226
16	1.21895139045026	1.21895141640816	1.21895141640815	1.21895141640814	1.21895141640813	1.21895141641440	1.21895141642691
18	1.21895140022042	1.21895141645514	1.21895141645514	1.21895141645513	1.21895141645513	1.21895141645809	1.21895141646400
20	1.21895140581060	1.21895141647568	1.21895141647568	1.21895141647568	1.21895141647568	1.21895141647719	1.21895141648023
22	1.21895140919440	1.21895141648549	1.21895141648549	1.21895141648549	1.21895141648548	1.21895141648632	1.21895141648798
24	1.21895141133897	1.21895141649051	1.21895141649051	1.21895141649051	1.21895141649051	1.21895141649099	1.21895141649196
26	1.21895141275111	1.21895141649324	1.21895141649324	1.21895141649324	1.21895141649324	1.21895141649353	1.21895141649412
28	1.21895141371149	1.21895141649480	1.21895141649480	1.21895141649480	1.21895141649480	1.21895141649498	1.21895141649535
30	1.21895141438295	1.21895141649573	1.21895141649573	1.21895141649572	1.21895141649573	1.21895141649585	1.21895141649609

Table no.(3) to calculate integration  $I = \int_1^2 \sqrt{x} dx = 1.21895141649746$  by simpson's rule with the triangulation methods of Al-tememe

n	Values of simpson's rule	$A^S_{\cos(h)}$	$A^S_{\sin(h)}$	$A^S_{\tan(h)}$	$A^S_{\sec(h)}$	$A^S_{\sin^2(\frac{h}{2})}$	$A^S_{\cos^2(\frac{h}{2})}$
2	0.69444444444444						
4	0.69325396825397	0.69317460075693	0.69317460236873	0.69317460478635	0.69317460559220	0.69317326511316	0.69317047191437
6	0.69316979316979	0.69314907301252	0.69314907310356	0.69314907324011	0.69314907328562	0.69314899821935	0.69314884669627
8	0.69315453065453	0.69314746627120	0.69314746628395	0.69314746630308	0.69314746630945	0.69314745581059	0.69314743474382
10	0.69315023068893	0.69314724751064	0.69314724751352	0.69314724751785	0.69314724751929	0.69314724514463	0.69314724038989
12	0.69314866220910	0.69314720125415	0.69314720125502	0.69314720125634	0.69314720125677	0.69314720053659	0.69314719909607
14	0.69314798387509	0.69314718829010	0.69314718829042	0.69314718829091	0.69314718829107	0.69314718802508	0.69314718749334
16	0.69314765281942	0.69314718387251	0.69314718387265	0.69314718387285	0.69314718387292	0.69314718375976	0.69314718353362
18	0.69314747598101	0.69314718213501	0.69314718213508	0.69314718213518	0.69314718213521	0.69314718208167	0.69314718197469
20	0.69314737466512	0.69314718137238	0.69314718137241	0.69314718137246	0.69314718137248	0.69314718134496	0.69314718128998
22	0.69314731327821	0.69314718100734	0.69314718100735	0.69314718100738	0.69314718100739	0.69314718099227	0.69314718096208
24	0.69314727434432	0.69314718081991	0.69314718081992	0.69314718081994	0.69314718081994	0.69314718081117	0.69314718079366
26	0.69314724869303	0.69314718071793	0.69314718071794	0.69314718071795	0.69314718071795	0.69314718071263	0.69314718070201
28	0.69314723124018	0.69314718065968	0.69314718065969	0.69314718065969	0.69314718065970	0.69314718065634	0.69314718064964
30	0.69314721903355	0.69314718062500	0.69314718062500	0.69314718062501	0.69314718062501	0.69314718062282	0.69314718061846
32	0.69314721028982	0.69314718060360	0.69314718060360	0.69314718060360	0.69314718060360	0.69314718060214	0.69314718059921
34	0.69314720389440	0.69314718058997	0.69314718058997	0.69314718058997	0.69314718058998	0.69314718058897	0.69314718058696
36	0.69314719912962	0.69314718058106	0.69314718058106	0.69314718058106	0.69314718058106	0.69314718058035	0.69314718057894
38	0.69314719552111	0.69314718057508	0.69314718057508	0.69314718057508	0.69314718057508	0.69314718057458	0.69314718057356
40	0.69314719274796	0.69314718057099	0.69314718057099	0.69314718057099	0.69314718057099	0.69314718057062	0.69314718056988
42	0.69314719058850	0.69314718056813	0.69314718056813	0.69314718056813	0.69314718056813	0.69314718056786	0.69314718056731
44	0.69314718888677	0.69314718056610	0.69314718056610	0.69314718056610	0.69314718056610	0.69314718056589	0.69314718056548
46	0.69314718753113	0.69314718056463	0.69314718056463	0.69314718056463	0.69314718056463	0.69314718056447	0.69314718056416
48	0.69314718644045	0.69314718056356	0.69314718056356	0.69314718056356	0.69314718056356	0.69314718056344	0.69314718056320

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50	0.69314718555495	0.69314718056276	0.69314718056276	0.69314718056276	0.69314718056276	0.69314718056266
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Table no.(1) to calculate integration  $I = \int_1^2 \frac{1}{x} dx = 0.69314718055995$  by simpson's rule with the triangulation methods of AI-tememe

**References**

- [1] Fox L. , “ Romberg Integration for a Class of Singular Integrands “ , comput .J.10,pp.87-93,196
- [2] D. Zwillinger, “Standard Mathematical Tables and Formulae”,31<sup>st</sup> edition, Boca Raton, London, New York Washington,