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HILBERT'S METAMATHEMATICAL PROBLEMS AND THEIR SOLUTIONS

by

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ABSTRACT

This dissertation examines several of the problems that Hilbert discovered in the foundations of mathematics, from a metalogical perspective. The problems manifest themselves in four different aspects of Hilbert's views: (i) Hilbert's axiomatic approach to the foundations of mathematics; (ii) His response to criticisms of set theory; (iii) His response to intuitionist criticisms of classical mathematics; (iv) Hilbert's contribution to the specification of the role of logical inference in mathematical reasoning. This dissertation argues that Hilbert's axiomatic approach was guided primarily by model-theoretical concerns. Accordingly, the ultimate aim of his consistency program was to prove the model-theoretical consistency of mathematical theories. It turns out that for the purpose of carrying out such consistency proofs, a suitable modification of the ordinary first-order logic is needed. To effect this modification, independence-friendly logic is needed as the appropriate conceptual framework. It is then shown how the model-theoretical consistency of arithmetic can be proved by using IF logic as its basic logic.

Hilbert's other problems, manifesting themselves as aspects (ii), (iii), and (iv)—most notably the problem of the status of the axiom of choice, the problem of the role of

the law of excluded middle, and the problem of giving an elementary account of quantification—can likewise be approached by using the resources of IF logic. It is shown that by means of IF logic one can carry out Hilbertian solutions to all these problems. The two major results concerning aspects (ii), (iii) and (iv) are the following: (a) The axiom of choice is a logical principle; (b) The law of excluded middle divides metamathematical methods into elementary and non-elementary ones. It is argued that these results show that IF logic helps to vindicate Hilbert's nominalist philosophy of mathematics. On the basis of an elementary approach to logic, which enriches the expressive resources of ordinary first-order logic, this dissertation shows how the different problems that Hilbert discovered in the foundations of mathematics can be solved.

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1. INTRODUCTION

This work aims at developing a new approach to Hilbert's philosophy of mathematics. According to this new approach, the true basis and functioning of Hilbert's metamathematical program in the foundations of mathematics is derived from model-theoretical conceptualizations. Hilbert's model-theoretical viewpoint roots from his abstract algebraic motivation in connection with the metatheoretical studies of the nineteenth century mathematics. These metatheoretical studies set the main motivation for Hilbert's further inquiry into the foundational organization of mathematical theories, by means of the axiomatic method.

One of the most important aspects of Hilbert's contributions to the foundations of mathematics (as well as to the foundations of different physical theories and logic) is his application of the axiomatic method. Hilbert's axiomatic approach, in line with the nineteenth century mathematical developments, is metatheoretically and model-theoretically oriented.[1] Such an approach excludes what might be called epistemological foundations of mathematics from foundational considerations. According to Hilbert, genuine foundational investigations should concern logical and mathematical methods only.

In chapters 2-10 different aspects of Hilbert's model-theoretical and metatheoretical approach to the foundations of mathematics are discussed. His non-epistemological motivation is indicated. The role of logical axiomatization in metatheoretical considerations is explained, and its importance is emphasized.

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[1] Cf. Hintikka 1988 and 1997.

Hilbert had different aims for the foundations of mathematics. In accordance with his metatheoretical concerns, Hilbert needed a theory of logic to use in his foundational investigations. It is known today that ordinary first-order logic as was (to a considerable extent) developed by Hilbert and Ackermann 1928 cannot serve as a suitable language to achieve Hilbert's aims. For example, the consistency of arithmetic cannot be proved on the basis of this logic. Also this logic does not throw much light on the foundational problems concerning the status of the axiom of choice, which Hilbert believed to be a logical principle, and the status of the law of excluded middle, which Hilbert believed to be an indispensable maintenance for the mathematician.

In chapter 11 the so-called independence-friendly (IF) logic is suggested as an improvement on ordinary first-order logic. Then it is shown in chapter 12 how a consistency proof for the model-theoretical foundations of arithmetic can be carried out on the basis of IF logic. In chapter 13 Hilbert's other problems in the foundations of mathematics are outlined. In chapters 13-20 it is discussed how these problems as well can be carried out, by using IF logic as the basic logic for the metatheoretical and model-theoretical foundations of mathematics. As a result, the axiom of choice becomes a logical principle (cf. chapter 20). And the restricted and unrestricted applications of the law of excluded middle, divides metamathematical methods into two, as elementary and non-elementary methods (cf. chapter 21). Thereby Hilbert's consistency program can be carried out by elementary means, that is to say, without assuming the law of excluded middle. So that Hilbert's approach is saved from being open to intuitionist criticisms.

Finally, in chapter 22 it is emphasized that proof-theoretical extensions of Hilbert's program—using Gentzen-type transfinite recursion—cannot be taken as solutions to Hilbert's different problems, as long as they do not provide an elementary account of our quantification theory. As is argued in this dissertation, IF logic provides an elementary account of quantification theory and provides solutions to Hilbert's different problems in the foundations of mathematics.

2. ABSTRACT MATHEMATICS

The foremost general novelty of the nineteenth century mathematics is its abstractions from the traditional conception of mathematics as the study of space and number.[2] The new level of abstractions signifies not only further development and systematization of the mother subjects of mathematics in algebra, analysis and geometry. It is also the birth and development of new theories in abstract algebra and topology. The new (and then forthcoming) developments in the study of surfaces, symmetries, manifolds, measurement, motion, combinatorics etc. in the nineteenth century signify a progressive departure from the traditional ways of doing mathematics. For example, the discovery of non-Euclidean geometries, group theory and set theory provided new tools and new structures for the working mathematician. They provided plenty of new structures, which seem to be unsustainable by the traditional one-space geometry and number system. Consequently, the task of mathematical theorizing was extended to include what we call meta-theories of different mathematical theories.

In so many ways all the newly discovered mathematical structures were (and are) of course connected to each other. So, surely, a mathematician could (and can) still argue that, on the basis of the interconnections of all the new theories, everything in mathematics ultimately reduces to the study of number and space. Nevertheless, if such an argument overlooked what was really the issue in the study of new mathematical structures, then it would fall short in providing a satisfactory account of its own day actual mathematics. The real issue in question is the investigation of *different* algebraic

[2] Cf. Mac Lane 1986, p. 119.

and geometric structures, their common properties, classifications and differences, which in practice included the metalogical study of the axiom systems for them.

The general significance of non-Euclidean geometries for example lies in the characterization of different spaces, and hence in forming new systems of objects, and the study of the different models of different systems of geometry. This does not in any sense have a claim to weaken the existence of the previous Euclidean models. In similar manner, the theory of groups has brought into mathematics the determination of common models for different types of physical and mathematical structures and operations, such as change of place, transformations, (and primarily) symmetries. In that sense what had been forthcoming then in the nineteenth century abstract algebra and different geometries was a model-theoretical view of mathematics, which requires a metatheoretical structural approach to the multiplicity of different mathematical models.[3]

Hilbert with his mathematical work was one of the high peaks of the new mathematics. His 1890 basis theorem which reduces every ideal in a given field of a ring of polynomials to a finite basis is considered as one of the first contributions to modern algebra. It is considered also as a direct path to provide the foundations of algebraic geometry.[4] The novelty in Hilbert's contribution was its structural metatheoretical approach to certain algebraic structures (e.g. number rings) and their invariant properties. In that sense, Hilbert's viewpoint is metatheoretical and model-theoretical in the same sense as that abstract algebra or category theory is metatheoretical and model-theoretical

[3] Cf. Corry 1996.
[4] Cf. Dreben and Kanamori 1997; also see Kline 1972, § 39.

today.[5] Also, Hilbert's work on the foundations of geometry has got a common feature with his basis theorem and his contribution to the theory of algebraic invariants. Broadly put, this common feature is again the establishment of metatheoretical connections between certain algebraic structures (e.g. ordered number fields) and geometric structures (e.g. the Euclidean space), as well as their substructures.

For Hilbert, the establishment of meta-theoretical interconnections between different parts of mathematics was a leading motivation concerning his general view of mathematics. As he points out in the closing paragraphs of his "Mathematical Problems" (Hilbert 1900a), for him

> Mathematical science … is an indivisible whole, an organism whose vitality is conditioned upon the connection of its parts. …the farther a mathematical theory is developed, the more harmoniously and uniformly does its construction proceed, and unsuspected relations are disclosed between hitherto separate branches of the science.[6]

Therefore, the importance of metatheoretical connections and hence the need for a metatheoretical structural approach to the multiplicity of different mathematical models for Hilbert is foremost and obvious; not only from his actual mathematical work, but also from his overall view of the development of mathematics.

The establishment of metatheoretical connections between algebraic and geometrical structures requires on the abstract level a language to express those connections and structures. Such language is sometimes considered to be the language of set theory. Whether this might be the right approach in mathematics is an open question.

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[5] Cf. Bernays 1967, p. 496.
[6] Hilbert 1900a, p. 436.

And affirmative answers to the question should account for the acceptability of the infinitistic methods that are allowed therein, by the application of the set theoretical language to different mathematical theories, as if it is foundationally the proper way to study mathematical structures and their meta-level interconnections.

Hilbert was a defender of set theory. A typical example of Hilbert's defense of set theory is Hilbert 1926. He considers there set theory as providing a suitable abstract language to study the mathematical infinite. This does not mean however that he considers set theoretical language as foundationally unproblematic.[7] What Hilbert defends is the fruitfulness of abstract set theory as the study of mathematical structures consisting of finite and infinite collections. Hilbert's approval is on the one hand enough to admit set theory as a suitable candidate for metatheoretical and model-theoretical considerations, since most basic mathematical concepts can be represented by finite or infinite sets. Functions can (apparently) be represented by sets of ordered pairs, natural numbers by sets of sets, integers by using pairs of naturals, rationals by a set of pairs, reals by sets of rationals, and so on. What this picture suggests is that one can build a model of most basic mathematical structures by using sets. Hilbert must have thought that by using the language of set theory one can study the (metatheoretical) properties of different mathematical models. All one has to do seems to be to define mathematical structures by using finite and infinite sets and then to make logical inferences from them, as well as to compare their model-theoretical properties. Nevertheless, Hilbert's approval of the abstract language of sets, on the other hand, is very cautious as to the limitations of

[7] See for example Hilbert 1926, pp. 375-376 and 392.

this language and to the foundational problems concerning the existence assumptions about infinite collections. The limitations, from Hilbert's viewpoint, are in need of logical (and combinatorial) undertaking. Therefore, abstract mathematical theories and their metatheoretical interconnections that are intended to be studied model-theoretically via set theory, however wealthy of useful tools for the working mathematician, should be given logical foundations, according to Hilbert.

3. THE AXIOMATIC APPROACH

The aim of abstract mathematics is to get hold of the totality of different models, by using the same type of logical and conceptual tools in different theories. In Hilbert's way of putting this:

> …with all the variety of mathematical knowledge, we are still clearly conscious of the similarity of the logical devices, the *relationship* of the *ideas* in mathematics as a whole and the numerous analogies in its different departments.[8]

That is why one suitable way to study the logical foundations and metatheoretical interconnections between algebraic and geometric structures (as well as to study the structures themselves) is to use the axiomatic method. What is peculiar to the axiomatic method after Hilbert is its service in determining different classes of models for different specific fields of mathematics. In the applications of the axiomatic method a unified view of different models for different mathematical theories is (or at least, is intended to be) captured. By way of locating the basic structural properties of a mathematical system interrelated—so as to explicitly define its intended models—an axiom system provides an overview of the system in question.

In order for such an overview to serve as a useful tool for mathematical purposes it has to have an uninformative inferential net. That is, an uninformative system of logical rules is needed. This enables the axiom system to admit any structure that satisfies the axioms as its subject matter. For that matter, once an axiom system is set up, all that one needs are purely logical consequence relations to derive new results from the axioms. It

[8] Hilbert 1900a, p. 436.

follows that all the new results that are reached in this way have a structural (model-theoretical) meaning. What is needed for a mathematical foundation then is to prove the model-theoretical consistency of the axiom system.

It is crucial here to note that (due to the uninformative character of logical inference) there is a difference between the axiomatization of logic and axiomatization of non-logical (mathematical and scientific) theories. The main difference stems from the role of non-logical constants occurring in the axioms of non-logical theories. With the help of the non-logical constants a non-logical axiom system specifies the class of its intended models. Then the notion of truth (in the axiom system) captures truth in the intended (and only in the intended) models. In the axiomatization of (parts of) logic, however, the aim is not to seek truth in specified intended models. It is rather to capture the notion of true in all possible models. Therefore, in a way axiomatization of logic and axiomatization of non-logical theories designate two different levels of codified information, i.e. information about all possible models and information about some specified models.[9]

Arguably, Hilbert's application of the axiomatic method, in the first place to geometry and then to other branches of mathematics, was the most perceptive application in history, with its sharp penetration of the logical and model-theoretical foundations of theories. Even if it may sound too strong here to say that Hilbert's application of the axiomatic method to the foundations of geometry (as well as to the other branches of mathematics) is the most perceptive one in history, it is quite true that Hilbert is the first

[9] Cf. Hintikka 2007.

mathematician "who moved to the 'metageometric' level" as Weyl once put it.[10] With its metatheoretical perspective Hilbert's axiomatic approach is unique.[11]

Hilbert's 1899 axiomatization of geometry is an application of the axiomatic method with a successful claim to provide the foundations of the entire field. What Hilbert accomplished in this work was to set up a list of axioms for Euclidean geometry and to work out their dependency relations as well as proving their model-theoretical consistency, by using countable arithmetical models.[12] Hilbert's main ideas for the development of the axiomatic method can be found in the 1899 work. Some of these ideas were improved and refined later. What is peculiar to the case of geometry is that Hilbert takes its axiomatization to be of central importance for the different applications and development of the axiomatic method.

Hilbert's axiomatization of geometry goes back to his early lectures in the 1890s.[13] Later, in his 1900 paper, Hilbert calls attention to the distinction between the genetic method in number theory and the axiomatic method in geometry. In the genetic approach to number theory as Hilbert states "the most general concept of real number is engendered by the successive extension of the simple concept of number". The case is different however in axiomatic geometry. There all the objects of the theory (points, lines, planes) are taken as being given. The purpose of the axioms is to define some basic relations between these objects. What one has to do is to prove the adequacy

[10] Cf. Reid 1970, p. 264.
[11] For more on this point, see Weyl 1944.
[12] Well before then with Beltrami and Klein, the consistency of non-Euclidean geometries had already been proved, relative to the consistency of Euclidean geometry.
[13] For these lectures see Hallett and Majer 2004.

(consistency, independence and completeness, in the first place) of the axioms. This investigation Hilbert maintains, namely the axiomatic method,

> ...despite the high pedagogic and heuristic value of the genetic method, for the final presentation and the complete logical grounding of our knowledge…deserves the first rank.[14]

Accordingly, Hilbert suggests using the axiomatic method in different branches of mathematics. In Hilbert and Bernays 1934-39 it is still the foundations of geometry that is taken as the paradigm case for different mathematical applications of the method. Hilbert and Bernays consider axioms of connection, axioms of order and the parallel axiom, with points as the individuals of the system. Accordingly, they formulate propositions of geometry. For example, Gr(x, y, z) means in their system that x, y and z lie on one line. Zw(x, y, z) means that x lies between y and z. Here, Gr and Zw can be treated like any two three-place predicates. And when they are treated as such, let us say as relations R and S respectively, whether they satisfy a given condition A, i.e. whether

(1) $A(R, S)$

or whether for any given R and S,

(2) $\neg A(R, S)$

[14] Hilbert 1900, p. 1093.

is true becomes the central question, which Hilbert and Bernays introduce as a decision problem.[15] Given this decision problem, consistency and independence problems for sets of axioms—this is not to say that they are subordinate to the decision problem—can also be formulated. For example, by asking whether a set of axioms Γ and a given axiom A are satisfiable by a domain, or whether A is independent in the sense that Γ and ¬A are consistent. These tasks can be accomplished as Hilbert and Bernays state, by formulating logical inferences and showing that a given axiom system AX is satisfied by a model M of things and relations. What Hilbert showed, in proving the consistency of his axioms in his 1899 book is essentially this kind of satisfiability.[16]

In addition to the consistency of the axioms also their completeness and independence have to be proved. Indeed Hilbert's main aim as he stated in Hilbert 1899 was to set up a simple and complete axiom system in which axioms are mutually independent from each other. What he meant by completeness was not explicit at the time. Hilbert adopted different views of completeness (i.e. different completeness axioms) in different (later) editions of his 1899 work. He developed his initial ideas further also in adopting completeness to the foundations of other mathematical fields, most notably to mathematical logic.[17] Yet it was clear that Hilbert's central aim in his 1899 book was to capture the intended characterization of the Euclidean space, by means of a finite list of axioms (together with their logical consequences).

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[15] The decision problem is often taken to be the problem of the mechanical derivability of (1) or (2). One has to note here that Hilbert and Bernays by no means restrict their view of the decision problem to mechanical derivability. See Hilbert and Bernays 1934, § 1

[16] Of course, Hilbert's proof was a relative consistency proof, assuming the consistency of the system of real numbers.

[17] See Moore 1997 on the connection between completeness in different foundational issues.

For that purpose Hilbert's axiomatic approach requires, in an axiomatized mathematical theory, the underlying logic to enable the axiom system capture the information codified by the axioms. So that it enables the axiom system to capture the intended class of models. This can be taken as twofold: First, the derivation of the theorems from the axioms must be formal. The derivations must be formal in the sense that no new information other than what was already codified by the system of axioms is used in them. Hilbert states this requirement in his 1917/18 lectures as follows:

> The system of axioms provide us with a procedure to carry out logical proofs strictly formally, i.e. in such a way that we need not be concerned at all with the meaning of the judgments that are represented by formulas, rather we just have to attend to the prescriptions contained in the rules.[18]

In other words, all we have as valid inference depend only on the logical form of the premises and the conclusion. This does not mean that the derivation of theorems has to be completely mechanical. The idea is, rather, that derivations must not introduce any new information into the reasoning. That is, inference rules are applied without changing the information content, but in order to obtain the model-theoretically correct results. So in order to judge the correctness of the results,

> We have to interpret the signs of our calculus when representing symbolically the premises from which we start and when understanding the results obtained by formal operations.[19]

Accordingly, Hilbert formulates the validity of a logical inference as non-refutability by

[18] Quoted in Sieg 1999, p. 18.
[19] Ibid.

an arithmetical model.[20] Based on such conception of logical validity (and of logical consequence) the question of completeness of the system of logic that is used as an auxiliary tool in axiomatic investigation arises as an adequacy requirement. After all,

> The goal of symbolic logic is to develop ordinary logic from the formalized assumptions. Thus it is essential to show that our axiom system suffices for the development of ordinary logic.[21]

For that purpose, second, the logic used in the derivations of theorems must be complete. Otherwise the axiom system does not capture the intended class of models.

It is relevant to note here a distinction between different notions of completeness. There are at least three relevant conceptions of completeness: descriptive, deductive and semantical completeness. [22] We say a non-logical axiom system is *descriptively complete* if and only if the models of the system include all and only intended models. We say a non-logical axiom system is *deductively complete* if and only if, for every sentence S of the system it can be logically proved either S or ¬S as a theorem of the system.[23] We say an axiomatization of logic is *semantically complete* if and only if it admits a recursive enumeration of the valid formulas.

Without giving a clear distinction between different notions of completeness, Hilbert recognized the first requirement about the need for the uninformative character of logical inference. He held that all logical inferences in an axiom system must be carried

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[20] Sieg 1999 quotes (p. 20) Hilbert from his talk to the International Congress of Mathematicians in Bologna (Hilbert 1929).
[21] Quoted in Sieg 1999, p. 20,.
[22] Cf. Hintikka 1996 chapter 5.
[23] As a result of Gödel 1930 this can be in two ways: in the sense based on proof-theoretical derivability and in the sense of model-theoretical consequence.

out in a formal way. And in this respect, his axiomatic approach is based on the idea that mathematical reasoning can be interpreted as being logical reasoning in a sense that implies uninformativeness. Once the axioms are set up in a logical manner, then all that one has to do is to apply the uninformative logical inferences and build the rest of the results purely formally (i.e. based on the logical form only, which includes the semantic notion of logical consequence). Therefore, the formality requirement is by no means to say that mathematics is a formal game with arbitrary rules. Hilbert's foundational aims were never arbitrary. They were guided by suitable criteria that underlie what might be called a natural philosophy.[24] Still the word "formal" might still appear tricky here. It has to be freed from a solely mechanistic understanding. One has to remind oneself that Hilbert's primary aim was to capture the intended models of axiom systems, without necessarily having commitment to any claim to show that his aim coincides with what can be achieved by axiomatization in a purely mechanical way. Even if what is captured as the intended models somehow coincided with the outcomes of mechanical procedures, this would by no means show that Hilbert was seeking for purely mechanical procedures in his axiomatism. The completeness issues were in an inadequate stage of development in logical theory, when Hilbert started to shape his foundational program. Therefore, it would be wrong to attribute any strict mechanist (or formalist) view to Hilbert about his aim to capture the intended models of axiom systems, until the impact of Gödel's incompleteness results on Hilbert's approach is rightly understood. (See further chapter 10)

[24] Cf. Corry 1997.

Nevertheless, Hilbert might have presupposed (at some point) the completeness of the underlying logic of mathematical systems:

> If it can be proved that the attributes assigned to the concept can never lead to a contradiction by the application of a finite number of logical inferences, I [Hilbert] say that the mathematical existence of the concept (for example, of a number of a function which satisfies certain conditions) is thereby proved.[25]

Here it might have been the case that Hilbert has taken for granted that deduction entails semantic consequence. In this he might have presupposed the semantical completeness of the underlying logic of axiomatization; since only if the underlying logic is semantically complete, then the deductive consistency of a system implies its model-theoretical consistency. Indeed, it might even count as historical evidence for Hilbert's inclination to such a presupposition, that the completeness of propositional logic was first proved by Bernays and Hilbert in 1918.[26] Then in 1930 Gödel proved the completeness of the usual first-order logic that had been formulated (to a considerable extent) in Hilbert and Ackerman 1928. Gödel's motivation for his completeness proof for the standard first-order logic thus seems to have been done partly due to Hilbert's influence.[27]

What Gödel proved is the semantical completeness of first-order (Hilbert-Ackermann) logic. And what he later proved in 1931 is the deductive incompleteness of arithmetic, which implies the descriptive incompleteness of arithmetic only if the

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[25] Hilbert 1900, par. 42.
[26] See chapter 2 of Zach 2001 for a historical discussion of completeness proofs in Hilbert's school.
[27] Cf. Dreben and Kanamori 1997.

underlying logic is semantically complete.[28] Therefore, if Hilbert really presupposed the semantical completeness of the underlying logic, then such presupposition would make his aim to specify the intended models of axiom systems unattainable. That is why, in the light of Gödel's completeness and incompleteness results, the first-order logic that was formulated by Hilbert and Ackermann has to be replaced by a suitable semantically incomplete alternative. Otherwise Hilbert's goals and requirements for axiomatic foundations are beyond the reach. And yet, if the underlying logic of axiomatization is semantically incomplete, then there may be descriptively complete characterizations of arithmetic by using that incomplete logic.[29]

[28] Gödel proved the deductive incompleteness of elementary arithmetic in two ways: (i) in the sense based on logical consequence, (ii) in the sense based on mechanical derivability. Now, (ii) generalizes to other systems of arithmetic whereas (i) does not. Cf. Hintikka 1996, chapter 5.
[29] Cf. Hintikka 1996, chapter 5.

4. METAMATHEMATICS

The two requirements mentioned in the previous chapter (i.e. about uninformativeness and the completeness), if fulfilled, give all the mathematical results that are obtained from the axioms a structural meaning. According to this structural meaning, in the derivation of theorems from the axioms the interpretation of the axioms does not matter. Any given domain that satisfies the axioms, can instantiate the structure. The plausibility of such structural approach was emphasized by Hilbert himself. As he once put it, in deriving theorems from geometrical axioms, one might as well speak of tables, chairs and beer mugs instead of points, lines and circles.[30] This has to be taken as a straightforward reminder that logical reasoning is valid modulo isomorphism. Hilbert in one of his letters to Frege summarizes the main idea of his approach as follows:

> …it is self-evident that every theory is merely a framework or schema of concepts together with their necessary relations to one another, and that the basic elements can be construed as one pleases. If I think of my points as some system or other of things, e.g. the system of love, of law, or of chimney sweeps…and then conceive of all my axioms as relations between these things, then my theorems, e.g. the Pythagorean one, will hold of these things as well. In other words, each and every theory can always be applied to infinitely many systems of basic elements. For one merely has to apply a univocal and reversible one-to-one transformation and stipulate that the axioms for the transformed things be correspondingly similar. Indeed, this is frequently applied, for example, in the principle of duality, etc; I also apply it in my independence-proofs.[31]

Since any domain that satisfies the axioms can be taken as to instantiate the considered mathematical structures, symbols and their combinations themselves can as well be taken as the objects of the models. So, in number theory Hilbert's view amounts to suggesting

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[30] Cf. Reid 1970, p. 57.
[31] Kluge 1971, pp. 13-14.

the following model-theoretical strategy: Since the particular selection of the domain of a model of number theory is arbitrary, we might think of this domain as consisting of symbols (or signs). We might think of them for instance as consisting of the number-theoretical symbols. The same statement holds for the individual objects that there are in the models of our theories we might think of. According to Hilbert, the objects one thinks of can be chosen (as one pleases) freely as the individuals that one's theory deals with. Therefore, in the case of number theory, what Hilbert is proposing is the study of the intended models of axiomatized number theory. Their domain can be taken as to consist of signs (or, symbols).

In fact a closer look at abstract algebra discloses the fact that there—for example in Galois' groups—is seen already the key ideas of Hilbert's structural approach. Algebraic manipulations can be thought of as having the same properties as other types of symmetries and transformations. This would essentially amount to the same technique as Hilbert's use of symbols as their own representations:

> In algebra…we consider the expressions formed with letters to be independent objects in themselves, and the contentual propositions of number theory are formalized by means of them. Where we had propositions concerning numerals, we now have formulas, which themselves are concrete objects….[32]

A forceful statement of a similar point is found in the introductory paragraphs of Hilbert's address to the International Congress of Mathematicians in Paris (Hilbert 1900a). There Hilbert says: "The arithmetical symbols are written diagrams and the geometrical figures are graphic formulas."

[32] Hilbert 1926, p. 379.

Practically, Hilbert's recommendation of using symbols for metatheoretical purposes amounted to proposing to use a kind of metalogical tool in proof theory. However, as a tactic for handling algebraic manipulations, it is not merely proof-theoretical. The underlying rationale of Hilbert's method is still model-theoretical. It is similar to the one used, for example, by Henkin in his completeness proof for first-order logic.[33] Henkin used as models for certain kinds of sets of formulas (symbol combinations) those very same sets of symbols themselves.[34] On similar lines, Hilbert's statements about signs as objects of mathematics are a recommendation of the application of logical inferences:

> ...as a precondition for the application of logical inferences and for the activation of logical operations, something must already be given in representation [in der Vorstellung]: certain extralogical discrete objects, which exist intuitively [anschaulich] as immediate experience before all thought.[35]

The condition in question is the domain of individuals, i.e. extra-logical discrete objects. What they are is not really important. Hilbert says that they could be chairs, tables and beer mugs. By the same token, they could be symbols and structures of symbols. Thereby, the reasons Hilbert gives for his alleged formalism, do not serve more than as practical reasons.

At this point Hilbert's practical reasons can be given further theoretical background. Wittgenstein's picture theory of meaning serves as an illustration of Hilbert's model-theoretical presuppositions. The picture in the sense of Wittgenstein serves as a

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[33] See Henkin 1949.
[34] Later Hintikka 1955 and Smullyan 1968 generalized Henkin's method. Henkin's idea was very close to the idea of Gödel numbering and indeed Hilbert's formalization of metamathematics.
[35] Hilbert 1922, par. 25.

model of reality.[36] There we admit a "form of representation" of the picture which fixes the things in reality in such a way to produce the same structure as that the elements of the picture provide. In that sense a picture is "like a scale applied to reality".[37] Symbols (or signs) which are meaningless themselves in Hilbert's terminology can likewise be thought of as elements of pictures that are made possible by their form of representation. The form of representation is what Hilbert seems to have assumed when he occasionally appeals to a harmony between thought and reality.[38] The gist of the harmony of thought and reality in question is the assumption that human thought can produce symbol structures as models of reality. These structures can at the same time be considered as isomorphic replications of what they represent. The same view can be traced at least back to Leibniz. In his dialogue on the connection between things and words, Leibniz says:

> …if characters can be used for ratiocination, there is in them a kind of complex natural relation [situs] or order which fits the things; if not in single words at least in their combination and inflection, although it is better if found in the single words themselves.[39]

One has to note here that it is a semantical phenomenon rather than an epistemological or an ontological one that it is possible to produce structures that are isomorphic with what they represent. It is an assumption concerning how language is seen to be related with reality. Although such assumption is philosophically in need of further investigation, the investigation is unnecessary for Hilbert's practical purposes to perform logical operations on given mathematical structures. On this point, the philosophical import of Hilbert's

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[36] Wittgenstein 1921, 2.12.
[37] Ibid, 2.1512.
[38] See Hilbert 1930, par. 13.
[39] Loemker 1970, p. 184.

views seems to be a part of his optimism concerning problem-solving.

Hilbert's optimism (under the assumption that the correct symbolism gives isomorphic replications of what it represents) consists of an attempt to reconstruct all mathematics in the shape of "inventory of formulas".[40] The inventory in question has two parts:

> …first, formulas to which contentual communication of finitary propositions [hence, in the main, numerical equations and inequalities] correspond and, second, further formulas that mean nothing in themselves and are the *ideal objects of our theory*.[41]

What is needed in order to give a logical foundation for the two parts of the inventory of formulas (i.e. the contentual (real) part and the ideal part) is a proof of consistency, which will show in turn that the application of ideal elements in mathematical symbolism is model-theoretically unproblematic.[42] So that the assumption that mathematical symbolism provides a characterization (up to isomorphism) of its subject matter—with arbitrary instantiations of mathematical structures—has safely been applied.

[40] Cf. Hilbert 1922.
[41] Hilbert 1926, p. 380.
[42] Cf. Hilbert 1926, p. 383.

5. IDEAL ELEMENTS

The arbitrary instantiations of mathematical models (structures) by using number theoretic symbols as instantiations (being a practical move) involves no epistemological or ontological aims. Hilbert's axiomatic approach is intended to have been freed from epistemological concerns. For sure the investigation of symbol structures is for the sake of providing a combinatorial justification of what Hilbert calls the use of ideal elements in mathematics. And it might be objected that the justification of ideal elements involves an epistemological (or an ontological) reduction of the ideal to the real. Such objection is based on a misconception about the use of ideal elements in mathematics.

Ideal elements are introduced into a mathematical system in order to extend the models of the initial system consisting of real elements. The purpose of extending the models and introducing ideal elements is to obtain (in a simple and fruitful way) new results which are not available when the system is restricted to the study of models with only the so-called real elements. All that is required for that purpose is the following: the addition of new (ideal) elements must be carried out without violating the consistency of the initial system. For the consistency, what is required is a model-theoretically consistent characterization of both the initial mathematical system of real elements and the extended system with the ideal elements.[43]

On this explanation of the role of ideal elements in mathematics, giving the question of ideal elements an ontological (or an epistemological) significance would be a misleading way to interpret Hilbert's views. Questions of what the mode of existence or

[43] Cf. Hilbert 1926, pp. 372-373.

the type of epistemic accessibility of those ideal elements is or may be would be off the mark. There are however epistemological interpretations of the distinction. Detlefsen 1986 for example argues that Hilbert's real-ideal distinction can be taken as to distinguish between the kind of epistemic value that the ideal and real statements in mathematics depend on (or derived from).[44] Perhaps this can be done. However, it has nothing to do with Hilbert's model-theoretical aims. In the way Hilbert introduces the notion of an ideal element there is no dependency on epistemic conditions.

It is thought by questioning epistemic and ontological aspects of the issue that ideal objects are somehow added in some fixed given domain of objects, even though they do not really exist (or, are not really known). From a model-theoretical point of view however, there is neither an ontological nor an epistemological shift from the use of real elements to the use of ideal elements. When we are studying a model or class of models modulo isomorphism, it is sometimes helpful to compare those possible structures to other structures. These are obtained by adding to the original structures some new elements. To repeat, the reason for adding new elements is simply that it might be easier then to prove certain theorems. This is what happens in the typical uses of ideal elements. Hilbert gives the familiar example from projective geometry. In projective geometry one can postulate a new geometrical entity, say an infinitely distant point, added to each model of the axioms of projective geometry. Then one can for instance show that in an extended model any two lines intersect in a point when produced indefinitely.[45] Similar

[44] See Detlefsen 1986, chapter 1. The same kind of interpretation can be found (as Detlefsen also points out in a footnote) in von Neumann 1931 and Kitcher 1976.
[45] Cf. Hilbert 1926, p. 372.

examples can be given for arithmetic. Hilbert for example wrote in his 1926 paper:

> Just as in geometry infinitely many straight lines, namely those that are parallel to one another, are used to define an ideal point, so in higher arithmetic certain systems of infinitely many numbers are combined into a *number ideal*….[46]

The real problem here is to find out what the properties of the extended models tell us about the models of the original axioms. This problem is practical. If philosophical, it should be discussed separately from Hilbert's foundational line of thought. It depends on the particular way in which the ideal elements are introduced. For Hilbert's purposes there are no philosophical problems about this procedure. The procedure is purely structural. And Hilbert's distinction between ideal and real elements has to be separated from possible external justifications to the use of this structural approach:

> The terminology of ideal elements thus properly speaking only has its justification from the point of view of the system one starts out from.[47]

The problems concerning ideal elements are model-theoretical, not epistemological. Questions (with philosophical emphasis) about whether ideal elements "really" exist, whether they have meaning, whether we can know them, or whether we can trust results obtained by their means are therefore misleading.

Misleading questions lead to misleading reading of Hilbert's foundational works. One misleading way of reading Hilbert's views has persisted in association with the formalist approach to mathematical theories. For example, Hilbert has been routinely

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[46] Hilbert 1926, p. 373
[47] Quoted by Hallett 1995 from Hilbert's 1919 lectures; Hallett 1995, p. 149.

called a formalist. Historically, this goes partly back to Brouwer's criticisms and partly to Weyl's. In his 1912 paper Brouwer made the distinction between intuitionism and formalism.[48] In his Cambridge Lectures he characterized Hilbert as the founder of the new formalist school in the foundations of mathematics.[49] The way Brouwer makes his characterization is perhaps not completely unfair to Hilbert's description of his own metamathematics. Brouwer seems to be aware of the fact that Hilbert does not reject elements extraneous to language. However the attribution he made easily leads to misleading readings of Hilbert. It is misleading in that Hilbert never claims that mathematical activity is restricted to the manipulation of formal symbols.[50] This might be attributed perhaps to what Brouwer calls the old formalist school that rejected all elements extraneous to language.[51]

Formalism is not the only philosophical view that has been wrongly attributed to Hilbert. There are other philosophical interpretations of Hilbert's foundational ideas (by way of finitist and instrumentalist epistemologies in particular) which lead to a seriously misleading reading of Hilbert.[52] For one main reason such interpretations turn one's set of foundational problems into an epistemological problem concerning the certainty of mathematics. This line of interpretation therefore, is rooted in the same misunderstandings as those which misled logicians to have taken Gödel's second incompleteness result as to imply the failure of Hilbert's metamathematical consistency

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[48] See Brouwer 1912 titled "Intuitionism and formalism".
[49] van Dalen 1981, pp. 3-4
[50] Cf. Kreisel 1983, pp. 207-208
[51] Brouwer acknowledges that Hilbertian metamathematics is an improvement on the old formalist view. See van Dalen 1981, p. 2.
[52] Here one should distinguish the mathematical contributions of finitist and instrumentalist approaches from the philosophically misleading part.

program; since one of the main reasons why Gödel's second incompleteness theorem has been taken to imply the failure of Hilbert's consistency program is that possible modifications of the so-called finitist standpoint has been considered to be a deviation from Hilbert's alleged epistemological drive in proposing his views.[53]

All the mentioned epistemological interpretations of Hilbert's views shift the emphasis from the model-theoretical foundations of mathematics to an irrelevant conceptual framework. On the instrumentalist interpretation, for example, Hilbert's views are taken to admit an anti-realistic conception of mathematical truth for some (i.e. ideal) part of mathematics. On this interpretation, mathematical statements involving ideal elements are admitted to have only instrumental value as opposed to statements concerning the real elements, which are thought to have "real" truth value. In similar manner, on the same type of interpretation the knowledge of real statements is counted to be the rock bottom foundation of mathematical knowledge, whereas statements about ideal elements are taken to have justification only by means of the epistemological ground framed by the real part. All this appear to have been connected with the so-called finitism of Hilbert. Real statements are considered to have finitary content whereas the ideal parts of mathematical models, for example the infinite parts, are thought to be lacking such content. The way we make the distinction between real and ideal as above is not a useful way for Hilbert's axiomatic purposes. From the axiomatist point of view what is misleadingly called Hilbert's instrumentalism and Hilbert's finitism consist only of practical metalogical strategies. The best way to characterize these strategies is by way

[53] The point here is not that finitist and instrumentalist results in the logical foundations of mathematics are useless or fruitless. Their epistemological interpretation that they attribute to Hilbert is misguided.

of studying the semantic features of humanly practicable operations of logic and mathematics. Therefore, there is no good reason to assume that the practicality in question will be incompatible with a realistic conception of mathematical truth.[54] Hence there is no good reason to put an anti-realistic emphasis on the interpretation of Hilbert's views either. The question of Hilbert's practical metalogical strategies states only a matter of deciding which applications of the logical operations to the so-called ideal elements are elementary:

> …since the ideal propositions…do not express finitary assertions…the logical operations cannot be applied to them in a contentual way, as they are to the finitary propositions.[55]

The logical operations in question here are what Hilbert refers as the laws and operations of the Aristotelian logic, most notably the law of excluded middle and negation.[56] The epistemological force that is tried to be attached to the metalogical standpoint such operations require is a distortion on Hilbert's viewpoint; since Hilbert had the idea of using the combinatorial model-theoretical properties of a formal language, as the basis of all the different applications of logical and mathematical operations. Such an idea supports a realistic conception of mathematical truth rather than an anti-realistic conception.

Missing Hilbert's implicit search for models might lead to further misleading characterizations of his axiomatic approach. For example, it might seem that for him the axioms are neither true nor false. Such an explanation would not capture Hilbert's idea.

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[54] Cf. Hintikka 1996 chapter 9, especially pp. 199-202.
[55] Hilbert 1926, p. 381.
[56] Cf. Hilbert 1926, p. 379.

For Hilbert one of the most important applications of the axiomatic method was its application to theories of physics.[57] In such applications, it is hard to say that the axioms are neither true nor false. One has to take into consideration whether the axioms are true or false. The crucial point here is the kind of purpose that axioms and the axiomatic approach were calculated to serve. Sometimes axioms are taken to be intuitive truths and logical deductions from them to serve as a means of establishing the truth of their consequences. As should be clear by now, such inference from intuitive truths is not the whole story in Hilbert's conception of inference from the axioms. For Hilbert an axiom system also provides an overview of our knowledge of certain types of structures. What is contained in such overview can be explained (in Hilbert's terms) by means of a "mapping from a domain of knowledge onto a framework of concepts" in an axiomatic theory;

> Through [such] mapping, the investigation becomes completely detached from concrete reality. The theory has nothing more to do with real objects or with the intuitive content of knowledge. It is a pure thought construction of which one can no longer say that it is true or false [in the actual world].[58]

Nevertheless, the detachment from concrete reality mentioned in the quotation obviously does not mean that the theory loses all contact with reality. It has a meaning concerning what we know about reality. That is, in Hilbert's words, "it presents a possible form of actual connections".[59] Therefore, true statements in such a framework are truths in a model. The purpose of the axiomatic method is to find out those truths, by means of

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[57] For a historical study of Hilbert's work on the axiomatization of physical theories, see Corry 1997.
[58] Quoted from Hilbert's 1921-22 lectures, in Hallett 1994, p. 168
[59] Ibid.

logical inferences from the axioms.

Physical systems as being idealizations of actual physical phenomena can also be considered as examples here:

> [In physics] we have to do ...predominantly with theories which do not reproduce the actual state of affairs completely, but represent a *simplifying idealization* of the state of affairs and have their meaning therein.[60]

Such idealizations are the models of the axiom systems. And the derivation of theorems from axioms amounts to a study of those models:

> What the physicist demands precisely of a theory is that particular propositions be derived from laws of nature or hypotheses solely by inferences, hence on the basis of a pure formula game, without extraneous considerations being adduced. Only certain combinations and consequences of the physical laws can be checked by experiment—just as in my proof theory only the real propositions are directly capable of verification.[61]

Here, it is obvious from the analogy between physics and proof theory that what Hilbert means by "pure formula game" includes model-theoretical thinking, which depends solely on the logical formation of—and hence not extreneus to—the axiomatic framework. In this regard it is not expected from Hilbert-style axiomatization to serve only a deductive (mechanical) purpose. However, this does not mean that it is expected from Hilbert-style axiomatizations to serve an epistemological purpose. Epistemological purposes, especially in the sense of leading to new truths or to new evidence for old ones—as well as other external purposes for justification of the choice of the axioms—must be sharply separated from the inherent working of the axiomatic method. Likewise,

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[60] Cf. Hilbert and Bernays 1934, pp. 2-3. (Translation by Kleene 1952)
[61] Hilbert 1928, p. 475.

epistemological purposes in the sense of describing or explaining the nature of any relation between (axiomatic) thought and reality must be sharply separated from Hilbert's foundational line of thought. As he points out in his 1922-23 lectures, the service of the axiomatic foundations is

> [to] have stressed a separation into the things of thought of the [axiomatic] framework and the real things of the actual world, and then to have carried this through.[62]

In addition, that Hilbert's aim is not to seek new truths by way of axiomatic method is obvious from the following:

> I [Hilbert] understand from the axiomatical exploration of a mathematical truth [or theorem] an investigation which does not aim at finding new or more general theorems being connected with this truth, but to determine the position of this theorem within the system of known truths in such a way that it can be clearly said which conditions are necessary and sufficient for giving a foundation of this truth.[63]

One might be tempted here to seek the epistemological ground of the necessary and sufficient conditions in question. However, that is exactly what Hilbert seems to have cut off, by not aiming at a search for new truths by means of axiomatization. In that sense Hilbert's method might be taken as epistemological perhaps only from a wider perspective in which it serves the purpose of exploring a class of models, and as such it prepares maps for an overview of those models.

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[62] Quoted in Hallett 1994, p. 167.
[63] Quoted in Peckhaus 2003, from Hilbert's 1902/03 lectures.

6. NO EPISTEMOLOGY

Partly because Hilbert had to take a stand against Brouwer's intuitionism in the nineteen-twenties, the whole issue of Hilbert's foundational views is often taken to have been a response on epistemological grounds. Brouwer's foundational worries were epistemological. Mathematics proper, for him, presupposed an indispensable epistemic element.[64] It was taken as to rest on and generated from a fundamental mathematical act of the mind. According to Brouwer, the derivation of mathematical truths by repeated mental acts take place as a generation of new knowledge from a previous source. The true foundation of mathematics we should seek, therefore, where the original source of the generation of repeated mental acts was activated. In that sense the true intuitionistic foundation is what Brouwer calls the first act of the mind towards mathematical knowledge. In Brouwer 1948 we read: "consciousness in its deepest home seems to oscillate slowly, will-lessly, and reversibly between stillness and sensation".[65] The creative subject departs from this will-less stage by a move of time. From that stage it passes to the combination of past and present moments of the ur-intuition of two-ity.[66] Iteration of this ur-intuition gives the creative subject, according to Brouwer, sequences. Brouwer calls them causal sequences. Mathematical activity with such sequences is called mathematical attention.

As if Hilbert's foundational terminology had to have conceptual commitment to Brouwer-like epistemological worries, many works in the philosophy of mathematics

[64] Cf. Hintikka 2001a.
[65] Cf. van Dalen 2000, p. 120
[66] Cf. Brouwer 1948, p. 1235

literature focus on the epistemological force of Hilbert's foundational views[67]. It is nevertheless misleading to do so in that Hilbert's model-theoretical concerns were prior to any search for an epistemological foundation. What is commonly misleading in the epistemological interpretations is the meaning assigned to questions like "What are signs?", "What is the epistemological status of finitary objects?", "What kind of intuition is the finitary intuition?" These questions, when they are asked as questions of epistemology, have no significant value for a better understanding of Hilbert's philosophy of mathematics; even though it is true that Hilbert himself sometimes speaks of the *a priori* intuition, and characterizes it as the "frame of the finitary mode of thought".[68] When Hilbert discusses the *a priori*, he does not do it for the sake of explaining his epistemological standpoint. Rather, he wants to emphasize the foundational import of certain mathematical or logical propositions. For example, when he says, in his 1931 paper:

> …there are…those propositions that are generally held to be *a priori*, but which cannot be achieved within the frame of the finite mode of thought—for example, the principle of *tertim non datur*, as well as the so-called transfinite statements generally.[69]

Here Hilbert is not (primarily) taking his epistemological stance and indicating where it differs from Kant's. Hilbert's main point is rather that the applications of the law of excluded middle, for instance, are in some cases not elementary (i.e. metamathematically problematic) applications. Such point concerns only the foundational (and metalogical)

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[67] Kitcher 1976, Giaquinto 1983, Detlefsen 1986, Parsons 1998, Tait 1981, and Zach 2001 are some examples.
[68] Cf. Hilbert 1931
[69] Hilbert 1931, p. 1150.

import of a certain logical principle.

Hilbert tried to clear his way from epistemological and metaphysical assumptions about the nature of mathematics. A sharp statement of Hilbert's nonepistemological view can be found, for example, in his 1917 lectures on the principles of mathematics. He says there that his axiomatic approach is not to overcome philosophical difficulties, but to "cut them off"[70] Therefore, the questions mentioned (in the beginning of the chapter) above should not be asked as epistemological questions, but rather be asked in association with a metalogical sense of the terms occurring in them. They must be treated like questions of metalogic and metamathematics, i.e. as a part of what Hilbert calls the simultaneous development of logical and mathematical methods.[71] Most importantly perhaps, Hilbert's so-called finitism (and hence his view of the *a priori*) should be taken as an object of metalogical investigation, i.e. by seeking an elementary account of the theory of logic.

Hilbert was interested in what was there in axiomatic mathematics as determination of models for the theories (as much as in their proof-theoretical structure). This is an immediate consequence of his conception of an axiomatic foundation for mathematics. Epistemological problems concerning the cognitive content of symbol structures are a completely different issue. This is not to say that there are no philosophical problems (epistemological or otherwise) concerning the existence and knowledge of the models. Nor it is to say that Hilbert ignores such problems. The point is that epistemological problems are of a different sort. They are of secondary importance for Hilbert's metatheoretical purposes. It was Brouwer, not Hilbert, who injected

[70] Cf. Sieg 1999, p. 11
[71] Cf. Hilbert 1905.

epistemology into the discussion of the foundations of mathematics.

The following statement of Hilbert sharply distinguishes his own approach from others' who favored epistemological (or metaphysical) primitives in their foundational views:

> …I should like to assert what the final outcome will be: mathematics is a presuppositionless science. To found it I do not need God, as does Kronecker, or the assumption of a special faculty of our understanding attuned to the principle of mathematical induction, as does Poincaré, or the primal intuition of Brouwer, or finally, as do Russell and Whitehead, axioms of infinity, reducibility, or completeness, which in fact are actual, contentual assumptions that cannot be compensated for by consistency proofs.[72]

Here the passage implies that Hilbert considered certain restricted forms of mathematical induction as non-informative. The roots of Hilbert's view of the problems concerning mathematical induction go back to Hilbert 1905. In his 1905 paper Hilbert considers object combinations (or, as he calls them, combinations of thought-objects) on the concrete level as a basis for showing how to set up an axiom system for number theory. He takes the objects of his system to be denoted by signs (1's) and their combinations. An inference in the system is of the form $A_1 \,\&\, A_2 \ldots \,\&\, A_n \mid B_1 \vee B_2 \ldots \vee B_n$. Existentially and universally quantified sentences are of the form $A(x^{(\vee)})$ and $A(x^{(\&)})$ respectively. The axioms[73] are:

1. $x = x$
2. $x = y \,\&\, A(x) \mid A(y)$

[72] Hilbert 1928, p. 479.
[73] Cf. Hilbert 1905, p. 133.

3. $f(ux) = u(f'x)$
4. $f(ux) = f(uy) \mid ux = uy$
5. $f(ux) \neq u1$

Here the intended meaning of Axiom 3 is that each element ux is followed by a definite object f(ux) which is equal to an element of the set u, namely the element $u(f'x)$. Axiom 4 means, if the same element follows two elements of the set u, these two elements themselves are equal. Axiom 5 means, there is no element in u that is followed by the element u1. Hilbert gives these axioms in order to point out that the formula $f(x^{(\vee)}) = u1$, which contradicts Axiom 5, cannot follow from Axioms 1-4. For that purpose Hilbert defines homogeneous equations of the form $\alpha = \alpha$. He indicates that from the Axioms 1-4 only homogeneous equations can follow as consequences, whereas the equation $f(x^{(\vee)}) = u1$ is not a homogeneous equation. On the basis of this observation Hilbert argues that there is no way that the given axioms can lead to contradictions, i.e. equations which are not homogeneous. That is, no counter-model construction for the logical inferences from the axioms can succeed.

In order to prove such consistency for the general case Hilbert needed to use the principle of mathematical induction, which had to be considered also as one of the axioms of number theory of which consistency was in question. Poincaré in his 1906 paper criticized Hilbert's argument on this point. He pointed out that Hilbert's appeal to mathematical induction in his proof was circular reasoning:

…the point at issue is reasoning by recurrence and the question of knowing whether a system of postulates is not contradictory

…

A demonstration is necessary.

The only demonstration possible is the proof by recurrence.

This is legitimate only if we regard it not as a definition but as a synthetic judgment.[74]

Poincaré also took Hilbert's line of thought to have assumed the principle of mathematical induction as a synthetic a priori principle:

…Hilbert's reasoning not only assumes the principle of induction, but it supposes that this principle is given us not as a simple definition, but as a synthetic judgment *a priori*.[75]

However, Hilbert's argument did not involve any epistemological concerns. In fact it was Hilbert's aim to eliminate epistemological presuppositions from the foundations of mathematics. Hilbert's later response to Poincaré makes this point sufficiently clear. The reason why Hilbert's argument appeared to have involved epistemic elements is presumably the then missing (then forthcoming) developments in logical theory. Hilbert's own later remark on Poincaré's challenge is that it was a mistake on Poincaré's part that he rejected Hilbert's theory in its "inadequate early stages".[76] The source of Poincaré's mistaken view, according to Hilbert, was that Poincaré did not distinguish between two different methods of induction:

Poincaré…denied from the outset the possibility of a consistency proof for the arithmetic axioms, maintaining that the consistency of the method of mathematical induction could never be proved except through the inductive method itself. But as my proof theory

[74] Poincaré 1906, p. 1058-1059.
[75] Ibid. p. 1059.
[76] Hilbert 1928, p. 473.

shows, two distinct methods that proceed recursively come into play when the foundations of arithmetic are established, namely, on the one hand, the intuitive construction of the integers as numeral (to which there also corresponds, in reverse the decomposition of any numeral, or the decomposition of any concretely given array constructed just as an array is), that is, *contentual* induction, and on the other hand, formal induction proper, which is based on the induction axiom and through which alone the mathematical variable can begin to play its role in the formal system.[77]

It can still be questioned whether or to what extent Poincaré was right in his general criticism based on the synthetic *a priori* character of mathematical induction. However, the epistemological force of Poincaré's criticism makes it uninteresting to run the discussion for Hilbertian purposes as they have been presented here.

On similar lines there is no need for an appeal to any basic intuition in our foundational theorizing, according to Hilbert. Foundations can be studied mathematically by improving the logical methods. What this means is that the exclusion of certain principles like the axiom of infinity, the axiom of reducibility[78] and the axiom of completeness is for the sake of showing that a logical axiomatic foundation without making contentual (existential) assumptions about mathematical infinity is possible. This immediately implies that Hilbert's preference is first-order level theorizing in logical theory, which can be applied to different mathematical domains without making actual assumptions about infinite totalities etc. On this point, epistemological interpretations of

[77] Hilbert 1928, pp. 472-473.

[78] Hilbert and Ackermann 1928 used (following Hilbert's lectures) Russell's ramified theory of types as what they considered to be the extended predicate calculus. This treatment included the definition of real numbers and an upper bound as a class of real numbers, which in turn required infinitely many types, since the upper bound (as a class of real numbers) of a set of real numbers has to be a real number of a higher type. Russell's solution for this problem was to introduce an axiom (viz. axiom of reducibility) which reduces the higher types to the lowest compatible type. Hilbert followed Russell's solution in his lectures and also in the Hilbert and Ackermann 1928 (first edition). Nevertheless, his ultimate aim was to eliminate the axiom of reducibility as a presupposition, which he considered to be an infinitistic assumption. In that sense, Hilbert's aim was still in line with his earlier claims and criticisms against Dedekind and Frege's presuppositions about the application of the universal quantifier (for these criticisms see chapters 13 and 14.

Hilbert's views are based on patent misconceptions about Hilbert's philosophy of mathematics. All that is needed for Hilbert's foundational purposes is first the determination of models by axiomatic analysis and then second model-theoretical consistency proofs for the axiomatizations.

7. APPLICATIONS OF THE AXIOMATIC METHOD

Hilbert's own mathematical practice is enough to show the importance of the different applications of the axiomatic method for him. In addition to his axiomatization of geometry, he worked on the axiomatization of different mathematical and physical theories.[79] Hilbert's sixth Paris problem was about axiomatization of physical theories.[80] He claims there that the investigations on the foundations of geometry suggest that physical sciences must be axiomatized. Especially those in which mathematics has a crucial role (probability and mechanics in the first place) were to be axiomatized.[81] As a later continuation of these programmatic aims, Hilbert's 1917 address "Axiomatic Thought" (Hilbert's 1918) is devoted to emphasizing the importance of different examples of the application of the axiomatic method to different mathematical and physical theories.[82]

Why was axiomatization so important for Hilbert? The main part of the answer has already been given in the previous chapters: As soon as an axiom system is set up, assuming that it has models, it offers the mathematician an overview on the class of its models. Thereby it is an overview on the information codified by the axiom system. By solving problems, the mathematician puts the information to use by exploring what there is to be found out about those models. Surely a mathematician can study the different particular aspects of a mathematical theory without the axiomatic method as well. There

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[79] Main examples include mechanics, thermodynamics, probability calculus, kinetic theory of gases and electromagnetics. Cf. Corry 1997, pp. 131-178.
[80] See Hilbert 1900.
[81] Notice here that Hilbert considered probability as a part of mathematical physics. This is interesting. It can be taken as a clear evidence for that the attribution of formalism to Hilbert would be seriously misleading. See further chapter 8 below.
[82] See Hilbert 1918.

is no unbreakable law that no mathematician can obtain a systematic overview of a mathematical theory without using the axiomatic method. Yet it must have seemed to Hilbert that axiomatic method prepares the best conditions for both the actual (foundational) mathematical work and its presentation for communicative purposes.

It would be an oversimplification to say that the axiomatic research was for Hilbert an end in itself without external philosophical justification. Yet one has to separate external justifications from the internal working of the axiomatic method.[83] Axiomatic method was a means to achieve a clearer understanding of mathematical theories. In fact, we see that Hilbert emphasizes that the method forced itself upon his research, rather than it flourished as a branch of Hilbert's personal foundational preferences.[84]

In the historical development of Hilbert's work as an axiomatist, it is plain to the eye that his different applications (as well as his approvals of others' axiomatizations) match with different periods of heated dispute on the foundations of different fields. Hilbert's axiomatization of geometry corresponds to that period of epistemological disputes on Euclidean and non-Euclidean geometries. His encouragement and approval of the axiomatization of set theory corresponds to the period of ontological disputes as a result of the set theoretical paradoxes. His call for the axiomatization of physical theories corresponds to those dates when theories of special and general relativity were about to shake the grounds. These examples are enough to show Hilbert's quickness to respond

[83] Cf. Gauthier 2002.
[84] This is plainly stated in Hilbert's Dec 29 1899 letter to Frege. See Frege 1980; also see Corry 1997, pp. 116-117.

different crisis periods.

What was common to different crisis periods in geometry, set theory and physics is that in each case there appeared epistemological and ontological issues which were taken to be reasons as to admit some of the theories as correct (true) and some of them incorrect (not true). This whole issue, it seems, according to Hilbert's viewpoint, was ill-formed. The main source of the ill-conceived issues (especially in mathematics, but also in physics) is due to the lack of appreciation of the model-theoretical viewpoint and of the absence of epistemological and ontological concerns in such a viewpoint.[85] Expanded briefly, the absence of epistemological and ontological (or otherwise empirical) concerns is due to a distinction we have to make—and Hilbert assumed implicitly—between uninterpreted axiom systems and interpreted axiom systems. The distinction can best be explained by means of Einstein's observation in his 1921 paper: "As far as the laws of mathematics refer to reality, they are not certain, and as far as they are certain, they do not refer to reality."[86] What Einstein means is simply that we separate "the logical-formal" from its "objective or intuitive content". Thereby we separate the uninterpreted axiomatizations from interpreted axiomatizations. Hence, by doing so, the applications of the axiomatic method (in its uninterpreted sense) provides the possibility of various foundational investigations which are freed from epistemological or ontological concerns; and hence from crises in the sciences.

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[85] The remarks here about Hilbert's model-theoretical viewpoint concern pure (uninterpreted) axiom systems for foundational purposes. Hilbert's views on experience and its relation to scientific theories must be excluded from these purposes.
[86] Einstein 1921, p. 147.

9. TRUTH, EXISTENCE AND CONSISTENCY

Hilbert's conception of truth and existence in mathematics indicates where the model-theoretical viewpoint cuts off the epistemological and ontological concerns. The information codified by an axiom system specifies the class of its models. So that it becomes a meaningful task to try to understand the contents of mathematical theories by means of axiomatic analysis. Hilbert's conception of truth and existence in mathematics are also along this line. They are envisioned from a model-theoretical viewpoint. In Hilbert 1900a we find a strong statement of this viewpoint:

> …the demonstration of the consistency of the axioms [of the real number system] is at the same time the proof of the mathematical existence of the totality of all real numbers or of the continuum. In fact, when the demonstration has been fully achieved, then all objections which hitherto have been raised against the existence of this totality will lose all justification.[87]

Also in Hilbert's 1899 letter to Frege we read:

> If the arbitrary chosen axioms do not contradict each other with all their consequences, then they are true and the things defined by the axioms exist. That for me is the criterion of truth and existence.[88]

Such point of view is almost a refutation of the formalist philosophy of mathematics, which is sometimes misleadingly attributed to Hilbert. To avoid misunderstandings on this point, Hilbert's approach must be put into a proper context. It has to be taken into consideration against the tacit assumption Hilbert seems to have made when he says that consistency implies existence. The consistency in question is model-theoretical

[87] Hilbert 1900a, p. 1105.
[88] Kluge 1971, p.12.

consistency. In line with his general model-theoretical outlook the tacit assumption that comes with Hilbert's criterion of truth (i.e. as the consistency of the axioms) seems to admit the determination of models of potential models for theories, viz. ultimately a model of *all* models. Indeed Hilbert's paradoxical sounding claim about truth and existence as implied by consistency is true in the model of all models. The (model-theoretical) consistency of a theory implies the existence of models for it in this model of models.[89]

It would be an oversimplification to assume that axiom systems are generated arbitrarily out of nowhere. New systems are in some way built up on and connected to the previous theories. For such building and connectedness the notion of a model of all potential models is very useful. In it quantification provides the same conception of mathematical existence (as well as truth) on the pre-theoretic level for different axiom systems. This kind of view is, for instance, implicit in Hilbert's following statement:

> The conception of the continuum, or equally the concept of the system of all functions, exists then in precisely the same sense as does the system of rational numbers or that of the higher *Cantorian* number-classes and powers.[90]

Purportedly the same sense of mathematical existence is obtained if the model-theoretical consistency of each axiom system is proved. In that sense what Hilbert envisions and hints at in the quoted passage is a uniting model of all models for different axiom systems.

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[89] The importance of the idea of "model of all models" in the foundations of set theory was emphasized by Hintikka 2004.
[90] Hilbert 1900a. p. 1105.

The tacit assumption Hilbert seems to have made here shares the same presuppositions as Husserl's notion of definite (complete) manifolds in which "the concepts of true and formal implication of the axioms are [considered to be] equivalent."[91] The ontology (of manifolds) in question involves a super-universe of potential models for the theory, a "model of all models". A similar ontology can be imagined in connection with Riemann's work on manifolds, for instance, as a chapter in what might be called a general study of forms of space; since a manifold by definition is a geometrical entity which is a structured totality of all possible solutions of a given (polynomial) equation.[92] Even though this sense of manifold is not necessarily the same as Husserl's, they are obviously familiar. Likewise, Cantorian universe of sets can be seen as an abstraction from Riemann's geometric notion of manifold.[93]

The so-called model of all models can be considered a natural presupposition of mathematical activity. After all what the mathematician does is to build (and connect to each other) different structures. The beginning stage of such activity requires the grasp of what might be called a particular relational structure. When the net of relations of such a structure is considered as basis, the task of understanding its models (as well as the task of extending its models) is a matter of application of structure-preserving rules. Such application presupposes consistency as a ground for its own justification. At that point the model-theoretical consistency of a particular axiom system suffices to justify mathematician's actual intentions to study what there is to be known in the models of the

[91] Husserl 1913, section 72
[92] Cf. Mac Lane 1986, chapter VIII.
[93] Cf. Hintikka 2004.

system.

Husserl presented a similar argument (to the one just has been sketched) in his Göttingen lecture at Hilbert's seminar in 1901.[94] Roughly, Husserl's argument goes as follows: Take two axiomatic systems AX_1 and AX_2. Let AX_1 be a subsystem of AX_2, in the sense that AX_2 is the extended system by additional axioms when AX_1 is considered as the original system. Two conditions must hold then, according to Husserl. One is that AX_1 must be a definite manifold. Two is that AX_2 must be consistent. Definite manifold means the intended model of the theory is determined completely. This is suggestive of descriptive completeness. If these two conditions are satisfied then we say AX_2 is a conservative extension of AX_1, in the sense that its models can as well be determined on the basis of the models of AX_1. Here, Husserl's major aim seems to have been to show how an axiom system determines its intended models in a definite way and to justify (if possible) different extensions of the theories by proving their consistency and completeness.[95]

Assuming that the (conservative) domain extension of the models of systems (say, from AX_1 to AX_n) reaches up to a uniquely determined universe of definite manifolds, the maximal extension that is obtained in the end of such domain-extension procedure can be considered as an analogue to what has been called above "the model of all models". That maximal model is what Husserl and Hilbert seems to have presupposed as a ground for their pretheoretical conception of mathematical truth, existence and

[94] See Husserl 1970, supplementary texts B, essay III.
[95] Husserl explicitly says that his aims are in line with Hilbert's foundational views. See Appendix III of supplementary texts B, essay III.

consistency. For sure, such conception has to be backed up by a proof of the (model-theoretical) consistency of the systems involved, most notably the continuum (or equivalently the system of all functions).

9. CONTINUITY AND COMPLETENESS

The mathematical investigation of the structure of real numbers falls under a major aspect of the idea of "all models". It requires an explication of the continuity (and completeness) assumptions for its logical axiomatic characterization. Historically, the continuity and completeness assumptions in defining the structure of real numbers find their proper treatment in the works of Cantor and Dedekind. Dedekind's characterization of the real line as a densely ordered system which is closed under algebraic operations (as well as under limit operations) is sometimes called *complete* in the sense that it determines a model (in a definite way) for the continuous number line.[96] The considered completeness of the real line is obtained by using what are known as Dedekind cuts. The intuitive idea behind Dedekind's cut-procedure is that the so-called cuts fill in all the gaps in the system of rational numbers. So that each bounded set of reals enjoys having a least upper bound. What is remarkable about completeness in the sense just mentioned is that it entails that the structure of real numbers (as imagined) is uncountable. This is what Cantor's diagonal argument showed.

The results that were reached by Cantor and Dedekind's works were very important discoveries of the nineteenth-century mathematics, according to Hilbert. Nevertheless, one of the main purposes of Hilbert's axiomatic foundations was still to explain how the so-called uncountable infinity can come about without making any assumptions concerning the actual existence of infinite totalities. For that purpose, the completeness and continuity assumptions that are intuitively appealed to in Dedekind's

[96] Cf. Mac Lane 1986, p. 15. This kind of completeness is in a different sense from the completeness of axiom systems. It is rather a metatheoretical property of the models.

characterization of the real line have to be made explicit with their logical dependence on the axiomatic system. The same task is needed to be accomplished also for understanding mathematical interconnections between different axiom systems and the structures they characterize. Most notably, between algebraic and geometrical structures, interconnections must be studied in the light of the continuity properties. This arises from the geometrical sense of the models in characterizing the real line either as an infinite set of points or as of line segments. On similar lines, to find out metatheoretical interconnections between the system of real numbers and the Euclidean space, and hence to establish the possibility of the determination of models of geometry, an investigation of their continuity properties is inescapable. By way of disclosing the continuity assumptions of an axiom system one can characterize the space and hence the same sense of existence and truth is obtained in all its models.

Partly to point out the role of continuity assumptions (in the above sense) in axiomatized geometry, in addition to the original treatment of the axiomatic foundations of geometry in his 1899 book, Hilbert (in Appendix IV) gives a different determination of the plane geometry. (It can also be generalized to the case of space.) Hilbert's determination of the plane is by way of analyzing (in an axiomatic way) the properties of manifold congruent motions based on the notion of transformation group.[97] Mainly by appealing to the notion of continuous transformation and some axioms of motion (e.g. axiom of the composition of two motions as to form a group) Hilbert presents a determination of a model for the plane.

[97] Cf. Appendix IV of Hilbert 1899 (Second and later editions) A brief survey of Hilbert's work on geometry can be found in Bernays 1967 and Toretti 1978.

The continuity assumptions for the characterization (in the general case) of the space are made at the very beginning. The aim in such analysis is, as Hilbert points out:

> …to determine the least number of conditions from which to obtain by the most extensive use of continuity the elementary figures of geometry (circle and line) and their properties necessary for the construction of geometry.[98]

The difference between the main approach in Hilbert 1899 and the group-theoretical approach in the appendix has to do mainly with the role of axioms of continuity in the complete determination of models. In the main axiomatization Hilbert's first four groups of axioms are arranged in such a way that "continuity is required *last*". This provides a way to clarify which logical consequences of the axioms are independent of the continuity assumptions.[99]

First one of Hilbert's continuity axioms is what is called the Archimedean axiom. This axiom says that given two line segments AB and CD, either one of them, let us say, AB can be extended by multiplied measure of the other segment CD such that it exceeds the length of CD. The algebraic structure that might be superimposed on space with the help of the Archimedean axiom here is obtained by reference to the system of coordinates that satisfies Hilbert's axioms of incidence, order and congruence, and the axiom of parallels. An instance of this algebraic structure is the system of algebraic numbers and rational operations on them with the exclusion of square roots.[100] Hence with the help of the Archimedean axiom, continuity is obtained only up to a point. An additional second

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[98] Hilbert 1899, p. 189 (second English edition)
[99] Cf. Hilbert 1899, p. 189 (second English edition)
[100] See Hilbert's Theorem 65 in Hilbert 1899 (second English edition).

axiom, which connects the geometric continuity to the real continuum, is necessary. That second continuity axiom is Hilbert's axiom of (line) completeness, which says:

> An extension of a set of points on a line with its order and congruence relations that would preserve the relations existing among the original elements as well as the fundamental properties of line order and congruence that follows from [the axioms of incidence, order and congruence] and from [the Archimedean axiom] is impossible.[101]

From this axiom Hilbert derives the theorem of completeness which states that the extension of the elements (points, lines, planes) of geometry is not possible without violating the axioms of incidence, order, congruence and Archimedes.[102]

The theorem of completeness provides the appropriate perspective to consider the foundations of analysis in relation to the foundations of geometry. In particular, Hilbert's consistency proof for the axioms of geometry, which is relative to the consistency of analysis, can be positioned in the proper foundational basis. Most notably, as Hilbert also points out, the existence of infinitely many geometries which satisfy the first four groups of Hilbert's axioms plus the Archimedean axiom is shown. And when the axiom of line completeness is added to the axioms, a uniquely determination of the Cartesian geometry is obtained.[103] This signifies almost a simultaneous development in the foundations of geometry and of analysis, which is due to the additional of the continuity axioms. They are added to the axioms of number theory in Hilbert 1900, as follows:

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[101] Hilbert 1899, p. 26.
[102] Ibid. p. 27; the axiom of line completeness is added to Hilbert's book in the later editions. In the first edition there is no axiom of completeness. In the second edition there is the axiom of completeness for the general case. Later the axiom of line completeness is added so as to suffice to prove what is referred above as the theorem of completeness. For different completeness axioms see Peckhaus 1990, pp. 29-35.
[103] Cf. Hilbert 1899, p. 32.

(Archimedean axiom) If $a > 0$ and $b > 0$ are two arbitrary numbers, then it is always possible to add a to itself so often that the resulting sum has the property that

$$a + a + \ldots + a > b$$

(Axiom of completeness) It is not possible to add to the system of numbers another system of things so that the axioms [of linking, calculation and ordering with the Archimedean axiom] are all satisfied in the combined system; in short, the numbers form a system of things which is incapable of being extended while continuing to satisfy all the axioms.[104]

Hilbert in his 1900 paper defines the system of real numbers as a complete ordered Archimedean field. And the models that he constructed in Hilbert 1899 to prove the consistency of geometry can be considered as the relevant subfields of the system of real numbers for different sets of geometry axioms.[105] In general terms, it seems fair to say that Hilbert's completeness axiom (or theorem) provides a way of translating Euclidean geometry to the Cartesian geometry.[106] By doing that it specifies an ordered Archimedean field, for which if there were a combinatorial way to show its consistency that would also lay the foundations of analysis. What is further needed for the consistency proof is to eliminate the appeal to arbitrary sets, for instance in the application of Dedekind cuts and correspondingly in making a combinatorial sense of arbitrary sets of points in the continuity axioms.[107]

For the same reasons as in geometry, continuity assumptions (and hence completeness) play a crucial role also in physics. To give an example, in his mechanics lectures Hilbert considers the addition of vectors as a continuous operation, in the sense

[104] Hilbert 1900a, par. 6 (p. 1094)
[105] Hilbert 1899, Chapter II.
[106] Cf. Bernays 1967.
[107] Cf. Kreisel 1976, p. 101.

of the Archimedean axiom.[108] For example, given a domain D around the vector sum A + B, one can always find other domains D_1, D_2… around the endpoints of A and B such that any considered sum of two vectors in these domains has endpoints falling inside the domain D. The intuitive idea here seems to be closer to the notion of connectedness. What Hilbert had in mind though about continuity is fairly easy to understand. The punchline of the assumed principle is that we can move from any point of the domain to any other point of it through a continuous line, which remains in the same domain. It is plain to the eye here that Hilbert's major aim is to specify a particular class of models for physical forces, i.e. which obeys the continuity axiom.

This does not mean that Hilbert's view excludes systems with certain discontinuities or systems without the Archimedean property; since an axiom system in Hilbert's sense does not express a *fixed* set of states of affairs. It only defines a "possible form of a system of connections, a system which is to be investigated according to its internal properties."[109] Hilbert's view simply suggests the study of different physical systems. In his 1900 Paris address, he states it straightforwardly:

> As he has in geometry, the mathematician will not merely have to take account of those theories coming near to reality, but also of all logically possible theories.[110]

All that matters here is the determination of models up to isomorphism. And hence what matters in a logical axiomatization is the model-theoretical consistency of the axiom system. And for that purpose as was indicated above the underlying logic must be

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[108] In the 1905/06 lectures; see Corry 1997
[109] Cf. Hallett 1995, p. 137
[110] Gray 2000, p. 258

capable of allowing the intended models in question to be captured completely. This is suggestive, in the first place, of a descriptive completeness. Nevertheless, if a deductive consistency proof could be achieved, that also could serve as a way to capture the intended models of the theory. Of course, provided that the underlying logical theory is semantically complete. Otherwise the deductive consistency of the theory does not imply its model-theoretical consistency.

As can be seen from the considerations up to this point, the interconnections between completeness, continuity and consistency properties of mathematical systems are closely related with their model-theoretical characterizations. If one uses a logical axiomatization these characterizations can be handled by means of the two requirements of the axiomatic method that were mentioned before: First, the purely logical character of inferences from axioms to the truths of the theories is needed. Second, a complete logic which provides means to obtain deductively or descriptively complete representations of the theories must be formulated.

At some point Hilbert might have assumed the semantic completeness of the underlying logic of axiomatization. Nevertheless, even if this is true, it does not mean that he was arguing for a mechanical procedure to prove the consistency of mathematical theories. The model-theoretical character of his viewpoint excludes such an approach as an ultimate foundational aim for Hilbert. Whatever “comes near to reality”, whatever is logically possible are at bottom all depending on their determination up to isomorphism and hence on the meta-theoretical level, on the model-theoretical consistency of the axiom systems.

As is presented in his sixth Paris problem, probability as part of physics provides a strong case for Hilbert's views. Hilbert considered probability as a part of the physical sciences and his main interest in the probability was the problem of how to avoid and eliminate observational errors in measuring physical magnitudes.[111] Hilbert's application of probabilistic reasoning to the physical measurement proves that the continuity assumptions for Hilbert—however appears to involve infinitistic operations—always had a combinatorial and model-theoretical basis:

> The validity of the Archimedean axiom in nature stands in just as much need of confirmation by experiment as does the familiar proposition about the sum of angles of a triangle.[112]

In this regard any view stating that the infinite (as well as the continuity assumptions about infinite systems) in mathematics is part of mere formal manipulations for Hilbert, misses the essential connection of Hilbert's mathematical ideas with his general model-theoretical view of physics and physical continuum:

> In general, I [Hilbert] shall like to formulate the axiom of continuity in physics as follows: 'If for the validity of a proposition of physics we prescribe any degree of accuracy whatsoever, then it is possible to indicate small regions within which the presuppositions that have been made for the proposition may vary freely, without the deviation of the proposition exceeding the prescribed degree of accuracy.' This axiom basically does nothing more than express something that already lies in the essence of experiment; it is constantly presupposed by the physicists, although it has not previously been formulated.[113]

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[111] Corry 1997, p. 160-161.
[112] Hilbert 1918, p. 1110.
[113] Ibid.

To considerable extent Hilbert's work on physics is devoted to the purpose of searching suitable ways of axiomatizing different theories. As also is seen in the statement of his sixth problem Hilbert's central emphasis is on the logical axiomatization of theories. As has been sketched here, the determination of models by means of logical axiomatization is obtained by investigating the continuity properties of the systems in consideration. Thereby, above all, the streamline of Hilbert's foundational investigations is to be found where the continuity assumptions for different mathematical and physical fields meet, viz. in the metatheoretical study of the system of real numbers and in its model-theoretical consistency. For that purpose development of the metatheory for logical axiomatization is also required; presumably, on the basis of suitable model-theoretical consideration of continuity and completeness properties of algebraic and geometrical structures on the metatheoretical level.

10. LOGICAL AXIOMATIZATION

The problem of model-theoretical consistency of analysis (and arithmetic) has to be approached by means of logical axiomatization. For the primary purposes of a logical axiomatization, it suffices that the theorems of arithmetic, for example, are all logical (semantic) consequences of the axioms. As has been pointed out, this does not require that these consequence relations can be implemented by mechanical rules of inference. Thus for example a second-order axiomatization can serve these primary purposes as well as a first-order one, even though second-order logical truths are not recursively enumerable. For this reason it cannot be conclusively said that Hilbert's consistency program was made impossible by Gödel's results.

As is well known, a crucial first step to achieve Hilbert's principal aims for the foundations of mathematics is to prove that the usual set of axioms of arithmetic is consistent. Gödel's second incompleteness result showed that if any such set of formal axioms AX (that can codify elementary arithmetic) is consistent, then the consistency of AX cannot be proved in AX. That is to say, the sentence coded in the language of AX, which says that AX is consistent, cannot be derived in the formal system AX. Gödel's argument implicitly assumes that ordinary first-order logic is used in the axiomatization. It also seems to assume that we are dealing solely with proof-theoretical consistency in metamathematics. This result led some logicians to immediately give up hope about Hilbert's program. However, Hilbert himself never admitted that it contradicted his conception of the problem of foundation. Hilbert was right in not giving up his foundational aims. One can base AX on a richer logic than the ordinary first-order logic,

and then a proof of the consistency of arithmetic which is acceptable by Hilbert's standards can be given. What one has to do is to find out whether there are elementary logical operations that can be formulated in a second-order axiomatization, to carry out a proof of the consistency of arithmetic. Presumably, Hilbert would not consider the underlying logic of an axiomatization elementary, if the logic allows quantification over all predicates without restriction. Yet this does not mean that parts of second-order logic which permit quantification over definable predicates (as well as their possible reductions to first-order level of reasoning) are excluded:

> We have to ask ourselves the question, what does it mean when we say "There is a predicate P"? In axiomatic set theory, the "there is" always refers to the domain B we take to be there at the foundation. In logic, we could think of the predicates as collected together in a domain. But this domain of predicates cannot be considered as something given from the beginning; rather it must be formed through logical operations. Only through the rules of logical construction is the predicate-domain subsequently determined.
>
> And now it becomes obvious that, in the rules of the logical construction of predicates, reference to the domain of predicates can be permitted.[114]

Therefore, it would be a mistake to think that Hilbert's model-theoretical aims are not realizable by means of semantically incomplete logics that are strong enough to codify mathematical reasoning. The idea of purely logical axiomatization does not necessarily presuppose that the underlying logic is semantically complete. What is necessarily presupposed is a demarcation between the logical and extra-logical structures. That does not require all valid formulas to be recursively enumerated, by deriving them from a recursive set of axioms. An axiomatization can be purely logical even when the

[114] Quoted from Hilbert's 1920 lectures, in Hallett 1995, p. 165.

derivation of theorems from axioms is carried out by semantically valid inferences instead of formal derivations. It can even be called "formal" in that semantically valid inferences depend only on the logical form of the premises and the conclusions.

To explicate this point further we can distinguish between the formalist view of mathematics, and the formal character of logical inference. Formalist view of mathematics is the view that mathematical reasoning consists primarily of the manipulation of formal symbols. It is a separate view from the doctrine of the formal character of logical inference. Formal character of a logical inference means that the inference from a sentence to another is independent of the non-logical constants occurring in them. That is, an inference from S_1 to S_2 depends only on the logical structure of S_1 and S_2. One way of seeing the difference between these two meanings of the term "formalist" is to imagine a framework in which philosophical formalism fails but formal character of logic is retained. Second-order logic provides such a framework. In second-order axiomatizations mathematical inferences cannot be reduced to the manipulation of formulas, such as mechanical deductions. Yet the validity of second-order inferences depends only on the logical structure of the inferences.

It can be safely said that most of actual mathematical reasoning can be thought of as being carried out in second-order logic.[115] And such an enterprise cannot be restricted to mechanical deduction. The reason is that there is no (semantically) complete axiomatization of second-order logic. Hence from the point of view of logical theory, philosophical formalism cannot yield an adequate account of mathematical reasoning.

[115] For an account of the second-order logical foundations of mathematics, see Shapiro 1985 and Väänänen 2001.

Deduction must be complemented by an additional of new principles of proof, presumably on the basis of suitable model-theoretical considerations, which fits into Hilbert's mould. The crucial point here is that deductive incompleteness does not make any difference to the formal character of relations of logical consequence. A sentence S_1 logically implies another one, say S_2, if and only if the same relation holds between any two sentences of the same logical form but with different non-logical constants. In this sense the formal character of logical reasoning is an obvious truth.

What Hilbertian formalization amounts to then is a reduction of all derivation of theorems from axioms to purely logical inferences. Such inference is formal only in the sense of being independent of the interpretation of the basic concepts of the axiom system. In this sense, all the proofs are intended to be independent of the domain of objects that is being considered. Here the fact that deduction is independent of interpretation is compatible with Hilbert's insistence that the choice of axioms is guided by the intuitive content of the concepts involved. That is why for example Hilbert and Bernays 1934 discuss two kinds of axiomatization: formal and contentual.[116] This point makes it conclusively clear that the attribution of formalism to Hilbert's foundational ideas is missing an essential distinction between form and content in Hilbert's axiomatic approach.

The distinction between form and content in mathematics, and the fact that inferences are independent of interpretation means that all mathematical results considered in an axiomatic framework are intended to have a structural meaning. This is

[116] See Hilbert and Bernays 1934, § 1.

an obvious truth from the model-theoretical viewpoint. As a problem of mathematical logic the topic arises in Hilbert's 1920 lectures:

> …we have to interpret our signs of our calculus when representing separately symbolically the premises from which we start and when understanding the results obtained by formal operations.
>
> The logical signs are interpreted as before according to the prescribed linguistic reading; and the occurrence of indeterminate statement-signs and function-signs in a formula is to be understood as follows: for arbitrary replacements by determinate statements and functions…the claim that results from the formula is correct.[117]

Here the leading idea is that a correct symbolism constitutes an isomorphic replication of what it represents. This is seen from the fact that Hilbert intends to obtain (in the quoted passage) arbitrary instantiations of the structures that are described by the logical axiomatization (with the prescribed linguistic reading of the logical signs) give (model-theoretically) correct results. In that sense the proof-theoretical analysis of mathematical inference is not enough for Hilbert's model-theoretical purposes. Correct interpretation of symbolic framework is essential:

> We have analyzed the language (of the logical calculus proper) in its function as a universal instrument of human reasoning and revealed the mechanism of argumentation. However the kind of viewpoint we have taken is incomplete in so far as the application of the logical calculus to a particular domain of knowledge requires an axiom system as its basis. I.e. a system (or several systems) of objects must be given and between them certain relations with particular assumed basic properties are considered.[118]

When Hilbert's starting point, i.e. that number symbols themselves as objects of number theory, is combined with the idea that correct symbolism is an isomorphic replication of

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[117] Cf. Sieg 1999, p. 18.
[118] Quoted from Hilbert's 1920 lectures, in Sieg 1999, p. 24

what it represents, models in Hilbert-style axiomatization can safely consist of any objects, including number symbols. What it brings about, as was indicated earlier, is that mathematical symbols themselves (for example number-theoretical symbols) can be used as the contentual (extra-logical) elements of the mathematical proofs. In that sense Hilbert's signs (or symbols), as in the case of algebraic manipulations and symmetries in abstract algebra, can share the same common models (up to isomorphism) with their objects, whatever those objects might be. Metamathematics in that sense can be seen as the combinatorial study of certain symbol structures. The so-called nineteenth century arithmetization of analysis can be included in that. Of course, this combinatorial study presupposes its own determination of models, and its own model-theoretical consistency.

Such a determination of models up to isomorphism requires that the underlying logic of axiomatization is semantically complete. As was shown by Gödel 1931—since it shows the impossibility of a categorical characterization of arithmetic by using first-order axiomatization—there is no hope for determining a unique model, and also no hope for proving the consistency of arithmetic, by using the ordinary-first-order logic as the underlying logic. That is the case, although the proof-theoretical consistency of an axiom system implies its model-theoretical consistency in virtue of the completeness of first-order logic. Hilbert's aim to prove the proof-theoretical consistency of arithmetic cannot be achieved due to the deductive incompleteness of first-order arithmetic (based on the ordinary first-order logic). This impossibility calls for an investigation of the possible continuity principles underlying Hilbert's assumption that structures of symbol combinations can be used as instantiations of mathematical structures. However, without

a proof of model-theoretical consistency such investigation would be a *petitio principii*. Therefore, in order to carry out the desired consistency proofs, by means of suitable alternative logical and algebraic techniques, it is more appropriate (as much as it is inevitable) that alternative continuity and completeness assumptions for these techniques must be introduced in tandem with those techniques.

11. IF LOGIC

It is a characteristic feature of some of the developments in the nineteen-twenties that quantifiers were considered to be closely related to choice functions.[119] In Skolem's work, for example, this was the case. According to Skolem quantifiers serve no better than choice functions.[120] Like Skolem, Hilbert recognized the close interrelation between quantifiers and choice functions. In fact he realized that the basic idea underlying the axiom of choice and quantification was one and the same:

> We have not yet addressed the question of the applicability of these concepts ["all" and "there is"] to infinite totalities. …The objections…are directed against the choice principle. But they should likewise be directed against "all" and "there is" which are based on the same basic idea.[121]

Later this basic idea is outlined in Hilbert and Bernays 1934 as that a finitistic interpretation of a universal statement is an assertion about any given object (from a domain), whereas an existential statement amounts to a series of operations that have a definite bound. So, for example,

(0) $(\forall x)\ (A(x) \supset (\exists y)\ B(x, y))$

means a series of operations, which for any given x that is A makes it possible to find a y (on the basis of x) that is related to x by B.[122] Later developments in logic makes it

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[119] Goldfarb 1979, p. 357.
[120] Goldfarb 1979, p. 357-358.
[121] Zach 2001 quotes Hilbert; Zach 2001, pp. 70-71.
[122] Hilbert and Bernays 1934, pp. 32-33.

sufficiently clear that the operations Hilbert and Bernays considered are based on the idea of operating with choice functions.

In his 1961 paper, Henkin introduced the first-order (partially-ordered) branching quantifiers, e.g.:

(1) $$\begin{matrix} (\forall x)\ (\exists y) \\ (\forall z)\ (\exists u) \end{matrix} \quad A(x, y, z, u)$$

If we use Skolem functions, (1) is equivalent with:

(2) $(\exists f)(\exists g)(\forall x)(\forall y)\ A(x, f(x), z, g(y))$

If quantifiers are interpreted as choice functions (like Hilbert also seems to have done), Henkin's quantifiers amount to expressing different dependency relations between quantified objects (compare (2) and (4)) from the linearly-ordered quantified versions—such as of (1):

(3) $(\forall x)(\exists y)(\forall z)(\exists u)\ A(x, y, z, u)$

If we use Skolem functions (3) is equivalent with:

(4) $(\exists f)(\exists g)(\forall x)(\forall y)\ A(x, f(x), z, g(x,z))$

Here in (4) the choice of a value for u depends both on (∀x) and (∀z), whereas in (1) u depends only on (∀z). What is relevant here to Hilbert's views on quantification theory is that Henkin quantifiers unseals the connections between quantifiers (quantifier-dependence) and choice functions, when one comes to interpret their meaning.

Henkin suggested in his 1961 paper to treat the alternation between quantifiers as choices (dependently or independently) made from a domain.[123] Accordingly a given formula, say (1), can be evaluated by means of a procedure of choices made by two players. (In order for keeping with Hilbert's approach, one has to find the appropriate operations for the evaluation in the sense that infinitistic assumptions about quantifying "all" must be eliminated.) In the general case, say for all sequence of choices c_1, c_2, c_3,…, c_n, the existence of a function s (viz. a winning strategy) which is correlated to c_1, c_2, c_3,…, c_n in the given formula determines the winning (and hence truth).

Hintikka, in his game-theoretical semantics, generalized Henkin's idea.[124] The leading idea in Hilbert, Henkin, and Hintikka's approaches is the same. It is that the meaning of quantifiers is based on the same idea as that of choice functions. What is new in Henkin and especially in Hintikka's approach is that quantifier-dependence is taken into consideration more closely than it is taken in Hilbert. Hilbert seems to have missed the importance of quantifier dependence. In his 1921/22 lectures—as Sieg 1999 notes—he indicates that existential claims in logic and mathematics

[123] Henkin 1961, p. 179.
[124] See Hintikka and Sandu 1997.

…have sense only as a pointer to a search procedure which one possesses, but that ordinarily need not be elaborated because it suffices generally to know that one has it.[125]

Hilbert was probably right in that ordinarily a mathematician does not need to elaborate the search procedure in question. Probably, he was also right in that one does not need to elaborate it in order to provide epistemological ground for mathematical reasoning. However, for semantical purposes, one has to elaborate what Hilbert calls a search procedure and in fact when that is carried out—as in Henkin's case and in Hintikka's cse—the whole picture of the logic of quantifiers change. The tools that Henkin and Hintikka introduced are intended to capture the logic of quantifier-dependence and independence, and they were not (formally) available to Hilbert, since the basic logic Hilbert used was the ordinary first-order logic as was developed in Hilbert and Ackermann 1928.

IF logic can be considered as a correction to (as well as an improvement on) the ordinary first-order logic in the following way: The notation in ordinary first-order logic does not enable us to express all possible patterns of dependence and independence between variables. The notation used to express formal dependence is the use of scopes. A quantifier depends on another one if and only if it lies within its scope. Since scopes are in the usual notation nested, one can in this way express only asymmetric and transitive dependence, leaving others inexpressible. Such restrictions concerning quantifier dependence and independence are removed in IF logic by extending the notation. The new notation exempts a quantifier, say (Q_2y), from its semantical

[125] Quoted in Sieg 1999, p. 28.

dependence on another one, say (Q_1x), within whose syntactical scope it occurs. This can be expressed by writing (Q_2y/Q_1x).

With the help of the independence constant '/' in IF logic, Henkin's branching quantifiers have a new shape. For example, (1) is interdefinable with the IF formula

(5) $(\forall x)(\forall z)(\exists y/\forall z)(\exists u/\forall x)\ A(x, y, z, u)$

With the game-theoretical interpretation of quantifiers, truth of (5) can be evaluated by means of a series of choices made by two players, the initial falsifier and the initial verifier. In this series of choices verifier's choices do not depend on the prior choices made by the falsifier.

The falsifier chooses values for x and z from a given domain of a model M. And the verifier chooses values for y and u. If the verifier has a winning strategy, then (5) is true in M. A winning strategy for the verifier is defined as a sequence of functions, whose arguments are the objects that were previously chosen from the domain by the falsifier. In order to win, and hence to show that the sentence S in the game G(S) is true, the verifier has to make the right moves by keeping track of the falsifier's choices as long as they are available to his or her information. Thereof the truth of a sentence S is defined as: S is true if and only if there is a winning strategy for the verifier in G(S).

This does not mean that if the verifier has no winning strategy, S is false in M. There are IF-sentences for which neither the verifier nor the falsifier has a winning strategy. A simple example is:

(6) $(\forall x)(\exists y/\forall x)\ A(x, y)$

Here in (6) since the verifier's choices are independent from the falsifier's there is no winning strategy for the verifier. It is essentially not so much different from playing scissors, paper, and rock. Obviously there is no winning strategy for the falsifier either. What this means is that the law of excluded middle does not hold in IF logic. It is relevant to note here that the failure of the law of excluded middle is a meta-mathematically (in the Hilbert sense) appropriate feature for the underlying logic of axiomatized mathematical theories.

It has been noted that the existence of a winning strategy for the verifier amounts to the existence of a sequence of functions which have as arguments the objects of the previous choices of the falsifier. What this means is that in the simplest case the truth condition of a statement in the form

(7) $(\forall x)(\exists y)\ A(x, y)$

is the existence of the function which can pick a witness individual y depending on x such that A(x, y). That is, in order for (7) to be true

(8) $(\exists f)(\forall x)\ A(x, f(x))$

must be true. The step from (7) to (8) can be generalized so as to generate all formulas in the same form as (8), from given IF formulas, by translating all the existential quantifiers and disjunctions in them into Skolem functions. The aim of such generalization would be to see the extent of the applications of Skolem functions as truth-makers and to determine the bounds of their expressive power. In fact, when the generalization procedure is carried out, every resulting formula with Skolem functions is a Σ^1_1 second-order sentence. Also every Σ^1_1 sentence in turn can be translated into an IF-sentence.[126] This means that for any given IF first-order sentence, its (equivalent) game-theoretical truth condition can be formulated, without stepping beyond the expressive resources of IF means. What this shows is also that IF logic has the same expressive power as Σ^1_1 part of second-order logic. This is a significant result for Hilbertian purposes. Without quantifying over higher-order entities, a number of mathematical concepts, which cannot be expressed on the ordinary first-order level, can be expressed on the IF logic first-order level.

This expressive power involves the definition of a truth predicate in the same language (assuming that the IF language can express its own syntax).[127] For a Σ^1_1 truth predicate can be formed on the basis of game-theoretical truth conditions.[128] And this predicate can be translated to an IF sentence which is equivalent to it. By using this result, truth predicate T(x) can be defined for an IF-based arithmetic in the same

[126] Cf. Enderton 1970 and Walkoe 1970.
[127] Cf. Hintikka 1998 and Sandu 1998.
[128] See Hintikka 1998 and 2001. See also the papers on truth in Auxier and Hahn 2005 and Hintikka's replies to them.

arithmetic.[129] And a Gödel sentence ~T(**n**) (with Gödel number n), which says that the sentence with the Gödel number n is false, can be formed. If this sentence was true then it would be false. If it were false then it would be true. Therefore, it has to be neither true nor false. Neither the verifier nor the falsifier would have a winning strategy for such statements. This solves the liar paradox.

It might be pursued whether a stronger form of the liar paradox can be obtained here.[130] For example, by bringing in the question what if we have a sentence S which says that it is either false or neither true nor false? What such sentence would attempt to say is tantamount, in the first place, to that there is no winning strategy for the verifier in the G(S). The non-existence of a winning strategy for the verifier cannot be expressed in the IF language itself. This is what the game-theoretical conception of quantifiers (as choice functions) imply. If one tries to impose such an expression to the language of IF logic one has to bring in the law of excluded middle as well. (See below)

However, IF logic can be extended. The extension can be considered by means of adding a sentence-initial contradictory negation ¬ into the language, so as to capture the meaning of the non-existence of a winning strategy for the verifier. It can be done without allowing liar-type paradoxes. In the extended IF logic the contradictory negation ¬ is used only sentence initially (i.e. initial to closed sentences); since there are no game rules for it. Hence the liar-type sentences such as "I am either false or either true or false" or "I am not true" ("not" here is in the contradictory sense) are ill formed in IF logic.[131]

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[129] See Sandu 1998.
[130] Cf. Cook and Shapiro 1998
[131] Cf. Hintikka 2002.

What is brought in by means of the contradictorily negated IF sentences is the dual form of those IF-sentences (i.e. they can be translated into Π^1_1 sentences). That is, when IF logic is extended with the contradictory negation ¬ as described we get IF + Dual(IF), or equivalently $\Sigma^1_1 + \Pi^1_1$ part of second-order logic. (In a sense, trivially, Σ^1_1 and Π^1_1 parts of second-order logic are mirror images of each other.) With the help of contradictory negation ¬, therefore, both the expressive and deductive resources of IF logic are enriched further. The expressive richness has already been mentioned. The deductive resources are two-folded, due to the existence of a complete proof procedure for Π^1_1 part and due to the existence of a complete disproof procedure for Σ^1_1 part of second-order logic. Thereof the extended IF logic has a complete proof procedure for one half and complete disproof procedure for the other half of its resources.

The different uses of the two negations (~, ¬) in the extended IF logic can be extended even further, by allowing the law of excluded middle unrestrictedly in the language. This is done by allowing the contradictory negation ¬ to appear inside the scopes of quantified IF sentences. By doing this, the game-theoretical semantics of quantifiers has to be modified so as to be capable of handling contradictory negations inside the formulas. There are no defined rules for such procedures in IF logic, as it is usually defined.

A suitable modification is by considering a nesting of infinite semantical games. Given a sentence S_0 in which ¬ occurs within the scope of quantifiers, $G(S_0)$ is played until a closed sentence in the form $\neg S_1$ is reached. The truth value of $\neg S_1$ is assumed to

obey the law of excluded middle. This means that in order to account for the existence of a winning strategy for the verifier one has to assume that all the substitution instances of S_1 are available to verifier's information. In that sense verifier's winning strategies, if any, are infinitistic. They consist of verifier's back-tracking of the falsifier's moves infinitely often, in the rest of the game. That is to say, whether $\neg S_1$ is true or false depends on the next game $G(S_1)$ which is again infinite and played until a $\neg S_2$ is reached by way of infinite back-tracking of the falsifier's moves. Furthermore, the truth of $\neg S_1$ means the same as the non-existence of a winning strategy for the verifier. So in order to make sure that $\neg S_1$ is true or not, the existence of a winning strategy for the verifier must be definitely determined by trying all the substitution values in the relevant instantiations. This procedure is carried out until a sentence within the quantifiers of which $\neg$ does not occur at all. And if in the last game the verifier has a winning strategy, then S_0 is false. Otherwise it is true.

Nested infinite games bring in a non-elementary assumption to the quantification theory. What is needed for the nested semantical games—jointed with contradictory negations—to come to an end (after infinite number of moves) is to assume for each game $G(S_i)$ that the law of excluded middle holds for S_i. Such an assumption obviously amounts to going beyond what is humanly playable. It is an assumption concerning all possible values that the players can choose in S_i, including infinite domains. Hence from

the very beginning nesting of games partly presuppose infinitely many operations, viz. by means of requiring in a sense a substitutional semantics.[132]

On the basis of the distinction between infinite operations and finite ones a clear-cut distinction is available between the elementary and non-elementary applications of quantification. The former involves the usual IF logic and its game-theoretical interpretation of quantifiers, as well as the extended IF logic. The latter involves the further extensions and in the limit fully extended IF logic, which appeal to unrestricted uses of the law of excluded middle.

The nested hierarchy of semantical games is structured in such a way that at each level of complexity there are sentences with different complex structure of nested contradictory negations. Starting from the least complex case and moving down to the more complex cases, the structure that is obtained in order is the same structure as the Σ^1_n – Π^1_n hierarchy of second-order sentences.[133] This result is obtained by means of a generalization of the reduction of Σ^1_1 sentences to IF logical sentences. As was mentioned Σ^1_1 sentences can be translated to IF logical sentences. And the mirror-image sentence of each IF logical sentence S is obtained by taking its contradictory negation, viz. ¬S. Sentences of the form ¬S (where S is an IF logical sentence with no contradictory negations) has Π^1_1equivalents. Thus showing that Σ^1_n sentences can be translated to fully extended IF logical sentences with (n – 1) layers of contradictory negations, is enough to

[132] For further discussion on this point, see Hintikka 2006.
[133] Cf. Väänänen 2001 and Hintikka 2006.

provide that each Π^1_n sentence has an equivalent of the form ¬S. The translation of Σ^1_n sentences can be given by replacing the existential second-order quantifiers in them by independent quantifiers. The translation procedure can be sketched as follows[134]:

Suppose the given Σ^1_n sentence is

(9) $(\exists f)\ F(f)$

It is equivalent to:

(10) $(\forall x)(\forall y)(\exists z/\forall y)(\exists u/\forall x)((x = y \supset z = u)\ \&\ F^*[x,y,z,u])$

Here in (10), F* will be obtained by means of a nest of replacements. Subformulas of the form f(w) = v and A(f(w)) will be replaced by (x = w ⊃ z = v) and (x = w ⊃ A(z)), respectively. By applying such replacement procedure, the entire second-order logic can be reconstructed in IF logic and its extensions.[135]

[134] Cf. Hintikka 2006, pp. 211-213.

[135] Here nested functions will be translated as in the translation of Σ^1_1 sentences. And predicates will be translated by using their characteristic functions. See Väänänen 2001 and 2006 for further technical information on the reduction of second-order logic to the fully extenden IF logic.

12. CONSISTENCY OF ARITHMETIC

A consistency proof that is needed for the foundations of arithmetic has to be a direct proof, i.e. it has to be an absolute and not a relative consistency proof. This means that it must not appeal to further (infinite) mathematical domains in order to show that arithmetic axioms are satisfiable. Otherwise, the proof attempt can provide at most the relative consistency of arithmetic. Of course this does not mean that the consistency that is tried to be proved has to be restricted to proof-theoretical consistency; since what is needed for Hilbert's purposes is to decide "whether a system of the requisite sort is thinkable".[136] Roughly, a direct proof, in the way Hilbert needed, should be obtainable by using symbols and their combinations to instantiate the structures that are defined by the axioms of arithmetic. One can then show that no contradictory symbol combination can be derived from the axioms of arithmetic:

> …this is a task that fundamentally lies within the province of intuition just as much as does in contentual number theory the task, say, of proving the irrationality of $\sqrt{2}$, that is, of proving that it is impossible to find two numerals a and b satisfying the relation $a^2 = 2b^2$, a problem in which it must be shown that it is impossible to exhibit two numerals having a certain property. Correspondingly, the point for us is to show that it is impossible to exhibit a proof of a certain kind. But a formalized proof, like a numeral, is a concrete surveyable object. It can be communicated from beginning to end. That the end formula has the required structure, namely $0 \neq 0$, is also a property of the proof that can be concretely ascertained. The demonstration [that "$0 \neq 0$" is not a provable formula] can in fact be given, and this provides us with a justification for the introduction of our ideal propositions.[137]

That would be enough to prove the model-theoretical consistency of arithmetic, as Hilbert wanted. However, for such a proof to be carried out—beside other

[136] Hilbert 1922, Par. 8.
[137] Hilbert 1928, p. 471.

requirements—the underlying logic has to be semantically complete. Otherwise the axiom system might be deductively consistent in the sense that no contradictory formula can be derived from the axioms, but still it might not have any models. That is why the underlying logic must be complete. First-order logic as was developed in Hilbert and Ackermann 1928 is semantically complete. But if the underlying logic of axiomatization of arithmetic is semantically complete, the consistency of that arithmetic cannot be proved in the same arithmetic. This is what Gödel's second incompleteness result showed. Therefore, the consistency of arithmetic cannot be proved by means of the ordinary Hilbert-Ackermann first-order logic. A suitable extension of the resources of ordinary first-order logic is necessary.

IF logic provides the needed resources for a model-theoretical consistency proof for the IF-based number theory in the same number theory. [138] However, IF logic is semantically incomplete.[139] So at first it seems to destroy one's hope of carrying out a proof of the model-theoretical consistency of an axiom system, by proving that no inconsistent formula can be deductively proved from the axioms. Such a proof seems to presuppose the completeness of the logic being used in the axiomatization. If this logic is incomplete, then the axiom system might be deductively consistent, without being model-theoretically consistent. That is to say, the inconsistency that makes it impossible to have models might be hidden so deeply that it is not accessible by the incomplete deductive proof methods.

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[138] Cf. Hintikka and Karakadılar 2006.

[139] Standard first-order logic with the addition of branching quantifiers is incomplete in the same sense. See Krynicki and Lachlan 1979. It follows from this that IF logic is also incomplete; for it makes use of a generalization of branching quantifiers. Also see Hintikka 1996, pp. 66-68.

Then the question to be answered is: can inconsistencies be hidden so deeply, beyond the reaches of IF means? Assuming the law of excluded middle we can say that the inconsistency of a formula C is derivable if and only if its negation ¬C in the usual contradictory sense is provable. As was indicated such contradictory negation is not defined in IF logic. It can be used only in front of a closed sentence (cf. chapter 11). Hence there has to be other ways to decide the inconsistency of a formula. Under this limitation what is needed for the inconsistencies to be reachable is rather a recursive enumeration of contradictory formulas—i.e. a complete disproof procedure, instead of a complete proof procedure; since if the list of contradictory formulas can be recursively enumerated, all one has to do then in order to decide whether a given formula is inconsistent is to check whether it belongs to the list or not.

IF logic does have a complete disproof procedure. In order to seek a consistency proof by appealing to this procedure we must first show that the disproof procedure in a suitable elementary arithmetic based on IF logic cannot disprove every proposition.[140] To see whether this can be done let us first extend IF logic by allowing sentence-initial contradictory negation ¬ into the language. The resulting logic is called extended IF logic. It is equivalent with the Π^1_1 part of second-order logic, which consists of the duals (in the sense of contradictory negation) of Σ^1_1 sentences (cf. chapter 11). That is, within this extension there obtains a duality between IF sentences and their contradictory negations. For the former, there exists a complete disproof procedure but not a complete

[140] If a given axiomatic system is inconsistent, this means that any proposition can be proved and any proposition can be disproved in that system. If there are propositions which are not disprovable by arithmetic axioms based on IF logic the existence of such proposition will establish the deductive consistency of the axioms in question.

proof procedure. For the latter, there exists a complete proof procedure but not a complete disproof procedure.

Here what can be done in an elementary arithmetic based on extended IF logic can be seen by a comparison with Gödel's 1931 incompleteness argument. Gödel constructs a predicate Prov[x] in a self-applied number theory. The predicate expresses the provability of the sentence S with the Gödel number g(S) = x. Then Gödel applies a diagonal argument to the predicate ¬Prov[x] to find a sentence

(1) ¬Prov[**n**]

with the Gödel number n (here **n** is the numeral expressing n). Now if (1) is false the sentence with the Gödel number n is provable. But if the system of number theory is consistent in the (strong) sense that whatever is provable is true, then (1) is true. This contradicts the assumption of its falsity, wherefore it must be true. Consequently, it is not provable, because that is what it says.

Similar to the Gödelian argument, the IF predicate Disp[x] can be formed in the number theory based on IF logic. It says that the sentence with the Gödel number x is disprovable. A diagonal argument then produces a sentence of the form

(2) Disp[**n**]

where **n** is the numeral representing n and where the Gödel number of (2) is n. If (2) is

true, it is disprovable. Assuming that all disprovable sentences are false (i.e. that the disproof procedure is sound), (2) is then false. This contradicts the hypothesis that it is true and shows that (2) is not true. What it means is that the sentence with the Gödel number n is not disprovable. Hence there is at least one sentence that is not disprovable. This shows the deductive consistency of the IF elementary number theory, which can be demonstrated in the IF-logic-based number theory itself.

In the original case, Gödel had to assume that the system of number theory he was using (including the proof procedure it uses) is consistent in the sense that each provable sentence is true. For in his argument he had to argue that if the critical sentence ¬Prov[**n**] is false and Prov[**n**] is therefore true, then the sentence with the Gödel number n is in fact true. This presupposes both consistency and the law of excluded middle. Otherwise we cannot eliminate the possibility that the critical sentence is false. In the IF logic based argument, we have to assume only the soundness of the disproof procedure in the sense that each disprovable sentence is false. This does not depend on the law of excluded middle and is hence possible to prove elementarily.[141]

What can be proved in the way just described is the deductive consistency of the number theory in question. It means that the disproof procedure does not refute all formulas. As has been indicated, for Hilbert's purposes deductive consistency is only a mid-step in Hilbert's attempted proof of model-theoretical consistency. Therefore, we would have to prove not only the soundness but also the model-theoretical completeness of the disproof procedure. That is, we would have to prove that if a sentence of our

[141] At least if we can assume that it is the use of the law of the excluded middle that makes an argument non-elementary. See chapters 11 and 21.

number theory is not disprovable, it has a model. And such a proof must be carried out in the IF number theory itself.

Such proof cannot possibly be carried out in an elementary number theory based on ordinary first-order logic. For completeness proofs cannot be carried out there. The reason is that such a proof (for example using the tree method) relies on König's lemma, which says: if a tree branches finitely, then it is either finite or has an infinite branch. Since the notion of infinity cannot be expressed in the ordinary first-order logic, König's lemma cannot be expressed either, in an elementary number theory based on such logic.[142] Nevertheless, infinity can be expressed in IF logic.[143] Thereof the existence of an upper bound on the lengths of branches is expressible in IF logic. Hence the entire König's lemma as well can be so expressed.[144]

If so, the completeness of the disproof procedure that IF logic yields can be proved in our elementary number theory. In order to show in what sense this shows the existence of models for sentences that cannot be disproved; the following observations can be made: What König's lemma implies when applied to the attempted model set construction that for example the tree method provides, is that if the procedure does not yield a disproof, there exists an infinite branch which is a model set containing the sentence under scrutiny. We can interpret model sets (as sets of symbol combinations) as

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[142] Indeed appeals to König's lemma are often thought of as being the source of the infinitary character of completeness proofs for first-order logic. Cf. Beth 1962, section 38.

[143] For example, an infinite number of individuals satisfy a non-empty predicate A(x) if and only if the following IF proposition is true: $(\forall x)(\forall y)((A(x) \,\&\, A(y)) \supset (\exists z/\forall y)(\exists u/\forall x)(x \neq z \,\&\, y \neq u \,\&\, A(z) \,\&\, A(u) \,\&\, ((x=y) \leftrightarrow (z = u))))$. Here z is a function of x alone and u of y. And, if x = y, then z = u. Hence z is the same function f of x as u is of y, where f satisfies $(\forall x)(\forall y)(x \neq f(x) \,\&\, (A(x) \supset A(f(x)) \,\&\, ((x=y) \leftrightarrow (f(x) = f(y))))$ So that for any a satisfying A(a), a, f(a), f(f(a)),… are all different individuals.

[144] In fact it is an IF logical truth. It can be one of the deductive axioms of our elementary number theory. See further Chapter 20.

being their own models like Hilbert's signs, modulo isomorphism. Thus proof-theoretical IF-consistency of a sentence S implies the existence of a model in which S is not false. An argument to this effect can be carried out in the IF elementary number theory, which can therefore be proved model-theoretically consistent. Of course, what the consistency proof accomplishes here is showing how a model in which a certain sentence is not false, not one in which it is true. Essentially it provides a Hilbertian justification of Kreisel's no-counterexample interpretation, which was originally introduced as a variant of Hilbert's view.[145] This should complete a significant part of the task that Hilbert took on in his consistency program.[146]

How further can we get along with this consistency result? Some important arithmetical and logical concepts that cannot be expressed by means of the ordinary first-order logic can be expressed in terms of IF logic.[147] Hence a significant part of analysis falls within the purview of the consistency result reached by means of IF logic. Thereby what the above proof sketch for the consistency of arithmetic and its implications for the foundations of analysis suggest is that we have to reconsider the extent that Hilbert's different aims for the foundations of mathematics can be reached.

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[145] See Kreisel 1953-54.

[146] The consistency of the entire analysis however cannot be proved by means of IF logic. One way of seeing this is to realize that IF logic is not as strong as the full second-order logic. It is equivalent only to the Σ_1^1 fragment of second-order logic. And there is clearly no hope of extending the kind of consistency proof that was outlined above to the entire second-order logic. Since this logic is presumably needed in analysis, a consistency proof for the entire analysis along Hilbert's lines seems impossible.

[147] For the IF characterizable structures, see Väänänen 2006.

13. HILBERT'S AIMS

Hilbert's different aims for the foundations of mathematics can be considered in the following four different aspects of his views: His axiomatic approach to the foundations of mathematics, his response to criticisms of set theory, his response to intuitionist criticisms of mathematics, and his metalogical work for the specification of the role of logical inference in mathematical reasoning. These four aspects overlap and are closely interrelated in their historical development. However, they specify different sets of problems for the foundations of mathematics. Some of the main problems are:

(i) Proving the model-theoretical consistency of arithmetic

(ii) Clarifying the status of the axiom of choice

(iii) Clarifying the status of the law of excluded middle

(iv) Giving an elementary (humanly practicable) account of quantification

As was sketched in Chapter 12, (i) can be carried out by using IF logic as the basic logic of axiomatization. It will be argued that (ii), (iii), and (iv) can also be carried out by using IF logic as basic, going beyond ordinary first-order logic.

All the mentioned problems and the different aspects of Hilbert's aims are strongly connected to Hilbert's mathematical style and his overall view of mathematics. Hilbert describes mathematics as "an organism whose vitality is conditioned upon the connection of its parts".[148] Investigations concerning this organism, according to Hilbert,

[148] Hilbert 1900a, p. 436.

should be carried out by means of new problems (and new theories for the solution of the problems):

> Just as every human undertaking pursues certain objects, so also mathematical research requires its problems. It is by the solution of problems that the investigator tests the temper of his steel; he finds new methods and new outlooks, and gains a wider and freer horizon.[149]

Such a view requires (to a certain degree) optimism in problem-solving. Hilbert was an optimist in a characteristic way. That is why he disliked restrictive approaches to mathematics. That is why he did not believe in the *ignoramus et ignorabimus* in mathematics.

On this broad description of Hilbert's overall view, it is easy to see how the above mentioned foundational problems are connected with Hilbert's actual mathematical work. Hilbert's application of the axiomatic method to the foundations of geometry can provide enough case-study for that. According to Hilbert, the axiomatic method "guarantees maximum flexibility in research".[150] Therefore, it is indispensable for an optimist problem-solver. In this regard, it is not surprising that Hilbert attacks the problem of consistency of mathematical theories in the light of the axiomatic (and hence model-theoretical) conceptualizations. The IF logical approach to the consistency problem follows Hilbert's optimistic style, in finding a suitable way out from the limitations drawn by Gödel's completeness and incompleteness theorems, which is necessary in order to pursue Hilbert's foundational aims.

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[149] Ibid, p. 407
[150] Hilbert 1922, Par. 14.

Similarly, in Hilbert's work in proof theory (including the epsilon technique), the problems concerning the status of the axiom of choice and the status of the law of excluded middle were aimed to be carried out as parts of an optimist mathematical self-defense against restrictive approaches. Same is true, of course, for the pure existence proofs. In defense of pure existence proofs, Hilbert argued that "brevity and economy of thought" are their *raison d'etre*.[151] This line of thought is followed up to the best extent, as will be shown in the following chapters, in the IF logical approach.

According to Hilbert, restrictive criticisms against classical methods in mathematics "were not put into effect at the right place in a unified front".[152] Therefore, in order to give complete justification of his attitude against restrictive approaches, Hilbert had to pursue the solutions of his different problems against "a vast domain of difficult epistemological questions".[153] The foremost such domain concerns the development of mathematical logic in association with the problem of specifying the role of logical inference in mathematical reasoning. A closer look at the problems of mathematical logic makes it sufficiently clear that the needed advances in the foundations of mathematics (then and now) has to be in complete agreement with Hilbert's overall view of mathematics.

In Hilbert 1918, the following problems are listed as necessary to be investigated: (1) solvability in principle of every mathematical question, (2) checkability of the results of a mathematical investigation, (3) criterion of simplicity for mathematical proofs, (4)

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[151] Hilbert 1928, p. 475.
[152] Hilbert 1926, p. 375.
[153] Hilbert 1918, Par. 43.

relationship between content and form, (5) decidability of a mathematical question in a finite number of operations. Then it is pointed out,

> We cannot rest content with the axiomatization of logic until all questions of this sort and their interconnections have been understood and cleared up.[154]

Here, it is clear enough that, in searching for the solutions of these problems (1-5) Hilbert was projecting into the logical deepening of the foundations of his optimism. However, it is also clear that one has to have a similar optimistic (rather than restrictive) attitude in approaching these very problems themselves. From Hilbert's point of view, what is needed is "new methods and new outlooks" and "wider freer horizon".[155] We are not satisfied with seeing only a short distance ahead, even if we can see plenty there that needs to be done.[156]

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[154] Ibid.
[155] Hilbert 1900a, p. 407.
[156] Cf. Turing 1950.

14. AXIOMATIZATION OF SET THEORY

The reconstruction of second-order logic by means of IF logical resources suggests that questions of set-theoretical validity can be put into a purely logical context by means of IF logic. The expressive power of second-order logic is known to be as strong as the expressive power of set theory.[157] What this means is that both the second-order axiomatizations and set-theoretical foundations are dispensable in favor of first-order level IF formalizations in mathematics. The higher-order modes of reasoning in second-order logic and set theory can be reduced to the IF first-order level. This is suggestive of a reconsideration of Hilbert's aims for the foundations of set theory and its logical axiomatization.[158] It also suggests reconsidering Hilbert's views on higher-order and first-order levels of reasoning and quantification.

Historically, after the discovery of logical paradoxes about infinite sets it was clear that some refinement to certain techniques and modes of reasoning were necessary, if contradictions were to be avoided in mathematics. The question "What kind of restrictions to which techniques was necessary?" surfaced different philosophical approaches about the existence of mathematical entities. It led mathematicians to reconsider different aspects of mathematics critically. Some of the disputed matters appeared to have created a crisis in mathematics. Among the leading figures of the historical disputation, Hilbert was arguably the most optimistic one. He argued that in

[157] See Väänänen 2001.
[158] Hintikka 2006, p. 213.

general terms there were no unsolvable problems and hence no possible crisis in mathematics.[159]

Hilbert's axiomatic approach, which—as was argued— is intended to avoid epistemological and ontological issues in mathematics and its foundations, supports this optimism. It strongly suggests that philosophical worries about the existence of infinite sets (or given infinite totalities) can be removed from mathematics by means of a logical axiomatization of set theory. This is clear from Hilbert's blaming the traditional ways of thinking in logic rather than the development of set theoretical analysis, which might appear to involve contradictions or to create paradoxes:

> [Paradoxes] led me to the conviction that traditional logic is inadequate and the theory of concept-formation needs to be sharpened and refined. … What is decisive is the recognition that the axioms that define [their own subject matter] are free from contradictions.[160]

Paradoxes hence were not problems to be worried about for Hilbert, as long as they were remedied by logical axiomatization and the model-theoretical consistency of the axiom system. What was required for that purpose was the development of logical methods.

In his 1905 paper, Hilbert after a brief criticism of different approaches to the foundations of arithmetic and analysis, suggests that a "simultaneous development of logic and of arithmetic is required". Hilbert criticizes first Kronecker's approach, calls him a dogmatist for accepting integers as the real foundation of arithmetic and for not

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[159] Kronecker's and Poincaré's approaches are two of the most pessimistic examples from the nineteenth century. In the early twentieth century Weyl's constructivist argument in his 1918 book and his later intuitionistic views, as well as Brouwer's 1907 dissertation and his further development of the intuitionist philosophy of mathematics are other approaches in similar spirit to Kronecker and Poincaré. They also required considerable amount of restrictions to the classical techniques in analysis.

[160] Frege 1980, p. 51

considering further possible foundations. Hilbert then criticizes Helmholtz's empiricist view. He claims that the empiricist position cannot give an adequate account of arbitrarily large numbers. He mentions Christoffel and other opponents of Kronecker. According to Hilbert these mathematicians could not provide a refutation of Kronecker, even though they were right in pointing out that Kronecker's approach leads to serious limitations in the methods of analysis. Beside these, Hilbert lists Frege, Dedekind and Cantor among who he thinks had a better understanding of foundational problems concerning integers. However, their views were also limited and insufficient in providing a foundation for arithmetic, according to Hilbert.

Clearly Hilbert favored logical foundations. However, he found Frege's and Dedekind's logical approaches inadequate and even transcendental in applying (universal) quantification without giving an elementary (humanly practicable) account of it. For this reason alone, according to Hilbert, Frege's and Dedekind's approaches were prone to contradictions and paradoxes. Hilbert excludes from his criticisms Cantor in that Cantor realized the difficulties in dealing with "all sets", and distinguished between consistent and inconsistent sets. But still, Cantor did not give an objective ground for his distinction, and this is the weakness of his views. According to Hilbert there were further problems concerning the infinite which are lurking in the foundations. In order to resolve these problems and to defend the fruitful methods against restrictive criticisms such as Kronecker's, Hilbert demands a logical clarification of the notion of the infinite. Hilbert points out that in Weierstrass's analysis infinity comes into play by way of logical quantifiers in the form of "*all* real numbers" or "*there exist* real numbers" etc. So the

logical clarification of the notion of the infinite in association with the applications of quantifiers became Hilbert's main topic of interest.[161] By following an axiomatic approach, Hilbert proposed, the difficulties with the notion of "all" or "every" can be taken care of and also Cantor's distinction between consistent and inconsistent sets can be given a firm footing.

Hilbert encouraged Zermelo to axiomatize set theory. Zermelo gave the first axiomatization of set theory in his 1908a paper.[162] What he accomplished was to give a characterization of the structure of the so-called cumulative hierarchy of sets in a suitable axiomatic framework. Such characterization was the first essential step to avoid paradoxes and uncertainties, according to Hilbert. Hilbert approved it as an appropriate way to avoid paradoxes and uncertainties:

> By setting up appropriate axioms which in a precise way restricted both the arbitrariness of the definitions of sets and admissibility of statements about their elements, Zermelo succeeded in developing set theory in such a way that the contradictions disappear, but the scope and applicability of set theory remain the same.[163]

Zermelo's 1908 axiomatization was Hilbertian in spirit. However, it involved impredicative and higher-order elements.[164] Hence it was open to predicativist criticisms. From Hilbert's point of view thus further investigation was needed for its foundations (see further chapters 14, 15). Yet from a purely practical (mathematical) point of view the

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[161] Cf. Hilbert 1926, pp. 369-370

[162] Here "axiomatization of set theory" might seem ambiguous. The axiomatization that go back to Zermelo are in our days taken to be first-order. Zermelo did not interpret his axiomatization in this way. For more discussion see Kanamori 2004.

[163] Hilbert 1918, par.

[164] Impredicative elements in Zermelo's system were the impredicative subset-formation, the power-set operation and the union-set operation. Of course, today's Zermelo-Fraenkel first-order system does not involve such elements. See Hallett 1984, p. 251.

apparent problems such as paradoxes and contradictions (within certain limits) had been avoided.

Even though Hilbert encouraged Zermelo to axiomatize set theory, he himself followed rather a Russellian line of thought in the foundational investigations for a solution of the paradoxes. In his 1917 address (Hilbert 1918) Hilbert refers to Russels's axiomatization of logic as the "crowning achievement" in the field.[165] This was partly on the practical level where presumably Hilbert saw the "promise of success". On the theoretical level further work was required.

In his 1917/18 lectures follows Russell and Whitehead's axiomatization of logic (and hence according to their view, of mathematics). In Hilbert and Ackermann 1928 (first edition) the extended calculus that Hilbert and Ackermann consider is the ramified theory of types. This theory was a way to avoid paradoxes such as the Zermelo-Russell paradox, the Richard paradox etc. What were not suitable for Hilbert's purposes in this theory were the axiom of infinity and the axiom of reducibility.[166]

Yet, for the development of mathematical logic and for considerations of foundational problems which are related to set theory Russell and Whitehead's theory seems to have provided a suitable framework for some of Hilbert's problems such as how to avoid the paradoxes on the logical level, to supply models for all mathematical

[165] Hilbert 1918, par. 40

[166] Ramsey showed in his 1926 paper that the axiom of reducibility was not a necessary assumption in the theory, if the aim was to avoid the paradoxes. He distinguished between logico-mathematical paradoxes and semantical paradoxes. Ramsey showed that the latter kind was inessential to mathematics. In order to avoid logico mathematical-paradoxes, it was enough to have kept the theory of types unramified, simply as defining a hierarchy of types of entities. The critical idea in Ramsey's paper was the standard interpretation of higher-order quantifiers. Arguably, it was also shared by Hilbert (as a requirement of his aim to provide models for all axiomatic theories; cf. chapter 8), although Hilbert worked and lectured on the ramified theory of types.

theories, as well as how to pursue the solution of his first Paris problem, i.e. proving or disproving Cantor's continuum hypothesis. Nevertheless, avoiding the paradoxes by using type theory could not be the leading motivation for Hilbert. It is clear enough, both from his lectures and his explicit statement in Hilbert 1918, that the paradoxes, according to Hilbert, were avoided by Zermelo's axiomatization of set theory. What was needed was further meta-theoretical investigation in mathematical logic, and distinguishing it from (extra-logical) mathematical content. The axiom of infinity, in that regard was a significant part of such mathematical content. It had to be eliminated from logical theory. For this reason alone, it was clear by then that Russell and Whitehead's approach, although it provided useful tools for Hilbert, could not obtain any foundation of mathematics.

Avoidance of the paradoxes was important. But in order to give conclusive answers to the foundational questions about set-theoretical modes of reasoning, further developments in logical theory were needed. Zermelo's axiomatization did not have an explicit underlying logic in its formulation. There was no first-order logic as distinguished from second-order logic in 1908. As a matter of fact, it is not clear which of the logics that different mathematicians would choose, if there was a multiplicity of mathematical logics.[167] Therefore, the historical situation was such that the further development of logical methods was necessary, as Hilbert seems to have foreseen since early 1900s.

[167] See Kanamori 2004.

15. FIRST-ORDER AXIOMATIZATION

First-order quantification theory was separated from higher-order in Hilbert and Ackermann 1928. In the book, although higher-order logic is introduced with an indication that it is "the appropriate means of expressing the modes of inference of mathematical analysis"[168], higher-order reasoning does not fit into Hilbert's mould for axiomatic purposes. For example in defining the notion of an upper bound for sets of real numbers one has to quantify (like Weierstrass did, for example) over real numbers. But for that purpose one has to have an elementary (humanly practicable) account of such quantification:

> …the infinite still appears in the infinite number sequences that define the real numbers, and, further, in the notion of the real number system, which we conceive to be an actually given totality, complete and closed.
>
> The forms of logical inference in which this conception finds its expression—namely, those that we employ when, for example, we deal with *all* real numbers having a certain property or assert that *there exist* real numbers having a certain property—are called upon quite without restriction….[169]

Hilbert's general objection to the unrestricted quantification over real numbers by using higher-order reasoning calls for a criticism of second-order quantification, which is closely related to Hilbert's point in the quotation above. The basic objection to second-order quantification is that the use of second-order entities as objects leads to impredicative definitions. This comes about because in a second-order language the meaning of expressions is determined by a totality of propositions, properties or relations. In this sense they must be considered as being given. And this is tantamount to

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[168] Hilbert and Ackerman 1928, p. 163
[169] Hilbert 1926, p. 369-370.

considering what is called as "given closed infinite totalities". [170] This very objection is emphasized for example in Hilbert's 1917/18 lectures. There Hilbert says:

In the original function calculus, we took a system or several systems (species) as given from the beginning, and by referring to these totalities of objects, the operation with the variables...was given a significance. The extension of the calculus now consisted in regarding statements, predicates and relations as types of object, and according to this, allowing symbolic expressions whose logical significance demands reference to the totality of statements respectively functions.[171]

The extension of the "original function calculus" Hilbert here refers to is second-order logical calculus. He explains the grounds for his objection about impredicative definitions in the following paragraph of the same paper:

This procedure is in fact dubious in the following way. Those expressions which obtain their content through reference to the totality of statements respectively functions, while on the other hand, before we can refer to the totality of statements or functions the statements resp. functions must be considered as determined from the beginning. Here there is a kind of logical circle, and we have grounds for the assumption that this circle is the cause of paradoxes.[172]

As a solution to the problem here, it might be suggested that first-order modes of reasoning are suitable means to avoid infinitistic operations and operations with higher-order entities in quantification theory. A similar suggestion was made by Hilbert. It occurs in Hilbert's 1922 paper:

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[170] Nevertheless, as was seen in the case of fully extended IF logic, the closed infinite totalities are not necessarily introduced by quantification over higher-order entities. They can be introduced instead on the IF first-order level by unrestricted use of the law of excluded middle. Conversely, the usual meta-theory of first-order logic can already involve appeals to unrestricted use of the law of excluded middle. For example, as it is the case in Tarski-type truth definitions.

[171] Quoted by Hallett from the original; see Hallett pp. 218-220. Cf. also Sieg 1999, pp. 9-10 and 16.

[172] Ibid.

As we saw, abstract operation with general concept-scopes and contents has proved to be inadequate and uncertain. Instead, as a precondition for the application of logical inferences and for the activation of logical operations, something must already be given in representation [in der Vorstellung]: certain extralogical discrete objects, which exist intuitively as immediate experience before all thought.[173]

What Hilbert criticizes in the quoted passage is the operations with higher-order entities, i.e. concepts or their extensions. In their place Hilbert wants to place discrete individual objects that can be given to us intuitively and in immediate experience. This is a way to reconstruct the apparently higher-order modes of reasoning on the combinatorial level for concrete objects. Hence in general terms it is fair to say that Hilbert's aim was to be able to understand all mathematical reasoning taking-place on the first-order level. As has been seen such a reconstruction of second-order logic can be obtained by means of the fully extended IF logic. Hence Hilbert's first-order view is vindicated by IF logic.

Vindication of Hilbert's first-order view is one thing. Axiomatization of set theory on the first-order level is a different matter. Hilbert's first-order view suggests that he would prefer a first-order level axiomatization of set theory. This does not necessarily mean that the basic logic for such axiomatization has to be ordinary first-order logic. It only means that Hilbert preferred a first-order level combinatorial account for the foundations of set theory. However, at the time he considered first-order logic as suitable for axiomatization purposes:

The calculus [first-order logic] is well suited for the purpose [purpose of presenting theories] mainly for two reasons: one because its application prevents that—without being unnoticed—assumptions are used that have not been introduced as axioms, and

[173] Hilbert 1922, p. 202.

furthermore because the logical dependencies so crucial in axiomatic investigations are represented by the symbolism of the calculus in a particularly perspicuous way.[174]

This last point suggests that Hilbert's approval of Zermelo's axiomatization must have been mainly on the practical level; since Zermelo's axiomatization involved higher-order modes of reasoning. However, in order to answer the question whether those apparently higher-modes of reasoning can be reconstructed on the combinatorial (first-order) level, the logical methods had to be improved. And this is what Hilbert suggested from the very beginning: "a partly simultaneous development of the laws of logic and of arithmetic is required…."[175]

The purpose of such development is to see the extent that mathematical reasoning can be reconstructed on the first-order level:

… we also want to investigate the foundations of mathematical theories and examine what their relation to logic is and how far they can be built up from purely logical operations and concepts; and for this purpose the logical calculus [first-order logic] is to serve as an auxillary tool.[176]

The reconstruction of the entire second-order logic and also the set-theoretical modes of inferences by means of the IF resources can already be noted here as a realization of Hilbert's original aims then, for the foundations of set theory.

[174] Cf. Sieg 1999, p. 15.
[175] Hilbert 1905, p. 131.
[176] Quoted in Sieg 1999, p. 15

16. NOMINALISM

From the very beginning of his foundational studies, it was clear to Hilbert that even the first-order applications of quantifiers with the assumption of infinite operations is a problematic issue.[177] If one wants to clarify the nature of the infinite in mathematics and give a humanly practicable account of universal and existential quantification (i.e. without assuming infinite operations) one has to face the problem of quantification over infinite domains in mathematical reasoning. So not only higher-order reasoning must be reconstructed on the first-order level, but also first-order quantification must be given a practicable (elementary) account.

One of the nominalistic assumptions in the philosophy of mathematics is that only individuals are admissible as objects of quantification. In logical terminology this assumption amounts to permitting only to first-order quantification. Hilbert's line of thought is in keeping with such a view:

> If logical inference is to be certain , then these objects must be capable of being completely surveyed in all their parts, and their presentation, their difference, their succession (like the objects themselves) must exist for us immediately, intuitively, as something that cannot be reduced to something else.

In this sense Hilbert is defending here first-order logic, which accepts quantification only over individuals, in contrast to a higher-order one. Hilbert continues:

> Because I take this standpoint, the objects [Gegenstände] of number theory are for me—in direct contrast to Dedekind and Frege—the signs themselves, whose shape [Gestalt] can be generally and certainly recognized by us—independently of space and time, of their special conditions of the production of the sign, and of insignificant differences in

[177] See for example Hilbert 1905.

the finished product.

Hilbert criticizes thereof Frege and Dedekind on their quantification over concepts or their extensions in their logical language. This is in line with Hilbert's overall view on logic and logical reasoning. As was noted it was in Hilbert's school that first-order logic was separated from the higher-order quantification theories of Frege and Russell-Whitehead.[178]

Hence in a wider philosophical perspective Hilbert's opposition to Frege and Dedekind, and operations with general concept scopes is not an opposition of a formalist to a non-formalist. It is rather an opposition of a nominalist to conceptual realism. Under wrong interpretations of Hilbert's philosophical terminology—especially under the attribution of "finitism" and "formalism" to it—the real gist of Hilbert's "philosophical attitude" is poorly obtained. The rest of the passage in Hilbert's paper—what follows below—leads to serious misunderstanding when it is read out of its proper context:

> The solid philosophical attitude that I think is required for the grounding of pure mathematics—as well as for all scientific thought, understanding, and communication—is this: *In the beginning was the sign*.

The correct interpretation of this passage should be that Hilbert favored nominalism, and hence first-order quantification in contrast to a higher-order one. In this light, from

[178] In the later editions of their book, Hilbert and Ackermann consider second-order logic. Its incompleteness is pointed out. Its relation to set theory is briefly discussed. Higher-order logic is introduced with an indication by examples that it is "the appropriate means of expressing the modes of inference of mathematical analysis". However, just like types and the axiom of reducibility, higher-order quantification does not exactly fit into Hilbert's mould. It involves quantification over a domain of so-called "all" predicates. That is why he preferred first-order logic and tried to surpass the difficulties with universal quantification by means of his epsilon technique. It can be treated as a nominalistic account of quantification theory. (See further Chapter 20)

Hilbert's nominalistic point of view, Frege's conceptual realism was totally ill-advised:

> [Frege] fell to some extent into an extreme realism of concepts. …he believed he was entitled to take [concept scopes] unrestrictedly as things.[179]

All this is in accordance with Hilbert's concern for concrete content in metamathematics.[180] Salvageable domains of concrete objects (i.e. signs with their representative role) which are immediately given in mathematical practice should be the ground to rely on in foundational considerations.

From a wider historical perspective, Hilbert is against a commonly accepted view in the philosophy of logic and mathematics. According to this view, logic and mathematics deal with general concepts. And in the last analysis it is sense-perception that grasps particulars. Therefore, the justification of all instantiation and the introduction of particular (concrete) representatives of general concepts must be perceptual.[181] In their foundational works Frege, for example, follows such a view but Hilbert does not. When Frege is trying on the one hand to dispense with intuition, he is on the other trying to reduce number theory to what he takes to be the most general concepts and principles of reasoning. Hilbert notwithstanding treats logic preferably on the first-order level. He criticizes the reliance (especially by Dedekind and Frege) on general concept-scopes. He wants to formulate axiomatic foundations of mathematics in the study of the structures of concrete objects. Accordingly, Hilbert tries to practice his metamathematics in nominalistic terms. He believes that logic can cope fully with reasoning about (and with)

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[179] Hilbert 1922, Par. 21.
[180] Cf. Hilbert 1926, p. 377.
[181] Cf. Webb 2005.

particular objects, and on the first-order logical level. In this regard his epsilon-technique for example amounts to a method of instantiation (see Chapter 20). It aims to make systematic use of the particular instances of general concepts in nominalistic terms.

IF logic vindicates Hilbert's nominalistic approach. As was sketched in chapter 11, it provides a way to reconstruct apparently higher-order modes of reasoning by allowing only first-order quantification and cashing higher-order operations into complexes of choices of individuals as instantiation values.

17. FINITISM

Hilbert's finitism is sometimes seen as the view that the (apparently) actual infinitistic assumptions of mathematical reasoning can be given an epistemological foundation, by reference only to finitary content of mathematical statements (not by going beyond that). As has been pointed out in chapters 5 and 6, such conception of finitism makes misleading ways to understanding Hilbert. Hilbert's aim was to provide logical axiomatic foundations, rather than epistemological foundations. He hoped to have reached this aim by detaching the axiomatic investigation from epistemological concerns. In that sense Hilbert's aim amounts to finding out the appropriate logical treatment of the apparently infinitistic assumptions of mathematical reasoning, without permitting any infinitistic technique in the foundational practice. Here the problem is not with the epistemological admissibility of the techniques used. It is more appropriate to say that, in its axiomatic form, Hilbert's finitism amounts to a metalogical (as well as metamathematical) strategy. The right source to decide the admissibility of the techniques involved in this strategy is logical semantics, not epistemology. On this explanation, possible definitions of "Hilbert's finitism" in terms of epistemological or ontological primitives lead to wrong interpretations of Hilbert's ideas. The wrong interpretations are usually implied by the restriction of the so-called big problem about the infinite to that the infinite does not obviously correspond to anything in reality.[182] If the definition of the concept of finitism is restricted to a way out from the lack of correspondence between infinity and reality, then such restriction would lead to misunderstandings. Because, even though it is a part

[182] Cf. Simpson 1988, p. 358.

of the problem of foundations to explain how the infinite can come about in actual (real) mathematics, this is not an epistemological concern, according to Hilbert. Its treatment should be accordingly. Otherwise the same mistakes that were made by mathematicians like Poincaré, Weyl and Brouwer would be made.[183]

Hilbert's nominalism was for the sake of eliminating "dubious or problematic modes of inference" from foundational studies.[184] "Finitism" is the name he gave his strategy to cope with infinitistic operations in mathematical reasoning. The question here of what the so-called finitistic operations consist of is therefore a tricky one. Yet it should be clear that nothing relevant to Hilbert's views can come out of it, if it is asked as an epistemological question concerning the admissibility of certain recursion techniques. Two well known attempts to explain finitism are due to Tait 1981 and Parsons 1998. Tait considers finitism to cover a minimal kind of reasoning presupposed by all reasoning about number.[185] Parsons, on the other hand, argues that finitism determines the domain of intuitive evidence. Thereby, Parsons admits a basic intuition of finite objects.[186] Both approaches try to give an account of epistemic primacy and certainty of finitist mathematical reasoning. From Hilbert's point of view, such an enterprise is pointless. The problem is not how to come up with criteria for an epistemically safe beginning to mathematical reasoning. The criteria are needed rather for metalogical purposes. For the same reason, asking for example whether Hilbert's "finitistic intuition" is the Kantian space-time intuition or it is something else, is a seriously misguided way of approaching

[183] Cf. Hilbert 1928, p. 479.
[184] Hilbert 1924, p. 1139.
[185] Cf. Zach 2001, chapter 4.
[186] For a discussion of Tait's and Parsons' views, see Zach 2001, chapter 4.

the foundational problems. It is being neglected in such mode of questioning that the set of problems concerning finitism and quantification has to be detached from epistemological concerns. From Hilbert's point of view, the solution of foundational problems cannot be dependent on any epistemological preferences.

In logical theory and metamathematics no reference to finitude is necessary. We do not need to commit to the finitude of the domain of objects we are dealing with. The characterization of finitistic methods can be maintained entirely in terms of salvageable objects of mathematics[187]:

> If logical inference is to be certain, then … objects [of mathematics] must be capable of being completely surveyed in all their parts, and their presentation, their difference, their succession (like the objects themselves) must exist for us immediately intuitively as something that cannot be reduced to something else.[188]

Now basic operations of elementary arithmetic are in principle finite and salvageable (surveyable). The infinitistic element, as Hilbert seems to have assumed, comes in when we use quantifiers. The central question here is: what kind of operations do we need to clear the quantifiers from committing to infinitistic assumptions (and salvage the entities that are quantified over)? Hilbert considered these operations to be what might be called finitistic operations. In his 1926 paper he states:

> …the modes of inference employing the infinite must be replaced generally by finite processes that have precisely the same results, that is, that permit us to carry out proofs along the same lines and to use the same methods of obtaining formulas and theorems.[189]

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[187] Here we find the word "salvageable" more in agreement with the game-theoretical meaning of quantifiers as choice functions than the word "surveyable".
[188] Hilbert 1922, p. 202.
[189] Hilbert 1926, p. 370.

Hilbert's aim here is to find out suitable operations that give the same results as those modes of reasoning which appear to have employed the actual infinite in mathematical reasoning. In this sense Hilbert's aim does not involve any epistemologically restrictive (i.e. finitist) condition at all. In Kreisel's way of saying: the eliminability of the infinitistic assumptions "is thought of as a fact (to be discovered), not a doctrinaire restriction".[190] The epistemologically problematic modes of reasoning concerning the infinite can be taken care of by applying logically unproblematic techniques, without making existence claims about any extra-logical (mathematical) entities, other than the ones that are immediately given to our intuitions. For that purpose, all one has to do is to search for logically admissible modes of reasoning that can replace the figure of speech of the apparent infinitism in mathematics. On this point Hilbert remarks sharply in his 1926 paper; he, refers to a certain jargon in mathematics and says playfully:

> …if mathematics is to be rigorous, only a finite number of inferences is admissible in a proof—as if anyone had ever succeeded in carrying out an infinite number of them.[191]

What is crucial to Hilbert's purposes is contentual logical inference as he emphasizes in the same paper:

> Contentual logical inference is indispensable. It has deceived us only when we accepted arbitrary abstract notions, in particular those which infinitely many objects are subsumed. What we did, then, was merely to use contentual logical inference in an illegitimate way….

The task is then to find out the legitimate operations of logical inference, to be used in

[190] Kreisel 1976, p. 98.
[191] Hilbert 1926, p. 370.

handling mathematical notions which subsume infinitely many objects. These operations are provided, as has been seen, in IF logic by the game-theoretical interpretation of quantifiers (as choice functions). The way quantifiers operate in IF logical foundations of mathematics is elementary. They are practicable even if the domain of discourse is infinite. Semantic games are played without invoking any given infinite totalities, and hence are suitable for "finitistic" purposes in Hilbert's sense. (See Chapter 18 for further considerations) In a sense, the quantified entities in mathematical reasoning are salvaged by means of the game-theoretical uses of quantifiers.

18. ELEMENTARY OPERATIONS

It is usually tacitly assumed that the infinity of one's universe of discourse makes the use of quantification non-elementary. As has been mentioned, partly Hilbert assumed it too. Nevertheless, he tried to overcome it by interpreting quantifiers as choice functions. It was clear to him that the so-called infinitary character of the semantics of quantifiers leaves unexplained how it is the case that quantifiers are used so easily in mathematical practice. The assumption in question concerning quantification into infinite domains relies on the view that in mathematics quantifiers are used as if they are ranging over a class of values.[192] This might be a class of objects or a class of substitution instances. The difficulty with such view is the following: If the idea of "ranging over" is taken to exhaust the logic of quantifiers, any application of quantifiers to an infinite domain is received as to capture the range of infinite totalities. Thereby it encounters the impossibility of replacing infinitistic modes of inference by finite processes. In order to surpass this difficulty one has to detach the question of the infinity of the domain from the question whether the underlying reasoning is elementary.

IF logic provides a viewpoint in which infinity of the domain of discourse does not affect the elementary character of the semantics of quantifiers. In IF logic quantifiers are taken to operate as choice functions. Their application to infinite domains does not involve infinite operations.[193] The game-theoretical truth condition for example does not presuppose infinite closed totalities. It says that a quantificational sentence S is true if and only if there exists a winning strategy for the initial verifier in the correlated semantical

[192] Hintikka 2006, section 14.7.
[193] See further Hintikka 1996, and Hintikka and Sandu 1996

game G(S). This truth definition does not involve quantification over the domain. It involves rather quantification over the initial verifier's strategies. Their totality is for sure dependent on the domain. But this totality is never invoked in the course of actually playing a semantical game.[194] Here we might be tempted to think that quantifying over a set of strategies (strategy functions) presupposes this set as a closed totality. However, the apparent quantification over infinite sets of strategies (Skolem functions) is cashed in by reference to complexes of choices of particular objects, in IF logic.

It might be objected that game-theoretical truth definition relies on quantification over higher-order entities too. That is to say, the truth of a sentence S means the existence of a winning strategy in the correlated game G(S) for the verifier. However, in asserting S one does not make an infinite choice. One merely asserts the existence of choice. Thereby no commitment to infinite operation is necessary. Infinite operations come about only in the non-elementary extensions of IF logic.

Infinite operations come about only in the non-elementary extensions of IF logic. IF logic is distinguished from its non-elementary extension by virtue of its game-theoretical semantics, as follows: Due to semantic independence of quantifiers in IF-sentences their game-theoretical truth condition brings about indeterminacy of the existence of winning strategies in some semantical games. As was pointed out, this indeterminacy means a failure of the law of excluded middle, i.e. S being neither true nor false in the associated game G(S). As a result, the law of excluded middle holds only for sentences in the fragment of IF logic consisting of the ordinary first-order logic. For the

[194] Cf. Hintikka 2006, section 14.7.

rest the law does not hold.[195] Accordingly there are two negations with different uses in IF-logic: (i) the strong (dual) negation ~, (ii) the contradictory negation ¬. Recall that the contradictory negation expresses the non-existence of a winning strategy for the verifier in a semantical game. And it is not allowed inside the IF-sentences.

When the contradictory negation is allowed inside the IF-sentences, the elementary character of the logic is lost. Allowing more and more complex nested contradictory negations, as has been indicated in chapter 11, results in at the limit what might be called fully-extended IF logic.[196] The properties of fully-extended IF logic can be studied by means of a generalized Skolem form. As also was indicated, a generalized IF sentence S, can be translated into its Skolem form. The resulting translation is an ordinary second-order sentence.[197] In it arbitrary occurrences of the contradictory negation ¬ can be interpreted by means of nesting of infinite games, or equivalently, by repeated uses of a substitutional truth-condition.[198] The resulting substitutional semantics, which allows application of the law of excluded middle, gives the non-elementary character of the resulting logic. As was indicated, this logic has the same expressive

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[195] Cf. Hintikka 1996.

[196] For more discussion on the fully extended IF logic, see Hintikka 2006.

[197] In such translation it is assumed that the strong negations ~ are pushed as deep into S as they can go. If an expression occurs within the scope of an occurrence of the contradictory negation ¬, but not within the scope of any ¬ in whose scope $¬_1$ does not also occur, we will say that the expression is in the immediate scope of $¬_1$. The translation rule says that any expression (∃x) F[x] in S is replaced by $F[f(y_1, y_2,\ldots)]$ where (Q_1y_1), (Q_2y_2), … are all the quantifiers in S on which (∃x) depends and f is a new function variable. At the same time the second-order quantifier (∃f) is inserted to follow immediately the contradictory negation sign ¬ in whose immediate scope (∃x) F[x] occurs. If there is no such sign, (∃f) precedes the entire sentence S. Since ~ has been pushed into formulas as far as it goes, we have in the translation strong negations only in the combinations ~A and ¬~A, where A is atomic. In mathematics, we can assume that the law of excluded middle holds for atomic sentences. Hence ~A and ¬~A reduce to ¬A and A, respectively, and all strong negations are eliminated.

[198] This amounts to the same as treating them by using infinite games. See Hintikka 2006, sections 14.6–14.8

power as the entire second-order logic (with standard interpretation).[199] What this shows is, all classical second-order reasoning, and therefore, virtually all mathematical reasoning can be codified by means of quantifiers whose values are individuals. This is in keeping with Hilbert's nominalistic attitude, in that it shows that all of mathematics can in principle be done on the first-order level. Higher-order entities are not quantified over. On this level mathematics can be described as the study of all possible configurations—one might qualify these as combinatorial facts[200]—of particular objects.

Moreover, based on the substitutional interpretation of the unlimited use of ¬ in IF sentences the expressive power of second-order logic comes not from the use of quantification over predicates, but from the use of unrestricted law of excluded middle. Therefore, what makes the crucial difference between first-order and higher-order is not the ontological (i.e. type-theoretical) status of the entities quantified over. The difference comes from how freely the law of excluded middle is being applied.[201] The elementary character of IF logic is thus shown by the way it avoids the deceits of the law of excluded middle, infinite semantical games, (and substitutional interpretation of quantifiers) and higher-order quantifiers.

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[199] Cf. Väänänen 2001 and Hintikka 2006. Note that unrestricted second-order logic is more than sufficiently strong for coding the entire classical mathematics. See further Shapiro 1985.
[200] Cf. Kreisel 1983.
[201] Cf. Hintikka 2006, pp. 212-213.

19. FIRST-ORDER SET THEORY

To a considerable extent IF logic removes the problem of infinitistic assumptions from quantification theory. And in its fully extended form it can replace first-order axiomatic set theory.[202] So first-order axiomatic set theory is dispensable in the logical foundations of mathematics. On the other hand, ordinary first-order logic should not serve as the basic logic for the axiomatization of set theory. The reason is that admittedly there are (ordinary) first-order set-theoretically true sentences which do not have Skolem functions.[203] When Skolem functions are considered to be truth-makers this is a serious problem.

In order to see the problem, following Hintikka's analysis of truth in set theory[204], the following observations can be made: Let AX be some first-order axiomatic set theory. Suppose elementary arithmetic and the syntax of AX itself can be formulated in AX. Also suppose $\varsigma\kappa(x)$ (structurally) describes the sentence SK that asserts the existence of a full array of Skolem functions for sentence S, as a function of the Gödel number x of S. (In the same sense, we can call $S^{sk}(x)$ as a function of Gödel number x of S.) Here $\varsigma(x)$ is S and $\varsigma\kappa(x)$ is SK, but expressed with their syntactic dependence on Gödel numbers. Then we can form a truth-definition for AX-sentences in AX itself:

(1) $(\forall x)\ (T(x) \leftrightarrow \varsigma\kappa(x))$

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[202] Cf. Hintikka 2004 and 2006.
[203] Cf. Hintikka forthcoming.
[204] For a detailed treatment of how truth and quantifiers should be studied from the IF logic perespective, see Hintikka forthcoming.

Here $\varsigma\kappa(x)$ is a translation of $\varsigma^{sk}(x)$ from second-order language into first-order set-theoretical language. Since it expresses the existence of Skolem functions this can be done. Now, (1) says that a set-theoretical sentence is true if and only if its Skolem functions exist. But this cannot serve as a truth definition here, due to Tarski's undefinability theorem. The following question arises: Is $\varsigma\kappa(x)$ a genuine truth predicate?

Now, if T(x) in (1) is really a set-theoretical truth predicate then (1) will be false. Since $S^{sk} \supset S$ can be proved in set theory, (1) must be false. This is enough to show the existence of set-theoretical truths which do not have Skolem functions. When Skolem functions are considered as to give the truth conditions of mathematical statements in a logical axiomatization, the conclusion of the above argument is that first-order axiomatic set theory cannot serve as a foundation of mathematics. IF logic is based on the idea that quantifiers are choice functions and those functions provide the truth conditions of the logical sentences. Hence from the IF perspective there has to be a way out from set-theoretical foundations.[205]

The above considerations about first-order axiomatizations of set theory are closely related to the so-called Skolem paradox implied by the Löwenheim-Skolem theorem: If the axioms of a (first-order) set theory have a model, then they have a countable model.[206] From this the following question arises: How can, for example, the power set of an infinite set (uncountable by definition) belong to a countable model? Expectedly, from Hilbert's optimistic point of view the so-called paradox here cannot be

[205] This observation was made and the point was emphasized by Hintikka 1996, where Hintikka calls first-order axiomatic set theory Fraenkelstein's monster.
[206] Cf. Skolem 1962, § 11.

a serious problem for the working mathematician. One suitable resolution is by way of the introduction of ideal elements: Let M be a countable model for an ordinary first-order set theory. In this theory one can prove that there is no bijection from the set of real numbers into the set of natural numbers. On the other hand, let M(R) and M(N) be countable models for the real and natural numbers respectively. Then there must be a bijection from M(R) into M(N). Now, we can say that there is no bijection from M(R) into M(N) defined in the domain of M. That is, there is no Skolem function for certain truths in M. On this point we can also say that the existence of a function which links M(R) and M(N) one-to-one can be given in an extended model ExtM. In M no such bijection is found. But in ExtM such a function can in principle be defined.[207] However, such domain extension cannot provide a firm foundation in Hilbert's combinatorial and model-theoretical sense. First of all it does not reach up to the standard interpretation of the set-theoretical universe. Skolem interprets this fact as to imply what might be called a set-theoreical relativism.[208] It provides at most a model-theoretically (descriptively) incomplete characterization. In other words, the domain extension by using an ExtM cannot provide a conclusive justification of the use of ideal elements and higher-order modes of reasoning that are allowed therein by way of extending the initial domain. From Hilbert's point of view a logical reconstruction of the higher-order modes of reasoning in such domain extension and hence a justification of the use of ideal elements by reference to the initial model M is needed. Such reconstruction, as has been seen, is possible by

[207] In fact this argument was part of Zermelo's response to Skolem's criticisms in the 1920s. See further Kanamori 2004.
[208] See Skolem 1962, § 11.

means of the resources of IF logic. IF logic can cope with higher-order reasoning on the first-order level. So from a Hilbertian point of view, the right approach in the logical axiomatization of mathematical theories can dispense with the use of ordinary first-order set theory and replace it by the fully extended IF logic.

As was pointed out fully extended IF logic is strong enough to replace second-order logic and hence axiomatic set theory. Hence set theory is dispensable as a foundation. This has no claim to weaken abstract set theory as a mathematical study of infinite structures. It only shows that the alternative idea of using choice functions as truth-makers can replace set-theoretical modes of reasoning in axiomatizations. Therefore, Hilbert's first-order view and nominalism is vindicated by IF logic in the sense that set-theoretical foundations of mathematics can be dispensed with, by means of a first-order level reconstruction of the apparently higher-order modes of reasoning.[209] Hilbert's aroused aim was not in any sense to dispense with set theory perhaps. But his aim was to investigate the extent that set-theoretical reasoning (just like other modes of mathematical reasoning) can be reconstructed on the first-order logical level. And as has been seen above IF logic provides such a reconstruction on Hilbertian line of thought, without appealing to problematic techniques of set theoretical reasoning.

[209] Cf. Hintikka 1997.

20. THE AXIOM OF CHOICE

The justification of the axiom of choice, which has been one of the primary debated issues in the foundations of mathematics, is a part of Hilbert's proposal to reconstruct the apparently higher-order modes of reasoning on the combinatorial (first-order) level. Indeed, Hilbert states his belief that the mode of inference underlying the axiom of choice was a logical principle:

> …the essential thought underlying the principle of choice is a general logical principle which is necessary and indispensable even for the most elementary rudiments of mathematical inference.[210]

Also in his 1922 paper Hilbert says that it must be possible to formulate the axiom of choice in such a way that it becomes as obvious as 2 + 2 = 4.[211]

As was indicated in Chapter 11, Hilbert recognized the close interrelation between quantifiers and choice functions. In fact he realized that the basic idea underlying the axiom of choice and quantification was one and the same. For example, Hilbert introduced his epsilon technique in order to capture the usual instatiation rules and the so-called axiom of choice. In the epsilon technique, an epsilon term $\varepsilon x A(x)$ stands for an individual x of which A(x) holds (if there are such individuals). And the logical axiom $A(x) \supset A(\varepsilon(A))$ contains according to Hilbert "the core of…the axiom of choice".[212]

Hilbert's aim to treat the axiom of choice and quantification in tandem has its roots in Hilbert's 1923 lectures. There he points out the close connection and his

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[210] Hilbert 1923, par. 4
[211] Hilbert 1922, par. 1
[212] Hilbert 1925, p. 382. One must note here that it is missing in Hilbert's axiom what the choice in question depends on. So it does not actually cover the core of the axiom of choice.

proposed solution (i.e. the epsilon technique) to the problems arising from quantification and choice:

> We have not yet addressed the question of the applicability of these concepts ["all" and "there is"] to infinite totalities. …The objections…are directed against the choice principle. But they should likewise be directed against "all" and "there is" which are based on the same basic idea.[213]

In line with his aim concerning the axiom of choice, Hilbert's main concern seems to have been to point out the need for a logic which is based on the same basic idea as the axiom of choice. In that sense the status of the axiom of choice is the paradigm case for Hilbertian foundations of mathematics.

As was mentioned Hilbert's approval of Zermelo's work was on the practical level and for the proper (mathematical) development of set theory. On the other hand there were problems concerning the logical foundations of the apparently set theoretical modes of reasoning and principles, which occupied Hilbert for his further foundational work through the following years. The status of the axiom of choice for the foundations of mathematics was one of those problems. It has to be reconstructed on the combinatorial (first-order) level.

When Zermelo introduced his axiom of choice in his 1904 and 1908 papers he argued that it was a self-evident and practically very fruitful principle. Zermelo's approach opened a debate on whether his axiom was mathematically acceptable. In the original formulation of the axiom of choice, Zermelo assumed that given a set of non-empty sets there is a function that takes each of the non-empty sets to one of its

[213] Quoted in Zach 2001, pp. 70-71.

elements.[214] The problem with this assumption according to the critics of Zermelo was that it was not determinate whether the outcome of the choice operation could be made out to accord with a finite definition (or a rule).[215] Thereof Zermelo's assumption was too strong, according to his critics. For example, Lebesgue raised the question:

How can Zermelo be certain that in the different parts of his argument he is always speaking of *the same* choice of distinguished elements, since he characterizes them in no way?[216]

On similar lines, Borel's main argument against Zermelo's assumption was based on considerations of reasoning about arbitrary choices:

[Zermelo's assumption] seems to me no better grounded than the following: 'To well-order a set *M*, it suffices to choose arbitrarily an element to which one assigns the rank 1, then another to which one assigns rank 2, and so on *transfinitely*, that is, until one has exhausted all the elements of *M* by the sequence of transfinite numbers.' Now, no mathematician could regard this latter reasoning as valid. It seems to me that the objection that one can raise against it apply equally well against any reasoning where one supposes an *arbitrary choice* to be made a non-denumerable infinity of times….[217]

Accordingly, Borel claimed that Zermelo's axiom might be acceptable only if its application is restricted to countable domains. Similarly, Peano, who pointed out that the

[214] Zermelo 1904, p. 184.
[215] In the original formulation of the axiom of choice, Zermelo assumed that given a set of non-empty sets there is a function that takes each of the non-empty sets to one of its elements. (Zermelo 1904, p. 184) It can be noted at this point that Lebesgue's question is answered by the use of Skolem functions in logical notation.
[216] Borel 1905b, 1086. (Here notice that Lebesgue's question is answered by the use of Skolem functions in logical notation.)
[217] Borel 1905a, pp. 1076-1077.

axiom of choice did not follow from logical principles, claimed that it had no place in mathematics.[218]

Zermelo's ultimate response to criticisms was a practical resolution:

Such an extensive use of a principle can be explained only by its *self-evidence*, which, of course, must not be confused with its provability. No matter if this self-evidence is to a certain degree subjective, even if it is not a tool of mathematical proofs, and Peano's assertion that it has nothing to do with mathematics fails to do justice to manifest facts.[219]

Zermelo further pointed out that the equivalents of his axiom were indispensably used by different mathematicians.[220]

Hilbert's approach to the subject was logically speaking more perceptive than Zermelo's. It was also a response to the objections. Hilbert believed that from a suitable point of view the reasoning behind Zermelo's axiom of choice could be justified. In fact he thought that it was closely related to the problems with applications of quantifiers to infinite domains:

The objections … are directed against the choice principle. But they should likewise be directed against "all" and "there is", which are based on the same basic idea.[221]

Hilbert tried to give the axiom of choice (as well as to the application of quantifiers) a firm footing by the "logical ε-axiom"[222]:

(1) $A(x) \supset A(\varepsilon(A))$

[218] Cf. Kennedy 1973, chapter XVIII
[219] Zermelo 1908, p. 187.
[220] See Zermelo 1908 and Moore 1982 for examples and further discussion.
[221] Hilbert's and Bernays' 1923 lectures. Zach's translation; cf. Zach 2001, pp. 70-71.
[222] See Hilbert 1926, Hilbert 1928, and Hilbert and Bernays 1934-39.

Hilbert put the ε-function (or strictly, ε-functional) to use for different purposes. His main goal was to use it in consistency proofs. By its means he defined universal and existential quantifiers:

(2) $(\forall x)\ A(x) \leftrightarrow A(\varepsilon(\neg A))$

(3) $(\exists x)\ A(x) \leftrightarrow A(\varepsilon(A))$

On this basis Hilbert formulated universal instantiation, and the law of excluded middle (as a quantifier rule):

(4) $(\forall x)\ A(x) \supset A(x)$

(5) $\neg\ (\forall x)\ A(x) \supset\ \ (\exists x) \neg\ A(x)$

The ε-function could also serve to pick witness individuals for those propositions which hold for one and only one individual. If A(x) is one such proposition, then there obtains

(6) $x = \varepsilon(A)$

Most notably the ε-function could take the role of a choice function.[223] In case A(x) holds for more than one object, ε(A) is one of those objects x of which A(x) holds. This is where Hilbert's logical ε-axiom was intended to cover the main idea behind the axiom of choice. At the same time it was also a tool for instantiation in the sense that the value of a ε-function for a predicate A is an individual for which A holds (if it holds for any).[224]

Here, based on Hilbert's epsilon definition of the existential quantifier (viz. (3) above) the following can be stated:

(7) $(\forall x)(\exists y)\ A(x, y) \supset (\forall x)\ A(x, \varepsilon(A(x, y)))$

Here (7) can be read as to capture a nominalistic formulation of the axiom of choice, since it asserts a choice from any given domain {y: A(x, y)}, where ε(A(x, y)) designates the (arbitrarily) chosen individual.

The problem with Hilbert's epsilon calculus is that it assumes (in its day) that ordinary first-order logic is the basic logic. (The definition of quantifiers and instantiation rules are given by Hilbert for the ordinary (Hilbert-Ackermann) first-order logic.)[225] More specifically, it assumes that an epsilon term depends on all the outside universal quantifiers; since an epsilon term does not indicate what it formally depends on. Because of this assumption epsilon functions, although they seem to capture the intended force of the axiom of choice in meaning, they cannot serve its intended purpose as the paradigm

[223] This is not completely true though; for it is not indicated in the epsilon term what the choice is based on.
[224] Cf. Hilbert and Bernays 1939, p. 12.
[225] See Hilbert 1926 and 1928.

case for developing a combinatorial interpretation of quantification theory that Hilbert seems to have aimed at.

Practically, Hilbert put his epsilon technique in use for several aims, including: formulating the axiom of choice as a logical principle, explaining applications of the quantifiers, and proving the consistency of mathematical theories. The fundamental idea of the epsilon technique for consistency proofs is to make use of epsilon functions in producing quantifier-free true formulas. Any consistency proof has to include a proof that each such quantifier-free formula is correct:

> In proving consistency for the ε-function the point is to show that from a given proof of $0 \neq 0$ the ε-function can be eliminated, in the sense that the arrays formed by means of it can be replaced by numerals in such a way that the formulas resulting from the logical axiom of choice by substitution, "the critical formulas", go over into "true" formulas in virtue of these replacements.[226]

Given a mathematical proof formulated in the epsilon calculus, each epsilon term occurring in the proof is assigned a numerical value. The aim of this procedure is to transform all the uses of epsilon axioms (as well as the axioms of AX of the theory in question) into quantifier-free formulas in finitely many steps. Since epsilon-terms are used in a proof finitely many times, this must have seemed to Hilbert to be possible.

However, values that are assigned to different epsilon terms depend on each other due to the nested structure of epsilon terms in some formulas. Since in the usual notation of first-order logic scopes are nested, quantifier dependence is eventually packed into epsilon dependence and it creates difficulties in assigning numerical terms for nested

[226] Hilbert 1928, p. 477.

terms. For example, the values that we assign to the inner epsilon terms might necessitate changes in the previous assignments. Later assignments might turn the correct formulas into incorrect formulas. Thereby the nested structure of the assignment process might divide into branches (and loops on the branches) so that the substitution procedure might never come to an end.

In any case, due to Gödel's second incompleteness result we cannot reach true (numerically correct) formulas by means of the epsilon technique. For if we could, then this would give us a consistency proof for the axioms of number theory; since the theorems of number theory would then also be numerically correct. Such a consistency proof (assuming that the underlying logic is the ordinary first-order logic) is impossible to carry out. Therefore, Hilbert's epsilon calculus cannot serve its intended purpose.

One can follow a similar line of thought to Hilbert's epsilon technique, in second-order logic too. Since the job of the epsilon functions can be done by Skolem functions as well. One can start with the general observation that for each choice value x a natural number y can be found such that y is correlated in some way to x:

(8) $(\forall x)\ (\exists y)\ A(x,y)$

This can be taken here as to imply the existence of a function f such that for every x, f produces a term out of x. Thereby one can obtain a second-order form of the axiom of choice:

(9) $(\forall x)\ (\exists y)\ A(x,y) \supset (\exists f)\ (\forall x)\ A(x,f(x))$

which is a second-order logically valid formula. In fact, it is also the same general formulation of the axiom of choice as in Hilbert and Bernays 1934.[227]

The same line of thought can be even traced back to operating (only) with first-order quantifiers. If we allow functional instantiation in (8) and write:

(10) $(\forall x)\ A(x, f(x))$

The step from (8) to (10) is enough in principle to capture Hilbert's main idea in putting epsilon functions in use. Just like Hilbert's epsilon function, any arbitrary function-name can be considered as to pick (ideally) an individual from a given domain.

In IF logic (and hence on the first-order level) it can be shown that such principle is in fact logically true. In the second-order formulation of a general choice principle such as (9) the existentially quantified function f can be cashed in by independent choices of individuals. That is, the second-order sentence $(\exists f)\ (\forall x)\ A(x,f(x))$ is translated to

(11) $(\forall x_1)(\forall x_2)(\exists y_1/\forall x_2)(\exists y_2/\forall x_1)(((x_1=x_2) \supset (y_1=y_2))\ \&\ A[x_1,y_1]\ \&\ A[x_2,y_2])$

Here if we use Skolem functions (11) is equivalent with:

[227] Hilbert and Bernays 1934, p. 41

(12) $(\exists f_1)(\exists f_2)(\forall x_1)(\forall x_2)\ (((x_1=x_2) \supset f_1(x_1) = f_2(x_2)) \supset A(x_1, f_1(x_1))\ \&\ A(x_2, f_2(x_2)))$

Thereby the choices which are expressed by Skolem functions in (12) are reduced to suitable operations by means of quantifiers and their dependency relations. Then we have the following IF formulation of the axiom of choice:

(13) $(\forall x)\ (\exists y)\ A(x,y) \supset (\forall x_1)(\forall x_2)(\exists y_1/\forall x_2)(\exists y_2/\forall x_1)(((x_1=x_2) \supset (y_1=y_2))\ \&$

$A[x_1,y_1]\ \&\ A[x_2,y_2])$

What has been achieved in (13) is the conclusion that the way quantifiers operate on the first-order level provides a suitable framework, as Hilbert seems to have thought, to place the reasoning behind the axiom of choice in its appropriate logical context. Thereof the apparently second-order reasoning behind the axiom of choice is translated to (combinatorial) first-order reasoning. To that extent Hilbert's aim to show that Zermelo's axiom of choice is as a logical truth can be thus achieved, although by a technique different from his.

In fact this result is the paradigm case for IF logic as much as it seems to have been for Hilbert's theory of quantification. The reason is that the existence of Skolem functions is dependent on the status of the axiom of choice. The argument that has been carried out shows that quantification and the axiom of choice are really based on the same basic idea, as Hilbert expressed.

21. THE LAW OF EXCLUDED MIDDLE

Hilbert's nominalism and his reasons for putting the epsilon calculus to use were for the sake of giving a combinatorial account of mathematical proofs. He wanted to construe all mathematical reasoning as concrete (algebraic) manipulation on the first-order level. However, such manipulations are not possible in ordinary first-order logic, which assumes the law of excluded middle. This can be explained by means of the applications of the tableau method to ordinary first-order reasoning. In a tableau proof (in the sense of Beth 1955) everything on the left side of the tableau is assumed to be given (known). Everything on the right side is assumed (for the sake of the argument) to be false. Hence the introduction of new individuals into the argument on the left means simply introducing an example of individuals known to exist. But the mirror-image instantiation on the right side means introducing an individual that only exists on the assumption that the desired conclusion is false — which is known to be not the case. Hence such a step cannot be interpreted as a concrete operation on the given objects. The problem here is due to the use of the law of excluded middle as the source of the need of the right side in a tableau proof. Assuming the ordinary first-order logic as basic, Hilbert's approach is subject to Brouwer's intuitionist criticism of the law of excluded middle.

Brouwer introduced his intuitionistic approach to the foundations of mathematics in his 1907 dissertation. He defended what might be called an intuitive genesis of mathematical objects and criticized different classical approaches to the foundations of mathematics. The targets of Brouwer's criticisms were including Hilbert's axiomatic approach and his early consistency program (mainly Hilbert 1900 and 1905). In his 1908

paper Brouwer argued against the unlimited applications of the law of excluded middle in mathematical proofs and rejected the law on intuitionistic grounds. In 1912 he began criticizing what he called the formalist approach with a stronger voice. Brouwer's views were recognized as truly revolutionary in some mathematical circles. Based on the criticism of the law of excluded middle—plus some epistemological presuppositions[228]—he and his followers (most notably Weyl) rejected a significant portion of classical mathematics in favor of intuitionistic mathematics. This was more than enough to disquiet Hilbert, who argued from the beginning that classical mathematics was on safe ground and this could be proved by his foundational program.

The status of the law of excluded middle as a problem concerning the admissibility of logical operations has a central role in the Hilbert-Brouwer controversy. Brouwer's rejection of several important results in classical mathematics was based on his criticism of the law of excluded middle. On the other hand, Hilbert tried to show through his consistency program that no restriction to our logic was necessary. According to him classical results were on safe ground.

Some of the combinatorial aspects— that Brouwer's criticism illuminates—of the problems with the unlimited use of the law of excluded middle can be outlined as follows: To prove, for example, whether a mathematical statement in the logical form

(1) $(\forall x)(\exists y)\ A[x, y]$

[228] See van Dalen 2000 for an introduction to Brouwer's intuitionistic ideas.

is a theorem, it is tempting to assume, by appealing to the law of excluded middle, that

(2) $(\forall x)(\exists y)\ A[x, y] \vee (\exists x)(\forall y)\ \neg A[x, y]$.

Then we could prove (1) by refuting the second disjunct of (2). Separating the cases as in (2), nevertheless is for the intuitionist, too strong an assumption to begin with, for we might still not have sufficient given information about A, for a procedure of finding y on the basis of x in (1). The problem is how to keep the application of the law of excluded middle on the concrete level of reasoning. Hence values to consider for x and y must be accessible by means of concretely admissible procedures.

Now, according to Brouwer, every mathematical assertion that is considered to be a finitely-bounded possible mathematical construction can be "judged" whether correct or incorrect. Therefore, there is no problem for Brouwer with the application of the law of excluded middle for such constructions. However, for properties for which there is no known way of finding out whether a mathematical object can have that property (assuming that for some objects the property is known to hold) the law of excluded middle is not applicable.

Along these lines, the law of excluded middle cannot be taken as to have unproblematic application over (intuitionistically) problematic entities. Brouwer in his Cambridge Lectures considers the decimal expansion of π to provide an example for his thesis. He asks whether there can be found a natural number x such that in the decimal expansion of π the digits x, x + 1, x + 2 … x + 8, x + 9 are identical with 0, 1, 2, …, 8, 9.

Brouwer claims that on intuitionistic grounds there is no method to decide this. Hence there is neither a positive nor a negative solution of the problem according to Brouwer. Therefore, the law of excluded middle fails and the statement that "…in the decimal expansion of π a sequence 0123456789 either does or does not occur" has no mathematical sense".[229]

Brouwer's rejection of several important results in classical mathematics was based on his criticism of the law of excluded middle. For him the law of excluded middle was a dogma that has its origin in finite mathematics and applied to infinite domains without justification:

> [It has its] origin in the practice of first abstracting the system of classical logic from the mathematics of subsets of a definite set, and then attributing to this system an a priori existence independent of mathematics, and finally applying it wrongly—on the basis of its reputed a priori nature—to the mathematics of the infinite sets.[230]

From Hilbert's point of view there is no good reason to follow Brouwer's line of thought. Instead one should try to show through consistency proofs that no restriction to the classical mathematical reasoning is necessary.[231] Hilbert formulated the law of excluded middle as a consequence of his logical ε-axiom. This was his way to avoid its unrestricted use in proof theory. However, as was pointed out, Hilbert's epsilon calculus cannot fulfill its purposes.

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[229] van Dalen 1981, p. 6

[230] Brouwer 1921, p. 27.

[231] Brouwer's investigation of the genesis of mathematical reasoning could have no mathematical significance for Hilbert. Brouwer was pushing it too far the informal inquiry concerning the foundations of a formal discipline like mathematics. Thereby he was confusing the job description of a mathematician with that of a philosopher. Therefore, from Hilbert's point of view, Brouwer's claims cannot be taken as a foundation for mathematics. Let alone a restriction to mathematical methods. Hilbert's axiomatic approach was strictly anti-intuitionistic in that sense.

On Brouwer's side, the efforts were made not for a justification of the law of excluded middle of course, but in order to avoid its wrong way of applications in mathematical reasoning. For example, in Brouwer's 1923 paper it is argued that "the principle that for every system the correctness of a property follows from the impossibility of the impossibility of this property" does not hold in every application.[232] Being faithful to his own philosophical views concerning the uselessness of logical symbolism, Brouwer did not appeal to symbolization in his argument. However, his paper opened up further possibilities for the interpretation of the so-called logic of intuitionism.[233] Brouwer approved Heyting's work as the authoritative account of what might be called intuitionistic logic.[234] Heyting in his 1930a and 1930b papers introduced the basics of intuitionistic logic and gave the list of formal rules of it.

According to Heyting 1930a, the intuitionistic conception of assertion is different from the classical conception. It is different in the sense that the classical assertion of a proposition corresponds to a semantic situation whereas the intuitionistic conception corresponds to an epistemic situation. For example, when we assert A, in the classical conception it has to do with whether A is true. For the intuitionist it has to do whether it is known to us how to prove A.[235] The intuitionist then, according to Heyting, take truth claims that are independent of our knowledge to be mathematically dubious.[236] This is usually taken to mean that in intuitionistic mathematics only provable propositions are

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[232] Brouwer 1923, p. 335.
[233] For Brouwer 1923 and some important contributions to the debate, see Mancosu 1998, part IV.
[234] Mancosu and van Stigt 1998, p. 277.
[235] Heyting 1930a, p. 307.
[236] Heyting 1974, p. 87.

truths.[237]

It is misleading to assume however that in Brouwerian intuitionism only provable propositions are truths. Brouwer in his 1954 corrigenda to Brouwer 1923 distinguishes between "testing" and "judging" mathematical assertions. The former corresponds to showing the contradictoriness or uncontradictoriness of an assertion. The latter corresponds to showing the presence or the absurdity of an entity. In the logical framework Brouwer's distinction has to be taken into consideration as a distinction between two different notions of mathematical truth.[238] It would simply be dismissive of the intended distinction to consider testing and judging both as proving.

Equating truth and provability would be misleading also for the following reason: The real gist of the problematic applications of the law of excluded middle (according to Brouwer) lies in our knowledge of combinatorial matters, not in merely whether A or ¬A can be proved. The main concern for the intuitionist is what we know and what we do not know about the existence of certain mathematical entities. If we partly adopt Heyting's wording, asserting the existence of certain mathematical entities, which are considered to be independent of our knowledge is intuitionistically problematic.

An intuitionist has to be careful in interpreting the meaning of quantifiers and connectives. Quantifiers operating as choice functions might already involve non-intuitionistic elements (as Hilbert also pointed out; see the relevant quotation in chapter 20). What this shows is that an enriched treatment of the logic that was postulated by Heyting is needed in order to capture the different ways of finding out and distinguishing

[237] Cf. Artemov and Beklemishev 2004.
[238] Cf. Brouwer 1923, § 1 and p. 341.

intuitionistically true propositions of mathematics. There is no guarantee that approaching only the formal axiom schema critically from an intuitionistic perspective will provide the correct logical resources for intuitionistic purposes. The meaning of quantifiers and connectives has to be reexamined.

Although the key role of the law of excluded middle was recognized both by the intuitionists and by Hilbert, its precise role seems to have been not fully understood. It is not clear that the precise limitations that Brouwer intended have been captured by any explicit logical formulation. On the other hand, Hilbert did not give a full satisfactory justification for the unproblematic uses of the law in mathematics.

The status of the law of excluded middle in IF logic, however, seems to clarify many points in the Hilbert-Brouwer disagreement. The law of excluded middle plays a key role in proceeding from (elementary) IF logic to its non-elementary extensions. This turns out to be fitting into Brouwer's diagnosis. That is, the source of the non-elementary character of classical mathematics is in the unrestricted use of the law of excluded middle. This point illustrates where the borderline goes between elementary and non-elementary methods in the foundations of mathematics.

On the other hand, recognizing the borderline clarifies conceptually what was wrong with the logic that Heyting formulated for intuitionistic purposes in 1930. It was shown independently by Gentzen 1933 and Gödel 1933 that classical first-order arithmetic could be reduced to Heyting arithmetic. Hence the following question arises: Is Heyting's intuitionistic arithmetic really deductively weaker than the classical one? If not, what type of non-intuitionistic elements might have survived in Heyting's

intuitionistic logic? The borderline that was drawn between IF logic and its non-elementary extensions shows that whatever nonintuitionistic resources were allowed unnoticed in Heyting's axioms, they have to do with the different meanings of finding out the truth value of a mathematical assertion. Given an interpreted first-order sentence, finding out its truth value may mean two different things: (i) It may mean to find out the witness individuals that make the sentence true. (ii) It may mean to find out the (Skolem) functions that guarantee the success of the procedure in (i).[239] It is not immediately clear whether this distinction amounts to making a similar distinction to that of Brouwer between testing and judging. Yet both distinctions are intended partly to clarify logically the combinatorial situation in different applications of quantifiers and connectives as well as the law of excluded middle. The distinction between (i) and (ii) is a direct consequence of the interpretation of quantifiers as choice functions. The intuitionist objections as Hilbert suggested must be to the meaning of quantifiers, not to the axiom of choice, for example.[240] Accordingly, criticisms of the law of excluded middle have to be reconsidered.

A formal comparison between extended IF logic and Heyting's intuitionistic logic supports this point. On the propositional level, the two logics share the same modal structure, viz. S4.[241] On the other hand, IF logic where the law of excluded middle is dispensed with is deductively weaker than Heyting's logic. It can be considered as more

[239] Cf. Hintikka 1998, pp. 334-335, and Hintikka 1996 Chapter 2.
[240] Cf. Zach 2001, pp. 70-71
[241] This is seen from the fact that extended IF logic is a Boolean algebra with an operator in the sense of Jonsson and Tarski 1951-52. See further Hintikka 2004.

faithful to Brouwer's ideas. Therefore, Hilbert's combinatorial approach to proof analysis can be carried out without being subject to Brouwer's criticisms.

Hilbert's preference of combinatorial methods was not merely for avoiding the criticisms; above all, it was to be able to understand the actual process of our mathematical reasoning:

> The fundamental idea of my proof theory is none other than to describe the activity of our understanding, to make a protocol of the rules according to which our thinking actually proceeds.[242]

In search for such a protocol, one of the key problems was to understand the role of negation in logical and mathematical reasoning:

> It is one of the most important tasks of proof theory to present clearly the sense and admissibility of negation: negation is a formal process, by means of which, from a statement S, another arises, which is bound to S by the axioms of negation mentioned above (essentially, the principle of contradiction and *tertium non datur*).

This point, viz. the connection between negation and the law of excluded middle and the law of contradiction has to be considered critically. On this point negation is to be considered as an ideal process and not a real one in mathematical reasoning:

> The process of negation is a necessary means of theoretical investigation; its unconditional application first makes possible the completeness and closure of logic. But in general the statement arises through negation is an ideal statement, and to take this ideal statement as being in itself a real statement would be to misunderstand the nature and essence of thought.[243]

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[242] Hilbert 1928, p. 475.
[243] Hilbert 1930, par. 38.

This line of thought is carried out in Hilbert and Bernays 1934. There Hilbert and Bernays state that for elementary statements either the statement itself or its negation is correct, since their value can be determined by "intuitive finding".[244] What about negation of universal and existential statements? Consider, for example:

(3) $\sim(\exists x)\ A(x)$

It means that one has no object available for indicating that it is A. For Hilbert and Bernays it has no objective meaning due to its "epistemological" condition, which has to be avoided anyway. Accordingly, one can say that an object *cannot* have the property A. This kind of move gives more to the meaning of negation, by adding the modality "cannot". Thereof Hilbert and Bernays consider it to bring in a "sharpened" negation. In this regard it is not considered any longer as the contradictory negation of an existential statement. The existential statement

(4) $(\exists x)\ A(x)$

amounts to saying that an object is obtainable by means of a (search) procedure. It follows from this that (3) and (4) signify different procedures. And since ~ is taken to be the sharpened negation (and not the contradictory one), the law of excluded middle fails in Hilbert and Bernays' approach.

[244] Hilbert and Bernays 1934, § 2

Hilbert and Bernays consider the failure of the law of excluded middle by distinguishing between the applications of the sharpened negation, namely, in finite and infinite domains. Here the difference of Hilbert and Bernays' approach and the IF logic approach is the following: From the IF logical point of view, Hilbert and Bernays were right in distinguishing between two negations (contradictory and sharpened negations; just like the contradictory and strong negation of IF logic). Game-theoretical interpretation of quantifiers in the IF logic shows that different applications of negation have to do rather with the notion of quantifier dependence, not with the finiteness or infinity of the domain of discourse. The game-theoretical interpretation of quantifiers keeps the law of excluded middle out of the ways in which quantifiers operate. Thereby the different uses of the two negations are sharply separated.

The situation changes when the law of excluded middle is applied without restrictions to logical analysis of mathematical proofs. If mathematical proofs are analyzed by using (fully extended) IF formalizations, where and how the law of excluded middle enters in the mathematical reasoning would be one central aspect of such analysis. In ordinary first-order logic, the law of excluded middle enters in reasoning in the use of cut rules. And all applications of cut rules can be eliminated in that logic.[245] What is missing in it is a method to investigate non-cut-free rules in mathematical reasoning. Non-cut-free rules can be studied directly however in the fully extended IF logic. By means of the reconstruction of second-order logic (by extending IF logic so as to capture all mathematical reasoning), it seems possible (at least in principle) to recognize the uses

[245] Cf. Gentzen 1935.

of the law of excluded middle (via the unrestricted uses of the contradictory negations) directly in (higher-order) mathematical reasoning. This task can be considered also as an extension of Hilbert's consistency program.

22. EXTENSIONS OF HILBERT'S PROGRAM

In 1936, Gentzen gave a proof of the consistency of arithmetic. Gentzen in his proof had to appeal to the principle of transfinite induction, which states that if for all ordinal numbers x preceeding y, x has the property A (A being a property defined for all ordinals), then y is A. Gentzen first showed how to assign ordinal numbers (less than ε_0) to each arithmetical proof so as to make a list of the proof figures. Then he showed, in case a contradictory formula $0 \neq 0$ occurs as the end formula of a proof, the well-ordering of the proof figures would guarantee that the ordinal assigned to the proof ending with $0 \neq 0$ has been kept out of the list. Gentzen's assignment of ordinals to proof figures is in order to give a measure for the complexity of arithmetical derivations.[246] And there, the well-ordering of the proof figures is "of a special kind", to put in Gentzen's words. This special character is seen as follows: Suppose one gets some formula of the form

(1) $(\forall x)\, F[x]$

(by induction) as the end-sequent of a proof figure. Such proof figures must be considered as more complex than its infinitely many particular (substitution) instances. Thereof the measure of complexity of those proof figures will be inevitably higher than what can be ordered by using ordinary induction. That is why transfinite induction is needed to measure the complexity of all arithmetical derivations. Ultimately, the needed

[246] Cf. Gentzen 1936, p. 186.

ordinals reach up to ε_0 (as Gentzen shows in his paper) in order to make a list of all the proof figures in elementary arithmetic.[247]

Gentzen claimed about his consistency proof that the induction principle he used in the proof—more specifically transfinite induction up to the ordinal ε_0—was harmless such that between the ordinals ω and ε_0 "nothing new ever happens"; i.e. every ordinal in between can be represented by means of primitive recursive relations.[248] As was later remarked by different logicians[249], about Gentzen's approach, one might find a high degree of intuitiveness about the induction on ε_0. However, this kind of intuition is hardly found acceptable here under Hilbert's insistence on (first-order) concrete content in logical and metamathematical methods, since "transfinite induction means always a detour via an infinite set", to put it in Gauthier's words.[250] With the help of a game-theoretical interpretation of Gentzen's proof, the non-elementary assumptions that are brought in by the application of transfinite induction can be made clear, as follows: Gentzen's reductions of given complex formulas to true atomic formulas—formulas are obtained by the usual natural deduction rules—can be considered as winning strategies in proof-games for the verifier, if we borrow some game-theoretical terminology.[251] Then a Gentzen-style proof game can be thought of as played between a verifier and a falsifier, as in the usual game-theoretical semantics. The difference is that in the game-theoretical semantics as it is usually defined for IF logic, the existence of winning strategies is not always determinate. This indeterminacy is due to the failure of the law of excluded

[247] See Gentzen 1936, § 15.
[248] Gentzen 1936, pp. 195-197.
[249] Especially Gödel, Bernays and Max Black .
[250] Gauthier 2002, 101.
[251] For a game-theoretical treatment of Gentzen's consistency proof, see Tait 2005.

middle. In Gentzen's case however, strategies are recursively determined. Due to the same reason as why that transfinite induction is needed for measuring proof figure complexity, the reductions in proof games (and hence winning strategies) will inevitably be consisting of verifier's infinite backtracking of falsifier's moves. This will in turn bring in the non-elementary assumption that all the substitution instances of quantified sentences in the proof games must be available to verifier's information and the verifier can try them infinitely often in order to obtain reductions. Essentially, this (infinitistic) reduction procedure is as non-elementary as the infinite semantical games (of chapter 11) for the fully extended IF logical sentences, where the unrestricted use of the law of excluded middle is allowed.

Beside its non-elementary assumptions, Gentzen's contribution has been taken as a reason to search for other means and techniques than were seen appropriate by Hilbert, to learn more about the proof-theoretical structure of mathematical inference. To extend the methods of proof theory, as Bernays claimed:

> Instead of restricting to finitist methods of reasoning, it was required only that the arguments be of a constructive character, allowing us to deal with general forms of inferences.[252]

The main aim by following Gentzen's strategy in pursuing a similar program to that of Hilbert's is to see in a mathematical theory, what derivability results (including whether or to what extent the consistency of the theory is derivable relative to another theory) we can establish by using other appropriate means than Hilbert's so-called finitistic

[252] Sieg 1999 quotes Bernays 1967.

methods.[253] What there is in the works of Kreisel, Feferman and Simpson is proposals which kept their main ideas on this line, for example by trying to reduce proof-theoretically what can be done in one deductive system to another.[254]

Proof-theoretical reductions provide proof analyses and certain derivability results. Analysis of proofs and the derivability results are relative to further mathematical theories. Admittedly they cannot go beyond establishing the relative consistency of a mathematical theory, which is (from the conceptual point of view) less than what Hilbert hoped to have reached.

Proof-theoretical reductions are (or are based on) important mathematical results. However they cannot be taken as realizations of Hilbert's aims, as long as they do not clarify the notion of quantification. As has been argued, Hilbert's primary aim was to formulate a humanly practicable account of the apparently infinitistic operations, by improving a theory of quantification better than the traditional conceptions. None of the proof-theoretical reductions have a claim to provide such clarification. Consequently, they do not provide explicit criteria for how one can achieve Hilbert's aims. As was argued in chapters 12-21 such criteria is provided by the resources of IF logic. Thereby, Hilbert's different problems in the foundations of mathematics can be carried out on the basis of IF logic.

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[253] Cf. Kreisel 1958, p. 155, Sieg 1988, p. 343.

[254] See Simpson 1988, Feferman 1988 and Kreisel 1983. There are also other proof-theorists such as Schütte and Takeuti who followed Gentzen's steps, by going up in the stronger mathematical systems and prove their consistency by appealing to the needed induction principle (i.e. up to the correspondent ordinal number). For a brief survey of such developments see Feferman 1988 and 2000.

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