Stability Analysis of Systems With Time-Varying Delay via Improved Lyapunov–Krasovskii Functionals

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***Abstract*—This paper is concerned with the delay-dependent stability analysis of linear systems with a time-varying delay. Two types of improved Lyapunov–Krasovskii functionals (LKFs) are developed to derive less conservative stability criteria. First, a new delay-product-type LKF, including single integral terms with time-varying delays as coefficients is developed, and two stability criteria with less conservatism due to more delay information included are established for different allowable delay sets. Second, the delay-product-type LKF is further improved by introducing several negative definite quadratic terms based on the idea of matrix-refined-function-based LKF, and two stability criteria with more cross-term information and less conservatism for different allowable delay sets are also obtained. Finally, a numerical example is utilized to verify the effectiveness of the proposed methods.**

***Index Terms*—Allowable delay set method, delay-product-type functional, stability, time delay system, time-varying delay.**

I. Introduction

TIME-VARYING delays widely exist in many systems, such as neural networks, power systems, genetic networks, and telerobotic systems [1]–[7]. In general, the existence of time delay would degrade the performance of systems and even cause instability. Stability is the fundamental requirement of systems, thus stability analysis of time-delay systems is of certainly theoretical and practical significance. For the time-delay systems, the main objective of stability analysis is to compute a maximal admissible delay region, within which the systems remain asymptotically stable [8]. As mentioned in [9], the Lyapunov–Krasovskii functional (LKF) method, which can lead to stability criteria in the form of linear matrix inequalities (LMIs), is the most popular and effective method for the analysis of systems with a time-varying delay. However, the stability criteria obtained are usually sufficient and have conservatism more or less. Moreover, it is well known that the conservatism of stability criteria mainly roots in two aspects: 1) the construction of the LKF and 2) the estimation of the derivative of the LKF [10]. Therefore, the stability analysis of time-delay systems mainly focuses on deriving less conservative delay-dependent criteria from those two aspects.

In order to obtain delay-dependent stability criteria, the LKF candidates are usually required to contain a double integral term, whose derivative would lead to a single integral term. In order to express the final stability criteria as tractable LMIs, the key issue is to estimate such integral term through necessary bounding techniques [11]–[14]. The model transformation method [15], the free-weighting matrix method [16], [17], and the integral inequality method [18] are three main types of bounding techniques, and the first two types were used in early research. Currently, the integral inequality-based direct bounding method has became the most popular technique, especially after the development of various reciprocally convex combination lemmas [19], [20]. Among them, the Jensen inequality [18] was widely used before 2013. After that the Wirtinger-based inequality [11], the auxiliary function-based inequality [21], and the free-matrix-based integral inequality [10] were proposed to improve the results. Later, the Bessel–Legendre inequality provided a general form of the single integral inequality [22], [23], which has potential to eliminate the estimation error of single integral terms, and it seems that there is limited room remained through developing tighter single integral inequalities to reduce the conservatism of criteria. The another idea of reducing conservatism is to make the LKF candidate more general in comparison to the simple one used in early researches. By improving different parts of the simple LKF, the delay-partitioning LKF approach [24], [25], the augmented LKF approach [26]–[31], and the multiple integral LKF approach [32]–[37] were proposed to improve the results by containing more information of system states and time delays. Those types of LKFs reduce the conservatism of the derived stability criteria to a certain extent. However, it is found that with the increase of the complexity of those LKFs, the reduction of conservatism becomes less obvious while the computational burden greatly increases.

Instead of simply complicating different parts of simple LKF like the aforementioned LKFs do, two novel ways for the LKF-construction were proposed very recently. In [38], a so-called delay-product-type LKF was developed by introducing

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several delay-product-type nonintegral terms, which have coefficient with information of time-varying delays. The derivative of this LKF will lead to several terms connected to both time delay and its derivative so as to well reflect the delay information, which in turn contributes to reduce the conservatism of the stability criteria. In [39]–[42], by introducing several coupled terms obtained by integral inequalities, a so-called matrix-refined-function-based LKF was developed. The coupled terms contain some without positive-definite requirement quadratic terms, the derivative of them leads to several cross terms, and both of these features make the LKF more effective to reduce the conservatism. Although those LKFs show the potential contribution to reduce conservatism, they are only studied in a few literature and are worth more deeply considering. This motivates this paper.

Based on the above discussion, this paper further investigates the delay-dependent stability analysis of the systems with a time-varying delay. By following and extending the idea of the delay-product-type LKF and the matrix-refined-function-based LKF, two new types of LKFs are developed. Those LKFs, together with an extended reciprocally convex matrix inequality and two types of allowable delay sets, lead to four stability criteria with less conservatism in comparison to the existing ones. The effectiveness of the proposed criteria is verified by a numerical example. The main contributions of this paper are the developments of two novel LKFs, summarized as follows.

1) A new LKF containing delay-product-type single integral terms is developed inspired by the authors' pervious work [38]. In the derivatives of the delay-product-type single integral terms, besides some delay and delay variation dependent cross terms, the delay derivative related single integral terms would also be included. Namely, the new LKF leads to novel type of single integral terms in its derivative. During the estimation of the derivative of the LKF, the delay derivative related single integral terms introduce many additional delay and its derivative dependent cross terms, but require less decision variables in comparison to delay-product-type quadratic terms. That is, the delay information can be reflected much well in the obtained criteria (see Remarks 2 and 3 for details).
2) By extending the idea of the matrix-refined-function-based LKF, an improved delay-product-type LKF is developed through introducing a negative definite quadratic term. In comparison to the existing matrix-refined-function-based LKFs [40]–[42], the proposed LKF has two advantages. First, instead of coupling the negative definite quadratic terms with the single integral terms, the negative definite quadratic term is introduced alone, which reduces the computation complexity. Moreover, the reciprocally convex lemma is first employed to construct the negative definite term, and it can provide extra freedom to the stability criteria (see Remark 4 for details).

Notations: Throughout this paper, the superscripts T and -1 mean the transpose and the inverse of a matrix, respectively; $\mathcal{R}^n$ denotes the n-dimensional Euclidean space; $\|\cdot\|$ refers to the Euclidean vector norm; $P > 0$ (≥ 0) means P is a real symmetric and positive-definite (semi-positive-definite) matrix; I and 0 stand for the identity matrix and the zero-matrix, respectively; symmetric term in the symmetric matrix is denoted by $*$; and $\mathrm{Sym}\{X\} = X + X^{\mathrm{T}}$.

II. Problem Formulation and Preliminaries

Consider the following linear system with a time-varying delay:

$$\begin{cases} \dot{x}(t) = Ax(t) + A_d x(t - d(t)), & t \geq 0 \\ x(t) = \phi(t), & t \in [-h, 0] \end{cases} \tag{1}$$

where $x(t) \in \mathcal{R}^n$ is the state vector; A and A_d are the known real constant matrices; $\phi(t)$ is the initial function; and $d(t)$ is the time-varying delay satisfying

$$0 \leq d(t) \leq h, \ \ |\dot{d}(t)| \leq \mu \tag{2}$$

with h and μ being constants.

This paper aims to develop improved delay-dependent stability criteria for analyzing the influence of the time-varying delay on the stability of system (1).

Remark 1: In order to judge the stability of system with the time delay satisfying (2), the criteria obtained are required to hold for all $(d(t), \dot{d}(t)) \in \mathcal{D} = \{d(t) \in [0, h], \dot{d}(t) \in [-\mu, \mu]\}$, which is impossible due to the time variation of $d(t)$ and $\dot{d}(t)$. In most literature, by using the convex combination lemma, the above requirement for all $(d(t), \dot{d}(t)) \in \mathcal{D}$ is modified to the requirement for the following polyhedral allowable delay set:

$$\mathcal{H}_1 : \left(d(t), \dot{d}(t)\right) \in \{(0, -\mu), (0, \mu), (h, -\mu), (h, \mu)\}. \tag{3}$$

It can be seen that the above transformation is reasonable for all time-varying delays satisfying (2). Currently, it was pointed out in [43] that $\dot{d}(t)$ would not be negative (or positive) when $d(t) = 0$ (or $d(t) = h$), which means the above transformation is too strict. Then, the allowable delay set $\mathcal{H}_1$ is relaxed as the following one:

$$\mathcal{H}_2 : \left(d(t), \dot{d}(t)\right) \in \{(0, 0), (0, \mu), (h, -\mu), (h, 0)\}. \tag{4}$$

It should be noted that it is just reasonable for some special time-varying delays described in [43]. For full comparisons, both allowable delay sets, $\mathcal{H}_1$ and $\mathcal{H}_2$, will be discussed in this paper.

The lemmas required to obtain main results of this paper are given as follows.

Lemma 1 [11], [18], [21]: For symmetric matrix $R > 0$, scalars a and b with $a < b$, and a differentiable signal x in $[a, b] \to \mathcal{R}^n$, if the integrals concerned are well defined, then the following inequalities hold:

$$\begin{cases} (b-a)\int_a^b \dot{x}^{\mathrm{T}}(s)R\dot{x}(s)ds \geq \chi_1^{\mathrm{T}} R \chi_1 \\ (b-a)\int_a^b x^{\mathrm{T}}(s)Rx(s)ds \geq \hat{\chi}_1^{\mathrm{T}} R \hat{\chi}_1 \end{cases} \tag{5}$$

$$\begin{cases} (b-a)\int_a^b \dot{x}^{\mathrm{T}}(s)R\dot{x}(s)ds \geq \chi_1^{\mathrm{T}} R \chi_1 + 3\chi_2^{\mathrm{T}} R \chi_2 \\ (b-a)\int_a^b x^{\mathrm{T}}(s)Rx(s)ds \geq \hat{\chi}_1^{\mathrm{T}} R \hat{\chi}_1 + 3\hat{\chi}_2^{\mathrm{T}} R \hat{\chi}_2 \end{cases} \tag{6}$$

$$(b-a)\int_a^b \dot{x}^{\mathrm{T}}(s)R\dot{x}(s)ds \geq \chi_1^{\mathrm{T}} R \chi_1 + 3\chi_2^{\mathrm{T}} R \chi_2 + 5\chi_3^{\mathrm{T}} R \chi_3 \tag{7}$$

where $\chi_1 = x(b) - x(a)$, $\chi_2 = x(b) + x(a) - [2/(b-a)]\int_a^b x(s)ds$, $\hat{\chi}_1 = \int_a^b x(s)ds$, $\hat{\chi}_2 = \int_a^b x(s)ds - [2/(b-a)]\int_a^b \int_\theta^b x(s)dsd\theta$, and $\chi_3 = x(b) - x(a) + [6/(b-a)]\int_a^b x(s)ds - (12/[(b-a)^2])\int_a^b \int_\theta^b x(s)dsd\theta$.

Lemma 2: For vectors β_1 and β_2, a real scalar $\alpha \in (0, 1)$, symmetric matrices $R_1 > 0$ and $R_2 > 0$, and any matrices S_1, S_2, and S, the following inequality holds [20]:

$$\begin{bmatrix} \frac{1}{\alpha}R_1 & 0 \\ * & \frac{1}{1-\alpha}R_2 \end{bmatrix} \geq \begin{bmatrix} R_1 + (1-\alpha)T_1 & (1-\alpha)S_1 + \alpha S_2 \\ * & R_2 + \alpha T_2 \end{bmatrix} \tag{8}$$

where $T_1 = R_1 - S_2 R_2^{-1} S_2^{\mathrm{T}}$ and $T_2 = R_2 - S_1^{\mathrm{T}} R_1^{-1} S_1$. And if the condition $\begin{bmatrix} R_1 & S \\ * & R_2 \end{bmatrix} \geq 0$ is required, the following inequality holds [11]:

$$\frac{1}{\alpha}\beta_1^{\mathrm{T}} R_1 \beta_1 + \frac{1}{1-\alpha}\beta_2^{\mathrm{T}} R_2 \beta_2 \geq \begin{bmatrix} \beta_1 \\ \beta_2 \end{bmatrix}^{\mathrm{T}} \begin{bmatrix} R_1 & S \\ * & R_2 \end{bmatrix} \begin{bmatrix} \beta_1 \\ \beta_2 \end{bmatrix}. \tag{9}$$

Lemma 3 [44]: For a given quadratic function $f(y) = a_2 y^2 + a_1 y + a_0$, where $a_0, a_1, a_2 \in \mathcal{R}$, if the following inequalities hold:

$$(i)\ f(0) < 0; \ \ (ii)\ f(h) < 0; \ \ (iii)\ -h^2 a_2 + f(0) < 0 \tag{10}$$

then $f(y) < 0$ for $\forall y \in [0, h]$.

III. Main Results

In this section, four delay-dependent stability criteria are established by developing two novel LKFs. First, a novel delay-product-type LKF containing two delay-product-type single integral terms is constructed, and two stability criteria are developed based on this LKF for two different allowable delay sets, $\mathcal{H}_1$ and $\mathcal{H}_2$, respectively. Then, this LKF is further improved based on the idea of the matrix-refined-function-based LKF, and two criteria are also developed for allowable delay sets $\mathcal{H}_1$ and $\mathcal{H}_2$, respectively.

A. Stability Criteria Based on New Delay-Product-Type LKF

For simplification, several related vectors are given below before deriving main stability criteria

$$\xi(t) = \big[x^{\mathrm{T}}(t), x^{\mathrm{T}}(t-d(t)), x^{\mathrm{T}}(t-h), \dot{x}^{\mathrm{T}}(t-d(t)), \dot{x}^{\mathrm{T}}(t-h), v_6^{\mathrm{T}}(t), v_7^{\mathrm{T}}(t), v_8^{\mathrm{T}}(t), v_9^{\mathrm{T}}(t)\big]^{\mathrm{T}} \tag{11}$$

$$\xi_1(s,t) = \left[x^{\mathrm{T}}(s), \dot{x}^{\mathrm{T}}(s), x^{\mathrm{T}}(t), x^{\mathrm{T}}(t-d(t)), x^{\mathrm{T}}(t-h), \int_s^t x(u)du\right]^{\mathrm{T}} \tag{12}$$

$$\xi_2(s,t) = \left[x^{\mathrm{T}}(s), \dot{x}^{\mathrm{T}}(s), x^{\mathrm{T}}(t), x^{\mathrm{T}}(t-d(t)), x^{\mathrm{T}}(t-h), \int_s^{t-d(t)} x(u)du\right]^{\mathrm{T}} \tag{13}$$

$$v_6(t) = \int_{t-d(t)}^{t} \frac{x(s)}{d(t)} ds, \quad v_7(t) = \int_{t-h}^{t-d(t)} \frac{x(s)}{h-d(t)} ds \tag{14}$$

$$v_8(t) = \int_{t-d(t)}^{t} \int_s^t \frac{x(u)}{d^2(t)} du ds \tag{15}$$

$$v_9(t) = \int_{t-h}^{t-d(t)} \int_s^{t-d(t)} \frac{x(u)}{(h-d(t))^2} du ds \tag{16}$$

$$e_i = \big[0_{n\times(i-1)n} \quad I_n \quad 0_{n\times(9-i)n}\big], \ i = 1, 2, \ldots, 9 \tag{17}$$

$$e_s = Ae_1 + A_d e_2. \tag{18}$$

First, by extending the idea of delay-product-type nonintegral terms, several delay-product-type single integral terms are developed, which together with double integral terms are expressed in Proposition 1. Those terms will be introduced for constructing the LKF candidate.

Proposition 1: For positive-definite matrices Z_2, Z_4, R_1, R_2, M_2, M_4 N_1, and N_2, and any symmetric matrices Z_1, Z_3, M_1, and M_3 satisfying $hZ_i + Z_{i+1} > 0$ and $hM_i + M_{i+1} > 0$, $i = 1, 3$, the following functions are positive definite:

$$\begin{aligned} V_a(t) &= \int_{t-d(t)}^{t} \dot{x}^{\mathrm{T}}(s)[d(t)Z_1 + Z_2]\dot{x}(s)ds \\ &\quad + \int_{t-h}^{t-d(t)} \dot{x}^{\mathrm{T}}(s)[(h-d(t))Z_3 + Z_4]\dot{x}(s)ds \\ V_b(t) &= \int_{-d(t)}^{0}\int_{t+\theta}^{t} \dot{x}^{\mathrm{T}}(s)R_1\dot{x}(s)dsd\theta \\ &\quad + \int_{-h}^{-d(t)}\int_{t+\theta}^{t} \dot{x}^{\mathrm{T}}(s)R_2\dot{x}(s)dsd\theta \\ V_p(t) &= \int_{t-d(t)}^{t} x^{\mathrm{T}}(s)[d(t)M_1 + M_2]x(s)ds \\ &\quad + \int_{t-h}^{t-d(t)} x^{\mathrm{T}}(s)[(h-d(t))M_3 + M_4]x(s)ds \\ V_q(t) &= \int_{-d(t)}^{0}\int_{t+\theta}^{t} x^{\mathrm{T}}(s)N_1 x(s)dsd\theta \\ &\quad + \int_{-h}^{-d(t)}\int_{t+\theta}^{t} x^{\mathrm{T}}(s)N_2 x(s)dsd\theta \end{aligned}$$

where the related vectors and scalars are defined in system (1). Let $V_{ab}(t) = V_a(t) + V_b(t)$ and $V_{pq}(t) = V_p(t) + V_q(t)$.

Proof: The positive definiteness of $V_a(t)$, $V_b(t)$, $V_p(t)$, and $V_q(t)$ is obvious, so the proof is omitted. ■

In order to clearly show the advantages of the proposed delay-product-type terms, the time derivatives of $V_b(t)$, $V_a(t)$, $V_p(t)$, and $V_q(t)$ are given as follows:

$$\begin{aligned} \dot{V}_a(t) &= \dot{d}(t)\int_{t-d(t)}^{t} \dot{x}^{\mathrm{T}}(s)Z_1\dot{x}(s)ds + \dot{x}^{\mathrm{T}}(t)[d(t)Z_1 + Z_2]\dot{x}(t) \\ &\quad - \big(1-\dot{d}(t)\big)\dot{x}^{\mathrm{T}}(t-d(t))[d(t)Z_1 + Z_2]\dot{x}(t-d(t)) \\ &\quad - \dot{d}(t)\int_{t-h}^{t-d(t)} \dot{x}^{\mathrm{T}}(s)Z_3\dot{x}(s)ds + \big(1-\dot{d}(t)\big)\dot{x}^{\mathrm{T}}(t-d(t)) \\ &\quad \times [(h-d(t))Z_3 + Z_4]\dot{x}(t-d(t)) \\ &\quad - \dot{x}^{\mathrm{T}}(t-h)[(h-d(t))Z_3 + Z_4]\dot{x}(t-h) \end{aligned} \tag{19}$$

$$\begin{aligned} \dot{V}_b(t) &= d(t)\dot{x}^{\mathrm{T}}(t)R_1\dot{x}(t) - \big(1-\dot{d}(t)\big)\int_{t-d(t)}^{t} \dot{x}^{\mathrm{T}}(s)R_1\dot{x}(s)ds \\ &\quad + (h-d(t))\dot{x}^{\mathrm{T}}(t)R_2\dot{x}(t) - \dot{d}(t)\int_{t-d(t)}^{t} \dot{x}^{\mathrm{T}}(s)R_2\dot{x}(s)ds \\ &\quad - \int_{t-h}^{t-d(t)} \dot{x}^{\mathrm{T}}(s)R_2\dot{x}(s)ds \end{aligned} \tag{20}$$

$$\begin{aligned} \dot{V}_p(t) &= \dot{d}(t)\int_{t-d(t)}^{t} x^{\mathrm{T}}(s)M_1 x(s)ds + x^{\mathrm{T}}(t)[d(t)M_1 + M_2]x(t) \\ &\quad - \big(1-\dot{d}(t)\big)x^{\mathrm{T}}(t-d(t))[d(t)M_1 + M_2]x(t-d(t)) \\ &\quad - \dot{d}(t)\int_{t-h}^{t-d(t)} x^{\mathrm{T}}(s)M_3 x(s)ds + \big(1-\dot{d}(t)\big) \\ &\quad \times x^{\mathrm{T}}(t-d(t))[(h-d(t))M_3 + M_4]x(t-d(t)) \\ &\quad - x^{\mathrm{T}}(t-h)[(h-d(t))M_3 + M_4]x(t-h) \end{aligned} \tag{21}$$

$$\dot{V}_q(t) = d(t)x^{\mathrm{T}}(t)N_1x(t) - \big(1-\dot{d}(t)\big)\int_{t-d(t)}^{t} x^{\mathrm{T}}(s)N_1x(s)ds + (h-d(t))x^{\mathrm{T}}(t)N_2x(t) - \dot{d}(t)\int_{t-d(t)}^{t} x^{\mathrm{T}}(s)N_2x(s)ds - \int_{t-h}^{t-d(t)} x^{\mathrm{T}}(s)R_2x(s)ds. \tag{22}$$

Based on (19) and (20), the derivative of $V_{ab}(t)$ can be derived

$$\begin{aligned}\dot{V}_{ab}(t) &= \dot{V}_a(t) + \dot{V}_b(t)\\ &= \dot{x}^{\mathrm{T}}(t)[d(t)R_1 + (h-d(t))R_2 + d(t)Z_1 + Z_2]\dot{x}(t)\\ &\quad + \big(1-\dot{d}(t)\big)\dot{x}^{\mathrm{T}}(t-d(t))\\ &\quad \times [(h-d(t))Z_3 + Z_4 - d(t)Z_1 - Z_2]\dot{x}(t-d(t))\\ &\quad - \dot{x}^{\mathrm{T}}(t-h)[(h-d(t))Z_3 + Z_4]\dot{x}(t-h)\\ &\quad - \int_{t-d(t)}^{t} \dot{x}^{\mathrm{T}}(s)T_1\big(\dot{d}(t)\big)\dot{x}(s)ds\\ &\quad - \int_{t-h}^{t-d(t)} \dot{x}^{\mathrm{T}}(s)T_2\big(\dot{d}(t)\big)\dot{x}(s)ds \end{aligned}\tag{23}$$

where

$$T_1\big(\dot{d}(t)\big) = \big(1-\dot{d}(t)\big)R_1 + \dot{d}(t)R_2 - \dot{d}(t)Z_1 \tag{24}$$

$$T_2\big(\dot{d}(t)\big) = R_2 + \dot{d}(t)Z_3. \tag{25}$$

Then, based on (21) and (22), the derivative of $V_{pq}(t)$ can be derived

$$\begin{aligned}\dot{V}_{pq}(t) &= \dot{V}_p(t) + \dot{V}_q(t)\\ &= x^{\mathrm{T}}(t)[d(t)N_1 + (h-d(t))N_2 + d(t)M_1 + M_2]x(t)\\ &\quad + \big(1-\dot{d}(t)\big)x^{\mathrm{T}}(t-d(t))\\ &\quad \times [(h-d(t))M_3 + M_4 - d(t)M_1 - M_2]x(t-d(t))\\ &\quad - x^{\mathrm{T}}(t-h)[(h-d(t))M_3 + M_4]x(t-h)\\ &\quad - \int_{t-d(t)}^{t} x^{\mathrm{T}}(s)T_3\big(\dot{d}(t)\big)x(s)ds\\ &\quad - \int_{t-h}^{t-d(t)} x^{\mathrm{T}}(s)T_4\big(\dot{d}(t)\big)x(s)ds \end{aligned}\tag{26}$$

where

$$T_3\big(\dot{d}(t)\big) = \big(1-\dot{d}(t)\big)N_1 + \dot{d}(t)N_2 - \dot{d}(t)M_1 \tag{27}$$

$$T_4\big(\dot{d}(t)\big) = N_2 + \dot{d}(t)M_3. \tag{28}$$

Remark 2: In [38], a delay-product-type quadratic term was proposed as follows:

$$V_Z(t) = d(t)\zeta_2^{\mathrm{T}}(t)P_1\zeta_2(t) + [h-d(t)]\zeta_3^{\mathrm{T}}(t)P_2\zeta_3(t)$$

where $\zeta_2(t) = [x^{\mathrm{T}}(t), x^{\mathrm{T}}(t-d(t)), v_6^{\mathrm{T}}(t)]^{\mathrm{T}}$ and $\zeta_3(t) = [x^{\mathrm{T}}(t), x^{\mathrm{T}}(t-d(t)), v_7^{\mathrm{T}}(t)]^{\mathrm{T}}$ with $v_6(t)$ and $v_7(t)$ being defined in (14). The derivative of $V_Z(t)$ is given as

$$\begin{aligned}\dot{V}_Z(t) &= \dot{d}(t)\zeta_2^{\mathrm{T}}(t)P_1\zeta_2(t) + 2d(t)\zeta_2^{\mathrm{T}}(t)P_1\dot{\zeta}_2(t)\\ &\quad - \dot{d}(t)\zeta_3^{\mathrm{T}}(t)P_2\zeta_3(t) + 2[h-d(t)]\zeta_3^{\mathrm{T}}(t)P_2\dot{\zeta}_3(t). \end{aligned}\tag{29}$$

As mentioned in [38], [41], and [42], the $d(t)$- and $\dot{d}(t)$-dependent terms above can well reflect the information of time delays so as to reduce the conservatism. In fact, it can be shown that the proposed delay-product-type integral terms, $V_a(t)$ and $V_p(t)$, will also introduce several similar terms into the derivative of the LKF candidate. In (19) and (21), except for some $d(t)$- and $\dot{d}(t)$-related cross terms, several delay-variation-related integral terms, such as $\dot{d}(t)\int_{t-d(t)}^{t}\dot{x}^{\mathrm{T}}(s)Z_1\dot{x}(s)ds$ and $-\dot{d}(t)\int_{t-h}^{t-d(t)}\dot{x}^{\mathrm{T}}(s)Z_2\dot{x}(s)ds$, are also produced by $V_a(t)$ and $V_p(t)$. In comparison to $V_Z(t)$, these $\dot{d}(t)$-related integral terms are newly introduced. And then, the estimation of the derivatives of $V_a(t)$ and $V_p(t)$ would lead to new and more $d(t)$- and $\dot{d}(t)$-dependent terms, but employs less decision variables than $V_Z(t)$ dose, which helps to reduce the conservatism and the computation complexity.

Remark 3: In Proposition 1, besides delay-product-type integral terms, several double integral terms, $V_b(t)$ and $V_q(t)$, are also given. On the one hand, by assuming $T_i(\dot{d}(t)) \geq 0, i = 1,2,3,4$, those terms are introduced for handling the delay-variation-related integral terms, such as $\dot{d}(t)\int_{t-d(t)}^{t}\dot{x}^{\mathrm{T}}(s)Z_1\dot{x}(s)ds$ and $-\dot{d}(t)\int_{t-h}^{t-d(t)}\dot{x}^{\mathrm{T}}(s)Z_2\dot{x}(s)ds$, which cannot be estimated directly by integral inequalities due to the existence of $\dot{d}(t)$. On the other hand, from (23) and (26), the estimation of those $T_i(\dot{d}(t))$-related integral terms via integral inequalities will further lead to many $\dot{d}(t)$-related cross terms.

By introducing the terms in Proposition 1 to construct the LKF candidate, stability criteria of system (1) can be established as follows.

Theorem 1: For given scalars h and μ, system (1) with a time-varying delay satisfying (2), under polyhedral allowable delay sets $\mathcal{H}_1$ (or $\mathcal{H}_2$), is asymptotically stable if the following condition C1 (or C2) holds

C1: If there exist $6n \times 6n$ matrices $Q_1 > 0, Q_2 > 0$, $n \times n$ matrices $Z_1, Z_2 > 0, Z_3, Z_4 > 0, R_1 > 0, R_2 > 0, M_1, M_2 > 0, M_3, M_4 > 0, N_1 > 0, N_2 > 0$, and $3n \times 3n$ matrices S_1 and S_2, the following LMIs hold:

$$\begin{bmatrix}\Psi_1(0,-\mu) & E_1^{\mathrm{T}}S_2\\ * & -h\hat{T}_2(-\mu)\end{bmatrix} < 0 \tag{30}$$

$$\begin{bmatrix}\Psi_1(0,\mu) & E_1^{\mathrm{T}}S_2\\ * & -h\hat{T}_2(\mu)\end{bmatrix} < 0 \tag{31}$$

$$\begin{bmatrix}\Psi_1(h,-\mu) & E_2^{\mathrm{T}}S_1^{\mathrm{T}}\\ * & -h\hat{T}_1(-\mu)\end{bmatrix} < 0 \tag{32}$$

$$\begin{bmatrix}\Psi_1(h,\mu) & E_2^{\mathrm{T}}S_1^{\mathrm{T}}\\ * & -h\hat{T}_1(\mu)\end{bmatrix} < 0 \tag{33}$$

$$\begin{bmatrix}\Psi_1(0,-\mu) - h^2\Gamma_0(-\mu) & E_1^{\mathrm{T}}S_2\\ * & -h\hat{T}_2(-\mu)\end{bmatrix} < 0 \tag{34}$$

$$\begin{bmatrix}\Psi_1(0,\mu) - h^2\Gamma_0(\mu) & E_1^{\mathrm{T}}S_2\\ * & -h\hat{T}_2(\mu)\end{bmatrix} < 0 \tag{35}$$

$$hZ_i + Z_{i+1} > 0,\ hM_i + M_{i+1} > 0,\ i = 1,3 \tag{36}$$

$$T_j(\mu) \geq 0, T_j(-\mu) \geq 0,\ j = 1,2,\ldots,4. \tag{37}$$

C2: If there exist $6n \times 6n$ matrices $Q_1 > 0, Q_2 > 0$, $n \times n$ matrices $Z_1, Z_2 > 0, Z_3, Z_4 > 0, R_1 > 0, R_2 > 0, M_1, M_2 > 0, M_3, M_4 > 0, N_1 > 0, N_2 > 0$, and $3n \times 3n$ matrices S_1 and S_2, the following LMIs hold:

$$\begin{bmatrix}\Psi_1(0,0) & E_1^{\mathrm{T}}S_2\\ * & -h\hat{T}_2(0)\end{bmatrix} < 0 \tag{38}$$

$$\begin{bmatrix}\Psi_1(0,\mu) & E_1^{\mathrm{T}}S_2\\ * & -h\hat{T}_2(\mu)\end{bmatrix} < 0 \tag{39}$$

$$\begin{bmatrix}\Psi_1(h,-\mu) & E_2^{\mathrm{T}}S_1^{\mathrm{T}}\\ * & -h\hat{T}_1(-\mu)\end{bmatrix} < 0 \tag{40}$$

$$\begin{bmatrix}\Psi_1(h,0) & E_2^{\mathrm{T}}S_1^{\mathrm{T}}\\ * & -h\hat{T}_1(0)\end{bmatrix} < 0 \tag{41}$$

$$\begin{bmatrix} \Psi_1(0,0) - h^2\Gamma_0(0) & E_1^{\mathrm{T}} S_2 \\ * & -h\hat{T}_2(0) \end{bmatrix} < 0 \quad (42)$$

$$\begin{bmatrix} \Psi_1(0,\mu) - h^2\Gamma_0(\mu) & E_1^{\mathrm{T}} S_2 \\ * & -h\hat{T}_2(\mu) \end{bmatrix} < 0 \quad (43)$$

$$hZ_i + Z_{i+1} > 0,\ \ hM_i + M_{i+1} > 0,\ \ i = 1, 3 \quad (44)$$

$$T_j(\mu) \geq 0, T_j(-\mu) \geq 0,\ \ j = 1, 2, \ldots, 4 \quad (45)$$

where

$$\Psi_1\big(d(t), \dot{d}(t)\big) = \sum_{i=1}^{3} \Phi_i\big(d(t), \dot{d}(t)\big) \quad (46)$$

$$\begin{aligned}\Phi_1\big(d(t), \dot{d}(t)\big) &= F_1^{\mathrm{T}} Q_1 F_1 - \big(1 - \dot{d(t)}\big) F_2^{\mathrm{T}} Q_1 F_2 - F_4^{\mathrm{T}} Q_2 F_4 \\ &\quad + \big(1 - \dot{d(t)}\big) F_3^{\mathrm{T}} Q_2 F_3 + \mathrm{Sym}\big\{F_5^{\mathrm{T}} Q_1 F_6 + F_7^{\mathrm{T}} Q_2 F_8\big\}\end{aligned} \quad (47)$$

$$F_1 = \big[e_1^{\mathrm{T}}, e_s^{\mathrm{T}}, e_1^{\mathrm{T}}, e_2^{\mathrm{T}}, e_3^{\mathrm{T}}, 0\big]^{\mathrm{T}}$$

$$F_2 = \big[e_2^{\mathrm{T}}, e_4^{\mathrm{T}}, e_1^{\mathrm{T}}, e_2^{\mathrm{T}}, e_3^{\mathrm{T}}, d(t)e_6^{\mathrm{T}}\big]^{\mathrm{T}} \quad (48)$$

$$F_3 = \big[e_2^{\mathrm{T}}, e_4^{\mathrm{T}}, e_1^{\mathrm{T}}, e_2^{\mathrm{T}}, e_3^{\mathrm{T}}, 0\big]^{\mathrm{T}} \quad (49)$$

$$F_4 = \big[e_3^{\mathrm{T}}, e_5^{\mathrm{T}}, e_1^{\mathrm{T}}, e_2^{\mathrm{T}}, e_3^{\mathrm{T}}, (h - d(t))e_7^{\mathrm{T}}\big]^{\mathrm{T}} \quad (50)$$

$$F_5 = \Big[d(t)e_6^{\mathrm{T}}, e_1^{\mathrm{T}} - e_2^{\mathrm{T}}, d(t)e_1^{\mathrm{T}},\ \ d(t)e_2^{\mathrm{T}}, d(t)e_3^{\mathrm{T}}, d^2(t)e_8^{\mathrm{T}}\Big]^{\mathrm{T}} \quad (51)$$

$$F_6 = \big[0, 0, e_s^{\mathrm{T}}, \big(1 - \dot{d(t)}\big)e_4^{\mathrm{T}}, e_5^{\mathrm{T}}, e_1^{\mathrm{T}}\big]^{\mathrm{T}} \quad (52)$$

$$\begin{aligned}F_7 = \Big[&(h - d(t))e_7^{\mathrm{T}}, e_2^{\mathrm{T}} - e_3^{\mathrm{T}}, (h - d(t))e_1^{\mathrm{T}}, \\ &(h - d(t))e_2^{\mathrm{T}}, (h - d(t))e_3^{\mathrm{T}}, (h - d(t))^2 e_9^{\mathrm{T}}\Big]^{\mathrm{T}}\end{aligned} \quad (53)$$

$$F_8 = \big[0, 0, e_s^{\mathrm{T}}, \big(1 - \dot{d(t)}\big)e_4^{\mathrm{T}}, e_5^{\mathrm{T}}, \big(1 - \dot{d(t)}\big)e_2^{\mathrm{T}}\big]^{\mathrm{T}} \quad (54)$$

$$\begin{aligned}\Phi_2\big(d(t), \dot{d}(t)\big) &= \Phi_{2a}\big(d(t), \dot{d}(t)\big) - \frac{1}{h}\begin{bmatrix} E_1 \\ E_2 \end{bmatrix}^{\mathrm{T}} \\ &\quad \times \begin{bmatrix} \frac{2h - d(t)}{h}\hat{T}_1 & \frac{h - d(t)}{h}S_1 + \frac{d(t)}{h}S_2 \\ * & \frac{h + d(t)}{h}\hat{T}_2 \end{bmatrix} \begin{bmatrix} E_1 \\ E_2 \end{bmatrix}\end{aligned} \quad (55)$$

$$\begin{aligned}\Phi_{2a}\big(d(t), \dot{d}(t)\big) &= e_s^{\mathrm{T}}[d(t)R_1 + (h - d(t))R_2 + d(t)Z_1 + Z_2]e_s \\ &\quad + \big(1 - \dot{d}(t)\big)e_4^{\mathrm{T}}[(h - d(t))Z_3 + Z_4 - d(t)Z_1 - Z_2]e_4 \\ &\quad - e_5^{\mathrm{T}}[(h - d(t))Z_3 + Z_4]e_5\end{aligned} \quad (56)$$

$$\hat{T}_1 = \begin{bmatrix} T_1 & 0 & 0 \\ 0 & 3T_1 & 0 \\ 0 & 0 & 5T_1 \end{bmatrix},\ \hat{T}_2 = \begin{bmatrix} T_2 & 0 & 0 \\ 0 & 3T_2 & 0 \\ 0 & 0 & 5T_2 \end{bmatrix} \quad (57)$$

$$T_1\big(\dot{d}(t)\big) = \big(1 - \dot{d}(t)\big)R_1 + \dot{d}(t)R_2 - \dot{d}(t)Z_1 \quad (58)$$

$$T_2\big(\dot{d}(t)\big) = R_2 + \dot{d}(t)Z_3 \quad (59)$$

$$E_1 = \begin{bmatrix} e_1 - e_2 \\ e_1 + e_2 - 2e_6 \\ e_1 - e_2 + 6e_6 - 12e_8 \end{bmatrix},\ E_2 = \begin{bmatrix} e_2 - e_3 \\ e_2 + e_3 - 2e_7 \\ e_2 - e_3 + 6e_7 - 12e_9 \end{bmatrix} \quad (60)$$

$$\Phi_3\big(d(t), \dot{d}(t)\big) = \Phi_{3a}\big(d(t), \dot{d}(t)\big) - d(t)E_3^{\mathrm{T}}\hat{T}_3 E_3 - (h - d(t))E_4^{\mathrm{T}}\hat{T}_4 E_4 \quad (61)$$

$$\begin{aligned}\Phi_{3a}\big(d(t), \dot{d}(t)\big) &= e_1^{\mathrm{T}}[d(t)N_1 + (h - d(t))N_2 + d(t)M_1 + M_2]e_1 \\ &\quad + \big(1 - \dot{d}(t)\big)e_2^{\mathrm{T}} \\ &\quad \times [(h - d(t))M_3 + M_4 - d(t)M_1 - M_2]e_2 \\ &\quad - e_3^{\mathrm{T}}[(h - d(t))M_3 + M_4]e_3\end{aligned} \quad (62)$$

$$\hat{T}_3 = \begin{bmatrix} T_3 & 0 \\ 0 & 3T_3 \end{bmatrix},\ \hat{T}_4 = \begin{bmatrix} T_4 & 0 \\ 0 & 3T_4 \end{bmatrix} \quad (63)$$

$$T_3\big(\dot{d}(t)\big) = \big(1 - \dot{d}(t)\big)N_1 + \dot{d}(t)N_2 - \dot{d}(t)M_1 \quad (64)$$

$$T_4\big(\dot{d}(t)\big) = N_2 + \dot{d}(t)M_3 \quad (65)$$

$$E_3 = \begin{bmatrix} e_6 \\ e_6 - 2e_8 \end{bmatrix},\ E_4 = \begin{bmatrix} e_7 \\ e_7 - 2e_9 \end{bmatrix} \quad (66)$$

$$\begin{aligned}\Gamma_0 &= -\big(1 - \dot{d(t)}\big)\hat{F}_2^{\mathrm{T}} Q_1 \hat{F}_2 - \hat{F}_4^{\mathrm{T}} Q_2 \hat{F}_4 \\ &\quad + \mathrm{Sym}\Big\{\hat{F}_5^{\mathrm{T}} Q_1 F_6 + \hat{F}_7^{\mathrm{T}} Q_2 F_8\Big\}\end{aligned} \quad (67)$$

$$\hat{F}_2 = \big[0, 0, 0, 0, 0, e_6^{\mathrm{T}}\big]^{\mathrm{T}} \quad (68)$$

$$\hat{F}_4 = \big[0, 0, 0, 0, 0, -e_7^{\mathrm{T}}\big]^{\mathrm{T}} \quad (69)$$

$$\hat{F}_5 = \big[0, 0, 0, 0, 0, e_8^{\mathrm{T}}\big]^{\mathrm{T}} \quad (70)$$

$$\hat{F}_7 = \big[0, 0, 0, 0, 0, e_9^{\mathrm{T}}\big]^{\mathrm{T}}. \quad (71)$$

Proof: Consider the following LKF candidate:

$$V_1(t) = V_0(t) + V_{ab}(t) + V_{pq}(t) \quad (72)$$

where $V_{ab}(t)$ and $V_{pq}(t)$ are defined in Proposition 1 to be positive definite, and

$$V_0(t) = \int_{t-d(t)}^{t} \xi_1^{\mathrm{T}}(s,t) Q_1 \xi_1(s,t) ds + \int_{t-h}^{t-d(t)} \xi_2^{\mathrm{T}}(s,t) Q_2 \xi_2(s,t) ds$$

with Q_1, Q_2 being positive-definite matrices. It can be found that the above LKF candidate satisfies $V_1(t) \geq \epsilon_1 \|x(t)\|^2$ for a sufficient small scalar $\epsilon_1 > 0$.

$V_0(t)$ is similar to the augmented terms in [27] and [29]. Through the same deriving procedures of proof of [27] and [29], the derivative of $V_Q(t)$ can be obtained

$$\dot{V}_0(t) = \xi^{\mathrm{T}}(t)\Phi_1\big(d(t), \dot{d}(t)\big)\xi(t) \quad (73)$$

where $\Phi_1(d(t), \dot{d}(t))$ is defined in (47).

Based on (23), the derivative of $V_{ab}(t)$ can be derived as

$$\begin{aligned}\dot{V}_{ab}(t) &= \xi^{\mathrm{T}}(t)\Phi_{2a}\big(d(t), \dot{d}(t)\big)\xi(t) - \int_{t-d(t)}^{t} \dot{x}^{\mathrm{T}}(s) T_1\big(\dot{d}(t)\big)\dot{x}(s) ds \\ &\quad - \int_{t-h}^{t-d(t)} \dot{x}^{\mathrm{T}}(s) T_2\big(\dot{d}(t)\big)\dot{x}(s) ds\end{aligned} \quad (74)$$

where $\Phi_{2a}(d(t), \dot{d}(t))$, $T_1(\dot{d}(t))$, and $T_2(\dot{d}(t))$ are defined in (56), (58), and (59), respectively.

Based on (26), the derivative of $V_{pq}(t)$ can be derived as

$$\begin{aligned}\dot{V}_{pq}(t) &= \xi^{\mathrm{T}}(t)\Phi_{3a}\big(d(t), \dot{d}(t)\big)\xi(t) - \int_{t-d(t)}^{t} x^{\mathrm{T}}(s) T_3\big(\dot{d}(t)\big)x(s) ds \\ &\quad - \int_{t-h}^{t-d(t)} x^{\mathrm{T}}(s) T_4\big(\dot{d}(t)\big)x(s) ds\end{aligned} \quad (75)$$

where $\Phi_{3a}(d(t), \dot{d}(t))$, $T_3(\dot{d}(t))$, and $T_4(\dot{d}(t))$ are defined in (62), (64), and (65), respectively.

Based on the convex combination technique, (37) or (45) ensures the positive definiteness of $T_1(\dot{d}(t))$ and $T_2(\dot{d}(t))$, then applying inequality (7) to estimate $T_1(\dot{d}(t))$ and $T_2(\dot{d}(t))$ dependent terms in (74) yields

$$\begin{aligned}&- \int_{t-d(t)}^{t} \dot{x}^{\mathrm{T}}(s) T_1(\dot{d}(t))\dot{x}(s) ds - \int_{t-h}^{t-d(t)} \dot{x}^{\mathrm{T}}(s) T_2\big(\dot{d}(t)\big)\dot{x}(s) ds \\ &\quad \leq -\xi^{\mathrm{T}}(t)\begin{bmatrix} E_1 \\ E_2 \end{bmatrix}^{\mathrm{T}} \begin{bmatrix} \frac{1}{d(t)}\hat{T}_1 & 0 \\ 0 & \frac{1}{h-d(t)}\hat{T}_2 \end{bmatrix} \begin{bmatrix} E_1 \\ E_2 \end{bmatrix}\xi(t)\end{aligned} \quad (76)$$

where $\hat{T}_i$ and $E_i, i = 1, 2$ are defined in (57) and (60), respectively.

Similarly, the positive definiteness of $T_3(\dot{d}(t))$ and $T_4(\dot{d}(t))$ is guaranteed by (37) or (45), then $T_3(\dot{d}(t))$ and $T_4(\dot{d}(t))$ dependent terms in (75) can be estimated by inequality (6)

$$
\begin{aligned}
&-\int_{t-d(t)}^{t} x^{\mathrm{T}}(s)T_3\big(\dot{d}(t)\big)x(s)ds - \int_{t-h}^{t-d(t)} x^{\mathrm{T}}(s)T_4\big(\dot{d}(t)\big)x(s)ds \\
&\quad \le -\xi^{\mathrm{T}}(t)\begin{bmatrix} E_3 \\ E_4 \end{bmatrix}^{\mathrm{T}}\begin{bmatrix} d(t)\hat{T}_3 & 0 \\ 0 & (h-d(t))\hat{T}_4 \end{bmatrix}\begin{bmatrix} E_3 \\ E_4 \end{bmatrix}\xi(t) \quad (77)
\end{aligned}
$$

where $\hat{T}_i$ and $E_i, i = 3, 4$ are defined in (63) and (66), respectively. Then, by inequality (8), it follows from (76) that:

$$
\begin{aligned}
&-\int_{t-d(t)}^{t} \dot{x}^{\mathrm{T}}(s)T_1\big(\dot{d}(t)\big)\dot{x}(s)ds - \int_{t-h}^{t-d(t)} \dot{x}^{\mathrm{T}}(s)T_2(\dot{d}(t))\dot{x}(s)ds \\
&\quad \le -\frac{1}{h}\xi^{\mathrm{T}}(t)\begin{bmatrix} E_1 \\ E_2 \end{bmatrix}^{\mathrm{T}}\begin{bmatrix} \frac{2h-d(t)}{h}\hat{T}_1 & \frac{h-d(t)}{h}S_1 + \frac{d(t)}{h}S_2 \\ * & \frac{h+d(t)}{h}\hat{T}_2 \end{bmatrix}\begin{bmatrix} E_1 \\ E_2 \end{bmatrix}\xi(t) \\
&\qquad + \xi^{\mathrm{T}}(t)\Pi_1\xi(t) \quad (78)
\end{aligned}
$$

where

$$
\Pi_1 = \frac{h-d(t)}{h^2}E_1^{\mathrm{T}}S_2\hat{T}_2^{-1}S_2^{\mathrm{T}}E_1 + \frac{d(t)}{h^2}E_2^{\mathrm{T}}S_1^{\mathrm{T}}\hat{T}_1^{-1}S_1E_2.
$$

It can be checked that the obtained inequalities (77) and (78) hold for all $d(t) \in [0, h]$.

Thus, (74) and (75) can be bounded as

$$
\dot{V}_{ab}(t) \le \xi^{\mathrm{T}}(t)\big(\Phi_2\big(d(t), \dot{d}(t)\big) + \Pi_1\big)\xi(t) \quad (79)
$$

$$
\dot{V}_{pq}(t) \le \xi^{\mathrm{T}}(t)\Phi_3\big(d(t), \dot{d}(t)\big)\xi(t) \quad (80)
$$

where $\Phi_2(d(t), \dot{d}(t))$ and $\Phi_3(d(t), \dot{d}(t))$ are defined in (55) and (61), respectively.

By combining (73), (79), and (80), the $\dot{V}_1(t)$ can be estimated as

$$
\begin{aligned}
\dot{V}_1(t) &\le \xi^{\mathrm{T}}(t)\left(\sum_{i=1}^{3}\Phi_i\big(d(t), \dot{d}(t)\big) + \Pi_1\right)\xi(t) \\
&= \xi^{\mathrm{T}}(t)\tilde{\Psi}_1\big(d(t), \dot{d}(t)\big)\xi(t) \quad (81)
\end{aligned}
$$

where

$$
\tilde{\Psi}_1\big(d(t), \dot{d}(t)\big) = \Psi_1\big(d(t), \dot{d}(t)\big) + \Pi_1 \quad (82)
$$

with $\Psi_1(d(t), \dot{d}(t))$ being defined in (46).

Note that there are some $d^2(t)$-dependent terms in $\tilde{\Psi}_1(d(t), \dot{d}(t))$. In fact, $\tilde{\Psi}_1(d(t), \dot{d}(t))$ can be rewritten as the following form:

$$
\tilde{\Psi}_1\big(d(t), \dot{d}(t)\big) = d^2(t)\Gamma_0\big(\dot{d}(t)\big) + d(t)\Gamma_1\big(\dot{d}(t)\big) + \Gamma_2\big(\dot{d}(t)\big)
$$

where $\Gamma_0(\dot{d}(t))$ is defined in (67), and $\Gamma_i(\dot{d}(t)), i = 1, 2$ are $d(t)$ itself independent symmetric matrices. By Lemma 3, $\tilde{\Psi}_1(d(t), \dot{d}(t)) < 0$ holds for all $d(t) \in [0, h]$ if the following inequalities hold:

$$
\begin{aligned}
&\tilde{\Psi}_1\big(0, \dot{d}(t)\big) < 0, \ \tilde{\Psi}_1\big(h, \dot{d}(t)\big) < 0 \\
&-h^2\Gamma_0\big(\dot{d}(t)\big) + \tilde{\Psi}_1\big(0, \dot{d}(t)\big) < 0.
\end{aligned}
$$

For the first polyhedra allowable delay set $\mathcal{H}_1$, $\tilde{\Psi}_1(d(t), \dot{d}(t)) < 0$ holds for all $(d(t), \dot{d}(t)) \in \mathcal{H} = \{d(t) \in [0, h], \dot{d}(t) \in [-\mu, \mu]\}$ if the following inequalities hold:

$$
\begin{aligned}
&\tilde{\Psi}_1(0, -\mu) < 0, \tilde{\Psi}_1(0, \mu) < 0 \\
&\tilde{\Psi}_1(h, -\mu) < 0, \tilde{\Psi}_1(h, \mu) < 0 \\
&-h^2\Gamma_0(-\mu) + \tilde{\Psi}_1(0, -\mu) < 0, \ -h^2\Gamma_0(\mu) + \tilde{\Psi}_1(0, \mu) < 0.
\end{aligned}
$$

Moreover, the above inequalities can be guaranteed if (30)–(35) hold based on Schur complement. Therefore, if (30)–(37) hold, then for a sufficient small scalar $\epsilon_2 > 0$, $\dot{V}_1(t) \le -\epsilon_2\|x(t)\|^2$ holds, which shows the asymptotical stability of system (1). This completes the proof of Theorem 1.C1.

Similarly, for the second polyhedra allowable delay set $\mathcal{H}_2$, $\tilde{\Psi}_1(d(t), \dot{d}(t)) < 0$ holds for all $(d(t), \dot{d}(t)) \in \mathcal{H} = \{d(t) \in [0, h], \dot{d}(t) \in [-\mu, \mu]\}$ if the following inequalities hold:

$$
\begin{aligned}
&\tilde{\Psi}_1(0, 0) < 0, \tilde{\Psi}_1(0, \mu) < 0 \\
&\tilde{\Psi}_1(h, -\mu) < 0, \tilde{\Psi}_1(h, 0) < 0 \\
&-h^2\Gamma_0(0) + \tilde{\Psi}_1(0, 0) < 0, \ -h^2\Gamma_0(\mu) + \tilde{\Psi}_1(0, \mu) < 0.
\end{aligned}
$$

Moreover, the above inequalities can be guaranteed by (38)–(43) based on Schur complement. Thus, if (38)–(45) hold, then for a sufficient small scalar $\epsilon_2 > 0$, $\dot{V}(t) \le -\epsilon_2\|x(t)\|^2$ holds, which shows the asymptotical stability of system (1). This completes the proof of Theorem 1.C2. ■

B. Stability Criteria Based on the Improved Delay-Product-Type LKF

Inspired by the idea of the matrix-refined-function-based LKF, the delay-product-type LKF (72) is further improved by introducing a negative term, shown as follows:

$$
V_N(t) = -\frac{1}{h}\eta_1^{\mathrm{T}}(t)Z_M\eta_1(t) \quad (83)
$$

where

$$
\begin{aligned}
\eta_1(t) = \big[&x^{\mathrm{T}}(t) - x^{\mathrm{T}}(t-d(t)), d(t)v_6^{\mathrm{T}}(t) \\
&d(t)\big(v_6^{\mathrm{T}}(t) - 2v_8^{\mathrm{T}}(t)\big), x^{\mathrm{T}}(t-d(t)) - x^{\mathrm{T}}(t-h) \\
&(h-d(t))v_7^{\mathrm{T}}(t), (h-d(t))\big(v_7^{\mathrm{T}}(t) - 2v_9^{\mathrm{T}}(t)\big)\big]^{\mathrm{T}} \quad (84)
\end{aligned}
$$

$$
Z_M = hZM_{13} + ZM_{24} \quad (85)
$$

$$
ZM_{13} = \begin{bmatrix} ZM_1 & 0 \\ 0 & ZM_3 \end{bmatrix} \quad (86)
$$

$$
ZM_{24} = \begin{bmatrix} ZM_2 & S_3 \\ * & ZM_4 \end{bmatrix} \ge 0 \quad (87)
$$

$$
ZM_i = \begin{bmatrix} Z_i & 0 & 0 \\ 0 & M_i & 0 \\ 0 & 0 & 3M_i \end{bmatrix}, \quad i = 1, 2, 3, 4 \quad (88)
$$

and other related vectors and scalars are defined in Proposition 1.

In order to satisfy the positive definite requirement of LKF candidate after introducing the above negative term, the negative term is combined with the delay-dependent single integral terms in (72) to obtain positive terms, shown in the following proposition.

Proposition 2: For any symmetric matrices $Z_1, Z_2 > 0, Z_3, Z_4 > 0, M_1, M_2 > 0, M_3, M_4 > 0$, and any matrix S_3 satisfying condition (87), the following function is positive definite:

$$
V_{apN}(t) = V_a(t) + V_p(t) + V_N(t) \quad (89)
$$

where $V_a(t)$ and $V_p(t)$ are defined in (72), and $V_N(t)$ is given in (83).

Proof: By using inequality (5), $V_a(t)$ is estimated as

$$V_a(t) \geq \eta_{11}^{\mathrm{T}}(t)\left[Z_1 + \frac{1}{d(t)}Z_2\right]\eta_{11}(t) + \eta_{14}^{\mathrm{T}}(t)\left[Z_3 + \frac{1}{h-d(t)}Z_4\right]\eta_{14}(t). \tag{90}$$

And by using inequality (6), $V_p(t)$ is estimated as

$$V_p(t) \geq \begin{bmatrix}\eta_{12}\\ \eta_{13}\end{bmatrix}^{\mathrm{T}}\begin{bmatrix} M_1 + \frac{1}{d(t)}M_2 & 0 \\ 0 & 3M_1 + \frac{3}{d(t)}M_2\end{bmatrix}\begin{bmatrix}\eta_{12}\\ \eta_{13}\end{bmatrix} + \begin{bmatrix}\eta_{15}\\ \eta_{16}\end{bmatrix}^{\mathrm{T}}\begin{bmatrix} M_3 + \frac{1}{h-d(t)}M_4 & 0 \\ 0 & 3M_1 + \frac{3}{h-d(t)}M_4\end{bmatrix}\begin{bmatrix}\eta_{15}\\ \eta_{16}\end{bmatrix} \tag{91}$$

where $\eta_1(t) = [\eta_{11}^{\mathrm{T}}, \eta_{12}^{\mathrm{T}}, \ldots, \eta_{16}^{\mathrm{T}}]^{\mathrm{T}}$ and $\eta_1(t)$ is given in (84).

Then, based on (90) and (91), the following holds:

$$V_a(t) + V_p(t) \geq \eta_1^{\mathrm{T}}(t)\begin{bmatrix} ZM_1 & 0 \\ 0 & ZM_3\end{bmatrix}\eta_1(t) + \eta_1^{\mathrm{T}}(t)\begin{bmatrix} \frac{1}{d(t)}ZM_2 & 0 \\ 0 & \frac{1}{h-d(t)}ZM_4\end{bmatrix}\eta_1(t) \tag{92}$$

where $ZM_i,\ i = 1, 2, 3, 4$ are defined in (88).

By employing inequality (9), for any matrix S_3 satisfying $ZM_{24} = \begin{bmatrix} ZM_2 & S_3 \\ * & ZM_4\end{bmatrix} \geq 0$, it follows from (92) that:

$$\begin{aligned} V_a(t) + V_p(t) &\geq \eta_1^{\mathrm{T}}(t)ZM_{13}\eta_1(t) + \frac{1}{h}\eta_1^{\mathrm{T}}(t)ZM_{24}\eta_1(t) \\ &= \frac{1}{h}\eta_1^{\mathrm{T}}(t)Z_M\eta_1(t) \end{aligned} \tag{93}$$

which shows the positive definiteness of $V_{apN}(t)$. This completes the proof. ■

Remark 4: Although the negative quadratic terms $V_N(t)$ are proposed by extending the idea of the matrix-refined-function LKF given in [41], there are differences between $V_N(t)$ and the matrix-refined-function LKF. For easy comparison, the related terms in [41] are given as follows:

$$V_A(t) = \int_{t-d(t)}^{t} \dot{x}^{\mathrm{T}}(s)N\dot{x}(s)ds + \int_{t-h}^{t-d(t)} \dot{x}^{\mathrm{T}}(s)M\dot{x}(s)ds - \frac{1}{h^2}\big(d(t)\eta_3^{\mathrm{T}}(t)N_d\eta_3(t) + (h-d(t))\eta_4^{\mathrm{T}}(t)M_d\eta_4(t)\big)$$

where

$$\eta_3(t) = \begin{bmatrix} x(t) - x(t-d(t)) \\ x(t) + x(t-d(t)) - 2v_6(t)\end{bmatrix}$$

$$\eta_4(t) = \begin{bmatrix} x(t-d(t)) - x(t-h) \\ x(t-d(t)) + x(t-h) - 2v_7(t)\end{bmatrix}$$

$$N_d = \begin{bmatrix} N & 0 \\ 0 & 3N\end{bmatrix} \geq 0,\ M_d = \begin{bmatrix} M & 0 \\ 0 & 3M\end{bmatrix} \geq 0.$$

First, in [41], positive definite $V_A(t)$ is introduced to improve original LKF candidate and it contains both single integral terms and negative quadratic terms, while, in this paper, just several negative quadratic terms in $V_N(t)$ is introduced to improve LKF (72) to reduce the computational complexity, and the positive definiteness of the LKF candidate is guaranteed by combining the added negative terms and the original single integral terms. Second, for guaranteeing the positive definiteness of $V_A(t)$, $(1/d(t))$ and $(1/[h-d(t)])$ are directly enlarged into $(d(t)/h^2)$ and $([h-d(t)]/h^2)$, respectively; while, instead of such direct enlargement, the reciprocally convex combination lemma (9) is used to guarantee the positive definiteness of $V_{apN}(t)$. That is, the conditions of the positive definite requirement are relaxed. Third, due to the usage of reciprocally convex combination lemma, S_3-dependent cross terms are introduced, which can provide extra freedom.

By introducing $V_N(t)$ into $V_1(t)$, a new LKF candidate is constructed, and then stability criteria of system (1) can be established as follows.

Theorem 2: For given scalars h and μ, system (1) with a time-varying delay satisfying (2), under polyhedral allowable delay sets $\mathcal{H}_1$ (or $\mathcal{H}_2$), is asymptotically stable if the following condition C1 (or C2) holds.

C1: If there exist $6n \times 6n$ matrices $Q_1 > 0, Q_2 > 0$, $n \times n$ matrices $Z_1, Z_2 > 0, Z_3, Z_4 > 0, R_1 > 0, R_2 > 0, M_1, M_2 > 0, M_3, M_4 > 0, N_1 > 0, N_2 > 0$, and $3n \times 3n$ matrices S_1, S_2, and S_3, the following LMIs hold:

$$\begin{bmatrix} \Psi_2(0,-\mu) & E_1^{\mathrm{T}}S_2 \\ * & -h\hat{T}_2(-\mu)\end{bmatrix} < 0 \tag{94}$$

$$\begin{bmatrix} \Psi_2(0,\mu) & E_1^{\mathrm{T}}S_2 \\ * & -h\hat{T}_2(\mu)\end{bmatrix} < 0 \tag{95}$$

$$\begin{bmatrix} \Psi_2(h,-\mu) & E_2^{\mathrm{T}}S_1^{\mathrm{T}} \\ * & -h\hat{T}_1(-\mu)\end{bmatrix} < 0 \tag{96}$$

$$\begin{bmatrix} \Psi_2(h,\mu) & E_2^{\mathrm{T}}S_1^{\mathrm{T}} \\ * & -h\hat{T}_1(\mu)\end{bmatrix} < 0 \tag{97}$$

$$\begin{bmatrix} \Psi_2(0,-\mu) - h^2\Gamma_0(-\mu) & E_1^{\mathrm{T}}S_2 \\ * & -h\hat{T}_2(-\mu)\end{bmatrix} < 0 \tag{98}$$

$$\begin{bmatrix} \Psi_2(0,\mu) - h^2\Gamma_0(\mu) & E_1^{\mathrm{T}}S_2 \\ * & -h\hat{T}_2(\mu)\end{bmatrix} < 0 \tag{99}$$

$$hZ_i + Z_{i+1} > 0,\ hM_i + M_{i+1} > 0,\ i = 1, 3 \tag{100}$$

$$T_j(\mu) \geq 0, T_j(-\mu) \geq 0,\ j = 1, 2, \ldots, 4 \tag{101}$$

$$ZM_{24} = \begin{bmatrix} ZM_2 & S_3 \\ * & ZM_4\end{bmatrix} \geq 0. \tag{102}$$

C2: If there exist $6n \times 6n$ matrices $Q_1 > 0, Q_2 > 0$, $n \times n$ matrices $Z_1, Z_2 > 0, Z_3, Z_4 > 0, R_1 > 0, R_2 > 0, M_1, M_2 > 0, M_3, M_4 > 0, N_1 > 0, N_2 > 0$, and $3n \times 3n$ matrices S_1, S_2, and S_3, the following LMIs hold:

$$\begin{bmatrix} \Psi_2(0,0) & E_1^{\mathrm{T}}S_2 \\ * & -h\hat{T}_2(0)\end{bmatrix} < 0 \tag{103}$$

$$\begin{bmatrix} \Psi_2(0,\mu) & E_1^{\mathrm{T}}S_2 \\ * & -h\hat{T}_2(\mu)\end{bmatrix} < 0 \tag{104}$$

$$\begin{bmatrix} \Psi_2(h,-\mu) & E_2^{\mathrm{T}}S_1^{\mathrm{T}} \\ * & -h\hat{T}_1(-\mu)\end{bmatrix} < 0 \tag{105}$$

$$\begin{bmatrix} \Psi_2(h,0) & E_2^{\mathrm{T}}S_1^{\mathrm{T}} \\ * & -h\hat{T}_1(0)\end{bmatrix} < 0 \tag{106}$$

$$\begin{bmatrix} \Psi_2(0,0) - h^2\Gamma_0(0) & E_1^{\mathrm{T}}S_2 \\ * & -h\hat{T}_2(0)\end{bmatrix} < 0 \tag{107}$$

$$\begin{bmatrix} \Psi_2(0,\mu) - h^2\Gamma_0(\mu) & E_1^{\mathrm{T}}S_2 \\ * & -h\hat{T}_2(\mu)\end{bmatrix} < 0 \tag{108}$$

$$hZ_i + Z_{i+1} > 0,\ hM_i + M_{i+1} > 0,\ i = 1, 3 \quad (109)$$

$$T_j(\mu) \geq 0, T_j(-\mu) \geq 0,\ j = 1, 2, \ldots, 4 \quad (110)$$

$$ZM_{24} = \begin{bmatrix} ZM_2 & S_3 \\ * & ZM_4 \end{bmatrix} \geq 0 \quad (111)$$

where

$$\Psi_2\big(d(t), \dot{d}(t)\big) = \sum_{i=1}^{3} \Phi_i\big(d(t), \dot{d}(t)\big) + \Phi_N\big(d(t), \dot{d}(t)\big) \quad (112)$$

$$\Phi_N\big(d(t), \dot{d}(t)\big) = -\mathrm{Sym}\left\{\frac{1}{h} E_5^{\mathrm{T}} Z_M E_6\right\} \quad (113)$$

$$E_5 = \big[e_1^{\mathrm{T}} - e_2^{\mathrm{T}}, d(t)e_6^{\mathrm{T}}, d(t)(e_6 - 2e_8)$$
$$e_2^{\mathrm{T}} - e_3^{\mathrm{T}}, (h - d(t))e_7^{\mathrm{T}}, (h - d(t))\big(e_7^{\mathrm{T}} - 2e_9^{\mathrm{T}}\big)\big]^{\mathrm{T}} \quad (114)$$

$$E_6 = \big[e_s^{\mathrm{T}} - \big(1 - \dot{d}(t)\big)e_4^{\mathrm{T}}, e_1^{\mathrm{T}} - \big(1 - \dot{d}(t)\big)e_2^{\mathrm{T}}$$
$$- e_1^{\mathrm{T}} - \big(1 - \dot{d}(t)\big)\big(e_2^{\mathrm{T}} - 2e_6^{\mathrm{T}}\big) + 2\dot{d}(t)e_8^{\mathrm{T}}$$
$$\big(1 - \dot{d}(t)\big)e_4^{\mathrm{T}} - e_5^{\mathrm{T}}, \big(1 - \dot{d}(t)\big)e_2^{\mathrm{T}} - e_3^{\mathrm{T}}$$
$$- \big(1 - \dot{d}(t)\big)e_2^{\mathrm{T}} - e_3^{\mathrm{T}} + 2e_7^{\mathrm{T}} - 2\dot{d}(t)e_9^{\mathrm{T}}\big]^{\mathrm{T}}. \quad (115)$$

Z_M and ZM_{24} are given in (85) and (87), respectively, and other notations are defined in Theorem 1.

Proof: Consider the following LKF candidate:

$$V_2(t) = V_1(t) + V_N(t) \quad (116)$$

where $V_1(t)$ and $V_N(t)$ are defined in (72) and (83), respectively.

Based on Proposition 2, the sum of the negative definite terms in $V_N(t)$ and the single integral terms in $V_1(t)$ is positive definite if $Z_2 > 0$, $Z_4 > 0$, $M_2 > 0$, $M_4 > 0$, and (100) hold. Moreover, the other terms of $V_1(t)$ are positive definite if $Q_i > 0$, $R_i > 0$, and $N_i > 0$, $i = 1, 2$. Thus, $V_2(t) \geq \epsilon_3 \|x(t)\|^2$ for a sufficient small scalar $\epsilon_3 > 0$.

Taking the time derivative of $V_N(t)$ along the trajectory of system (1) yields

$$\dot{V}_N(t) = -\mathrm{Sym}\left\{\frac{1}{h}\eta_1^{\mathrm{T}}(t) Z_M \dot{\eta}_1(t)\right\}$$
$$= \xi^{\mathrm{T}}(t)\Phi_N\big(d(t), \dot{d}(t)\big)\xi(t) \quad (117)$$

where Z_M is given in (85), and $\Phi_N(d(t), \dot{d}(t))$ is given in (113).

Based on the proof of Theorem 1, if (101) or (110) holds, the derivative of $V_1(t)$ can be estimated as (81). By combining (81) and (117), the derivative of $V_2(t)$ can be estimated as

$$\dot{V}_2(t) = \dot{V}_1(t) + \dot{V}_N(t)$$
$$\leq \xi^{\mathrm{T}}(t)\tilde{\Psi}_2\big(d(t), \dot{d}(t)\big)\xi(t) \quad (118)$$

where

$$\tilde{\Psi}_2\big(d(t), \dot{d}(t)\big) = \tilde{\Psi}_1\big(d(t), \dot{d}(t)\big) + \Phi_N\big(d(t), \dot{d}(t)\big) \quad (119)$$

with $\tilde{\Psi}_1(d(t), \dot{d}(t))$ being given in (81).

Similar to $\dot{V}_1(t)$, some $d^2(t)$-dependent terms also appear in $\tilde{\Psi}_2(d(t), \dot{d}(t))$, and $\tilde{\Psi}_2(d(t), \dot{d}(t))$ can be rewritten as the following form:

$$\tilde{\Psi}_2\big(d(t), \dot{d}(t)\big) = d^2(t)\Gamma_0\big(\dot{d}(t)\big) + d(t)\hat{\Gamma}_1\big(\dot{d}(t)\big) + \hat{\Gamma}_2\big(\dot{d}(t)\big)$$

where $\Gamma_0(\dot{d}(t))$ is defined in (67), and $\hat{\Gamma}_i(\dot{d}(t)), i = 1, 2$ are $d(t)$ itself independent symmetric matrices.

By Lemma 3, $\tilde{\Psi}_2(d(t), \dot{d}(t)) < 0$ holds for all $d(t) \in [0, h]$ if the following inequalities hold:

$$\tilde{\Psi}_2\big(0, \dot{d}(t)\big) < 0,\ \tilde{\Psi}_2\big(h, \dot{d}(t)\big) < 0$$
$$-h^2\Gamma_0\big(\dot{d}(t)\big) + \tilde{\Psi}_2\big(0, \dot{d}(t)\big) < 0.$$

For the first polyhedra allowable delay set $\mathcal{H}_1$, $\tilde{\Psi}_2(d(t), \dot{d}(t)) < 0$ holds for all $(d(t), \dot{d}(t)) \in \mathcal{H} = \{d(t) \in [0, h], \dot{d}(t) \in [-\mu, \mu]\}$ if the following inequalities hold:

$$\tilde{\Psi}_2(0, -\mu) < 0,\ \tilde{\Psi}_2(0, \mu) < 0$$
$$\tilde{\Psi}_2(h, -\mu) < 0,\ \tilde{\Psi}_2(h, \mu) < 0$$
$$-h^2\Gamma_0(-\mu) + \tilde{\Psi}_2(0, -\mu) < 0,\ -h^2\Gamma_0(\mu) + \tilde{\Psi}_2(0, \mu) < 0.$$

Based on Schur complement, the above inequalities can be guaranteed if (94)–(99) hold. Therefore, if (94)–(102) hold, then for a sufficient small scalar $\epsilon_4 > 0$, $\dot{V}_2(t) \leq -\epsilon_4\|x(t)\|^2$ holds, which shows the asymptotical stability of system (1). This completes the proof of Theorem 2.C1.

Similarly, for the second polyhedra allowable delay set $\mathcal{H}_2$, $\tilde{\Psi}_2(d(t), \dot{d}(t)) < 0$ holds for all $(d(t), \dot{d}(t)) \in \mathcal{H} = \{d(t) \in [0, h], \dot{d}(t) \in [-\mu, \mu]\}$ if the following inequalities hold:

$$\tilde{\Psi}_2(0, 0) < 0,\ \tilde{\Psi}_2(0, \mu) < 0$$
$$\tilde{\Psi}_2(h, -\mu) < 0,\ \tilde{\Psi}_2(h, 0) < 0$$
$$-h^2\Gamma_0(0) + \tilde{\Psi}_2(0, 0) < 0,\ -h^2\Gamma_0(\mu) + \tilde{\Psi}_2(0, \mu) < 0.$$

Based on Schur complement, the above inequalities can be guaranteed by (103)–(108). Thus, if (103)–(111) hold, then for a sufficient small scalar $\epsilon_4 > 0$, $\dot{V}_2(t) \leq -\epsilon_4\|x(t)\|^2$ holds, which shows the asymptotical stability of system (1). This completes the proof of Theorem 2.C2. ∎

IV. Numerical Example

In this section, a numerical example is given to illustrate the superiority of the proposed methods. The computed maximal admissible delay upper bounds (MADUBs), which reveal the conservatism of stability criteria, are given for two different allowable delay sets $\mathcal{H}_1$ and $\mathcal{H}_2$, respectively. At the same time, the number of decision variables (NoV) which represents the computational complexity of stability criteria is also summarized.

Example 1: Consider system (1) with the following parameters:

$$A = \begin{bmatrix} -2 & 0 \\ 0 & -0.9 \end{bmatrix},\ A_d = \begin{bmatrix} -1 & 0 \\ -1 & -1 \end{bmatrix}.$$

In Table I, the MADUBs for the polyhedral allowable delay set $\mathcal{H}_1$ with respect to μ computed by Theorems 1.C1 and 2.C1, as well the ones in some latest literatures, are summarized. Based on the listed results, some observations can be made.

1) The MADUBs produced by Theorems 1.C1 and 2.C1 are bigger than the existing results. Furthermore, the NoV of Theorem 1.C1 is less than that of [10], [29], [30], [40], and [42], and Theorem 2.C1 employs less NoV than [30], [40], and [42] do. Thus, the superiority of the proposed LKFs is demonstrated.

TABLE I
MAUBs for Various $\mu = -\mu_1 = \mu_2$ ($\mathcal{H}_1$)

Method	μ				NoVs
	0.1	0.2	0.5	0.8	
Theorem 7 [21]	4.703	3.834	2.420	2.137	$10n^2+3n$
Theorem 1 [10]	4.788	4.060	3.055	2.615	$65n^2+11n$
Theorem 2.C2 [38]	4.809	4.091	3.109	2.710	$25n^2+7n$
Theorem 1 [40]	4.829	4.139	3.155	2.730	$114n^2+18n$
Theorem 3 [42]	4.844	4.141	3.117	2.698	$70n^2+12n$
Theorem 4 [30]	4.906	4.201	3.178	2.715	$131.5n^2+12.5n$
Proposition.1 [27]	4.910	-	3.233	2.789	$54.5n^2+6.5n$
Theorem 2 [29]	4.936	-	3.273	2.848	$67n^2+7.5n$
Theorem 1.C1	4.937	4.256	3.284	2.862	$60n^2+12n$
Theorem 2.C1	4.940	4.262	3.304	2.877	$69n^2+12n$

TABLE II
MAUBs for Various $\mu = -\mu_1 = \mu_2$ ($\mathcal{H}_2$)

Method	μ				
	0	0.1	0.2	0.5	0.8
Theorem 8 $N=1$ [43]	6.168	6.168	6.154	4.83	3.70
Theorem 1.C2	6.168	6.168	6.168	5.692	4.804
Theorem 2.C2	6.168	6.168	6.168	5.752	5.002

2) Except for the proposed delay-product-type terms $V_{ab}(t)$ and $V_{pq}(t)$, the augmented term $V_Q(t)$ and bounding techniques employed by Theorem 1.C1 are similar to these used by [27] and [29], while, Theorem 1.C1 provides bigger MADUBs than [27] and [29] do. Namely, the introduction of delay-product-type terms $V_{ab}(t)$ and $V_{pq}(t)$ can reduce the conservatism.
3) Since the introduction of $V_N(t)$ to $V_2(t)$, Theorem 2.C1 provides further improved MADUBs in comparison to Theorem 1.C1. Moreover, the introduction of $V_N(t)$ is motivated by [40] and [42], but Theorem 2.C1 is less conservative than the criteria of [40] and [42] and requires less NoV than them. Thus, the advantages stated in Remark 4 are verified.

In Table II, the MADUBs for the polyhedral allowable delay set $\mathcal{H}_2$ with respect to μ computed by Theorems 1.C2 and 2.C2, and [43, Th. 8] are shown. First, the proposed Theorems 2.C1 and 2.C2 provide less conservative results than [43, Th. 8] dose. Therefore, the newly proposed methods can provide less conservative results. Besides, it can be found that the results given for $\mathcal{H}_2$ in Table II are greater than the ones for $\mathcal{H}_1$ in Table I, that is, the allowable delay set is also related to the conservatism, thus special attentions should be paid to the allowable delay set during the stability analysis of time-delay systems.

V. Conclusion

This paper has studied the stability of time-delay systems by constructing novel delay-product-type LKFs. First, a new type of LKF containing delay-product-type single integral terms has been proposed, based on which two stability criteria, including more delay information have been developed for two allowable delay sets, $\mathcal{H}_1$ and $\mathcal{H}_2$, respectively. Second, in order to introduce more cross-term information, the delay-product-type LKF has been further improved by introducing some negative definite quadratic terms based on the idea of matrix-refined-function-based LKF, and then two stability criteria have been also established for, $\mathcal{H}_1$ and $\mathcal{H}_2$, respectively. Finally, a numerical example has been given to illustrate the validity of the proposed methods.

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